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FORWARD-LOOKING PAIRWISE STABILITY IN NETWORKS WITH EXTERNALITIES ACROSS COMPONENTS

by

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Forward-looking Pairwise Stability in Networks with Externalities across Components[*]

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Abstract

We consider cooperation situations where players have network relations. Networks evolve according to a stationary transition probability matrix and at each moment in time players receive payoffs from a stationary allocation rule. Players discount the future by a common factor. The pair formed by an allocation rule and a transition probability matrix is called a forward-looking network formation scheme if, first, the probability that a link is created is positive if the discounted, expected gains to its two participants are positive, and if, second, the probability that a link is eliminated is positive if the discounted, expected gains to at least one of its two participants are positive. The main result is the existence, for all discount factors and all value functions, of a forward-looking network formation scheme. Furthermore, we can always find a forward-looking network formation scheme such that (i) the allocation rule is component balanced and (ii) the transition probabilities increase in the difference in payoffs for the corresponding players responsible for the transition. We use this dynamic solution concept to explore the tension between efficiency and stability.

Keywords: Allocation Rules for Networks, Markov Chain, Strategic Network Formation, Pairwise Stability

JEL Classification: C71, C79.

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1 Introduction

When considering networks generating a value, it is commonly understood that this value arises as the result of some economic activity (1) in which individuals are engaged and (2) for which the network serves as a structure or organization. Given that network structures have an impact on economic results and, therefore, on welfare, it is essential for economists to understand processes of network formation undertaken by self-interested individuals, and how this relates to the way the value is distributed among them. The purpose of this paper is to contribute to the development of foundational theoretical models that can serve to analyze simultaneously network formation and value allocation. These are closely related problems since the payoff a player gets in each possible network gives incentives for creating certain links and severe others.

A social network here is represented as a set of bilateral and reciprocal relations connecting individuals in a society (called players). If these network relations have an effect on some economic activity, they are social-economic networks. Job contact networks, R+D bilateral agreements among firms, and crime networks are examples of social-economic networks that have recently appeared in the literature.[1] In what follows, I will call them simply networks.

The dynamics of the network formation process are represented by means of a stationary transition probability matrix. At any moment in time players receive payoffs according to a stationary or stage-wise allocation rule, depending on the current network but not on the moment in time. Players discount the future by some common discount factor. Because the transition probability matrix represents the dynamics induced by self-interest individuals, it should be somehow consistent with the allocation rule at hand. With this in mind, we define the notion of discounted, expected pairwise stability based on the notion of pairwise stability by Jackson and Wolinsky (1996), where players take the whole stream of payoffs into account. In particular, players anticipate (with some decay) the inter-temporal repercussions of their own linking decisions.

The system or scheme formed by a stage-wise allocation rule and a transition probability matrix is called a forward-looking network formation scheme if, first, the probability that a link is created is positive if the discounted, expected gains to its two participants are positive, and if, second, the probability that a link is eliminated is positive if the discounted, expected gains to at least one of its two participants are positive. The main result is the existence, for all discount factors and all value functions, of a forward-looking network formation scheme. Furthermore, we can always find a forward-looking network formation scheme such that the allocation rule is component balanced,[2] and the transition probabilities increase in the difference in payoffs of the corresponding agents responsible for the transition.

The definition of a forward-looking network formation scheme captures some best-response dynamics as in the seminal paper by Jackson and Watts (2002), themselves based on the notion

[1] See Jackson (2008) for a review.

[2] An allocation rule component balanced or component efficient if it always distributes the total value of a maximal connected subnetwork among its participants

of pairwise stability introduced by Jackson and Wolinsky (1996). Pairs of players are called to play with a certain probability at each point in time. When called to play, players decide wether to create a link if they are not directly connected in the current network, or to severe it if they are, or to leave the network unchanged. Dutta et al (2005), Page et al (2005) and Herings et al (2009) have, as we do here, proposed dynamic processes of network formation where players are forward looking, extending therefore Jackson and Wolinsky (1996) and Jackson and Watts (2002) setting. While all these works focus on the problem of network formation and whether farsightedness can help resolve the tension between stability and efficiency already pointed out by Jackson and Wolinsky (1996), our intention is to provide a solution concept to analyze the problem of network value allocation at the same time as the problem of network formation. We derive very general results, namely existence for any discount factor, by leaving the conditions on the allocation rule open, and considering very mild conditions on the transition probabilities. Our analytical framework is flexible enough and leaves room for further research, namely to analyze forward-looking agents that are at the same time forming the network and deciding how to distribute its value. As an example, in a recent working paper, Navarro (2013) considers forward-looking pairwise network formation schemes where the allocation rule satisfies a forward-looking version of equal balance contributions (or fairness) and component balance or component efficiency. Unfortunately, such a scheme is guaranteed to exist only for discount factors small enough.

Currarini and Morelli (2000) and Slikker and van den Nouweland (2000) also analyze network value allocation and network formation simultaneously. In these articles both the value allocation and the network formation are the result of a multilateral (non cooperative) bargaining procedure written as a one-shot game. The setting we introduce here allows us to acknowledge the bilateral nature of network formation, as the notion of pairwise stability does, and allows to explore the tensions among different assumptions on the way the value is allocated.

We finally identify the conditions on the value function w that guarantee the existence of at least one forward-looking network formation scheme such that the dynamics of network formation converge to an efficient structure with probability one. Similarly to Dutta et al. (2005), a condition of monotonicity of the stationary or stage-wise allocation rule guarantees that the process will converge to the complete graph. This is so as the creation of links are at any time always beneficial. The questions to explore are, first, whether in any situation where the complete network is efficient we can find a stationary allocation rule that is monotonic, and second, whether anytime we can find a stationary allocation rule that is monotonic with respect to the addition of own links the complete network is efficient. The answer is obviously not to both, but we can identify conditions on the value structure for the first question to be positive (therefore obtaining an efficient result). Finally, we also show that the presence of externalities across components makes the efficiency results more difficult to obtain. Better said, if the value structure is such that there are externalities across components, our requirement of increasing returns to link creation (with respect to the value to be allocated) is more restricted than when there are not externalities across components.

This paper proceeds as follows. Section 2 introduces the setting and main definitions. Section 3 states the results. Section 4 illustrates the existence result and the limits of the efficiency result by means of examples. Finally, Section 5 concludes.

2 Definitions

2.1 Players, Coalitions and Networks

Let $N = \{1, ..., n\}$ be a finite set of players. A subset $S \subseteq N$ is called a coalition. Let 2^N be the set of all possible coalitions in N. There are network relations among the players in N, formally represented by an undirected graph. Here, an undirected graph g is a set of unordered pairs, denoted ij, with i, $j \in N$, and $i \neq j$. In what follows, each element ij in a graph g will be referred to as a link.

Let g^N be the complete graph on the set N. Let $g \cup ij$ be the graph resulting from adding the link ij to the existing graph g, and let $g \backslash ij$ be the graph resulting from eliminating the link ij from the graph g. A coalition $T \subseteq N$ is a connected component of N in g if: (1) for each pair of players in T, there exists a path, i. e., a set of consecutive links, in g that connects them, and (2) for each player i in T and each player j not in T, there is no path in g that connects them. Let $N|g$ be the set of connected components of N in g. Note that $N|g$ is a partition of N.

A link ij is called *critical* in g, where $ij \in g$, if the component $T \in N|g$ to which i and j belong splits into two components in $g \backslash ij$. Formally, $|N|(g \backslash ij)| = |N|g| + 1$, where $|N|g|$ denotes the cardinality of the set $N|g$.

For every graph g and coalition $S \subseteq N$ we define the restriction of g to S as the subset of links in g directly connecting players in S to players in S, formally $g|_S = \{ij \in g : i \in S \text{ and } j \in S\}$. Note that $g|_S \subseteq g$, $g|_N = g$, and $g = \bigcup_{T \in N|g} g|_T$, for any $g \subseteq g^N$.

A pair of graphs g and g' in G are called *adjacent graphs* (Jackson and Watts, 2002) if $g = g'$ or if there exists a link $ij \in g^N$ with $g' = g \backslash ij$ or $g' = g \cup ij$.

Let G be the set of all possible graphs over N, and $|G|$ its cardinality. Note that $|G| = 2^{\frac{n(n-1)}{2}}$ because the number of links in g^N is equal to $\frac{n(n-1)}{2}$.

2.2 Values, Allocation Rules, Stability and Efficiency

Assume now that for every graph $g \in G$ and for every connected component $T \in N|g$, there is a value $w(T, g)$ that can be perfectly distributed among the players in T. A function w, which to every graph g and every connected component T in $N|g$ assigns a value $w(T, g)$, is called a *value function*. Let $\mathcal{W}$ be the set of all possible value functions, once the set of players is equal to N.

Let $v(g)$ be the total value available to distribute among players in N when the latter organize as in g, namely

$$v(g) = \sum_{T \in N|g} w(T, g). \tag{1}$$

A value function w is called component additive if $w(T, g) = w(T, g|_T)$ for any $g \in G$ and any $T \in N|g$. This means that the value of a component of the network does not depend on the structure of the network outside the component. In such a case, we can compute the total value $v(g)$ that can be distributed from the sum of the value of each component taken in isolation, namely

$$v(g) = \sum_{T \in N|g} w(T, g|_T). \tag{2}$$

We will not assume component additivity (absence of externalities across components) in general, and whenever used, it will be stated explicitly.

A value function is *link monotonic* if for any $T \in N|g$ and any $ij \in g|_T$ the following two conditions hold.

1. Assume ij is critical in g, so that T splits into two components T_i and T_j when ij is removed from g. Formally, $T_i \in N|(g \backslash ij)$ with $i \in T_i$ and $T_j \in N|(g \backslash ij)$ with $j \in T_j$, and $T = T_i \cup T_j$. Then, it has to be that $w(T, g) > w(T_i, g \backslash ij) + w(T_j, g \backslash ij)$.

2. Assume ij is not critical in g, so that T is also a component in $g \backslash ij$. Then, it has to be that $w(T, g) > w(T, g \backslash ij)$.

A value function satisfies *strong critical-link monotonicity*[3] if it is link monotonic and if for every graph g, any $T \in N|g$ and any $ij \in g|_T$ critical $\frac{w(T,g)}{|T|} > \max\{\frac{w(T_i, g\backslash ij)}{|T_i|}, \frac{w(T_j, g\backslash ij)}{|T_j|}\}$, where T_i and T_j are, as before, the two components in which T splits when ij is removed from g. Note that critical-link monotonicity requires that the per-capita value in the component increases as the component gets more connected or larger (as opposed to link monotonicity, where the *total* value in the component increases).

A graph $g \in G$ is called *strongly efficient* if its total value $v(g)$ is maximized in g, namely, if $v(g) \geq v(g')$ for any $g' \in G$. Note that if a value function is link-monotonic and component additive, then the complete graph is strongly efficient. Section 4 (Examples) shows an example of a value function that is link monotonic but not component additive and such that the empty graph is strongly efficient.

An *allocation rule* y is a function that assigns to every value function w in $\mathcal{W}$ a payoff recommendation $y_{i,g}(w)$ for every player $i \in N$ and every graph $g \in G$. From now on we will omit w in brackets in all notation that follows, as we will consider w to be fixed. Abusing

[3] The definition of critical link monotonicity in Jackson and Wolinsky (1996) applies to a pair formed by a graph g and a value function v. Here, I take critical link monotonicity to be applied to value functions w for all graphs g and it is a stronger condition than in their article.

notation we will also denote the resulting payoff vector from an allocation rule y by $y \in \Re^{n \times |G|}$, for the sake of simplicity. An allocation rule y is called *component efficient* (Myerson, 1977) if for every graph $g \in G$ and for every connected component $T \in N|g$

$$\sum_{i \in T} y_{i,g} = w(T, g).$$

Since Jackson and Wolinsky (1996) this property is also referred to as *component balance*.

Aa allocation rule is *monotonic with respect to own links* if for any graph $g \in G$ and any link $ij \in g$ we have that $y_{i,g} - y_{i,g \setminus ij} > 0$ and $y_{j,g} - y_{j,g \setminus ij} > 0$. Based on a previous result by Jackson and Wolinsky (1996) we can show that if the value function is component additive and link monotonic then we can guarantee the existence of at least one allocation rule that is component efficient and monotonic with respect to own links (see Lemma 3.6 in the Results section). Unfortunately, such a result does not hold if the value function is not component additive. In Section 4 (Examples) we show a value function that is link monotonic, but not component additive, and for which we cannot find an allocation rule that is component efficient and monotonic with respect to own links. The reason is that such a value function does not satisfy strong critical-link monotonicity.

Given an allocation rule y, a graph $g \in G$ is called *pairwise stable* (Jackson and Wolinsky 1996) if

1. For any link $ij \in g$:
$$y_{i,g} - y_{i,g \setminus ij} \geq 0,$$
and
$$y_{j,g} - y_{j,g \setminus ij} \geq 0.$$
2. For any link $ij \notin g$: if
$$y_{i,g} - y_{i,g \cup ij} < 0,$$
then
$$y_{j,g} - y_{j,g \cup ij} > 0.$$

2.3 Dynamics and Forward-Looking Stability

Suppose we have an infinite number of stages, and at each stage t there is a transition probability from the existing graph g_t to another graph g_{t+1} in the next stage. If we assume that these probabilities do not depend on time t, the graph at each stage follows a stationary Markov chain of infinite length with transition probabilities that are given by the matrix

$$P = \left[P\left(g'|g\right)\right]_{g',g \in G},$$

where $P(g'|g)$ is the probability to arrive at g' conditional on g being the current graph.

The transition probability matrix represents the way the agents in N build the network over time. Accordingly, the only transitions happening with positive probability are the ones corresponding to the creation or deletion of at most one link at a time. Formally, $P(g'|g) > 0$, for any pair of graphs g, g' in G, only if g and g' are adjacent. Let Π denote the set of all possible transition probability matrices from graphs in G to graphs in G, and such that $P(g'|g) > 0$ only if g and g' are adjacent.

Suppose that players receive stage payoffs according to an allocation rule y and they discount the future by some common factor $0 < \delta < 1$. Define

$$P^{\delta,\infty} = \sum_{t=0}^{\infty} \delta^t P^t.$$

Here, $P^{\delta,\infty}(g'|g)$ can be interpreted as the total discounted probability of arriving at the end of the process at graph g' when starting from graph g.

For any allocation rule y, and any initial graph g, the discounted, expected payoff to player i is given by

$$x_{i,g}(y,\delta,P) = \sum_{g'} P^{\delta,\infty}\left(g'|g\right) y_{i,g'}.$$

For any $\delta \in [0,1)$ the matrix $(I - \delta P)$, where I is the identity matrix, has an inverse. It is easy to see that $(I-\delta P)^{-1} = P^{\delta,\infty}$. Let $x(y,\delta,P) = [x_{i,g}(y,\delta,P)]_{\{i\in N,\ g\in G\}}$. From now on, we will simply write $x(y,P)$ instead of $x(y,\delta,P)$, for δ will be fixed.[4]

We first adapt the notion of pairwise stability to this dynamic context, assuming players care about the whole stream of discounted, expected payoffs. Given a pair (y,P), where y is a (stage-wise) allocation rule and P is a transition probability matrix, a graph g is called *pairwise stable in discounted, expected terms* if

1. For any link $ij \in g$:
$$x_{i,g}(y,P) - x_{i,g\setminus ij}(y,P) \geq 0,$$
and
$$x_{j,g}(y,P) - x_{j,g\setminus ij}(y,P) \geq 0.$$

2. For any link $ij \notin g$: if
$$x_{i,g}(y,P) - x_{i,g\cup ij}(y,P) < 0,$$
then
$$x_{j,g}(y,P) - x_{j,g\cup ij}(y,P) > 0.$$

[4]See Meyer (2001) for further reference on matrix algebra or on Markov chains.

Since the transition probability matrix represents a network formation process, we can impose some conditions on P, related to the notion of pairwise stability. The pair (y, P), where y is a stage-wise allocation rule and P is a transition probability matrix, is called a *forward-looking network formation scheme* if:

1. For any link $ij \in g$, $P(g\backslash ij|g) = 0$ if
$$x_{i,g}(y, P) - x_{i,g\backslash ij}(y, P) \geq 0,$$
and
$$x_{j,g}(y, P) - x_{j,g\backslash ij}(y, P) \geq 0.$$

2. For any link $ij \notin g$, $P(g \cup ij|g) = 0$ if whenever
$$x_{i,g}(y, P) - x_{i,g\cup ij}(y, P) < 0,$$
we have that
$$x_{j,g}(y, P) - x_{j,g\cup ij}(y, P) > 0,$$
or if
$$x_{i,g}(y, P) - x_{i,g\cup ij}(y, P) \geq 0 \text{ and } x_{j,g}(y, P) - x_{j,g\cup ij}(y, P) \geq 0.$$

Note that if (y, P) is a forward-looking network formation scheme and if a graph g is pairwise stable in discounted, expected terms with respect to (y, P), then $P(g|g) = 1$. When a graph g satisfies that $P(g|g) = 1$ we will say that g is a stationary state of the Markov chain defined by P. If, furthermore, graph g is reached in the long run dynamics given by P with probability equal to 1 from any initial graph we will call it an absorbing state or absorbing graph.

3 Results

Proposition 3.1 *Fix any value function w and take any allocation rule y. Then, for any $\delta \in [0, 1)$ there exists a transition probability P such that (y, P) is a forward-looking network formation scheme.*

Proof of Proposition 3.1. The proof makes use of Brower's fixed point theorem. We will define a continuous function F that maps transition probability matrices in Π into transition probability matrices in Π, meaning that $F(P)$ is also a transition probability matrix where only transitions to adjacent graphs can happen with positive probability. The function F will be defined such that its fixed point P^* induces a forward-looking network formation scheme (y, P^*). Note that Π is a compact, convex set in the space of square matrices of dimension $|G|$.

Fix any $\delta \in [0,1)$ and any $w \in \mathcal{W}$. For any allocation rule y, we abuse notation and use $y \in \Re^{n|G|}$ as the payoff scheme or stage-wise payoff vector induced by the allocation rule applied to w. We proceed to define the function $F : \Pi \to \Pi$ as a composition of functions.

First, let $x(y,P)$ be the discounted, expected payoff vector for the players if the stage-wise payoff vector is equal to y and the transition probability matrix is equal to $P \in \Pi$. For each player $i \in N$ let us denote by $y_i \in \Re^{|G|}$ and $x_i(y,P) \in \Re^{|G|}$ the vector of stage-wise payoffs and the vector of total discounted, expected payoffs, respectively, that this player i obtains in each possible graph. Formally, $y_i = (y_{i,g})_{g \in G}$ and $x_i(y,P) = (x_{i,g}(y,P))_{g \in G}$. Then, by definition,

$$x_i(y,P) = (I - \delta P)^{-1} y_i.$$

Note that $x_i(y,P)$ is continuous in P given any δ or y because $\delta < 1$.

For each graph $g \in G$ and each link $ij \in g^N$ we can define the *truncated differences* in expected payoffs $d_i(y,P)(g,ij) \in \Re_+$ and, equivalently, $d_j(y,P)(g,ij) \in \Re_+$ as follows.

Case i. Assume $ij \in g$. Then

$$d_i(y,P)(g,ij) = \begin{cases} x_{i,g\setminus ij}(y,P) - x_{i,g}(y,P), & \text{if } x_{i,g\setminus ij}(y,P) - x_{i,g}(y,P) > 0, \\ 0, & \textit{otherwise.} \end{cases} \tag{3}$$

Case ii. Assume $ij \notin g$. Then

$$d_i(y,P)(g,ij) = \begin{cases} x_{i,g\cup ij}(y,P) - x_{i,g}(y,P), & \text{if } x_{i,g\cup ij}(y,P) - x_{i,g}(y,P) > 0, \\ 0, & \textit{otherwise.} \end{cases} \tag{4}$$

Note that the truncated differences $d_i(y,P)(g,ij)$ and $d_j(y,P)(g,ij)$ are continuous in $x_i(y,P)$ and $x_j(y,P)$, respectively.

Fix any $\beta \in (0,1)$. For each graph $g \in G$ and each link $ij \in g^N$ we can define the function $h(g,ij)$ assigning to each pair of truncated differences $(d_i(y,P)(g,ij), d_j(y,P)(g,ij))$ a transition probability $h(g,ij)[(d_i(y,P)(g,ij), d_j(y,P)(g,ij))] \in [0,1]$, for simplicity denoted $h(g,ij)$, as follows

$$h(g,ij) = \begin{cases} 1 - \frac{1}{2}\left[\beta^{d_i(g,ij)} + \beta^{d_j g,ij}\right], & \text{if } ij \in g \\ \left(1 - \beta^{d_i(g,ij)}\right)\left(1 - \beta^{d_j(g,ij)}\right) & \text{if } ij \notin g \end{cases}$$

Finally, let $F(P) = P'$, where $P' \in \Pi$ verifies the following.

1. $P'(g\setminus ij|g) = h(g,ij)$ for $ij \in g$,
2. $P'(g \cup ij|g) = h(g,ij)$ for $ij \notin g$, and
3. $P'(g|g) = 1 - \sum\limits_{ij \in g} h(g,ij) - \sum\limits_{ij \notin g} h(g,ij)$.

Hence, $P'(g'|g) = 0$ if g' and g are not adjacent graphs, which means that $P' \in \Pi$.

By definition as a composition of continuous functions, the function F is continuous in Π. By Brouwer's Fixed Point Theorem, we know that there exists at least one fixed point for the function $F : \Pi \to \Pi$ defined just above, denoted P^*. It is easy to check that any fixed point P^* satisfy that the pair (y, P^*) is a forward-looking network formation scheme, by definition of $P' = P^* = F(P^*)$. This completes the proof of Theorem 3.1. $\square$

From the proof of Proposition 3.1 we can conclude that not only a forward-looking network formation scheme always exists, but that such a forward-looking network formation scheme verifies that the transition probabilities are increasing in the corresponding differences in payoffs. We state it formally in the following corollary. No proof is necessary, as the function $h(g, ij)$ satisfies such a monotonic property.

Corollary 3.2 *Fix any value function w and take any allocation rule y. Then, for any $\delta \in [0, 1)$ there exists a transition probability P such that (y, P) is a forward-looking network formation scheme, satisfying the following. For any $g \in G$, the transition probability $P(g \cup ij|g)$, for any $ij \notin g$, and the transition probability $P(g \backslash ij|g)$, for any $ij \in g$, are increasing in the differences of payoffs to the participating agents i and j, namely $x_{i,g\cup ij}(y, P) - x_{i,g}(y, P)$ and $x_{j,g\cup ij}(y, P) - x_{j,g}(y, P)$ if $ij \notin g$, and $x_{i,g\backslash ij}(y, P) - x_{i,g}(y, P)$ and $x_{j,g\backslash ij}(y, P) - x_{j,g}(y, P)$ if $ij \in g$, whenever those differences are positive.*

Proposition 3.1 and Corollary 3.2 indicate that we can fix any allocation rule y that is component efficient and find a transition probability matrix that will form a forward-looking network formation scheme with it. In particular, we can fix any allocation rule that is component efficient, as a component efficient allocation rule always exists. We can state it formally in the following corollary.

Corollary 3.3 *Fix any value function w. Then, for any $\delta \in [0, 1)$ there exists a forward-looking network formation scheme (y, P) such that the allocation rule y is component efficient or balanced, and the transition probability matrix P satisfies the following. For any $g \in G$, the transition probability $P(g \cup ij|g)$, for any $ij \notin g$, and the transition probability $P(g \backslash ij|g)$, for any $ij \in g$, are increasing in the differences of payoffs to the participating agents i and j, namely $x_{i,g\cup ij}(y, P) - x_{i,g}(y, P)$ and $x_{j,g\cup ij}(y, P) - x_{j,g}(y, P)$ if $ij \notin g$, and $x_{i,g\backslash ij}(y, P) - x_{i,g}(y, P)$ and $x_{j,g\backslash ij}(y, P) - x_{j,g}(y, P)$ if $ij \in g$, whenever those differences are positive.*

The reader should note that all these results do not mean that for each allocation rule we find one and the same transition probability matrix that will form a forward-looking network formation scheme for any value of δ. The results state that given δ we can find a transition probability matrix with the desired properties. In general, the transition probability matrix depends on the value of δ (see Section 4 for an illustration of this fact.)

To conclude this section, we will explore the implications of our solution concept in terms of resolving the tension between efficiency and stability. First, we identify conditions for the strongly efficient graph to be an absorbing state in the Markov chain given by P and discuss the implications of the presence of externalities across components.

Proposition 3.4 *Let the value function w be component additive and link-monotonic. Then, there exists a forward-looking network formation scheme (y, P) such that y is component efficient with respect to w and the complete graph is the absorbing state in P for any $\delta \in [0, 1)$.*

Proof of Proposition 3.4. We will prove this proposition by means of two lemmas. Lemma 3.5 identifies conditions on the stage-wise allocation rule that will guarantee the existence of a network formation scheme with the desired characteristics. Lemma 3.6 states that if the value function is link-monotonic and component additive then we can always find at least one stage-wise allocation rule with the required conditions for Lemma 3.5 to be applied.

Lemma 3.5 *Let the value function $w \in \mathcal{W}$ be given and suppose that there is a stage-wise allocation rule y that is monotonic with respect to own links and component efficient with respect to w. Then there is a transition probability matrix P such that (y, P) is a forward-looking network formation procedure for any $\delta \in [0, 1)$ where the dynamics given by P lead to g^N with probability one from any initial graph $g \in G$.*

Proof of Lemma 3.5. For each link $l \in g^N$ we can define a probability $p(l)$ such that $\sum\limits_{l \in g^N} p(l) \leq 1$. Consider the transition probability matrix P assigning for each graph g a transition probability to another graph g' as follows

$$P(g'|g) = \begin{cases} p(l), & \text{if } g' = g \cup l, \\ 1 - \sum\limits_{l \in (g^N \backslash g)} p(l), & \text{if } g' = g \\ 0, & \text{otherwise.} \end{cases}$$

Note that in such a Markov chain the only transitions that happen with positive probability are the ones where there is creation of links. No destruction of links take place. Fix y, an allocation rule that is monotonic with respect to own links and component efficient with respect to w. To check that (y, P) is a forward-looking network formation scheme we check that any two players involved in the creation of a link at any point in time gain in discounted, expected terms from the creation of it for any $\delta \in (0, 1]$. This is to say that $x_{i,g}(y, P) - x_{i,g\backslash ij}(y, P) \geq 0$ and $x_{j,g}(y, P) - x_{j,g\backslash ij}(y, P) \geq 0$, for any $ij \in g$. We prove such a statement by induction on the links in $g^N \backslash g$.

Let us start the induction by supposing that $g = g^N$, so that $g^N \backslash g = \emptyset$. Given that $P(g^N|g^N) = 1$, we have that $x_{i,g^N} = \frac{1}{1-\delta} y_{i,g^N}$ for any $i \in N$. Take any $ij \in g^N$. By definition of our P matrix again, we have that

$$x_{i,g^N \backslash ij} = \frac{1}{1 - \delta + \delta p(ij)} y_{i,g^N \backslash ij} + \frac{\delta p(ij)}{1 - \delta + \delta p(ij)} x_{i,g^N},$$

and

$$x_{j,g^N\setminus ij} = \frac{1}{1-\delta+\delta p(ij)} y_{j,g^N\setminus ij} + \frac{\delta p(ij)}{1-\delta+\delta p(ij)} x_{j,g^N}.$$

Rearranging terms we can conclude that

$$x_{i,g^N} - x_{i,g^N\setminus ij} = \frac{1}{1-\delta+\delta p(ij)} \left(y_{i,g^N} - y_{i,g^N\setminus ij} \right),$$

and

$$x_{j,g^N} - x_{j,g^N\setminus ij} = \frac{1}{1-\delta+\delta p(ij)} \left(y_{j,g^N} - y_{j,g^N\setminus ij} \right).$$

Given that we have chosen a stage-wise allocation rule that is monotonic we respect to own links, we have shown that $x_{i,g^N}(y,P) - x_{i,g^N\setminus ij}(y,P) \geq 0$ and $x_{j,g^N}(y,P) - x_{j,g^N\setminus ij}(y,P) \geq 0$.

Now take another graph g and assume that for any link $l \in g^N\setminus g$ it was true that $x_{i,g\cup l} - x_{i,(g\cup l)\setminus ij} \geq 0$ and $x_{j,g\cup l} - x_{j,(g\cup l)\setminus ij} \geq 0$, for any link $ij \in g$. By definition of our P matrix again, we now have that

$$x_{i,g} = \frac{1}{1-\delta+\delta \sum_{l\in(g^N\setminus g)} p(l)} \left(y_{i,g} + \delta \sum_{l\in(g^N\setminus g)} p(l) x_{i,g\cup l} \right),$$

for any $i \in N$. Take any $ij \in g$. We have that

$$x_{i,g\setminus ij} = \frac{1}{1-\delta+\delta p(ij)+\delta \sum_{l\in(g^N\setminus g)} p(l)} \left(y_{i,g\setminus ij} + \delta p(ij) x_{i,g} + \delta \sum_{l\in(g^N\setminus g)} p(l) x_{i,g\setminus ij\cup l} \right),$$

and

$$x_{j,g\setminus ij} = \frac{1}{1-\delta+\delta p(ij)+\delta \sum_{l\in(g^N\setminus g)} p(l)} \left(y_{j,g\setminus ij} + \delta p(ij) x_{j,g} + \delta \sum_{l\in(g^N\setminus g)} p(l) x_{j,g\setminus ij\cup l} \right).$$

Rearranging terms we can conclude that

$$x_{i,g} - x_{i,g\setminus ij} = \frac{y_{i,g} - y_{i,g\setminus ij} + \delta \sum_{l\in(g^N\setminus g)} p(l) \left(x_{i,g\cup l} - x_{i,g\setminus ij\cup l} \right)}{1-\delta+\delta p(ij)+\delta \sum_{l\in(g^N\setminus g)} p(l)},$$

and

$$x_{j,g} - x_{j,g\setminus ij} = \frac{y_{j,g} - y_{j,g\setminus ij} + \delta \sum_{l\in(g^N\setminus g)} p(l) \left(x_{j,g\cup l} - x_{j,g\setminus ij\cup l} \right)}{1-\delta+\delta p(ij)+\delta \sum_{l\in(g^N\setminus g)} p(l)}.$$

By induction hypothesis, $x_{i,g\cup l} - x_{i,g\setminus ij\cup l} \geq 0$ for any $l \in (g^N\setminus g)$, because $g\setminus ij \cup l = (g\cup l)\setminus ij$ for any $l \in g^N\setminus g$ and $ij \in g$. As before, the stage-wise allocation rule is monotonic we respect to own links. These two facts indicate that $x_{i,g}(y,P) - x_{i,g\setminus ij}(y,P) \geq 0$ and $x_{j,g}(y,P) - x_{j,g\setminus ij}(y,P) \geq 0$, and hence the proof of Lemma 3.5 is complete.

Lemma 3.6 *Let the value function $w \in \mathcal{W}$ be component additive and link monotonic. Then there always exists an allocation rule y that is monotonic with respect to own links and component efficient with respect to w.*

Proof of Lemma 3.6. Let us fix the Myerson value (Myerson, 1977), adapted by Jackson and Wolinsky (1996) to the setting of value functions that are component additive, as a stage-wise allocation rule Y.[5] Jackson and Wolinsky (1996) have shown that such an allocation rule always exists and it is unique for any value function w that is component additive.[6] By definition, the Myerson value or equal bargaining power rule is component efficient or component balance. We make use of the first part of Corollary to Theorem 4, p.65, from Jackson and Wolinsky (1996), which statement is as follows (only first part has been taken).

Corollary 3.7 *Let Y be the equal bargaining power rule, and consider a component additive w and any g and $ij \in g$. If for all $g' \subset g$, $v(g) \geq v(g \backslash ij)$, then $Y_{i,g} \geq Y_{i,g \backslash ij}$.*

Recall that $v(g) = \sum_{T \in N|g} w(T, g)$. It is easy to see that if w is component additive and link monotonic then, for any $g \in G$ and any $g' \subseteq g$, $v(g)$ satisfies that $v(g') \geq v(g' \backslash ij)$. This means by the Jackson and Wolinsky's result above that the Myerson value of w is link-monotonic with respect to own links. This completes the proof of Lemma 3.6.

From Lemmas 3.5 and 3.6 we can conclude that the statement in Proposition 3.4 holds for any $\delta \in [0, 1)$. $\square$

Note that Proposition 3.4 requires the value function w to be component additive. To help us better understand the role played by the absence of externalities across components, specially in Lemma 3.6, Section 4 (Examples) shows a value function that is link monotonic but not component additive, and for which we cannot find an allocation rule that is both component efficient and monotonic with respect to own links. The following proposition identifies a condition on the value function w that is stronger than link monotonicity but allows us to drop the condition of component additivity.

Proposition 3.8 *Let the value function w satisfy strong critical-link monotonicity. Then, there exists a forward-looking network formation scheme (y, P) such that y is component efficient with respect to w and the complete graph is the only absorbing state in P for any $\delta \in [0, 1)$.*

Proof of Proposition 3.8. We will prove that if the value function w satisfies critical-link monotonicity, we can always find at least one stage-wise allocation rule that is component efficient and monotonic with respect to own links. Hence, Lemma 3.5 can be applied because

[5] Jackson and Wolinsky refer to this allocation rule as the equal bargaining power rule, p. 64

[6] See Theorem 4, p.65.

Lemma 3.5 only imposes conditions on the stage-wise allocation rule, but not on the value function w. Let us consider the component-wise egalitarian allocation rule. Recall that critical link-monotonicity imposes further conditions on value functions w that are already link monotonic. For every $i \in N$ and every $g \in G$ denote by $T_i(g)$ the component in $N|g$ to which i belongs. Then the component-wise egalitarian allocation rule is defined as

$$y_{i,g} = \frac{w(T_i(g), g)}{|T_i(g)|},$$

for every $i \in N$ and any $g \in G$. It is easy to see that the component-wise egalitarian allocation rule is monotonic with respect to own links when the value function w is link-monotonic and satisfies critical-link monotonicity. Consider first the case when the link $ij \in g$ is not critical. This means that $T_i(g\backslash ij) = T_i(g)$ and $T_j(g\backslash ij) = T_j(g) = T_i(g)$ and, because w is link monotonic, $w(T_i(g), g) > w(T_i(g), g\backslash ij)$, and therefore

$$y_{i,g} - y_{i,g\backslash ij} = y_{j,g} - y_{j,g\backslash ij} = \frac{w(T_i(g), g) - w(T_i(g), g\backslash ij)}{|T_i(g)|} > 0.$$

Consider now the case when the ink $ij \in g$ is critical. By definition of critical-link monotonicity the component-wise allocation rule is increasing in own links when they are critical links for the current network. This completes the proof of Proposition 3.8.□

When the value function w is strong critical-link monotonic, we can guarantee the existence of an allocation rule that is component efficient and monotonic with respect to own links, namely the component-wise egalitarian allocation rule. With such a stage-wise allocation rule, agents will always find profitable to create links at each point in time. The condition of strong critical-link monotonicity guarantees that the complete network is the strong efficient network, but link monotonicity does not. Last example in next section will show a value function w that is link monotonic, but not component additive, such that the empty network is the efficient network. The reason is that the value function w is not strong critical-link monotonic. We provide the proof of the following lemma.

Lemma 3.9 *Let w satisfy strong critical-link monotonicity. Then, the complete network is the strong efficient network given w.*

Proof of Lemma 3.9. First note that if w satisfies critical-link monotonicity then

$$\frac{w(N, g^N)}{|N|} > \frac{w(T, g)}{|T|},$$

for any other $g \subsetneq g^N$ and any $T \in N|g$. Note that if g is such that there is only one component in the network, i.e., $|N|g| = 1$, then $\frac{w(N,g^N)}{n} > \frac{w(N,g)}{n}$ because strong critical-link monotonicity requires link monotonicity. Assume now by contradiction that there is a $\tilde{g} \subsetneq g^N$ and a component $\tilde{T} \in N|\tilde{g}$ with $\tilde{T} \subsetneq N$ such that

$$\max_{g \in G,\ T \in N|g} \frac{w(T,g)}{|T|} = w(\tilde{T},\tilde{g}).$$

Take a player $i \in \tilde{T}$ and another player $j \notin \tilde{T}$ and add the link ij to $\tilde{g}$. Note that ij does not belong to $\tilde{g}$ because i and j belong to two different components in $\tilde{g}$. At the new graph $\tilde{g} \cup ij$, $\tilde{T}$ merges to the component in $\tilde{g}$ including player j, formally there is now a $T' \in N|(\tilde{g} \cup ij)$ such that $T' = \tilde{T} \cup T_j$, where $T_j \in N|g$ and $j \in T_j$. By construction, ij is a critical link in $\tilde{g} \cup ij$. By strong critical-link monotonicity, $\frac{w(T',\tilde{g})}{T'} > \frac{w(\tilde{T},\tilde{g})}{|\tilde{T}|}$, a contradiction with $(\tilde{T},\tilde{g})$ maximizing the component per capita value.

Finally note that if (N, g^N) maximizes the per capita value in a component, then g^N maximizes $v(g)$ for all $g \subseteq g^N$. As before, if g is such that there is only one component in the network, i.e., $|N|g| = 1$, then $v(g^N) = w(N, g^N) > v(g) = w(N,g)$ because strong critical-link monotonicity requires link monotonicity. Assume now by contradiction that there is a $\tilde{g} \subsetneq g^N$ such that $v(\tilde{g}) > v(g^N)$. This means that

$$\sum_{T \in N|\tilde{g}|} w(T,\tilde{g}) > w(N, g^N).$$

Dividing by $|N| = n$ left and right,

$$\sum_{T \in N|\tilde{g}} \frac{w(T,\tilde{g})}{n} > \frac{w(N,g^N)}{n},$$

and multiplying and dividing each term of the summation on the left by $|T|$ we obtain that

$$\sum_{T \in N|\tilde{g}|} \frac{|T|}{n} \frac{w(T,\tilde{g})}{|T|} > \frac{w(N,g^N)}{n}.$$

The left-hand side is a convex combination of the component per capita values in $\tilde{g}$ (each component weighed by its size relative to total size of the population), while the right-hand side is the per capita value of the complete network. For the inequality to hold we need that at least one $T \in N|\tilde{g}$ the per capita value $\frac{w(T,g)}{|T|}$ is higher than the per capita value in g^N. But this is a contradiction with strong critical-link monotonicity, as the latter implies that the component per capita value is maximized in (N, g^N). $\square$

As a final comment, it is important to stress that the existence results mean that, in general, the transition probability matrix depends on the value of δ. The efficiency results, on the contrary, state that we can find a forward-looking network formation scheme with the desired characteristics that is always a forward-looking formation scheme for any value of δ.

4 Examples

Let us consider three players, 1, 2, and 3. There are eight possible network structures:

1. The complete network
2. Three structures with two connections each:
   (a) Players 1 and 2 both directly connected to 3, no direct connection between them
   (b) Players 1 and 3 both directly connected to 2, no direct connection between them
   (c) Players 2 and 3 both directly connected to 1, no direct connection between them
3. Three structures with one connection each:
   (a) Players 1 and 3 directly connected, 2 disconnected
   (b) Players 2 and 3 directly connected, 1 disconnected
   (c) Players 1 and 2 directly connected, 3 disconnected
4. The empty network

Let us consider the following value function w

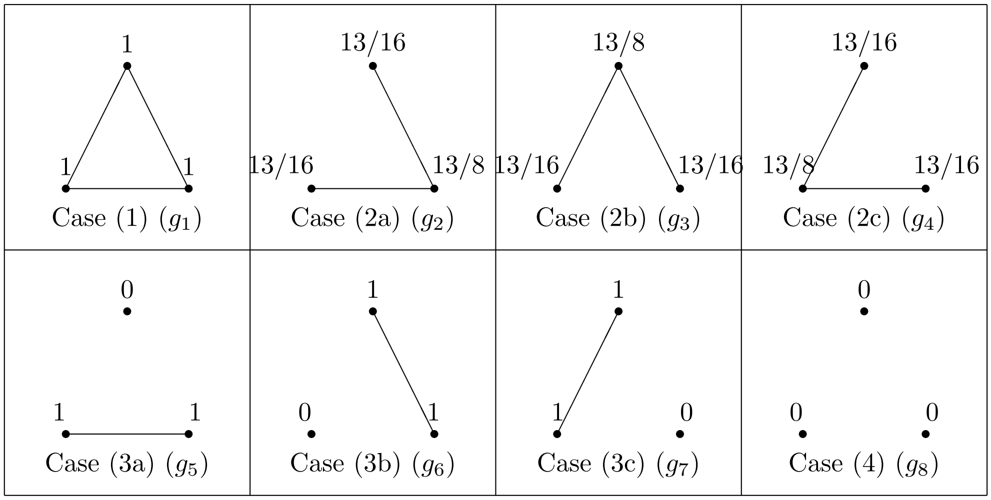
$$w(T,g) = \begin{cases} 3, & \text{if } g \text{ is the complete graph} \\ \frac{13}{4}, & \text{if } g \text{ has two links} \\ 2, & \text{if } g \text{ has one link and } T \text{ contains the two directly connected agents} \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

Note that w is component additive but it is not link monotonic, so that the statement in Proposition 3.1 cannot be applied. Lemma 3.5 on the other hand states that if we can find a stage-wise allocation rule that is monotonic in own links, then a forward-looking network formation procedure such that the complete graph g^N is an absorbing state is guaranteed to exist. This does not automatically mean that the complete graph is efficient. We will show that we can find at least one stage-wise allocation rule that is monotonic with respect to own links for the value function w stated just above. Nevertheless, the strongly efficient networks are the ones with two links each, yielding a value of 13/4.

Example 1. An allocation rule that is monotonic with respect to own links and an inefficient absorbing graph.

Consider the following stage-wise allocation rule, given by Figure 1 below. We have noted the payoff recommendations next to the nodes representing players. Player 1 is placed bottom left, player 2 on top and player 3 bottom right. Note that the stage-wise allocation rule in Figure

Figure 1: A proportional stage-wise allocation rule



1 is component efficient and allocates the value of a component according to the number of links a player holds as a share of the total links in the component (divided by two, as two players participate in each of the links). Furthermore, it is monotonic with respect to own links.

Let $p \in [0,1]$, $q \in \left[0, \frac{1}{2}\right]$ and $r \in \left[0, \frac{1}{3}\right]$. Let us consider a transition probability matrix as the one below, where the order of rows respects the order in Figure 1 top left to top right first and bottom left to bottom right afterwards. In other words, first row and column of the transition probability matrix corresponds to the complete network, second row and column corresponds to the network where players 1 and 2 are both directly connected to 3, and so on and so forth. Last row and column corresponds then to the empty network.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p & 1-p & 0 & 0 & 0 & 0 & 0 & 0 \\ p & 0 & 1-p & 0 & 0 & 0 & 0 & 0 \\ p & 0 & 0 & 1-p & 0 & 0 & 0 & 0 \\ 0 & q & 0 & q & 1-2q & 0 & 0 & 0 \\ 0 & q & q & 0 & 0 & 1-2q & 0 & 0 \\ 0 & 0 & q & q & 0 & 0 & 1-2q & 0 \\ 0 & 0 & 0 & 0 & r & r & r & 1-3r \end{bmatrix} \tag{6}$$

It is easy to see that the complete network is an absorbing state of the Markov chain given by P. Table 1 shows the expected, discounted payoffs $x(y,P)$, where y is the allocation rule in

Figure 1, and P given by transition probability matrix in (6).

Table 1: The expected discounted payoffs $x(y,P)$ obtained from matrix in (6)

$$x_{1,g_1} = x_{2,g_1} = x_{3,g_1} = \frac{1}{1-\delta}$$

$$x_{1,g_2} = x_{2,g_2} = \frac{13(1-\delta)+16\delta p}{16(1-\delta)(1-\delta+\delta p)}, \text{ and } x_{3,g_2} = \frac{13(1-\delta)+8\delta p}{8(1-\delta)(1-\delta+\delta p)}$$

$$x_{1,g_3} = x_{3,g_3} = \frac{13(1-\delta)+16\delta p}{16(1-\delta)(1-\delta+\delta p)}, \text{ and } x_{2,g_3} = \frac{13(1-\delta)+8\delta p}{8(1-\delta)(1-\delta+\delta p)}$$

$$x_{2,g_4} = x_{3,g_4} = \frac{13(1-\delta)+16\delta p}{16(1-\delta)(1-\delta+\delta p)}, \text{ and } x_{1,g_4} = \frac{13(1-\delta)+8\delta p}{8(1-\delta)(1-\delta+\delta p)}$$

$$x_{1,g_5} = x_{3,g_5} = \frac{16(1-\delta)(1-\delta+\delta p)+\delta q(39(1-\delta)+32\delta p)}{16(1-\delta)(1-\delta+\delta p)(1-\delta+2\delta q)}, \text{ and } x_{2,g_5} = \frac{\delta q(13(1-\delta)+16\delta p)}{8(1-\delta)(1-\delta+\delta p)(1-\delta+2\delta q)}$$

$$x_{2,g_6} = x_{3,g_6} = \frac{16(1-\delta)(1-\delta+\delta p)+\delta q(39(1-\delta)+32\delta p)}{16(1-\delta)(1-\delta+\delta p)(1-\delta+2\delta q)}, \text{ and } x_{1,g_6} = \frac{\delta q(13(1-\delta)+16\delta p)}{8(1-\delta)(1-\delta+\delta p)(1-\delta+2\delta q)}$$

$$x_{1,g_7} = x_{2,g_7} = \frac{16(1-\delta)(1-\delta+\delta p)+\delta q(39(1-\delta)+32\delta p)}{16(1-\delta)(1-\delta+\delta p)(1-\delta+2\delta q)}, \text{ and } x_{3,g_7} = \frac{\delta q(13(1-\delta)+16\delta p)}{8(1-\delta)(1-\delta+\delta p)(1-\delta+2\delta q)}$$

$$x_{1,g_8} = x_{2,g_8} = x_{3,g_8} = \delta r \frac{4(1-\delta)(1-\delta+\delta p)+\delta q(13(1-\delta)+12\delta p)}{2(1-\delta)(1-\delta+\delta p)(1-\delta+2\delta q)(1-\delta+3\delta r)}$$

Let us define the differences A, B, C and D as follows.

$$A = x_{1,g_1} - x_{1,g_2} = x_{1,g_1} - x_{1,g_3} = x_{2,g_1} - x_{2,g_4} = \frac{3}{16(1-\delta+\delta p)}$$

$$B = x_{1,g_2} - x_{1,g_6} = x_{1,g_3} - x_{1,g_6} = x_{1,g_4} - x_{1,g_7} = \frac{13(1-\delta)+16\delta p}{16(1-\delta+\delta p)(1-\delta+2\delta q)}$$

$$C = x_{3,g_2} - x_{3,g_6} = x_{2,g_3} - x_{2,g_6} = x_{3,g_4} - x_{3,g_7} = \frac{10(1-\delta)+13\delta q}{16(1-\delta+\delta p)(1-\delta+2\delta q)}$$

$$D = x_{1,g_5} - x_{1,g_8} = x_{2,g_6} - x_{2,g_8} = x_{1,g_7} - x_{1,g_8} =$$

$$= \frac{16(1-\delta+\delta p)(1-\delta+\delta r)+13\delta q(3(1-\delta)+\delta r)+32\delta^2 pq}{16(1-\delta+\delta p)(1-\delta+2\delta q)(1-\delta+3\delta r)}$$

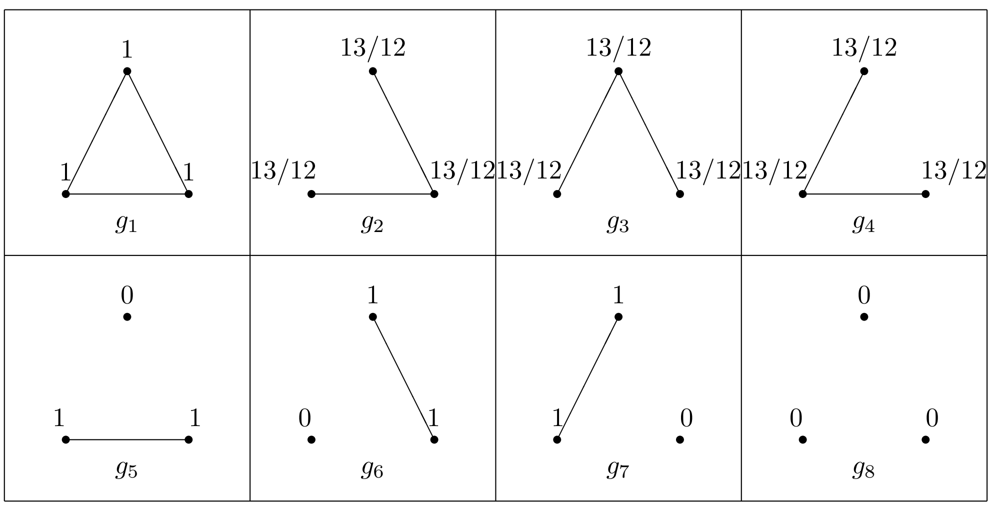
Given that all differences A, B, C and D are greater or equal to zero, we have that the pair (y, P) is indeed a forward-looking network formation scheme.

Nevertheless, since our notion of forward-looking network formation scheme is not unique, we can find for the same value function w another forward-looking network formation scheme such that the stage-wise allocation rule is component efficient and the dynamics given by the transition probability matrix converge to an efficient network with probability equal to one, from any initial network.

Example 2: The component-wise allocation rule with the dynamics converging to the efficient networks.

Let us fix the stage-wise allocation rule to be equal to the component-wise egalitarian allocation rule. The component-wise egalitarian allocation rule is stated in Figure 2.

Figure 2: The component-wise egalitarian allocation rule



Let $p \in \left[0, \frac{1}{3}\right]$, $q \in \left[0, \frac{1}{2}\right]$ and $r \in \left[0, \frac{1}{3}\right]$. The transition probability matrix presented below follows the same order of rows as in (6).

$$P = \begin{bmatrix} 1-3p & p & p & p & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & q & 1-2q & 0 & 0 & 0 \\ 0 & q & q & 0 & 0 & 1-2q & 0 & 0 \\ 0 & 0 & q & q & 0 & 0 & 1-2q & 0 \\ 0 & 0 & 0 & 0 & r & r & r & 1-3r \end{bmatrix} \tag{7}$$

Table 2 shows the expected, discounted payoffs $x(y^{CW-E}, P)$ obtained from y^{CW-E}, the component-wise egalitarian allocation rule, and P given by transition probability matrix in (7).

Table 2: The expected discounted payoffs $x(y, P)$ obtained from matrix in (7)

$$x_{1,g_1} = x_{2,g_1} = x_{3,g_1} = \frac{4(1-\delta)+13\delta p}{4(1-\delta)(1-\delta+3\delta p)}$$

$$x_{i,g_2} = x_{i,g_3} = x_{i,g_4} = \frac{13}{12(1-\delta)}, \text{ for } i = 1, 2, 3$$

$$x_{1,g_5} = x_{3,g_5} = \frac{6(1-\delta)+13\delta q}{6(1-\delta)(1-\delta+2\delta q)}, \text{ and } x_{2,g_5} = \frac{13\delta q}{6(1-\delta)(1-\delta+2\delta q)}$$

$$x_{2,g_6} = x_{3,g_6} = \frac{6(1-\delta)+13\delta q}{6(1-\delta)(1-\delta+2\delta q)}, \text{ and } x_{1,g_6} = \frac{13\delta q}{6(1-\delta)(1-\delta+2\delta q)}$$

$$x_{1,g_7} = x_{2,g_7} = \frac{6(1-\delta)+13\delta q}{6(1-\delta)(1-\delta+2\delta q)}, \text{ and } x_{3,g_7} = \frac{13\delta q}{6(1-\delta)(1-\delta+2\delta q)}$$

$$x_{1,g_8} = x_{2,g_8} = x_{3,g_8} = \delta r \frac{4(1-\delta)+13\delta q}{2(1-\delta)(1-\delta+2\delta q)(1-\delta+3\delta r)}$$

We can similarly define the differences A, B and C as follows.[7]

$$A = x_{1,g_2} - x_{1,g_1} = x_{1,g_3} - x_{1,g_1} = x_{2,g_4} - x_{2,g_1} = \tfrac{1}{12(1-\delta+3\delta p)}$$

$$B = x_{1,g_2} - x_{1,g_6} = x_{1,g_3} - x_{1,g_6} = x_{1,g_4} - x_{1,g_7} = \tfrac{13}{12(1-\delta+2\delta q)}$$

$$C = x_{3,g_2} - x_{3,g_6} = x_{2,g_3} - x_{2,g_6} = x_{3,g_4} - x_{3,g_7} = \tfrac{1}{12(1-\delta+2\delta q)}$$

$$D = x_{1,g_5} - x_{1,g_8} = x_{2,g_6} - x_{2,g_8} = x_{1,g_7} - x_{1,g_8} =$$

$$= \tfrac{6(1-\delta+\delta r)+13\delta q}{6(1-\delta+2\delta q)(1-\delta+3\delta r)}$$

Given that all differences A, B, C and D are greater or equal to zero, we have that the (y^{CW-E}, P) is indeed a forward-looking network formation scheme.

Up to now the dynamics represented by P seem to coincide with the static or myopic notion of pairwise stability, when we take the stage-wise allocation rule as given. We show now another forward-looking network formation scheme for the same value function w such that the stage-wise allocation rule is fixed but the dynamics given by P are different from the dynamics induced by the myopic notion of pairwise stability when δ is big enough.

Example 3: Given one allocation rule, the transition probability matrix can differ for different values of δ.

Let us consider an allocation rule defined by weights as follows. Assume each player $i = 1, 2, 3$ has a weight λ_i and that the stage-wise allocation rule allocates the value of a component according to the player's weight relative to the total weight of the players participating in the component. In other words, for any $i \in N$ and any $g \in G$, and denoting by T_i the component in $N|g$ to which player i belongs, the stage-wise allocation rule yields

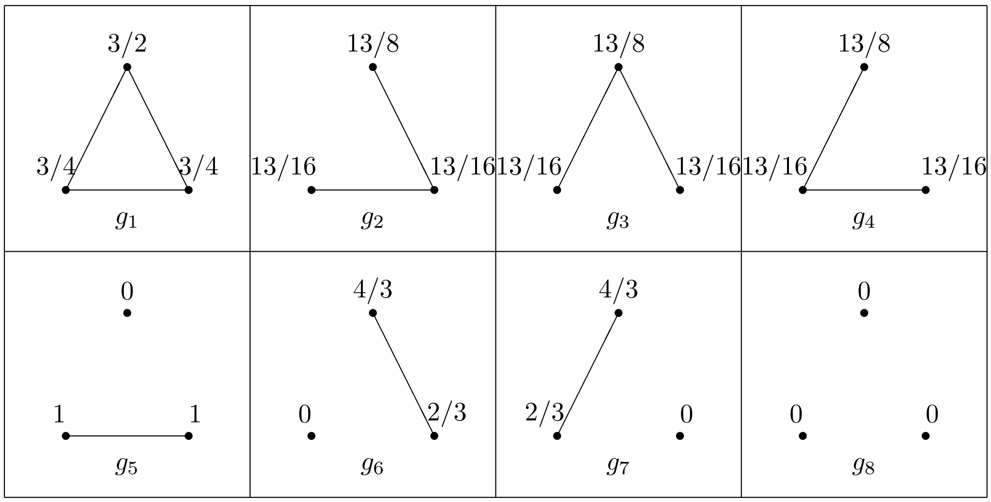
$$y_{i,g} = \frac{\lambda_i}{\sum_{j \in T_i} \lambda_j} w(T_i, g).$$

Figure 3 shows such an allocation rule based on weights, where $\lambda_1 = \lambda_3 = \frac{1}{2}\lambda_2$.

If $\delta = 0$ players create and delete links according to pairwise stability. Such a process can be represented by a transition probability matrix as follows. Let $p_1 \geq 0$ and $p_2 \geq 0$ be such that

[7]Note that A is the negative of the difference A before, since the transition that takes place is in the opposite direction.

Figure 3: A stage-wise allocation rule based on weights



$2p_1 + p_2 \leq 1$, $q \in [0, 1]$, $r_1 \geq 0$, $r_2 \geq 0$ with $r_1 + r_2 \leq 1$, and $s_1 \geq 0$, $s_2 \geq 0$ with $s_1 + 2s_2 \leq 1$. The transition probability matrix presented below follows the same order of rows as in (6).

$$P = \begin{bmatrix} 1-2p_1-p_2 & p_1 & p_2 & p_1 & 0 & 0 & 0 & 0 \\ 0 & 1-q & 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-q & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & r_1 & r_2 & 0 & 0 & 1-r_1-r_2 & 0 & 0 \\ 0 & 0 & r_2 & r_1 & 0 & 0 & 1-r_1-r_2 & 0 \\ 0 & 0 & 0 & 0 & s_1 & s_2 & s_2 & 1-s_1-2s_2 \end{bmatrix} \quad (8)$$

Table 3 shows the expected, discounted payoffs $x(y, P)$ obtained when y is as given by Figure 3 above, and P given by transition probability matrix in (8).

Note that players 1 and 3, respectively, are responsible for the transitions from g_1 to g_2 and from g_1 to g_4, both with probability p_1, because they gain from breaking their link with player 2. The corresponding differences in payoffs $x_{1,g_2} - x_{1,g_1} = x_{3,g_4} - x_{3,g_1}$ are equal to $\frac{1-\delta+4\delta q}{16(1-\delta+\delta q)(1-\delta+2\delta p_1)}$, which is positive for any value of $\delta \in [0, 1)$. Those same players are also responsible for the transitions from g_2 to g_5 or from g_4 to g_5 by deleting again their link with player 2. The corresponding differences in payoffs $x_{3,g_5} - x_{3,g_2} = x_{1,g_5} - x_{1,g_4}$ are equal to $\frac{3}{16(1-\delta+\delta q)}$, which is again positive for any value of $\delta \in [0,1)$. Concerning r_1, transitions from g_6 to g_2 or from g_7 to g_4 are governed by players 1 and 3 creating their link 13. The corresponding differences $x_{1,g_2} - x_{1,g_6} = x_{3,g_4} - x_{3,g_7}$ and $x_{3,g_2} - x_{3,g_6} = x_{1,g_4} - x_{1,g_7}$ are respectively equal to $\frac{13(1-\delta)^2+16\delta q(1-\delta)+3\delta^2 q r_2}{16(1-\delta)(1-\delta+\delta q)(1-\delta+\delta r_1+\delta r_2)}$ and $\frac{13(1-\delta)^2+22\delta q(1-\delta)+9\delta^2 q r_2}{48(1-\delta)(1-\delta+\delta q)(1-\delta+\delta r_1+\delta r_2)}$, which are positive for any value of $\delta \in [0,1)$. The transition from g_8 to g_5 is also governed by players 1 and 3 creating their link 13, with probability s_1. The corresponding differences in payoffs $x_{1,g_5} - x_{1,g_8} = x_{3,g_5} - x_{3,g_8}$ are equal to

$$\frac{1}{1-\delta+\delta s_1+\delta s_2}\left(1+\delta s_2 \frac{32(1\delta+\delta q)+9\delta(r_1+r_2)+\frac{57}{2}}{24(1-\delta+\delta q)(1-\delta+\delta r_1+\delta r_2)}\right),$$

which is positive for any value of $\delta \in [0,1)$. The rest of transitions, namely the ones governed by probabilities p_2, r_2 and s_2, correspond to a forward-looking network formation scheme only if δ is small enough.

Concerning p_2, the transition from g_1 to g_3 is governed by players 1 or 3 severing their link. The corresponding differences in payoffs $x_{1,g_3} - x_{1,g_1} = x_{3,g_3} - x_{3,g_1}$ are equal to $\frac{(1-\delta)(1-\delta+\delta q)-6\delta^2 p_1 q}{16(1-\delta)(1-\delta+\delta q)(1-\delta+2\delta p_1 \delta p_2)}$, which is positive only for δ small enough. Concerning r_2, the transitions from g_6 to g_3 or from g_7 to g_3 are governed by either player 1 or player 3 creating their link with player 2. The corresponding differences in payoffs $x_{1,g_3} - x_{1,g_6} = x_{3,g_3} - x_{3,g_7}$ and $x_{2,g_3} - x_{2,g_6} = x_{2,g_3} - x_{2,g_7}$ are respectively equal to $\frac{13(1-\delta)^2+13\delta q(1-\delta)-3\delta^2 r_1 q}{16(1-\delta)(1-\delta+\delta q)(1-\delta+\delta r_1 \delta r_2)}$ and $\frac{7(1-\delta)(1-\delta+\delta q)+39\delta^2 r_1 q}{24(1-\delta)(1-\delta+\delta q)(1-\delta+\delta r_1 \delta r_2)}$, the latter being positive for all $\delta \in [0,1)$ but the former being positive only for δ small enough (note that both being non negative is required). Concerning s_2, the transitions from g_8 to g_6 or from g_8 to g_7 are governed by either player 1 or player 3 creating their link with player 2. The corresponding differences in payoffs $x_{1,g_7} - x_{1,g_8} = x_{2,g_6} - x_{2,g_8}$ and $x_{2,g_6} - x_{2,g_8} = x_{2,g_7} - x_{2,g_8}$ are respectively equal to

$$\frac{1}{(1-\delta)(1-\delta+\delta s_1+2\delta s_2)}\left((1-\delta)(1-\delta+\delta r_1)x_{1,g_7}+\frac{4}{3}(1-\delta)\delta s_2-\delta s_1\right),$$

and

$$\frac{1-\delta+\delta s_1}{(1-\delta+\delta s_1+2\delta s_2)(1-\delta+\delta r_1+\delta r_2)}\left(\frac{4}{3}+\frac{13\delta r_1}{8(1-\delta+\delta q)}+\frac{13\delta r_2}{8(1-\delta)}\right),$$

the latter being positive for all $\delta \in [0,1)$ but the former being positive only for δ small enough.

As a conclusion if δ is big enough the scheme above formed by the allocation rule in Figure 3 and the transition probability matrix given by matrix (8) cannot be a forward-looking network formation scheme. We can simply revert the direction of the transitions given by p_2, r_2 and s_2 as follows. Consider the following family of transition probabilities, where p_1, p_2, q, r_1, r_2, s_1, and s_2 take values such that P is a transition probability matrix.

$$P = \begin{bmatrix} 1-2p_1 & p_1 & 0 & p_1 & 0 & 0 & 0 & 0 \\ 0 & 1-q & 0 & 0 & q & 0 & 0 & 0 \\ p_2 & 0 & 1-p_2-2r_2 & 0 & 0 & r_2 & r_2 & 0 \\ 0 & 0 & 0 & 1-q & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 & 0 & 1-r_1-s_2 & 0 & s_2 \\ 0 & 0 & 0 & r_1 & 0 & 0 & 1-r_1-s_2 & s_2 \\ 0 & 0 & 0 & 0 & s_1 & 0 & 0 & 1-s_1 \end{bmatrix} \tag{9}$$

Table 4 shows the expected, discounted payoffs $x(y,P)$ obtained when y is as given by Figure 3 above, and P given by transition probability matrix in (9).

To better understand the forward looking scheme for δ big enough, let us give numerical values to the transition probability function in (9) as follows.

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} \end{bmatrix} \tag{10}$$

Table 5 shows the expected, discounted payoffs $x(y,P)$ obtained when y is as given by Figure 3 above, and P given by transition probability matrix in (10).

As before, players 1 and 3, respectively, are responsible for the transitions from g_1 to g_2 or from g_1 to g_4, both with probability $\frac{1}{3}$ this time, because they gain from breaking their link with player 2. The corresponding differences in payoffs $x_{1,g_2} - x_{1,g_1} = x_{3,g_4} - x_{3,g_1}$ are equal to $\frac{9+15\delta}{16(3-\delta)}$, which is positive for any value of $\delta \in [0,1)$. Those same players are also responsible for the transitions from g_2 to g_5 or from g_4 to g_5 by deleting again their link with player 2. The corresponding differences in payoffs $x_{3,g_5} - x_{3,g_2} = x_{1,g_5} - x_{1,g_4}$ are equal to $\frac{9}{16(3-\delta)}$, which is again positive for any value of $\delta \in [0,1)$. Transitions from g_6 to g_2 or from g_7 to g_4 are governed by players 1 and 3 creating their link 13. The corresponding differences $x_{1,g_2} - x_{1,g_6} = x_{3,g_4} - x_{3,g_7}$ and $x_{3,g_2} - x_{3,g_6} = x_{1,g_4} - x_{1,g_7}$ are respectively equal to $\frac{117+18\delta}{16(3-\delta)^2}$ and $\frac{21+50\delta}{16(3-\delta)^2}$, which are positive for any value of $\delta \in [0,1)$. The transition from g_8 to g_5 is also governed by players 1 and 3 creating their link 13, with probability $\frac{1}{3}$. The corresponding differences in payoffs $x_{1,g_5} - x_{1,g_8} = x_{3,g_5} - x_{3,g_8}$ are equal to $\frac{3}{3-\delta}$, which is positive for any value of $\delta \in [0,1)$. The rest of transitions, namely the transitions from g_3 to g_1, g_6, and g_7, governed by probabilities equal to $\frac{1}{3}$ each, correspond to a forward-looking network formation scheme if δ is big enough.

The transition from g_3 to g_1 is governed by players 1 and 3 creating their link. The corresponding differences in payoffs $x_{1,g_1} - x_{1,g_3} = x_{3,g_1} - x_{3,g_3}$ are equal to $\frac{91\delta+65\delta(1+\delta)-27}{48(3-\delta)^2)}$, which is positive only for δ big enough. The transitions from g_3 to g_6 or to g_7 are governed by either player 1 or player 3 deleting their link with player 2 in g_3. The corresponding difference in payoffs $x_{1,g_6} - x_{1,g_3} = x_{3,g_7} - x_{3,g_3}$ are equal to $\frac{82\delta+65(1+\delta)-351}{48(3-\delta)^2}$, positive only for δ big enough.

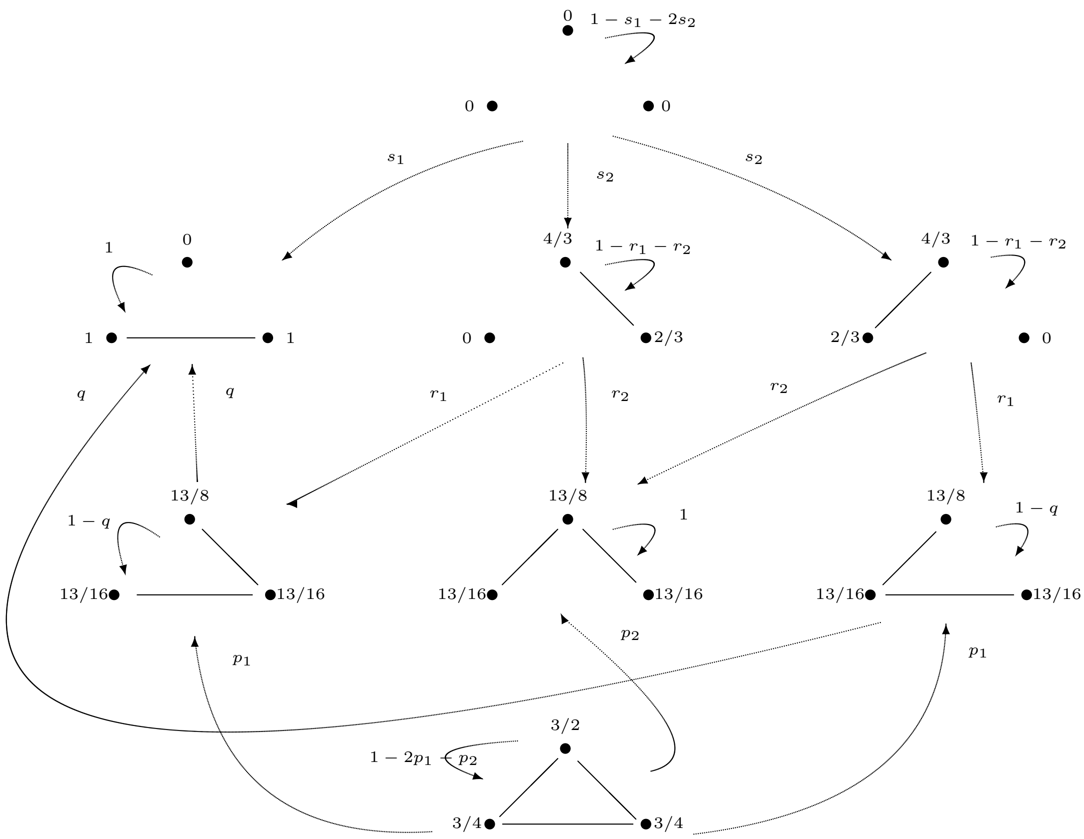
Figure 4 visualizes the dynamics in the forward-looking network formation scheme when the stage-payoffs are given by the allocation rule in Figure 3 and δ is small. Figure 5 visualizes the dynamics in the forward-looking network formation when the stage-payoffs are given by the allocation rule in Figure 3 and δ is big. Numbers next to the nodes indicate the payoff that players obtain in each given graph. Transitions from one graph to another are indicated by means of arrows, and the letters next to each arrow indicate the corresponding probability. Only positive transition probabilities are indicated.

When players are impatient (δ small), network g_5, where 1 and 3 are directly connected and disconnected from 2, and network g_3, where 1 and 3 do not connect directly to each other but only to 2, are both stationary states of the Markov chain, and no one is absorbing. But when players are patient, only network g_5, where 1 and 3 are directly connected, is a stationary and absorbing state of the Markov chain. Indeed, player 2 always gets double the amount of what player 1 or 3 get. Starting at g_3 players 1 and 3 are willing to create their link and lose money in the short run only when they are patient enough, because by creating their link they lose a payoff of $\frac{13}{16} - \frac{3}{4}$ immediately, but the process will lead to g_5, where they would obtain a payoff of $1 > \frac{13}{16}$. If players 1 and 3 are impatient, the loss of $\frac{13}{16} - \frac{3}{4}$ immediately is more important than obtaining 1 in the future, because the latter is strongly discounted by δ.

Furthermore, when players 1 and 3 are patient enough, they delete their direct connection to 2, whenever this connection is the only one present in the network. Like that, they obtain zero immediately, sacrificing $\frac{2}{3}$, but they get compensated by obtaining 1 in the future because they anticipate that they will create their link 13 once the empty network is reached. When they are impatient, players 1 and 3 will not disconnect their link with player 2 in g_6 or g_7, and they will instead choose to create the link 13 directly, a strategy that gives a (stage-wise) higher payoff than passing through the empty graph.

To conclude this section, we would like to illustrate the limits of our last results, stated in Proposition 3.4 and in Proposition 3.8, and the role that externalities across components play to contribute to the tension of stability and efficiency. In order to do that, we introduce other two value functions w that do not satisfy component additivity, but such that they are both link-monotonic. In the first case, we cannot find an allocation rule y that is component efficient and monotonic with respect to own links. In the second case, the empty graph, and not the complete graph, is strongly efficient. Both effects are due to the un-balances created by the externalities across components.

Figure 4: The forward-looking network formation scheme when δ is small

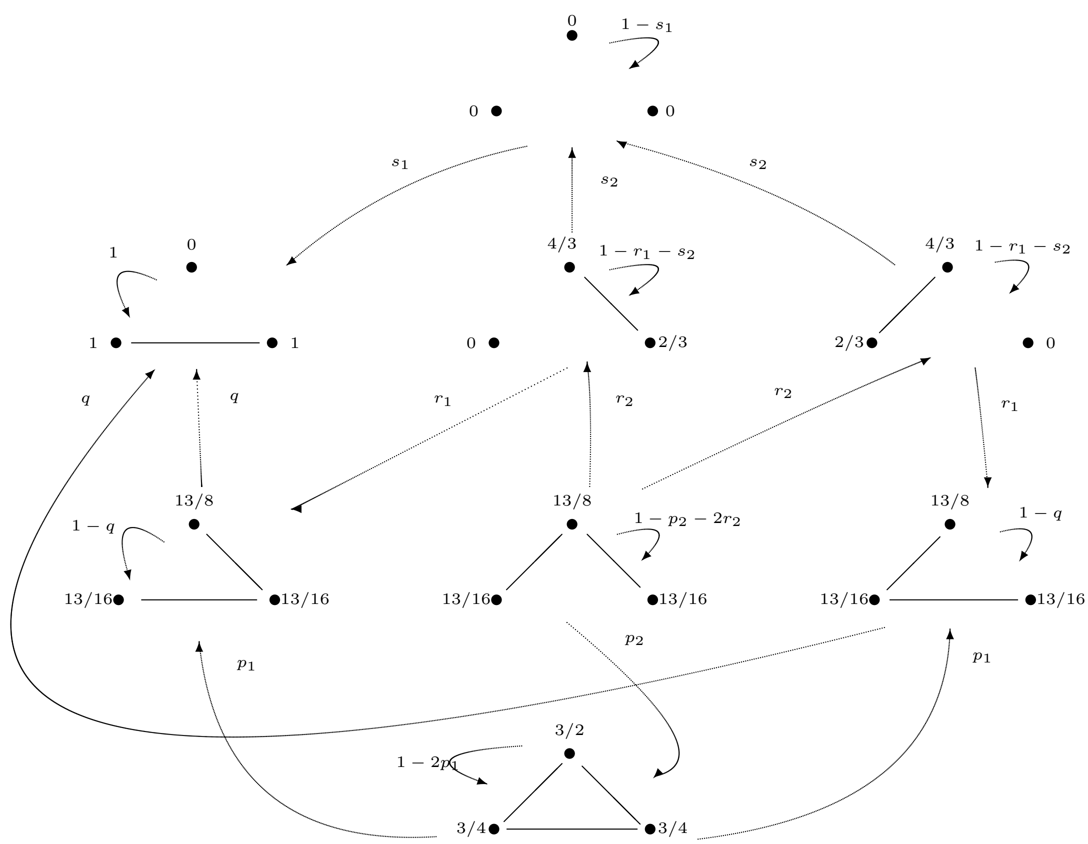


Example 4: A link-monotonic value function that that is not component additive and for which no allocation rule that is component efficient and monotonic with respect to own links exists.

Let us consider the value function w indicated in Figure 6 below. The number next to each component indicates the value.

Let us see why we cannot build a payoff vector such that it is component efficient and it is monotonic with respect to own links. Assume we can find one. Then any player participating in the link present in each of the one-link graphs, namely g_5, g_6, and g_7, should obtain a positive payoff, as by monotonicity with respect to own links they should earn more than in g_8. This in turn means that the player participating in two links in each of the two-link graphs, namely the player at bottom right in g_2, the player at the top in g_3, and the player at bottom left

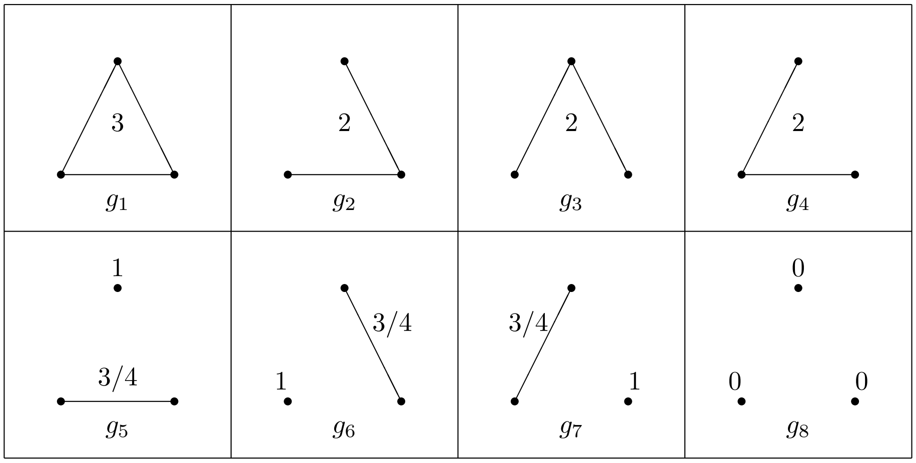
Figure 5: The forward-looking network formation scheme when δ is big



in g_4, should obtain a positive payoff too, as they should obtain more than when they are participating in only one link. Unfortunately, players that participate in only one link in those two-link graphs should obtain more than a payoff of 1, as that is the payoff they obtain when they are disconnected in a one-link network. This means that the value of 2, which is the value of a two-link graph, is not enough to pay all players more than what they would obtain by removing any of their links. We cannot find an allocation rule that is component efficient and satisfies monotonicity with respect to own links, therefore there is no guarantee to find a forward-looking network formation scheme such that the complete graph is an absorbing state of the dynamics given by the transition probability matrix.

Example 5: A link-monotonic value function that is not component additive and for which the empty graph is strongly efficient.

Figure 6: Non existence of an allocation rule that is monotonic with respect to own links



Let us consider the value function w indicated in Figure 7. The number next to each component indicates the value.

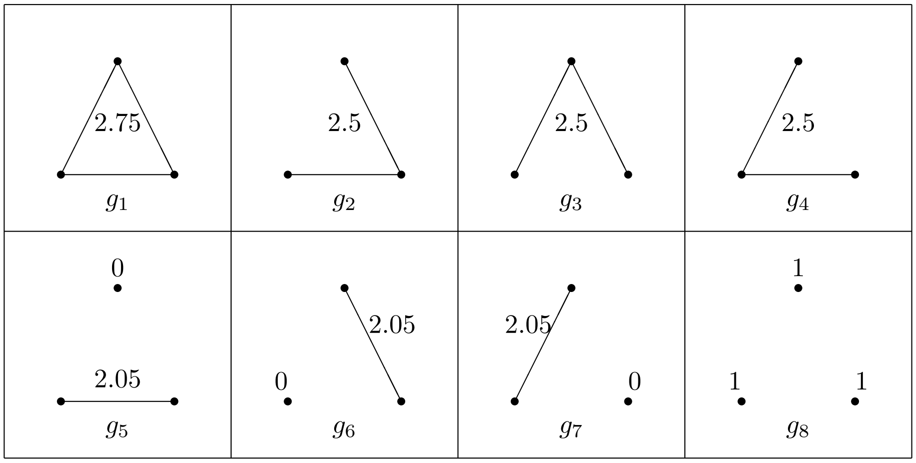
Note that the value function w is link monotonic, because connecting two players yields a value of 2.05, higher than $1 + 1$, the sum of the individual values at the empty network. Given that there are externalities across components, connecting two players yields a negative externality on the player that remains disconnected, a fact that prevents the one-link network to obtain a higher total value than the empty network. If disconnected agents in the one-link network suffered no externality, their stand-alone value would again be equal to 1, and by link monotonicity of the value function, we would have that the two-link networks and the complete network yield a higher value than the one-link networks, and by transitivity, than the empty network. The negative externality across components breaks the strong efficiency of the complete network when the value function is link monotonic.

5 Concluding Comments

The purpose of this paper is to to provide a solution concept to analyze the problem of network value allocation at the same time as the problem of network formation. These are closely related problems since the payoff a player gets in each possible network gives incentives for creating certain links and severe others. We derive very general results by leaving the conditions on the allocation rule open, and keeping very mild conditions on the transition probabilities.

Similarly to Dutta et al. (2005), a condition of monotonicity of the stationary or stage-wise

Figure 7: A link-monotonic value function with the empty network being strongly efficient



allocation rule guarantees that the process will converge to the complete graph. This is so as the creation of links are at any time always beneficial. It is not true in general that for any value function such that the complete network is efficient we can find a stationary allocation rule that is component efficient and monotonic with respect to own links. It is not true in general either that for any value function such that we can find a component efficient allocation rule that is monotonic with respect to own links the complete network will be strongly efficient. Nevertheless, when there are no externalities across components and the value function is link monotonic, we can find a forward-looking network formation scheme such that the allocation rule is monotonic with respect to own links and the complete graph is an absorbing state of the transition probability matrix. If the value function is not component additive, then the value function being link monotonic is not strong enough to obtain an efficiency result. In such a case, the condition is a stronger version of critical-link monotonicity, in which the per-capita value of a component is increasing as the component gets larger and more connected.

Our analytical framework calls for further research on forward-looking agents that are at the same time forming the network and deciding how to allocate its value. An interesting avenue would be to impose conditions on the allocation rule that depend on the transition probability matrix, capturing the idea that the value allocation is a result of some bargaining among forward-looking players. We will expect the existence result to hold for a smaller interval of discount factors, or for value functions with more structure, as done by Navarro (2013) in a recent working paper, where the implications of imposing a forward-looking version of equal balance contributions (or fairness) are explored.

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Table 3: The expected discounted payoffs $x(y)$ obtained from allocation in Figure 3 and matrix in (8)

$$x_{1,g_1} = x_{3,g_1} = \frac{1}{1-\delta+2\delta p_1+\delta p_2}\left(\frac{3}{4} + \delta p_1 \frac{13(1-\delta)+16\delta q}{8(1-\delta)(1-\delta+\delta q)} + \delta p_2 \frac{13}{16(1-\delta)}\right), \text{ and}$$

$$x_{2,g_1} = \frac{1}{1-\delta+2\delta p_1+\delta p_2}\left(\frac{3}{2} + \delta p_1 \frac{13}{4(1-\delta+\delta q)} + \delta p_2 \frac{13}{8(1-\delta)}\right)$$

$$x_{1,g_2} = x_{3,g_2} = \frac{13(1-\delta)+16\delta q}{16(1-\delta)(1-\delta+\delta q)}, \text{ and}$$

$$x_{2,g_2} = \frac{13}{8(1-\delta+\delta q)}$$

$$x_{1,g_3} = x_{3,g_3} = \frac{13}{16(1-\delta)}, \text{ and } x_{2,g_3} = \frac{13}{8(1-\delta)}$$

$$x_{1,g_4} = x_{3,g_4} = \frac{13(1-\delta)+16\delta q}{16(1-\delta)(1-\delta+\delta q)}, \text{ and } x_{2,g_4} = \frac{13}{8(1-\delta+\delta q)}$$

$$x_{1,g_5} = x_{3,g_5} = \frac{1}{1-\delta}, \text{ and } x_{2,g_5} = 0$$

$$x_{1,g_6} = \frac{1}{16(1-\delta)(1-\delta+\delta r_1+\delta r_2)}\left(13\delta(r_1+r_2) + \frac{3\delta^2 q r_1}{1-\delta+\delta q}\right),$$

$$x_{2,g_6} = \frac{1}{8(1-\delta)(1-\delta+\delta r_1+\delta r_2)}\left(\frac{32}{3}(1-\delta) + 13\delta r_1 1-\delta 1-\delta+\delta q + 13\delta r_2\right), \text{ and}$$

$$x_{3,g_6} = \frac{1}{16(1-\delta)(1-\delta+\delta r_1+\delta r_2)}\left(\frac{32}{3}(1-\delta) + 13\delta(r_1+r_2) + \frac{3\delta^2 q r_1}{1-\delta+\delta q}\right)$$

$$x_{1,g_7} = \frac{1}{16(1-\delta)(1-\delta+\delta r_1+\delta r_2)}\left(\frac{32}{3}(1-\delta) + 13\delta(r_1+r_2) + \frac{3\delta^2 q r_1}{1-\delta+\delta q}\right),$$

$$x_{2,g_7} = \frac{1}{8(1-\delta)(1-\delta+\delta r_1+\delta r_2)}\left(\frac{32}{3}(1-\delta) + 13\delta r_1 1-\delta 1-\delta+\delta q + 13\delta r_2\right), \text{ and}$$

$$x_{3,g_7} = \frac{1}{16(1-\delta)(1-\delta+\delta r_1+\delta r_2)}\left(13\delta(r_1+r_2) + \frac{3\delta^2 q r_1}{1-\delta+\delta q}\right)$$

$$x_{1,g_8} = x_{3,g_8} = \frac{\delta}{(1-\delta+\delta s_1+2\delta s_2)}\left[\frac{s_1}{1-\delta} + s_2 s_2\left(x_{1,g_6} + x_{2,g_7}\right)\right] \text{ and}$$

$$x_{2,g_8} = \frac{\delta s_2}{4(1-\delta)(1-\delta+\delta s_1+2\delta s_2)}\left[\frac{32}{3}(1-\delta) + 13\delta r_1 \frac{1-\delta}{1-\delta+\delta q} + 13\delta r_2\right]$$

Table 4: The expected discounted payoffs $x(y)$ obtained from allocation in Figure 3 and matrix in (9)

$$x_{1,g_1} = x_{3,g_1} = \frac{1}{1-\delta+2\delta p_1}\left(\frac{3}{4} + \delta p_1 \frac{13(1-\delta)+16\delta q}{8(1-\delta)(1-\delta+\delta q)}\right), \text{ and}$$

$$x_{2,g_1} = \frac{1}{1-\delta+2\delta p_1}\left(\frac{3}{2} + \delta p_1 \frac{13}{4(1-\delta+\delta q)}\right)$$

$$x_{1,g_2} = x_{3,g_2} = \frac{13(1-\delta)+16\delta q}{16(1-\delta)(1-\delta+\delta q)}, \text{ and}$$

$$x_{2,g_2} = \frac{13}{8(1-\delta+\delta q)}$$

$$x_{1,g_3} = x_{3,g_3} = \frac{1}{1-\delta+\delta p_2+2\delta r_2}\left(\frac{13}{16} + \delta p_2 x_{1,g_1} + \delta r_2\left(x_{1,g_6} + x_{1,g_7}\right)\right), \text{ and}$$

$$x_{2,g_3} = \frac{1}{1-\delta+\delta p_2+2\delta r_2}\left(\frac{13}{8} + \delta p_2 x_{2,g_1} + \delta r_2\left(x_{2,g_6} + x_{2,g_7}\right)\right)$$

$$x_{1,g_4} = x_{3,g_4} = \frac{13(1-\delta)+16\delta q}{16(1-\delta)(1-\delta+\delta q)}, \text{ and}$$

$$x_{2,g_4} = \frac{13}{8(1-\delta+\delta q)}$$

$$x_{1,g_5} = x_{3,g_5} = \frac{1}{1-\delta}, \text{ and } x_{2,g_5} = 0$$

$$x_{1,g_6} = \frac{1}{(1-\delta)(1-\delta+\delta r_1+\delta s_2)}\left(\delta r_1 \frac{13(1-\delta)+16\delta q}{16(1-\delta+\delta q)} + \frac{\delta^2 s_1 s_2}{1-\delta+\delta s_1}\right),$$

$$x_{2,g_6} = \frac{1}{1-\delta+\delta r_1+\delta s_2}\left(\frac{4}{3} + \frac{13\delta r_1}{8(1-\delta+\delta q)}\right), \text{ and}$$

$$x_{3,g_6} = \frac{1}{(1-\delta)(1-\delta+\delta r_1+\delta s_2)}\left(\frac{2}{3}(1-\delta) + \delta r_1 \frac{13(1-\delta)+16\delta q}{16(1-\delta+\delta q)} + \frac{\delta^2 s_1 s_2}{1-\delta+\delta s_1}\right)$$

$$x_{1,g_7} = \frac{1}{(1-\delta)(1-\delta+\delta r_1+\delta s_2)}\left(\frac{2}{3}(1-\delta) + \delta r_1 \frac{13(1-\delta)+16\delta q}{16(1-\delta+\delta q)} + \frac{\delta^2 s_1 s_2}{1-\delta+\delta s_1}\right),$$

$$x_{2,g_7} = \frac{1}{1-\delta+\delta r_1+\delta s_2}\left(\frac{4}{3} + \frac{13\delta r_1}{8(1-\delta+\delta q)}\right), \text{ and}$$

$$x_{3,g_7} = \frac{1}{(1-\delta)(1-\delta+\delta r_1+\delta s_2)}\left(\delta r_1 \frac{13(1-\delta)+16\delta q}{16(1-\delta+\delta q)} + \frac{\delta^2 s_1 s_2}{1-\delta+\delta s_1}\right)$$

$$x_{1,g_8} = x_{3,g_8} = \frac{\delta s_1}{(1-\delta)(1-\delta+\delta s_1)} \text{ and } x_{2,g_8} = 0$$

Table 5: The expected discounted payoffs $x(y)$ obtained from allocation in Figure 3 and matrix in (10)

$$x_{1,g_1} = x_{3,g_1} = \frac{54-33\delta+11\delta^2}{8(1-\delta)(3-\delta)^2}, \text{ and } x_{2,g_1} = \frac{54+21\delta}{4(3-\delta)^2}$$

$$x_{1,g_2} = x_{3,g_2} = \frac{39-7\delta}{16(1-\delta)(3-\delta)}, \text{ and } x_{2,g_2} = \frac{39}{8(3-\delta)}$$

$$x_{1,g_3} = x_{3,g_3} = \frac{13}{16} + \frac{\delta(51-29\delta+26\delta^2)}{12(1-\delta)(3-\delta)^2}, \text{ and } x_{2,g_3} = \frac{13}{8} + \frac{\delta(75+14\delta)}{6(3-\delta)^2}$$

$$x_{1,g_4} = x_{3,g_4} = \frac{39-7\delta}{16(1-\delta)(3-\delta)}, \text{ and } x_{2,g_4} = \frac{39}{8(3-\delta)}$$

$$x_{1,g_5} = x_{3,g_5} = \frac{1}{1-\delta}, \text{ and } x_{2,g_5} = 0$$

$$x_{1,g_6} = \frac{39\delta+25\delta^2}{16(1-\delta)(3-\delta)^2}, \; x_{2,g_6} = \frac{96+7\delta}{8(3-\delta)^2}, \text{ and } x_{3,g_6} = \frac{96-89\delta+57\delta^2}{16(1-\delta)(3-\delta)^2}$$

$$x_{1,g_7} = \frac{96-89\delta+57\delta^2}{16(1-\delta)(3-\delta)^2}, \; x_{2,g_7} = \frac{96+7\delta}{8(3-\delta)^2}, \text{ and } x_{3,g_7} = \frac{39\delta+25\delta^2}{16(1-\delta)(3-\delta)^2}$$

$$x_{1,g_8} = x_{3,g_8} = \frac{2\delta}{(1-\delta)(3-\delta)} \text{ and } x_{2,g_8} = 0$$