SUPPLEMENTARY INFORMATION

for

"A quantitative witness for Greenberger-Horne-Zeilinger entanglement"

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A. Two-qubit GHZ-symmetric states

The twirling operation equation (2) in the main text defines a family of GHZ-symmetric mixed states for each qubit number $N \geq 2$. The simplest case is that of two qubits. The symmetrization $\rho^{\mathrm{S}}(\rho)$ of an arbitrary two-qubit state ρ is characterised by two real parameters for which we choose the following parametrization[20]

$$x(\rho) = \frac{1}{2}\left(\rho_{00,11} + \rho_{11,00}\right) \tag{S1}$$

$$y(\rho) = \frac{1}{\sqrt{2}}\left(\rho_{00,00} + \rho_{11,11} - \frac{1}{2}\right) \quad . \tag{S2}$$

We emphasise that these coordinates (as well as those in equations (4), (5) in the main text) are defined for normalised density matrices. The corresponding states form a triangle in the xy plane, see Supplementary Fig. S1.

The entanglement monotone considered here is the concurrence $C(\psi) = |\langle\psi^*|\,\sigma_y \otimes \sigma_y\,|\psi\rangle|$. Its convex-roof extension[15] is defined in analogy with equation (9) in the main text via

$$C(\rho) = \min_{\text{all decomp.}} \sum p_j\ C(\psi_j) \quad , \tag{S3}$$

i.e., the average concurrence minimised over all possible pure-state decompositions $\{p_j, \psi_j\}$ of the two-qubit state $\rho = \sum p_j\,|\psi_j\rangle\langle\psi_j|$. The concurrence of GHZ-symmetric two-qubit states is a function of the coordinates[20]

$$C(x, y) = \max\left(0, 2|x| + \sqrt{2}y - \frac{1}{2}\right) \quad . \tag{S4}$$

We can rewrite this formula in terms of the matrix elements of the original state ρ using equations (S1), (S2), keeping in mind that symmetrization cannot increase the concurrence:

$$C(\rho) \geq \max\left(0, |\rho_{00,11} + \rho_{11,00}| + \rho_{00,00} + \rho_{11,11} - 1\right) \tag{S5}$$

i.e., we obtain equation (3) of the main text. The analogy with some of the equations in Ref.[11] is remarkable, in particular with equation (6), if we use $\mathcal{W}_2 = \frac{1}{2}\mathbb{1}_4 - |\Phi^+\rangle\langle\Phi^+|$ as the

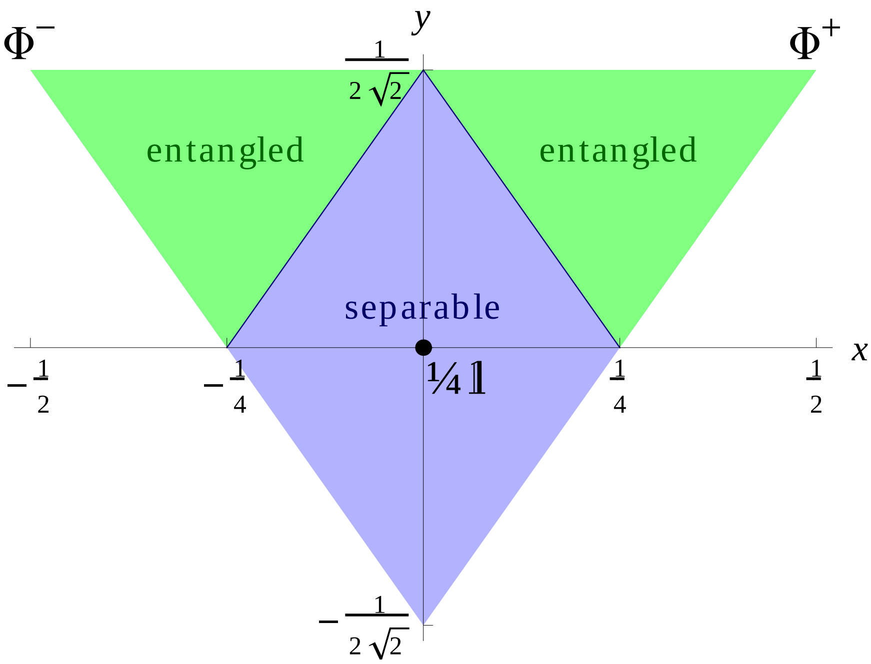

Figure S1. The geometric representation of two-qubit GHZ-symmetric states[20]. The upper corners are defined by the Bell states $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ (left) and $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ (right). The lower corner represents the mixture $\frac{1}{2}(|\Psi^+\rangle\langle\Psi^+| + |\Psi^-\rangle\langle\Psi^-|)$ with $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$. The blue region shows the separable states whereas the states in the green region have non-vanishing concurrence.

only witness (with the optimal slope $r = -2$ and the offset $c = 0$). It arises due to the fact that the concurrence of GHZ-symmetric two-qubit states is a linear function, and the linear one-witness approximation in Ref.[11] becomes exact. We note also that our concurrence formula in the main text, $C(\rho) = \max(0, 2f - 1)\,\mathrm{tr}\,\rho^{\mathrm{NF}}$, is reminiscent of the so-called *fully entangled fraction*[4]. However, the optimisation of the fully entangled fraction includes only local unitaries while our approach allows for general SLOCC operations.

B. Normal form of two-qubit states

According to Verstraete *et al.* it is always possible to obtain a Bell-diagonal (renormalised) normal form ρ^{NF} for two-qubit states[7,18]. That is, ρ^{NF} can be written as a mixture

$$\rho^{\mathrm{NF}} = \sum_{j=1}^{4} \lambda_j \, |\phi_j\rangle\langle\phi_j|$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq 0$, $\sum \lambda_j = 1$ and $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ is a permutation of the four Bell states $\{\Phi^+, \Psi^+, \Psi^-, \Phi^-\}$. Evidently the Bell-diagonal form with maximum $\tilde{\rho}^{\rm NF}_{00,11}$ is one where $\phi_1 = \Phi^+$ and $\phi_4 = \Phi^-$. It is not difficult to see that by applying appropriate combinations of the local operations $\mathbb{1}_2$, σ_x, σ_y, σ_z as well as

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

to the qubits in $\rho^{\rm NF}$, it is always possible to achieve the correct permutation of the ϕ_j[19]. Note that this implies that there cannot be another Bell-diagonal normal form derived from the original two-qubit state ρ with a concurrence larger than that of $\tilde{\rho}^{\rm NF}$.

Bennett *et al.* have demonstrated that the concurrence of a Bell-diagonal two-qubit density matrix depends only on its largest eigenvalue[4]. Therefore $C(\tilde{\rho}^{\rm NF}) = \max(0, 2\lambda_1 - 1)$ can be determined exactly without reference to the Wootters-Uhlmann method[8,9] and does not change on applying the symmetrization operation equation (2) in the main text.

We mention that in the two-qubit case the different optimisation criteria for step *(2)* (*i.e.*, maximal concurrence of $\tilde{\rho}^{\rm NF}$, maximal fidelity with Φ^+, maximal Re $\tilde{\rho}^{\rm NF}_{00,11}$, and minimal Hilbert-Schmidt distance from Φ^+) are equivalent.

C. Entanglement loss in three-qubit symmetrization

In the main text we have mentioned that for three qubits one may not expect to find the exact three-tangle for arbitrary mixed states, and that in general entanglement is lost in the symmetrization. This statement is illustrated by the mixtures

$$\rho_1 = p\,|\mathrm{GHZ}_+\rangle\langle \mathrm{GHZ}_+| + (1-p)\,|W\rangle\langle W|$$

versus

$$\rho_2 = p\,|\mathrm{GHZ}_+\rangle\langle \mathrm{GHZ}_+| + \frac{1-p}{2}\left(|W\rangle\langle W| + |\bar{W}\rangle\langle \bar{W}|\right)$$

where $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$, $|\bar{W}\rangle = \sigma_x^{\otimes 3}\,|W\rangle$. It is known[21,22] that ρ_1 has non-vanishing three-tangle for $p \gtrsim 0.627$, as opposed to ρ_2 which is GHZ-entangled only for $p > 3/4$. Note that ρ_2 is already given in the normal form. For ρ_1 the normal form can be calculated analytically.

In the range $0.70 < p < 0.74$ the exact three-tangle of $\rho_1(p)$ is $0.19 < \tau_3(\rho_1(p)) < 0.31$[17] while $\tau_3(\rho_2(p)) = 0$. The optimisation leaves $\rho_1^{\rm NF}/\operatorname{tr}\rho_1^{\rm NF}$ and ρ_2 practically unchanged. The corresponding points in the xy plane are located close to each other and still in the region of W states, *i.e.*, we obtain the estimates $\tau_3(\rho^{\rm S}(\rho_1^{\rm NF})) = \tau_3(\rho^{\rm S}(\rho_2)) = 0$. Hence, the GHZ entanglement in $\rho_1^{\rm NF}$ will be underrated while that of ρ_2 is determined exactly.

D. Relation between projective GHZ witness and quantitative witness

In the discussion part of the main text we mention that the standard projective GHZ witness $\mathcal{W}_3 = \frac{3}{4}\mathbb{1}_8 - |{\rm GHZ}_+\rangle\langle{\rm GHZ}_+|$ can, in modified form, be used as a quantitative witness. Here we explain this fact in more detail.

The standard witness $\mathcal{W}_3$ detects the GHZ-type entanglement in an arbitrary three-qubit state ρ: it is a GHZ-class state if $\operatorname{tr}(\mathcal{W}_3\rho) < 0$. Our aim is to elucidate that $\mathcal{W}_3' = -4\mathcal{W}_3$ is a quantitative witness for ρ, *i.e.*, that

$$\tau_3(\rho) \;\geq\; \operatorname{tr}(\mathcal{W}_3'\rho) \;=\; -4\operatorname{tr}(\mathcal{W}_3\rho) \tag{S6}$$

is a lower bound to $\tau_3(\rho)$ for arbitrary three-qubit states ρ.

It appears obvious that, in order to obtain a non-optimal witness, it is not necessary to use the GHZ/W line which is difficult to handle analytically. The solution of the two-qubit case suggests the following simpler alternative: We start at the end point of the GHZ/W line $P = (x = \frac{3}{8}, y = \frac{\sqrt{3}}{6}, \tau_3 = 0)$ and consider the straight line which contains this point and is parallel to the lower-left border of the triangle. Its equation is $y_P = \left(-2x + \frac{5}{4}\right)/\sqrt{3}$. It crosses the triangle only in the GHZ part, that is, its points lie above the GHZ/W line. For all the states $\rho_P^{\rm S}$ which correspond to triangle points on this line the Hilbert-Schmidt scalar product with GHZ$_+$ equals $({\rm GHZ}_+, \rho_P^{\rm S}) \equiv \frac{1}{2}\operatorname{tr}\left(|{\rm GHZ}_+\rangle\langle{\rm GHZ}_+|\,\rho_P^{\rm S}\right) = \frac{3}{8}$.

From Supplementary Fig. S2 it is easy to see that a plane which contains this line and the point $(x_{{\rm GHZ}_+} = \frac{1}{2}, y_{{\rm GHZ}_+} = \frac{\sqrt{3}}{4}, \tau_3 = 1)$ represents a lower bound to the three-tangle of GHZ-symmetric three-qubit states. It is straightforward to check that the function $\rho^{\rm S} \mapsto \tau_3^P(\rho^{\rm S})$ corresponding to the points of that plane is given by

$$\tau_3^P(\rho^{\rm S}) \;=\; -4\operatorname{tr}\left(\mathcal{W}_3\rho^{\rm S}\right) \quad . \tag{S7}$$

As the plane $\tau_3^P(\rho^{\rm S})$ lies below the exact $\tau_3(\rho^{\rm S})$ we have also

$$\tau_3^P(\rho^{\rm S}) \;\leq\; \tau_3(\rho^{\rm S}) \quad .$$

Further, the operator $\mathcal{W}_3$ has GHZ symmetry so that for an arbitrary state ρ

$$\mathrm{tr}\left(\mathcal{W}_3\rho\right) \;=\; \mathrm{tr}\left(\mathcal{W}_3\rho^{\mathrm{S}}(\rho)\right) \quad .$$

By combining the preceding relations and the conclusions from the Section "Results" in the main text we obtain

$$-4\,\mathrm{tr}\left(\mathcal{W}_3\rho\right) = -4\,\mathrm{tr}\left(\mathcal{W}_3\rho^{\mathrm{S}}(\rho)\right) \leq \tau_3\left(\rho^{\mathrm{S}}(\rho)\right) \leq \tau_3\left(\rho^{\mathrm{S}}(\tilde{\rho}^{\mathrm{NF}})\right) \leq \tau_3(\rho) \tag{S8}$$

which confirms the desired result, equation (S6). We mention that also here there is a certain freedom whether or not one wants to optimise the state ρ before symmetrizing it. One may note the relation between this type of equation deriving from our method and some of the findings in Sections 3.4–3.6 of Ref.[12], as well as those in Ref.[23]

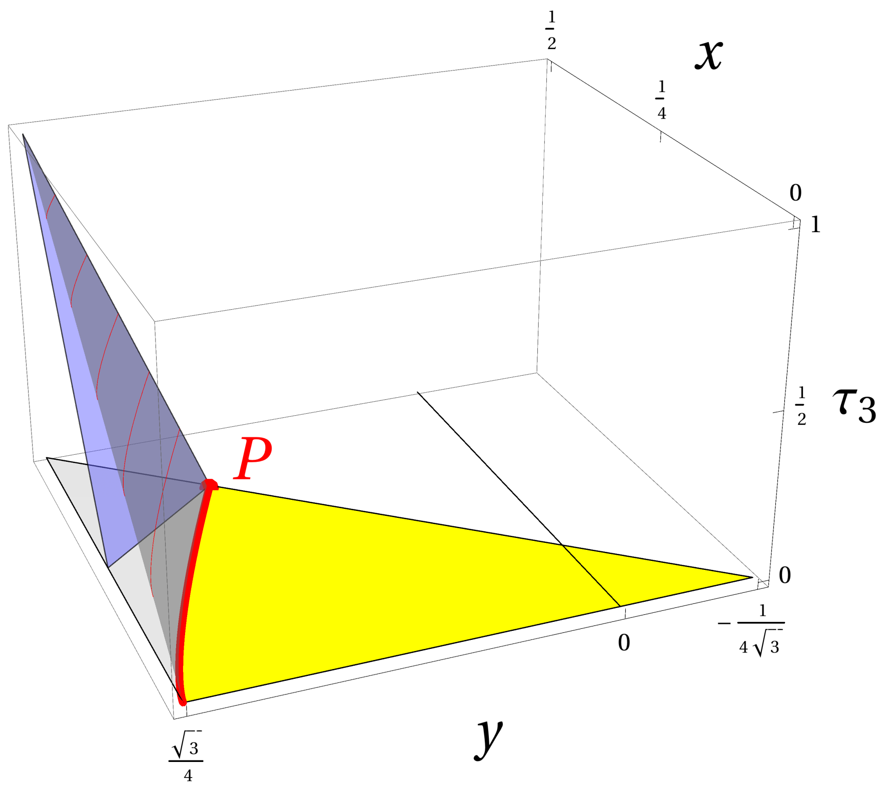

Figure S2. The three-tangle for three-qubit GHZ-symmetric states as in equation (6) (grey surface) compared to the non-optimal (but easy-to-handle) quantitative witness, supplementary equation (S7), blue triangle (see text).

From these remarks one might feel tempted to conclude that our method is a mere extension to the standard witness approach as it detects GHZ entanglement in a given state

ρ more or less according to the fidelity of the GHZ state and assigns a number to it. To clarify this point consider the example

$$\rho_3 \;=\; p\,|\mathrm{GHZ}_+\rangle\langle\mathrm{GHZ}_+| + (1-p)\,|001\rangle\langle 001| \quad .$$

The exact three-tangle is $\tau_3(\rho_3) = p$, that is, the state contains GHZ entanglement for arbitrarily small p. While the standard witness would not detect entanglement for $p < 3/4$ our approach produces the correct value (with a relative error $< 10^{-2}$) for values as small as $p \sim 10^{-5}$.