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AN ALGEBRAIC PRESERVATION THEOREM FOR $\aleph_0$-CATEGORICAL QUANTIFIED CONSTRAINT SATISFACTION

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Abstract. We prove an algebraic preservation theorem for positive Horn definability in $\aleph_0$-categorical structures. In particular, we define and study a construction which we call the *periodic power* of a structure, and define a *periomorphism* of a structure to be a homomorphism from the periodic power of the structure to the structure itself. Our preservation theorem states that, over an $\aleph_0$-categorical structure, a relation is positive Horn definable if and only if it is preserved by all periomorphisms of the structure. We give applications of this theorem, including a new proof of the known complexity classification of quantified constraint satisfaction on equality templates.

1. Introduction

Model checking – deciding if a logical sentence holds on a structure – is a basic computational problem which is in general intractable; for example, model checking first-order sentences on finite structures is well-known to be PSPACE-complete. In the context of model checking, fragments of first-order logic based on restricting the connectives $\{\wedge, \vee, \neg\}$ and quantifiers $\{\exists, \forall\}$ have been considered in a variety of settings. For instance, the problem of model checking *primitive positive* sentences, sentences formed using $\{\wedge, \exists\}$, is a NP-complete problem that is a formulation of the *constraint satisfaction problem (CSP)*, and admits a number

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of other natural characterizations, as shown in the classical work of Chandra and Merlin [16]. The problem of model checking *positive Horn* sentences, sentences formed using $\{\wedge, \exists, \forall\}$, is known as the *quantified constraint satisfaction problem (QCSP)*, and is PSPACE-complete; indeed, certain cases of this problem are canonical complete problems for PSPACE [39, Chapter 19]. Another natural fragment consists of the *existential positive* sentences, which are formed from $\{\wedge, \vee, \exists\}$.

Such syntactically restricted fragments of first-order logic can be naturally parameterized by the structure [38]. As examples, consider the following problems for a structure $\mathfrak{A}$:

- $\mathsf{CSP}(\mathfrak{A})$: decide the primitive positive theory of $\mathfrak{A}$.
- $\mathsf{QCSP}(\mathfrak{A})$: decide the positive Horn theory of $\mathfrak{A}$.
- $\mathsf{EXPOS}(\mathfrak{A})$: decide the existential positive theory of $\mathfrak{A}$.
- $\mathsf{EFPOS}(\mathfrak{A})$: decide the equality-free positive theory of $\mathfrak{A}$.

Via this parameterization, one obtains four *families* of problems, and is prompted with classification programs: for each of the families, classify the problems therein according to their computational complexity. On finite structures, comprehensive classifications are known for the families $\mathsf{EXPOS}(\mathfrak{A})$ and $\mathsf{EFPOS}(\mathfrak{A})$. Each problem $\mathsf{EXPOS}(\mathfrak{A})$ is either in L or NP-complete [5], and each problem $\mathsf{EFPOS}(\mathfrak{A})$ is either in L, NP-complete, coNP-complete, or PSPACE-complete [37]. Moreover, each of these two classifications is effective in that for each, there exists an algorithm that, given a finite structure, tells what the complexity of the corresponding problem is. For the family of problems $\mathsf{CSP}(\mathfrak{A})$, Feder and Vardi [25] famously conjectured that there is a dichotomy in the finite: for each finite structure $\mathfrak{A}$, the problem $\mathsf{CSP}(\mathfrak{A})$ is either polynomial-time tractable or NP-complete. Investigation of the complexity-theoretic properties of the problem families $\mathsf{CSP}(\mathfrak{A})$ and $\mathsf{QCSP}(\mathfrak{A})$, on finite structures, is a research theme of active interest [18, 1, 33, 2, 15, 29, 20, 21].

At the heart of the work on these classification programs are *algebraic preservation theorems* which state that, relative to a finite structure, the relations definable in a given fragment are precisely those preserved by a suitable set of operations. As an example, one such theorem states that a relation is primitive positive definable on a finite structure $\mathfrak{A}$ if and only if all polymorphisms of $\mathfrak{A}$ are polymorphisms of the relation [28, 14]. (A polymorphism of a structure $\mathfrak{A}$ is a homomorphism from a finite power $\mathfrak{A}^k$ to $\mathfrak{A}$ itself.) On finite structures there are analogous preservation theorems connecting positive Horn definability to surjective polymorphisms [15], existential positive definability to endomorphisms [32], and equality-free positive definability to so-called surjective hyper-endomorphisms [36]. For the purposes of complexity classification, these preservation theorems are relevant in that they allow one to pass from the study of structures to the study of algebraic objects. For instance, it follows from the preservation theorem for primitive positive definability that two finite structures $\mathfrak{A}, \mathfrak{B}$ having the same polymorphisms are primitive positively interdefinable, from which it readily follows that the problems $\mathsf{CSP}(\mathfrak{A})$ and $\mathsf{CSP}(\mathfrak{B})$ are interreducible and share the same complexity (under many-one logspace reduction); thus, insofar as one is interested in CSP complexity, one can focus on investigating the polymorphisms of structures.

Given the import and reach of these algebraic preservation theorems for finite structures, a natural consideration is to generalize them to infinite structures. Although it is known that these preservation theorems do not hold on *all* infinite structures (see the discussion in [6] as well as [8, Theorem 4.7]), Bodirsky and Nešetřil [13, Theorem 5.1] established that the preservation theorem characterizing primitive positive definability via polymorphisms does hold on $\aleph_0$-categorical structures, which have countably infinite universes. An $\aleph_0$-categorical structure is "finite-like" in that for each fixed arity, there are a finite number of first-order definable relations; indeed, this is one of the characterizations of $\aleph_0$-categoricity given by the classical theorem of Ryll-Nardzewski. The class of $\aleph_0$-categorical structures includes many structures of computational interest, including those whose relations are first-order definable over one of the following structures: equality on a countable universe, the ordered rationals $(\mathbb{Q}, <)$, and the countable random graph; see [4] for a survey.

In this paper, we present an algebraic preservation theorem for positive Horn definability on $\aleph_0$-categorical structures. This theorem characterizes positive Horn definability by making use of a construction which we call the *periodic power*. In particular, we define a *periomorphism* of a structure $\mathfrak{A}$ as a homomorphism from the periodic power of $\mathfrak{A}$ to $\mathfrak{A}$ itself, and show that a relation is positive Horn definable over an $\aleph_0$-categorical structure $\mathfrak{A}$ if and only if all surjective periomorphisms of $\mathfrak{A}$ are periomorphisms of the relation.

The periodic power of a structure $\mathfrak{A}$ is the substructure of $\mathfrak{A}^{\mathbb{N}}$ whose universe is the set of all periodic tuples in $\mathfrak{A}^{\mathbb{N}}$; a tuple $(a_0, a_1, \ldots)$ is periodic if there exists an integer $k \geq 1$ such that the tuple *repeats mod* k, by which is meant $a_n = a_{n \bmod k}$ for all $n \in \mathbb{N}$. As we discuss in the paper, the periodic power arises as the direct limit of an appropriately defined system of embeddings. Despite the extremely natural character of the periodic power, we are not aware of previous work where this construction has been explicitly considered. We believe that it could be worthwhile to seek applications of the periodic power in other areas of mathematics. One basic fact that we demonstrate is that the positive Horn theory of a structure holds in the structure's periodic power; this readily implies that the class of groups is closed under periodic powers, and likewise for other classes of classical algebraic structures such as rings, lattices, and Boolean algebras. Our introduction and study of the periodic power also forms a contribution of this paper.

A direct corollary of our preservation theorem is that for two $\aleph_0$-categorical structures $\mathfrak{A}, \mathfrak{B}$ with the same universe, if $\mathfrak{A}$ and $\mathfrak{B}$ have the same surjective periomorphisms, then the structures $\mathfrak{A}$ and $\mathfrak{B}$ are positive Horn interdefinable, and the computational problems $\mathsf{QCSP}(\mathfrak{A})$ and $\mathsf{QCSP}(\mathfrak{B})$ are interreducible (under many-one logspace reductions). This permits the use of surjective periomorphisms in the study of the complexity of the QCSP on $\aleph_0$-categorical structures. As an application of our preservation theorem and the associated theory that we develop, we give a new proof of the known complexity classification of *equality templates*, which are structures whose relations are first-order definable over the equality relation on a countable set.

Related work. An algebraic preservation theorem for positive Horn definability via surjective polymorphisms was shown for the special case of equality templates [9]. The presented proof crucially depends on results on the clones of equality templates given there and in [11].

In model theory, there are *classical preservation theorems* that show that a sentence is equivalent to one in a given fragment if and only if its model class satisfies some suitable closure properties. Such theorems have been shown for positive Horn logic. A well-known instance is Birkhoff's HSP theorem characterizing universally quantified equations. And in 1955, Bing [3] showed that a positive sentence is preserved by direct products if and only if it is equivalent to a positive Horn sentence. Later, assuming the continuum hypothesis

(CH), Keisler proved[1] that a sentence is equivalent to a positive Horn sentence if and only if it is preserved (in the parlance of [41, 27]) by the following binary relation: relate $\mathfrak{A}$ to $\mathfrak{B}$ when $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}^{\mathbb{N}}$ [30, Corollary 3.8] (see also [17, Section 6.2]). Absoluteness considerations can be used to eliminate the assumption of CH when one has ZFC provability of the stated closure property. More recently, Madelaine and Martin [35, Theorem 1] showed, without relying on CH, that Keisler's result holds when one considers preservation under the relation defined as above, but where $\mathfrak{B}$ is required to be finite.

In some cases, an algebraic preservation theorem can be derived from a corresponding classical preservation theorem. Such a derivation has been given for Bodirsky and Nesetril's theorem in [4], and Bodirsky and Junker [7] derived algebraic preservation theorems for existential positive definability and positive definability in $\aleph_0$-categorical structures from well-known classical preservation theorems of Lyndon. Roughly speaking, these methods need the preservation relation to be PC_Δ (cf. [27] or [41, p.103]) and thus cannot be applied to Keisler's classical preservation theorem mentioned above. To the best of our knowledge, prior to this work no algebraic preservation theorem for positive Horn formulas on $\aleph_0$-categorical structures has been known (neither in the presence nor absence of CH).

2. Preliminaries from model theory

2.1. First-order logic. Throughout the paper, L will denote a countable first-order language. If not explicitly stated otherwise, by a structure (formula) we always mean an L-structure (first-order L-formula). Throughout, we use the letters $\mathfrak{A}$, $\mathfrak{B}$, etc. to denote structures with universes A, B, etc.; we use φ, ψ, χ, etc. to denote formulas. For a structure $\mathfrak{A}$ and a (finite) tuple $\overline{a}$ from A, by $(\mathfrak{A}, \overline{a})$ we denote, as usual, the expansion of $\mathfrak{A}$ interpreting new constants by the components of $\overline{a}$. We do not distinguish between constants outside L and variables. For a formula $\varphi = \varphi(\overline{x})$ and a structure $\mathfrak{A}$, writing $(\mathfrak{A}, \overline{a}) \models \varphi(\overline{x})$ or $\mathfrak{A} \models \varphi(\overline{a})$ (with $\overline{x}$ clear from context) means that $\mathfrak{A}$ satisfies $\varphi(\overline{x})$ under the assignment $\overline{a}$ to $\overline{x}$. By $\varphi(\mathfrak{A})$ we denote the relation $\{\overline{a} \mid \mathfrak{A} \models \varphi(\overline{a})\}$ on A; this relation is said to be *defined by* φ *in* $\mathfrak{A}$. A relation is *first-order (positive Horn, primitive positively) definable in* $\mathfrak{A}$ if it is defined by some first-order (positive Horn, primitive positive) formula φ in $\mathfrak{A}$ (see Section 3 for definitions of positive Horn and primitive positive).

Let L' be another first-order language, $\mathfrak{B}$ an L'-structure and $\mathfrak{A}$ an L-structure such that $A = B$. Then $\mathfrak{B}$ is *first-order (positive Horn, primitive positively) definable in* $\mathfrak{A}$ if for every atomic L'-formula φ the relation $\varphi(\mathfrak{B})$ is (positive Horn, primitive positively) definable in $\mathfrak{A}$.

2.2. Direct products. For a family of (L-)structures we denote its direct product by $\prod_{i\in I} \mathfrak{A}_i$. Recall that this structure

- has universe $\prod_{i\in I} A_i$, which is the set of functions mapping each $i \in I$ into the universe A_i of $\mathfrak{A}_i$;
- interprets a k-ary relation symbol $R \in L$ by those k-tuples $(\vec{a}_0, \dots, \vec{a}_{k-1})$ from $\prod_{i\in I} A_i$ such that $\mathfrak{A}_i \models R\vec{a}_0(i)\cdots\vec{a}_{k-1}(i)$ for all $i \in I$; and
- interprets a k-ary function symbol $f \in L$ by the function mapping a k-tuple $(\vec{a}_0, \dots, \vec{a}_{k-1})$ from $\prod_{i\in I} A_i$ to the element $\vec{a} \in \prod_{i\in I} A_i$ having the property that $\mathfrak{A}_i \models f(\vec{a}_0(i), \dots, \vec{a}_{k-1}(i)) = \vec{a}(i)$ for all $i \in I$.

[1]In fact, Keisler could do assuming only the existence of some cardinal $\kappa \geq \aleph_0$ such that $2^\kappa = \kappa^+$.

We write $\mathfrak{A}^I$ for $\prod_{i\in I}\mathfrak{A}_i$ with all $\mathfrak{A}_i = \mathfrak{A}$; we write $\mathfrak{A}^k$ to indicate $\mathfrak{A}^I$ when $I = \{0,\dots,k-1\}$ for $k \in \mathbb{N}, k > 0$. We consider $\mathfrak{A}^k$ to have universe A^k, the set of k-tuples over A. We do not distinguish between 1-tuples and elements, that is, $\mathfrak{A}^1 = \mathfrak{A}$. The direct product of two structures $\mathfrak{A}$ and $\mathfrak{B}$ is denoted $\mathfrak{A}\times\mathfrak{B}$ and considered to have universe $A \times B$.

2.3. Direct limits. We recall the definitions associated with direct limits. Let $(I,\prec)$ be a strict partial order that is directed: every two elements in I have a common upper bound. An $(I,\prec)$-*system of embeddings (homomorphisms)* is a family of embeddings (homomorphisms) $e_{(i,j)} : \mathfrak{A}_i \to \mathfrak{A}_j$ for $i \prec j$ such that $e_{(i,k)} = e_{(j,k)} \circ e_{(i,j)}$ for all $i \prec j \prec k$. A *cone* of the system is a family of *limit embeddings (homomorphisms)* $e^*_i : \mathfrak{A}_i \to \mathfrak{A}^*$ such that $e^*_j \circ e_{(i,j)} = e^*_i$. It is known that, for a system, there exists a cone satisfying the following universal property: for every other cone, say given by $\tilde{\mathfrak{A}}$ and $(\tilde{e}_i)_{i\in I}$, there exists a unique embedding (homomorphism) $e : \mathfrak{A}^* \to \tilde{\mathfrak{A}}$ such that $e \circ e^*_i = \tilde{e}_i$. A structure $\mathfrak{A}^*$ with this universal property is unique up to isomorphism and called the *direct limit* of the system; if $(I,\prec)$ and the $e_{(i,j)}$s are clear from context, it is denoted by $\lim_i \mathfrak{A}_i$.

2.4. $\aleph_0$-categoricity. A structure $\mathfrak{A}$ is $\aleph_0$-*categorical* if it is countable and every countable structure $\mathfrak{B}$ that satisfies the same first-order sentences as $\mathfrak{A}$ is isomorphic to $\mathfrak{A}$. We assume basic familiarity with $\aleph_0$-categoricity as covered by any standard course in model theory (see for example [17]). Here, we briefly recall some facts that we are going to use.

The theorem of *Ryll-Nardzewski* states that a countable structure $\mathfrak{A}$ is $\aleph_0$-categorical if and only if for every $k \in \mathbb{N}$ there are at most finitely many k-ary relations that are first-order definable in $\mathfrak{A}$. It is straightforward to verify that this implies that for an $\aleph_0$-categorical structure $\mathfrak{A}$, when $\overline{a}$ is an arbitrary finite-length tuple from A, the structure $(\mathfrak{A},\overline{a})$ is also $\aleph_0$-categorical. Further, it implies that for an $\aleph_0$-categorical structure $\mathfrak{A}$, the structure $\mathfrak{A}^k$ is $\aleph_0$-categorical for any $k \in \mathbb{N}$; in fact, every structure that is first-order interpretable in an $\aleph_0$-categorical structure is also $\aleph_0$-categorical.

Another easy consequence of this theorem is that $\aleph_0$-categorical structures are $\aleph_0$-*saturated*, by which is meant that for every finite tuple $\overline{a}$ from A and every set of formulas $\Phi = \Phi(x)$ in the language of $(\mathfrak{A},\overline{a})$ (that is, having constants for $\overline{a}$) one has: if every finite subset of $\Phi(x)$ is satisfiable in $(\mathfrak{A},\overline{a})$, then so is $(\mathfrak{A},\overline{a})$.

Finally, we mention the fact that for an $\aleph_0$-categorical structure $\mathfrak{A}$, a relation over A is first-order definable if and only if it is preserved by all automorphisms of $\mathfrak{A}$ (see Section 3.3 for the definition of preservation).

3. Preliminaries from constraint satisfaction

3.1. Positive Horn formulas. As noted in the introduction, a *positive Horn* formula is a first-order formula built from atoms, conjunction, and the two quantifiers. Existential such formulas are *primitive positive*. For simplicity, we assume that first-order logic contains a propositional constant $\bot$ for falsehood; formally, $\bot$ is a 0-ary relation symbol always interpreted by $\emptyset$. Note that $\bot$ is a positive atomic sentence. If any positive Horn sentence true in $\mathfrak{A}$ is also true in $\mathfrak{B}$, we write $\mathfrak{A} \Rrightarrow_{\mathrm{pH}} \mathfrak{B}$.

A formula $\varphi(\overline{x})$ is *preserved by direct products* if it holds in $(\mathfrak{A},\overline{a}) \times (\mathfrak{B},\overline{b})$ whenever it holds in both $(\mathfrak{A},\overline{a})$ and $(\mathfrak{B},\overline{b})$. Positive Horn formulas are preserved by direct products, in fact, the following is straightforward to verify.

Lemma 3.1. *Let* $(\mathfrak{A}_i)_{i\in I}$ *be a family of structures. A positive Horn sentence holds in* $\prod_{i\in I}\mathfrak{A}_i$ *if and only if it holds in every* $\mathfrak{A}_i, i \in I$. □

3.2. Quantified constraints. The quantified constraint satisfaction problem (QCSP) on a structure $\mathfrak{A}$, denoted by $\mathsf{QCSP}(\mathfrak{A})$, is the problem of deciding the positive Horn theory of $\mathfrak{A}$. The following proposition relates positive Horn definability to the complexity of the QCSP.

Proposition 3.2. *Let* $\mathfrak{A}$ *be an* L*-structure and* $\mathfrak{B}$ *be an* L_0*-structure for some finite first-order language* L_0*. If* $\mathfrak{B}$ *is positive Horn definable in* $\mathfrak{A}$*, then the problem* $\mathsf{QCSP}(\mathfrak{B})$ *many-one logspace reduces to* $\mathsf{QCSP}(\mathfrak{A})$.

Proof. For every function symbol $f \in L_0$, constant $c \in L_0$ and relation symbol $R \in L_0$ choose some fixed positive Horn L-formulas $\psi_f(\overline{x}, y), \psi_c(x), \psi_R(\overline{x})$ that respectively define, in $\mathfrak{A}$, the relations given by the formulas $f(\overline{x}) = y, x = c, R\overline{x}$ interpreted over $\mathfrak{B}$. Let φ be an instance of $\mathsf{QCSP}(\mathfrak{B})$, that is, a positive Horn sentence in the language L_0. In a first step, compute in logspace an equivalent sentence φ^* in which every atomic subformula contains at most one symbol from L_0, that is, has the form $x = y$, $f(\overline{x}) = y$ or $R\overline{x}$. This can be done by successively replacing atomic subformulas of φ, for example, replacing $Rxcf(f(x))$ by

$$\exists y_0y_1y_2(Rxy_0y_2 \wedge y_0 = c \wedge f(y_1) = y_2 \wedge f(x) = y_1)$$

In a second step, replace in φ^* every atomic subformula that mentions $s \in L_0$ by the formula ψ_s (with the right choice of variables). This can also be done in logspace: note that we may hardwire the finite list of the formulas ψ_s into the algorithm. Finally, recall that the composition of two logspace algorithms can be implemented in logspace. □

Remark 3.3. In the literature, the CSP and QCSP are typically defined in *relational* first-order logic. We take a more general stance and allow the language to contain function symbols if not explicitly stated otherwise. In particular, our preservation theorem (Theorem 6.1) holds in the presence of function symbols.

3.3. Preservation. Let A be a set, I a nonempty set and h a partial function from A^I to A. If h is defined on all of A^I (and I is finite), it is called a *(finitary) operation* on A. Then h is said to *preserve* an r-ary relation $R \subseteq A^r$ if it is a partial homomorphism from $(A, R)^I$ to (A, R). This means the following: whenever $\vec{a}_0, \ldots, \vec{a}_{r-1}$ are in the domain of h and $(\vec{a}_0(i), \ldots, \vec{a}_{r-1}(i)) \in R$ for all $i \in I$, then $(h(\vec{a}_0), \ldots, h(\vec{a}_{r-1})) \in R$. Further, relative to a structure $\mathfrak{A}$ with universe A, we say that h preserves a formula φ if it preserves the relation $\varphi(\mathfrak{A})$.

3.4. Clones and Polymorphisms. A *clone on* A is a set of finitary operations on A that is closed under composition and contains all projections. A set F of operations on A *interpolates* an operation g on A if for all finite sets B there exists an operation $f \in F$ such that $f \restriction B = g \restriction B$. A set of operations is *locally closed* if it contains every operation that it interpolates.

A *polymorphism of* $\mathfrak{A}$ is a homomorphism from $\mathfrak{A}^k$ to $\mathfrak{A}$ where k is a positive integer called the *arity* of the polymorphism. Equivalently, a polymorphism of $\mathfrak{A}$ is a finitary operation on A that preserves each $\mathfrak{A}$-relation, $\mathfrak{A}$-constant, and graph of an $\mathfrak{A}$-function; or, a polymorphism of $\mathfrak{A}$ is a finitary operation on A that preserves all atomic formulas. It is straightforward to verify that the set of polymorphisms of any structure $\mathfrak{A}$ forms a locally closed clone on A.

An operation $h : A^k \to A$ is a polymorphism of a relation $R \subseteq A^\ell$ if h is a polymorphism of the structure (A, R). In a picture, this means the following. If every column of

$$\begin{matrix} a_0^0 & a_1^0 & \cdots & a_{k-1}^0 \\ a_0^1 & a_1^1 & \cdots & a_{k-1}^1 \\ \vdots & \vdots & \ddots & \vdots \\ a_0^{\ell-1} & a_1^{\ell-1} & \cdots & a_{k-1}^{\ell-1} \end{matrix}$$

is a tuple contained in R, then so is the ℓ-tuple obtained by applying h to each row.

We have the following polymorphism-based characterization of primitive positive definability.

Theorem 3.4 ([13])**.** *Let $\mathfrak{A}$ be $\aleph_0$-categorical. A relation R over A is primitive positively definable in $\mathfrak{A}$ if and only if it is preserved by all polymorphisms of $\mathfrak{A}$.* □

4. Periodic powers

In this section, we present the notion of the *periodic power* of a structure, and identify some basic properties thereof. We also discuss how the periodic power arises as the direct limit of a system of embeddings. Throughout this section, we use $\mathfrak{A}, \mathfrak{B}$ to denote structures.

Definition 4.1. A function $\vec{a} : \mathbb{N} \to A$ is *periodic* if there exists $k \in \mathbb{N}, k > 0$ such that for all $i \in \mathbb{N}$, it holds that $\vec{a}(i) = \vec{a}(i \bmod k)$; in this case the function $\vec{a}$ is said to be *k-periodic*, and we write $\langle \vec{a}(0) \cdots \vec{a}(k-1) \rangle$ to denote $\vec{a}$. The set of periodic functions A^{per} carries a substructure in $\mathfrak{A}^{\mathbb{N}}$: the set A^{per} is nonempty and closed under all $\mathfrak{A}^{\mathbb{N}}$-interpretations of function symbols. We define the *periodic power* of $\mathfrak{A}$, denoted $\mathfrak{A}^{\text{per}}$, to be the substructure of $\mathfrak{A}^{\mathbb{N}}$ induced on A^{per}.

When $\overline{\vec{a}} = \vec{a}_0 \cdots \vec{a}_{\ell-1}$ is a tuple from A^{per} and $i \in \mathbb{N}$, we let $\overline{\vec{a}}(i)$ denote the tuple $\vec{a}_0(i) \cdots \vec{a}_{\ell-1}(i)$ from A.

Lemma 4.2. *Assume that $\varphi(\overline{x})$ is a positive Horn formula. Then $(\mathfrak{A}^{\text{per}}, \overline{\vec{a}}) \models \varphi(\overline{x})$ if and only if $(\mathfrak{A}, \overline{\vec{a}}(i)) \models \varphi(\overline{x})$ for all $i \in \mathbb{N}$.*

Proof. Call a formula φ *good* if it satisfies the claimed equivalence. Clearly, conjunctions of atoms are good. Assume $\varphi(\overline{x}, y)$ is good. It is easy to see that also $\forall y \varphi(\overline{x}, y)$ is good. We show that $\exists y \varphi(\overline{x}, y)$ is good, via the following equivalences.

$$\begin{aligned} &(\mathfrak{A}^{\text{per}}, \overline{\vec{a}}) \models \exists y \varphi(\overline{x}, y) \\ &\iff \exists \vec{b} \in A^{\text{per}} : \ (\mathfrak{A}^{\text{per}}, \overline{\vec{a}}, \vec{b}) \models \varphi(\overline{x}, y) \\ &\iff \exists \vec{b} \in A^{\text{per}}\ \forall i \in \mathbb{N} : \ (\mathfrak{A}, \overline{\vec{a}}(i), \vec{b}(i)) \models \varphi(\overline{x}, y) \qquad (4.1) \\ &\iff \forall i \in \mathbb{N}\ \exists b \in A : \ (\mathfrak{A}, \overline{\vec{a}}(i), b) \models \varphi(\overline{x}, y) \qquad (4.2) \\ &\iff \forall i \in \mathbb{N} : \ (\mathfrak{A}, \overline{\vec{a}}(i)) \models \exists y \varphi(\overline{x}, y). \end{aligned}$$

The second equivalence follows from $\varphi(\overline{x}, y)$ being good. The rest being trivial, we show that (4.2) implies (4.1). By (4.2) there is a function $\vec{b} : \mathbb{N} \to A$ such that $(\mathfrak{A}, \overline{\vec{a}}(i), \vec{b}(i)) \models \varphi(\overline{x}, y)$ for all $i \in \mathbb{N}$. For every component $\vec{a}$ of $\overline{\vec{a}}$ choose $n_{\vec{a}} \in \mathbb{N}$ such that $\vec{a}$ is $n_{\vec{a}}$-periodic, and let $n \in \mathbb{N}$ be a common multiple of the $n_{\vec{a}}$s. Then any component of $\vec{a}$ is n-periodic and, in particular,

$$\overline{\vec{a}}(i) = \overline{\vec{a}}(i \bmod n)$$

for all $i \in \mathbb{N}$. Define $\vec{b}^* : \mathbb{N} \to A$ by

$$\vec{b}^*(i) := \vec{b}(i \bmod n).$$

Then $\vec{b}^* \in A^{\mathrm{per}}$ and $(\mathfrak{A}, \overline{\vec{a}}(i), \vec{b}^*(i)) \models \varphi(\overline{x}, y)$ for all $i \in \mathbb{N}$; this is (4.1). □

Consider the following embeddings.

– The function $e_1 : \mathfrak{A} \to \mathfrak{A}^{\mathrm{per}}$ defined by $e_1(a) := \langle a \rangle$, that is, the function mapping each $a \in A$ to the constant sequence $(a)_{i \in \mathbb{N}}$, is a canonical embedding of $\mathfrak{A}$ into $\mathfrak{A}^{\mathrm{per}}$.
– More generally, for each $k > 0$, the function $e_k : \mathfrak{A}^k \to \mathfrak{A}^{\mathrm{per}}$ defined by $e_k((a_0, \dots, a_{k-1})) := \langle a_0 \cdots a_{k-1} \rangle$ is a canonical embedding from $\mathfrak{A}^k$ into $\mathfrak{A}^{\mathrm{per}}$.

In the following proposition we identify $a \in A$ with $e_1(a) \in A^{\mathrm{per}}$ for notational simplicity. We use $\mathfrak{A} \preceq_{\mathrm{pH}} \mathfrak{B}$ to indicate that $\mathfrak{A} \subseteq \mathfrak{B}$ (i.e. $\mathfrak{A}$ is a substructure of $\mathfrak{B}$) and that for every positive Horn formula $\varphi(\overline{x})$ and all tuples $\overline{a}$ from A, it holds that

$$(\mathfrak{A}, \overline{a}) \models \varphi(\overline{x}) \iff (\mathfrak{B}, \overline{a}) \models \varphi(\overline{x}).$$

Lemmas 3.1 and 4.2 imply:

Proposition 4.3. $\mathfrak{A} \preceq_{\mathrm{pH}} \mathfrak{A}^{\mathrm{per}} \preceq_{\mathrm{pH}} \mathfrak{A}^{\mathbb{N}}$. □

The next two propositions explain how the periodic power relates to finite powers.

Proposition 4.4. *Let $k \in \mathbb{N}, k > 0$. Then $\mathfrak{A}^{\mathrm{per}} \cong (\mathfrak{A}^k)^{\mathrm{per}}$ via an isomorphism that maps $\langle a_0 \cdots a_{k-1} \rangle$ to $\langle (a_0, \dots, a_{k-1}) \rangle$ for all $a_0, \dots, a_{k-1} \in A$.*

To make clear the notation used in the statement of this proposition, let us look at an example: the notation $\langle ab \rangle$ denotes the 2-periodic sequence $ababab \cdots \in A^{\mathrm{per}}$, whereas the notation $\langle (a, b) \rangle$ denotes the constant, 1-periodic sequence $(a, b)\ (a, b)\ (a, b) \cdots \in (A^2)^{\mathrm{per}}$.

Proof of Proposition 4.4. Define the map $f : \mathfrak{A}^{\mathrm{per}} \to (\mathfrak{A}^k)^{\mathrm{per}}$ to map $\vec{a} \in \mathfrak{A}^{\mathrm{per}}$ to

$$i \mapsto (\vec{a}(ik), \dots, \vec{a}((i+1)k - 1))$$

The map f is clearly injective. For $j < k$ let π^k_j denote the projection of k-tuples to their $(j+1)$th component. An element $\vec{b} \in (\mathfrak{A}^k)^{\mathrm{per}}$ has

$$i \mapsto \pi^k_{i \bmod k}(\vec{b}(\lfloor i/k \rfloor))$$

as preimage under f, so f is surjective. It is straightforward to verify that f is an isomorphism. □

Proposition 4.5. *Let $k \in \mathbb{N}, k > 1$. Then $\mathfrak{A}^{\mathrm{per}} \cong (\mathfrak{A}^{\mathrm{per}})^k$.*

The proof relies on the following observation.

Lemma 4.6. $\mathfrak{A}^{\mathrm{per}} \times \mathfrak{B}^{\mathrm{per}} \cong (\mathfrak{A} \times \mathfrak{B})^{\mathrm{per}}$.

Proof. Map a pair of functions $(\vec{a}, \vec{b}) \in A^{\mathrm{per}} \times B^{\mathrm{per}}$ to $((\vec{a}(i), \vec{b}(i)))_{i \in \mathbb{N}}$; note this function is nm-periodic whenever $\vec{a}$ and $\vec{b}$ are n- and m-periodic respectively. The map is clearly injective. It is surjective as $((a_i, b_i))_{i \in \mathbb{N}} \in (A \times B)^{\mathrm{per}}$ has preimage $((a_i)_{i \in \mathbb{N}}, (b_i)_{i \in \mathbb{N}}) \in A^{\mathrm{per}} \times B^{\mathrm{per}}$. To see that it is an isomorphism, let α be an atom. For simplicity assume $\alpha = \alpha(x, y)$, and let

$(\vec{a},\vec{b}),(\vec{a}',\vec{b}') \in A^{\mathrm{per}} \times B^{\mathrm{per}}$. Then

$$\begin{aligned}
&(\mathfrak{A}^{\mathrm{per}} \times \mathfrak{B}^{\mathrm{per}}, (\vec{a},\vec{b}),(\vec{a}',\vec{b}')) \models \alpha(x,y)\\
&\iff (\mathfrak{A}^{\mathrm{per}}, \vec{a},\vec{a}') \models \alpha(x,y) \text{ and } (\mathfrak{B}^{\mathrm{per}}, \vec{b},\vec{b}') \models \alpha(x,y)\\
&\iff \forall i \in \mathbb{N}: \ (\mathfrak{A}, \vec{a}(i), \vec{a}'(i)) \models \alpha(x,y) \text{ and } (\mathfrak{B}, \vec{b}(i), \vec{b}'(i)) \models \alpha(x,y)\\
&\iff \forall i \in \mathbb{N}: \ (\mathfrak{A}\times\mathfrak{B}, (\vec{a}(i),\vec{b}(i)), (\vec{a}'(i),\vec{b}'(i))) \models \alpha(x,y)\\
&\iff ((\mathfrak{A}\times\mathfrak{B})^{\mathrm{per}}, (\vec{a}(i),\vec{b}(i))_{i\in\mathbb{N}}, ((\vec{a}'(i),\vec{b}'(i)))_{i\in\mathbb{N}}) \models \alpha(x,y),
\end{aligned}$$

where the first and third equivalence hold by definition of direct products, and the second and fourth equivalence hold by Lemma 4.2. ☐

Proof of Proposition 4.5 by induction on k: we have the isomorphisms

$$(\mathfrak{A}^{\mathrm{per}})^{k+1} = (\mathfrak{A}^{\mathrm{per}})^{k} \times \mathfrak{A}^{\mathrm{per}} \cong \mathfrak{A}^{\mathrm{per}} \times \mathfrak{A}^{\mathrm{per}} \cong (\mathfrak{A}^2)^{\mathrm{per}} \cong \mathfrak{A}^{\mathrm{per}}$$

by induction, the previous lemma and Proposition 4.4. ☐

Observe that for $n,m > 0$ there is a natural embedding $e_{(n,m)} : \mathfrak{A}^n \to \mathfrak{A}^m$ whenever $n < m$ and n divides m, namely the embedding that maps the n-tuple $\overline{a} \in A^n$ to the m-tuple

$$e_{(n,m)}(\overline{a}) = \underbrace{\overline{a}\,\overline{a}\cdots\overline{a}}_{m/n \text{ times}} \in A^m.$$

Clearly, these embeddings are compatible in the sense that $e_{(\ell,m)} \circ e_{(n,\ell)} = e_{(n,m)}$ whenever $n < \ell < m$, n divides ℓ and ℓ divides m. In other words, the $e_{(n,m)}$s determine an $(I,\prec)$-system of embeddings where $I = \mathbb{N}_{>0}$ and $\mathfrak{A}_n := \mathfrak{A}^n$ and $\prec$ denotes divisibility.

Proposition 4.7. $\mathfrak{A}^{\mathrm{per}} \cong \lim_n \mathfrak{A}^n$.

Proof. Let $(e_n^*)_{n>0}$ denote the limit homomorphisms into the direct limit $\lim_n \mathfrak{A}^n$ of the directed system of embeddings given by the embeddings $e_{(n,m)}$ (for $n < m$ and n divides m). Observe that the embeddings e_n from $\mathfrak{A}^n$ into $\mathfrak{A}^{\mathrm{per}}$ satisfy the requirement for limit embeddings, so these embeddings e_n are also a cone of the directed system. By the universal property of $\lim_n \mathfrak{A}^n$ there is an embedding $e : \lim_n \mathfrak{A}^n \to \mathfrak{A}^{\mathrm{per}}$ such that $e \circ e_n^* = e_n$ for all $n > 0$. But every element of A^{per} is in the image of some e_n, so e has to be surjective and thus is an isomorphism. ☐

Recall, an $\forall\exists$-sentence is a sentence of the form $\forall\overline{x}\exists\overline{y}\psi$ with ψ quantifier free. Propositions 4.3 and 4.7 imply:

Corollary 4.8. *Every positive Horn sentence true in* $\mathfrak{A}$ *and every* $\forall\exists$*-sentence true in all finite powers of* $\mathfrak{A}$*, is true in* $\mathfrak{A}^{\mathrm{per}}$. ☐

5. Periomorphisms

In this section, we introduce and study the notion of *periomorphism*. Throughout this section, let $\mathfrak{A}$ be a structure.

Definition 5.1. A *periomorphism of* $\mathfrak{A}$ is a homomorphism from $\mathfrak{A}^{\mathrm{per}}$ to $\mathfrak{A}$.

In other words, a periomorphism of $\mathfrak{A}$ is a partial function from $A^{\mathbb{N}}$ to A with domain A^{per} that preserves all atomic formulas. The following lemma follows straightforwardly from the definitions.

Lemma 5.2. *A periomorphism h of $\mathfrak{A}$ preserves a relation $R \subseteq A^\ell$ if and only if for any choice of finitely many tuples $\overline{a}_0 = (a_0^0, \dots, a_0^{\ell-1}), \dots, \overline{a}_{k-1} = (a_{k-1}^0, \dots, a_{k-1}^{\ell-1})$ from R, we have*

$$\big(h(\langle a_0^0 a_1^0 \cdots a_{k-1}^0 \rangle), \dots, h(\langle a_0^{\ell-1} a_1^{\ell-1} \cdots a_{k-1}^{\ell-1} \rangle)\big) \in R.$$

Proof: The forward direction is trivial. Conversely assume the right hand side of the claimed equivalence and let $\vec{a}_0, \dots, \vec{a}_{\ell-1} \in \mathfrak{A}^{\mathrm{per}}$ be such that for all $i \in \mathbb{N}$, $(\vec{a}_0(i), \dots \vec{a}_{\ell-1}(i)) \in R$. We claim $h(\vec{a}_0) \cdots h(\vec{a}_{\ell-1}) \in R$. Choose a sufficiently large $k \in \mathbb{N}$ such that all $\vec{a}_j$ are k-periodic, that is, $\vec{a}_j = \langle \vec{a}_j(0) \cdots \vec{a}_j(k-1) \rangle$ for all $j < \ell$. Applying the assumption yields the claim. □

To see the lemma's statement with a picture, let h be a periomorphism of $\mathfrak{A}$, and consider the following.

$$\begin{matrix} \langle a_0^0 & a_1^0 & \cdots & a_{k-1}^0 \rangle \\ \langle a_0^1 & a_1^1 & \cdots & a_{k-1}^1 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle a_0^{\ell-1} & a_1^{\ell-1} & \cdots & a_{k-1}^{\ell-1} \rangle \end{matrix}$$

The right hand side of the lemma states that if the columns $\overline{a}_i = (a_i^0, \dots, a_i^{\ell-1})$ are contained in R for all $i < k$, then so is the ℓ-tuple $\overline{b}$ obtained by applying h to each row.

For later use we introduce the following mode of speech.

Definition 5.3. In the situation above, if h is a *surjective* periomorphism of the structure under study, then we call $\overline{b}$ *a surjective periomorphic image of the tuples* $(\overline{a}_i)_{i<k}$.

Proposition 5.4. *Every positive Horn formula is preserved by all surjective periomorphisms of $\mathfrak{A}$.*

Proof: Let $\varphi(\overline{x})$ be a positive Horn formula and h be a surjective periomorphism of $\mathfrak{A}$. For notational simplicity assume $\overline{x} = xx'$ and let $a_0 a_0', \dots, a_{k-1} a_{k-1}'$ be any finitely many pairs in $\varphi(\mathfrak{A})$. We have to show that $\varphi(xx')$ is true in $(\mathfrak{A}, h(\langle a_0 \cdots a_{k-1} \rangle), h(\langle a_0' \cdots a_{k-1}' \rangle))$; see the previous lemma. But $\varphi(xx')$ is true in $(\mathfrak{A}^{\mathrm{per}}, \langle a_0 \cdots a_{k-1} \rangle, \langle a_0' \cdots a_{k-1}' \rangle)$ by Lemma 4.2, and, being positive, is preserved by surjective homomorphisms. □

The periomorphisms and the polymorphisms of a structure contain the same information. If one knows the periomorphisms of a structure, then one also knows its polymorphisms – and vice-versa. Why is this? For $k \in \mathbb{N}, k > 0$ define $\pi_{<k} : A^{\mathrm{per}} \to A^k$ by

$$\pi_{<k}(\vec{a}) := (\vec{a}(0), \dots, \vec{a}(k-1)).$$

This operation is clearly a homomorphism from $\mathfrak{A}^{\mathrm{per}}$ to $\mathfrak{A}^k$. Now, if someone hands us an operation $h : A^k \to A$, we can decide if it is a polymorphism of $\mathfrak{A}$ by checking if

$$h^{\mathrm{per}} := h \circ \pi_{<k}.$$

is a periomorphism of $\mathfrak{A}$. For, if h is a polymorphism of $\mathfrak{A}$, then by composing homomorphisms, we have that h^{per} is a periomorphism of $\mathfrak{A}$; and, if h^{per} is a periomorphism of $\mathfrak{A}$, by composing homomorphisms, we have that $h^{\mathrm{per}} \circ e_k$, which is equal to h, is a homomorphism from $\mathfrak{A}^k$ to $\mathfrak{A}$.

Going the other way, suppose that someone places in our hands an operation $h : A^{\mathrm{per}} \to A$. It can be seen from Lemma 5.2 that h is a periomorphism of $\mathfrak{A}$ if and only if each of the operations

$$h_{<k} := h \circ e_k. \tag{5.1}$$

is a polymorphism of $\mathfrak{A}$.

It is thus no surprise that preservation by periomorphisms coincides with preservation by polymorphisms. Preservation by *surjective* periomorphisms, however, is an a priori stronger property than preservation by surjective polymorphisms.

Proposition 5.5. *Let φ be a formula. Then*

(1) *φ is preserved by all periomorphisms of $\mathfrak{A}$ if and only if φ is preserved by all polymorphisms of $\mathfrak{A}$;*
(2) *if φ is preserved by all surjective periomorphisms of $\mathfrak{A}$, then φ is preserved by all surjective polymorphisms of $\mathfrak{A}$.*

Proof. To see the forward directions, observe that if h is a (surjective) polymorphism of $\mathfrak{A}$ that does not preserve φ, then h^{per} is a (surjective) periomorphism of $\mathfrak{A}$ that does not preserve φ. For the converse direction in (1) assume h is a periomorphism that does not preserve $\varphi = \varphi(x_0, \ldots, x_{\ell-1})$. Then by Lemma 5.2 we have that there are $k \in \mathbb{N}$ and $(a_0^0, \ldots, a_0^{\ell-1}), \ldots, (a_{k-1}^0, \ldots, a_{k-1}^{\ell-1}) \in \varphi(\mathfrak{A})$ such that

$$(h(\langle a_0^0 a_1^0 \cdots a_{k-1}^0 \rangle), \ldots, h(\langle a_0^{\ell-1} a_1^{\ell-1} \cdots a_{k-1}^{\ell-1} \rangle)) \notin \varphi(\mathfrak{A}),$$

that is,

$$\big(h(e_k(a_0^0, a_1^0, \ldots a_{k-1}^0)), \ldots, h(e_k(a_0^{\ell-1}, a_1^{\ell-1}, \ldots a_{k-1}^{\ell-1}))\big) \notin \varphi(\mathfrak{A}).$$

Hence, $h_{<k}$ is a k-ary polymorphism of $\mathfrak{A}$ that does not preserve φ. □

Remark 5.6. The converse of (2) is true in case $\mathfrak{A}$ satisfies the following condition: for every surjective periomorphism h of $\mathfrak{A}$ there exists $k \in \mathbb{N}$ such that $h_{<k}$ is surjective. For example, finite structures satisfy this condition.

We saw that a periomorphism h gives rise to a sequence of polymorphisms $(h_{<k})_{k>0}$. In fact, this gives a one-to-one correspondence with those polymorphism sequences that satisfy the following property.

Definition 5.7. A sequence $(g_k)_{k>0}$ is a *cone of polymorphisms of* $\mathfrak{A}$ if every g_k is a k-ary polymorphism of $\mathfrak{A}$ and $g_\ell = g_k \circ e_{(\ell,k)}$ whenever $\ell < k$ and ℓ divides k.

Proposition 5.8. *A sequence $(g_k)_{k>0}$ is a cone of polymorphisms of $\mathfrak{A}$ if and only if there is a periomorphism h of $\mathfrak{A}$ such that $h_{<k} = g_k$ for all $k > 0$.*

Proof. For the backward direction, let h be a periomorphism of $\mathfrak{A}$. Clearly, $(h_{<k})_{k>0}$ is a sequence of polymorphisms of $\mathfrak{A}$ – and it is a cone:

$$h_{<\ell} = h \circ e_\ell = h \circ (e_k \circ e_{(\ell,k)}) = h_{<k} \circ e_{(\ell,k)}.$$

Here, the second equality follows from the e_ℓs being limit embeddings (see the previous section).

Conversely, assume that $(g_k)_{k>0}$ is a cone of polymorphisms of $\mathfrak{A}$. Then this is a cone of the directed system given by the $e_{(n,m)}$s (viewed as a directed system of homomorphisms). By the universal property of limits we get a homomorphism h from $\mathfrak{A}^{\text{per}} \cong \lim_n \mathfrak{A}^n$ into $\mathfrak{A}$ such that $h \circ e_k = g_k$. □

Intuitively speaking, just as the periodic power is a cone of finite powers, any periomorphism "is" a cone of (finitary) polymorphisms.

6. Preservation theorem

Theorem 6.1 (Main). *Let $\mathfrak{A}$ be an $\aleph_0$-categorical structure. A relation R over A is positive Horn definable in $\mathfrak{A}$ if and only if it is preserved by all surjective periomorphisms of $\mathfrak{A}$.*

The following is a straightforward generalization of Proposition 5.4.

Proposition 6.2. *If $\mathfrak{A}$ and $\mathfrak{B}$ are structures such that there is a surjective homomorphism from $\mathfrak{A}^{\mathrm{per}}$ onto $\mathfrak{B}$, then $\mathfrak{A} \Rrightarrow_{\mathrm{pH}} \mathfrak{B}$.* □

The main lemma in the proof of Theorem 6.1 states that a converse of this proposition holds true in the $\aleph_0$-categorical case:

Lemma 6.3. *If $\mathfrak{A}$ and $\mathfrak{B}$ are $\aleph_0$-categorical structures such that $\mathfrak{A} \Rrightarrow_{\mathrm{pH}} \mathfrak{B}$, then there is a surjective homomorphism from $\mathfrak{A}^{\mathrm{per}}$ onto $\mathfrak{B}$.*

Proof: Let I be the set of finite partial functions f from $\mathfrak{A}^{\mathrm{per}}$ to $\mathfrak{B}$ such that

$$(\mathfrak{A}^{\mathrm{per}}, \overline{\vec{a}}) \Rrightarrow_{\mathrm{pH}} (\mathfrak{B}, \overline{b}). \tag{6.1}$$

where $\overline{\vec{a}}$ is a (finite) tuple from $\mathfrak{A}^{\mathrm{per}}$ listing all elements of the domain of f and $\overline{b}$ is a tuple from $\mathfrak{B}$ such that f maps $\overline{\vec{a}}$ to $\overline{b}$.

Observe that $\mathfrak{A}^{\mathrm{per}}$ is countable. Hence, by a standard back and forth argument, it suffices to verify the following two claims.

Claim 1. For all $f \in I$ and $\vec{a} \in A^{\mathrm{per}}$ there is $b \in B$ such that $f \cup \{(\vec{a}, b)\} \in I$.

Claim 2. For all $f \in I$ and $b \in B$ there is $\vec{a} \in A^{\mathrm{per}}$ such that $f \cup \{(\vec{a}, b)\} \in I$.

Proof of Claim 1. Given $f \in I$ choose a tuples $\overline{\vec{a}}$ and $\overline{b}$ as above. Let $\vec{a} \in A^{\mathrm{per}}$ be arbitrary. It sufficies to find $b \in B$ such that

$$(A^{\mathrm{per}}, \overline{\vec{a}}, \vec{a}) \Rrightarrow_{\mathrm{pH}} (\mathfrak{B}, \overline{b}, b) \tag{6.2}$$

Note in particular that $x = y$ is positive Horn, so (6.2) implies that $f \cup \{(\vec{a}, b)\}$ is a function. To find such b consider the set $\Delta(x)$ of all positive Horn formulas $\psi(x)$ (in the language of $(\mathfrak{A}^{\mathrm{per}}, \overline{\vec{a}})$) satisfied by $\vec{a}$ in $(\mathfrak{A}^{\mathrm{per}}, \overline{\vec{a}})$. It suffices to show this set is satisfiable in $(\mathfrak{B}, \overline{b})$. Since $\mathfrak{B}$ is $\aleph_0$-categorical, it is $\aleph_0$-saturated (recall Section 2.4), and hence it suffices to show that every finite subset of $\Delta(x)$ is satisfiable in $(\mathfrak{B}, \overline{b})$. But for a finite $\Delta_0(x) \subsetneqq \Delta(x)$ the positive Horn sentence $\exists x \bigwedge \Delta_0(x)$ is true in $(\mathfrak{A}^{\mathrm{per}}, \overline{\vec{a}})$, so it is also true in $(\mathfrak{B}, \overline{b})$ by (6.1). Hence $(\mathfrak{B}, \overline{b})$ contains some b satisfying $\Delta_0(x)$. ⊣

Proof of Claim 2. Let $f \in I$ and again choose $\overline{\vec{a}}$ and $\overline{b}$ as above; let k denote the length of these tuples. Again, it suffices given any $b \in B$ to find some $\vec{a} \in A^{\mathrm{per}}$ such that (6.2) holds. As $\mathfrak{A}$ is $\aleph_0$-categorical by Ryll-Nardzewski there are up to equivalence in $\mathfrak{A}$ only finitely many formulas in the variables $\overline{y}x$ where $\overline{y}$ is a tuple of k variables. Let

$$\psi_0(\overline{y}, x), \ldots, \psi_{m-1}(\overline{y}, x)$$

list, up to equivalence in $\mathfrak{A}$, all positive Horn formulas $\psi(\overline{y}, x)$ such that

$$\mathfrak{B} \not\models \psi(\overline{b}, b). \tag{6.3}$$

In particular, for every $j < m$ we have $(\mathfrak{B}, \overline{b}) \not\models \forall x \psi_j(\overline{y}, x)$ and because $f \in I$ also $(\mathfrak{A}^{\text{per}}, \vec{a}) \not\models \forall x \psi_j(\overline{y}, x)$. By Lemma 4.2 there are $i_0 \in \mathbb{N}$ and $a_0 \in A$ such that

$$(\mathfrak{A}, \overline{\vec{a}}(i_0)) \not\models \psi_0(\overline{y}, a_0).$$

Similarly, there are $i_1 \in \mathbb{N}$ and $a_1 \in A$ such that

$$(\mathfrak{A}, \overline{\vec{a}}(i_1)) \not\models \psi_1(\overline{y}, a_1). \tag{6.4}$$

Moreover, we can choose i_1 such that $i_1 > i_0$ by periodicity: if $i_1 \leq i_0$ replace it by $i_1 + i_0 \cdot n$ where $n \in \mathbb{N}$ is large enough such that all components of $\vec{a}$ are n-periodic; then $\overline{\vec{a}}(i_1) = \overline{\vec{a}}(i_1 + i_0 \cdot n)$ and (6.4) remains true.

Continuing in this manner we get sequences $i_0 < i_1 < \cdots < i_{m-1}$ and $a_0, a_1, \ldots, a_{m-1}$ such that for all $j < m$

$$(\mathfrak{A}, \overline{\vec{a}}(i_j)) \not\models \psi_j(\overline{y}, a_j). \tag{6.5}$$

Choose a periodic $\vec{a} : \mathbb{N} \to A$ such that for all $j < m$

$$\vec{a}(i_j) = a_j. \tag{6.6}$$

We verify (6.2) for this $\vec{a}$: let $\psi(\overline{y}, x)$ be a positive Horn formula such that $(\mathfrak{B}, \overline{b}) \not\models \psi(\overline{y}, b)$. Then there exists $j < m$ such that $\psi(\overline{y}, x)$ is in $\mathfrak{A}$ equivalent to $\psi_j(\overline{y}, x)$. By (6.5) and (6.6) we get $(\mathfrak{A}, \overline{\vec{a}}(i_j)) \not\models \psi_j(\overline{y}, \vec{a}(i_j))$ and hence $(\mathfrak{A}, \overline{\vec{a}}(i_j)) \not\models \psi(\overline{y}, \vec{a}(i_j))$. By Lemma 4.2 we conclude $(\mathfrak{A}^{\text{per}}, \overline{\vec{a}}) \not\models \psi(\overline{y}, \vec{a})$. □

Proof of Theorem 6.1: The forward direction follows from Proposition 5.4 (note the $\aleph_0$-categoricity of $\mathfrak{A}$ is not needed).

Conversely, assume that a relation $R \subseteq A^\ell$ is preserved by all surjective periomorphisms of $\mathfrak{A}$. By Proposition 5.5 (2) it is preserved by all surjective polymorphisms, and in particular by all automorphisms of $\mathfrak{A}$. Since $\mathfrak{A}$ is $\aleph_0$-categorical, R is first-order definable in $\mathfrak{A}$ (recall Section 2.4). Let $\varphi_R(\overline{x}) = \varphi_R(x_0, \ldots, x_{\ell-1})$ be a formula such that $R = \varphi_R(\mathfrak{A})$.

By Ryll-Nardzewski there is a finite list of positive Horn formulas

$$\psi_0(\overline{x}), \ldots, \psi_{m-1}(\overline{x})$$

in the free variables $\overline{x} = x_0 \cdots x_{\ell-1}$ such that every such formula is in $\mathfrak{A}$ equivalent to one from the list. Some of these formulas are implied by $\varphi_R(\overline{x})$ (in $\mathfrak{A}$) and others not, and we may suppose that precisely the first k are not:

$$\begin{aligned} &\forall i < k \ \exists \overline{a}_i \in A^\ell \ : \ \overline{a}_i \in \varphi_R(\mathfrak{A}) \setminus \psi_i(\mathfrak{A}); \\ &\forall k \leq j < m \ : \ \varphi_R(\mathfrak{A}) \subseteq \psi_j(\mathfrak{A}). \end{aligned} \tag{6.7}$$

We can assume that $k \neq 0$ as otherwise $(\varphi_R \leftrightarrow \bot)$ holds in $\mathfrak{A}$ and then we are done. We claim that the positive Horn formula $\bigwedge_{k \leq j < m} \psi_j(\overline{x})$ is equivalent to $\varphi_R(\overline{x})$ in $\mathfrak{A}$. Therefore, it suffices to show

$$\mathfrak{A} \models \forall \overline{x} \big(\textstyle\bigwedge_{k \leq j < m} \psi_j(\overline{x}) \to \varphi_R(\overline{x}) \big).$$

So we assume that $\overline{b}$ satisfies $\bigwedge_{k \leq j < m} \psi_j(\overline{x})$ in $\mathfrak{A}$ and have to show that $\overline{b} \in \varphi_R(\mathfrak{A})$.

Choose for $i < k$ a tuple $\overline{a}_i \in A^\ell$ according to (6.7).

Claim. $\prod_{i<k} (\mathfrak{A}, \overline{a}_i) \Rrightarrow_{\text{pH}} (\mathfrak{A}, \overline{b})$.

Proof of the claim. Let $\psi(\overline{x})$ be a positive Horn formula that is not satisfied by $\overline{b}$ in $\mathfrak{A}$. Choose $i < m$ such that $\psi_i(\overline{x})$ is equivalent to $\psi(\overline{x})$ in $\mathfrak{A}$. Then $\overline{b}$ does not satisfy $\psi_i(\overline{x})$ in $\mathfrak{A}$, so $i < k$. But then $(\mathfrak{A}, \overline{a}_i) \not\models \psi_i(\overline{x})$ by (6.7) and thus $(\mathfrak{A}, \overline{a}_i) \not\models \psi(\overline{x})$. As $\psi(\overline{x})$ is positive Horn, $\prod_{i<k} (\mathfrak{A}, \overline{a}_i) \not\models \psi(\overline{x})$ by Lemma 3.1. ⊣

Write $\overline{a}_i = a_i^0 \cdots a_i^{\ell-1}$ for $i < k$. Then $\prod_{i<k}(\mathfrak{A}, \overline{a}_i)$ equals

$$\big(\mathfrak{A}^k, (a_0^0, \dots, a_{k-1}^0)(a_0^1, \dots, a_{k-1}^1) \cdots (a_0^{\ell-1}, \dots, a_{k-1}^{\ell-1})\big).$$

With $\mathfrak{A}$ also $(\mathfrak{A}, \overline{b})$ is $\aleph_0$-categorical. Further, the structure $\big(\mathfrak{A}^k, (a_0^0, \dots, a_{k-1}^0) \cdots\big)$ is $\aleph_0$-categorical, because $\mathfrak{A}^k$ is (see Section 2.4). By the claim we can thus apply Lemma 6.3 and conclude that there is a surjective homomorphism

$$h : \big(\mathfrak{A}^k, (a_0^0, \dots, a_{k-1}^0) \cdots (a_0^{\ell-1}, \dots, a_{k-1}^{\ell-1})\big)^{\mathrm{per}} \twoheadrightarrow (\mathfrak{A}, \overline{b}).$$

By Proposition 4.4 there is an isomorphism g from the left hand side structure onto

$$\big(\mathfrak{A}^{\mathrm{per}}, \langle a_0^0 \cdots a_{k-1}^0\rangle \cdots \langle a_0^{\ell-1} \cdots a_{k-1}^{\ell-1}\rangle\big).$$

Then $h \circ g^{-1}$ is a surjective homomorphism from $\mathfrak{A}^{\mathrm{per}}$ onto $\mathfrak{A}$, i.e. a surjective periomorphism of $\mathfrak{A}$, such that

$$h \circ g^{-1}(\langle a_0^0 \cdots a_{k-1}^0\rangle) \cdots h \circ g^{-1}(\langle a_0^{\ell-1} \cdots a_{k-1}^{\ell-1}\rangle) = \overline{b}.$$

By (6.7) we have $\overline{a}_i \in \varphi_R(\mathfrak{A})$ for all $i < k$. By Lemma 5.2 and the assumption that R and hence $\varphi_R(\overline{x})$ is preserved by surjective periomorphisms of $\mathfrak{A}$, we conclude $\overline{b} \in \varphi(\mathfrak{A})$, as was to be shown. □

Theorem 6.4. *For a finite language L_0, let $\mathfrak{B}$ be an L_0-structure and $\mathfrak{A}$ an L-structure on the same universe. If every surjective periomorphism of $\mathfrak{A}$ is a periomorphism of $\mathfrak{B}$, then the problem* QCSP($\mathfrak{B}$) *many-one logspace reduces to* QCSP($\mathfrak{A}$).

Proof: If $\varphi(\overline{x})$ is an atomic L_0-formula, then $\varphi(\mathfrak{B})$ is preserved by all polymorphisms of $\mathfrak{B}$, hence also by all periomorphisms of $\mathfrak{B}$ (by Proposition 5.5 (1)), and hence by all surjective periomorphisms of $\mathfrak{A}$ (by assumption). By the Main Theorem 6.1 the relation $\varphi(\mathfrak{B})$ is positive Horn definable in $\mathfrak{A}$. Hence $\mathfrak{B}$ is positive Horn definable in $\mathfrak{A}$. Now apply Proposition 3.2. □

7. Characterization of the pH-hull

A central tool in constraint complexity is the description of the smallest primitive positive definable relation containing a given relation R as the smallest relation that contains all polymorphic images of R; this description follows readily from Theorem 3.4. Here we provide a similar tool for quantified constraint complexity. The proof of this uses most of the results we established so far.

Recall Definition 5.3.

Theorem 7.1. *Let $\mathfrak{A}$ be $\aleph_0$-categorical and let R be a relation over A. Then*

$$\{\overline{a} \mid \exists k \in \mathbb{N}\ \exists \overline{a}_0, \dots \overline{a}_{k-1} \in R : \overline{a} \text{ is a surjective periomorphic image of } (\overline{a}_i)_{i<k}\}$$

is the smallest positive Horn definable relation containing R.

Proof: For notational simplicity, we assume that R is binary. It is easy to see that the displayed relation $\tilde{R}$ contains R. We have to show

(i) $\tilde{R} \subseteq \psi(\mathfrak{A})$ for any positive Horn formula ψ such that $R \subseteq \psi(\mathfrak{A})$;
(ii) $\tilde{R}$ is positive Horn definable in $\mathfrak{A}$.

To show (i) let $aa' \in \tilde{R}$. Choose $a_i a'_i, i < k$, in R such that some surjective periomorphism of $\mathfrak{A}$ maps $\langle a_0 \cdots a_{k-1}\rangle\langle a'_0 \cdots a'_{k-1}\rangle$ to aa'. Then $a_i a'_i \in \psi(\mathfrak{A})$ as $R \subseteq \psi(\mathfrak{A})$, so $aa' \in \psi(\mathfrak{A})$ by Proposition 5.4 as ψ is positive Horn.

We now prove (ii). By Theorem 6.1 it suffices to show that $\tilde{R}$ is preserved by all surjective periomorphisms of $\mathfrak{A}$. We use Lemma 5.2, so let $a_i a'_i, i < k$, be k tuples in $\tilde{R}$ and h be a surjective periomorphism that maps $\langle a_0 \cdots a_{k-1}\rangle\langle a'_0 \cdots a'_{k-1}\rangle$ to aa'. We have to show that $aa' \in \tilde{R}$.

For $i < k$ choose ℓ_i pairs $b_{ij} b'_{ij}, j < \ell_i$, in R such that there is a surjective periomorphism h_i that maps $\langle b_{i0} \cdots b_{i(\ell_i-1)}\rangle\langle b'_{i0} \cdots b'_{i(\ell_i-1)}\rangle$ to $a_i a'_i$. Letting the h_is act componentwise we get a surjective homomorphism

$$h' : \prod_{i<k}(\mathfrak{A}^{\mathrm{per}}, \langle b_{i0} \cdots b_{i(\ell_i-1)}\rangle\langle b'_{i0} \cdots b'_{i(\ell_i-1)}\rangle) \twoheadrightarrow \prod_{i<k}(\mathfrak{A}, a_i a'_i). \tag{7.1}$$

By Proposition 4.4 the left hand side structure is isomorphic to

$$\prod_{i<k}(\mathfrak{A}^{\ell_i}, (b_{i0} \cdots b_{i(\ell_i-1)})(b'_{i0} \cdots b'_{i(\ell_i-1)}))^{\mathrm{per}}$$

and thus by Lemma 4.6 to the periodic power of

$$\mathfrak{B} := \left(\mathfrak{A}^{\sum_{i<k}\ell_i}, (b_{00} \cdots b_{(k-1)(\ell_{k-1}-1)}), (b'_{00} \cdots b'_{(k-1)(\ell_{k-1}-1)})\right).$$

By (7.1) and Proposition 6.2 we get

$$\mathfrak{B} \Rrightarrow_{\mathrm{pH}} \prod_{i<k}(\mathfrak{A}, a_i a'_i). \tag{7.2}$$

By Proposition 4.4 the structure $(\prod_{i<k}(\mathfrak{A}, a_i a'_i))^{\mathrm{per}}$ is isomorphic to the structure

$$(\mathfrak{A}^{\mathrm{per}}, \langle a_0 \cdots a_{k-1}\rangle, \langle a'_0 \cdots a'_{k-1}\rangle)$$

which maps surjectively onto $(\mathfrak{A}, aa')$ by h. Hence, by Proposition 6.2 again,

$$\prod_{i<k}(\mathfrak{A}, a_i a'_i) \Rrightarrow_{\mathrm{pH}} (\mathfrak{A}, aa'). \tag{7.3}$$

By (7.2) and (7.3) we conclude $\mathfrak{B} \Rrightarrow_{\mathrm{pH}} (\mathfrak{A}, aa')$. But these two structures are $\aleph_0$-categorical (by Ryll-Nardzewski), so Lemma 6.3 applies and there is a surjective homomorphism

$$h'' : \mathfrak{B}^{\mathrm{per}} \twoheadrightarrow (\mathfrak{A}, aa').$$

By Proposition 4.4, $\mathfrak{B}^{\mathrm{per}}$ is isomorphic to

$$\left(\mathfrak{A}^{\mathrm{per}}, \langle b_{00} \cdots b_{(k-1)(\ell_{k-1}-1)}\rangle\langle b'_{00} \cdots b'_{(k-1)(\ell_{k-1}-1)}\rangle\right),$$

so aa' is a surjective periomorphic image of the $\sum_{i<k}\ell_i$ many pairs

$$b_{00}b'_{00}, \ldots, b_{(k-1)(\ell_{k-1}-1)}b'_{(k-1)(\ell_{k-1}-1)} \in R.$$

Thus $aa' \in \tilde{R}$, as was to be shown. $\square$

8. Equality templates

Fix a countably infinite set A and define an *equality template* to be a relational structure $\mathfrak{A}$ that is first-order definable in (A), the structure interpreting the empty language; that is, every relation of $\mathfrak{A}$ is definable by a pure equality formula. A complexity classification of the QCSPs of equality templates was given in previous work [9] (see Theorem 8.9 below): it was shown that each such QCSP is either in L, NP-complete or coNP-hard. In this section, we re-examine this classification theorem. Based on our Main Theorem 6.1 we give a new proof of this classification which is, in our view, shorter, more modular, and conceptually cleaner than the original proof.

8.1. Clone analysis. Our proof follows the algebraic approach to constraint complexity and thereby relies on an analysis of the polymorphism clones of equality templates. Such clones are locally closed and contain all permutations, as every permutation of A is an automorphism of $\mathfrak{A}$. Bodirsky, Chen, and Pinsker [11], building on the work of Bodirsky and Kara [12], performed a study of these clones. Here we state only what we shall need from their analysis.

We define an operation to be *elementary* if it is contained in the smallest locally closed clone containing all permutations; a set of operations is *elementary* if each of its operations is elementary. Let us say that an operation f *generates* another operation g if g is contained in the smallest locally closed clone that contains f and all permutations of A. Note, an operation is elementary if and only if it is generated by the identity on A. Finally, recall that an *essentially unary* operation is one that can be written as the composition of a unary operation and a projection; and, an *essential* operation is one that is not essentially unary.

Lemma 8.1 (Clone analysis)**.**

(1) *A non-elementary operation generates either a binary injective operation or a unary constant operation.*

(2) *An operation with infinite image that does not preserve $\neq$ generates all unary operations.*

(3) *Let $k \geq 3$. An essential operation with image size k generates all operations with image size at most k.*

Proof. The lemma can be derived from results in [12, 11] as follows. To prove (1), let f be a non-elementary operation. If f is essentially unary, then f generates a unary non-elementary operation h. The operation h is not injective, since all unary injective operations can be interpolated by permutations. By the proof of [12, Lemma 10], h generates a unary constant operation.

Now suppose that f is essential. By [12, Lemma 12], f generates an essential binary operation. By [12, Theorem 13], f generates either a unary constant operation or a binary injective operation.

Statement (2) follows from [11, Lemma 38] and statement (3) is [11, Lemma 36]. □

8.2. Classification. We now start the proof of the classification theorem for equality templates.

Theorem 8.2. *Let $\mathfrak{A}$ be an equality template such that $\neq$ is not positive Horn definable in $\mathfrak{A}$. Then every unary operation on A is a polymorphism of $\mathfrak{A}$.*

Proof: If $\neq$ is not positive Horn definable in $\mathfrak{A}$, then, by our Main Theorem 6.1, the relation $\neq$ is not preserved by some surjective periomorphism h of $\mathfrak{A}$. Recall that according to (5.1) with h there is a naturally associated sequence of polymorphisms $(h_{<k})_{k\geq 1}$. Because h does not preserve $\neq$, there exists k_0 such that $h_{<k_0}$ does not either. Suppose there exists some k_1 such that $h_{<k_1}$ has infinite image. Then $h_{<k_0 \cdot k_1}$ does not preserve $\neq$ and has infinite image. Then our claim follows from Lemma 8.1 (2). We thus assume that all $h_{<k}$ have finite image. By local closure it suffices to show:

Claim. For every $k \in \mathbb{N}$ every partial unary operation $g : A \to A$ that is defined on k points can be extended to a (unary) polymorphism of $\mathfrak{A}$.

We prove the claim by induction on k. For $k = 0$ there is nothing to show. Suppose that the claim is true for k and let g be a unary operation defined on $k + 1$ points. If g has image size $k + 1$, then there exists a permutation g' extending g, and the claim follows; recall that all permutations are automorphisms of $\mathfrak{A}$. So suppose that g has image of size at most k.

It suffices to show that the polymorphism clone of $\mathfrak{A}$ contains a unary operation that has finite image of size $\geq k$, for this implies that the clone contains a unary operation that maps $k + 1$ points to k points; by composing this unary operation with itself and suitable permutations, one obtains the claim.

Since h has infinite image, there exists $\ell > 0$ such that $h_{<\ell}$ has image size $\geq k$. Let $\bar{a}_0, \dots, \bar{a}_{k-1} \in A^\ell$ be k many ℓ-tuples on which $h_{<\ell}$ is injective. Assume for the sake of notation that $0, \dots, k-1 \in A$. Consider the maps $u_0, \dots, u_{\ell-1}$ defined on $\{0, \dots, k-1\}$ such that u_j maps each $i < k$ to the jth component of $\bar{a}_i$. Note that $u_0(i) \cdots u_{\ell-1}(i) = \bar{a}_i$. By induction every u_j can be extended to a polymorphism u'_j of $\mathfrak{A}$. Define $u : A \to A$ to map $a \in A$ to $h_{<\ell}(u'_0(a), \dots, u'_{\ell-1}(a))$. Then $u(i) = h_{<\ell}(\bar{a}_i)$ for every $i < k$, so u is injective on the set $\{0, \dots, k-1\}$. Thus the image of u has size $\geq k$ and is finite because it is contained in the image of $h_{<\ell}$. □

The following simple lemma will be useful. It appears as Lemma 11 in [12]; we supply a proof for self-containment.

Lemma 8.3. *Let $\mathfrak{A}$ be an equality template. Either $\mathfrak{A}$ has a constant polymorphism, or the relation $\neq$ is primitive positively definable in $\mathfrak{A}$.*

Proof. Suppose that $\mathfrak{A}$ does not have a constant polymorphism. Then there is a relation $R^{\mathfrak{A}}$ that is non-empty and does not contain the constant tuple. Let k be the arity of $R^{\mathfrak{A}}$. Let us say that an equivalence relation σ on $\{0, \dots, k-1\}$ is *realized* if there exists a tuple $(a_0 \dots, a_{k-1}) \in R^{\mathfrak{A}}$ such that $a_i = a_j$ if and only if $(i, j) \in \sigma$. (Note that if there exists one tuple in $R^{\mathfrak{A}}$ satisfying the given condition, then all tuples satisfying the given condition are in $R^{\mathfrak{A}}$.) Let τ be a coarsest realized equivalence relation. Consider the relation defined in $\mathfrak{A}$ by the primitive positive formula

$$\varphi(x_0, \dots, x_{k-1}) := Rx_0 \cdots x_{k-1} \wedge \bigwedge_{(i,j)\in\tau} x_i = x_j;$$

in this relation, τ is realized, and it is the only equivalence relation that is realized. Since $R^{\mathfrak{A}}$ does not contain the constant tuple, τ contains more than one equivalence class. Fix $i, j \in \{0, \dots, k-1\}$ to be values such that $(i, j) \notin \tau$. The formula $\psi(x_i, x_j)$ derived from φ by existentially quantifying all variables other than x_i and x_j defines the relation $\neq$. □

Let us say that a relation over A is *negative* if it is definable as the conjunction of *(i)* equalities and *(ii)* disjunctions of disequalities; by a disequality, we mean a formula of the form $\neg x = y$. Let us say that a relation is *positive* if it is definable using equalities and the binary connectives $\{\wedge, \vee\}$. We call an equality template *negative* or *positive* if each of its relations is negative or positive respectively.

Example 8.4. The ternary relation $P \subseteq A^3$ defined by the formula $\varphi_P(x, y, z) := (x = y \vee y = z)$ in (A) is positive; it can be verified from the definition that it is not negative.

Example 8.5. The ternary relation $I \subseteq A^3$ defined by the formula $\varphi_I(x, y, z) := (x = y \rightarrow y = z)$ in (A) is neither positive not negative; this can be verified from the definitions.

Positivity can be characterized algebraically as follows. This has been shown in [9, Proposition 7.3].

Proposition 8.6. *Let $\mathfrak{A}$ be an equality template, and fix f to be any non-injective surjective unary operation on A. The following are equivalent:*
- *$\mathfrak{A}$ is positive.*
- *Every unary operation is a polymorphism of $\mathfrak{A}$.*
- *The operation f is a polymorphism of $\mathfrak{A}$.* □

We have the following fact.

Corollary 8.7.
(1) *If $\mathfrak{A}$ is a positive equality template, then every positive Horn definable relation in $\mathfrak{A}$ is positive.*
(2) *If $\mathfrak{A}$ is a negative equality template, then every positive Horn definable relation in $\mathfrak{A}$ is negative.*

Proof. By Proposition 8.6 we have that for any fixed non-injective surjective unary operation f, a relation is positive if and only if it is preserved by f; this characterization of positivity implies (1).

Likewise, (2) follows from the fact that negativity can be characterized by preservation by a surjective operation (see [11, Proposition 68]). □

The following is known ([9, Lemma 8.8]):

Lemma 8.8. *If R is a relation over A that is not negative and is preserved by a binary injective operation, then I is primitive positively definable in $(A, R, \neq)$.* □

We are ready to state and prove the classification.

Theorem 8.9 ([9])**.** *Let $\mathfrak{A}$ be an equality template.*
(1) *If $\mathfrak{A}$ is negative, then* $\mathsf{QCSP}(\mathfrak{A})$ *is in* L.
(2) *If $\mathfrak{A}$ is not negative but positive, then the relation P is positive Horn definable in $\mathfrak{A}$ and* $\mathsf{QCSP}(\mathfrak{A})$ *is* NP*-complete.*
(3) *If $\mathfrak{A}$ is neither negative nor positive, then the relation I is positive Horn definable in $\mathfrak{A}$ and* $\mathsf{QCSP}(\mathfrak{A})$ *is* coNP*-hard.*

Proof: We take as given the following complexity results: it is shown in [9] that a negative template $\mathfrak{A}$ has $\mathsf{QCSP}(\mathfrak{A})$ in L, that $\mathsf{QCSP}((A, P))$ is NP-hard, and that $\mathsf{QCSP}((A, I))$ is coNP-hard; and, it follows from [31] that a positive template $\mathfrak{A}$ has $\mathsf{QCSP}(\mathfrak{A})$ in NP. By Proposition 3.2 and Corollary 8.7, it thus suffices to show that for an equality template $\mathfrak{A}$ one of the following three conditions holds:
(i) $\mathfrak{A}$ is negative.
(ii) $\mathfrak{A}$ is positive and P is positive Horn definable in $\mathfrak{A}$.

(iii) I is positive Horn definable in $\mathfrak{A}$.

Let $\mathfrak{A}$ be an equality template and let $[\mathfrak{A}]_{\mathrm{pH}}$ denote its expansion by all relations that are positive Horn definable in $\mathfrak{A}$. Further, let C denote the clone of polymorphisms of $[\mathfrak{A}]_{\mathrm{pH}}$. By Lemma 8.1 (1), the following three cases are exhaustive.

Case 1: C is elementary. Then C preserves I, so this relation is primitive positively definable in $[\mathfrak{A}]_{\mathrm{pH}}$ by Theorem 3.4 and hence positive Horn definable in $\mathfrak{A}$.

Case 2: C contains a constant operation. Then $\neq$ is not contained in $[\mathfrak{A}]_{\mathrm{pH}}$, since $\neq$ is not preserved by a constant operation. Applying Theorem 8.2 to $[\mathfrak{A}]_{\mathrm{pH}}$, we obtain that C contains all unary operations. Proposition 8.6 implies that $[\mathfrak{A}]_{\mathrm{pH}}$ (and hence $\mathfrak{A}$) is positive. We claim that either $[\mathfrak{A}]_{\mathrm{pH}}$ (and hence $\mathfrak{A}$) is negative or P is positive Horn definable in $\mathfrak{A}$.

Case 2.1: Suppose that there exists a surjective periomorphism h of $\mathfrak{A}$ and a $k > 0$ such that the polymorphism $h_{<k}$ is essential. We claim that in this case C contains all operations. It is known (and easy to verify) that each relation preserved by this clone can be defined by a conjunction of equalities, so then $[\mathfrak{A}]_{\mathrm{pH}}$ will be negative. By local closure, it suffices to show that C contains all finite image operations. Hence, by Lemma 8.1 (3), it suffices to show that C contains a sequence of polymorphisms that is *desirable* in the sense that each polymorphism is essential and has finite image, and that the sequence has unbounded image size. Now, $(h_{<\ell\cdot k})_{\ell>0}$ is such a desirable sequence in case each $h_{<\ell\cdot k}$ has finite image. And otherwise there is $\ell_0 > 0$ such that $h_{<\ell_0\cdot k}$ has infinite image, and then one obtains a desirable sequence $(u_i \circ h_{<\ell_0\cdot k})_{i>0}$ for suitable unary operations u_i (recall that all unary operations are in C).

Case 2.2: Suppose otherwise that for every surjective periomorphism h and all $k > 0$ the polymorphism $h_{<k}$ is essentially unary. We claim that then the relation P is positive Horn definable in $\mathfrak{A}$. By our Main Theorem 6.1 it suffices to show that P is preserved by all surjective periomorphisms of $\mathfrak{A}$. But if a surjective periomorphism h of $\mathfrak{A}$ does not preserve P, then there exists $k > 0$ such that $h_{<k}$ does not preserve P. Since $h_{<k}$ is essentially unary, this is impossible.

Case 3: C contains a binary injective operation and does not contain a constant operation. In this case, $[\mathfrak{A}]_{\mathrm{pH}}$ contains $\neq$ by Lemma 8.3. It follows immediately from Lemma 8.8 that either $[\mathfrak{A}]_{\mathrm{pH}}$ (and hence $\mathfrak{A}$) is negative or I is primitive positively definable in $[\mathfrak{A}]_{\mathrm{pH}}$ and hence positive Horn definable in $\mathfrak{A}$. □

9. Discussion

Bing's theorem [3] involves a clever, technical argument that allows us to strengthen our main preservation theorem for structures that are isomorphic to their finite powers. Such structures have gained some attention in constraint complexity [10, 6]. We have the following theorem.

Theorem 9.1. *Let $\mathfrak{A}$ be a countable $\aleph_0$-categorical structure such that $\mathfrak{A} \cong \mathfrak{A}^2$. Then a formula $\varphi(\overline{x})$ is equivalent to a positive Horn formula in $\mathfrak{A}$ if and only if it is preserved by all surjective polymorphisms of $\mathfrak{A}$.*

Proof. Let $\mathfrak{A}$ accord the assumption of the theorem. We only prove the backward direction. Assume $\varphi(\overline{x})$ is preserved by all surjective polymorphisms of $\mathfrak{A}$. In particular, $\varphi(\overline{x})$ is preserved by all surjective homorphisms from $\mathfrak{A}$ to $\mathfrak{A}$. It is not hard to see that Lyndon's

Theorem implies that there exists a positive formula $\varphi^+(\overline{x})$ such that $\varphi(\mathfrak{A}) = \varphi^+(\mathfrak{A})$ (see [7, Proposition 2 (c)] for details). We can assume that φ^+ has the form of some quantifier prefix followed by a quantifier free formula

$$\psi = \bigwedge_{i\in I}\bigvee_{j\in J}\alpha_{ij},$$

where the α_{ij}s are atoms. For each $f \in J^I$ write

$$\psi_f := \bigwedge_{i\in I}\alpha_{if(i)}.$$

Bing's argument. Let $\overline{Q}\overline{y}$ be an arbitrary quantifier prefix. Assume for every $f \in J^I$ the tuple $\overline{a}_f$ in $\mathfrak{A}$ is an assignment to the free variables in $\overline{Q}\overline{y}\psi$ such that $\prod_{f\in J^I}(\mathfrak{A},\overline{a}_f) \models \overline{Q}\overline{y}\psi$. Then there exists $f \in J^I$ such that $(\mathfrak{A},\overline{a}_f) \models \overline{Q}\overline{y}\psi_f$.

Proof of Bing's argument. This can be proved by a straightforward induction on the length of $\overline{Q}\overline{y}$. See [3, Lemma 3] for details. ⊣

Write $\varphi^+(\overline{x}) = \overline{Q}\overline{y}\psi(\overline{y},\overline{x})$.

Claim. There exists $f \in J^I$ such that $\mathfrak{A} \models \forall\overline{x}(\varphi^+(\overline{x}) \to \overline{Q}\overline{y}\psi_f(\overline{y},\overline{x}))$.

Proof of Claim. Otherwise we find for every $f \in J^I$ an $\overline{a}_f \in \varphi^+(\mathfrak{A})$ such that

$$(\mathfrak{A},\overline{a}_f) \not\models \overline{Q}\overline{y}\psi_f(\overline{y},\overline{x}).$$

Then $\prod_{f\in J^I}(\mathfrak{A},\overline{a}_f) \not\models \varphi^+(\overline{x})$ by Bing's argument. As $\mathfrak{A} \cong \mathfrak{A}^2$, there is an isomorphism

$$h : \mathfrak{A}^{J^I} \cong \mathfrak{A}.$$

Write $\overline{x} = x_0 \cdots x_{\ell-1}$ and $\overline{a}_f = a_f^0 \cdots a_f^{\ell-1}$. Then

$$\begin{aligned} h : \prod_{f\in J^I}(\mathfrak{A},\overline{a}_f) &= \big(\mathfrak{A}^{J^I}, (a_f^0)_{f\in J^I},\ldots,(a_f^{\ell-1})_{f\in J^I}\big)\\ &\cong \big(\mathfrak{A}, h((a_f^0)_{f\in J^I}),\ldots,h((a_f^{\ell-1})_{f\in J^I})\big).\end{aligned}$$

Since h is an isomorphism, $\varphi^+(\overline{x})$ is false in the right hand side structure. Hence h is (up to a renaming of indices) a surjective polymorphism of $\mathfrak{A}$ that does not preserve $\varphi(\overline{x})$, a contradiction. ⊣

Since $(\overline{Q}\overline{y}\psi_f \to \varphi^+)$ is logically valid, the claim implies that φ^+ is equivalent in $\mathfrak{A}$ to the positive Horn formula $\overline{Q}\overline{y}\psi_f$. □

Examples 9.2. An example of a structure satisfying the assumption of the theorem is the countable atomless Boolean algebra (cf. [4, Section 5.2]). This template is of central importance for spatial reasoning in artificial intelligence. Another example is an infinite dimensional vectorspace over some finite field (cf. [15, Example 2.10], [4, Section 5.3]). More generally, it is easy to see that every countable $\aleph_0$-categorical structure $\mathfrak{A}$ whose theory is Horn axiomatizable satisfies $\mathfrak{A} \cong \mathfrak{A}^2$.

We conclude with some remarks and questions.

• Very recently, Bodirsky, Hils and Martin [6] explored the possibilities to extend the algebraic machinery for constraint satisfaction to structures that are not necessarily $\aleph_0$-categorical; they established a variant of the preservation theorem for primitive positive definability via ω-polymorphisms for structures that are in a certain sense sufficiently saturated. (An ω-polymorphism of a structure $\mathfrak{A}$ is a homomorphism from $\mathfrak{A}^{\mathbb{N}}$ to $\mathfrak{A}$.)

• The first author showed [19, Lemma 7.5] that, in finite structures, positive Horn definability coincides with Π_2 positive Horn definability (see [35, 22] for a related result). Using the method of the proof, one can infer that Boolean QCSPs with quantifier alternation rank restricted to some even $t \geq 2$ are either Π_t^{P}-complete or in P (cf. [19, Theorem 7.2]). An open issue is to study $\aleph_0$-categorical QCSPs with bounded alternation rank.

One can ask the following concrete question. Let $\mathfrak{A}$ be a $\aleph_0$-categorical structure and φ a Π_t formula that is preserved by the surjective periomorphisms of $\mathfrak{A}$. Is φ equivalent to a positive Horn formula that is also Π_t?

• A related question is posed by Y. Chen and Flum in [24]. They ask for an alternation rank preserving version of Lyndon's preservation theorem: is any Π_t sentence that is preserved by surjective homomorphisms equivalent to a positive Π_t sentence? This is known to be true for $t \leq 2$ [40]. By a well-known trick of Lyndon [34] (see also Fefermann's survey [26]) a positive answer would follow from a proof of the following: any implication between Π_t formulas has a Π_t Lyndon-interpolant. The usual argument constructs an interpolant by recursion on a cut-free proof of the given implication. But again for $t > 3$ there seems to be no control on the alternation rank of an interpolant constructed in this way.

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