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A quantitative witness for Greenberger-Horne-Zeilinger entanglement

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Along with the vast progress in experimental quantum technologies there is an increasing demand for the quantification of entanglement between three or more quantum systems. Theory still does not provide adequate tools for this purpose. The objective is, besides the quest for exact results, to develop operational methods that allow for efficient entanglement quantification. Here we put forward an analytical approach that serves both these goals. We provide a simple procedure to quantify Greenberger-Horne-Zeilinger–type multipartite entanglement in arbitrary three-qubit states. For two qubits this method is equivalent to Wootters' seminal result for the concurrence. It establishes a close link between entanglement quantification and entanglement detection by witnesses, and can be generalised both to higher dimensions and to more than three parties.

It is a fundamental strength of physics as a science that most of its basic concepts have quantifiability built into their definition. Just think of, *e.g.*, length, time, or electrical current. Their quantifiability allows to measure and compare them in different contexts, and to build mathematical theories with them[1]. There is no doubt that entanglement is a key concept in quantum theory, but it seems to resist in a wondrous way that universal principle of quantification. The reason for this is, in the first place, that entanglement comes in many different disguises related to its resource character, *i.e.*, what one would like to do with it. In principle, there are numerous task-specific entanglement measures[2,3]. However, most of them cannot be calculated easily (nor measured or estimated) for generic mixed quantum states, and therefore it is difficult to use them.

There are notable exceptions, the concurrence[4] and the negativity for bipartite systems[5]. These measures have already provided deep insight into the nature of entanglement, but they also have their shortcomings. The concurrence is strictly applicable only to two-qubit systems while for the negativities it is not known how to distinguish entanglement classes. The generalisations of the concurrence (such as the residual tangle[6]) do quantify task-specific entanglement even for multipartite systems but again it is not known how to estimate them for general mixed quantum states.

There is another difficulty. An N-qubit density matrix ρ is characterised mathematically by $2^{2N} - 1$ real parameters. Reducing it to its so-called normal form[7]—which contains the essential entanglement information—removes $6N$ parameters. The entanglement measure is determined by the remaining exponentially many parameters which need to be processed to calculate the precise value. Even an operational method similar to that of Wootters-Uhlmann[8,9] would quickly reach its limits with increasing N. Therefore it is desirable to develop methods which provide useful approximate answers even for larger systems. If one asks for mere entanglement detection, witnesses[10] are such a tool because here the number of required parameters (both for measurement and processing) can be reduced substantially. There are also estimates of entanglement measures using witness operators[11,12] which, however, have not yet produced practical methods for entanglement quantification.

Here we develop an easy-to-handle quantitative witness for Greenberger-Horne-Zeilinger (GHZ) entanglement[13] in arbitrary three-qubit states. It yields the exact three-tangle for the family of GHZ-symmetric states[14], and those states which are locally equivalent to them. For all other states, the method gives an optimised lower bound to the three-tangle. Due to this feature we call the approach a *witness*.

We start by defining the GHZ symmetry[14] and stating our central result. Then we prove the validity of the statement for two qubits. We obtain a method equivalent to that of Wootters-Uhlmann, *i.e.*, it gives the exact concurrence for arbitrary density matrices. Subsequently we explain the extension of the approach to arbitrary three-qubit states.

Results

The procedure. The N-qubit GHZ state in the computational basis is defined as $|\mathrm{GHZ}\rangle \equiv \frac{1}{\sqrt{2}}(|00\ldots0\rangle + |11\ldots1\rangle)$. It is invariant under: *(i)* Qubit permutation. *(ii)* Simultaneous spin flips *i.e.*, application of $\sigma_x^{\otimes N}$. *(iii)* Correlated local z rotations:

$$U_N = \mathrm{e}^{\mathrm{i}\varphi_1\sigma_z} \otimes \mathrm{e}^{\mathrm{i}\varphi_2\sigma_z} \otimes \ldots \mathrm{e}^{-\mathrm{i}\left(\sum_1^{N-1}\varphi_j\right)\sigma_z} \tag{1}$$

where σ_x, σ_y, σ_z are Pauli matrices. An N-qubit state is called *GHZ symmetric* and denoted by ρ^{S} if it remains invariant under the operations *(i)–(iii)*. An arbitrary N-qubit state ρ can be symmetrized by the operation

$$\rho^{\mathrm{S}}(\rho) = \int \mathrm{d}U_{\mathrm{GHZ}} U_{\mathrm{GHZ}} \rho U_{\mathrm{GHZ}}^{\dagger} \tag{2}$$

where the integral denotes averaging over the GHZ symmetry group including permutations and spin flips. Notably, the GHZ-symmetric N-qubit states form a convex subset of the space of all N-qubit states.

Observation: *If an appropriate entanglement measure μ is known exactly for GHZ-symmetric N-qubit states ρ^{S}, it can be employed to quantify GHZ-type entanglement in arbitrary N-qubit states ρ. Here, $\mu(\psi)$ is a positive $\mathrm{SL}(2,\mathbb{C})^{\otimes N}$-invariant function of homogeneous degree 2 in the coefficients of a pure quantum state ψ, and $\mu(\rho)$ is its convex-roof extension*[15]. *The estimate for $\mu(\rho)$ is found in the following sequence of steps:*

(1) Given a state ρ, derive a normal form $\rho^{\mathrm{NF}}(\rho)$, i.e., apply local filtering operations so that all local density matrices are proportional to the identity[7] *(see Section Methods). If $\rho^{\mathrm{NF}}(\rho) = 0$ the procedure terminates here, and $\mu(\rho) = 0$.*

(2) Renormalise $\rho^{\mathrm{NF}}/\mathrm{tr}\,\rho^{\mathrm{NF}}$ and transform it using local unitaries $V \in \mathrm{SU}(2)^{\otimes N}$ to obtain the state

$$\tilde{\rho}^{\mathrm{NF}}(\rho) = V \frac{\rho^{\mathrm{NF}}}{\mathrm{tr}\,\rho^{\mathrm{NF}}} V^{\dagger}$$

according to appropriate criteria (*see below*) *so that the entanglement of $\rho^{\mathrm{S}}(\rho^{\mathrm{NF}}/\mathrm{tr}\,\rho^{\mathrm{NF}})$ is enhanced.*

(3) Project the state onto the GHZ-symmetric states $\tilde{\rho}^{\mathrm{NF}}(\rho) \mapsto \rho^{\mathrm{S}}(\tilde{\rho}^{\mathrm{NF}})$. The estimate for $\mu(\rho)$ is obtained after renormalisation

$$\mu\left(\rho^{\mathrm{S}}\left(\tilde{\rho}^{\mathrm{NF}}\right)\right)\mathrm{tr}\,\rho^{\mathrm{NF}} \le \mu(\rho).$$

Two qubits. For two qubits the entanglement measure under consideration is the concurrence $C(\rho)$ (Refs. 4,8). From the symmetrization $\rho^{\mathrm{S}}(\rho)$ of an arbitrary two-qubit state ρ we find (for details see Supplementary Information):

$$C(\rho) \ge \max\left(0, |\rho_{00,11} + \rho_{11,00}| + \rho_{00,00} + \rho_{11,11} - 1\right). \tag{3}$$

In the symmetrization entanglement may be lost, as illustrated by the state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ for which inequality (3) gives the poor estimate $C(\Psi^-) \ge 0$. Therefore, the optimisation steps *(1)* and *(2)* are necessary to avoid unwanted entanglement loss in the symmetrization *(3)*. The goal is to augment the right-hand side of inequality (3) up to the point that equality is reached. We will show now that for two qubits this can indeed be achieved.

It is fundamental that the maximum of an $\mathrm{SL}(2,\mathbb{C})^{\otimes N}$-invariant function $\mu(\rho)$ under general local operations can be reached by applying the optimal transformation $\rho \mapsto A\rho A^{\dagger}/\mathrm{tr}\,A\rho A^{\dagger}$ where $A = A_1 \otimes \ldots \otimes A_N$ and $A_j \in \mathrm{SL}(2,\mathbb{C})$ is an invertible local operation[7]. Consider first the normal form $\rho^{\mathrm{NF}}(\rho)$ which is obtained from ρ by iterating determinant-one local operations[7] (see also Methods). Such operations (represented by $\mathrm{SL}(2,\mathbb{C})$ matrices) describe stochastic local operations and classical communication (SLOCC). Consequently, the normal form is locally equivalent to the original state ρ, that is, it lies in the entanglement class of ρ. Note that the iteration leading to the normal form *minimises* the trace of the state. Subsequent renormalisation *increases* the absolute values of all matrix elements in equation (3). Here, the correct rescaling of the mixed-state entanglement measure is crucial. This is why homogeneity degree 2 of $\mu(\psi)$ is required[16,17].

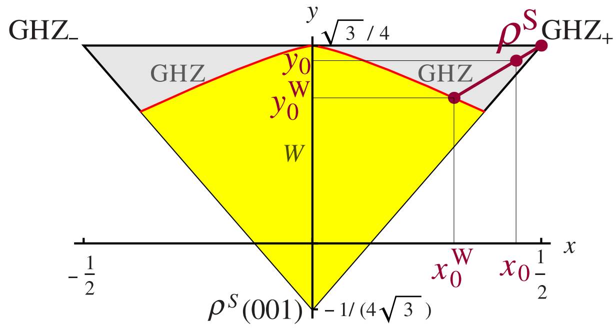

Figure 1 | The triangle of GHZ-symmetric three-qubit states. The upper corners correspond to $|\mathrm{GHZ}_{\pm}\rangle \equiv \frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle)$ and the lower corner to $\rho^{\mathrm{S}}(001)$, cf. Ref. 14. The grey area shows GHZ-class states ($\tau_3 > 0$) whereas the yellow area comprises states with vanishing τ_3 ("W"). The border between GHZ-class and W-class states is the GHZ/W line, equation (10) (red solid line). We also show a state $\rho^{\mathrm{S}}(x_0, y_0)$ together with the point (x_0^W, y_0^W) that is required to determine the three-tangle $\tau_3(x_0, y_0)$, equation (6).

Hence, transforming ρ to its normal form increases the moduli of $\rho_{00,00}$, $\rho_{00,11}$, $\rho_{11,00}$, $\rho_{11,11}$ (and also the concurrence) as much as possible for a state that is SLOCC equivalent with ρ. The sum of the off-diagonal matrix elements in equation (3) reaches its maximum if $\rho_{00,11}$ is real and positive. As this can be achieved by a z rotation on one qubit we may consider it part of finding the normal form and drop the absolute value bars in equation (3). Then, the sum of matrix elements equals, up to a factor 1/2, the fidelity of $\rho^{\mathrm{NF}}/\mathrm{tr}\,\rho^{\mathrm{NF}}$ with the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The question is how large this fidelity may become.

To find the answer we transform $\rho^{\mathrm{NF}}/\mathrm{tr}\,\rho^{\mathrm{NF}}$ to a Bell-diagonal form using local unitaries (this is always possible[7,18,19]). If then $\rho^{\mathrm{NF}}_{00,11} < |\rho^{\mathrm{NF}}_{01,10}|$ we apply another $\mathrm{SU}(2)^{\otimes 2}$ operation to maximise $\rho^{\mathrm{NF}}_{00,11}$ (see Supplementary Information). The result is a Bell-diagonal $\tilde{\rho}^{\mathrm{NF}}$ with maximum real off-diagonal element $\tilde{\rho}^{\mathrm{NF}}_{00,11}$ (please note that $\tilde{\rho}^{\mathrm{NF}}$ denotes a normalised state, whereas ρ^{NF} is not normalised). However, Bell-diagonal two-qubit density matrices with this property can be made GHZ symmetric *without* losing entanglement[4] (see also Supplementary Information).

Hence, our optimised symmetrization procedure *(1)–(3)* leads to the exact concurrence for arbitrary two-qubit states ρ. In passing, we have demonstrated that the concurrence is related via $C(\rho) = \max(0, 2f - 1) \cdot \mathrm{tr}\,\rho^{\mathrm{NF}}$ to the maximum fidelity $f = \langle\Phi^+|\tilde{\rho}^{\mathrm{NF}}|\Phi^+\rangle$ that can be achieved by applying invertible local operations to ρ.

Three qubits. For three qubits, the GHZ-symmetric states are described by two parameters[14] and therefore form a two-dimensional submanifold in the space of all three-qubit density matrices. It turns out that it has the shape of a flat isosceles triangle, see Fig. 1. A convenient parametrisation is

$$x(\rho) = \frac{1}{2}\left(\rho_{000,111} + \rho_{111,000}\right) \tag{4}$$

$$y(\rho) = \frac{1}{\sqrt{3}}\left(\rho_{000,000} + \rho_{111,111} - \frac{1}{4}\right) \tag{5}$$

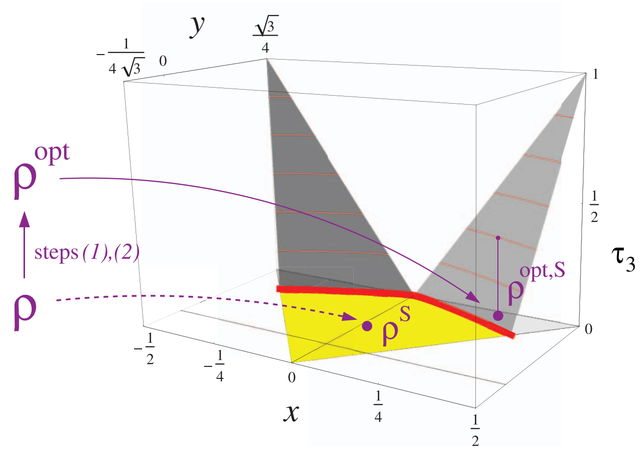

Figure 2 | Illustration of the procedure for finding the three-tangle of a general mixed three-qubit state ρ. In the xy plane, there is the triangle of GHZ-symmetric states while on the vertical axis, the three-tangle for each GHZ-symmetric state (cf. equation (6)) is shown. Simple projection $\rho \mapsto \rho^{S}$ generates a non-optimal GHZ-symmetric state. The optimisation steps *(1), (2)* move the symmetrization image to $\rho^{\text{opt,S}} \equiv \rho^{S}(\tilde{\rho}^{\text{NF}})$ with enhanced three-tangle.

as it makes the Hilbert-Schmidt metric in the space of density matrices coincide with the Euclidean metric. This way geometrical intuition can be applied to understand the properties of this set of states. All entanglement-related properties of GHZ-symmetric states are symmetric under sign change $x \leftrightarrow -x$ as this is achieved by applying σ_z to one of the qubits.

The GHZ-class entanglement of three-qubit states is quantified by the three-tangle τ_3 (Refs. 6,17, see also Methods). For GHZ-symmetric three-qubit states $\rho^{S}(x_0, y_0)$ the exact solution for the three-tangle[20] (see also Methods) is

$$\tau_3(x_0,y_0) = \begin{cases} 0 & \text{for } x_0 < x_0^W \text{ and } y_0 < y_0^W \\ \dfrac{x_0 - x_0^W}{\frac{1}{2} - x_0^W} = \dfrac{y_0 - y_0^W}{\frac{\sqrt{3}}{4} - y_0^W} & \text{otherwise} \end{cases} \qquad (6)$$

where $x_0 \geq 0$ and (x_0^W, y_0^W) are the coordinates of the intersection of the GHZ/W line with the direction that contains both GHZ$_+$ and $\rho^{S}(x_0, y_0)$ (cf. Fig. 1). The grey surfaces in Fig. 2 illustrate this solution.

Now we turn to constructing a quantitative witness for the three-tangle of arbitrary three-qubit states by using the solution in equation (6). As before, the main idea is that an arbitrary state can be symmetrized according to equation (2) and thus is projected into the GHZ-symmetric states. Again, we assume $\rho_{000,111}$ real and nonnegative, so that $x(\rho) \geq 0$. From Figs. 1 and 2 it appears evident that the entanglement of the symmetrization image $\rho^{S}(\rho)$ can be improved by moving its point $(x(\rho), y(\rho))$ closer to GHZ$_+$. More precisely, the entanglement measure is enhanced upon increasing one of the coordinates without decreasing the other (cf. equations (3) and (6)).

In this spirit, finding the normal form in step *(1)* is appropriate as it yields the largest possible three-tangle for a state $\rho^{\text{NF}}/\text{tr}\,\rho^{\text{NF}}$ locally equivalent to the original ρ (cf. Ref. 7). As the normal form is unique only up to local unitaries it does not automatically give the state with minimum entanglement loss in the symmetrization. Therefore, the unitary optimisation step *(2)* is required to generate the best coordinates.

In the symmetrization the information contained in various matrix elements is lost. For two qubits, however, the concurrence of the optimised Bell-diagonal states depends only on $\tilde{\rho}^{\text{NF}}_{00,00} + \tilde{\rho}^{\text{NF}}_{00,11}$ and the loss of $\tilde{\rho}^{\text{NF}}_{01,10}$ in the symmetrization does not harm. In contrast, the three-qubit normal form depends on 45 parameters. We may not expect that $\tau_3(\rho)$ depends only on two of them and, hence, entanglement loss in the symmetrization *(3)* is inevitable (cf. Supplementary Information). Consequently, steps *(1)–(3)* lead to a lower bound for the three-tangle that coincides with the exact $\tau_3(\rho)$ at least for those states which are locally equivalent to a GHZ-symmetric state. The most straightforward optimisation criterion in step *(2)* is to maximise $\mu(\rho^{S}(\tilde{\rho}^{\text{NF}}))$. Alternative criteria which generally do not give the best $\tau_3(\rho)$ but can be handled more easily (possibly analytically) are maximum fidelity $\langle \text{GHZ}_+ | \rho^{S}(\tilde{\rho}^{\text{NF}}) | \text{GHZ}_+ \rangle$, minimum Hilbert-Schmidt distance of $\rho^{S}(\tilde{\rho}^{\text{NF}})$ from GHZ$_+$, or maximum Re $\tilde{\rho}^{\text{NF}}_{0\ldots0,1\ldots1}$.

Discussion

Evidently this approach can be generalised. Therefore we conclude with a discussion of some of its universal features. The essential ingredients are an exact solution of the entanglement measure for a sufficiently general family of states with suitable symmetry, and the entanglement optimisation for a given arbitrary state ρ via general local operations. The former determines the border where the entanglement vanishes. The latter ensures an appropriate fidelity of the image $\rho^{S}(\rho)$ with the maximally entangled state. This reveals a remarkable relation between entanglement quantification through SL(2, $\mathbb{C}$) invariants and the standard entanglement witnesses which we briefly explain in the following.

A well-known witness for two-qubit entanglement is $\mathcal{W}_2 = \frac{1}{2}\mathbb{1}_4 - |\Phi^+\rangle\langle\Phi^+|$. It detects the entanglement of an arbitrary normalised two-qubit state $\rho^{2\text{qb}}$ if

$$0 > \text{tr}\left(\rho^{2\text{qb}}\mathcal{W}_2\right) = \frac{1}{2} - \langle\Phi^+|\rho^{2\text{qb}}|\Phi^+\rangle.$$

On the other hand, from our concurrence result

$$\begin{aligned} C\left(\rho^{2\text{qb}}\right) &= \max\left(0, \max_{A=A_1\otimes A_2}\left[2\langle\Phi^+|A\rho^{2\text{qb}}A^\dagger|\Phi^+\rangle - \text{tr}\left(A\rho^{2\text{qb}}A^\dagger\right)\right]\right) \\ &\geq 2\langle\Phi^+|\rho^{2\text{qb}}|\Phi^+\rangle - \text{tr}\left(\rho^{2\text{qb}}\right) \\ &= -2\text{tr}\left(\rho^{2\text{qb}}\mathcal{W}_2\right) \end{aligned} \qquad (7)$$

we see, by dropping the optimisation over SLOCC $A = A_1 \otimes A_2$, that $\mathcal{W}'_2 = -2\mathcal{W}_2$ is a (non-optimised) quantitative witness for two-qubit entanglement. In other words, $\mathcal{W}'_2$ yields one of the many possible lower bounds to the exact result. Analogously it is straightforward to establish the relation between the standard GHZ witness $\mathcal{W}_3 = \frac{3}{4}\mathbb{1}_8 - |\text{GHZ}_+\rangle\langle\text{GHZ}_+|$ and the non-optimal quantitative witness $\mathcal{W}'_3 = -4\mathcal{W}_3$. The latter represents a linear lower bound to the three-tangle obtained via the optimisation steps *(1)–(3)* (see Supplementary Information).

Finally we mention that our approach can be used without optimisation, *i.e.*, either without step *(1)*, or *(2)*, or both. This renders the witness less reliable but more efficient. At best it requires only four matrix elements (for *any* N). We note that, if we apply the witness to a tomography outcome the measurement effort can be reduced by using the prior knowledge of the state and choosing the local measurement directions such that the fidelity with the expected GHZ state is measured directly. This implements optimisation step *(2)* right in the measurement.

Methods

Normal form of an N-qubit state. The normal form of a multipartite quantum state is a fundamental concept that was introduced by Verstraete *et al.*[7]. It applies to arbitrary (finite-dimensional) multi-qudit states. Here we focus on N-qubit states only.

In the normal form of an N-qubit state ρ, all local density matrices are proportional to the identity. Therefore the normal form is unique up to local unitaries. Remarkably, the normal form can be obtained by applying an appropriate *local filtering operation*

$$\rho^{NF} = (A_1 \otimes \ldots \otimes A_N)\rho(A_1 \otimes \ldots \otimes A_N)^\dagger$$

where $A_j \in SL(2,\mathbb{C})$. Therefore ρ^{NF} is locally equivalent to the original state ρ. The normal form ρ^{NF} is peculiar since it has the *minimal* norm of all states in the orbit of ρ generated by local filtering operations. Practically, the normal form can be found by a simple iteration procedure described in Ref. 7. It is worth noticing that GHZ-symmetric states – which play a central role in our discussion – are naturally given in their normal form.

Three-tangle of three-qubit GHZ-symmetric states. The pure-state entanglement monotone that needs to be considered for three-qubit states is the three-tangle $\tau_3(\psi)$, *i.e.*, the square root of the residual tangle introduced by Coffman *et al.*[6]:

$$\begin{aligned}\tau_3(\psi) &= 2\sqrt{|d_1 - 2d_2 + 4d_3|},\\ d_1 &= \psi_{000}^2\psi_{111}^2 + \psi_{001}^2\psi_{110}^2 + \psi_{010}^2\psi_{101}^2 + \psi_{011}^2\psi_{100}^2\\ d_2 &= \psi_{000}\psi_{001}\psi_{110}\psi_{111} + \psi_{000}\psi_{010}\psi_{101}\psi_{111} +\\ &\quad + \psi_{000}\psi_{011}\psi_{100}\psi_{111} + \psi_{001}\psi_{010}\psi_{101}\psi_{110} +\\ &\quad + \psi_{001}\psi_{011}\psi_{100}\psi_{110} + \psi_{010}\psi_{011}\psi_{100}\psi_{101}\\ d_3 &= \psi_{000}\psi_{110}\psi_{101}\psi_{011} + \psi_{100}\psi_{010}\psi_{001}\psi_{111}.\end{aligned} \tag{8}$$

Here ψ_{jkl} with $j,k,l \in \{0,1\}$ are the components of a pure three-qubit state in the computational basis. The three-tangle becomes an entanglement measure also for mixed states $\rho = \sum_j p_j \left|\psi_j\right\rangle\left\langle\psi_j\right|$ via the convex-roof extension[15]

$$\tau_3(\rho) = \min_{\text{all decomp.}} \sum p_j \tau_3\left(\psi_j\right), \tag{9}$$

i.e., the minimum average three-tangle taken over all possible pure-state decompositions $\{p_j, \psi_j\}$. In general it is difficult to carry out the minimisation procedure in equation (9), but there exist various approaches for special families of states[16,17,20–23]. For GHZ-symmetric three-qubit states, the convex roof of the three-tangle can be calculated exactly (see equation (6)). This solution is shown in Fig. 2 and can be understood as follows. The border between the W and the GHZ states is the GHZ/W line which has the parametrised form[14]

$$x^W = \frac{v^5 + 8v^3}{8(4 - v^2)}, \quad y^W = \frac{\sqrt{3}}{4}\frac{4 - v^2 - v^4}{4 - v^2} \tag{10}$$

with $-1 \le v \le 1$. The solution for the convex roof is obtained by connecting each point of the GHZ/W line (x^W, y^W, $\tau_3 = 0$) with the closest of the points ($x_{GHZ_\pm} = \pm\frac{1}{2}$, $y_{GHZ_\pm} = \frac{\sqrt{3}}{4}$, $\tau_3 = 1$). That is, the three-tangle is nothing but a linear interpolation between the points of the border between GHZ and W states, and the maximally entangled states $GHZ_\pm$.

1. v. Helmholtz, H. *Zählen and Messen, vom erkenntnistheoretischen Standpunkt aus betrachtet* (Fues's Verlag, Leipzig, 1887).
2. Plenio, M. B. & Virmani, S. An introduction to entanglement measures. *Quant. Inf. Comput.* **7**, 1 (2007).
3. Horodecki, R., Horodecki, P., Horodecki, M. & Horodecki, K. Quantum entanglement. *Rev. Mod. Phys.* **81**, 865 (2009).
4. Bennett, C. H., DiVincenzo, D. P., Smolin, J. A. & Wootters, W. K. Mixed-state entanglement and quantum error correction. *Phys. Rev. A* **54**, 3824 (1996).
5. Vidal, G. & Werner, R. F. Computable measure of entanglement. *Phys. Rev. A* **65**, 032314 (2002).
6. Coffman, V., Kundu, J. & Wootters, W. K. Distributed entanglement. *Phys. Rev. A* **61**, 052306 (2000).
7. Verstraete, F., Dehaene, J. & De Moor, B. Normal forms and entanglement measures for multipartite quantum states. *Phys. Rev. A* **68**, 012103 (2003).
8. Wootters, W. K. Entanglement of formation of an arbitrary state of two qubits. *Phys. Rev. Lett.* **80**, 2245 (1998).
9. Uhlmann, A. Fidelity and concurrence of conjugated states. *Phys. Rev. A* **62**, 032307 (2000).
10. Gühne, O. & Tóth, G. Entanglement detection. *Phys. Rep.* **474**, 1 (2009).
11. Gühne, O., Reimpell, M. & Werner, R. F. Estimating entanglement measures in experiments. *Phys. Rev. Lett.* **98**, 110502 (2007).
12. Eisert, J., Brandão, F. G. S. L. & Audenaert, K. Quantitative entanglement witnesses. *New J. Phys.* **9**, 46 (2007).
13. Dür, W., Vidal, G. & Cirac, J. I. Three qubits can be entangled in two different ways. *Phys. Rev. A* **62**, 062314 (2000).
14. Eltschka, C. & Siewert, J. Entanglement of three-qubit Greenberger-Horne-Zeilinger–symmetric states. *Phys. Rev. Lett.* **108**, 020502 (2012).
15. Uhlmann, A. Entropy and optimal decompositions of states relative to a maximal commutative subalgebra. *Open Sys. & Inf. Dyn.* **5**, 209 (1998).
16. Gour, G. Evolution and symmetry of multipartite entanglement. *Phys. Rev. Lett.* **105**, 190504 (2010).
17. Viehmann, O., Eltschka, C. & Siewert, J. Rescaling multipartite entanglement measures for mixed states. *Appl. Phys. B* **106**, 533 (2012).
18. Verstraete, F., Dehaene, J. & De Moor, B. Local filtering operations on two qubits. *Phys. Rev. A* **64**, 010101(R) (2001).
19. Leinaas, J. M., Myrheim, J. & Ovrum, E. Geometrical aspects of entanglement. *Phys. Rev. A* **74**, 012313 (2006).
20. Siewert, J. & Eltschka, C. Quantifying tripartite entanglement of three-qubit generalized Werner states. *Phys. Rev. Lett.* **108**, 230502 (2012).
21. Eltschka, C., Osterloh, A., Siewert, J. & Uhlmann, A. Three-tangle for mixtures of generalised GHZ and generalised W states. *New J. Phys.* **10**, 043014 (2008).
22. Jung, E., Hwang, M. R., Park, D. & Son, J. W. Three-tangle for rank-three mixed states: Mixture of Greenberger-Horne-Zeilinger, W, and flipped-W states. *Phys. Rev. A* **79**, 024306 (2009).
23. Lee, S.-S. & Sim, H.-S. Quantifying entanglement by optimal entanglement witnesses. *Phys. Rev. A* **85**, 022325 (2012).

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Author contributions

The authors contributed equally to this work.

Additional information

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