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Universidad del País Vasco / Euskal Herriko Unibertsitatea

# The forward problem for the electromagnetic Helmholtz equation

Miren Zubeldia Plazaola

PhD Thesis

University of the Basque Country
UPV/EHU
Bilbao, 2012

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PhD Thesis

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by

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Memoria realizada bajo la dirección del catedrático de Análisis Dr. D. Luis Vega González, para optar al grado de Doctor en Ciencias, Sección de Matemáticas, por la Universidad del País Vasco - Euskal Herriko Unibertsitatea.

Cuando emprendes tu viaje hacia Ítaca
debes rogar que el viaje sea largo,
lleno de peripecias, lleno de experiencias.
No has de temer ni los lestrigiones ni a los cíclopes,
ni la cólera del airado Posidón.
Nunca tales monstruos hallarás en tu ruta
si tu pensamiento es elevado, si una exquisita
emoción penetra en tu alma y en tu cuerpo.
Los lestrigones y los cíclopes
y el feroz Posidón no podrán encontrarte
si tú no los llevas ya dentro, en tu alma,
si tu alma no los conjura ante ti.
Debes rogar que el viaje sea largo,
que sean muchos los días de verano;
que te vean arriba con gozo, alegremente,
a puertos que tú antes ignorabas.
Que puedas detenerte en los mercados de Fenicia,
madreperlas, coral, ébano y ámbar,
y perfumes placenteros de mil clases.
Acude a muchas ciudades de Egipto
para aprender, y aprender de quienes saben.
Conserva siempre en tu alma la idea de Ítaca:
llegar allí, he aquí tu destino.
Mas no hagas con prisa tu camino;
mejor será que dure muchos años,
y que llegues, ya viejo, a la pequeña isla,
rico de cuanto habrías ganado en el camino.
No has de esperar que Ítaca te enriquezca:
Ítaca te ha concedido ya un hermoso viaje.
Sin ellas, jamás habrías partido;
mas no tiene otra cosa que ofrecerte.
Y si lo encuentras pobre, Ítaca no te ha engañado.
Y siendo ya tan viejo, con tanta experiencia,
sin duda sabrás qué significan las Ítacas.

**Ítaca. Konstantínos Kaváfis.**

# Abstract

We consider the Helmholtz equation in $\mathbb{R}^d$, $d \geq 3$, with electric and magnetic potentials. The aim of this thesis is to study the direct problem of such an equation for potentials that decay at infinity, but also have singularities at the origin. We use integration by parts to achieve this. A main tool is a multiplier method in spirit of the so called Morawetz estimate.

We prove that there exists a unique solution of this equation such that satisfies some a-priori estimates together with some Sommerfeld radiation condition. We then deduce some extra information about the behavior of the solution for different classes of electric and magnetic potentials, which allows us to derive some applications related to the spectral properties for the magnetic Schrödinger operator.

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# Introduction

The classical partial differential equations of mathematical physics, formulated and intensively studied by the great mathematicians of the nineteenth century as for example, D'Alembert, Euler and Lagrange, remain the foundation of investigations into waves, heat conductions, electromagnetism and other physical problems. Although the issue of existence and uniqueness of solutions of ordinary differential equations has a very satisfactory answer with the Picard-Lindelöf theorem, that is far from the case for partial differential equations.

Problems involving stationary phenomena, i.e. phenomenas which are independent of time, can be reduced to equations of elliptic type. The Helmholtz equation is a second order elliptic partial differential equation

$$\Delta u(x) + k^2 u(x) = f(x), \tag{0.0.1}$$

where $\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ is the Laplacian, $k$ is the wave number and $u$, $f$ are real or complex valued functions on the euclidean space $\mathbb{R}^d$. This equation is encountered in many branches of mathematical physics as in the theory of elasticity or the theory of electromagnetic waves. For $k = 0$ it coincides with Poisson's equation

$$\Delta u(x) = f(x). \tag{0.0.2}$$

The theory of Helmholtz's equation is close to that of Poisson's equation or of the Laplace operator, but there are a few peculiarities concerning the uniqueness of solution (for $k^2 > 0$). It is well known that the solution of Poisson's equation in a whole space $\mathbb{R}^d$ is unique in the class of generalized functions and tends to zero at infinity. However, this statement is not true for Helmholtz's equation. For example, when $d = 3$

$$u(x) = -\frac{\sin k|x|}{4\pi|x|} \tag{0.0.3}$$

is a non zero solution of the corresponding homogeneous equation

$$\Delta u(x) + k^2 u(x) = 0. \tag{0.0.4}$$

This also happens in all dimensions $d \geq 1$. In order to isolate the class of unique solutions for Helmholtz's equation, there must be additional restrictions on the behavior of the solution at infinity. These restrictions are the so-called Sommerfeld radiation conditions, which are typically read by

$$\lim_{|x|\to+\infty} |x|^{\frac{d-1}{2}} \left( \frac{\partial u(x)}{\partial |x|} \mp iku(x) \right) = 0 \tag{0.0.5}$$

uniformly in $\frac{x}{|x|} \in S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$. The condition with the minus sign corresponds to divergent waves (outgoing to infinity), while the plus sign is related to convergent waves (incoming from infinity).

The limiting absorption principle provides one way for isolating the unique solution of the Helmholtz equation (0.0.1). One shall add the absorption term $i\varepsilon u$ to the left-hand side of Helmholtz's equation obtaining

$$\Delta u_\varepsilon + (k^2 + i\varepsilon) u_\varepsilon = f(x). \tag{0.0.6}$$

Then from the self-adjointness of the Laplace operator on $\mathbb{R}^d$, we have that its spectrum is real and thus when $\varepsilon \neq 0$, it follows that for any generalized function $f$ with compact support, there exists a solution of the equation (0.0.6) in the set of all generalized functions of slow growth. Uniqueness of solution follows from the study of the corresponding homogeneous equation

$$\Delta u_\varepsilon + (k^2 + i\varepsilon) u_\varepsilon = 0. \tag{0.0.7}$$

Looking at the Fourier side of the above identity, we get

$$(-|\xi|^2 + k^2 + i\varepsilon)\mathcal{F}[u_\varepsilon] = 0, \tag{0.0.8}$$

which makes it obvious that $u_\varepsilon = 0$ for $\varepsilon \neq 0$.

As a consequence, the uniform limit in $x$ of $u_\varepsilon \to \pm 0$, that we will denote by

$$u_\pm = R(k^2 \pm i0) f = \lim_{\varepsilon \to \pm 0} u_\varepsilon, \tag{0.0.9}$$

produces the unique solution of the non homogeneous Helmholtz equation (0.0.1) satisfying the corresponding Sommerfeld radiation condition (0.0.5). In the three dimensional case, these solutions can be expressed by

$$u = \frac{e^{\pm ik|x|}}{4\pi |x|} * f, \tag{0.0.10}$$

where $\Phi(x) = \frac{e^{ik|x|}}{4\pi i}$ is the fundamental solution of the Helmholtz equation (0.0.1). For a higher dimensions, the fundamental solution of the Helmholtz equation in $\mathbb{R}^d$ can be given by Hankel's function as follows

$$\Phi(x) = c_d \frac{k^{\frac{d-2}{2}} H^{(1)}_{(d-1)/2}(k|x|)}{|x|^{\frac{(d-2)}{2}}}, \quad \text{where} \quad c_d = \frac{1}{2i(2\pi)^{\frac{d-2}{2}}}. \tag{0.0.11}$$

Hence the asymptotic behavior of Hankel's functions allows us to deduce the behavior of $\Phi$ at infinity. Therefore, we can conclude that the outgoing solution of the equation (0.0.1) satisfies

$$u(x) = c_d k^{\frac{d-1}{2}} \frac{e^{ik|x|}}{|x|^{\frac{d-1}{2}}} u_\infty \left(k, \frac{x}{|x|}\right) + o(|x|^{-\frac{(d-1)}{2}}) \qquad \text{as} \qquad |x| \to \infty. \tag{0.0.12}$$

The function $u_\infty$ is known as the far field pattern or scattering amplitude of $u$ and is given by the following limit

$$u_\infty(k,\omega) = \lim_{|x|\to\infty} c_{d,k} |x|^{\frac{d-1}{2}} e^{-ik|x|} u(\omega|x|) \tag{0.0.13}$$

in $L^2(S^{d-1})$ where $\omega = \frac{x}{|x|}$, provided that the above limit exists. The scattering amplitudes are the data used in inverse scattering problems. From these data one tries to reconstruct the inner structure of any system ruled by any equation which is the Helmholtz equation in some exterior domains. In some cases, what is observable are the scattering cross-sections, i.e., the absolute values of the far field pattern. The problem of how to obtain the phases of the scattering amplitudes from these datas is highly nontrivial and in general, does not have a unique solution (see [KoSc] and the references given there).

The Helmholtz equation can also be understood as an eigenvalue problem of the Laplace operator. In this case, one would be interested in the study of the resolvent operator

$$R(k^2) = (\Delta + k^2 + i0)^{-1}. \tag{0.0.14}$$

Resolvent estimates are crucial in order to prove the limiting absorption principle for the corresponding operator. Among the family of a-priori estimates for the Helmholtz equation, the most celebrated and widely used one is due to Agmon [A], where it is proved that

$$\|R(k^2)f\|_{L^2_{-\delta}} \leq \frac{C(\delta)}{|k|} \|f\|_{L^2_\delta} \tag{0.0.15}$$

with

$$\|f\|_{L^2_\delta} := \|(1+|x|^2)^{\frac{\delta}{2}} f\|_{L^2}, \tag{0.0.16}$$

for $\delta > 1$. From this, it may be concluded that $u = R(k^2)f$ is the unique outgoing solution of the equation (0.0.1) satisfying

$$\lim_{|x|\to+\infty} |x|^{\frac{d-1}{2}} \left(\frac{\partial u}{\partial |x|} - iku\right) = 0. \tag{0.0.17}$$

Later on, Agmon and Hörmander [AH] showed that estimate (0.0.15) held with the $L^2_\delta$ norms replaced by the norms

$$|||u|||_{R_0} := \sup_{R>R_0} \left(\frac{1}{R}\int_{|x|\leq R} |u(x)|^2 dx\right)^{1/2}$$

and

$$N_{R_0}(f) := \sum_{j>J} \left( 2^{j+1} \int_{C(j)} |f(x)|^2 dx \right)^{1/2} + \left( R_0 \int_{|x|\le R_0} |f(x)|^2 dx \right)^{1/2}$$

with $R_0 = 1$, where $C(j) = \{x \in \mathbb{R}^d : 2^{j-1} \le |x| \le 2^j\}$ and $J$ is defined by $2^{J-1} \le R_0 \le 2^J$. The norms $|||u|||_1$ and $N_1(f)$ are known as Agmon-Hörmander norms. We drop the index $R_0$ if $R_0 = 0$, getting then the Morrey-Campanato norm and its dual,

$$|||u||| := \sup_{R>0} \left( \frac{1}{R} \int_{|x|\le R} |u(x)|^2 dx \right)^{1/2} \tag{0.0.18}$$

$$N(f) := \sum_{j\in\mathbb{Z}} \left( 2^{j+1} \int_{C(j)} |f(x)|^2 dx \right)^{1/2}. \tag{0.0.19}$$

Note that it is satisfied

$$\begin{aligned} \int fg &\le \sum_{j\in\mathbb{Z}} \left[ 2^j \int_{C(j)} |f|^2 \frac{1}{2^j} \int_{C(j)} |g|^2 \right]^{1/2} \\ &\le |||g|||N(f). \end{aligned} \tag{0.0.20}$$

The Agmon-Hörmander estimate was improved by Kenig, Ponce and Vega [KPV] to the Morrey-Campanato norm in their study of the nonlinear Schrödinger equation. In fact, they proved

$$|||u||| \le Ck^{-1}N(f). \tag{0.0.21}$$

This estimate plays a fundamental role in solving Schrödinger evolution equations with nonlinear first order terms.

It is of interest to know whether the limiting absorption principle, the existence of the far field pattern or the resolvent estimates are still true when the free operator $H_0 = \Delta$ is perturbed by an external potential. First we consider a zero order perturbation of $H_0$ given by the differential operator $\Delta + V(x)$, where $V(x)$ is a real function, also called an electric potential. In the free case $V \equiv 0$, the seminal papers by Agmon and Hörmander [A], [AH], inspired a huge literature (see for example [A], [AH], [Be], [Ik], [Is], [Mou] [MU], [Ka3],. . . ) which has been produced in order to obtain weighted $L^2$-estimates for solutions of Helmholtz equations. As it is well known, apart from the limiting absorption principle, one of the consequences of the Agmon-Hörmander estimate is related to the spectral properties of $\Delta + V$ which have applications to scattering theory.

The classical work of Agmon [A] shows the limiting absorption principle for short range perturbations of $H_0$. Fourier analysis is involved as a crucial tool in the proofs strategy; however, Fourier transform does not permit in general to treat neither rough potentials nor the case in which the same problems are settled in domains that are different from the whole space. For this reason, a great effort has been spent in order to develop multiplier methods which work directly on the equation, inspired by the techniques introduced by Morawetz in [Mo] for the Klein-Gordon equation. We point out that when $V(x) = O(|x|^{-1-\mu})$, $\mu > 0$, the phase of the Sommerfeld radiation condition that satisfies the solution of the equation $(H_0 + V + \lambda)u = f$ is given by $\lambda^{1/2}\frac{x}{|x|}$.

Multiplier techniques are also used by Saito [S1], [S2], Isozaki [Is] or Mochizuki and Uchiyama [MU] among others, in order to study existence and uniqueness of solution of the Helmholtz equation

$$\Delta u + Vu + \lambda u = f \tag{0.0.22}$$

with long range electric potential $V$. In all of them the Sommerfeld radiation condition that the solution $u$ satisfies is given by

$$\int |\nabla u - i(\nabla K)u|^2 \frac{1}{(1+|x|)^{1-\delta}} < +\infty, \tag{0.0.23}$$

being $K = K(x,\lambda)$ an exact or approximate solution of the eikonal equation

$$|\nabla K|^2 = \lambda + V(x), \tag{0.0.24}$$

such that $\nabla K$ has the form

$$\nabla K(x,\lambda) = \Phi(x,\lambda)\frac{x}{|x|} + (\text{lower order terms}). \tag{0.0.25}$$

We emphasize that in general smoothness conditions on $V$ are required for solving the eikonal equation (0.0.24).

In order to prove the existence of the far field pattern of the solution $u$ of the equation (0.0.22) with long range potential, it is also necessary to solve an eikonal equation. This can be found in [Is]. Here Isozaki proves the existence of the limit

$$\lim_{|x|\to\infty} |x|^{\frac{d-1}{2}} e^{-iK(x,\lambda)} R(\lambda + i0) f(|x|\omega) \tag{0.0.26}$$

in $L^2(S^{d-1})$, where $\omega = \frac{x}{|x|}$, $R(z) = (\Delta + V + z)^{-1}$ and $K(x,\lambda)$ is an approximate solution of the equation (0.0.24). For this purpose, the limiting absorption principle for $\Delta + V$ is essential. In addition, the spectral representation theorem for Schrödinger operators with long range potentials is obtained by considering the limit (0.0.26).

The references related to the resolvent estimates for the zero order perturbations of the Laplacian in this thesis are due to Perthame and Vega ([PV1], [PV2]). They study the Helmholtz equation in an inhomogeneous medium of refraction index $n(x) = \lambda + V(x)$, generalizing the estimate (0.0.21) to a variable case. They use a multiplier method with appropriate weights as those used for the wave, Schrödinger or kinetic equations by Morawetz [Mo], Lin and Strauss [LS] or Lions and Perthame [LP], respectively. This direct method permits them to treat coefficients with very low regularity and also cases in which $V$ does not vanish at infinity. We point out that the estimates are uniform for any $\lambda \geq 0$ and have the right scaling. Similar results but not scaling invariant were obtained in [JP] and [Zh1]. The scaling plays a fundamental role in the applications to nonlinear Schrödinger equation [KPV] and in the high frequency limit for Helmholtz equations [BCKP], [CPR].

Let us pass now to consider first order perturbations of the Laplacian. We denote a general perturbed Hamiltonian by

$$H = H_0 + V(x, D), \tag{0.0.27}$$

where $D_j = -i\frac{\partial}{\partial x_j}$. Limiting absorption principle, resolvent estimates and scattering theory related to the forward problem for $H$ have been studied by Hörmander in a very general framework for regular potentials. Hörmander's approach is perturbative; hence, any special structure of $V(x, D)$ plays no role. We refer to ([H], Chapter XIV) for more details in this approach.

In this manuscript we are interested in a very precise first order perturbation of the Laplacian. More concretely, we wish to investigate the electromagnetic Schrödinger operator

$$H_A = \Delta + V(x, D),$$

where

$$V(x, D) = 2iA(x) \cdot \nabla + i\nabla \cdot A(x) - A(x) \cdot A(x) + V(x). \tag{0.0.28}$$

Here $A : \mathbb{R}^d \to \mathbb{R}^d$ is the magnetic vector potential and $V : \mathbb{R}^d \to \mathbb{R}$ is the electric scalar potential. The standard covariant form of the electromagnetic Schrödinger hamiltonian is

$$H_A = \nabla_A^2 + V \tag{0.0.29}$$

with

$$\nabla_A = \nabla + iA. \tag{0.0.30}$$

The magnetic potential $A$ describes the interaction of a free particle with an external magnetic field. The magnetic field that corresponds to a magnetic potential $A$ is given by the $d \times d$ anti-symmetric matrix defined by

$$B = (DA) - (DA)^t, \quad B_{kj} = \left(\frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}\right) \quad k, j = 1, \ldots, d. \tag{0.0.31}$$

In geometric terms, it is given by the 2-form $dA$ as

$$dA = \sum_{k,j=1}^{d} B_{kj}\, dx^k \wedge dx^j. \tag{0.0.32}$$

In dimension $d = 3$, $B$ is uniquely determined by the vector field $curl\,A$ via the vector product

$$Bv = curl\,A \times v, \quad \forall v \in \mathbb{R}^3. \tag{0.0.33}$$

We also define the trapping component of $B$ as

$$B_\tau(x) = \frac{x}{|x|}B(x), \qquad\qquad (B_\tau)_j = \sum_{k=1}^{d} \frac{x_k}{|x|}B_{kj} \tag{0.0.34}$$

and we say that $B$ is non-trapping if $B_\tau = 0$. Observe that in dimension $d = 3$ it coincides with

$$B_\tau(x) := \frac{x}{|x|} \times curl\,A(x).$$

Hence, $B_\tau(x)$ is the projection of $B$ on the tangential space in $x$ to the sphere of radius $|x|$, for $d = 3$. Observe also that $B_\tau \cdot x = 0$ for any $d \geq 2$, therefore $B_\tau$ is a tangential vector field in any dimension and we call it the tangential component of the magnetic field $B$.

Related to the magnetic hamiltonian $H_A$, several papers are devoted to the study of the existence of a unique solution of the electromagnetic Helmholtz equation

$$(\nabla + iA(x))^2u + V(x)u + \lambda u = f(x), \quad x \in \mathbb{R}^d. \tag{0.0.35}$$

The first result goes back to the work of Eidus [E1] in 1962, where it is showed that there exists a unique solution $u(\lambda, f)$ of the equation (0.0.35) in $\mathbb{R}^3$ with the radiation condition

$$\lim_{r\to\infty} \int_{|x|=r} \left| \frac{\partial u}{\partial |x|} - i\lambda^{1/2}u \right|^2 d\sigma(r) = 0. \tag{0.0.36}$$

Here $A_j(x)$ is assumed to vanish close to infinity and the electric potential satisfies $V(x) = O(|x|^{-2-\alpha})$ with $\alpha > \frac{1}{6}$ at infinity.

In 1972, Ikebe and Saito [IS] extend the above result to any $d \geq 3$ for potentials $V$ that are the sum of a long-range potential $V_1$, being $V_1(x) = O(|x|^{-\mu})$, $\frac{\partial V_1}{\partial |x|} = O(|x|^{-1-\mu})$ at infinity and a short-range potential $V_2$ such that $V_2(x) = O(|x|^{-1-\mu})$, for $\mu > 0$ when $|x| \to \infty$. Concerning the magnetic part, they require that $A_j \in C^1(\mathbb{R}^d)$ such that each component of the magnetic field holds $|B_{kj}| \leq C(1 + |x|)^{-1-\mu}$ for some $C > 0$, $\mu > 0$, $|x| \geq 1$. By integration by parts they solve the electromagnetic Helmholtz equation

$$(\nabla + A)^2u + V_1u + V_2u + \lambda u = f \tag{0.0.37}$$

in a $L^2(\mathbb{R}^d)$-weighted space with the spherical radiation condition

$$\int_{|x|\geq 1} \left| \nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u \right|^2 \frac{1}{(1 + |x|)^{1-\delta}} < +\infty, \tag{0.0.38}$$

and a weighted $L^2$ a-priori estimate

$$\int \frac{|u|^2}{(1+|x|)^{1+\delta}} < +\infty, \tag{0.0.39}$$

where $0 < \delta < 1$ is a fixed constant.

In 1987, Saito [S] considers more general long-range potentials and proves the limiting absorption principle for the equation (0.0.37) with a nonspherical radiation condition

$$\int_{|x|\geq 1} \left|\nabla_A u - i\sqrt{\lambda}\nabla K u\right|^2 \frac{1}{(1+|x|)^{1-\delta}} < +\infty, \tag{0.0.40}$$

in the sense that $\nabla K$ is the outward normal of a surface which is not a sphere in general and satisfies the eikonal equation

$$|\nabla K|^2 = 1 + \frac{V_1}{\lambda}.$$

Moreover, the same $L^2$-weighted estimate (0.0.39) as in [IS] for the solution $u$ is also proved.

There are not many works regarding to the far field pattern for the magnetic case. We should mention the paper by Iwatsuka [Iw] where the author proves the existence of the limit

$$\lim_{n\to\infty} r_n^{\frac{d-1}{2}} e^{-ik(r_n\omega,\lambda)} u(r_n\omega) \tag{0.0.41}$$

in $L^2(S^{d-1})$, being $r_n$ a sequence tending to infinity as $n \to \infty$ and $k(x,\lambda)$ is a real-valued function of the form $\lambda^{1/2}|x| - m(x)$. The function $m(x)$ depends on the magnetic potential $A(x)$ and is constructed as

$$m(x) = \sum_{j=1}^{d} x_j \int_0^1 A_j(tx)dt, \qquad (x \in \mathbb{R}^d). \tag{0.0.42}$$

It is closely related to the gauge transformation which changes the magnetic potential $A$ into $A - \nabla m$, but does not change the magnetic field. For this result, $V$ is assumed to be short range and $A_j(x) \in C^2(\mathbb{R}^d)$ such that $|B_{jk}(x)| \leq C_0(1+|x|)^{-3/2-\mu}$, for some $C_0, \mu > 0$. However, singular magnetic potentials $A$, that are related to the tangential component of the magnetic field $B$ are not considered, see section 1.6.

The literature about resolvent estimates related to the magnetic Schrödinger operator is more extensive. We are mainly interested in giving a-priori estimates for solutions $u$ of the equation (0.0.37) imposing conditions on the trapping component of the magnetic field $B$, instead of on the magnetic potential $A$. The quantity $B_\tau$ was introduced by Fanelli and Vega [FV] in which it is proved that weak dispersion for the magnetic Schrödinger and wave equation holds, for example, for non-trapping potentials , i.e., $B_\tau = 0$. This is also what happens in the stationary case, as it is shown in [F]. Following [PV1], Fanelli generalizes the uniform a-priori estimate (0.0.21) to the magnetic case. This estimate has several consequences about the so called Kato smoothing effects for solutions of the evolutions problems which in general do not hold for long range potentials (see among others [BRV], [DF], [Ka4], [KaYa], [LP], [LS1]). The uniform resolvent estimate

$$\int \frac{|u|^2}{|x|^2} \leq C \int |x|^2 |f|^2 \tag{0.0.43}$$

also plays a fundamental role for dispersive estimates on the time dependent Schrödinger operator, as for the study of the Strichartz estimates for the Schrödinger equation with electromagnetic potential, see for example [DFVV], [FG], [M2], [M3]. Thus it is of interest to investigate this estimate.

# The aim of this thesis

The main purpose of this thesis is to study the forward problem for the Helmholtz equation with electromagnetic potential, using multiplier techniques and integration by parts as main tools. Our first concern will be the study of the limiting absorption principle for $H_A$ with singular potentials and rather mild conditions on the potentials at infinity. We will see that the behavior of the solution $u$ of the equation

$$(H_A + \lambda)u = f$$

changes depending on the classes of potentials we work with. We will also provide resolvent estimates and Sommerfeld radiation conditions that will derive some applications related to the cross-section and spectral properties for the Schrödinger operator $H_A$.

To this end, the self-adjointness of $H_A$ with singular potentials in the Hilbert space $L^2(\mathbb{R}^d)$ will be essential. We will show that under some local integrability conditions on the potentials $A$ and $V$, $H_A$ is self-adjoint operator in $L^2(\mathbb{R}^d)$. Thus it may be concluded that the spectrum of this operator is real and consequently, we obtain the existence of solution of the equation

$$H_A u + (\lambda \pm i\varepsilon)u = f \qquad \text{in} \qquad L^2(\mathbb{R}^d),$$

with $\varepsilon \neq 0$ (see section 1.2 below).

In this dissertation we will follow carefully the above mentioned works [IS], [S], [PV1] and [PV2].

## Limiting absorption principle with singular potential

The first goal of the thesis is to prove the limiting absorption principle for the electromagnetic Helmholtz equation with singular potentials. Let us decompose the electric potential $V$ as

$V = V_1 + V_2$ and let us consider the corresponding magnetic Schrödinger operator

$$L = (\nabla + iA)^2 + V_1 + V_2. \tag{0.0.44}$$

Under appropriate assumptions on the potentials, we will construct the correct outgoing solution of the resolvent equation

$$(L + \lambda)u(x) = f(x), \quad x \in \mathbb{R}^d \tag{0.0.45}$$

in some Hilbert space. Let $z = \lambda \pm i\varepsilon$ and $R(z) = (L + z)^{-1}$ be the resolvent of $L$ defined for $z \notin \sigma(L)$. Then we will state that the limit operators

$$R(\lambda \pm i0) = \lim_{\varepsilon \to 0} R(\lambda \pm i\varepsilon) \tag{0.0.46}$$

are well defined as bounded operators in some weighted $L^2$ space. The solutions $u_\pm = R(\lambda \pm i0)f$ obtained by using this method satisfy the equation (0.0.45) and the corresponding radiation condition at infinity.

For this purpose we follow [IS], where existence of a unique solution of the equation (0.0.45) is asserted imposing some asymptotic condition on $A_j$ at infinity, while $V_1$ is long range and $V_2$ is short range. More concretely, Ikebe and Saito require that there exist positive constants $c, \mu > 0$ and $r_0 \geq 1$ such that

($V_1$) for $|x| \geq r_0$, $|V_1(x)| \leq c|x|^{-\mu}$ and the radial derivative $\frac{\partial V_1}{\partial |x|}$ exists with $\frac{\partial V_1}{\partial |x|} \leq c|x|^{-1-\mu}$.

($V_2$) $|V_2(x)| \leq c|x|^{-1-\mu}$, for $|x| \geq r_0$.

(B) $A_j(x)$ is a real-valued $C^1$-function with $|B_{jk}(x)| \leq c|x|^{-1-\mu}$ for $|x| \geq r_0$, $j, k = 1, \ldots, d$, where $B_{jk} = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}$.

(UC) The unique continuation property holds for the differential operator $L$ in $\mathbb{R}^d$.

Then, using the integration by parts method under these assumptions the following is proved:

**Theorem 0.0.1.** *([IS]) Let $K$ be an open set in the upper half-plane of $\mathbb{C}$ of the form*

$$K = \{z = \lambda + i\varepsilon \in \mathbb{C} : \lambda \in (a, b), \varepsilon \in (0, \alpha)\}, \tag{0.0.47}$$

*where $0 < a < b < \infty$ and $0 < \alpha < \infty$. Let $\delta$ be a constant such that $0 < \delta \leq \frac{\mu}{2}$ and $\delta < 1$.*

*(1) For $\lambda \in K$, there exists a strong limit*

$$s - \lim_{\varepsilon \to 0} R(\lambda \pm i\varepsilon) = R(\lambda \pm i0) \in B\left(L^2_{\frac{(1+\delta)}{2}}, L^2_{-\frac{(1+\delta)}{2}}\right). \tag{0.0.48}$$

(2) *For* $f \in L^2_{\frac{(1+\delta)}{2}}$, $u_\pm = R(\lambda \pm i0)f \in L^2_{-\frac{(1+\delta)}{2}}$ *is the unique solution of the equation (0.0.45) with the Sommerfeld radiation condition*

$$\int_{|x|\geq 1} \left| \nabla_A u_\pm \mp i\lambda^{1/2} \frac{x}{|x|} u_\pm \right|^2 \frac{1}{(1+|x|)^{1-\delta}} < +\infty \tag{0.0.49}$$

*and the a-priori estimate*

$$\int \frac{|u_\pm|^2}{(1+|x|)^{1+\delta}} \leq C \int (1+|x|)^{1+\delta} |f|^2. \tag{0.0.50}$$

One of our goals will be to improve the a-priori estimate (0.0.50) by showing that

$$\lambda |||u|||_1^2 \leq C(N_1(f))^2,$$

for $\lambda \geq \lambda_0 > 0$, which is stringer than the first one. We should mention here the paper by Perthame and Vega [PV1] in which they prove the Morrey-Campanato type estimates for the Helmholtz equation in an inhomogeneous medium of refraction index $n(x) > 0$ with very low regularity and some growth at infinity. They prove that for $d \geq 3$ the solution of the equation

$$\Delta u + n(x)u = -f(x)$$

satisfies

$$\begin{aligned} &|||\nabla u|||^2 + |||n^{1/2}u|||^2 + \sup_{R>0} \frac{1}{R^2} \int_{|x|=R} |u|^2 d\sigma_R \\ &+ (d-3) \int \frac{|u|^2}{|x|^3} + \int \frac{|\nabla^\perp u|^2}{|x|} \leq C(N(f))^2. \end{aligned} \tag{0.0.51}$$

These a-priori estimates were extended by Fanelli [F] to the magnetic case.

Using multiplier method and integration by parts, in Chapter 2 of this dissertation we are able to strongly improve the result by Ikebe and Saito in several ways. First of all, we will consider potentials that have the same decay as in [IS] at infinity, and thanks to the extra a-priori estimates that will be proved following the ideas of [PV1] and [F], we can also permit some singularities on them. Moreover, we can extend the range of the frequency $\lambda$ from $\lambda \in (\lambda_0, \lambda_1)$ with $0 < \lambda_0 < \lambda_1$ to any $\lambda \geq \lambda_0 > 0$. Finally, one of the most important tasks is that we are able to extend the range of the exponent $\delta$ which appears in the Sommerfeld radiation condition (0.0.38) from $0 < \delta < 1$ to $0 < \delta < 2$. This fact is a motivation of some open questions related to the resolvent estimate

$$\int \frac{|u|^2}{|x|^2} \leq C \int |x|^2 |f|^2$$

and the existence of the far-field pattern for the magnetic Schrödinger operator which will be studied in Chapter 4.

## Energy concentration and explicit Sommerfeld radiation condition

In order to prove the limiting absorption principle for the electromagnetic Helmholtz equation (0.0.45) with more general long range potential, we will see that it is necessary to solve the eikonal equation

$$|\nabla K|^2 = 1 + \frac{V_1}{\lambda}. \tag{0.0.52}$$

To this end, although less decay on the potential $V_1$ than in [IS] is required, more regularity on it will be essential. More concretely, in 1987 Saito [S] proved that under the same assumptions on the potential $V_2$ and the magnetic field $B$ as in [IS], if moreover the potential $V_2$ admits the singularity

$$\int_{|x-y|\le 1} \frac{|V_2(x)|}{|x-y|^{d-4+\nu}} dy < \infty, \qquad \nu > 0 \tag{0.0.53}$$

and $V_1$ is a bounded real-valued function belonging to $C^2(\mathbb{R}^d \backslash \{0\})$ such that

$$|\partial^\alpha V_1(x)| \le C|x|^{-|\alpha|}, \quad |\alpha| \le 2, \tag{0.0.54}$$

then there exists a unique solution of the Helmholtz equation (0.0.45) satisfying the Sommerfeld radiation condition

$$\int_{|x|\ge 1} |\nabla_A u - i\sqrt{\lambda}\nabla K u|^2 (1+|x|)^{\delta-1} < \infty \tag{0.0.55}$$

for all $0 < \delta < 1$. Here $K = K(x,\lambda)$ is the solution of the eikonal equation (0.0.52) and has the form

$$K(x,\lambda) = |x| g(x,\lambda), \tag{0.0.56}$$

where for sufficiently large $|x|$ and $\lambda$, there exist $0 < c_0 < c_1 < \infty$ such that

$$c_0 \le g(x,\lambda) \le c_1.$$

See [B] for more details.

In 2007, Perthame and Vega [PV2] showed that under some extra restriction on the potential $V_1$, then the solution of the Helmholtz equation

$$\Delta u + n(x) u = f \tag{0.0.57}$$

with $n(x) = \lambda + V_1(x)$ that can have an angular dependency like

$$n(x) \quad \to \quad n_\infty\left(\frac{x}{|x|}\right) \qquad \text{as} \qquad |x| \to \infty, \tag{0.0.58}$$

satisfies the Sommerfeld condition under the explicit form

$$\int \left| \nabla u - i n_\infty^{1/2} \frac{x}{|x|} u \right|^2 \frac{1}{|x|} < \infty. \tag{0.0.59}$$

It is a striking and unexpected feature that the index $n_\infty$ appears in this formula and not the gradient of the phase as established by Saito [S]. This apparent contradiction is clarified by the existence of an extra estimate on the energy decay deducing that the behavior of the solution of the equation (0.0.57) can be very different to the one exhibited by free solutions.

In [PV2] Perthame and Vega study the inhomogeneous Helmholtz equation

$$\Delta u + n(x)u + i\varepsilon u = -f(x), \qquad \varepsilon > 0 \tag{0.0.60}$$

with $n(x) > 0$ such that

$$n = n_1 + n_2 \qquad \text{with} \qquad n_2 \in L^\infty, \tag{0.0.61}$$

$$\|n_1^{1/2}u\|_{L^2} \leq (1-c_0)\|\nabla u\|_{L^2} \quad \text{for some } c_0 > 0, \tag{0.0.62}$$

$$2\sum_{j\in\mathbb{Z}} \sup_{C(j)} \frac{(x\cdot\nabla n(x))_-}{n(x)} := \beta < 1, \tag{0.0.63}$$

where $C(j) = \{x \in \mathbb{R}^d : 2^{j-1} \leq |x| \leq 2^j\}$ and $(a)_-$ denotes the negative part of $a \in \mathbb{R}$ given by $(a)_- = -\min\{0, a\}$. On the one hand, the authors provide the a-priori estimate

$$|||\nabla u|||^2 + |||n^{1/2}u|||^2 + \int \frac{|\nabla^\perp u|^2}{|x|} < \infty \tag{0.0.64}$$

for solutions $u$ of the equation (0.0.60). From this, if the index of refraction $n(x)$ has a slow and only radial decay to a constant $n_\infty\left(\frac{x}{|x|}\right)$ at infinity, they show that $u$ also satisfies the energy estimate

$$\int_{|x|\geq 1} |\nabla_\omega n_\infty(\omega)|^2 \frac{|u|^2}{|x|} < \infty, \tag{0.0.65}$$

where $\omega = \frac{x}{|x|}$. This inequality uses in a strong way the estimate for the tangential part of the gradient of the solution in (0.0.64) and says that the points where $|\nabla_\omega n_\infty(\omega)|$ vanishes on the sphere are the concentration directions for the energy $|u|^2$. In other words, the Sommerfeld condition hides the main physical effect arising for a variable $n$ at infinity; energy concentration on lines rather than dispersion in all directions.

The main goal of Chapter 3 of the thesis is to extend this result by Perthame and Vega [PV2] to the magnetic case. Under certain hypotheses on $n(x)$ and the magnetic field $B$, we will show the energy estimate (0.0.65) for solutions of the magnetic Helmholtz equation

$$\nabla_A^2 u + n(x)u + i\varepsilon u = f. \tag{0.0.66}$$

Furthermore, we are also interested in the study of the limiting absorption principle for the equation (0.0.66) with $\varepsilon = 0$. However, we are not able to do that assuming the conditions considered for proving the energy estimate. In order to accomplish this task, we follow

Saito [S] and by the same method as in the previous section, we allow singularities on the potentials at the origin. Finally, a combination of the both results will permit to deduce the explicit Sommerfeld condition

$$\int \left|\nabla_A u - i n_\infty \frac{x}{|x|} u\right|^2 \frac{1}{1+|x|} < \infty \tag{0.0.67}$$

for solutions obtained from the limiting absorption principle.

## Resolvent estimates and Applications

The last part of the thesis concerns with new estimates which imply some applications related to the spectral properties of the magnetic Schrödinger operator

$$H_A = \nabla_A^2 + V$$

with potentials that have strong singularity. Under smallness conditions on the trapping component of the magnetic field $B$ and on the electric potential $V$, we give uniform resolvent estimates for the solution $u$ of the equation

$$(H_A + \lambda + i\varepsilon)u = f \tag{0.0.68}$$

for any $\lambda \in \mathbb{R}$, $\varepsilon > 0$. More concretely, we prove that the solution $u \in H_A^1(\mathbb{R}^d)$ of the equation (0.0.68) satisfies the following resolvent estimate for any $\lambda \in \mathbb{R}$

$$\int \frac{|u|^2}{|x|^2} \leq C \int |x|^2 |f|^2, \tag{0.0.69}$$

where $C > 0$ is independent of $\lambda$ and $\varepsilon$.

Furthermore, we sharpen the Sommerfeld radiation condition proved in the previous chapters for all $\lambda \geq \lambda_0 > 0$. In fact, we prove the radiation condition

$$\sup_{R\geq 1} R \int_{|x|\geq R} |\nabla_A (e^{-i\lambda^{1/2}|x|} u)|^2 \leq C \int (1+|x|)|x|^2|f|^2 \tag{0.0.70}$$

for any $\lambda \geq \lambda_0 > 0$, where $C = C(\lambda_0) > 0$ is independent of $\varepsilon$.

Estimate (0.0.69) implies on the one hand, the a-priori estimate

$$\lambda |||u|||_1^2 \leq C \int |x|^2 |f|^2. \tag{0.0.71}$$

Moreover, combining with the uniqueness result of the equation

$$(H_A + \lambda)u = f, \tag{0.0.72}$$

we will derive the limiting absorption principle considering potentials with the singularity $\nu|x|^{-2}$ and $\nu$ sharp at the origin. On the other hand, estimates (0.0.70) and (0.0.71) imply existence of the cross-section of the solution $u$ of the equation (0.0.72). In addition, if the estimate (0.0.69) is also true, then we are able to deduce uniqueness of the cross-section.

Having disposed of the above results, our next goal will be to see some spectral properties of the magnetic Schrödinger operator $H_A$. To this end, the formula which relates the resolution of the identity for the self-adjoint operator $H_A$ with its resolvent will be fundamental. Let $R(z) = (H_A + z)^{-1}$ denote the resolvent of $H_A$, $\Delta = (\lambda_1, \lambda_2)$ with $0 < \lambda_1 < \lambda_2 < \infty$ and $E(\Delta)$ the spectral measure associated with $H_A$. By the well known formula (see section 1.5 below)

$$(E(\Delta)f, f) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \lim_{\nu \to 0} \int_{\lambda_1 - \nu}^{\lambda_2 + \nu} (R(\lambda - i\varepsilon)f - R(\lambda + i\varepsilon)f, f)\, d\lambda, \tag{0.0.73}$$

we will show that $H_A$ is absolutely continuous on $(0, \infty)$ and we will give the spectral representation of $H_A$ through the cross-section of the solution $u = R(\lambda + i0)f$ of the equation (0.0.72).

It is worth pointing out that although our hypotheses on the potentials will be given in a more general setting, the ones that we should keep in mind for obtaining the resolvent estimates are, on the one hand, small inverse square potentials $V = \frac{\nu_1}{|x|^2}, |B_\tau| = \frac{\nu_2}{|x|^2}$ with sharp constants related to Hardy's inequality. On the other hand, we have Coulomb type electric potentials and long range magnetic potential $A$ such that $B_\tau = 0$ or $B_\tau$ is small. (See section 1.6 below). Nevertheless, in order to get the Sommerfeld radiation condition (0.0.70), one can preserve the same kind of singularity at the origin, but needs to require more decay at infinity. In fact, we will assume

$$|B_\tau| + |V| \leq \begin{cases} \frac{c}{|x|^2} & \text{if } |x| \leq 1 \\ \frac{c}{|x|^{5/2+\alpha}} & \text{if } |x| \geq 1. \end{cases} \tag{0.0.74}$$

for some $c > 0$, $\alpha > 0$.

The proofs of the results of Chapter 4 are based on integration by parts. We emphasize that in order to prove existence of the cross-section it is crucial to show that

$$|\mathcal{F}(\lambda, r)f(\omega)| \in H^1(S^{d-1}), \tag{0.0.75}$$

where

$$\mathcal{F}(\lambda, r)f(\omega) = C(\lambda) r^{\frac{d-1}{2}} e^{-i\lambda^{1/2} r} u(r\omega).$$

Property (0.0.75) follows from the Sommerfeld condition (0.0.70) combining with the a-priori estimate (0.0.71). The existence of the far field pattern for the magnetic Schrödinger operator $H_A$ would assert if we proved that

$$\mathcal{F}(\lambda, r)f(\omega) \in H^1(S^{d-1}). \tag{0.0.76}$$

By our approach, this could be done by putting some restriction to the magnetic potential $A$. Another way for that would be to follow Iwatsuka [Iw] where the definition of $\mathcal{F}(\lambda, r)f$ is replaced by the next one

$$\mathcal{F}(\lambda, r)f(\omega) = r^{\frac{d-1}{2}} e^{-ik(r\omega,\lambda)} R(\lambda + i0) f(r\omega). \tag{0.0.77}$$

Here $k(x, \lambda) = \lambda^{1/2}|x| - m(x)$, where $m(x)$ is a certain function depending on the magnetic potential $A$ and is constructed by using the trapping component of the magnetic field $B$, see Lemma 1.6.1 below. However, this topic exceeds the scope of this thesis and we propose to study it in the future.

To facilitate access to the individual topics, the chapters of this dissertation are rendered as self-contained as possible. The next chapter constitutes sufficient preparation for following the main results of the thesis which are given and developed in Chapters 2, 3 and 4.

## Notation

Throughout the thesis, $C$ denotes an arbitrary positive constant and $\kappa$ stands for a small positive constant. In most of the cases, $\kappa$ will come from the inequality $ab \le \kappa a^2 + \frac{1}{4\kappa} b^2$, which is true for arbitrary $\kappa > 0$. In the integrals where we do not specify the integration space we mean that we are integrating in the whole $\mathbb{R}^d$ with respect to the Lebesgue measure $dx$, i.e. $\int = \int_{\mathbb{R}^d} dx$.

# Chapter 1

# Preliminaries

The purpose of this chapter is to provide a general background for the thesis to follow. We review some of the standard facts related to the magnetic Hamiltonian $H_A$ and we introduce the techniques that will be used to prove the main results of the manuscript.

## 1.1 $H_A^1$ space and the diamagnetic inequality

In differential geometry it is often necessary to consider connections, which are more complicated derivatives than $\nabla$. The simplest example is a connection on a U(1) bundle over $\mathbb{R}^d$, which merely means acting on complex-valued functions $f$ by $(\nabla + iA(x))$, with $A(x) : \mathbb{R}^d \to \mathbb{R}^d$ being some preassigned, real vector field. The same operator occurs in the quantum mechanics of particles in external magnetic fields (with $d = 3$). As we have already seen, a magnetic field $B : \mathbb{R}^3 \to \mathbb{R}^3$ in quantum mechanics involves replacing $\nabla$ by $\nabla + iA(x)$ where $A$ is called a magnetic vector potential and satisfies

$$curl A = B.$$

Unlike in the differential geometry setting, $A$ need not be smooth, because we could add an arbitrary gradient to $A$, $A \to A + \nabla\psi$, and still get the same magnetic field. This is called gauge invariance. The problem is that $\psi$ (and hence $A$) could be a wild function even if $B$ is well behaved.

For these reasons we want to find a large class of $A$'s for which we can make (distributional) sense of $(\nabla + iA(x))$ and $(\nabla + iA(x))^2$ when acting on a suitable class of $L^2(\mathbb{R}^3)$ functions. For general dimension $d$, the appropriate condition on $A$, which we assume henceforth, is

$$A_j \in L^2_{loc}(\mathbb{R}^d) \quad \text{for} \quad j = 1, \ldots, d. \tag{1.1.1}$$

Because of this condition the functions $A_j f$ are in $L^1_{loc}(\mathbb{R}^d)$ for every $f \in L^2_{loc}(\mathbb{R}^d)$. Therefore the expression $(\nabla + iA)f$ called the covariant derivative (with respect to $A$) of $f$, is a distribution for every $f \in L^2_{loc}(\mathbb{R}^d)$.

Hence, for each $A : \mathbb{R}^d \to \mathbb{R}^d$ satisfying (1.1.1), one could define the space $H^1_A(\mathbb{R}^d)$ which consists of all functions $f : \mathbb{R}^d \to \mathbb{C}$ such that

$$f \in L^2(\mathbb{R}^d) \quad \text{and} \quad (\partial_j + iA_j)f \in L^2(\mathbb{R}^d) \quad \text{for} \quad j = 1, \ldots, d. \tag{1.1.2}$$

Observe that we do not assume that $\nabla f$ or $Af$ are separately in $L^2(\mathbb{R}^d)$. The inner product in this space is

$$(f_1, f_2)_A = (f_1, f_2) + \sum_{j=1}^{d} ((\partial_j + iA_j)f_1, (\partial_j + iA_j)f_2) \tag{1.1.3}$$

where $(\cdot, \cdot)$ is the usual $L^2(\mathbb{R}^d)$ inner product. The second term on the right side of (1.1.3), in the case that $f_1 = f_2 = f$, is called the kinetic energy of $f$. It is to be compared to the usual kinetic energy $\|\nabla f\|_2^2$. As in the case of $H^1(\mathbb{R}^d)$, $H^1_A(\mathbb{R}^d)$ is complete.

If $\psi \in H^1_A(\mathbb{R}^d)$, then $(\nabla + iA)\psi$ is an $\mathbb{R}^d$-valued $L^2(\mathbb{R}^d)$-function. Hence

$$(\nabla + iA)^2\psi = \Delta\psi + 2iA \cdot \nabla\psi + i\nabla \cdot A\psi - A \cdot A\psi$$

makes sense as a distribution. But if $f \in H^1_A(\mathbb{R}^d)$, it is not necessarily true that $f \in H^1(\mathbb{R}^d)$. However, $|f|$ is always in $H^1(\mathbb{R}^d)$ as the following shows.

**Theorem 1.1.1.** *([LL])(diamagnetic inequality) Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be in $L^2_{loc}(\mathbb{R}^d)$ and let $f$ be in $H^1_A(\mathbb{R}^d)$. Then $|f|$, the absolute value of $f$, is in $H^1(\mathbb{R}^d)$ and the diamagnetic inequality,*

$$|\nabla|f|(x)| \leq |(\nabla + iA)f(x)|, \tag{1.1.4}$$

*holds pointwise for almost every $x \in \mathbb{R}^d$.*

It is called the diamagnetic inequality because it says that removing the magnetic field (A=0) allows us to decrease the kinetic energy by replacing $f(x)$ by $|f(x)|$ (and at the same time leaving $|f(x)|^2$ unaltered).

Throughout the thesis we will mainly have functions $u(x)$ that belongs to $H^1_A(\mathbb{R}^d)$. However, at some point it will be necessary to show that $u \in H^1_{loc}(\mathbb{R}^d)$. For this purpose, we will make the following assumption.

$$\int_{|x|\leq R} |Au|^2 \leq C_R \int |\nabla u|^2 \tag{1.1.5}$$

for any $R > 0$ and some $C_R > 0$. Combining this condition with the diamagnetic inequality (1.1.4), since

$$|\nabla u|^2 = |\nabla_A u|^2 - |Au|^2 + 2\Im A\bar{u} \cdot \nabla u, \tag{1.1.6}$$

we conclude that if $\nabla_A u \in L^2(\mathbb{R}^d)$ then $\nabla u \in L^2_{loc}(\mathbb{R}^d)$.

## 1.2 Self-adjointness of $H_A = \nabla_A^2 + V$

For various reasons the property of being self-adjoint is a fundamental information in quantum mechanics. So it is natural to ask whether this expression determines a self-adjoint operator or not in a suitable Hilbert space. In this work, recalling that the spectrum of a self-adjoint operator is real, the self-adjointness of $H_A$ will ensure the existence of solution of the equation

$$H_A u(x) + (\lambda \pm i\varepsilon)u(x) = f(x), \qquad \varepsilon \neq 0 \tag{1.2.1}$$

in $\mathbb{R}^d$ for any $f \in L^2(\mathbb{R}^d)$ and $u$ belonging to the Hilbert space $H_A^1(\mathbb{R}^d)$ introduced above.

Let us recall some definitions and basic results to this issue. (See [RS1])

**Definition 1.2.1.** *A densely defined operator $T$ on a Hilbert space is called* symmetric *(or Hermitian) if $T \subset T^*$, that is, if their domains satisfy $\mathcal{D}(T) \subset \mathcal{D}(T^*)$ and $T\varphi = T^*\varphi$ for all $\varphi \in \mathcal{D}(T)$. Here $T^*$ denotes the adjoint of $T$. Equivalently, $T$ is* symmetric *if and only if*

$$(T\varphi, \psi) = (\varphi, T\psi) \quad \textit{for all} \quad \varphi, \psi \in \mathcal{D}(T).$$

**Definition 1.2.2.** *An operator $T$ is called* self-adjoint *if $T = T^*$, that is, if and only if $T$ is symmetric and $\mathcal{D}(T) = \mathcal{D}(T^*)$.*

**Definition 1.2.3.** *A symmetric operator $T$ is called* essentially self-adjoint *if its closure $\overline{T}$ is self-adjoint.*

An equivalent characterization of essential self-adjointness is that there exists a unique self-adjoint extension of the operator to a larger domain.

When we deal with the magnetic Schrödinger operator we typically take $L^2(\mathbb{R}^d)$ as the Hilbert space mentioned in the previous definitions. The aim of this section is to show the self-adjointness of $H_A$, under some local integrability conditions on $A$ and $V$, for the kind of potentials we deal with in this dissertation (including singular potentials). Many authors like Ikebe and Kato [IK], Leinfelder and Simander [LS], Avron, Herbst and Simon [AHS] or Cycon, Froese, Kirsch and Simon [CFKS] among others has investigated the essential self-adjointness of the electromagnetic Schrödinger operator $H_A$ in a suitable Hilbert space.

We will prove that $H_A$ is self-adjoint by using the definition and also the Riesz representation theorem. From now on we assume that

$$\int V|u|^2 < 1^- \int |\nabla u|^2, \tag{1.2.2}$$

$$A_j \in L^2_{loc}, \qquad\qquad V \in L^1_{loc}. \tag{1.2.3}$$

and define the domain

$$D(H_A) := \{\phi \in L^2(\mathbb{R}^d) : \int |\nabla_A \phi|^2 - \int V|\phi|^2 < \infty\}. \tag{1.2.4}$$

**Remark 1.2.4.** *Potentials $V$ that belong to the Morrey Space $L^{2-\frac{d}{p},p}$ with $p > 1$ and smallness condition on $\|V\|_{L^{2-\frac{d}{p},p}}$ satisfy (1.2.2). Recall that for an open $\Omega \subset \mathbb{R}^d$, $1 \leq p \leq \infty$ and $\lambda \geq 0$, the Morrey space $L^{\lambda,p}$ is defined as*

$$L^{\lambda,p}(\Omega) = \{u \in L^p(\Omega) : \exists B < \infty,\ r^{-\lambda} \int_{\Omega_r(x_0)} |u|^p \leq B^p \, \forall x_0 \in \Omega, r > 0\},$$

*with norm defined as the least such constant, i.e.*

$$\|u\|_{L^{\lambda,p}} = \left( \sup_{x_0 \in \Omega, r>0} r^{-\lambda} \int_{\Omega_r(x_0)} |u|^p \right)^{1/p} \tag{1.2.5}$$

*where $\Omega_r(x) = B_r(x) \cap \Omega$ and $B_r$ denotes a ball of radio $r$.*

**Remark 1.2.5.** *Observe that by (1.2.2) and the diamagnetic inequality (1.1.4), we have*

$$\int |\nabla_A \phi| - \int V|\phi|^2 \geq \nu \int |\nabla_A \phi|^2 > 0,$$

*for some $0 < \nu < 1$. Hence, if $\phi \in D(H_A)$ we get that $\phi \in H^1_A(\mathbb{R}^d)$.*

**Lemma 1.2.6.** *Let $A_j$, $V$ be real-valued functions satisfying (1.2.3). Then, $\mathcal{H} := D(H_A)$ is a Hilbert space with the inner product*

$$(\phi, \varphi)_{\mathcal{H}} = (\phi, \varphi)_{L^2} + (\nabla_A \phi, \nabla_A \varphi)_{L^2} - (V\phi, \varphi)_{L^2} \tag{1.2.6}$$

*and the norm*

$$\|\phi\|^2_{\mathcal{H}} = \|\phi\|^2_{L^2} + \|\nabla_A \phi\|^2_{L^2} - \int V|\phi|^2. \tag{1.2.7}$$

*Proof.* Since $H^1_A(\mathbb{R}^d)$ is a Hilbert space, from Remark 1.2.5 it is very easy to check that $(\phi, \varphi)_{\mathcal{H}}$ defined as above is an inner product.

We have to see that $\mathcal{H}$ is complete. Let $\phi_n$ be a Cauchy sequence in $\mathcal{H}$, then so is in $L^2$ and there exist $\phi \in L^2$, $\psi \in L^2$ and $\varphi \in L^2$ such that

$$\lim_{n\to\infty} \|\phi_n - \phi\|_{L^2} = 0 \tag{1.2.8}$$

and

$$\lim_{n\to\infty} \|\nabla_A \phi_n - \psi\|_{L^2} = 0,$$
$$\lim_{n\to\infty} \int V|\phi_n|^2 = \int |\varphi|^2.$$

We claim that $\psi = \nabla_A \phi$ and $\varphi$ such that $|\varphi|^2 = V|\phi|^2$. By the local integrability assumptions on the potentials, since $\phi_n \to \phi$ in $L^2$, we get

$$\nabla_A \phi_n \quad \to \quad \nabla_A \phi \qquad \text{and} \qquad V\phi_n \quad \to \quad V\phi \tag{1.2.9}$$

in the distributional sense. Hence, $\phi_n \to \phi$ in $\mathcal{H}$, which completes the proof. $\square$

Now we can state our main result of this section.

**Theorem 1.2.7.** *Under the hypotheses of Lemma 1.2.6 the Schrödinger operator $H_A$ defined on $L^2(\mathbb{R}^d)$ is self-adjoint. Moreover, it is the unique self-adjoint extension of $H_A$ on $C_0^\infty(\mathbb{R}^d)$ such that the domain is contained in $L^2(\mathbb{R}^d)$.*

*Proof.* We begin by proving the self-adjointness of $H_A$. To this end, let us first prove that

$$\forall f \in L^2(\mathbb{R}^d) \quad \exists! u \in D(H_A) \quad \text{s.t.} \quad H_A u - u = -f. \tag{1.2.10}$$

Let us fix $f \in L^2(\mathbb{R}^d)$ and define the linear functional $T_f : \mathcal{H} \to \mathbb{C}$ such that

$$T_f(\nu) = (f, \nu)_{L^2}, \qquad \forall \nu \in \mathcal{H}. \tag{1.2.11}$$

Then, by (1.2.2) we have that $T_f$ is bounded, i.e.

$$|T_f(\nu)| \leq \|f\|_{L^2} \|\nu\|_{L^2} \leq C\|\nu\|_{\mathcal{H}}. \tag{1.2.12}$$

Hence, by using the Riesz Representation Theorem, we conclude that there exists a unique $\phi \in \mathcal{H}$ such that

$$(\phi, \nu)_{\mathcal{H}} = T_f(\nu) \qquad \forall \nu \in \mathcal{H}. \tag{1.2.13}$$

In fact,

$$-(\nabla_A^2 u + Vu - u, \nu)_{L^2} = (f, \nu)_{L^2}, \tag{1.2.14}$$

for any test function $\nu$. But since $f \in L^2$, the functional

$$\nu \quad \to \quad (\nabla_A^2 + V - 1, \nu)_{L^2} \tag{1.2.15}$$

extends uniquely to a continuous functional on $L^2(\mathbb{R}^d)$. Hence, we get that for $f \in L^2$ there exists a unique $u \in D(H_A) \subset L^2(\mathbb{R}^d)$ satisfying

$$\nabla_A^2 u + Vu - u = -f, \tag{1.2.16}$$

which is our first claim.

Let $J = (H_A - I)^{-1}$. Our next goal is to show that $J$ is self-adjoint. Since $J \in \mathcal{L}(L^2(\mathbb{R}^d))$, it is sufficient to show that

$$(Ju, v)_{L^2} = (u, Jv)_{L^2} \qquad \forall u, v \in L^2(\mathbb{R}^d). \tag{1.2.17}$$

Let $u_1 = Ju$, $v_1 = Jv$ such that

$$\begin{aligned} H_A u_1 - u_1 &= u, \\ H_A v_1 - v_1 &= v. \end{aligned}$$

Since $(u_1, H_A v_1) = (H_A u_1, v_1)$ (i.e. $H_A$ is symmetric), then $(u_1, v) = (u, v_1)$, and (1.2.17) is proved.

As $H_A$ is symmetric, we shall have established the self-adjointness of $H_A$ by showing that $D(H_A^*) \subset D(H_A)$. Let $u \in D(H_A^*)$ and set $f = H_A^* u - u$. Then

$$(f, v) = (u, H_A v - v) \qquad \forall v \in D(H_A), \tag{1.2.18}$$

that is,

$$(f, Jw) = (u, w) \qquad \forall w \in L^2. \tag{1.2.19}$$

As a consequence, $u = Jf$ which implies $u \in D(H_A)$. Thus $D(H_A^*) = D(H_A)$ and the self-adjointness follows.

The proof is completed by showing the uniqueness part of the theorem, since the symmetry is trivial. Assume that there exists another self-adjoint extension such that for any $\phi \in D'(H_A) \supset C_0^\infty(\mathbb{R}^d)$, then $\phi \in L^2(\mathbb{R}^d)$. Since $H_A$ is self-adjoint on $D'(H_A)$,

$$(\tilde{\phi}, (\nabla_A^2 + V)\phi)_{L^2} = ((\nabla_A^2 + V)\tilde{\phi}, \phi) \tag{1.2.20}$$

for all $\tilde{\phi} \in C_0^\infty$. This means that $\phi \in L^2$, $V^{1/2}\phi \in L^2$ and $\nabla_A \phi \in L^2$ in the distributional sense. Thus, $\phi \in D(H_A)$, i.e. $D'(H_A) \subset D(H_A)$ and $D(H_A)^* \subset (D'(H_A))^*$. Now since $H_A$ is self-adjoint in $D(H_A)$ and $D'(H_A)$, $D(H_A) = D'(H_A)$.

□

## 1.3 Multiplier method and integration by parts

The multiplier method technique has become an indispensable tool in the study of partial differential equations due to its versatile technical power as well as due to its global nature, which allows for obtaining results on existence or uniqueness for linear and nonlinear problems in the language of suitable integrals.

By a multiplier method, we mean that given a partial differential equation

$$Lu = f \quad \text{in} \quad \Omega \subset \mathbb{R}^d, \tag{1.3.1}$$

where $L$ is, for example, a second order elliptic linear differential equation with sufficiently smooth coefficients in $\Omega$ and $f$ a given function of $x$, one seeks to:

1. Multiply (1.3.1) by a suitable multiplier $M[u]$.
2. Integrate over $\Omega$.
3. Manipulate the expression using integration by parts and eventual boundary conditions to arrive at a potentially useful integral identity.

There is much artistry in both the choice of the multiplier $M[u]$ as well as in the choice of a suitable function space for $u$.

In this section, we consider the electromagnetic Helmholtz equation

$$(\nabla + iA(x))^2 u + V(x)u + \lambda u + i\varepsilon u = f(x), \quad x \in \mathbb{R}^d \tag{1.3.2}$$

where $\lambda \in \mathbb{R}$, $\varepsilon > 0$ and we derive such a integral identities basing on the standard technique of Morawetz multipliers. This was introduced in [Mo] for the Klein Gordon equation and then used in several other contexts (dispersive equations, kinetic equations, helmholtz equation, etc.) We should mention here [PV1] as our reference work about the relation between Morawetz methods and Morrey-Campanato estimates for the Helmholtz equation (when $A \equiv 0$) and its generalization to the magnetic case due to Fanelli [F].

We remark that the idea of integrating by parts with the covariant form $\nabla_A$ is to use the Leibnitz formula

$$\nabla_A(fg) = (\nabla_A f)g + f(\nabla g), \tag{1.3.3}$$

putting all the dissorted derivatives on the solution and the straight derivatives on the multiplier.

In order to carry out the integration by parts argument below, we need some regularity in the solution $u$. In general, it is enough to know that $u \in H^1_A(\mathbb{R}^d)$. Moreover, since we are including singularities in our potentials, it is necessary to put some restrictions on them to check that the contributions of these terms make sense. To this end, apart from the condition (1.2.2), the following assumptions on $V$ and $B_\tau$ will be needed throughout the thesis.

$$\int (r\partial_r V)|u|^2 \leq C \int |\nabla u|^2, \tag{1.3.4}$$

$$\int |x|^2 |B_\tau|^2 |u|^2 \leq C \int |\nabla u|^2, \tag{1.3.5}$$

for some $C > 0$, where $r = |x|$ and $\partial_r V = \frac{x}{|x|} \cdot \nabla V$ denotes the radial derivative of $V$.

Before stating the integral identities, we need some notation. We denote the radial derivative and the tangential component of the gradient by

$$\nabla_A^r u = \frac{x}{|x|} \cdot \nabla_A u, \qquad |\nabla_A^\perp u|^2 = |\nabla_A u|^2 - |\nabla_A^r u|^2, \tag{1.3.6}$$

respectively. We drop the index $A$ if $A = 0$, and in this case we write the radial derivative as $\partial_r$ and the tangential one as $\nabla^\perp$ or $\partial_\tau$. In addition, the following a-priori estimates are also necessary.

**Lemma 1.3.1.** *Let $\varphi \in C^\infty$ a real-valued radial function so that there exist $C > 0$, $k_0 \geq 0$ where $|\varphi^{(k_0+1)}| \leq C$. Then, for a suitable $f$, the solution of the Helmholtz equation (1.3.2)* *satisfies*

$$\varepsilon \int \varphi^{(k_0)}|u|^2 \leq \int |\varphi^{(k_0+1)}||\nabla_A^r u||u| + \int |f||\varphi^{(k_0)}||u| \tag{1.3.7}$$

$$\begin{aligned}\int \varphi^{(k_0)}|\nabla_A u|^2 \leq & \int (\lambda + V)\varphi^{(k_0)}|u|^2 + \int |\varphi^{(k_0+1)}||\nabla_A^r u||u| \\ & + \int |f||\varphi^{(k_0)}||u|.\end{aligned} \tag{1.3.8}$$

*Proof.* We just need to multiply the equation (1.3.2) by $\varphi^{(k_0)}\bar{u}$ and integrate it over $\mathbb{R}^d$. Then, the imaginary part gives (1.3.7) and (1.3.8) follows by taking the real part. □

**Corollary 1.3.2.** *In particular, the solution of the Helmholtz equation (1.3.2) satisfies the a-priori estimates*

$$\varepsilon \int |u|^2 \leq \int |f||u| \tag{1.3.9}$$

$$\int |\nabla_A u|^2 \leq \int (\lambda + V)|u|^2 + \int |f|||u|. \tag{1.3.10}$$

**Remark 1.3.3.** *Note that under appropriate assumption on the potential $V$, if there exist $C > 0$, $k_0 \geq 0$ such that $|\varphi^{(k_0+1)}| \leq C$, by induction on $k$ we get*

$$\int \varphi^{(k)}(|u|^2 + |\nabla_A u|^2) < +\infty \qquad \forall k \leq k_0, \tag{1.3.11}$$

*for $u \in H_A^1(\mathbb{R}^d)$ and suitable $f$.*

Now we are ready to formulate the key equalities of this manuscript.

**Lemma 1.3.4.** *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be regular enough. Then, the solution $u \in H_A^1(\mathbb{R}^d)$ of the Helmholtz equation (1.3.2) satisfies*

$$\begin{aligned}&\int \varphi\lambda|u|^2 - \int \varphi|\nabla_A u|^2 + \int \varphi V|u|^2 - \Re \int \nabla\varphi \cdot \nabla_A u\bar{u} \\ &= \Re \int \varphi f\bar{u},\end{aligned} \tag{1.3.12}$$

$$\varepsilon \int \varphi|u|^2 - \Im \int \nabla\varphi \cdot \nabla_A u\bar{u} = \Im \int \varphi f\bar{u}, \tag{1.3.13}$$

*Proof.* Let us multiply the equation (1.3.2) by the symmetric multiplier $\varphi u$ in the $L^2$-sense, obtaining

$$\int \nabla_A^2 u\varphi\bar{u} + (\lambda + i\varepsilon)\int \varphi|u|^2 + \int \varphi V|u|^2 = \int f\varphi\bar{u}. \tag{1.3.14}$$

Note that integration by parts gives

$$\int \nabla_A^2 u\varphi\bar{u} = -\int \nabla\varphi\cdot\nabla_A u\bar{u} - \int \varphi|\nabla_A u|^2.$$

Thus taking the real part of (1.3.14) we obtain (1.3.12). The imaginary part gives (1.3.13). □

**Lemma 1.3.5.** *Let $\psi : \mathbb{R}^n \longmapsto \mathbb{R}$ be regular enough. Then, any solution $u \in H_A^1(\mathbb{R}^d)$ of the equation (1.3.2) satisfies*

$$\begin{aligned}
&\int \nabla_A u\cdot D^2\psi\cdot\overline{\nabla_A u} + \Re\frac{1}{2}\int \nabla(\Delta\psi)\cdot\nabla_A u\bar{u} + \varepsilon\Im\int \nabla\psi\cdot\overline{\nabla_A u}u \\
&- \Im\int\sum_{j,k=1}^{d}\frac{\partial\psi}{\partial x_k}B_{kj}(\nabla_A)_j u\bar{u} - \frac{1}{2}\int \Delta\psi V|u|^2 - \Re\int V\nabla\psi\cdot\nabla_A u\bar{u} \\
&= -\Re\int f\nabla\psi\cdot\overline{\nabla_A u} - \frac{1}{2}\Re\int f\Delta\psi\bar{u},
\end{aligned} \tag{1.3.15}$$

*where $D^2\psi$ denotes the Hessian of $\psi$, while $(\nabla_A)_j = \partial_j + iA_j$.*

*Proof.* Let us consider the anti-symmetric multiplier

$$\mathcal{A} = \nabla\psi\cdot\nabla_A + \frac{1}{2}\Delta\psi. \tag{1.3.16}$$

We multiply the equation (1.3.2) by $\mathcal{A}u$ in the $L^2$-sense. Observe that since $\mathcal{A}$ is anti-symmetric then

$$\Re(u, \mathcal{A}u) = 0.$$

In addition, the operator $\Delta\psi$ is symmetric, then $\Re i\varepsilon(u, \Delta\psi u) = 0$. Thus taking the real part yields

$$\Re(\nabla_A^2 u, \mathcal{A}u) + \Re(V, \mathcal{A}u) - \varepsilon\Im(u, \nabla\psi\cdot\nabla_A u) = \Re(f, \mathcal{A}u).$$

Now, let us write

$$\begin{aligned}
\Re(\nabla_A^2 u, \mathcal{A}u) &= \Re\int \nabla_A^2 u\nabla\psi\cdot\overline{\nabla_A u} + \frac{1}{2}\Re\int \nabla_A^2 u\Delta\psi\bar{u} \\
&\equiv I_1 + I_2.
\end{aligned}$$

On the one hand, integration by parts gives

$$I_2 = -\frac{\Re}{2}\int \nabla(\Delta\psi)\cdot\nabla_A u\bar{u} - \frac{1}{2}\int |\nabla_A u|^2\Delta\psi. \tag{1.3.17}$$

On the other hand, we have

$$\begin{aligned}
I_1 &= \sum_{k,j=1}^{d}\Re\int\left(\frac{\partial}{\partial x_j}+iA_j\right)\left(\frac{\partial u}{\partial x_j}+iA_ju\right)\frac{\partial\psi}{\partial x_k}\left(\frac{\partial\bar{u}}{\partial x_k}-iA_k\bar{u}\right)\\
&= -\sum_{k,j=1}^{d}\Re\int\left(\frac{\partial u}{\partial x_j}+iA_ju\right)\frac{\partial\psi}{\partial x_k}\left[\left(\frac{\partial}{\partial x_j}-iA_j\right)\left(\frac{\partial\bar{u}}{\partial x_k}-iA_k\bar{u}\right)\right]\\
&- \int\nabla_A u\cdot D^2\psi\cdot\overline{\nabla_A u}\\
&\equiv I_{11} - \int\nabla_A u\cdot D^2\psi\cdot\overline{\nabla_A u}.
\end{aligned}$$

Note that

$$\left(\frac{\partial}{\partial x_j}+iA_j\right)\left(\frac{\partial u}{\partial x_k}+iA_ku\right) = \left(\frac{\partial}{\partial x_k}+iA_k\right)\left(\frac{\partial u}{\partial x_j}+iA_ju\right) - iB_{kj}u, \tag{1.3.18}$$

with $B_{kj}=\frac{\partial A_j}{\partial x_k}-\frac{\partial A_k}{\partial x_j}$. Hence, by (1.3.18) and integration by parts we obtain

$$\begin{aligned}
I_{11} &= \sum_{k,j=1}^{d}\Re\int\overline{\left(\frac{\partial u}{\partial x_j}+iA_ju\right)}\frac{\partial\psi}{\partial x_k}\left(\frac{\partial}{\partial x_k}+iA_k\right)\left(\frac{\partial u}{\partial x_j}+iA_ju\right)\\
&+\int|\nabla_A u|^2\Delta\psi + \Im\sum_{k,j=1}^{d}\int\left(\frac{\partial u}{\partial x_j}+iA_ju\right)\frac{\partial\psi}{\partial x_k}B_{kj}\bar{u}.
\end{aligned}$$

Therefore, we conclude

$$I_{11} = \frac{1}{2}\int|\nabla_A u|^2\Delta\psi + \Im\sum_{k,j=1}^{d}\int\frac{\partial\psi}{\partial x_k}B_{kj}\left(\frac{\partial u}{\partial x_j}+iA_ju\right)\bar{u}.$$

As a consequence, we get

$$\begin{aligned}
\Re(\nabla_A^2 u, \mathcal{A}u) &= -\int\nabla_A u\cdot D^2\psi\cdot\overline{\nabla_A u} - \frac{1}{2}\Re\int\nabla(\Delta\psi)\cdot\nabla_A u\bar{u}\\
&+\Im\int\sum_{j,k=1}^{d}\frac{\partial\psi}{\partial x_k}B_{jk}(\nabla_A u)_j\bar{u},
\end{aligned}$$

where $(\nabla_A u)_j = \partial_j u + iA_ju$. This combining with (1.3.17) gives (1.3.15). $\square$

**Remark 1.3.6.** *The integration by parts gives very precise information about the relevant quantities related to the electromagnetic field. It is of a particular interest the part concerning the magnetic potential $A$. Note that in the above identities only appear the quantity $B_{jk}$. Moreover, if we consider radial multipliers, then since $\frac{\partial\psi}{\partial x_k} = \frac{x_k}{|x|}\psi'$, by (0.0.34) one could rewrite the magnetic term as*

$$\begin{aligned}\Im \sum_{k,j=1}^{d} \int \frac{\partial \psi}{\partial x_k} B_{kj} \left( \frac{\partial u}{\partial x_j} + iA_j u \right) \bar{u} &= \Im \sum_{j=1}^{d} \int \psi'(B_\tau)_j \left( \frac{\partial u}{\partial x_j} + iA_j u \right) \bar{u} \\ &= \Im \int \psi' B_\tau \cdot \nabla_A u \bar{u}.\end{aligned}$$

*Thus the tangential component of the magnetic field is the only term that appears related to the magnetic potential $A$.*

**Remark 1.3.7.** *Observe that if $u \in H^1_A(\mathbb{R}^d)$ and the multipliers are radial, by Remark 1.3.3 all terms of the above identities that do not contain any potential are finite. In addition, if $|\varphi| \leq C$ and $|\psi'(r)| \leq r$, then by (1.2.2), (1.3.4), (1.3.5), the potential terms are controlled. It is easy to check that the terms containing the potential $V$ in the identity (1.3.15) can be rewritten as*

$$\frac{1}{2}\int \psi' \partial_r V |u|^2 = -\frac{1}{2}\int V \Delta\psi |u|^2 - \Re \int V \nabla\psi \cdot \nabla_A u \bar{u}. \tag{1.3.19}$$

**Remark 1.3.8.** *Note that we can consider the anti-symmetric multiplier as*

$$\mathcal{A} = E \cdot \nabla_A + \frac{1}{2} divE, \tag{1.3.20}$$

*where $E$ is a real vector field.*

These integral identities will be used to prove the main results of the thesis. A suitable multipliers must be chosen in order to estimate from above and below the corresponding terms of $(Lu, M[u])$. One needs to show the positivity of the terms that wants to control and estimate the remaining terms. Moreover, a precise combination of them implies the following key equality that will be used in the next three chapters and plays a fundamental role for proving the different versions of the Sommerfeld radiation condition of this manuscript.

**Proposition 1.3.9.** *Let $\psi : \mathbb{R}^d \longmapsto \mathbb{R}$ a regular radial function so that there exist $C > 0$, $k \geq 0$ such that $|\psi^{(k)}| \leq C$. Then, any solution $u \in H^1_A(\mathbb{R}^d)$ of the Helmholtz equation*

*(1.3.2) satisfies*

$$
\begin{aligned}
&\frac{1}{2}\int \psi''|\nabla_A^r u - i\lambda^{1/2}u|^2 + \int \left(\frac{\psi'}{|x|} - \frac{\psi''}{2}\right)|\nabla_A^\perp u|^2 \\
&+ \Re\frac{(d-1)}{2}\int \nabla\left(\frac{\psi'}{|x|}\right)\cdot\nabla_A u\bar{u} + \frac{\varepsilon}{2\lambda^{1/2}}\int \psi'\left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2 \\
&- \Im\int \psi'\bar{u}B_\tau\cdot\nabla_A u - \frac{d-1}{2}\int\frac{\psi'}{|x|}V|u|^2 - \Re\int V\psi'\nabla_A^r u\bar{u} \\
&+ \frac{\varepsilon}{2\lambda^{1/2}}\Re\int\psi''\nabla_A^r u\bar{u} - \frac{\varepsilon}{2\lambda^{1/2}}\int\psi' V|u|^2 \\
&= -\frac{\varepsilon}{2\lambda^{1/2}}\Re\int\psi' f\bar{u} - \Re\int f\psi'(\nabla_A^r\bar{u} + i\lambda^{1/2}\bar{u}) - \frac{(d-1)}{2}\Re\int\frac{\psi'}{|x|}f\bar{u}.
\end{aligned}
\tag{1.3.21}
$$

*Proof.* The proof consists in the combination of the above identities. We first compute

$$
(1.3.15) + (1.3.12) + \lambda^{1/2}(1.3.13)
$$

putting $\varphi = \frac{1}{2}\psi''$ in (1.3.12). Then since $\psi$ is radial, letting $r = |x|$ yields

$$
\nabla_A u\cdot D^2\psi\cdot\overline{\nabla_A u} = \psi''|\nabla_A^r u|^2 + \frac{\psi'}{r}|\nabla_A^\perp u|^2 \tag{1.3.22}
$$

and

$$
\Delta\psi = \frac{(d-1)\psi'}{r} + \psi''. \tag{1.3.23}
$$

Thus by Remark 1.3.6 we obtain

$$
\begin{aligned}
&\frac{1}{2}\int\psi''(|\nabla_A^r u|^2 + \lambda|u|^2) - \Im\lambda^{1/2}\int\psi''\nabla_A^r u\bar{u} + \int\left(\frac{\psi'}{r} - \frac{\psi''}{2}\right)|\nabla_A^\perp u|^2 \\
&+ \Re\frac{d-1}{2}\int\nabla\left(\frac{\psi'}{r}\right)\cdot\nabla_A u\bar{u} + \varepsilon\lambda^{1/2}\int\psi'|u|^2 - \varepsilon\Im\int\psi'\frac{x}{|x|}\cdot\nabla_A u\bar{u} \\
&- \frac{d-1}{2}\int\frac{\psi'}{r}V|u|^2 - \Re\int V\psi'\frac{x}{|x|}\cdot\nabla_A u\bar{u} - \Im\int\psi'\bar{u}B_\tau\cdot\nabla_A u \\
&= -\Re\int f\psi'\frac{x}{|x|}\cdot\nabla_A\bar{u} - \frac{(d-1)}{2}\Re\int\frac{\psi'}{|x|}f\bar{u} + \Im\lambda^{1/2}\int\psi' f\bar{u}.
\end{aligned}
$$

Let us subtract now the identity (1.3.12) multiplied by $\varepsilon$ from the above equality with the choice of the test function

$$
\varphi = \frac{1}{2\lambda^{1/2}}\psi'. \tag{1.3.24}
$$

Hence, from the fact that

$$
\left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2 = |\nabla_A u|^2 + \lambda|u|^2 - 2\lambda^{1/2}\Im\frac{x}{|x|}\cdot\nabla_A u\bar{u}, \tag{1.3.25}
$$

we get the square related to $\varepsilon$, and we conclude (1.3.21). □

**Remark 1.3.10.** *For $V \in C^1(\mathbb{R}^d)$, by integration by parts the electric terms in (1.3.21) can be rewritten as*

$$\frac{1}{2}\int (\psi'' V + \psi' \partial_r V)|u|^2 = \frac{1}{2}\int \left[\left(\psi'' - \frac{\psi'}{r}\right) V + \frac{\psi'}{r}\partial_r(rV)\right]|u|^2. \tag{1.3.26}$$

**Remark 1.3.11.** *Noting that $|\nabla_A u|^2 = |\nabla_A^r u|^2 + |\nabla_A^\perp u|^2$, in the case that $\frac{\psi'}{r} - \frac{\psi''}{2} \geq \nu\psi''$ for some $\nu > 0$, from (1.3.25) the first four terms of the equality (1.3.21) gives the control of the so-called Sommerfeld square*

$$\int \psi'' \left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2. \tag{1.3.27}$$

## 1.4 Magnetic Hardy type inequalities

In the course of providing a new proof of Hilbert's double series theorem, G. H. Hardy [Ha] discovered the inequality

$$\int_0^\infty \frac{u(x)^2}{x^2}dx \leq 4\int_0^\infty |u'(x)|^2 dx, \tag{1.4.1}$$

which is valid for absolutely continuous $u$ such that $u(0) = 0$ and $u' \in L^2(0,\infty)$. The generalization of this inequality to higher dimensions (see [OK] for historical background) reads as follows

$$\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2}dx \leq \frac{4}{(d-2)^2}\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx, \tag{1.4.2}$$

for $u \in H^1(\mathbb{R}^d)$, $d \geq 3$.

In this section we introduce the magnetic version of the Hardy inequality (1.4.2) that we will use throughout this work. We first give the same type of inequality in a ball of radius $R > 0$ and then we derive the standard one on the whole $\mathbb{R}^d$. In addition, we give a variant of this inequality with a different weight.

**Theorem 1.4.1.** *Let $d \geq 3$, $A : \mathbb{R}^d \to \mathbb{R}^d$, $\nabla_A = \nabla + iA$. Then, for any $f \in H_A^1(\mathbb{R}^d)$ and any $R > 0$ the following inequality holds:*

$$\int_{|x|\leq R} \frac{|f|^2}{|x|^2} \leq \frac{4}{(d-2)^2}\int_{|x|\leq R} |\nabla_A f|^2 + \frac{2}{(d-2)R}\int_{|x|=R} |f|^2 d\sigma_R. \tag{1.4.3}$$

*Proof.* We only need to prove (1.4.3) for $f \in C_0^\infty$, then we conclude by density. Let us observe that, for all $\alpha \in \mathbb{R}$

$$\begin{aligned} 0 &\leq \left|\nabla_A f + \alpha\frac{x}{|x|^2}f\right|^2 \\ &= |\nabla_A f|^2 + \alpha^2\frac{|f|^2}{|x|^2} + 2\alpha\Re\frac{x}{|x|^2}\cdot\nabla_A f\bar{f}. \end{aligned} \tag{1.4.4}$$

Let us integrate the above identity over $\{|x| \leq R\}$. Thus by integration by parts, using the Leibnitz formula (1.3.3) we have

$$2\alpha\Re\int_{|x|\leq R}\frac{x}{|x|^2}\cdot\nabla_A f\bar{f} = -(d-2)\alpha\int_{|x|\leq R}\frac{|f|^2}{|x|^2} + \alpha\int_{|x|=R}\frac{|f|^2}{|x|}d\sigma_R.$$

Combining this with (1.4.4) we get

$$\{-\alpha^2 + (d-2)\alpha\}\int_{|x|\leq R}\frac{|f|^2}{|x|^2} \leq \int_{|x|\leq R}|\nabla_A f|^2 + \frac{\alpha}{R}\int_{|x|=R}|f|^2 d\sigma_R \tag{1.4.5}$$

for all $\alpha \in \mathbb{R}$. Finally, observe that

$$\max_{\alpha\in\mathbb{R}}\{\, -\alpha^2 + (d-2)\alpha\} = \frac{(d-2)^2}{4} \tag{1.4.6}$$

when $\alpha = \frac{d-2}{2}$ and this completes the proof. $\square$

**Remark 1.4.2.** *Observe that in the same manner we can see that*

$$\int_{|x|\geq R}\frac{|f|^2}{|x|^2} + \frac{2}{(d-2)R}\int_{|x|=R}|f|^2 d\sigma_R \leq \frac{4}{(d-2)^2}\int_{|x|\geq R}|\nabla_A f|^2. \tag{1.4.7}$$

**Remark 1.4.3.** *Letting $R \to \infty$ in (1.4.3), we obtain the standard version of the magnetic Hardy inequality*

$$\int\frac{|f|^2}{|x|^2} \leq \frac{4}{(d-2)^2}\int|\nabla_A f|^2, \tag{1.4.8}$$

*which will be used very often along this thesis.*

Finally, we will give another Hardy type inequality for the magnetic case.

**Theorem 1.4.4.** *Let $d \geq 3$, $A : \mathbb{R}^d \to \mathbb{R}^d$, $\nabla_A = \nabla + iA$. Then, for any $f \in D(H_A)$ the following inequality holds:*

$$\int\frac{|f|^2}{|x|} \leq \frac{4}{(d-1)^2}\int|x||\nabla_A f|^2. \tag{1.4.9}$$

*Proof.* This inequality can be proved in much the same way, considering the following integral

$$\begin{aligned} 0 &\leq \int\left||x|^{1/2}\nabla_A f + \alpha\frac{x}{|x|^{3/2}}f\right|^2 \\ &= \int|x||\nabla_A f|^2 + \alpha^2\int\frac{|f|^2}{|x|} + 2\alpha\Re\int\frac{x}{|x|}\cdot\nabla_A f\bar{f}. \end{aligned} \tag{1.4.10}$$

$\square$

## 1.5 Spectral representation

The spectral theory of operators is very important for an understanding of the operators themselves.

Let $T$ be a linear transformation defined everywhere over a general Banach space. A complex number $z = \lambda + i\varepsilon$ is said to be in the resolvent set $\rho(T)$ of $T$ if $zI - T$ is a bijection with a bounded inverse. $R_z(T) = (zI - T)^{-1}$ is called the resolvent of $T$ at $z$. If $z \notin \rho(T)$, then $z$ is said to be in the spectrum $\sigma(T)$ of $T$. In the case that $T$ is a self-adjoint operator on a Hilbert space $\mathcal{H}$, then it follows that $\sigma(T)$ is a subset of $\mathbb{R}$.

Our main interest in this section is to present a formula that gives the resolution of the identity for a self-adjoint operator $T$ on the Hilbert space $\mathcal{H}$ in terms of its resolvent. If $B$ is a Borel subset of $\mathbb{R}$ and $1_B$ is the indicator function of $B$, then $1_B(T)$ is a self-adjoint projection on $\mathcal{H}$. Then the associated spectral measure

$$E : B \to E(B) = 1_B(T) \tag{1.5.1}$$

is a projection-valued measure called the resolution of the identity for the self-adjoint operator $T$. The identity operator can be expressed as the spectral integral $I = \int 1 dE$ and the operator $T$ can be represented as

$$T = \int_{-\infty}^{\infty} \lambda dE(\lambda). \tag{1.5.2}$$

Working with specific operators it is important to have a method for calculating the spectral measure $dE(\lambda)$. The following theorem gives a way of doing it for a self-adjoint operator $T$ in terms of its resolvent $R(z) = (zI - T)^{-1}$. Note that since the spectrum of a self-adjoint operator is real, $R(z)$ is defined for all non-real $z$.

**Theorem 1.5.1.** *([DS]) If $E$ is the resolution of the identity for the self-adjoint operator $T$ and $B = (a, b)$ is an open interval such that $a < \lambda < b$, then in the strong operator topology*

$$(E(B)f, f) = \lim_{\nu\to 0}\lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int_{a+\nu}^{b-\nu} (R(\lambda - i\varepsilon)f - R(\lambda + i\varepsilon)f, f)\, d\lambda, \tag{1.5.3}$$

*where $(\cdot,\cdot)$ denotes the inner product in the Hilbert space $\mathcal{H}$.*

## 1.6 Examples of magnetic potentials

This section is intended to give some magnetic potentials such that the results of this thesis are applicable. Recall that the magnetic field associated to the vector potential $A$ is a $d \times d$ anti-symmetric matrix defined by

$$B = DA - (DA)^t, \qquad B_{jk} = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \tag{1.6.1}$$

and is the quantity which is physically measurable. As we have already mentioned (see Remark 1.3.6), using integration by parts one only needs to consider the components $B_{jk}$ of the magnetic field and moreover, if we take radial multipliers, we can restrict our attention to the tangential component of the magnetic field given by

$$B_\tau = \frac{x}{|x|}B. \tag{1.6.2}$$

This issue allows us to consider singular magnetic potentials. Thus potentials $A$ that produce non-trapping magnetic fields ($B_\tau \equiv 0$) are particularly interesting for us. Small $B_\tau$ also plays an important role in the study of the dispersive estimates for magnetic Schrödinger equation (see for example [DFVV], [FG], [FV]). Hence, we will focus on the understanding of the relation between the magnetic potential $A$ and the quantity $B_\tau$.

Let us first recall the conditions that are required to the potential $A$ in the known results related to the magnetic Schrödinger hamiltonian $H_A$ that have been mentioned in Introduction. On the one hand, the theory of Agmon and Hörmander [AH] where the forward problem for perturbations of general elliptic operators is studied, is valid for short range magnetic potentials $A$, i.e.

$$|A(x)| \leq \frac{C}{(1+|x|)^{1+\mu}}, \qquad C, \mu > 0. \tag{1.6.3}$$

On the other hand, in the work of Ikebe and Saito [IS] where the multiplier techniques are used for proving the limiting absorption principle for the magnetic Schrödinger equation, the authors require that $A_j$ is a real-valued $C^1$ function satisfying

$$|B_{jk}(x)| \leq \frac{C}{(1+|x|)^{1+\mu}}. \tag{1.6.4}$$

Finally, concerning the far field pattern of the solution of the magnetic Schrödinger equation, Iwatsuka [Iw] assumes regular magnetic potentials such that the magnetic field decays more than short range one. More concretely, in [Iw] $A_j(x)$ are real-valued $C^2$ functions and

$$|B_{jk}| \leq C(1+|x|)^{-\frac{3}{2}-\mu}. \tag{1.6.5}$$

The regularity of the magnetic potential $A$ is fundamental in order to construct a function $m(x)$ that permits to deduce the desired conclusion. This function $m(x)$ were used by Kuroda [Ku5] and is closely related to the gauge transformation which changes the magnetic potential $A$ into $A - \nabla m$, but does not change the magnetic field.

**Lemma 1.6.1.** *([Iw])*

$$m(x) = x\int_0^1 A(tx)\, dt. \tag{1.6.6}$$

$$\nabla m(x) = A(x) + \int_0^1 txB(tx)dt. \tag{1.6.7}$$

*Proof.*

$$\begin{aligned}\partial_j m(x) &= \sum_{k=1}^{d}\left[\int_0^1 A_j(tx)dt + x_k\int_0^1 t\partial_j A_k(tx)dt\right]\\ &= \sum_{k=1}^{d}\left[\int_0^1 A_j(tx)dt + x_k\int_0^1 tB_{jk}(tx)dt + x_k\int_0^1 t\partial_k A_j(tx)dt.\right]\end{aligned}$$

Now, since

$$\begin{aligned}\int_0^1 tx_k\partial_k A_j(tx)dt &= \int_0^1 t\frac{d}{dt}A_j(tx)dt\\ &= A_j(x) - \int_0^1 A_j(tx)dt,\end{aligned}$$

(1.6.7) follows. □

**Remark 1.6.2.** *Note that if we take $\tilde{A} = A - \nabla m$, then by (1.6.2) we get*

*(i)* $\tilde{A} = -\int_0^1 |x|tB_\tau(tx)dt.$

*(ii)* $x\cdot\tilde{A} = x\cdot A - x\cdot\nabla m = -\int_0^1 txB(tx)tx\frac{dt}{t} = 0.$

*Therefore, if $A$ is regular enough, we always can restrict to magnetic potentials $A$ such that $x\cdot A = 0$ (transversality condition) and*

$$A(x) = -\int_0^1 txB(tx)dt. \tag{1.6.8}$$

**Remark 1.6.3.** *Observe that in the case that $B_\tau = 0$, it follows that $A = \nabla m$, which implies $B = 0$. Thus when the magnetic potential $A$ is regular, potentials such that $B_\tau = 0$ are not interesting.*

Let us see what happens if one considers singular magnetic potentials $A$. Note that if $A(x)$ has singularities at the origin, the integral (1.6.6) is not well defined.

**Remark 1.6.4.** *In the case that the magnetic potential $A$ has singularities at the origin, we could define the function $m(x)$ as*

$$m(x) = -x\int_1^\infty A(tx)dt. \tag{1.6.9}$$

*Then the argument of Iwatsuka will work if*

$$\lim_{t\to\infty} tA_j(tx)dt = 0. \tag{1.6.10}$$

*This condition is satisfied if $A(x)$ is short range.*

When the magnetic potential $A(x)$ is singular at the origin, we define

$$m_\varepsilon(x) = x \int_\varepsilon^1 A(tx)dt \tag{1.6.11}$$

and observe that under the condition

$$\lim_{\varepsilon \to 0} \varepsilon A_j(\varepsilon x) = 0, \tag{1.6.12}$$

we get the same conclusions as Iwatsuka does. This suggests that potentials $A$ that do not satisfy condition (1.6.12) are interesting. In particular, magnetic potentials $A$ that are homogeneous of degree $-1$ need a special attention.

A general example of potential $A$ such that $B_\tau$ is nonzero is the following:

**Example 1.6.5.** Let $h(x)$ be a homogeneous function of the zero order ($h(x) = h(\lambda x)$, $x \cdot \nabla h(x) = 0$) and we define

$$A(x) = x\varphi(|x|)h(x). \tag{1.6.13}$$

Then we have

$$B_{jk} = [x_j \partial_k h(x) - x_k \partial_j h(x)]\, \varphi(|x|) \tag{1.6.14}$$

and it follows that

$$\sum_{j=1}^{d} \frac{x_j}{|x|} B_{jk} = |x|\varphi(|x|)\partial_k h. \tag{1.6.15}$$

As a consequence, we obtain

$$B_\tau = |x|\varphi(|x|)\nabla h(x). \tag{1.6.16}$$

Note that $B_\tau$ is short-range if $\varphi(|x|) = O(|x|^{-1-\mu})$, $\mu > 0$. Thus $A(x) = O(|x|^{-\mu})$.

From this example, choosing the function $\varphi(|x|)$ appropriately, one can generate singular magnetic potentials such that $B_\tau$ is small.

**Example 1.6.6.** Let $A(x) = x\varphi(|x|)h(x)$ as above and take $\varphi(|x|) = \frac{1}{|x|^2}$. Then $A(x)$ is homogeneous of degree $-1$ and we have $B_\tau = \frac{\nabla h}{|x|}$. Since $\nabla h$ is homogeneous of order $-1$, it follows that

$$B_\tau \sim \frac{\nu}{|x|^2}, \qquad \nu > 0. \tag{1.6.17}$$

In this manuscript we can handle with this kind of potentials with $\nu$ small and we are able to prove the limiting absorption principle and uniform resolvent estimate for it. In addition, we prove the existence of the cross-section for singular magnetic potentials such that $B_\tau \equiv 0$. Note that as $A(x)$ is singular, Iwatsuka's approach does not work for this case and the question related to the existence of the far field pattern is unanswered. We propose this as an interesting problem for studying in the future.

Therefore, we are interested in giving singular magnetic potentials such that $B_\tau \equiv 0$. Some explicit examples in the three dimensional case are the following:

**Example 1.6.7.** We consider some singular potentials. Let us take

$$A = \frac{1}{x^2+y^2+z^2}(-y,x,0) = \frac{1}{x^2+y^2+z^2}(x,y,x)\times(0,0,1). \tag{1.6.18}$$

One can easily check that

$$\nabla\cdot A = 0, \qquad B = -2\frac{z}{(x^2+y^2+z^2)^2}(x,y,z), \qquad B_\tau = 0. \tag{1.6.19}$$

**Example 1.6.8.** Another (more singular) example is

$$A = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right) = \frac{1}{x^2+y^2}(x,y,z)\times(0,0,1). \tag{1.6.20}$$

Here we have $B=(0,0,\delta)$, where $\delta$ is the Dirac's delta function and we obtain $B_\tau = 0$. We should point out that in this case $A_j \notin L^2_{loc}$. Hence, we can not guaranteed the self-adjointness of $\nabla^2_A$ for $d \geq 3$. In fact, this is a question still unanswered. However, it is considered as an interesting example introduced by Aharonov-Bohm. The two dimensional is done by Felli, Ferrero and Terracini [FFT].

These two examples are particular cases of the following more general singular magnetic potentials with $B_\tau = 0$.

**Lemma 1.6.9.** *Let $x \neq 0$ and assume that*

*(i)* $\lambda A(\lambda x) = A(x)$.

*(ii)* $x\cdot A(x) = 0$.

*Then $B_\tau \equiv 0$.*

*Proof.* From (i), we have

$$A_j(x) + x_k\partial_k A_j(x) = 0 \qquad \forall j,k. \tag{1.6.21}$$

Condition (ii) implies

$$\delta_{jk}A_k + x_k\partial_j A_k = 0. \tag{1.6.22}$$

Thus we obtain

$$(xB)_k = x_j(\partial_j A_k - \partial_k A_j) = 0. \tag{1.6.23}$$

□

# Chapter 2

# Limiting absorption principle with singular potentials

The purpose of this chapter is to study the limiting absorption principle for the Schrödinger operator

$$L = \sum_{j=1}^{d} (\nabla_j + iA_j)^2 + V_1 + V_2 \tag{2.0.1}$$

in the Hilbert space $L^2(\mathbb{R}^d)$, $d \geq 3$, with potentials that admit singularities at a point. $A_j$ denotes the $j$-component of the vector potential, $V_1$ is a long-range potential and $V_2$ is a short-range one. Regarding to the magnetic part, our goal is to only impose conditions on the magnetic field $B$. However, in order to ensure the self-adjointness of $L$ we need to assume some local integrability condition on the magnetic potential $A$. In the remainder of this chapter we require that

$$\int (V_1 + V_2)|u|^2 < 1^- \int |\nabla u|^2, \qquad A_j \in L^2_{loc}, \qquad V_1, V_2 \in L^1_{loc}. \tag{2.0.2}$$

Thus we have that $L$ is self-adjoint on $L^2(\mathbb{R}^d)$ with form domain

$$D(L) = \{f \in L^2(\mathbb{R}^d) : \int |\nabla_A f|^2 - \int (V_1 + V_2)|f|^2 < \infty\}. \tag{2.0.3}$$

See section 1.2 for more details.

Under suitable assumptions on the potentials, our goal is to prove that there exists a unique solution of the resolvent equation

$$(L + \lambda)u = f, \qquad \lambda > 0 \tag{2.0.4}$$

satisfying a specific Sommerfeld radiation condition together with some a-priori estimates. We will construct this solution $u$ from the solution of the equation

$$(L + \lambda + i\varepsilon)u_\varepsilon = f, \qquad \varepsilon > 0, \tag{2.0.5}$$

whose existence is established by the self-adjointness of $L$. In fact, $u$ will be the limit of $u_\varepsilon$ in the Morrey-Campanato space. We point out that we need two main ingredients for this purpose. On the one hand, the a-priori estimates and Sommerfeld radiation condition for any solution $u_\varepsilon \in H^1_A(\mathbb{R}^d)$ of (2.0.5) will be needed. On the other hand, we shall assert uniqueness of solution of the equation (2.0.4) if such a radiation condition is satisfied.

It is a simple matter to show the uniqueness result for (2.0.5). Letting $f = 0$, we only need to multiply the corresponding equation by $u$ in the $L^2$-sense and take the imaginary part. Thus we get $\varepsilon\|u\|^2 = 0$ and so $u = 0$. Uniqueness criterion for the equation (2.0.4) presents a more delicate problem. In this case, we shall study the homogeneous electromagnetic Helmholtz equation

$$(\nabla + iA)^2 u + (V_1 + V_2)\, u + \lambda u = 0 \tag{2.0.6}$$

and show that if $u \in (H^1_A)_{loc}(\mathbb{R}^d)$ is a solution of (2.0.6), then $u$ is identically zero. The proof of this result is adapted from [M1] or [Z]. Nevertheless, as far as we know, it does not seem to appear in the literature for potentials as the one we can treat. Using the multiplier method we prove that $u = 0$ in $\Omega = \{x \in \mathbb{R}^d : |x| \geq R\}$ for $R > 0$ large enough. Then we apply the unique continuation property to deduce that $u$ vanishes in $\mathbb{R}^d$. Hence, in order to accomplish this task, we need that the unique continuation property holds for $L$. The best reference here is due to Regaboui [R], where it is proven that if $u \in H^1_{loc}$ satisfies

$$|P(x, D)u| \leq C_1|x|^{-2}|u| + C_2|x|^{-1}|\nabla u|, \tag{2.0.7}$$

with $C_2 > 0$ small enough and $P(x, D) = \sum_{j,k=1}^d a_{jk} D_j D_k$ is an elliptic operator with Lipschitz coefficients such that $a_{jk}(0)$ is real in a connected open subset $\Omega$ of $\mathbb{R}^d$ containing 0, then $u \equiv 0$ in $\Omega$. Thus for using this result, we will write the magnetic Schrödinger operator $L$ as a first order perturbation of the Laplacian,

$$L = \Delta + 2iA \cdot \nabla + i\nabla \cdot A - A \cdot A + V_1 + V_2 \tag{2.0.8}$$

and note that $u$ satisfies

$$|\Delta u + \lambda u| \leq 2|A||\nabla u| + \left(|\nabla \cdot A| + |V_1| + |V_2| + |A|^2\right)|u| \tag{2.0.9}$$

if $|x| \leq 1$.

The crux of the limiting absorption principle are certain $L^2$-weighted a-priori estimates for the operator $(L + z)^{-1}$, $z = \lambda + i\varepsilon$, such that are preserved after the limiting procedure. The classical result on the free resolvent case, which is usually denoted by

$$R_0(z) = (\Delta + z)^{-1}, \tag{2.0.10}$$

is due to Agmon [A] and states that the limits

$$R_0(\lambda \pm i0) = \lim_{\varepsilon \to 0} R_0(\lambda \pm i\varepsilon) \tag{2.0.11}$$

exist in the norm of bounded operators from $L^2_s(\mathbb{R}^d)$ to $L^2_{-s}(\mathbb{R}^d)$ for any $s > 1$, where

$$\|u\|_{L^2_s} = \|(1+|x|)^s u\|_{L^2}. \tag{2.0.12}$$

The convergence is uniform for $\lambda$ belonging to any compact subset of $]0+\infty[$, and the following estimate holds

$$\|R_0(\lambda \pm i0)f\|_{L^2_{-s}} \le \frac{C(s)}{\sqrt{\lambda}}\|f\|_{L^2_s}, \quad \lambda > 0, s > 1. \tag{2.0.13}$$

There is an analogue result for the electromagnetic case due to Ikebe and Saito [IS], where the authors assert the existence of a unique solution of the equation (2.0.4) in which they impose on $A_j$ some asymptotic growth condition at infinity, $V_1(x)$ is a long range potential and $V_2(x)$ a short range one. In fact, they require that there exist positive constants $c, \mu > 0$ and $r_0 \ge 1$ such that

($V_1$) for $|x| \ge r_0$, $|V_1(x)| \le c|x|^{-\mu}$ and the radial derivative $\frac{\partial V_1}{\partial |x|}$ exists with $\frac{\partial V_1}{\partial |x|} \le c|x|^{-1-\mu}$.

($V_2$) $|V_2(x)| \le c|x|^{-1-\mu}$, for $|x| \ge r_0$.

(B) $A_j(x)$ is a real-valued $C^1$-function with $|B_{jk}(x)| \le c|x|^{-1-\mu}$ for $|x| \ge r_0$, $j,k = 1,\dots,d$, where $B_{jk} = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}$.

(UC) The unique continuation property holds for the differential operator $L$ in $\mathbb{R}^d$.

The general idea of the proof of the main theorem, that largely improves the result by Ikebe and Saito [IS] (see Theorem 0.0.1), is based on the multiplier technique and integration by parts. We follow their ideas combined with the ones used by Perthame and Vega [PV1].

## 2.1 Assumptions and main results

Let us consider the inhomogeneous Helmholtz equation

$$(\nabla + iA)^2 u + V_1 u + V_2 u + \lambda u + i\varepsilon u = f \tag{2.1.1}$$

in $\mathbb{R}^d$, where $\lambda, \varepsilon > 0$ and $f$ is a suitable function on $\mathbb{R}^d$. We work with potentials that decay at infinity and can have singularities at a point that we will take to be at the origin. We will use a multiplier method based on radial multipliers. Thus just information for the tangential component of the magnetic field $B$ (see Remark 1.3.6) will be needed. Nevertheless, in order to assert the unique continuation property, it is necessary to put some restrictions on the whole $B$ when we are close to the origin.

One question still unanswered is whether the unique continuation property is satisfied assuming only the decay on the tangential part of $B$. We will not develop this point here, but we propose to study it in the future. A possible approach would be to show an appropriate Carleman estimate for the magnetic Hamiltonian $L$. In section 2.5 we start the analysis of this problem and we obtain partial results for $B_\tau = 0$. We also are interested in the case when $B_\tau$ is small, proposing its study for the future.

From now on, we assume that the magnetic potential $A$ satisfy

$$|\nabla \cdot A| \leq \frac{C}{|x|^2}, \tag{2.1.2}$$

for some $C > 0$. We point out that this condition is only needed for the unique continuation property.

We may now state our main assumptions on the potentials.

**Assumption 2.1.1.** *Let $V_1(x)$, $A_j(x)$, $j = 1, \ldots, d$, $V_2(x)$ be real-valued functions, $r_0 \geq 1$ and $\mu > 0$. For $d \geq 3$, if $|x| \geq r_0$ we assume*

$$\frac{|V_1(x)|}{|x|} + (\partial_r V_1(x))_- + |B_\tau(x)| + |V_2(x)| \leq \frac{c}{|x|^{1+\mu}}, \tag{2.1.3}$$

*for some $c > 0$, where $\partial_r V_1$ is considered in the distributional sense and $(\partial_r V_1)_-$ denotes the negative part of $\partial_r(V_1)$. On the other hand, we require*

$$V_1(x) = (\partial_r V_1(x))_- = 0 \quad \text{if} \quad |x| \leq r_0, \tag{2.1.4}$$

*and*

$$|V_2(x)| \leq \frac{c}{|x|^{2-\alpha}} \quad \text{if} \quad |x| \leq r_0, \quad \alpha > 0, \tag{2.1.5}$$

*for some $c > 0$.*
*If $d > 3$, we consider*

$$|B| \leq \frac{\mathcal{C}^*}{|x|^2} \qquad |x| \leq r_0, \tag{2.1.6}$$

*for some $\mathcal{C}^* > 0$ small enough. Finally, in dimension $d = 3$ we assume*

$$|B| \leq \frac{c}{|x|^{2-\alpha}} \qquad |x| \leq r_0, \quad \alpha > 0, \tag{2.1.7}$$

*for some $c > 0$.*

**Remark 2.1.2.** *Note that the requirements on the magnetic field $B$ at the origin differ depending on the dimension. This is due to the fact that we give an extra a-priori estimate for the solution $u$ of the equation (2.0.4) when $d > 3$, see (2.1.14) below.*

**Remark 2.1.3.** *For $d > 3$ we may allow the potential $V_2(x)$ to be more singular. Moreover, we can also permit some singularity on the potential $V_1(x)$ and its repulsive part $(\partial_r V_1(x))_-$. When $|x| \leq r_0$, one can actually require*

$$|V_2(x)| \leq \frac{\mathcal{C}^{**}}{|x|^2} \tag{2.1.8}$$

*and*

$$(\partial_r V_1(x))_- \leq \frac{\mathcal{C}^{**}}{|x|^3}, \qquad \frac{|V_1(x)|}{|x|} \leq \frac{\mathcal{C}^{***}}{|x|^3} \tag{2.1.9}$$

*for sufficiently small $\mathcal{C}^{**} > 0$ and for some $\mathcal{C}^{***} > 0$. See section 2.4 for more details.*

**Remark 2.1.4.** *Observe that in order to use the unique continuation result ([R]), by (2.0.9) we need to verify that*

$$|\nabla \cdot A| \leq C_1 |x|^{-2} \tag{2.1.10}$$

*and*

$$|A| \leq C_2 |x|^{-1} \tag{2.1.11}$$

*provided that $C_2 > 0$ is small. On the one hand, note that condition (2.1.2) gives (2.1.10). On the other hand, from (2.1.6) when $d > 3$ with $\mathcal{C}^*$ small enough and (2.1.7) when $d = 3$, by the Biot-Savart law it may be concluded that (2.1.11) holds. It is worth pointing out that condition (2.1.2) is only required in order to assure that this result is applicable.*

Our first theorem is the uniqueness result.

**Theorem 2.1.5.** *Let $d \geq 3$, $\lambda \geq \lambda_0 > 0$ and assume (2.1.3)-(2.1.5) and (2.1.6) or (2.1.7). Let $u \in (H_A^1)_{loc}$ be a solution of (2.0.6) such that*

$$\liminf \int_{|x|=r} (|\nabla_A u|^2 + \lambda |u|^2) d\sigma(x) \to 0, \quad \text{as} \quad r \to \infty. \tag{2.1.12}$$

*Then $u \equiv 0$. Moreover, if for some $\delta > 0$*

$$\int_{|x|\geq 1} \left| \nabla_A u - i\lambda^{1/2} \frac{x}{|x|} u \right|^2 \frac{1}{(1+|x|)^{1-\delta}} < \infty \tag{2.1.13}$$

*is satisfied, then (2.1.12) holds.*

The uniqueness result allows us to state the main result of this chapter.

**Theorem 2.1.6.** *Let $\mathcal{C}^*$ small enough, $\lambda_0 > 0$, $f \in L^2_{\frac{1+\delta}{2}}$ and assume that one of the following two conditions is satisfied:*

*(i) $d > 3$, with (2.1.3)-(2.1.5) and (2.1.6)*

*(ii)* $d = 3$, *with (2.1.3)-(2.1.5) and (2.1.7).*

*Then, for all* $\lambda \geq \lambda_0$ *there exists a unique solution* $u \in (H^1_A)_{loc}(\mathbb{R}^d)$ *of the Helmholtz equation (2.0.4) satisfying*

$$\lambda |||u|||_1^2 + |||\nabla_A u|||_1^2 + \int \frac{|\nabla_A^\perp u|^2}{|x|} + \sup_{R>1} \frac{1}{R^2} \int_{|x|=R} |u|^2 d\sigma_R$$
$$+ (d-3) \int \frac{|u|^2}{|x|^3} \leq C(N_1(f))^2 \tag{2.1.14}$$

*and the radiation condition*

$$\int_{|x|\geq 1} \left| \nabla_A u - i\lambda^{1/2} \frac{x}{|x|} u \right|^2 \frac{1}{(1+|x|)^{1-\delta}} \leq C \int_{\mathbb{R}^d} (1+|x|)^{1+\delta} |f|^2, \tag{2.1.15}$$

*for all* $0 < \delta < 2$ *such that* $\delta < \mu$, *where* $C = C(\lambda_0) > 0$.

**Remark 2.1.7.** *The smallness of the constant* $\mathcal{C}^*$ *is required for the unique continuation property proved by Regbaoui [R]. As we said, this constant is not explicit.*

**Remark 2.1.8.** *The case* $B_\tau = 0$ *is particularly interesting as we saw in section 1.6. In the Appendix B, Theorem 2.5.2, we give a unique continuation result for this kind of potentials. As a consequence, if* $V_1 = V_2 = 0$ *Theorems 2.1.5 and 2.1.6 will hold in this case.*

**Remark 2.1.9.** *In order to prove the Morrey-Campanato type estimates (2.1.14), condition (2.1.6) can be replaced by*

$$|B_\tau| \leq \frac{(d-1)(d-3)}{|x|^2}. \tag{2.1.16}$$

Theorem 2.1.6 extends a similar result proved by Ikebe and Saito in the 70's (see Theorem 0.0.1). Firstly, our estimates are not only for $\lambda \in (\lambda_0, \lambda_1)$ with $0 < \lambda_0 < \lambda_1 < \infty$ as in [IS], but also for all $\lambda \geq \lambda_0 > 0$. We also extend the Sommerfeld radiation condition (0.0.49) from $\delta \in (0,1)$ to the range $\delta \in (0,2)$ and we are able to give a more general one containing the limit case $\delta = 0$ (see Remark 2.2.9). Concerning the a-priori estimates, note that (2.1.14) is stronger than (0.0.50) in the sense that it gives more information about the solution and improves the $L^2$-weighted estimate from the $L^2_{-\frac{(1+\delta)}{2}}$ norm to the Agmon-Hörmander norm. More importantly, we are able to consider singular potentials and the estimate (2.1.14) is uniform on $\lambda$ for $\lambda \geq \lambda_0 > 0$. This permits to prove the $L^p$-$L^q$ estimates for the electromagnetic Helmholtz equation with singular potentials. (See [G1], chapter 2 and [G]).

In order to recover the a-priori estimates in the full frequency range $\lambda \geq 0$, a stronger decay on the potentials is needed. In 2009, Fanelli [F] proved (2.1.14) for any $\lambda \geq 0$ in $\mathbb{R}^d$ and very recently, Barceló, Fanelli, Ruiz and Vilela [BLRV] also get the analogous estimates for the Helmholtz equation with electromagnetic-type perturbations in the exterior of a domain. In fact, if such an estimate holds for $\lambda \geq 0$, it would imply as a by product the absence of zero-resonances (in a suitable sense) for the operator $L$. This is in general false with our type of potentials. For example, if we reduce to the case $\Delta u + Vu = 0$, the natural decay at infinity for the non-existence of zero-resonances is $|x|^{-(2+\delta)}$, $\delta > 0$.(See for example [BRV], Section 3 and [F] Remark 1.3).

The general outline for proving the main result consists of the following steps:

1. We take a sufficiently large $\lambda_1 (> \lambda_0)$ and we derive the Morrey-Campanato type estimates for any $\lambda \geq \lambda_1$ proceeding as in [PV1].

2. We prove that for any $\lambda \geq \lambda_0$ the Sommerfeld radiation condition is true if the Morrey-Campanato type estimates hold.

3. We use a compactness argument (in the spirit of [IS]) to deduce the result for all $\lambda \geq \lambda_0$.

4. From the estimates proved in the previous steps and by the uniqueness theorem, we prove the limiting absorption principle for the Schrödinger operator $L$ satisfying Assumption 2.1.1.

# 2.2 Proof of Theorem 2.1.6

According to the steps given above, the proof will be divided into four parts.

## 2.2.1 A priori estimates for $\lambda$ large enough ($\lambda \geq \lambda_1$)

We begin by proving the Morrey-Campanato type estimates for solutions of the equation (2.1.1) for $\lambda$ large enough. Since our assumptions on the magnetic field differ depending on the dimension, we first give a detailed proof of the result for $d > 3$. Then the three dimensional case follows by the same method.

**Theorem 2.2.1.** *For dimension $d > 3$, let $\varepsilon > 0$, $f$ such that $N(f) < \infty$. Let $\mathcal{C}^* < \sqrt{(d-1)(d-3)}$. Assume that (2.1.3)-(2.1.5) and*

$$|B_\tau| \leq \frac{\mathcal{C}^*}{|x|^2} \qquad \text{if} \qquad |x| \leq r_0 \tag{2.2.1}$$

*hold. Then there exists $\lambda_1 > 0$ such that for any $\lambda \geq \lambda_1$ the solution $u \in H^1_A(\mathbb{R}^d)$ of the Helmholtz equation (2.1.1) satisfies*

$$\begin{aligned} \lambda |||u|||^2 + |||\nabla_A u|||^2 + \int \frac{|\nabla_A^\perp u|^2}{|x|} + \sup_{R>0} \frac{1}{R^2} \int_{|x|=R} |u|^2 d\sigma_R + \int \frac{|u|^2}{|x|^3} \\ \leq C(1+\varepsilon)(N(f))^2, \end{aligned} \tag{2.2.2}$$

*where $C = C(\lambda_1) > 0$ is independent of $\varepsilon$.*

*Proof.* The proof is based on the identities which are established in Chapter 1 (section 1.3). Note that in this case we take $V = V_1 + V_2$. Hence, for any $\varphi, \psi$ real-valued radial functions, adding up (1.3.12) and (1.3.15), by Remark 1.3.6 together with Remark 1.3.7 we have

$$\begin{aligned}
&\int \nabla_A u \cdot D^2\psi \cdot \overline{\nabla_A u} - \int \varphi |\nabla_A u|^2 - \Re \int \left(\nabla\varphi - \frac{\nabla(\Delta\psi)}{2}\right) \cdot \nabla_A u \bar{u} \\
&+ \int \lambda |u|^2 + \int_{\mathbb{R}^d} \varphi V_1 |u|^2 + \frac{1}{2}\int \nabla V_1 \cdot \nabla\psi |u|^2 - \Im \int \psi' B_\tau \cdot \nabla_A u \bar{u} \\
&+ \int \left(\varphi - \frac{\Delta\psi}{2}\right) V_2 |u|^2 - \Re \int V_2 \nabla\psi \cdot \nabla_A u \bar{u} \\
&= \varepsilon \Im \int \nabla\psi \cdot \nabla_A u \bar{u} + \Re \int \left(\varphi - \frac{\Delta\psi}{2}\right) f \bar{u} - \Re \int f \nabla\psi \cdot \overline{\nabla_A u}.
\end{aligned} \tag{2.2.3}$$

Let us define for $R > 0$ the function $\psi(x) = \int_0^{|x|} \psi'(s) ds$, where

$$\psi'(x) = \begin{cases} \frac{|x|}{2R} + M & \text{if } |x| \le R, \\ \left(M + \frac{1}{2}\right) & \text{if } |x| \ge R, \end{cases} \tag{2.2.4}$$

for arbitrary $M > 0$,

$$\varphi(x) = \begin{cases} \frac{1}{4R} & \text{if } |x| \le R, \\ 0 & \text{if } |x| \ge R, \end{cases} \tag{2.2.5}$$

and we put these multipliers into (2.2.3).

First, note that since $N(f) < \infty$ then $f \in L^2(\mathbb{R}^d)$. Thus it is guaranteed the existence of solution of (2.1.1) in $H^1_A(\mathbb{R}^d)$. As a consequence, the terms on the right-hand side of (2.2.3) are finite. It is easy to check that

$$\begin{aligned}
&\left| \varepsilon \Im \int \nabla\psi \cdot \nabla_A u \bar{u} + \Re \int \left(\varphi - \frac{\Delta\psi}{2}\right) f \bar{u} - \Re \int f \nabla\psi \cdot \overline{\nabla_A u} \right| \\
&\le C \left( \|f\|^2_{L^2} + \|\nabla_A u\|^2_{L^2} + \|u\|^2_{L^2} \right) < \infty.
\end{aligned}$$

Let us show the positivity of the left-hand side of (2.2.3) with the above choice of the multipliers. By (1.3.22), it follows easily that

$$\int \nabla_A u \cdot D^2\psi \cdot \overline{\nabla_A u} - \int \varphi |\nabla_A u|^2 > \frac{1}{4R}\int_{|x|\le R} |\nabla_A u|^2 + M \int \frac{|\nabla_A^\perp u|^2}{|x|},$$

$$\int \varphi \lambda |u|^2 = \frac{1}{4R}\int_{|x|\le R} \lambda |u|^2.$$

In addition, since $\varphi$ and $\psi''$ are discontinuous in $\{|x| = R\}$, note that integrating by parts the term

$$-\Re\int\left(\nabla\varphi - \frac{\nabla(\Delta\psi)}{2}\right)\cdot\nabla_A u\bar{u} \tag{2.2.6}$$

gives a surface integral. In fact, after substituting our test functions in (2.2.6), we get

$$\begin{aligned}-\Re\int\left(\nabla\varphi - \frac{\nabla(\Delta\psi)}{2}\right)\cdot\nabla_A u\bar{u} > {}& \frac{M(d-1)(d-3)}{4}\int\frac{|u|^2}{|x|^3}\\ &+\frac{(d-1)}{8R^2}\int_{|x|=R}|u|^2 d\sigma_R.\end{aligned}$$

Let us analyze the terms containing the potentials. In what follows, $\sigma = \sigma(c,\mu,r_0,\alpha,M)$ denotes a positive constant where the parameters $c,\mu,r_0,\alpha$ have been introduced in Assumption 2.1.1 and $M > 0$ is related to the multipliers. For simplicity of notation, we use the same letter $\sigma$ for all constants related to the potentials. Moreover, we will use $\kappa$ for a small positive constant coming from the inequality $ab \le \kappa a^2 + \frac{1}{4\kappa}b^2$.

In order to estimate the term involving the magnetic field, observe that since $B_\tau$ is a tangential vector to the sphere, we have

$$B_\tau\cdot\nabla_A u = B_\tau\cdot\nabla_A^\perp u. \tag{2.2.7}$$

Hence,

$$\begin{aligned}\Im\int\psi' B_\tau\cdot\nabla_A u\bar{u} &\le (M+1/2)\int_{|x|\le r_0}|B_\tau||\nabla_A^\perp u||u|\\ &+(M+1/2)\int_{|x|\ge r_0}|B_\tau||\nabla_A^\perp u||u|\\ &\equiv B_1 + B_2,\end{aligned}$$

where by (2.1.3), (2.2.1) and Cauchy-Schwarz inequality, yields

$$B_1 \le \mathcal{C}^*(M+1/2)\left(\int_{|x|\le r_0}\frac{|\nabla_A^\perp u|^2}{|x|}\right)^{1/2}\left(\int_{|x|\le r_0}\frac{|u|^2}{|x|^3}\right)^{1/2} \tag{2.2.8}$$

and

$$\begin{aligned}B_2 &\le (M+1/2)\int_{|x|\ge r_0}|x|^{1/2}|B_\tau|\frac{|\nabla_A^\perp u||u|}{|x|^{1/2}}\\ &\le \frac{M}{2}\int_{|x|\ge r_0}\frac{|\nabla_A^\perp u|^2}{|x|} + \frac{c(M+1/2)^2}{2M}\sum_{j\ge j_0}2^{-2\mu j}\int_{C(j)}2^j\frac{|u|^2}{2^j}\\ &\le \frac{M}{2}\int_{|x|\ge r_0}\frac{|\nabla_A^\perp u|^2}{|x|} + \sigma|||u|||^2.\end{aligned}$$

We next turn to estimate the $V_1$ terms. Similarly, by (2.1.3) and (2.1.4), we get

$$-\int \varphi V_1|u|^2 \leq \frac{1}{4}\int_{r_0\leq|x|\leq R}\frac{|V_1||u|^2}{|x|}$$
$$\leq \frac{1}{4}\sum_{j\geq j_0}\int_{C(j)}\frac{|V_1||u|^2}{2^j} \leq \sigma|||u|||^2 \tag{2.2.9}$$

and

$$-\frac{1}{2}\int \nabla\psi\cdot\nabla V_1|u|^2 \leq \frac{1}{2}\int(\nabla\psi\cdot\nabla V_1)_-|u|^2$$
$$\leq \frac{(M+1/2)}{2}\sum_{j\geq j_0}\int_{C(j)}(\partial_r V_1)_-|u|^2$$
$$\leq \sigma|||u|||^2. \tag{2.2.10}$$

As far as the potential $V_2$ is concerned, let us first take $j_1 = j_1(\alpha) < j_0$ such that

$$c\sum_{j\leq j_1}2^{\alpha j} < \eta, \qquad\qquad c\sum_{j\leq j_1}2^{2\alpha j} < \eta \tag{2.2.11}$$

where $\eta > 0$ stands for a small constant, being $c$ and $\alpha$ are as in (2.1.5). To simplify notation, we continue to write $\eta$ for any small constant related to the potentials. We fix $r_1 < r_0$ by $2^{j_1-1} \leq r_1 \leq 2^{j_1}$. Then, by (2.1.3), (2.1.5) and Cauchy-Schwarz inequality, we have

$$\Re\int V_2\nabla\psi\cdot\nabla_A u\bar{u} \leq (M+1/2)\int_{|x|\leq r_0}|V_2||\nabla_A u||u|$$
$$+ (M+1/2)\int_{|x|\geq r_0}|V_2||\nabla_A u||u|$$
$$\equiv V_{21}+V_{22}. \tag{2.2.12}$$

Let us make now the following observation.

$$\sum_{j\leq 0}\int_{C(j)}\frac{|u|^2}{2^{j(3-\gamma)}} \leq \sum_{j\leq 0}\int_{2^{j-1}}^{2^j}\int_{|x|=r}\frac{|u|^2}{r^2}2^{-j(1-\gamma)}$$
$$\leq \sup_{R\leq 1}\frac{1}{R^2}\int_{|x|=R}|u|^2\sum_{j\leq 0}\int_{2^{j-1}}^{2^j}2^{j(\gamma-1)}$$
$$\leq \sup_{R>0}\frac{1}{R^2}\int_{|x|=R}|u|^2\sum_{j\leq 0}2^{j\gamma} \tag{2.2.13}$$

and $\sum_{j\le 0} 2^{\gamma j} < \infty$ if $\gamma > 0$. According to the above remark, using again the Cauchy-Schwarz inequality and the relation $ab \le \frac{1}{16}a^2 + 4b^2$, we have

$$V_{21} \le c(M+1/2)\left[\sum_{j\le j_1}\int_{C(j)} \frac{|\nabla_A u||u|}{2^{j(2-\alpha)}} + \sum_{j=j_1}^{j_0} 2^{-j(1-\alpha)}\int_{C(j)} \frac{|\nabla_A u||u|}{2^j}\right]$$
$$\le \eta\left(\sup_{R>0}\frac{1}{R}\int_{|x|\le r_1}|\nabla_A u|^2\right)^{\frac{1}{2}}\left(\sup_{R>0}\frac{1}{R^2}\int_{|x|=R}|u|^2\right)^{\frac{1}{2}}$$
$$+\frac{1}{16}\sup_{R>0}\frac{1}{R}\int_{r_1\le|x|\le R}|\nabla_A u|^2 + \sigma|||u|||^2$$

and

$$V_{22} \le (M+1/2)\sum_{j\ge j_0}\left(\int_{C(j)}\frac{|\nabla_A u|^2}{2^j}\right)^{1/2}\left(\int_{C(j)}\frac{|u|^2}{2^{j(1+2\mu)}}\right)^{1/2}$$
$$\le \frac{1}{16}\sup_{R>0}\frac{1}{R}\int_{r_1\le|x|\le R}|\nabla_A u|^2 + \sigma|||u|||^2.$$

Analysis similar to the above gives

$$-\int \varphi V_2|u|^2 \le \frac{c}{4}\int_{|x|\le r_0}\frac{|u|^2}{|x|^{3-\alpha}} + \frac{1}{4}\sum_{j\ge j_0}\int_{C(j)}\frac{|V_2||u|^2}{|x|}$$
$$\le \eta\sup_{R\le r_1}\frac{1}{R^2}\int_{|x|=R}|u|^2 d\sigma_R + \sigma|||u|||^2$$

and

$$\frac{1}{2}\int \Delta\psi V_2|u|^2 \le \left(\frac{d}{4}+\frac{M(d-1)}{2}\right)\int_{\mathbb{R}^d}\frac{|V_2||u|^2}{|x|}$$
$$\le \eta\left(\frac{d}{4}+\frac{M(d-1)}{2}\right)\sup_{R\le r_1}\frac{1}{R^2}\int_{|x|=R}|u|^2 d\sigma_R$$
$$+\sigma|||u|||^2.$$

In order to simplify the reading, let us introduce

$$a_1 = \left(\sup_{R>0}\frac{1}{R}\int_{|x|\le r_1}|\nabla_A u|^2\right)^{1/2}, \qquad a_2 = \left(\int_{|x|\le r_0}\frac{|\nabla_A^{\perp} u|^2}{|x|}\right)^{1/2},$$
$$a_3 = \left(\int_{|x|\le r_0}\frac{|u|^2}{|x|^3}\right)^{1/2}, \qquad a_4 = \left(\sup_{R>0}\frac{1}{R^2}\int_{|x|=R}|u|^2 d\sigma_R\right)^{1/2}.$$

Therefore, it turns out that the potential terms on (2.2.3) are lower bounded by

$$
\begin{aligned}
&-\frac{M}{2}\int_{|x|\geq r_0}\frac{|\nabla_A^{\perp}u|^2}{|x|}-\mathcal{C}^*(M+1/2)a_2a_3-(M+1/2)\eta a_1a_4-\eta a_4^2\\
&-\frac{1}{8}\sup_{R>0}\frac{1}{R}\int_{r_1\leq|x|\leq R}|\nabla_A u|^2-\sigma|||u|||^2.
\end{aligned}
$$

Our next step is to estimate the right-hand side of (2.2.3). Let us start by the $\varepsilon$ term. From the a-priori estimate (1.3.10), by the assumptions (2.1.3)-(2.1.5) and the Hardy inequality (1.4.8), it may be concluded that

$$
\int|\nabla_A u|^2\leq\lambda\int|u|^2+\sigma\int|u|^2+\int_{\mathbb{R}^d}|f||u|. \tag{2.2.14}
$$

Recall that $\sigma$ denotes a positive constant related to the potentials. Hence combining (2.2.14) with (1.3.9), by Cauchy-Schwarz inequality and the fact that $\int|f||u|\leq N(f)|||u|||$, we obtain

$$
\begin{aligned}
\varepsilon\Im\int\nabla\psi\cdot\nabla_A u\bar{u}&\leq(M+1/2)\varepsilon\int|\nabla_A u||u|\\
&\leq(M+1/2)\varepsilon^{1/2}\left(\varepsilon\int|u|^2\right)^{1/2}\left(\int|\nabla_A u|^2\right)^{1/2}\\
&\leq(M+1/2)\varepsilon^{1/2}\int|f||u|\\
&+(M+1/2)\varepsilon^{\frac{1}{2}}\left(\int|f||u|\right)^{\frac{1}{2}}\left((\lambda+\sigma)\int|u|^2\right)^{\frac{1}{2}}\\
&\leq(M+1/2)(\varepsilon^{1/2}+(\lambda+\sigma)^{1/2})\int|f||u|\\
&\leq\kappa(1+\lambda)|||u|||^2+C(1+\varepsilon)(N(f))^2.
\end{aligned}\tag{2.2.15}
$$

It remains to estimate the terms containing $f$ which can be handled in much the same way as the rest. In fact, it follows that

$$
\begin{aligned}
\Re\int f\left(\varphi-\frac{1}{2}\Delta\psi\right)\bar{u}&\leq\frac{M+1}{2}\int\frac{|f||u|}{|x|}\\
&\leq\frac{(M+1)}{2}\sum_{j\in\mathbb{Z}}\left(2^j\int_{C(j)}|f|^2\right)^{\frac{1}{2}}\left(\int_{C(j)}\frac{|u|^2}{2^{3j}}\right)^{\frac{1}{2}}\\
&\leq\kappa\sup_{R>0}\frac{1}{R^2}\int_{|x|=R}|u|^2d\sigma_R+C(N(f))^2
\end{aligned}
$$

and

$$\Re \int f\nabla\psi \cdot \overline{\nabla_A u} \leq (M+1/2)\int |f||\nabla_A u| \\ \leq \kappa |||\nabla_A u|||^2 + C(N(f))^2.$$

Finally, due to the freedom on the choice of $R$, let us take the supremum over $R>0$ on the both sides of the inequality. Thus from the above estimates, we obtain

$$\begin{aligned}
&\left(\frac{\lambda}{4}-\sigma\right)|||u|||^2 + \frac{M}{2}\int_{|x|\geq r_0}\frac{|\nabla_A^{\perp}u|^2}{|x|} + \left(\frac{d-1}{8}-\nu\right)a_4^2 + \frac{a_1^2}{4} - \nu a_1 a_4 \\
&+ \frac{1}{8}\sup_{R>0}\frac{1}{R}\int_{r_1\leq|x|\leq R}|\nabla_A u|^2 + \frac{M(d-1)(d-3)}{4}\int_{|x|\geq r_0}\frac{|u|^2}{|x|^3} \\
&+ Ma_2^2 + \frac{M(d-1)(d-3)a_3^2}{4} - \mathcal{C}^*(M+1/2)a_2a_3 \\
&\leq \kappa\left[(1+\lambda)|||u|||^2 + |||\nabla_A u|||^2 + a_4^2\right] + C(\varepsilon+1)\left(N(f)\right)^2.
\end{aligned}$$

Observe that we need

$$Ma_2^2 + \frac{M(d-1)(d-3)}{4}a_3^2 - \mathcal{C}^*(M+1/2)a_2a_3 > 0, \tag{2.2.16}$$

which is satisfied if

$$\frac{1}{(d-1)(d-3)}(\mathcal{C}^*)^2\frac{(M+1/2)^2}{M^2} < 1. \tag{2.2.17}$$

Letting $M\to\infty$, we obtain

$$(\mathcal{C}^*)^2 < (d-1)(d-3), \tag{2.2.18}$$

which is our assumption.

Consequently, taking $\kappa$, $\nu$ small enough and $\lambda_1 = \lambda_1(\sigma,\kappa,j_1) > 0$ large enough, we conclude (2.2.22), which is our claim. □

The result is slightly different in the 3d-case.

**Theorem 2.2.2.** *For dimension $d=3$, let $\varepsilon>0$, $f$ such that $N(f)<\infty$ and assume that (2.1.3)-(2.1.5), (2.1.7) hold. Then there exists $\lambda_1>0$ so that for any $\lambda\geq\lambda_1$ the solution $u\in H_A^1(\mathbb{R}^d)$ of the Helmholtz equation (2.1.1) satisfies*

$$\lambda|||u|||^2 + |||\nabla_A u|||^2 + \int\frac{|\nabla_A^{\perp}u|^2}{|x|} + \sup_{R>0}\frac{1}{R^2}\int_{|x|=R}|u|^2 d\sigma_R \\ \leq C(1+\varepsilon)(N(f))^2, \tag{2.2.19}$$

*being $C$ independent of $\varepsilon$.*

*Proof.* The proof follows by the same method as in the $d > 3$ case. We will use the same multipliers as in the previous theorem fixing $M = 1/2$. The main difference is that when $d = 3$ we do not get the term related to $\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^3}$ on the left-hand side of the inequality. Therefore, it is not possible to estimate the magnetic term as in (2.2.8). This requires the assumption (2.1.7) on the magnetic field $B$. Thus in this case, using the same notation as in the previous theorem we obtain

$$\begin{aligned} B_1 &\leq \int_{|x|\leq r_1} |B_\tau||u||\nabla_A^\perp u| + \int_{r_1\leq|x|\leq r_0} |B_\tau||\nabla_A^\perp u||u| \\ &\leq \eta \left(\int_{|x|\leq r_1} \frac{|\nabla_A^\perp u|^2}{|x|}\right)^{1/2} \left(\sup_{R\leq r_1} \int_{|x|=R} |u|^2\right)^{1/2} \\ &+ \frac{1}{4}\int_{r_1\leq|x|\leq r_0} \frac{|\nabla_A^\perp u|^2}{|x|} + \sigma|||u|||^2. \end{aligned}$$

The rest of the proof runs as before. □

**Remark 2.2.3.** *Note that if we did not take $\lambda$ big enough, we would obtain*

$$\begin{aligned} |||\nabla_A u|||^2 + \int \frac{|\nabla_A^\perp u|^2}{|x|} + (d-3)\int \frac{|u|^2}{|x|^3} + \sup_{R>0}\frac{1}{R^2}\int_{|x|=R} |u|^2 d\sigma_R \\ \leq C(1+\varepsilon)\{|||u|||^2 + (N(f))^2\}. \end{aligned}$$

**Remark 2.2.4.** *If we denote*

$$m(x) = \frac{|V_1(x)|}{|x|} + (\partial_r V_1(x))_- + |B_\tau(x)| + |V_2(x)|, \qquad (2.2.20)$$

*in order to prove the previous two results, the requirement on $m$ is that*

$$\sum_{j\geq j_0} 2^j \sup_{|x|\sim 2^j} m(x) < +\infty. \qquad (2.2.21)$$

*The notation $|x| \sim 2^j$ means that $x \in C(j) = \{x \in \mathbb{R}^d : 2^{j-1} \leq |x| \leq 2^j\}$ and $j_0$ is such that $2^{j_0-1} \leq |x| \leq 2^{j_0}$.*

**Remark 2.2.5.** *Observe that from the above results it follows that*

$$\begin{aligned} \lambda|||u|||_1^2 + |||\nabla_A u|||_1^2 + \int \frac{|\nabla_A^\perp u|^2}{|x|} + \sup_{R>1}\frac{1}{R^2}\int_{|x|=R} |u|^2 d\sigma_R + \int \frac{|u|^2}{|x|^3} \\ \leq C(1+\varepsilon)(N_1(f))^2, \end{aligned} \qquad (2.2.22)$$

**Remark 2.2.6.** *Since singularities on the potentials at the origin are allowed, we reduce to the case $d \geq 3$. When $d = 1, 2$, the problems come from the terms (2.2.6) and (2.2.15). Similar results to those in [PV1], section 5 could be obtained for weaker singularities in dimension $d = 2$.*

### 2.2.2 Sommerfeld radiation condition

Our next goal is to quantify the Sommerfeld radiation condition proving that it is upper bounded by the Agmon-Hörmander norm of the solution. To this end, the basic idea is to build the full form of the Sommerfeld terms, using the integral identities proved in Lemmas 1.3.4 and 1.3.5. We will proceed analogously to the proof of Proposition 1.3.9. We emphasize that since the Sommerfeld condition is applied at infinity, it is sufficient to know the behavior of the potentials for $|x| \geq R$, $R$ big enough.

**Proposition 2.2.7.** *For $d \geq 3$, let $\lambda_0 > 0$, $\varepsilon > 0$, $f \in L^2_{\frac{1+\delta}{2}}$ and suppose that (2.1.3) holds. Then, there exists a positive constant $C = C(\lambda_0, r_0, \mu)$ such that for all $\lambda \geq \lambda_0$ the solution $u \in H^1_A(\mathbb{R}^d)$ of the equation (2.1.1) satisfies*

$$\int_{|x|\geq 1} \left| \nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u \right|^2 \left( \frac{1}{(1+|x|)^{1-\delta}} + \varepsilon(1+|x|)^\delta \right)$$
$$\leq C(1+\varepsilon)\left[ |||u|||_1^2 + (N_1(f))^2 + \int_{|x|\geq 1} (1+|x|)^{1+\delta}|f|^2 \right], \tag{2.2.23}$$

*for all $0 < \delta < 2$ such that $\delta < \mu$, where $\mu$ is given in Assumption 2.1.1.*

*Proof.* The proof consists in the construction of the squares of the left hand side of (2.2.23). We use a combination of the identities of the lemmas 1.3.4 and 1.3.5, following the ideas of the proof of Proposition 1.3.9. The main difference is that in this case we put a cut-off function in all test functions.

Let us denote $r = |x|$ and we define a radial function $\Psi : \mathbb{R}^d \to \mathbb{R}$ by

$$\Psi(x) = \int_0^{|x|} \Psi'(s)ds,$$

with

$$\Psi'(r) = (1+r)^\delta, \quad 0 < \delta < 2. \tag{2.2.24}$$

Let us consider a cut off function $\theta \in C^\infty(\mathbb{R})$ such that $0 \leq \theta \leq 1$, $d\theta/dr \geq 0$ with

$$\theta(r) = \begin{cases} 1 & \text{if } r \geq 2 \\ 0 & \text{if } r \leq 1, \end{cases}$$

and set $\theta_{r_0}(x) = \theta\left(\frac{|x|}{r_0}\right)$, where $r_0$ is given in Assumption 2.1.1.

Let us compute

$$(1.3.15) + \frac{1}{2}(1.3.12) + \lambda^{1/2}(1.3.13) - \frac{\varepsilon}{2\lambda^{1/2}}(1.3.12)$$

with the following choice of the multipliers

$$\psi'(x) = \Psi'(r)\theta_{r_0}(x)$$
$$\varphi(x) = \Psi''(r)\theta_{r_0}(x)$$
$$\varphi(x) = \Psi'(r)\theta_{r_0}(x)$$
$$\varphi(x) = \Psi'(r)\theta_{r_0}(x),$$

respectively.

Note that by (2.2.24) we have

$$\frac{\Psi'}{r} - \frac{\Psi''}{2} > \frac{(2-\delta)}{2\delta}\Psi''. \tag{2.2.25}$$

Thus since $0 < \delta < 2$, letting $\nu = \frac{2-\delta}{2\delta} > 0$ we obtain

$$\begin{aligned}
&\frac{\delta}{2}\int (1+|x|)^{\delta-1}|\nabla_A^r u - i\lambda^{1/2}u|^2\theta_{r_0} + \delta\nu\int (1+|x|^{\delta-1})|\nabla_A^\perp u|^2\theta_{r_0} \\
&+ \frac{1}{2}\int (1+|x|)^\delta \left(\frac{\theta'}{r_0}|\nabla_A^r u - i\lambda^{1/2}u|^2 + \frac{\varepsilon}{\lambda^{1/2}}\left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2\theta_{r_0}\right) \\
&\leq \Re\int \nabla\left(\frac{\Psi'\theta'}{r_0} + \frac{(d-1)\Psi'\theta_{r_0}}{|x|}\right)\cdot\nabla_A u\bar{u} - \frac{\varepsilon\Re}{2\lambda^{1/2}}\int \nabla(\Psi'\theta_{r_0})\cdot\nabla_A u\bar{u} \\
&+ \Im\int \Psi' B_\tau\cdot\nabla_A^\perp u\bar{u}\theta_{r_0} + \frac{1}{2}\int (\Psi'' V_1 + \partial_r V_1\Psi')|u|^2\theta_{r_0} \\
&+ \frac{1}{2}\int \left(\frac{(d-1)\Psi'\theta_{r_0}}{|x|} + \frac{\Psi'\theta'}{r_0}\right)V_2|u|^2 + \Re\int V_2\Psi'\nabla_A^r u\bar{u}\theta_{r_0} \\
&- \Re\int f\Psi'\left\{\left[\theta_{r_0}\left(\frac{d-1}{2|x|} + \frac{\varepsilon}{2\lambda^{1/2}}\right) + \frac{\theta'}{r_0}\right]\bar{u} + (\nabla_A^r\bar{u} + i\lambda^{1/2}\bar{u})\theta_{r_0}\right\}.
\end{aligned}$$

Let us now estimate the right hand-side of the above inequality applying similar arguments and using the same notation as in the proof of Theorem 2.2.1. Since

$$\Re\nabla_A^r u\bar{u} = \Re(\nabla_A^r u - i\lambda^{1/2}u)\bar{u} \tag{2.2.26}$$

and $\delta < 2$, the first term can be upper bounded

$$\kappa\int |\nabla_A^r u - i\lambda^{1/2}u|^2\left((1+|x|)^{\delta-1}\theta_{r_0} + \frac{(1+|x|)^\delta\theta'}{r_0}\right) + C(\kappa)|||u|||_1^2, \tag{2.2.27}$$

for any $\kappa > 0$. Concerning the $\varepsilon$ term, note that by integration by parts and the a-priori estimate (1.3.9), we have

$$-\frac{\varepsilon\Re}{2\lambda^{1/2}}\int\nabla(\Psi'\theta_{r_0})\cdot\nabla_A u\bar{u} = \frac{\varepsilon}{4\lambda^{1/2}}\int\Delta(\Psi'\theta_{r_0})|u|^2$$
$$\leq\frac{C\varepsilon}{\lambda^{1/2}}\int_{|x|\geq\frac{r_0}{2}}\frac{|u|^2}{(1+|x|)^{2-\delta}}$$
$$\leq\frac{C}{\lambda_0^{1/2}}N_1(f)|||u|||_1. \tag{2.2.28}$$

We now pass to the terms containing the potentials. By (2.1.3) it follows easily that for $\delta<\mu$ yields

$$\frac{1}{2}\int\left[(\Psi''V_1+\partial_r V_1\Psi')\theta_{r_0}+\frac{(d-1)\Psi'\theta_{r_0}}{|x|}V_2+\frac{\Psi'\theta'}{r_0}V_2\right]|u|^2\leq C|||u|||_1^2.$$

If moreover, we apply tha Cauchy-Schwarz inequality, then we get

$$\Im\int\Psi' B_\tau\cdot\nabla_A^\perp u\bar{u}\theta_{r_0}\leq C\left(\int|\nabla_A^\perp u|^2(1+|x|)^{\delta-1}\theta_{r_0}\right)^{1/2}|||u|||_1$$

and combining with (2.2.26), gives

$$\Re\int V_2\Psi'\nabla_A^r u\bar{u}\theta_{r_0}\leq C\left(\int|\nabla_A^r u-i\lambda^{1/2}u|^2(1+|x|)^{\delta-1}\theta_{r_0}\right)^{1/2}|||u|||_1.$$

Thus the potential terms can be estimated by

$$\kappa\int(1+|x|)^{\delta-1}\left(|\nabla_A^\perp u|^2+|\nabla_A^r u-i\lambda^{1/2}u|^2\right)\theta_{r_0}+C|||u|||_1^2. \tag{2.2.29}$$

Finally, applying the same reasoning to the terms containing $f$, we obtain that they are upper bounded by

$$\kappa\int(1+|x|)^{\delta-1}|\nabla_A^r u-i\lambda^{1/2}u|^2\theta_{r_0}+C(\kappa)\int(1+|x|)^{1+\delta}|f|^2\theta_{r_0}$$
$$+C|||u|||_1\left(\int(1+|x|)^{1+\delta}|f|^2\theta_{r_0}\right)^{1/2}$$
$$+\frac{C\varepsilon}{\lambda^{1/2}}|||u|||_1^{1/2}(N_1(f))^{1/2}\left(\int(1+|x|)^{1+\delta}|f|^2\theta_{r_0}\right)^{1/2}.$$

As a consequence, choosing $\kappa$ small enough, we deduce (2.2.23) and the proof is over.

□

**Remark 2.2.8.** *Note that if $m(x)$ was as in (2.2.20), it would be enough that the potentials satisfy*

$$\sum_{j\geq j_0} 2^j \sup_{x\in C(j)} m(x)\Psi'(|x|) < +\infty. \tag{2.2.30}$$

**Remark 2.2.9.** *Observe that the previous proof does not work neither for the $\delta = 0$ case, nor for the $\delta = 2$ case. When $\delta = 0$, $\Psi'(|x|) = 1$ and $\Psi''(|x|) = 0$. Then, we would not obtain the main square in the left hand side of the inequality. On the other hand, when $\delta = 2$, the problem comes from the term*

$$\Re \int \nabla \left( \frac{\Psi'\theta'}{r_0} + \frac{(d-1)\Psi'\theta_{r_0}}{|x|} \right) \cdot \nabla_A u\bar{u}. \tag{2.2.31}$$

*If $\Psi'(r) = (1+r)^2$ one needs to estimate the term $\int_{|x|\geq r_0/2} \frac{|u|^2}{|x|}$, which is not upper bounded by $|||u|||_1^2$. Moreover, we do not get the estimate for the tangential component of the magnetic gradient and thus we are not able to absorb the term containing the magnetic field. The $\delta = 2$ case is particularly interesting and needs special attention so that it will be studied in the last chapter.*

*As we mentioned in the previous section, the above result can be extended for a more general case which contains the $\delta = 0$ one. For this purpose, we will make more general assumptions on the potentials. According to the above Remark 2.2.4, it is required that*

$$m(x) \leq h(x) \quad for \quad |x| \geq r_0, \tag{2.2.32}$$

*where $h(x) = h(|x|)$ is such that*

$$\sum_{j\geq j_0} 2^j \sup_{|x|\sim 2^j} h(|x|) < +\infty. \tag{2.2.33}$$

*In particular,*

$$\int_{r_0}^{+\infty} h(r)dr < +\infty. \tag{2.2.34}$$

*From (2.2.33), one can deduce that there exists $g(x) = g(|x|)$ such that*

$$g(x) \to \infty \quad as \quad |x| \to \infty,$$

*and*

$$\sum_{j\geq j_0} 2^j \sup_{|x|\sim 2^j} h(|x|) \int_0^{|x|} \frac{g(r)}{r} dr < +\infty. \tag{2.2.35}$$

*It suffices to define*

$$F(x) = -\int_{|x|}^{+\infty} h(t)dt \tag{2.2.36}$$

*and choose*

$$g(x) = -ah(|x|)|x|\,(F(x))^{-a-1} \quad \textit{where} \quad 0 < a < 1. \tag{2.2.37}$$

*We will only consider* $g(|x|)$ *such that*

$$g(|x|) \leq 2(1-\nu)g_1(|x|), \quad 0 < \nu < 1, \tag{2.2.38}$$

*where*

$$g_1(|x|) = \int_0^{|x|} \frac{g(r)}{r} dr. \tag{2.2.39}$$

*We also require*

$$\sum_{j \geq j_0} 2^{-2j} \sup_{|x| \sim 2^j} g_1(|x|) < +\infty, \tag{2.2.40}$$

*which fails if* $g(|x|) = |x|^2$*. From this, we get*

$$\int_{|x| \geq r_0} \frac{g(|x|)}{|x|} |\nabla_A^r u - i\lambda^{1/2} \frac{x}{|x|} u|^2 + \varepsilon \int_{|x| \geq r_0} g_1(|x|) |\nabla_A u - i\lambda^{1/2} \frac{x}{|x|} u|^2$$
$$+ \nu \int_{|x| \geq r_0} \frac{|\nabla_A^\tau u|^2}{|x|} g_1(|x|) \leq C \int_{|x| \geq \frac{r_0}{2}} \frac{|x|(g_1(|x|))^2}{g(|x|)} |f|^2. \tag{2.2.41}$$

### 2.2.3 Compactness argument when $\lambda \in [\lambda_0, \lambda_1]$

Our next objective is to show that for any $\lambda \in [\lambda_0, \lambda_1]$,

$$\lambda |||u|||_1^2 \leq C(N_1(f))^2. \tag{2.2.42}$$

In order to get this estimate, we begin by proving the following a-priori estimate.

**Lemma 2.2.10.** *Under the hypotheses of Theorem 2.1.6, for each* $R > 0$ *any solution* $u \in H_A^1(\mathbb{R}^d)$ *of the equation (2.1.1) satisfies*

$$\int_{|x| \leq R} |\nabla_A u|^2 \leq C(1+\lambda) \int_{|x| \leq R+1} |u|^2 + \int_{|x| \leq R+1} |f|^2. \tag{2.2.43}$$

*Proof.* Let $\psi \in C_0^\infty$ such that $0 \leq \psi \leq 1$ and

$$\psi(x) = \begin{cases} 1 & \text{if } |x| \leq R, \\ 0 & \text{if } |x| \geq R+1. \end{cases} \tag{2.2.44}$$

Note that $u \in H_A^1(\mathbb{R}^d)$ satisfies

$$(\nabla_A^2 + V_1 + V_2 + \lambda + i\varepsilon)(\psi u) = \psi f + u\Delta\psi + 2\nabla_A u \cdot \nabla\psi. \tag{2.2.45}$$

Let us multiply the above identity by $\psi\bar{u}$, integrate over $\mathbb{R}^d$ and take the real part. Hence, by integration by parts we get

$$\int |\nabla_A(\psi u)|^2 \leq \lambda \int_{|x|\leq R+1} |u|^2 + \int |V_1+V_2||\psi u|^2 + \int_{|x|\leq R+1} |f||u| \\ + \int |\nabla_A(\psi u)||\nabla\psi||u|.$$

Now by the assumption (2.0.2) on the potentials $V_1, V_2$ and the diamagnetic inequality (1.1.4) we have

$$\int |V_1+V_2||\psi u|^2 < \int |\nabla_A(\psi u)|^2.$$

Hence, by Cauchy-Schwarz inequality it follows that

$$\int |\nabla_A(\psi u)|^2 \leq C(1+\lambda) \int_{|x|\leq R+1} |u|^2 + \int_{|x|\leq R+1} |f|^2, \tag{2.2.46}$$

which gives (2.2.43) and the lemma follows.

□

**Remark 2.2.11.** *Note that since*

$$\int |\nabla(\psi u)|^2 \leq C \int (|\nabla_A(\psi u)|^2 + |A\psi u|^2), \tag{2.2.47}$$

*applying the condition (1.1.5) on the magnetic potential $A$ to $|u|$, then by the diamagnetic inequality (1.1.4) it follows that*

$$\int |\nabla(\psi u)|^2 \leq C \int |\nabla_A(\psi u)|^2. \tag{2.2.48}$$

*This combined with (2.2.46) gives the well known elliptic a-priori estimate*

$$\int_{|x|\leq R} |\nabla u|^2 \leq C(1+\lambda) \int_{|x|\leq R+1} |u|^2 + \int_{|x|\leq R+1} |f|^2 \tag{2.2.49}$$

*for solutions of the equation (2.1.1).*

Having disposed of this preliminary step, we can return to show (2.2.42).

**Proposition 2.2.12.** *For $d \geq 3$, under the assumptions of Proposition 2.2.7 above, let $\lambda_0 > 0$, $\lambda \in [\lambda_0, \lambda_1]$, with $\lambda_1 > \lambda_0$ and $\varepsilon \in (0, \varepsilon_1)$. Then, the solution of the equation (2.1.1) satisfies*

$$\lambda |||u|||_1^2 + |||\nabla_A u|||_1^2 \leq C(1+\varepsilon)(N_1(f))^2, \tag{2.2.50}$$

*where $C$ is independent of $\varepsilon$.*

*Proof.* Our proof starts recalling that

$$\left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2 = |\nabla_A u|^2 + \lambda|u|^2 - 2\Im\lambda^{1/2}\frac{x}{|x|}\cdot\nabla_A u\bar{u}.$$

Let us integrate the above identity over the sphere $S_r := \{|x| = r\}$, obtaining

$$\begin{aligned}\int_{S_r}(\lambda|u|^2 + |\nabla_A u|^2)d\sigma_r = &\int_{S_r}|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u|^2 d\sigma_r \\ &+ 2\Im\lambda^{1/2}\int_{S_r}\frac{x}{|x|}\cdot\nabla_A u\bar{u}d\sigma_r.\end{aligned} \tag{2.2.51}$$

Let us multiply now equation (2.1.1) by $\bar{u}$, integrate it over the ball $B_r := \{|x| \le r\}$ and take the imaginary part. Since $\varepsilon > 0$, it follows that

$$\Im\int_{S_r}\frac{x}{|x|}\cdot\nabla_A u\bar{u}d\sigma_r \le \Im\int_{B_r}f\bar{u}. \tag{2.2.52}$$

Combining this with (2.2.51) yields

$$\int_{S_r}(\lambda|u|^2 + |\nabla_A u|^2)d\sigma_r \le \int_{S_r}|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u|^2 d\sigma_r + 2\Im\lambda^{1/2}\int_{B_r}f\bar{u}. \tag{2.2.53}$$

Now, let $R > \rho \ge r_0$ and denote $j_0$ and $j_1$ by $2^{j_0-1} < \rho < 2^{j_0}$ and $2^{j_1-1} < R < 2^{j_1}$, respectively. Let us multiply both sides of (2.2.53) by $\frac{1}{R}$ and integrate from $\rho$ to $R$ with respect to $r$. Then we have

$$\begin{aligned}\frac{1}{R}\int_{\rho\le|x|\le R}(\lambda|u|^2 + |\nabla_A u|^2) \le &\frac{1}{R}\sum_{j=j_0}^{j_1}\int_{C(j)}\left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2 \\ &+ \kappa\lambda|||u|||_1^2 + C(\kappa)(N_1(f))^2 \\ \equiv &\, I_1 + I_2,\end{aligned} \tag{2.2.54}$$

for $\kappa > 0$ and by (2.2.23) we get

$$\begin{aligned}I_1 &\le \frac{1}{R}\sum_{j=j_0}^{j_1}(1+2^j)^{1-\delta}\int_{C(j)}\frac{1}{(1+2^j)^{1-\delta}}\left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2 \\ &\le C(1+\varepsilon)\sum_{j=j_0}^{j_1}\frac{(1+2^j)^{1-\delta}}{2^{j_1}}(|||u|||_1^2 + (N_1(f))^2) \\ &+ C(1+\varepsilon)\sum_{j=j_0}^{j_1}\frac{(1+2^j)^{1-\delta}}{2^{j_1}}\int_{2^j\ge\frac{r_0}{2}}(1+2^j)^{1+\delta}|f|^2 \\ &\le C(1+\varepsilon)\left[\sum_{j=j_0}^{j_1}2^{-\delta j}|||u|||_1^2 + \left(1 + \sum_{j=j_0}^{j_1}\frac{1+2^j}{2^{j_1}}\right)(N_1(f))^2\right].\end{aligned} \tag{2.2.55}$$

As a consequence, from (2.2.54) and (2.2.55), taking $\kappa$ small enough and $\rho$ big enough, we deduce

$$\frac{1}{R}\int_{\rho\le|x|\le R}(\lambda|u|^2+|\nabla_A u|^2)\le\frac{\lambda}{2}|||u|||_1^2+C(1+\varepsilon)(N_1(f))^2.$$

It remains to prove that

$$\int_{|x|\le\rho}(\lambda|u|^2+|\nabla_A u|^2)\le C(N(f))^2. \tag{2.2.56}$$

Let us assume that (2.2.56) is false. Then, for each $n\in\mathbb{N}$, there exist $\varepsilon_n\in(0,\varepsilon_1)$ with $0<\varepsilon_1<\infty$, $\lambda_n\in[\lambda_0,\lambda_1]$ and $u_n, f_n$ such that

$$(\nabla+iA)^2u_n+(V_1+V_2)u_n+\lambda_n u_n+i\varepsilon_n u_n=f_n, \tag{2.2.57}$$

with

$$\int_{|x|\le\rho}(\lambda_n|u_n|^2+|\nabla_A u_n|^2)=1 \tag{2.2.58}$$

and

$$N_1(f_n)\le\frac{1}{n}\qquad\left(\lim_{n\to\infty}N_1(f_n)=0\right). \tag{2.2.59}$$

Since $\lambda_n\in[\lambda_0,\lambda_1]$ and $\varepsilon_n\in(0,\varepsilon_1)$, we may assume with no loss of generality that $\lambda_n\to\lambda^0$ and $\varepsilon_n\to\varepsilon^0$ where $\lambda^0\in[\lambda_0,\lambda_1]$, $\varepsilon^0\in[0,\varepsilon_1]$, as $n$ tends to $\infty$.

On the other hand, from (2.2.58) and condition (1.1.5) on $A$, one can easily deduce that $\{u_n\}$ is a bounded sequence in $H^1_{loc}$. Hence, by the Rellich-Kondrachov theorem, one can conclude that there exists a subsequence of $u_n$, $u_{n_p}$, such that

$$u_{n_p}\to u\quad\text{in}\quad L^2_{loc},\quad\text{as}\quad p\to\infty,\quad\text{with}\quad u\in L^2_{loc}, \tag{2.2.60}$$

which implies

$$\sup_{R>1}\frac{1}{R}\int_{|x|\le R}|u_{n_p}-u|^2dx\qquad\to\qquad 0. \tag{2.2.61}$$

Moreover, from Lemma 2.2.10, if we denote $v_n=u_{n_p}-u$, noting that

$$\begin{aligned}g_p&\equiv(H_A+\lambda+i\varepsilon)v_n\\&=i(\varepsilon^0-\varepsilon)u+(\lambda^0-\lambda)u-f_n\quad\to\quad 0\qquad\text{in}\qquad L^2_{loc}\end{aligned} \tag{2.2.62}$$

as $p\to\infty$, one can also deduce

$$\int_{|x|\le R}|\nabla_A v_n|^2\le C(1+\lambda)\int_{|x|\le R+1}|v_n|^2+\int_{|x|\le R+1}|g_p|^2.$$

Hence,

$$\nabla_A u_{n_p}\to\nabla_A u\quad\text{in}\quad L^2_{loc},\quad\text{as}\quad p\to\infty,\quad\text{with}\quad\nabla_A u\in L^2_{loc}. \tag{2.2.63}$$

As a consequence, by (2.2.58) $u$ satisfies

$$\int_{|x|\le\rho}(\lambda|u|^2+|\nabla_A u|^2)=1. \tag{2.2.64}$$

In addition, it follows that

$$(u,(L+\lambda+i\varepsilon^0)\varphi)=(0,\varphi)\quad \forall\varphi\in C_0^\infty \tag{2.2.65}$$

and therefore, $u$ also satisfies

$$(\nabla+iA)^2u+(V_1+V_2)u+\lambda^0u+i\varepsilon^0u=0 \tag{2.2.66}$$

in the distributional sense. Thus by uniqueness of solution of the equation (2.2.66), we conclude that $u\equiv 0$, which contradicts (2.2.64).

We have thus proved that for $R>1$

$$\frac{1}{R}\int_{|x|\le R}(\lambda|u|^2+|\nabla_A u|^2)\le\frac{\lambda}{2}|||u|||_1^2+C(1+\varepsilon)(N_1(f))^2.$$

Taking the supremum over R, we get (2.2.50) and the proof is complete.

□

**Remark 2.2.13.** *The same reasoning applies to the more general case mentioned in Remark 2.2.9, using the Sommerfeld radiation condition (2.2.41). The details are omitted.*

### 2.2.4 Limiting absorption principle

Our next concern will be the existence of solution of the equation (2.0.4), which is stated in the following lemma.

**Lemma 2.2.14.** *Let $\lambda>0$, $\{u_n\}$ be a sequence such that for any $\rho>0$*

$$\int_{|x|\le\rho}(\lambda|u_n|^2+|\nabla_A u_n|^2)<+\infty \tag{2.2.67}$$

*and let $\varepsilon_n\in(0,1)$ be a convergent sequence with $\varepsilon_n\to 0$ as $n\to\infty$, $f$ such that $N(f)<\infty$. Assume that*

$$(H_A+\lambda+i\varepsilon_n)u_n=f$$

*and $\{u_n\}$ satisfies the radiation condition*

$$\int_{|x|\ge 1}\left|\nabla_A u_n-i\lambda^{1/2}\frac{x}{|x|}u_n\right|^2(1+|x|)^{\delta-1}<+\infty \tag{2.2.68}$$

*for some $\delta > 0$ and for all $n = 1, 2, \ldots$. Then, $\{u_n\}$ has a strong limit $u$ in $(H^1_A)_{loc}$ such that satisfies*

$$\begin{aligned}
&(H_A + \lambda)u = f \\
&\int_{|x|\le\rho} (\lambda|u|^2 + |\nabla_A u|^2) < +\infty \\
&\int_{|x|\ge 1} \left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2 (1+|x|)^{\delta-1} < +\infty,
\end{aligned}$$

*for $\delta > 0$.*

*Proof.* This follows in much the same way as in the proof of Proposition 2.2.12 by the compactness argument. Since $\varepsilon_n \to 0$ as $n \to \infty$, the same reasoning applies to this case and we deduce that there exists a subsequence of $u_n$, $u_{n_p}$, such that $u_{n_p} \to u$ in $(H^1_A)_{loc}$ as $p \to \infty$ where $u \in (H^1_A)_{loc}$ and satisfies

$$\begin{aligned}
&(\nabla + iA)^2 u + (V_1 + V_2)u + \lambda u = f, \\
&\int_{|x|\le\rho} (\lambda|u|^2 + |\nabla_A u|^2) < \infty.
\end{aligned}$$

In addition, if we denote $\mathcal{D}u = \nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u$, we also get that $\mathcal{D}u_{n_p}$ converges to $\mathcal{D}u$ in $L^2(E_1)_{loc}$, where $E_1 = \{|x| \ge 1\}$. As a consequence, we obtain $\mathcal{D}u_{n_p} \to \mathcal{D}u$ in $L^2_{\frac{\delta-1}{2}}(E_1)$ satisfying

$$\int_{|x|\ge 1} \left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2 (1+|x|)^{\delta-1} < \infty. \tag{2.2.69}$$

Finally, we shall show that the sequence $\{u_n\}$ itself converges in $(H^1_A)_{loc}$ to the $u$ obtained above, which in turn implies that $\{\mathcal{D}u_n\}$ converges to $\{\mathcal{D}u\}$ in $L^2_{loc}(E_1)$. In fact, let us assume that there exists a subsequence $\{n_q\}$ of $\{n\}$ such that

$$\|u - u_{n_q}\|_{L^2_{loc}} + \|\nabla_A u - \nabla_A u_{n_q}\|_{L^2_{loc}} \ge \gamma \quad (q = 1, 2, \ldots) \tag{2.2.70}$$

with some $\gamma > 0$. Then, proceeding as above, we can find a subsequence $\{n'_q\}$ of $\{n_q\}$ which satisfies

$$u_{n'_q} \to u' \quad \text{in} \quad (H^1_A)_{loc}, \tag{2.2.71}$$

$u'$ being a solution in $(H^1_A)_{loc}$ of

$$\nabla_A^2 u' + \lambda u' + (V_1 + V_2)u' = f \tag{2.2.72}$$

such that $\int_{|x|\geq 1}\left|\nabla_A u' - i\lambda^{1/2}\frac{x}{|x|}u'\right|^2 (1+|x|)^{\delta-1} < +\infty$. Finally, by Theorem 2.1.5 we assert that $u'$ obtained above is unique which implies that $u$ and $u'$ must coincide. Hence, from (2.2.71) it follows that

$$u_{n_q} \to u \quad \text{in} \quad (H_A^1)_{loc},$$

which contradicts (2.2.70). Thus $\{u_n\}$ converges to $u$ in $(H_A^1)_{loc}$ and the lemma follows. □

Finally, the preceding lemma together with the uniqueness result for (2.0.4) (Theorem 2.1.5) allows us to construct the unique solution $u = u(\lambda, f)$ as the limit of a sequence of solutions $\{u_n = u(\lambda + i\varepsilon_n, f)\}$ $(\varepsilon_n \to 0)$ obtained above.

**Theorem 2.2.15** (Limiting absorption principle)**.** *Under the assumptions of Theorem 2.1.6, let* $\{\varepsilon_n\} \subset (0,1)$ *be a sequence tending to* $0$*. Let* $u_n = u(\lambda + i\varepsilon_n, f)$*. Then* $\{u_n\}$ *converges in* $(H_A^1)_{loc}$ *to a* $u$ *such that*

$$\lambda|||u|||_1^2 + |||\nabla_A u|||_1^2 \leq C(N_1(f))^2, \tag{2.2.73}$$

*where* $C = C(\lambda_0) > 0$ *and solves* $(H_A + \lambda)u = f$*.*

*The limit* $u = u(\lambda, f)$ *is independent of the choice of the sequence* $\{\varepsilon_n\}$ *and is determined as the unique solution of the equation (2.0.4) that satisfies the Sommerfeld radiation condition*

$$\int_{|x|\geq 1}(1+|x|)^{\delta-1}\left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2 \leq C\int (1+|x|)^{1+\delta}|f|^2,$$

*for any* $0 < \delta < 2$*, being* $C = C(\lambda_0) > 0$*.*

*Proof.* Let $f \in L^2_{\frac{1+\delta}{2}}$. Take $\{\varepsilon_n\} \subset (0,1)$ such that $\varepsilon_n \to 0$ as $n \to \infty$. We know that there exists a unique solution $u_n \in H_A^1$ of the equation $(L + \lambda + i\varepsilon_n)u = f$ satisfying

$$\begin{aligned} &\lambda|||u_n|||_1^2 + |||\nabla_A u_n|||_1^2 \leq C(\varepsilon_n + 1)(N_1(f))^2 \\ &\|\mathcal{D}u_n\|_{L^2_{\frac{\delta-1}{2}}(E_1)} \leq C\|f\|^2_{\frac{1+\delta}{2}} \end{aligned}$$

for all $n = 1, 2, \ldots$, where $\mathcal{D}u = \nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u$ and $E_1 = \{|x| \geq 1\}$. Then one can see from Lemma 2.2.14 that $\{u_n\}$ has a strong limit in $(H_A^1)_{loc}$ which is a solution of the equation $(L + \lambda)u = f$ and it is easy to check that satisfies

$$\lambda|||u|||_1^2 + |||\nabla_A u|||_1^2 \leq C(N_1(f))^2 \tag{2.2.74}$$

$$\|\mathcal{D}u\|_{L^2_{\frac{\delta-1}{2}}(E_1)} \leq C\|f\|^2_{\frac{1+\delta}{2}}. \tag{2.2.75}$$

By the uniqueness result (see Theorem 2.1.5), it follows that the $u$ obtained above is a unique solution of $(L + \lambda)u = f$ satisfying (2.2.75) and the proof is complete. □

## 2.3 Proof of Theorem 2.1.5

The proof is based on multiplier method and integration by parts. It will be divided into three steps.

Let $R > r_0 \geq 1$, $r_0$ being as in Assumption 2.1.1. Our first goal is to show that there exists $\mu > 0$ such that

$$\int_{|x|>R} (|\nabla_A u|^2 + |u|^2) \leq \frac{C}{R^{1+\mu}} \int_{\frac{R}{2}\leq|x|\leq R} |u|^2. \tag{2.3.1}$$

For this purpose, we multiply the equation (2.0.6) by

$$\nabla\psi \cdot \overline{\nabla_A u} + \frac{1}{2}\Delta\psi\bar{u} + \varphi\bar{u}, \tag{2.3.2}$$

where $\psi, \varphi$ are regular radial real-valued functions, and we integrate it over the ball $\{|x| < R_1\}$, being $R_1 > R$. Applying similar arguments as in the proofs of Lemmas 1.3.4 and 1.3.5 we get

$$\begin{aligned}
&\int_{|x|<R_1} \nabla_A u \cdot D^2\psi \cdot \overline{\nabla_A u} - \int_{|x|<R_1} \varphi|\nabla_A u|^2 + \int_{|x|<R_1} \varphi\lambda|u|^2 \\
&= \frac{1}{4}\int_{|x|<R_1} (\Delta^2\psi - 2\Delta\varphi)|u|^2 - \int_{|x|<R_1} \varphi V_1|u|^2 - \int_{|x|<R_1} \varphi V_2|u|^2 \\
&- \frac{1}{2}\int_{|x|<R_1} \nabla V_1 \cdot \nabla\psi|u|^2 + \Im \sum_{k,m=1}^{d} \int_{|x|<R_1} \frac{\partial\psi}{\partial x_k} B_{km}(\nabla_A)_m u\bar{u} \\
&+ \frac{1}{2}\int_{|x|<R_1} V_2\Delta\psi|u|^2 + \Re\int_{|x|<R_1} V_2\nabla\psi \cdot \nabla_A u\bar{u} - \Re\int_{|x|=R_1} \nabla_A^r u\varphi\bar{u} \\
&+ \frac{1}{4}\int_{|x|=R_1} (\nabla(\Delta\psi) - 2\nabla\varphi) \cdot \frac{x}{|x|}|u|^2 + \frac{1}{2}\Re\int_{|x|=R_1} \overline{\nabla_A^r u}\Delta\psi u \\
&- \frac{1}{2}\int_{|x|=R_1} \frac{x}{|x|} \cdot \nabla\psi|\nabla_A u|^2 + \frac{1}{2}\int_{|x|=R_1} (\lambda + V_1)\frac{x}{|x|} \cdot \nabla\psi|u|^2.
\end{aligned} \tag{2.3.3}$$

Let us consider a cut off function $\theta$ with

$$\theta(r) = \begin{cases} 1 & \text{if } r \geq 1 \\ 0 & \text{if } r < \frac{1}{2}, \end{cases}$$

$\theta' \geq 0$ for all $r$, and set $\theta_R(x) = \theta\left(\frac{|x|}{R}\right)$. Then, for $R$ such that $\frac{R}{2} > r_0 \geq 1$ and $R < R_1$ we define the multiplier $\psi$ such that

$$\nabla\psi(x) = \frac{x}{R}\theta_R(x) \tag{2.3.4}$$

and $\varphi$ by

$$\varphi(x) = \frac{1}{2R}\theta_R(x). \tag{2.3.5}$$

Let us insert (2.3.4) and (2.3.5) into the identity (2.3.3). Hence, by (1.3.22) the left-hand side can be lower bounded by

$$\begin{aligned}\int_{|x|<R_1} \nabla_A u \cdot D^2\psi \cdot \overline{\nabla_A u} - \int_{|x|<R_1} \varphi|\nabla_A u|^2 + \int_{|x|<R_1} \varphi\lambda|u|^2 \\ > \frac{1}{2R}\int_{|x|<R_1} \left(|\nabla_A u|^2 + \lambda|u|^2\right)\theta_R.\end{aligned} \tag{2.3.6}$$

Regarding to the right-hand side of (2.3.3), first note that

$$\frac{1}{4}\int_{|x|<R_1} (\Delta^2\psi - 2\Delta\varphi)|u|^2 \leq \frac{C}{R^3}\int_{\frac{R}{2}<|x|<R} |u|^2. \tag{2.3.7}$$

In order to analyze the terms containing the potentials, here and subsequently, we will use $\eta = \eta(R)$ to denote a positive constant depending on $R$ that tends to 0 as $R$ tends to infinity. Thus by (2.1.3) and the Cauchy-Schwarz inequality we have

$$\begin{aligned}\Im\sum_{j,k=1}^{d}\int_{|x|<R_1} \frac{\partial\psi}{\partial x_k}B_{km}(\nabla_A)_m u\bar{u} &\leq \int_{|x|<R_1} \frac{|B_\tau||x|}{R}|u||\nabla_A u| \\ &\leq \sum_{j=j_1}^{j_2} 2^{-j\mu}\int_{|x|<R_1} \theta_R|u||\nabla_A u| \\ &\leq \eta(R)\int_{|x|<R_1} (|u|^2 + |\nabla_A u|^2)\theta_R.\end{aligned}$$

Similarly,

$$\Re\int_{|x|<R_1} V_2\nabla\psi\cdot\nabla_A u\bar{u} \leq \eta(R)\int_{|x|<R_1} (|u|^2 + |\nabla_A u|^2)\theta_R, \tag{2.3.8}$$

$$-\int_{|x|<R_1} \left(\frac{\nabla V_1\cdot\nabla\psi}{2} + \varphi V_1\right)|u|^2 \leq \eta(R)\int_{|x|<R_1} |u|^2\theta_R. \tag{2.3.9}$$

Finally, since $supp\,\theta'_R \subset \{\frac{R}{2} < |x| < R\}$, yields

$$\begin{aligned}\int_{|x|<R_1} \left(\frac{\Delta\psi}{2} - \varphi\right)V_2|u|^2 \leq \eta(R)\int_{|x|<R_1} |u|^2\theta_R \\ + \frac{c}{2R^{2+\mu}}\int_{\frac{R}{2}<|x|<R} |u|^2.\end{aligned} \tag{2.3.10}$$

Let us analyze now the surface integrals of the equality (2.3.3). An easy computation shows that by (2.3.4), (2.3.5) and condition (2.1.3) applying to $V_1$, the boundary terms are upper bounded by

$$\frac{1}{R}\int_{|x|=R_1}|u||\nabla_A^r u| + \frac{1}{2}\int_{|x|=R_1}(|\nabla_A u|^2+\lambda|u|^2) + \frac{1}{2R_1^{\mu}}\int_{|x|=R_1}|u|^2. \tag{2.3.11}$$

As a consequence, from (2.3.6)-(2.3.11) yields

$$\begin{aligned}\frac{1}{2R}\int_{|x|<R_1}(|\nabla_A u|^2+\lambda|u|^2)\theta_R &\le \eta(R)\int_{|x|<R_1}(|u|^2+|\nabla_A u|^2)\theta_R\\ &+\frac{C}{R^{2+\mu}}\int_{\frac{R}{2}<|x|<R}|u|^2\\ &+C(\lambda_0)\int_{|x|=R_1}\left\{|\nabla_A u|^2+\lambda|u|^2)\right\}.\end{aligned}$$

Now, taking $R$ large enough such that

$$\frac{\min(1,\lambda)}{2}-\eta(R)>0, \tag{2.3.12}$$

it follows that

$$\begin{aligned}\frac{1}{R}\int_{R<|x|<R_1}(|\nabla_A u|^2+|u|^2) &\le \frac{C}{R^{2+\mu}}\int_{\frac{R}{2}<|x|<R}|u|^2\\ &+C\int_{S_{R_1}}(|\nabla_A u|^2+\lambda|u|^2).\end{aligned} \tag{2.3.13}$$

Letting $R_1\to\infty$ in the above inequality, by (2.1.12) we conclude that (2.3.1), which is our claim.

Our next step is to prove that for $R>r_0\ge 1$ and any $m\ge 0$, then

$$\int_{|x|>R}|x|^m(|\nabla_A u|^2+|u|^2)<+\infty. \tag{2.3.14}$$

We do it by induction. Let $\gamma=1+\mu$ and first note that from the first step one can easily deduce that for any $R\ge 1$ holds

$$\begin{aligned}\int_{|x|\ge 2R}|x|^{\gamma}(|u|^2+|\nabla_A u|^2) &\le \sum_{j\ge J}(2^{j\gamma})\int_{2^{j-1}\le|x|\le 2^j}(|u|^2+|\nabla_A u|^2)\\ &\le C\sum_{j\ge J}\int_{2^{j-2}\le|x|\le 2^{j-1}}|u|^2 \le C\int_{|x|\ge R}(|u|^2+|\nabla_A u|^2)\\ &\le \frac{C}{R^{\gamma}}\int_{\frac{R}{2}\le|x|\le R}|u|^2,\end{aligned}$$

being $J$ such that $2^{J-1} \leq 2R \leq 2^J$. The same conclusion can be drawn for any $m \geq 0$. Indeed, assuming that

$$\int_{|x|\geq R} |x|^m(|u|^2 + |\nabla_A u|^2) \leq \frac{C}{R^{1+\mu}} \int_{\frac{R}{2}\leq|x|\leq R} |u|^2, \tag{2.3.15}$$

it follows that (2.3.15) is true when $m$ is replaced by $m+\gamma$. Thus we obtain (2.3.14).

We next claim the exponential decay. Let us multiply again the equation (2.0.6) by (2.3.2), but instead of integrating over a ball, we do it over the whole $\mathbb{R}^d$. Note that this is equivalent to adding the identities (1.3.12) and (1.3.15) with $f=0$. Thus we get the identity (2.2.3) with the right-hand side equals to 0. Let us now choose the multipliers as

$$\nabla\psi(x) = |x|^{m+1}\frac{x}{|x|}\theta_R(x),$$

$$\varphi(x) = \frac{1}{2}|x|^m\theta_R(x),$$

for $R \geq 2r_0 \geq 1$, $m \geq 1$ and $\theta_R$ being as above.
For simplicity of notation, we continue to write $\eta = \eta(R)$ for a function depending on $R$ such that $\eta(R) \to 0$ as $R \to \infty$. Thus analysis similar to that in the first step shows that taking $R$ large enough such that

$$\frac{\min\{1,\lambda\}}{2} - \eta(R) > 0, \tag{2.3.16}$$

we get

$$\begin{aligned}\int |x|^m(|\nabla_A u|^2 + |u|^2)\theta_R \leq &\int \left(\eta(R)m|x|^{m-1} + Cm^3|x|^{m-2}\right)|u|^2\theta_R \\ &+ \left(\frac{Cm^2}{R^2} + \frac{c}{2R^{1+\mu}}\right)\int_{\frac{R}{2}<|x|<R} |x|^m|u|^2.\end{aligned}$$

Let us take now $m = \delta l$ with $0 < \delta < 2/3$ and multiply both sides of the above inequality by $\frac{t^l}{l!}$, $t \geq 1$ and $l \geq 3$. Making the sum with respect to l from 3 to $\infty$ we have

$$\begin{aligned}&\left(1 - \frac{2t}{3}R^{\delta-1}\eta(R) - \frac{9}{2}R^{3\delta-2}t^3\right)\int e^{|x|^\delta t}(|\nabla_A u|^2 + |u|^2)\theta_R \\ &\leq \int (|\nabla_A u|^2 + |u|^2)\left(1 + t|x|^\delta + \frac{t^2}{2}|x|^{2\delta}\right)\theta_R \\ &+ \left(CR^{2(\delta-1)}t^2 + \frac{c}{2R^{1+\mu}}\right)\int_{\frac{R}{2}<|x|<R} e^{|x|^\delta t}|u|^2. \end{aligned}\tag{2.3.17}$$

Fix $t \geq 1$ and $0 < \delta < \frac{2}{3}$. Then, for sufficiently large $R = R(t)$ such that

$$\frac{2t}{3} R^{\delta-1}\eta(R) + \frac{9}{2} t^3 R^{3\delta-2} < 1,$$

by (2.3.14) we conclude that

$$\int_{|x|>R} e^{|x|^\delta t}(|\nabla_A u|^2 + |u|^2) < +\infty. \tag{2.3.18}$$

Therefore,

$$\int e^{|x|^\delta t}(|\nabla_A u|^2 + |u|^2) < +\infty \tag{2.3.19}$$

We are now in a position to show that $u = 0$ almost everywhere in $\{|x| > 2R\}$. Set $v = e^{t|x|^\delta/2}u$ with $t \geq 1$ and $0 < \delta < 2/3$. Then, by a direct computation $v$ satisfies the equation

$$\begin{aligned} &\nabla_A^2 v + [\lambda + V_1 + V_2]v - \delta t |x|^{\delta-1}\frac{x}{|x|}\cdot\nabla_A v \\ &+ \left[\frac{\delta^2 t^2 |x|^{2(\delta-1)}}{4} - \frac{\delta(\delta+d-2)t|x|^{\delta-2}}{2}\right] v = 0. \end{aligned} \tag{2.3.20}$$

We multiply (2.3.20) by

$$|x|\frac{x}{|x|}\cdot\overline{\nabla_A v} + \frac{d-1}{2}\bar{v}$$

(the combination of the symmetric and the antisymmetric multipliers, (2.3.2) with $\nabla\psi = x$ and $\varphi = -1/2$), integrate it over $\{|x| > R\}$ for some $R > 2r_0$ and take the real part. Hence, it follows that

$$\begin{aligned} &\frac{\min\{1,\lambda\}}{2}\int_{|x|>R}(|\nabla_A v|^2 + |v|^2) + \frac{(2\delta-1)\delta^2 t^2}{4}\int_{|x|>R}|x|^{2\delta-2}|v|^2 \\ &+ \delta t\int_{|x|>R}|x|^\delta\left|\nabla_A^r v\right|^2 \leq \frac{\delta t(d+\delta-2)}{2}\left(\frac{3d-5}{2}+\delta\right)\int_{|x|>R}|x|^{\delta-2}|v|^2 \\ &+ \eta(R)\int_{|x|>R}(|v|^2 + |\nabla_A v|^2) + \frac{1}{2}\int_{S_R}\lambda|x||v|^2 \\ &+ \left(\frac{d-1}{4} + \frac{R}{2} + \eta(R) + \frac{\delta^2 t^2 R^{2\delta-1}}{8}\right)\int_{S_R}(|v|^2 + |\nabla_A v|^2). \end{aligned}$$

Consequently, combining the right-hand side of the above inequality with the left-hand side, for $R$ large enough and for any $t \geq 1$, $0 < \delta < 2/3$, $\lambda \geq \lambda_0$, it follows that

$$\int_{|x|\geq R}|v|^2 \leq C_\delta\left(t^2 + R(1+\lambda)\right)\int_{S_R}(|v|^2 + |\nabla_A v|^2), \tag{2.3.21}$$

which implies

$$\int_{|x|>2R} |u|^2 \leq C_\delta e^{-tR^\delta}\left(1+\lambda+\frac{t^2}{R}\right), \tag{2.3.22}$$

being $C_\delta$ independent of $t$. Thus letting $t \to \infty$, we obtain that $u = 0$ almost everywhere in $\{|x| > 2R\}$. The unique continuation property ([R]) implies then $u = 0$ almost everywhere in $\mathbb{R}^d$.

Finally assume that the Sommerfeld radiation condition (2.1.15) holds. Moreover, observe that solutions of (2.0.6) satisfy (just multiply by $\bar{u}$ and integrate over a ball of radius R),

$$\Im \int_{|x|=R} \frac{x}{|x|} \cdot \nabla_A u\bar{u} = 0.$$

Hence, we have

$$\int_{|x|=R} (|\nabla_A u|^2 + \lambda |u|^2) d\sigma(x) = \int_{|x|=R} \left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2 d\sigma(x),$$

which together with (2.1.15) establishes (2.1.12). The proof of the theorem is complete.

## 2.4 Appendix A: More singularities on the potentials

In this section it is shown that the main result is still true if we consider singularities on the potential $V_1$. Nevertheless, we will see that more singularity on the potential $V_2$ breaks the uniformity of the Morrey-Campanato estimates on $\lambda$. We will restrict our attention to the case $d > 3$.

### 2.4.1 More singularities for $V_2$

As we have mentioned in remark 2.1.3, one can allow more singularities on the potential $V_2$. In fact, one may require

$$|V_2(x)| \leq \frac{\mathcal{C}^{**}}{|x|^2}, \quad \text{if} \quad |x| \leq r_0, \tag{2.4.1}$$

for sufficiently small $\mathcal{C}^{**} > 0$. However, in this case the constant $\mathcal{C}^{**}$ will depend on $\lambda$ and the uniformity of the estimate (2.2.22) on $\lambda$ for $\lambda \geq \lambda_1$ breaks down.

To see this, we follow the proof of Theorem 2.2.1 (with the same notation) and observe that the only problem appears in the analysis of the term $\Re \int V_2 \nabla\psi \cdot \nabla_A u\bar{u}$. In this case, we need the following a-priori estimate.

**Lemma 2.4.1.** *Assume that (2.1.3), (2.1.4), (2.1.8) hold. Then the solution* $u \in H^1_A(\mathbb{R}^d)$ *of the Helmholtz equation (2.1.1) satisfies*

$$\int_{|x|\le r_0} \frac{|\nabla_A u|^2}{|x|} \le \left(\lambda + \mathcal{C}^{**} + \frac{d-3}{2} + \kappa\right) \int_{|x|\le r_0} \frac{|u|^2}{|x|^3} + \sigma|||u|||^2 \\ + C(N(f))^2, \tag{2.4.2}$$

*where* $\sigma > 0$, $C > 0$ *are independent of* $\lambda$ *and* $\varepsilon$.

*Proof.* It suffices to define the test function $\varphi$ as

$$\varphi(x) = \begin{cases} \frac{1}{|x|} & \text{if } |x| \le r_0, \\ 0 & \text{if } |x| \ge 2r_0 \end{cases} \tag{2.4.3}$$

and put into the identity (1.3.12). □

Thus going back to the proof of Theorem 2.2.1, by (2.2.12) and using the same notation, we get

$$\begin{aligned} \Re \int V_2 \nabla\psi_R \cdot \nabla_A u \bar{u} \le & \left(M + \frac{1}{2}\right) \mathcal{C}^{**} \left(\int_{|x|\le r_0} \frac{|\nabla_A u|^2}{|x|}\right)^{\frac{1}{2}} \left(\int_{|x|\le r_0} \frac{|u|^2}{|x|^3}\right)^{\frac{1}{2}} \\ & + \sigma|||\nabla_A u||| \, |||u||| \\ & \equiv V_{21} + V_{22}. \end{aligned}$$

Let us estimate $V_{21}$ and $V_{22}$. On the one hand, by (2.4.2) we have

$$\begin{aligned} V_{21} \le & \, \mathcal{C}^{**}\left(M + \frac{1}{2}\right)\left(\lambda + \mathcal{C}^{**} + \frac{d-3}{2} + \kappa\right)^{1/2} \int_{|x|\le r_0} \frac{|u|^2}{|x|^3} \\ & + \mathcal{C}^{**}\sigma|||u|||\left(\int_{|x|\le r_0} \frac{|u|^2}{|x|^3}\right)^{\frac{1}{2}} + \kappa \int_{|x|\le r_0} \frac{|u|^2}{|x|^3} + C(\kappa)(N(f))^2. \end{aligned}$$

On the other hand, yields

$$V_{22} \le \frac{1}{8}|||\nabla_A u|||^2 + \sigma|||u|||^2. \tag{2.4.4}$$

As a consequence, writing

$$a = \left(\int_{|x|\le r_0} \frac{|\nabla_A^\perp u|^2}{|x|}\right)^{1/2}, \qquad b = \left(\int_{|x|\le r_0} \frac{|u|^2}{|x|^3}\right)^{1/2},$$

in the same manner we obtain

$$\begin{aligned}&\frac{\lambda}{4}|||u|||^2+\left(\frac{1}{8}-\kappa_2\right)|||\nabla_A u|||^2+Ma^2+\frac{M(d-1)(d-3)}{4}b^2\\&+\frac{M}{2}\int_{|x|\geq r_0}\frac{|\nabla_A^{\perp}u|^2}{|x|}+\frac{M(d-1)(d-3)}{4}\int_{|x|\geq r_0}\frac{|u|^2}{|x|^3}\\&+\left(\frac{d-1}{8}-\kappa_2\right)\sup_{R>0}\frac{1}{R^2}\int_{|x|=R}|u|^2d\sigma_R\leq C^*(M+1/2)ab\\&+\sigma(\mathcal{C}^{**},\lambda)b^2+\sigma|||u|||^2+C(N(f))^2.\end{aligned}$$

Taking $\lambda_1=\lambda_1(M,d,\sigma,\kappa)>0$ large enough it suffices to show that for $\lambda\geq\lambda_1$

$$Ma^2-\mathcal{C}^*(M+1/2)ab+\left[\frac{M(d-1)(d-3)}{4}-\sigma(\mathcal{C}^{**},\lambda)\right]b^2>0. \tag{2.4.5}$$

This is true if

$$\frac{1}{(d-1)(d-3)}\left[(\mathcal{C}^*)^2\frac{(M+1/2)^2}{M^2}+\frac{4\sigma(\mathcal{C}^{**},\lambda)}{M}\right]<1. \tag{2.4.6}$$

Letting $M\to\infty$ in (2.4.6), we choose $\mathcal{C}^*$ and $\mathcal{C}^{**}$ such that

$$\frac{1}{(d-1)(d-3)}\left[(\mathcal{C}^*)^2+\sigma(\mathcal{C}^{**},\lambda)\right]<1.$$

Observe that when $\mathcal{C}^{**}=0$, we recover (2.2.18).

### 2.4.2 Singularities for $V_1$

Neither the hypothesis nor the conclusion of the Theorem 2.1.6 when $d>3$ is affected if we assume $V_1$ to be singular. If we consider

$$|(\partial_r V_1)_-|\leq\frac{\mathcal{C}^{**}}{|x|^3},\quad |V_1|\leq\frac{\mathcal{C}^{***}}{|x|^2}\quad\text{when}\quad |x|\leq r_0, \tag{2.4.7}$$

a slight change in the proof of Theorem 2.2.1 shows that the a-priori estimate (2.1.14) is still satisfied. In fact, we only need to replace the definition of the symmetric multiplier $\varphi$ (2.2.5) by the following one.

$$\varphi(x)=\begin{cases}\frac{\beta}{4R} & \text{if } |x|\leq R,\\ 0 & \text{if } |x|\geq R,\end{cases} \tag{2.4.8}$$

with $0<\beta\leq 1$. As a consequence, the condition (2.2.16) is also modified and in this case it follows that

$$\begin{aligned}&Ma^2+\left[\frac{M(d-1)(d-3)}{4}-\frac{(M+1/2)\mathcal{C}^{**}}{2}-\frac{\beta}{4}\mathcal{C}^{***}\right]b^2\\&-\mathcal{C}^*(M+1/2)ab>0\end{aligned} \tag{2.4.9}$$

for any $a, b > 0$. Since $\beta$ is arbitrary in the definition (2.4.8) of $\varphi$, we can choose $\beta \in (-\gamma, \gamma)$, for $\gamma > 0$ arbitrarily small. Thus we can neglect the term containing $\beta$, and (2.4.9) is satisfied if

$$\frac{1}{(d-1)(d-3)} \left[ \frac{(M+1/2)^2}{M^2} (\mathcal{C}^*)^2 + 2\frac{(M+1/2)}{M} \mathcal{C}^{**} \right] < 1. \tag{2.4.10}$$

Finally, observe that

$$\inf_{M>0} \frac{(M+1/2)^2}{M^2} = \inf_{M>0} \frac{(M+1/2)}{M} = 1$$

and the infimum is reached in the limit as $M \to \infty$. Since $M$ is also arbitrary in the definition of $\psi$ we can optimize in terms of $\mathcal{C}^*, \mathcal{C}^{**}$ and conclude that the last condition is

$$(\mathcal{C}^*)^2 + 2\mathcal{C}^{**} < (d-1)(d-3), \tag{2.4.11}$$

which is in fact the same condition as in [F] (Assumption (1.19)). When $(\partial_r V_1)_- = 0$, we recover (2.2.18).

## 2.5 Appendix B: Unique continuation and Carleman estimates

Consider a partial differential operator $P(x, D)$ in an open connected set $G \subset \mathbb{R}^d$, $d \geq 2$. Then, the classical unique continuation problem can be formulated as follows:

- Let $u$ be a solution to $P(x, D)u = 0$ such that $u = 0$ in an open subset of $G$. Then $u \equiv 0$ in $G$.

The most common way of proving unique continuation results is by using Carleman estimates.

The Carleman estimates are weighted energy type estimates with some large parameter $\tau$, which were first introduced by Carleman in 1939 to prove uniqueness of the continuation results for elliptic systems with nonanalytic coefficients on the plane. His idea turned out to be extremely fruitful, and in 1950-70s it was applied to many important partial differential equations. At present, there are many (and in some cases complete) results on Carleman estimates and on unique continuation property of second order equations, including elliptic, parabolic, Schrödinger type, and hyperbolic equations. See [Ta].

The simplest Carleman estimate for the Laplace operator has the form

$$2\lambda\tau \int |x|^\tau |u|^2 \leq \int |(\Delta + \lambda)u|^2 |x|^{\tau+2}, \tag{2.5.1}$$

for $u \in C_0^\infty(\mathbb{R}^d \backslash \{0\})$ where $\lambda > 0$, $\tau > 0$. (see [H], Proposition 14.7.1). One could give a very easy and short proof of (2.5.1) by a multiplier method and integration by parts.

## $1^{st}$ proof of (2.5.1):

Let us define

$$v = |x|^\tau u.$$

Then,

$$\begin{aligned} |x|^\tau(\Delta u + \lambda u) &= \Delta v - 2\tau |x|^{-1}\frac{x}{|x|}\cdot\nabla v + \tau(\tau+1)|x|^{-2}v - \tau(d-1)|x|^{-2}v \\ &= \Delta v + \tau^2|x|^{-2}v - \frac{2\tau}{|x|^2}\left(x\cdot\nabla v + \frac{(d-2)}{2}v\right) + \lambda v. \end{aligned} \tag{2.5.2}$$

Let us now multiply (2.5.2) by $g(v) = x\cdot\nabla\bar{v} + \frac{d-2}{2}\bar{v}$ and integrate over the whole $\mathbb{R}^d$. Thus we get

$$\begin{aligned} \int |x|^\tau(\Delta u + \lambda u)g(v) = &\int (\Delta v + \tau^2|x|^{-2}v)g(v) - 2\tau\int\frac{1}{|x|^2}|g(v)|^2 \\ &+ \lambda\int vg(v). \end{aligned}$$

After some integration by parts, by Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \lambda\int|v|^2 + 2\tau\int\frac{|g(v)|^2}{|x|^2} &\le \int |x|^\tau|\Delta u + \lambda u||g(v)| \\ &\le \left(\int |x|^{2\tau+2}|\Delta u + \lambda u|^2\right)^{\frac{1}{2}}\left(\int\frac{1}{|x|^2}|g(v)|^2\right)^{\frac{1}{2}} \\ &\le \frac{1}{4\tau}\int|x|^{2\tau+2}|\Delta u + \lambda u|^2 + \tau\int\frac{1}{|x|^2}|g(v)|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} 2\lambda\tau\int|x|^\tau|u|^2 + \frac{\tau}{2}\int|x|^{\tau-2}&\left|\frac{(\tau+d-2)}{2}u + x\cdot\nabla u\right|^2 \\ &\le \int|x|^{\tau+2}|\Delta u + \lambda u|^2. \end{aligned}$$

In particular, we have (2.5.1).

There is also another simple version of this proof:

## $2^{nd}$ proof of (2.5.1):

Let us denote

$$M = \int |x|^{2\tau+2}|\Delta u + \lambda u|^2$$

Let

$$L_1 = \Delta + \tau^2|x|^{-2} + \lambda, \qquad L_2 = -2\tau|x|^{-2}\left(x\cdot\nabla + \frac{d-2}{2}\right).$$

Note that $L_1$ is symmetric and $L_2$ is skew symmetric. It follows that

$$\begin{aligned} M &= \int |x|^2|\Delta v + \tau^2|x|^{-2}v - \frac{2\tau}{|x|^2}\left(x\cdot\nabla v + \frac{(d-2)}{2}v\right) + \lambda|v|^2 \\ &= \int |x|^2|L_1v + L_2v|^2 \\ &= \||x|L_1v\|^2 + \||x|L_2v\|^2 + 2\Re\int |x|^2 L_1vL_2\bar{v}. \end{aligned}$$

Now (2.5.1) follows since

$$2\Re\int |x|^2 L_1vL_2\bar{v} = 4\tau\lambda\|v\|^2$$

and $\|v\|^2 = \int |x|^{2\tau}|u|^2$.

Estimate (2.5.1) leads inmediately to a uniqueness theorem for a resolvent equation of a zero order perturbation of the Laplacian, as Hörmander shows in [H].

**Theorem 2.5.1.** *([H], Theorem 14.7.2) Assume that $u$ is a solution of the equation*

$$(\Delta + \lambda + V)u = 0 \tag{2.5.3}$$

*where $\lambda > 0$ and $V$ is multiplication by a function $V(x)$ satisfying*

$$|V(x)| \leq \frac{C}{|x|}. \tag{2.5.4}$$

*If $(1+|x|)^\tau D^\alpha u \in L^2$ for all $\tau$ when $|\alpha| \leq 1$, it follows that $u = 0$.*

Therefore, one could think that replacing the Laplacian $\Delta$ in (2.5.1) by the magnetic Laplacian $(\nabla + iA)^2$, we would get the uniqueness theorem for the electromagnetic Helmholtz equation (2.0.6) with the corresponding potentials.

The same reasoning as in the $2^{nd}$ proof of (2.5.1) applies to the magnetic case and we get the following:

## Magnetic case

Let us denote in this case

$$M = \int |x|^{2\tau+2}|(\nabla + iA)^2u + \lambda u|^2 \tag{2.5.5}$$

and

$$L_1 = \nabla_A^2 + \tau^2|x|^{-2} + \lambda, \qquad L_2 = -2\tau|x|^{-2}\left(x \cdot \nabla_A + \frac{d-2}{2}\right).$$

Hence, we have

$$\begin{aligned} M &= \int |x|^2 \left|\nabla_A^2 v + \tau^2|x|^{-2}v - \frac{2\tau}{|x|^2}\left(x\cdot\nabla_A v + \frac{(d-2)}{2}v\right) + \lambda v\right|^2 \\ &= \int |x|^2|L_1 v + L_2 v|^2 \\ &= \||x|L_1 v\|^2 + \||x|L_2 v\|^2 + 2\Re\int |x|^2 L_1 v L_2 \bar{v}. \end{aligned}$$

Now, since

$$2\Re\int |x|^2 L_1 v L_2 \bar{v} = 4\tau\lambda\|v\|^2 - 4\tau\Im\int |x|B_\tau\cdot\nabla_A v\bar{v}$$

and $\|v\|^2 = \int |x|^{2\tau}|u|^2$, it follows that

$$\lambda\int |v|^2 \le \frac{1}{4\tau}\int |x|^{2\tau+2}|(\nabla + iA)^2 u + \lambda u|^2 + \int |x||B_\tau||v||\nabla_A v|. \tag{2.5.6}$$

Observe that if $B_\tau = 0$, then we obtain the corresponding Carleman estimate for the magnetic Hamiltonian. Consequently, following [H] (see Theorem 2.5.1 above) we deduce the following result.

**Theorem 2.5.2.** *Assume that $u$ is a solution of the equation*

$$(\nabla_A^2 + \lambda)u = 0, \tag{2.5.7}$$

*such that $(1+|x|)^\tau D^\alpha u \in L^2$ for all $\tau$ and $|\alpha| \le 1$. If $B_\tau = 0$, then $u = 0$.*

Nevertheless, in the general case, one needs to control the magnetic gradient $\nabla_A$ of the solution $u$ in order to absorb the magnetic term. Working with polynomial weights we have not been able to obtain that. Looking at the literature and talking to some experts in the field, it seems that one should go further and try with exponential weights. The idea would be to set

$$v = w^\tau u, \tag{2.5.8}$$

where $w = e^{-|x|^\beta}$, $\beta > 0$. Then, we could expect some Carleman estimate that involves $v$ and $\nabla_A v$ with exponential weights in the left-hand side of the inequality. We propose to study this in the future.

# Chapter 3

# Energy concentration and explicit Sommerfeld radiation condition

This chapter is devoted to the study of the following Helmholtz equation

$$(\nabla + iA(x))^2 u(x) + n(x)u(x) = f(x), \quad x \in \mathbb{R}^d \tag{3.0.1}$$

with magnetic vector potential $A : \mathbb{R}^d \to \mathbb{R}^d$, where $n(x) = \lambda(1 + \tilde{V}_1(x))$ with $\tilde{V}_1 : \mathbb{R}^d \to \mathbb{R}$ is a variable index of refraction that does not necessarily converge to a constant at infinity, but can have an angular dependency like

$$n(x) \to n_\infty \left(\frac{x}{|x|}\right) \quad \text{as} \quad |x| \to \infty. \tag{3.0.2}$$

We are interested in the study of the existence and uniqueness of solution of the equation (3.0.1) by the limiting absorption method such that satisfies an appropriate radiation condition. In addition, our goal is to show a new energy estimate for the solution $u$ of the equation

$$(\nabla + iA(x))^2 u(x) + n(x)u(x) + i\varepsilon u(x) = f(x), \quad \varepsilon > 0 \tag{3.0.3}$$

that characterizes the behavior of the solution at infinity. This estimate will be obtain by integration by parts and is given by

$$\int \left|\nabla_\omega n_\infty \left(\frac{x}{|x|}\right)\right|^2 \frac{|u|^2}{1 + |x|} < +\infty, \tag{3.0.4}$$

where $\omega = \frac{x}{|x|}$. Thus we see that in this case the behavior of the solution at infinity can be very different to the one exhibited by free solutions.

In the case that $A \equiv 0$, the existence of the energy estimate (3.0.4) has been established by Perthame and Vega [PV2]. They consider the Helmholtz equation

$$\Delta u + n(x)u + i\varepsilon u = -f(x), \qquad \varepsilon > 0 \tag{3.0.5}$$

with $n(x) > 0$ such that

$$2\sum_{j\in\mathbb{Z}} \sup_{C(j)} \frac{(x\cdot\nabla n(x))_-}{n(x)} < 1 \tag{3.0.6}$$

is studied. Here $C(j)$ denotes the annulus $\{2^{j-1} \leq |x| \leq 2^j\}$, while $(a)_- = -\min\{0, a\}$ is the negative part of $a \in \mathbb{R}$. We point out that this condition remains unchanged under the transformation $n \to \lambda n$. By (3.0.6) it may be concluded that the solution of the equation (3.0.5) satisfies the a-priori estimate

$$|||\nabla u|||^2 + |||n^{1/2}u|||^2 + \int \frac{|\nabla^\perp u|^2}{|x|} < \infty. \tag{3.0.7}$$

In addition, if there exists $n_\infty \in C^3(S^{d-1})$ such that $n_\infty(\omega) \geq n_0 > 0$ and

$$|n(x) - n_\infty(\omega)| \leq n_\infty(\omega)\frac{\Gamma}{|x|}, \quad \Gamma > 0, \quad n > 0, \tag{3.0.8}$$

it follows that $u$ also satisfies the energy estimate

$$\int_{|x|\geq 1} |\nabla_\omega n_\infty(\omega)|^2 \frac{|u|^2}{|x|} < \infty, \tag{3.0.9}$$

where for a function $n(\omega) \in C^1(S^{d-1})$ with $\omega = \frac{x}{|x|}$,

$$\nabla_\omega n(\omega) = \frac{\partial}{\partial\omega} n(\omega) := |x|\frac{\partial}{\partial\tau} n\left(\frac{x}{|x|}\right)$$

and

$$\frac{\partial}{\partial\tau}u(x) = \nabla u(x) - \frac{x}{|x|}\frac{\partial}{\partial r}u(x), \quad \frac{\partial}{\partial r}u(x) = \partial_r u(x) := \frac{x}{|x|}\cdot\nabla u(x).$$

The estimate (3.0.9) says that the points where $|\nabla_\omega n_\infty(\omega)|$ vanishes on the sphere are the concentration directions for the energy $|u|^2$. In other words, energy is not dispersed in all directions but concentrated on those given by the critical points of $\nabla n(\omega)$.

The role played by the critical points of $n_\infty$ was already pointed out by Herbst in [He], where it is considered potentials $V(x)$ such that

$$V(x) = |x|^{-\sigma} V\left(\frac{x}{|x|}\right) \quad 0 < \sigma < 2, \quad \forall x \in \mathbb{R}^d\backslash\{0\}. \tag{3.0.10}$$

This potential is also studied in [GVV] and [HS], for the study of the counterexamples of Strichartz inequalities for Schrödinger equations with repulsive potentials and the existence and completeness of the wave operator, respectively.

One of the main contributions of this chapter is the extension of the energy estimate (3.0.9) to the magnetic case. To this end, we will first prove the following a-priori estimate

$$|||n^{1/2}u|||^2 + |||\nabla_A u|||^2 + \int \frac{|\nabla_A^{\perp} u|^2}{|x|} \leq C\left(N\left(\frac{f}{n^{1/2}}\right)\right)^2. \tag{3.0.11}$$

We emphasize that the estimate for the tangential component of the magnetic gradient turns out to be fundamental. For this purpose, we will require that $n(x)$ and the tangential component of the magnetic field satisfy the condition

$$2\sum_{j\in\mathbb{Z}} \sup_{C(j)} \frac{(x\cdot\nabla n(x))_- + 2^{2j}|B_\tau|^2}{n(x)} < 1. \tag{3.0.12}$$

Note that when $B_\tau = 0$, we get (3.0.6). Thus we will recover the a-priori estimate (3.0.7) of [PV2]. For the energy estimate (3.0.9), it will be necessary to put some further restriction to $n(x)$ as in [PV2] and also to the magnetic field $B$. Nevertheless, it does not seem that these conditions are sufficient to prove the limiting absorption principle for the equation (3.0.1). Observe that the conditions that we are assuming here are weaker than the ones in the previous chapter.

In order to get the limiting absorption principle for the electromagnetic Helmholtz equation (3.0.1) we will follow Saito [S]. This will allow us to add a short range potential $V_2$ to the equation (3.0.1). Then, under suitable hypotheses on $\tilde{V}_1$ and $V_2$, we will prove the existence of a unique solution of the equation

$$\nabla_A^2 u + \lambda(1+\tilde{V}_1)u + V_2 u = f \tag{3.0.13}$$

for $\lambda \in [\lambda_0, \lambda_1]$ with $0 < \lambda_0 < \lambda_1 < \infty$ satisfying the a-priori estimate

$$\lambda|||u|||_1^2 + |||\nabla_A u|||_1^2 \leq C(N_1(f))^2 \tag{3.0.14}$$

and the radiation condition

$$\int_{|x|\geq 1} \left|\nabla_A u - i\sqrt{\lambda}\nabla K u\right|^2 \frac{1}{(1+|x|)^{1-\delta}} < +\infty, \tag{3.0.15}$$

where $K$ is the solution to the eikonal equation

$$|\nabla K|^2 = \frac{1}{\lambda} n(x), \qquad n(x) = \lambda(1+\tilde{V}_1(x)). \tag{3.0.16}$$

In fact, we will require that $\tilde{V}_1$ satisfies

$$|\partial^\alpha \tilde{V}_1(x)| \leq C^*|x|^{-|\alpha|} \qquad |\alpha| \leq 2, \tag{3.0.17}$$

for a small constant $C^* > 0$, while $V_2$ will be as in Chapter 2 including singularities at the origin. Regarding to the magnetic potential, in this case we need to impose conditions on each component $B_{jk}$ of the magnetic field. It is worth pointing out that we improve the result by Saito showing the estimate (3.0.14) instead of the $L^2$-weighted one (0.0.50) that is showed in [S]. However, (3.0.14) is not enough for proving the energy estimate (3.0.9); as we mentioned above, the estimate

$$\int \frac{|\nabla_A^\perp u|^2}{|x|} < \infty \tag{3.0.18}$$

is necessary in our approach.

Note that assumptions needed to obtain the energy estimate and those for the limiting absorption principle are different and not comparable. On the one hand, if $n = n_\infty$ and regular, (3.0.12) is trivially fulfilled. On the other hand, condition (3.0.17) with $n = \lambda(1+\tilde{V}_1)$ does not imply the existence of the limit $n_\infty$. In addition, condition (3.0.8) does not need any regularity assumption on $n(x)$ as in (3.0.17). It is easy to see that if besides (3.0.17), we require

$$|\partial_r \tilde{V}_1(x)| \leq c_2 |x|^{-1-\mu}, \qquad |x| \geq 1, \tag{3.0.19}$$

for some $c_2 > 0$, $\mu > 0$, then the index of refraction $n(x)$ admits a radial limit $n_\infty\left(\frac{x}{|x|}\right)$ as $|x| \to \infty$. Moreover, it follows that

$$|n(r\omega) - n_\infty(\omega)| \leq \Gamma |x|^{-\mu} \tag{3.0.20}$$

for $\Gamma > 0$, where $r = |x|$ and $\omega = \frac{x}{|x|}$ (see [PV2] for more details).

According to the above, this chapter will be divided into two parts. Firstly, we will extend the work by Perthame and Vega [PV2] to the magnetic case, giving the a-priori estimate (3.0.11) and the new energy estimate (3.0.9) for solutions $u \in H^1_A(\mathbb{R}^d)$ of the magnetic Helmholtz equation

$$(\nabla + iA)^2 u + n(x)u + i\varepsilon u = f, \qquad \varepsilon > 0, \tag{3.0.21}$$

with $n(x) = \lambda(1 + \tilde{V}_1(x))$. Secondly, basing on the paper by Saito [S], we will prove the limiting absorption principle for the equation (3.0.13). A combination of the both results will permit to deduce an explicit Sommerfeld condition

$$\int \left| \nabla_A u - i n_\infty^{1/2} \frac{x}{|x|} u \right|^2 \frac{1}{1+|x|} < +\infty \tag{3.0.22}$$

for solutions obtained from the limiting absorption principle. It is a very striking and unexpected feature that the index $n_\infty$ appears in this formula and not the gradient of the phase as established by Saito [S]. This apparent contradiction is clarified by the existence of the extra estimate (3.0.9) on the energy decay. In other words, the Sommerfeld condition hides the main physical effect arising for a variable $n$ at infinity; energy concentration on lines rather than dispersion in all directions.

Observe that in our approach it will be necessary to solve the eikonal equation

$$|\nabla K|^2 = 1 + \tilde{V}_1(x). \tag{3.0.23}$$

Barles [B] proved that under the assumption (3.0.17) and for $C^*$ small enough, there exists a solution of the equation (3.0.23) for $|x| > R_0$ with $R_0$ large enough, see section 3.1. In general, one can not expect that the vector $\nabla K$ points at the direction $\frac{x}{|x|}$. An illustrative example given by Saito [S] is to consider

$$\tilde{V}_1(x) = -\frac{1}{\lambda}\frac{x_1}{|x|}. \tag{3.0.24}$$

Then $\tilde{V}_1(x)$ satisfies (3.0.17) for $\lambda$ large enough and

$$K(x) = a(\lambda)|x| - b(\lambda)x_1 \tag{3.0.25}$$

with

$$\begin{cases} a(\lambda) = \frac{1}{2}[(1+1/\lambda)^{1/2} + (1-1/\lambda)^{1/2}] \\ b(\lambda) = \frac{1}{2}[(1+1/\lambda)^{1/2} - (1-1/\lambda)^{1/2}] \end{cases} \tag{3.0.26}$$

is a solution of the eikonal equation (3.0.23). This boundary condition differs from ours in all points except when $\nabla n = 0$ ($n = \lambda + \tilde{V}_1$). Then the apparent contradiction is clarified thanks to the estimate (3.0.9) which applies for this example. Note that in the trivial case $\tilde{V}_1(x) = 0$, one can take $K(x,\lambda) = |x|$.

It is worth pointing out that the self-adjointness of the Schrödinger operator

$$T = \nabla_A^2 + \lambda\tilde{V}_1 + V_2 \tag{3.0.27}$$

is necessary for the limiting absorption principle. As in Chapter 2, in what follows we assume (2.0.2) with $V_1 = \lambda\tilde{V}_1$. Hence, we conclude that $T$ is self-adjoint on $L^2(\mathbb{R}^d)$ with form domain

$$D(T) = \{f \in L^2(\mathbb{R}^d) : \int |\nabla_A f|^2 - \int (\lambda\tilde{V}_1 + V_2)|f|^2 < \infty\}. \tag{3.0.28}$$

The remainder of this chapter is organized as follows. In the next section, we give a brief exposition of the eikonal equation and its properties that will be useful for the proofs of the main results. Section 3.2 will be concerned with the new energy estimate. We will state and prove the result that extends [PV2] to the magnetic case, see Theorem 3.2.3, showing first the appropriate a-priori estimates given in Theorem 3.2.1 that permits to deduce the desired conclusion. In section 3.3 we proceed with the study of the limiting absorption principle for the equation (3.0.13) that is established in Theorem 3.3.4. The same reasoning as in Chapter 2 applies to this case. Therefore, we will restrict our attention to show the Sommerfeld radiation condition (Proposition 3.3.5) and the a-priori estimates for the solution $u$ of the equation (3.0.13) with $\lambda$ replaced by $\lambda + i\varepsilon$, $\varepsilon > 0$ (Proposition 3.3.7). The uniqueness result will be also established and proved (Theorem 3.3.8) in this section in much the same way as in the proof of Theorem 2.1.5 following Mochizuki [Mo] and Zhang [Zh]. Finally, section 3.4 provides a detailed proof of the new explicit Sommerfeld condition for the electromagnetic Helmholtz equation (3.0.1) given in Theorem 3.4.1.

## 3.1 The Eikonal Equation

In order to determine the phase arising in the Sommerfeld radiation condition (3.0.15) and to conclude the explicit one (3.0.22), we need to solve the eikonal equation

$$|\nabla K|^2 = 1 + \tilde{V}_1(x), \quad x \in \mathbb{R}^d. \tag{3.1.1}$$

Setting the solution $K = K(x, C^*)$ in terms of the bounded function $g$ defined as

$$g(x, C^*) = |x|^{-1} K(x, C^*), \tag{3.1.2}$$

we derive the following Hamilton-Jacobi equation

$$|g|^2 + 2r\partial_r g + |x|^2 |\nabla g|^2 = 1 + \tilde{V}_1(x), \quad x \in \mathbb{R}^d \backslash \{0\}. \tag{3.1.3}$$

From (3.1.3), under assumption (3.0.17), G. Barles [B] showed that there exist $C_0 > 0$ and $r_0 > 0$ such that for any $C^* < C_0$ and for $|x| \geq r_0$ the equation (3.1.1) has a solution $K = K(x, C^*)$ satisfying

(i) $K(x, C^*)$ is a real-valued $C^3$ function.

(ii) Let $0 < c_0 < c_1 < \infty$. Then,
$$c_0 \leq g(x, C^*) \leq c_1. \tag{3.1.4}$$

(iii) When $C^* \to 0$
$$|x|^j (\partial^j g)(x, C^*) \longrightarrow \begin{cases} 1, & j = 0 \\ 0, & j = 1, 2, 3 \end{cases} \tag{3.1.5}$$
uniformly for $x \in \{x \in \mathbb{R}^d : |x| \geq r_0\}$.

Therefore, one can easily deduce the following identity that will be very useful in section 3.3.

**Lemma 3.1.1.** *([S], Lemma 2.5) For the solution $K$ of the eikonal equation (3.1.1) and for $1 \le i, j \le d$, the following identity holds*

$$\frac{\partial^2 K}{\partial x_i \partial x_j} = \frac{|\nabla K|^2}{K}\delta_{ij} - \frac{1}{K}\frac{\partial K}{\partial x_i}\frac{\partial K}{\partial x_j} + \frac{1}{K}F_{ij}(x, C^*), \tag{3.1.6}$$

*where $F_{ij}(x, C^*)$ is a bounded function of $x$ for $|x| \ge r_0$ such that*

$$\lim_{C^* \to 0} \sup_{|x| \ge r_0} |F_{ij}(x, C^*)| = 0 \quad (i, j = 1, \dots, d) \tag{3.1.7}$$

*and*

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \ne j. \end{cases}$$

*Proof.* Setting $K(x, C^*) = |x|g(x, C^*)$ and $\tilde{x}_k = \frac{x_k}{|x|}$, we have

$$\frac{\partial K}{\partial x_i} = \tilde{x}_i g + |x|\frac{\partial g}{\partial x_i}. \tag{3.1.8}$$

Then,

$$\frac{\partial^2 K}{\partial x_i \partial x_j} = \frac{\delta_{ij} g}{|x|} - \tilde{x}_i \tilde{x}_j \frac{g}{|x|} + \tilde{x}_i \frac{\partial g}{\partial x_j} + \tilde{x}_j \frac{\partial g}{\partial x_i} + |x|\frac{\partial^2 g}{\partial x_i \partial x_j}, \tag{3.1.9}$$

which can be written as

$$\frac{\partial^2 K}{\partial x_i \partial x_j} = \frac{\delta_{ij}}{K}g^2 - \frac{\tilde{x}_i \tilde{x}_j}{K}g^2 + \frac{1}{K}G_{ij}(x, C^*), \tag{3.1.10}$$

with

$$G_{ij}(x, C^*) = \left( \tilde{x}_i |x| \frac{\partial g}{\partial x_j} + \tilde{x}_j |x| \frac{\partial g}{\partial x_i} + |x|^2 \frac{\partial^2 g}{\partial x_i \partial x_j} \right). \tag{3.1.11}$$

On the other hand, from (3.1.8) it follows that

$$\begin{cases} \tilde{x}_i g = \frac{\partial K}{\partial x_i} - |x|\frac{\partial g}{\partial x_i}, \\ |\nabla K|^2 = g^2 + 2g|x|\tilde{x} \cdot \nabla g + |x|^2 |\nabla g|^2. \end{cases}$$

Thus we obtain

$$\frac{\tilde{x}_i \tilde{x}_j}{K}g^2 = \frac{1}{K}\frac{\partial K}{\partial x_i}\frac{\partial K}{\partial x_j} - \frac{1}{K}\left( |x|\frac{\partial g}{\partial x_i}\frac{\partial K}{\partial x_j} + |x|\frac{\partial K}{\partial x_i}\frac{\partial g}{\partial x_j} - |x|^2 \frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j} \right),$$

which together with (3.1.10), gives (3.1.6) with

$$\begin{aligned} F_{ij} = G_{ij} - \delta_{ij}(2|x|\tilde{x} \cdot \nabla g) + |x|^2 |\nabla g|^2) + |x|\frac{\partial g}{\partial x_i}\frac{\partial K}{\partial x_j} \\ + |x|\frac{\partial K}{\partial x_i}\frac{\partial g}{\partial x_j} - |x|^2 \frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j}. \end{aligned} \tag{3.1.12}$$

The relation (3.1.7) follows from (3.1.5) and the lemma is proved.

∎

From (3.0.15) and (3.0.18), in order to prove the explicit condition (3.0.22) we shall deduce (see section 3.4) the estimate

$$\int |\partial_\tau K u|^2 \frac{1}{1+|x|} < +\infty. \tag{3.1.13}$$

This is an energy estimate in itself which says that $u$ concentrates along the critical points of $\nabla_\tau K$. In fact, from the hypotheses (3.0.19) for the potential $\tilde{V}_1(x)$, it follows that these critical point coincide with those of $\nabla_\omega n_\infty$ establishing a relation between the energy estimate (3.0.9) and (3.1.13).

**Lemma 3.1.2.** *(Theorem 3.2, [PV2]) Under assumptions (3.0.17) and (3.0.19), the solution to (3.1.3) satisfies for $C^*$ small enough and $x \neq 0$*

$$|\partial_r g| \leq C^* r^{-1-\mu}, \tag{3.1.14}$$

*and $g\left(r\frac{x}{|x|}\right) \to g_\infty\left(\frac{x}{|x|}\right)$ as $r \to \infty$, a smooth solution to the equation*

$$g_\infty(\omega)^2 + |\nabla_\omega g_\infty(\omega)|^2 = n_\infty(\omega), \quad \omega \in S^{d-1}. \tag{3.1.15}$$

*Moreover,*

$$|\nabla^\perp K| = |\nabla_\omega g_\infty(\omega)| + O(r^{-\mu}) \tag{3.1.16}$$

*and*

$$0 < C_1 |\nabla_\omega g_\infty| \leq |\nabla_\omega n_\infty| \leq C_2 |\nabla_\omega g_\infty|. \tag{3.1.17}$$

Observe however that for the energy estimate (3.0.9) we do not need the existence of a solution to the eikonal equation (3.1.1) which could well not exist.

## 3.2 The energy estimate

The purpose of this section is to extend the result by Perthame and Vega [PV2] to the magnetic case. To be more precise, we are interested in proving the energy estimate

$$\int |\nabla_\omega n_\infty(\omega)|^2 \frac{|u|^2}{|x|} \leq C(1+\varepsilon)\left(N\left(\frac{f}{n^{1/2}}\right)\right)^2 \tag{3.2.1}$$

for solutions $u \in H^1_A(\mathbb{R}^d)$ of the magnetic Helmholtz equation

$$\nabla_A^2 u + n(x)u + i\varepsilon u = f(x), \qquad \varepsilon > 0, \tag{3.2.2}$$

with a variable index of refraction $n(x)$, with a slow and only radial decay to a constant $n_\infty\left(\frac{x}{|x|}\right)$ at infinity. The estimate (3.2.1) uses in a strong way the a-priori estimate for the

Morrey-Campanato norm of the solution $u$ of the equation (3.2.2) as well as the estimate for the tangential part of its magnetic gradient.

Let us consider $n(x) > 0$ such that

$$n = n_1 + n_2 \qquad \text{with} \qquad n_2 \in L^\infty, \tag{3.2.3}$$

$$\|n_1^{1/2} u\|_{L^2} \leq (1 - c_0)\|\nabla u\|_{L^2} \quad \text{for some } c_0 > 0, \tag{3.2.4}$$

$$2\sum_{j\in\mathbb{Z}} \sup_{C(j)} \frac{(x \cdot \nabla n(x))_- + 2^{2j}|B_\tau|^2}{n(x)} := \beta < 1, \tag{3.2.5}$$

where $C(j) = \{x \in \mathbb{R}^d : 2^{j-1} \leq |x| \leq 2^j\}$ and $(a)_-$ denotes the negative part of $a \in \mathbb{R}$.

Then it follows the following result.

**Theorem 3.2.1.** *Let $d \geq 3$ and assume that (3.2.3)-(3.2.5) hold. Then the solution to the Helmholtz equation (3.2.2) satisfies*

$$\begin{aligned} M^2 &:= |||\nabla_A u|||^2 + |||n^{1/2}u|||^2 + \int \frac{|\nabla_A^\perp u|^2}{|x|} \\ &\leq C(\varepsilon + \|n_2\|_{L^\infty}) \left( N\left(\frac{f}{n^{1/2}}\right)\right)^2, \end{aligned} \tag{3.2.6}$$

*where $C$ is independet of $\varepsilon$ and $n$.*

*Proof.* Let $R > 0$ and we consider the functions $\psi$ and $\varphi$ given by

$$\nabla\psi(x) = \begin{cases} \frac{|x|}{R} & \text{if } |x| \leq R, \\ \frac{x}{|x|} & \text{if } |x| \geq R, \end{cases} \tag{3.2.7}$$

$$\varphi(x) = \begin{cases} \frac{1}{2R} & \text{if } |x| \leq R, \\ 0 & \text{if } |x| \geq R. \end{cases} \tag{3.2.8}$$

Let us add the identity (1.3.15) to (1.3.12) with the above choices of the multipliers. Thus, analysis similar to that in the proof of Theorem 2.2.1 gives

$$\begin{aligned} &\frac{1}{2R}\int_{|x|\leq R} (|\nabla_A u|^2 + n(x)|u|^2) + \int_{|x|\geq R} \frac{|\nabla_A^\perp u|^2}{|x|} + \frac{(d-1)}{8R^2}\int_{|x|=R} |u|^2 \\ &\quad \leq \frac{1}{2}\int (\partial_r n)_-|u|^2 + \frac{1}{R}\int_{|x|\leq R} |x||B_\tau||\nabla_A u||u| + \int_{|x|\geq R} |B_\tau||\nabla_A^\perp u||u| \\ &\quad + \varepsilon\int |\nabla_A u||u| + 2\int |f||\nabla_A u| + C\int \frac{|f||u|}{|x|}. \end{aligned} \tag{3.2.9}$$

The terms related to $f$ can be treated as in the proof of Theorem 2.2.1, obtaining

$$\int \frac{|f||u|}{|x|} + \int |f||\nabla_A u| \leq \kappa \left( |||\nabla_A u|||^2 + \sup_{R>0} \frac{1}{R^2} \int_{|x|=R} |u|^2 \right) + C_\kappa (N(f))^2, \tag{3.2.10}$$

where $\kappa$ denotes an arbitrary positive small constant.

Let us study the potential terms. On the one hand, we have

$$\begin{aligned} \frac{1}{2} \int (\partial_r n)_- |u|^2 &\leq \frac{1}{2} \sum_{j \in \mathbb{Z}} \int_{C(j)} \frac{(x \cdot \nabla n)_-}{2^{j-1} n} n|u|^2 \\ &\leq \sum_{j \in \mathbb{Z}} \frac{(x \cdot \nabla n)_-}{n} |||n^{1/2} u|||^2. \end{aligned} \tag{3.2.11}$$

On the other hand, let $J$ such that $2^{J-1} \leq R \leq 2^J$. Then by Cauchy-Schwarz inequality yields

$$\begin{aligned} \frac{1}{R} \int_{|x| \leq R} |x||B_\tau||\nabla_A u||u| &\leq \frac{1}{R} \left( \int_{|x| \leq R} |\nabla_A u|^2 \right)^{\frac{1}{2}} \left( \int_{|x| \leq R} |x|^2 |B_\tau|^2 |u|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{4R} \int_{|x| \leq R} |\nabla_A u|^2 + \sum_{j \leq J} \frac{2^{2j} |B_\tau|^2}{n(x)} |||n^{1/2} u|||^2 \end{aligned}$$

and

$$\begin{aligned} \int_{|x| \geq R} |B_\tau||\nabla_A^\perp u||u| &\leq \left( \int_{|x| \geq R} \frac{|\nabla_A^\perp u|^2}{|x|} \right)^{1/2} \left( \int_{|x| \geq R} |x||B_\tau|^2 |u|^2 \right)^{1/2} \\ &\leq \frac{1}{4} \int_{|x| \geq R} \frac{|\nabla_A^\perp u|^2}{|x|} + \sum_{j \geq J} \frac{2^{2j} |B_\tau|^2}{n(x)} |||n^{1/2} u|||^2. \end{aligned}$$

Finally, let us analyze the $\varepsilon$ term. In this case, the a-priori estimate (1.3.10) reads as

$$\int |\nabla_A u|^2 \leq \int n|u|^2 + \int |f||u|,$$

which together with assumptions (3.2.3)-(3.2.4) implies

$$\int |\nabla_A u|^2 \leq C \left( \int n_2 |u|^2 + \int |f||u| \right).$$

Hence, by the same method as in (2.2.15) it follows that

$$\varepsilon \int |\nabla_A u||u| \leq \kappa |||n^{1/2} u|||^2 + C_\kappa (\varepsilon + \sup |n_2|) \left( N \left( \frac{f}{n^{1/2}} \right) \right)^2. \tag{3.2.12}$$

As a consequence, plugging (3.2.10)-(3.2.12) into (3.2.9) and taking the supremmum over $R$, by condition (3.2.5) we get (3.2.6), which is our claim. □

**Remark 3.2.2.** *The dimension two is a special case. In this case, with the above choice of multipliers it follows that*

$$\Delta(2\varphi - \Delta\psi) \leq -\frac{C}{|x|^3}. \tag{3.2.13}$$

*Because of this singularity at zero, we cannot recover the full result (3.2.6) for* $d = 2$ *and we cannot reach the right behavior close to* $0$*. With some modifications in the proof (see [PV1], section 5 for more details) and assuming that* $n > n_0 > 0$*, then in the two dimensional case it may be proved that for* $R_0 = n_0^{-1/2}$ *the solution satisfies*

$$|||\nabla_A u|||_{R_0}^2 + |||n^{1/2}u|||_{R_0}^2 + \int_{|x|\geq R_0} \frac{|\nabla_A^\perp u|^2}{|x|} \leq C(1+\varepsilon)\left(N_{R_0}\left(\frac{f}{n^{1/2}}\right)\right)^2.$$

The homogeneity of the above estimate makes it compatible with the high frequencies (replace $n$ by $\mu^2 n$). Moreover, (3.2.6) allows us to get the new energy estimate. As we have already said, the estimate of the tangential component of the magnetic gradient given in (3.2.6) turns out to be fundamental. In order to get it, we need the smallness assumption given in (3.2.5). However, the condition (3.2.5) is necessary and can not be relaxed to a Coulomb type of decay, even if smallness is added (see [PV2], Appendix for more details).

In order to prove the energy estimate (3.2.1), we need to impose some extra assumptions on $B$ and $n$. In fact, on the one hand we assume that

$$\sum_{j\geq 0} 2^{2j}|B_{jk}|^2 < \infty. \tag{3.2.14}$$

On the other hand, it is required that

$$\text{there exists} \quad n_\infty\left(\frac{x}{|x|}\right) \in C^\infty\left(S^{d-1}\right), \quad n_\infty\left(\frac{x}{|x|}\right) \geq n_0 > 0, \tag{3.2.15}$$

and

$$\left|n(x) - n_\infty\left(\frac{x}{|x|}\right)\right| \leq n_\infty\left(\frac{x}{|x|}\right)\frac{\Gamma}{|x|}, \qquad \Gamma > 0, \quad n > 0. \tag{3.2.16}$$

Note that from (3.2.15) and (3.2.16) it may be concluded that

$$|n| \leq C \qquad \text{and} \qquad n \geq \frac{n_0}{2} \quad \text{for} \quad |x| \quad \text{large enough.} \tag{3.2.17}$$

We may now state the main result of this section. Its interest relies on the bounds stated in Theorem 3.2.1.

**Theorem 3.2.3.** *For dimensions $d \geq 3$, we assume (3.2.5), (3.2.14)-(3.2.16) and use the notation of Theorem 3.2.1. Then the solution of the Helmholtz equation (3.2.2) satisfies, for $R \geq 1$ large enough*

$$\int_{|x|\geq R} \left|\nabla_\omega n_\infty \left(\frac{x}{|x|}\right)\right|^2 \frac{|u|^2}{|x|} \leq C \left[M^2 + \left(N\left(\frac{f}{n^{1/2}}\right)\right)^2\right], \tag{3.2.18}$$

*for some constant $C$ independent of $\varepsilon$ and $n$.*

*Proof.* The proof consists in using the basic identity (1.3.15) with a test function that depends on the behavior of $n(x)$ at infinity. We choose $R \geq 1$ such that (3.2.17) holds and define

$$\psi_q(x) = q\left(\frac{|x|}{R}\right) n_\infty \left(\frac{x}{|x|}\right) \tag{3.2.19}$$

for some non-decreasing smooth function

$$q(r) = \begin{cases} 0 & \text{for} \quad r \leq 1 \\ r & \text{for} \quad r \geq 2. \end{cases}$$

Let us put $\psi_q$ into (1.3.15), obtaining

$$\begin{aligned} \frac{1}{2}\int \lambda \nabla \tilde{V_1} \cdot \nabla \psi_q |u|^2 &= -\int \nabla_A u \cdot \nabla_A^2 \psi_q \cdot \overline{\nabla_A u} \\ -\frac{1}{2}\Re \int \nabla(\Delta\psi_q)\cdot \nabla_A u \bar{u} &+ \Im \sum_{j,k=1}^{d} \int \frac{\partial \psi_q}{\partial x_k} B_{jk} (\nabla_A)_j u \bar{u} \\ -\Re \int f \nabla \psi_q \cdot \overline{\nabla_A u} - \frac{1}{2}\int f \Delta \psi_q \bar{u} &+ \varepsilon \int \nabla\psi_q \cdot \nabla_A u \bar{u}. \end{aligned} \tag{3.2.20}$$

We simplify the notation using $q = q\left(\frac{|x|}{R}\right)$, $n_\infty = n_\infty(\omega)$. Observe that

$$\frac{\partial \psi_q}{\partial x_k} = \frac{q' n_\infty}{R}\frac{x_k}{|x|} + \frac{q}{|x|}\frac{\partial n_\infty}{\partial \omega_k}. \tag{3.2.21}$$

and

$$\Delta \psi_q = \frac{q''}{R^2} n_\infty + \frac{q'}{R}\frac{d-1}{|x|} n_\infty + \frac{q}{|x|^2}\Delta_\omega n_\infty. \tag{3.2.22}$$

The left hand side of the estimate (3.2.18) will come from the term

$$\int \lambda \nabla \tilde{V_1} \cdot \nabla \psi_q |u|^2 = \int \nabla n \cdot \nabla \psi_q |u|^2 \tag{3.2.23}$$

which can be written as follows

$$\int \nabla n \cdot \nabla \psi_q |u|^2 = \int q \left| \frac{\partial n_\infty}{\partial \omega} \right|^2 \frac{|u|^2}{|x|}$$
$$+ \int \partial_r n \frac{q'}{R} n_\infty |u|^2$$
$$+ \int q|x| \nabla_\tau (n - n_\infty) \frac{\partial n_\infty}{\partial \omega} \frac{|u|^2}{|x|^2}$$
$$\equiv I_1 + I_2 + I_3. \tag{3.2.24}$$

The first term on the right-hand side of (3.2.24) is the one that gives the lower bound of what we want to control. By (3.2.5) (here smallness is not necessary) and (3.2.15), we get

$$I_2 \leq \frac{C\|n_\infty\|_{L^\infty}}{R^2} |||n^{1/2}u|||^2. \tag{3.2.25}$$

On the other hand, after integration by parts, by the diamagnetic inequality (1.1.4) and by (3.2.16), we obtain

$$\begin{aligned} I_3 &= -\int \frac{q}{|x|^2}(n - n_\infty) \left( \Delta_\omega n_\infty |u|^2 + 2\frac{\partial n_\infty}{\partial \omega} |x| \nabla |u| |u| \right) \\ &\leq \frac{C}{R} \|n_\infty\|_{C^2} |||n^{1/2}u|||^2 + \kappa \int q \left| \frac{\partial n_\infty}{\partial \omega} \right|^2 \frac{|u|^2}{|x|^2} + \frac{C(\kappa)}{R} |||\nabla_A u|||^2, \end{aligned} \tag{3.2.26}$$

for $\kappa > 0$.

Let us estimate now the remaining terms of the identity (3.2.20). A straightforward computation gives

$$\begin{aligned} \nabla_A u \cdot D^2 \psi_q \cdot \overline{\nabla_A u} &= \frac{q''}{R^2} n_\infty |\nabla_A^r u|^2 + \frac{q'}{R|x|} |\nabla_A^\perp u|^2 n_\infty \\ &\quad + 2\Re \left( \frac{q'}{R|x|} - \frac{q}{|x|^2} \right) \overline{\nabla_A^r u} \frac{\partial n_\infty}{\partial \omega} \cdot \nabla_A^\perp u \\ &\quad + \frac{q}{|x|^2} \nabla_A^\perp u \cdot D_\omega^2 n_\infty \cdot \overline{\nabla_A^\perp u}. \end{aligned} \tag{3.2.27}$$

Thus since $q'$, $q''$ and $\left( \frac{q'}{R|x|} - \frac{q}{|x|^2} \right)$ are supported in the ball $\{|x| \leq 2R\}$, by the Cauchy-Schwarz inequality it follows that the absolute value of the above terms in the corresponding integral are bounded by

$$\frac{C\|n_\infty\|_{C^2}}{R} \left( \int \frac{|\nabla_A^\perp u|^2}{|x|} + |||\nabla_A u|||^2 \right). \tag{3.2.28}$$

Moreover, by (3.2.22) and (3.2.17) one can easily check that

$$\begin{aligned} Re\int \nabla(\Delta\psi_q)\cdot\nabla_A u\bar{u} &\leq C\|n_\infty\|_{C^3}\int\left(\frac{q}{|x|^3}+\frac{q'}{R|x|^2}\right)|\nabla_A u||u| \\ &\leq \frac{C\|n_\infty\|_{C^3}}{R}|||n^{1/2}u||||||\nabla_A u|||. \end{aligned}$$

As far as the term containing the magnetic potential is concerned, first note that by (3.2.21) and (2.2.7) yields

$$\sum_{j,k=1}^{d}\frac{\partial\psi_q}{\partial x_k}B_{jk}(\nabla_A)_j u=\frac{q'n_\infty}{R}B_\tau\cdot\nabla_A^\perp u+\frac{q}{|x|}\sum_{j,k=1}^{d}\frac{\partial n_\infty}{\partial\omega_k}B_{jk}(\nabla_A)_j u. \tag{3.2.29}$$

Thus by (3.2.5) and (3.2.14), we get

$$\begin{aligned} \Im\sum_{j,k=1}^{d}\int\frac{\partial\psi_q}{\partial x_k}B_{jk}(\nabla_A)_j u\bar{u} &\leq \frac{C\|n_\infty\|}{R}\left(\int_{|x|\geq R}\frac{|\nabla_A^\perp u|^2}{|x|}\right)^{1/2}|||n^{1/2}u||| \\ &+\frac{C\|n_\infty\|_{C^1}}{R}|||\nabla_A u||||||n^{1/2}u|||. \end{aligned} \tag{3.2.30}$$

We now turn to analyze the terms containing $f$. On the one hand, by (3.2.22), we have

$$\int|f||\Delta\psi_q||u|\leq\frac{C}{R^2}\int_{|x|>R}|f||u|\leq\frac{C}{R^2}N\left(\frac{f}{n^{1/2}}\right)|||n^{1/2}u|||.$$

On the other hand, from (3.2.21) it follows that

$$\begin{aligned} \int|f||\nabla\psi_q||\nabla_A u| &\leq \int\frac{q'}{R}|f||n_\infty||\nabla_A u|+\int\frac{q}{|x|}|f|\left|\frac{\partial n_\infty}{\partial\omega}\right||\nabla_A^\perp u| \\ &\leq \frac{C\|n_\infty\|_{C^1}}{R}N\left(\frac{f}{n^{1/2}}\right)\left(|||\nabla_A u|||+\left(\int_{|x|\geq R}\frac{|\nabla_A^\perp u|^2}{|x|}\right)^{1/2}\right). \end{aligned}$$

Finally, the last term to be bounded is

$$\varepsilon\int\nabla\psi_q\cdot\nabla_A u\bar{u}\leq\frac{C\|n_\infty\|_{C^1}}{R}\varepsilon\int|\nabla_A u||u|, \tag{3.2.31}$$

which can be done as in (3.2.12).

Therefore, from the above inequalities, taking $\kappa$ small enough yields

$$\begin{aligned} \int q\left|\frac{\partial n_\infty}{\partial\omega}\right|^2\frac{|u|^2}{|x|^2} &\leq \frac{C_1}{R}\left(|||n^{1/2}u|||^2+|||\nabla_A u|||^2+\int_{|x|\geq R}\frac{|\nabla_A^\perp u|^2}{|x|}\right) \\ &+\frac{C_2}{R}\left(N\left(\frac{f}{n^{1/2}}\right)\right)^2, \end{aligned} \tag{3.2.32}$$

which gives (3.2.18) and the proof of the theorem is over.

□

A combination of the above results asserts the desired energy estimate.

**Corollary 3.2.4.** *For dimensions $d \geq 3$, assume (3.2.3)-(3.2.5) and (3.2.14)-(3.2.16). Then the solution of the Helmholtz equation (3.2.2) satisfies, for $R \geq 1$ large enough*

$$\int_{|x|\geq R} \left| \nabla_\omega n_\infty \left(\frac{x}{|x|}\right)\right|^2 \frac{|u|^2}{|x|} \leq C \left(N\left(\frac{f}{n^{1/2}}\right)\right)^2, \tag{3.2.33}$$

*for some constant $C$ independent of $\varepsilon$.*

**Remark 3.2.5.** *From Remark 3.2.2 the same result holds for the two dimensional case.*

**Remark 3.2.6.** *Condition (3.2.16) can be largely relaxed if, for example, $n - n_\infty$ is radial. It can be instead assumed the alternative conditions*

$$\left| n(x) - n_\infty \left(\frac{x}{|x|}\right)\right| \leq n \frac{\Gamma}{|x|^\delta} \quad \textit{for} \quad |x| > R_0, \Gamma > 0, \delta > 0, R_0 > 1 \tag{3.2.34}$$

*and that there exist $\tilde{\beta} < 1$, $\delta > 0$ and $\tilde{\Gamma} > 0$ such that*

$$\left( |x| \nabla^\perp (n - n_\infty) \cdot \frac{\partial n_\infty}{\partial \omega} \right)_- \leq \tilde{\beta} \left| \frac{\partial n_\infty}{\partial \omega} \right|^2 + n(x) \frac{\tilde{\Gamma}}{|x|^\delta}. \tag{3.2.35}$$

*In particular, when $n - n_\infty$ is radial then (3.2.34) is sufficient.*

**Remark 3.2.7.** *Note that in order to prove the energy estimate we impose conditions in each component of the magnetic field $B_{jk}$ and not in the tangential component of $B$, as in the first result. This is due to the fact that the test function chosen in the proof of Theorem 3.2.3 is not radial (see (3.2.21) and (3.2.29) above).*

## 3.3 Limiting absorption principle

Our next goal is to prove the limiting absorption principle for the equation

$$\nabla_A^2 u + \lambda(1 + \tilde{V}_1) u + V_2 u = f \tag{3.3.1}$$

with $\lambda > 0$, $\varepsilon > 0$, where the given functions $\tilde{V}_1(x)$, $V_2(x)$, $A_j(x)$ hold the following assumptions:

**Assumption 3.3.1.** *Let $A_j(x)$, $j = 1, \dots, d$, $\tilde{V_1}(x)$, $V_2(x)$ be real-valued functions, $r_0 \geq 1$ and $\mu > 0$. Recall that we define each component of the magnetic field $B$ as*

$$B_{jk} = \frac{\partial A_j}{\partial x_k} - \frac{\partial A_k}{\partial x_j}, \qquad j, k = 1, \dots, d.$$

*We require that the magnetic potential satisfies the condition*

$$|\nabla \cdot A| \leq c|x|^{-2}, \tag{3.3.2}$$

*for some $c > 0$.*
*We assume that when $d \geq 3$*

$$|B_{jk}(x)| + |V_2(x)| \leq \frac{c}{|x|^{1+\mu}}, \quad \text{if} \quad |x| \geq r_0, \tag{3.3.3}$$

*and*

$$|V_2(x)| \leq \frac{c}{|x|^{2-\alpha}} \quad \text{if} \quad |x| \leq r_0, \quad 0 < \alpha < 2, \tag{3.3.4}$$

*for some $c > 0$.*
*If $d > 3$, we consider*

$$|B| \leq \frac{c_1}{|x|^2} \qquad |x| \leq r_0, \tag{3.3.5}$$

*for some $c_1 > 0$ small enough; in dimension $d = 3$ we require*

$$|B| \leq \frac{c}{|x|^{2-\alpha}} \qquad |x| \leq r_0, \quad 0 < \alpha < 2, \tag{3.3.6}$$

*for some $c > 0$.*
*As far as the potential $\tilde{V_1}(x)$ is concerned, let $\tilde{V_1}(x) \in C^2(\mathbb{R}^d \backslash \{0\})$ and we assume*

$$|\partial^\alpha \tilde{V_1}(x)| \leq C^* |x|^{-\alpha} \quad (|\alpha| \leq 2), \tag{3.3.7}$$

*where $\alpha = (\alpha_1, \dots, \alpha_d)$ is an arbitrary multi-index with nonnegative integers $\alpha_j$ ($1 \leq j \leq d$), $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ and $C^*$ is a positive small constant ($0 < C^* < 1$).*

**Remark 3.3.2.** *Observe that the assumptions required for $V_2$ coincide with the ones in the previous chapter. Regarding to the magnetic field $B$, the only difference is related to its behavior at infinity; in this case we need to impose the condition on $B_{jk}$ and not only on $B_\tau$ as in (2.1.3). Concerning the long range potential $\tilde{V_1}$, we require less decay at infinity, but more regularity in order to construct the solution of the corresponding eikonal equation.*

**Remark 3.3.3.** *As in the previous chapter, Assumption 3.3.1 makes the unique continuation result by Regaboui [R] applicable to the magnetic operator $T$ (3.0.27). Unique continuation property will be necessary for proving the uniqueness result for the equation (3.3.1).*

Then we state the main result of this section.

**Theorem 3.3.4.** *Let $c_1$, $C^*$ small enough, $\lambda_1, \lambda_0 > 0$ such that $\lambda_1 > \lambda_0$ and $f \in L^2_{\frac{1+\delta}{2}}$. Assume one of the following two conditions:*

*(i) $d > 3$, with (3.3.3), (3.3.4), (3.3.5), (3.3.7)*

*(ii) $d = 3$, with (3.3.3), (3.3.4), (3.3.6), (3.3.7).*

*Then, for any $\lambda \in [\lambda_0, \lambda_1]$ there exists a unique solution $u \in (H^1_A)_{loc}(\mathbb{R}^d)$ of the Helmholtz equation (3.3.1) satisfying*

$$\lambda |||u|||_1^2 + |||\nabla_A u|||_1^2 \leq C(N_1(f))^2, \tag{3.3.8}$$

*and the radiation condition*

$$\int_{|x|\geq 1} \left|\nabla_A u - i\lambda^{1/2}\nabla K u\right|^2 \frac{1}{(1+|x|)^{1-\delta}} \leq C \int_{|x|\geq 1/2} (1+|x|)^{1+\delta}|f|^2, \tag{3.3.9}$$

*for any $0 < \delta \leq 1$ such that $\delta < \mu$, where $C = C(\lambda_0)$.*

This theorem is an analogue result of the work by Saito [S]. We improve this result in the sense that we permit stronger singularities on the potentials at the origin and we give an estimate of the Agmon-Hörmander norm of the solution $u$, recovering then the $L^2$-weighted estimate (2.2.36) which is the one proved in [S] for $\delta > 0$.

To do this, we first prove the corresponding Sommerfeld radiation condition and a-priori estimates for the solution of the equation

$$\nabla_A^2 u + \lambda(1 + \tilde{V}_1)u + V_2 u + i\varepsilon u = f, \tag{3.3.10}$$

for $\lambda \in [\lambda_0, \lambda_1]$ with $0 < \lambda_0 < \lambda_1 < \infty$ and $\varepsilon > 0$. We next turn to show the uniqueness result related to this equation. Indeed, we will see that if $u$ satisfies (3.3.10) with $\varepsilon = 0$ and $f = 0$, then $u \equiv 0$. Consequently, we will be in a position to construct the unique solution of the equation (3.3.1) with the condition (3.3.9) at infinity. The detailed proof of this construction is given in the previous chapter, see subsection 2.2.4. Thus we will omit it.

We begin by proving that the Sommerfeld radiation condition holds if the Agmon-Hörmander norm of the solution of the electromagnetic Helmholtz equation (3.3.10) is bounded. Making use of this inequality, we deduce the a-priori estimates when $\lambda \in [\lambda_0, \lambda_1]$ by the compactness argument already used in section 2.2. Finally, we state and prove the uniqueness of solution of the equation (3.3.1).

Since the proofs are adapted from the ones of the main results of the previous chapter, we will mainly focus on the analysis of the new terms, that is to say, $\tilde{V}_1$.

### 3.3.1 Sommerfeld radiation condition

We proceed by proving the Sommerfeld condition in terms of the Agmon-Hörmander norm of the solution. This result may be proved in much the same way as Proposition 2.2.7 of Chapter 2.

**Proposition 3.3.5.** *For dimensions $d \geq 3$, let $\lambda_0 > 0$, $\varepsilon > 0$, $f \in L^2_{\frac{1+\delta}{2}}$ and assume that (3.3.3) holds. Then, there exists a positive constant $C = C(\lambda_0)$ such that for $\lambda \geq \lambda_0$ and $C^*$ small enough, any solution $u \in H^1_A(\mathbb{R}^d)$ of the equation (3.3.10) satisfies for all $R_1 \geq r_0$*

$$\begin{aligned}
&\int_{K \geq R_1} |\nabla_A u - i\sqrt{\lambda}\nabla K u|^2 \left( \frac{1}{(1+K)^{1-\delta}} + \varepsilon (1+K)^{\delta} \right) \\
&+ (1-\delta) \int_{K \geq R_1} \frac{|\nabla K|^2 |\nabla_A u|^2 - |\nabla K \cdot \nabla_A u|^2}{(1+K)^{1-\delta}} \\
&\leq C(1+\varepsilon) \left( |||u|||_1^2 + (N_1(f))^2 + \int_{K \geq R_1} (1+K)^{1+\delta} |f|^2 \right). \qquad (3.3.11)
\end{aligned}$$

*Proof.* The proof will be divided into three steps. As in the proof of Proposition 2.2.7, it consists in the construction of the Sommerfeld terms using the identities proved in Lemma 1.3.4 and Lemma 1.3.5. The main difference in this case is that one must choose the multipliers depending on the solution of the eikonal equation. Thus, by Remark 1.3.8 we consider the anti-symmetric multiplier

$$\mathcal{A} = E \cdot \nabla_A + \frac{1}{2} div E \qquad (3.3.12)$$

and we need to chose the vector field $E$ properly. By abuse of notation, we write $\nabla\psi$ instead of $E$.

Let $R_1 \geq r_0$. We take a cut off function $\theta \in C^\infty(\mathbb{R})$ such that $0 \leq \theta \leq 1$, $d\theta/dr \geq 0$ with

$$\theta(r) = \begin{cases} 1 & \text{if } r \geq R_1 + 1 \\ 0 & \text{if } r \leq R_1, \end{cases} \qquad (3.3.13)$$

and set $\theta(K) = \theta(K(x, C^*))$. We define $\Psi : \mathbb{R} \to \mathbb{R}$ such that

$$\Psi'(r) = (1+r)^\delta, \qquad 0 < \delta < 1$$

and we set $\Psi(K) = \Psi(K(x, C^*))$.

**Step 1.** Let us first compute

$$(1.3.15) + (1.3.12),$$

with the following choice of the multipliers

$$\begin{aligned}
E &= \nabla\psi = \Psi'(K)\nabla K \theta(K) \\
\varphi(x) &= \frac{\delta |\nabla K|^2}{2(1+K)^{1-\delta}} \theta(K),
\end{aligned}$$

respectively. Let us analyze all the terms of the resulting identity by the same method as in the proofs of Propositions 1.3.9 and 2.2.7. In what follows, $\kappa$ denotes an arbitrary positive small constant and we use the same letter $C$ for any positive constant.

On the one hand, by (3.1.6) and the fact that $\theta'$ is nonnegative, we get

$$\begin{aligned}&\int \nabla_A u \cdot D^2\psi \cdot \overline{\nabla_A u} - \int \varphi|\nabla_A u|^2 > \frac{\delta}{2}\int_{\mathbb{R}^d} \frac{|\nabla K|^2|\nabla_A u|^2}{(1+K)^{1-\delta}}\theta(K)\\ &+ \int \theta(K)\left(\frac{(1+K)^\delta}{K} - \frac{\delta}{(1+K)^{1-\delta}}\right)\{|\nabla K|^2|\nabla_A u|^2 - |\nabla K\cdot\nabla_A u|^2\}\\ &+ \int \frac{(1+K)^\delta}{K}\sum_{k,j=1}^d (\nabla_A)_k u F_{kj}\overline{(\nabla_A)_j u}\theta(K)\\ &\equiv I_1 + I_2 + I_3,\end{aligned} \tag{3.3.14}$$

where

$$I_2 \geq (1-\delta)\int \frac{\theta(K)}{(1+K)^{1-\delta}}\{|\nabla K|^2|\nabla_A u|^2 - |\nabla K\cdot\nabla_A u|^2\}. \tag{3.3.15}$$

On the other hand, observe that in order to get the term related to $|u|^2$ of the Sommerfeld square $|\nabla_A u - i\lambda^{1/2}\nabla K u|^2$, we need to use the eikonal equation (3.1.1). Indeed, we have

$$\int \varphi\lambda(1+\tilde{V_1})|u|^2 = \frac{\delta}{2}\int \frac{|\nabla K|^2\lambda|\nabla K|^2|u|^2}{(1+K)^{1-\delta}}\theta(K). \tag{3.3.16}$$

Moreover, by the eikonal equation $\tilde{V_1}(x) = |\nabla K|^2 - 1$ and (3.1.6), it follows that

$$\frac{\partial\tilde{V_1}}{\partial x_k} = 2\sum_{j=1}^d \frac{1}{K}F_{kj}\frac{\partial K}{\partial x_j} \quad \text{for all } k = 1,\dots,d. \tag{3.3.17}$$

Thus the other term involving the potential $\tilde{V_1}$ gives

$$-\frac{\lambda}{2}\int \nabla\tilde{V_1}\cdot\nabla\psi|u|^2 = -\lambda\sum_{k,j=1}^d \int \frac{(1+K)^\delta}{K}\frac{\partial K}{\partial x_k}F_{kj}\frac{\partial K}{\partial x_j}|u|^2\theta(K) \equiv I_4.$$

Let us treat now the terms containing the magnetic field $B$ and the potential $V_2$. Since $c \leq |\nabla K|^2 \leq \tilde{c}$ for some $c, \tilde{c} > 0$, by (3.3.3) and Cauchy-Schwarz inequality we get

$$\begin{aligned}\sum_{k,m=1}^d \int \frac{\partial\psi}{\partial x_k}B_{km}u\overline{(\nabla_A)_m u} &\leq C\int |B_{km}||\nabla_A u||u|(1+K)^\delta\theta(K)\\ &\leq \kappa\int |\nabla K|^2|\nabla_A u - i\sqrt{\lambda}\nabla K u|^2\frac{\theta(K)}{(1+K)^{1-\delta}}\\ &+ C_\kappa(\sqrt{\lambda}+1)|||u|||_1^2.\end{aligned} \tag{3.3.18}$$

Similarly, by (3.3.3) we have

$$\Re \int V_2 \nabla\psi \cdot \nabla_A u\bar{u} \leq \kappa \int \frac{|\nabla K|^2 |\nabla_A u - i\sqrt{\lambda}\nabla K u|^2 \theta(K)}{(1+K)^{1-\delta}} + C_\kappa(\sqrt{\lambda}+1)|||u|||_1^2.$$

In addition, since

$$\Delta\psi = \Psi''(K)|\nabla K|^2\theta(K) + \Psi'(K)\Delta K\theta(K) + \Psi'(K)|\nabla K|^2\theta'(K),$$

by (3.1.2), (3.1.5) it may be concluded that

$$-\int \varphi V_2 |u|^2 + \frac{1}{2}\int V_2(x)\Delta\psi|u|^2 \leq C|||u|||_1^2. \tag{3.3.19}$$

As a consequence, we get the inequality

$$\begin{aligned}
&\frac{\delta}{2}\int |\nabla K|^2(|\nabla_A u|^2 + \lambda|\nabla K|^2|u|^2)\frac{\theta(K)}{(1+K)^{1-\delta}} \\
&+ (1-\delta)\int \frac{\theta(K)}{(1+K)^{1-\delta}}\{|\nabla K|^2|\nabla_A u|^2 - |\nabla K\cdot\nabla_A u|^2\} \\
&- \varepsilon Im \int \theta(K)\Psi'(K)\nabla K\cdot\nabla_A u\bar{u} \leq -I_3 + I_4 \\
&+ 2\kappa\int \frac{|\nabla K|^2|\nabla_A u - i\sqrt{\lambda}\nabla K u|^2\theta(K)}{(1+K)^{1-\delta}} + C(\sqrt{\lambda}+1)|||u|||_1^2 \\
&- \Re\int f\left(\Psi'(K)\nabla K\cdot\overline{\nabla_A u} + \frac{1}{2}\Psi'(K)\Delta K\right)\theta(K)\bar{u} \\
&- \frac{\Re}{2}\int \Psi'(K)|\nabla K|^2\theta'(K)\bar{u}.
\end{aligned}$$

**Step 2.** In order to obtain the desired square, let us add to the above inequality the identity (1.3.13) with the choice of a test function

$$\varphi(x) = \sqrt{\lambda}|\nabla K|^2(1+K)^\delta\theta(K). \tag{3.3.20}$$

Hence, it follows that

$$\begin{aligned}
\Im\int \nabla\varphi\cdot\nabla_A u\bar{u} &= \delta\Im\sqrt{\lambda}\int \frac{\theta(K)}{(1+K)^{1-\delta}}|\nabla K|^2\nabla K\cdot\nabla_A u\bar{u} \\
&+ \sqrt{\lambda}\Im\int |\nabla K|^2\theta'(K)(1+K)^\delta\nabla K\cdot\nabla_A u\bar{u} \\
&+ 2\sqrt{\lambda}\Im\int \frac{(1+K)^\delta}{K}\sum_{k,j=1}^d (\nabla_A)_k u F_{kj}\frac{\partial K}{\partial x_j}\bar{u}\theta(K) \\
&\equiv I_5 + I_6 + I_7.
\end{aligned}$$

The term $I_5$ is used to complete the square $|\nabla_A u - i\sqrt{\lambda}\nabla K u|^2$; $I_6$ can be upper bounded by

$$\kappa\int |\nabla K|^2|\nabla_A u - i\sqrt{\lambda}\nabla K u|^2\frac{\theta(K)}{(1+K)^{1-\delta}} + C_{\kappa,R_1}(1+\lambda)|||u|||_1^2. \tag{3.3.21}$$

In addition, denoting $(D_K)_i u = (\nabla_A)_i u - i\sqrt{\lambda}\frac{\partial K}{\partial x_i}u$, by (3.1.12) it may be concluded that

$$\begin{aligned}-I_3 + I_4 + I_7 &= -\int \frac{(1+K)^\delta}{K}\sum_{k,j=1}^{d}(D_K)_k u F_{kj}\overline{(D_K)_j u}\theta(K)\\ &\le CC^*\int\frac{|\nabla K|^2|\nabla_A u - i\sqrt{\lambda}\nabla K u|^2\theta(K)}{(1+K)^{1-\delta}}.\end{aligned} \tag{3.3.22}$$

Therefore, we deduce

$$\begin{aligned}&\frac{\delta}{2}\int |\nabla K|^2|\nabla_A u - i\sqrt{\lambda}\nabla K u|^2\frac{\theta(K)}{(1+K)^{1-\delta}}\\ &+ (1-\delta)\int\frac{\theta(K)}{(1+K)^{1-\delta}}\{|\nabla K|^2|\nabla_A u|^2 - |\nabla K\cdot\nabla_A u|^2\}\\ &+ \varepsilon\sqrt{\lambda}\int |\nabla K|^2(1+K)^\delta|u|^2\theta(K) - \varepsilon\Im\int\theta(K)(1+K)^\delta\nabla K\cdot\nabla_A u\bar{u}\\ &\le C(\lambda+1)|||u|||^2 + (3\kappa + CC^*)\int\frac{|\nabla K|^2|\nabla_A u - i\sqrt{\lambda}\nabla K u|^2\theta(K)}{(1+K)^{1-\delta}}\\ &- \Re\int f(1+K)^\delta\nabla K\cdot(\overline{\nabla_A u} + i\lambda^{1/2}\nabla K\bar{u})\theta(K)\\ &- \frac{\Re}{2}\int f\Psi'(K)(\Delta K\theta(K) + |\nabla K|^2\theta'(K))\bar{u}.\end{aligned}$$

**Step 3.** Let us subtract the identity (1.3.12) multiplied by $\varepsilon$ to the above inequality choosing the test function

$$\varphi(x) = \frac{1}{2\sqrt{\lambda}}\Psi'(K)\theta(K),$$

so that we get

$$\frac{\varepsilon}{\lambda^{1/2}}\int |\nabla K|^2(1+K)^\delta|\nabla_A u - i\lambda^{1/2}\nabla K u|^2\theta(K). \tag{3.3.23}$$

In order to complete the estimate, by integration by parts and the a-priori estimate (1.3.9), we have

$$\begin{aligned}\varepsilon\Re\int\nabla\varphi\cdot\nabla_A u\bar{u} &= \frac{\varepsilon}{2}\int\Delta\varphi|u|^2\\ &\le C\varepsilon\int|u|^2 \le CN_1(f)|||u|||_1.\end{aligned} \tag{3.3.24}$$

Furthermore, by (3.3.3) we deduce

$$\begin{aligned}\varepsilon \int \varphi V_2 |u|^2 &\leq \frac{C\varepsilon}{\lambda^{1/2}} \int_{|x|\geq r_0} \frac{|u|^2}{(1+|x|)^{1+\mu-\delta}} \\ &\leq C\varepsilon |||u|||_1^2.\end{aligned}$$

Finally, let us estimate the terms containing $f$. On the one hand, we have

$$\begin{aligned}&-\Re \int f(1+K)^\delta \nabla K \cdot (\overline{\nabla_A u} + i\lambda^{1/2}\nabla K \bar{u})\theta(K) \\ &\qquad \leq \kappa \int |\nabla K|^2 |\nabla_A u - i\lambda^{1/2}\nabla K u|^2 \frac{\theta(K)}{(1+K)^{1-\delta}} \\ &\qquad + C(\kappa) \int (1+K)^{1+\delta}|f|^2\theta(K).\end{aligned}$$

By (3.1.5), we get

$$\begin{aligned}&-\frac{\Re}{2}\int \Psi'(K)(\Delta K\theta(K) + |\nabla K|^2\theta'(K)) f\bar{u} \\ &\qquad \leq C\left(|||u|||_1^2 + \int (1+K)^{1+\delta}|f|^2\theta(K)\right).\end{aligned}$$

By the a-priori estimate (1.3.9), yields

$$\begin{aligned}&-\frac{\varepsilon}{2\sqrt{\lambda}}\Re\int (1+K)^\delta f\bar{u}\theta(K) \\ &\qquad \leq \left(\frac{4\varepsilon}{\lambda}\int |f|^2(1+K)^{1+\delta}\theta(K)\right)^{1/2}\left(\varepsilon\int |u|^2\right)^{1/2} \\ &\qquad \leq C\left(\varepsilon\int |f|^2(1+K)^{1+\delta}\theta(K) + |||u|||_1^2 + (N_1(f))^2\right).\end{aligned}$$

Consequently, taking $\kappa > 0$ and $C^*$ small enough, we obtain (3.3.11) and the proof is complete. □

**Corollary 3.3.6.** *Under the assumption of Proposition 3.3.5, the solution $u \in H_A^1(\mathbb{R}^d)$ of the Helmholtz equation (3.3.10) satisfies*

$$\begin{aligned}&\int_{|x|\geq r_0} \frac{|\nabla_A u - i\lambda^{1/2}\nabla K u|^2}{(1+|x|)^{1-\delta}} + \varepsilon\int_{|x|\geq r_0}(1+|x|)^\delta|\nabla_A u - i\lambda^{1/2}\nabla K u|^2 \\ &\leq C(1+\varepsilon)\left(|||u|||_1^2 + (N_1(f))^2 + \int_{|x|\geq r_0}(1+|x|)^{1+\delta}|f|^2\right),\end{aligned} \tag{3.3.25}$$

*for $\lambda \geq \lambda_0$, $C^*$ small enough and $C = C(\lambda_0)$.*

*Proof.* We need only take $R_1 = c_0 r_0$ with $c_0, r_0$ given in section 3.1 and use (3.1.4). □

### 3.3.2 A priori estimates for $\lambda \in [\lambda_0, \lambda_1]$

Using the previous result, we are now in a position to prove the a-priori estimates for the frequency $\lambda$ varying in a compact set.

**Proposition 3.3.7.** *For $d \geq 3$, under the hypotheses of Proposition 3.3.5, let $\lambda_0 > 0$, $\lambda \in [\lambda_0, \lambda_1]$, with $\lambda_1 > \lambda_0$ and $\varepsilon \in (0, \varepsilon_1)$. Then, the solution $u \in H^1_A(\mathbb{R}^d)$ of the Helmholtz equation (3.3.10) satisfies*

$$\lambda|||u|||_1^2 + |||\nabla_A u|||_1^2 \leq C(1+\varepsilon)(N_1(f))^2, \tag{3.3.26}$$

*where $C = C(\lambda_0, \varepsilon_1)$.*

*Proof.* The proof is a combination of the proof of Proposition 3.1 in [S] and the proof of Proposition 2.2.12 in Chapter 2.

Let $B_T$ be the interior of the closed surface $\Sigma_T = \{x : K(x, C^*) = T\}$ with $T > r_0$ and $C^* < C_0$, where $r_0$ and $C_0$ are given constants related to the assumptions of the potentials and the solution to the eikonal equation, respectively (see Assumption 3.3.1 and section 3.1). Let us multiply the equation (3.3.10) by $\bar{u}$, integrate over $B_T$ and take the imaginary part, obtaining

$$\Im \int_{\Sigma_T} \frac{\nabla K}{|\nabla K|} \cdot \nabla_A u \bar{u} + \varepsilon \int_{B_T} |u|^2 = \Im \int_{B_T} f \bar{u}.$$

From this it follows that

$$2\sqrt{\lambda}\Im \int_{\Sigma_T} \frac{\nabla K}{|\nabla K|} \cdot \nabla_A u \bar{u} \leq 2\sqrt{\lambda}\Im \int_{B_T} f \bar{u}. \tag{3.3.27}$$

Let us integrate now the identity

$$\frac{|\nabla_A u|^2}{|\nabla K|} + \lambda |\nabla K||u|^2 = \frac{1}{|\nabla K|}|\nabla_A u - i\sqrt{\lambda}\nabla K u|^2 + 2\Im\sqrt{\lambda}\frac{\nabla K}{|\nabla K|} \cdot \nabla_A u \bar{u}$$

over the surface $\Sigma_T$. Then by (3.3.27) we get

$$\begin{aligned} \int_{\Sigma_T} \left( \frac{|\nabla_A u|^2}{|\nabla K|} + \lambda |\nabla K||u|^2 \right) \leq & \int_{\Sigma_T} \frac{1}{|\nabla K|}|\nabla_A u - i\sqrt{\lambda}\nabla K u|^2 \\ & + 2\sqrt{\lambda}N_1(f)|||u|||_1. \end{aligned} \tag{3.3.28}$$

Let $R > \frac{\rho c_0}{c_1}$, where $\rho \geq r_0$, being $c_0$, $c_1$ as in (3.1.4). Let us multiply both sides of (3.3.28) by $\frac{1}{R}$ and integrate from $\rho c_0$ to $R c_1$ with respect to $T$. Hence, as $|\nabla K|^2$ is lower bounded by a positive constant we have

$$\begin{aligned} \frac{1}{R}\int_{\rho c_0 \leq K \leq R c_1} (\lambda |u|^2 + |\nabla_A u|^2) \leq & \frac{1}{R}\int_{\rho c_0 \leq K \leq R c_1} |\nabla_A u - i\lambda^{1/2}\nabla K u|^2 \\ & + C\sqrt{\lambda}N(f)|||u|||. \end{aligned} \tag{3.3.29}$$

On the other hand, observe that since $K = |x|g$ and $c_0 \leq g \leq c_1$, yields

$$\{\rho \leq |x| \leq R\} \subset \{\rho c_0 \leq K \leq Rc_1\} \subset \left\{ \frac{\rho c_0}{c_1} \leq |x| \leq \frac{Rc_1}{c_0} \right\}.$$

Consequently, denoting $j_0$ and $j_1$ by $2^{j_0-1} \leq \frac{\rho c_0}{c_1} \leq 2^{j_0}$ and $2^{j_1-1} \leq \frac{Rc_1}{c_0} \leq 2^{j_1}$, respectively, we deduce

$$\begin{aligned} \frac{1}{R}\int_{\rho \leq |x| \leq R} (\lambda|u|^2 + |\nabla_A u|^2) \leq \frac{1}{R}\sum_{j=j_0}^{j_1} \int_{C(j)} |\nabla_A u - i\lambda^{1/2}\nabla K u|^2 \\ + \kappa\lambda|||u|||_1^2 + C(\kappa)(N_1(f))^2. \end{aligned} \tag{3.3.30}$$

Now, note that we are in the same situation as in (2.2.54) of the proof of Proposition 2.2.12. Therefore, by (3.3.9), repeating the same reasoning to this case, it may be concluded that for $R > 1$

$$\frac{1}{R}\int_{|x| \leq R} (\lambda|u|^2 + |\nabla_A u|^2) \leq \frac{\lambda}{2}|||u|||_1^2 + C(1+\varepsilon)(N_1(f))^2. \tag{3.3.31}$$

Thus taking the supremum over $R$, the proposition follows. $\square$

### 3.3.3 Uniqueness result

This paragraph deals with the uniqueness of solution of the equation (3.3.1). Let us consider the homogeneous Helmholtz equation

$$\nabla_A^2 u + \lambda(1 + \tilde{V_1})u + V_2 u = 0. \tag{3.3.32}$$

Then we formulate the uniqueness theorem as follows.

**Theorem 3.3.8.** *Let $d \geq 3$, $\lambda_0 > 0$ and assume (3.3.3). Let $u$ be a solution of the equation (3.3.32) with $u, \nabla_A u \in L^2_{loc}$ such that*

$$\liminf \int_{|x|=r} (|\nabla_A u|^2 + \lambda|u|^2) d\sigma(x) \to 0, \quad as \quad r \to \infty, \tag{3.3.33}$$

*for $\lambda \geq \lambda_0$. Then $u \equiv 0$.*
*Moreover, if for some $\delta > 0$ the condition*

$$\int_{|x| \geq 1} |\nabla_A u - i\lambda^{1/2}\nabla K u|^2 \frac{1}{(1+|x|)^{1-\delta}} < \infty \tag{3.3.34}$$

*is satisfied, then (3.3.33) holds.*

*Proof.* The proof follows by the same method as in the proof of Theorem 2.1.5. Although the analysis of the terms related to $\tilde{V}_1$ are slightly different, the same conclusion can be drawn for this case.

In order to deduce (3.3.33) from (3.3.34), first observe that solutions of (3.3.32) satisfy

$$\Im \int_{\Sigma_T} \frac{\nabla K}{|\nabla K|} \cdot \nabla_A u \bar{u} = 0,$$

just multiplying the equation by $\bar{u}$ and integrating over $B_T$, the inside of the closed surface $\Sigma_T = \{x : K(x, C^*) = T\}$. Hence, we have

$$\int_{\Sigma_T} (|\nabla_A u|^2 + \lambda |\nabla K|^2 |u|^2) d\sigma(x) = \int_{\Sigma_T} |\nabla_A u - i\sqrt{\lambda} \nabla K u|^2 d\sigma(x),$$

which together with (3.3.34) gives (3.3.33).

□

## 3.4 Explicit radiation condition

This section establishes the relation between the energy estimate (3.2.33) and the Sommerfeld condition (3.3.9). We will see that when the variable index of refraction $n(x) = \lambda(1 + \tilde{V}_1(x))$ has an angular dependency like $n(x) \to n_\infty\left(\frac{x}{|x|}\right)$ as $|x| \to \infty$, then the Sommerfeld condition (3.3.9) at infinity still holds under the explicit form

$$\int \left| \nabla_A u - i n_\infty^{1/2} \frac{x}{|x|} u \right|^2 < \infty. \tag{3.4.1}$$

Note that the spherical term

$$n_\infty^{1/2}(\omega) \frac{x}{|x|}$$

appears in this formula instead of the phase as in (3.3.9), where $\nabla K$ is the outward normal of the surface $|K(x, \lambda)| = \lambda$, which is not necessarily a sphere. This apparent contradiction can be explained by the extra estimate (3.2.33) on the energy decay. In fact, it can be interpreted as a concentration of the energy along the directions given by the critical points of $n_\infty$.

The following result complements that of Saito [S] when $V_2 = 0$ and extends the one given in [PV2] to the magnetic case. Furthermore, it asserts the connection between the previous two sections of this chapter.

**Theorem 3.4.1.** *For dimension $d \geq 3$, assume (3.2.5) and (3.3.7). Then for sufficiently small $C^* > 0$ and for any $\lambda \in [\lambda_0, \lambda_1]$ with $0 < \lambda_0 < \lambda_1 < \infty$, there exists a unique solution of the Helmholtz equation (3.0.1) satisfying*

$$\int \left| \nabla_A u - i n^{1/2}(x) \frac{x}{|x|} u \right|^2 \frac{1}{|x|} \leq C_\delta \int (1+|x|)^{1+\delta} |f|^2, \tag{3.4.2}$$

*for some $\delta > 0$. Moreover, if there exist $n_\infty$, $\Gamma > 0$ and $\mu > 0$ such that*

$$\left| n(x) - n_\infty \left( \frac{x}{|x|} \right) \right| \leq n(x) \frac{\Gamma}{|x|^\mu} \quad \text{for} \quad |x| \quad \text{large enough}, \tag{3.4.3}$$

*then it follows that*

$$\int_{|x|\geq 1} \left| \nabla_A u - i n_\infty^{1/2} \frac{x}{|x|} u \right|^2 \frac{1}{|x|} \leq C \int (1+|x|)^{1+\delta} |f|^2. \tag{3.4.4}$$

*Proof.* The proof follows [PV2]. Let us first recall the tangential estimate

$$\int \frac{|\nabla_A^\perp u|^2}{|x|} \leq C(N(f))^2 \tag{3.4.5}$$

proved in Theorem 3.2.1 above and observe that (3.3.9) provides

$$\int |\nabla_A u - i\lambda^{1/2} \nabla K u|^2 \frac{1}{1+|x|} \leq C \int (1+|x|)^{1+\delta} |f|^2 dx. \tag{3.4.6}$$

Hence, just looking at the tangential part of the above inequality, by (3.4.5) it follows easily that

$$\begin{aligned} \int_{|x|\geq r_0} \lambda |\nabla^\perp K u|^2 \frac{1}{1+|x|} &\leq C \int \frac{|\nabla_A^\perp u|^2}{1+|x|} + C \int (1+|x|)^{1+\delta} |f|^2 \\ &\leq C \int (1+|x|)^{1+\delta} |f^2|. \end{aligned} \tag{3.4.7}$$

Furthermore, since $n = \lambda(1+\tilde{V_1})$, from the eikonal equation (3.1.1) we have

$$n - \lambda |\partial_r K|^2 = |\lambda^{1/2} \nabla^\perp K|^2. \tag{3.4.8}$$

Now, according to the properties (3.1.5) related to $\nabla K$, it is easy to see that $\partial_r K = g(x) + O(C^*) > 0$. Thus we obtain

$$|\lambda^{1/2} \partial_r K - n^{1/2}| = \frac{|\lambda^{1/2} \nabla^\perp K|^2}{|\lambda^{1/2} \partial_r K + n^{1/2}|} \leq C |\lambda^{1/2} \nabla^\perp K|^2. \tag{3.4.9}$$

In addition, looking at the radial part in (3.4.6) we have

$$\int_{|x|\geq r_0} |\nabla_A^r u - i\lambda^{1/2}\partial_r K u|^2 \frac{1}{1+|x|} \leq C \int (1+|x|)^{1+\delta}|f|^2. \tag{3.4.10}$$

Consequently, by (3.4.7), (3.4.9), (3.4.10) and the fact that

$$\begin{aligned}\left|\nabla_A u - in^{1/2}\frac{x}{|x|}u\right|^2 \leq{}& |\nabla_A^r u - i\sqrt{\lambda}\partial_r K u|^2 + |\sqrt{\lambda}\partial_r K u - n^{\frac{1}{2}}u|^2 \\ &+ |\nabla_A^{\perp} u|^2,\end{aligned}$$

we get (3.4.2) which is our first claim.

Finally, assuming $|n - n_\infty| \leq C(1+|x|)^{-\delta}$ and using (3.2.6) we conclude that

$$\int \left|\nabla_A u - in_\infty^{1/2}\frac{x}{|x|}u\right|^2 \frac{1}{1+|x|} \leq C \int |f|^2(1+|x|)^{1+\delta} \tag{3.4.11}$$

and the proof is complete.

□

# Chapter 4

# Resolvent estimates and Applications

In the last chapter of the thesis we study more topics related to the forward problem of the magnetic Schrödinger operator

$$H_A = \nabla_A^2 + V \tag{4.0.1}$$

with potentials that have a strong singularity at the origin. Besides the limiting absorption principle and resolvent estimates for the solution $u$ of the electromagnetic Helmholtz equation

$$(\nabla + iA)^2 u + Vu + \lambda u = f, \tag{4.0.2}$$

under suitable assumptions on the trapping component of the magnetic field, i.e.

$$B_\tau = \frac{x}{|x|} B, \qquad (B_\tau)_j = \sum_{k=1}^{d} \frac{x_k}{|x|} B_{jk}$$

and the scalar potential $V$, we also are able to prove the existence and uniqueness of the cross-section of $u$ and some spectral properties of $H_A$. Thus all the results that will be showed here are true for singular magnetic potentials $A$ such that $B_\tau \equiv 0$, see section 1.6.

Under some smallness conditions on $B_\tau$ and $V$, in the first part of the chapter we will study the electromagnetic Helmholtz equation

$$(\nabla + iA)^2 + Vu + \lambda u + i\varepsilon u = f, \qquad \varepsilon > 0. \tag{4.0.3}$$

On the one hand, we will prove that the solution $u \in H_A^1(\mathbb{R}^d)$ of the equation (4.0.3) allowing the sharp singularity $|x|^{-2}$ of the electric potential $V$ at the origin, satisfies the following uniform estimate for any $\lambda > \varepsilon > 0$

$$\int |\nabla_A (e^{-i\lambda^{1/2}|x|} u)|^2 \leq C \int |x|^2 |f|^2. \tag{4.0.4}$$

and

$$\int |\nabla_A u|^2 \leq C \int |x|^2 |f|^2 \tag{4.0.5}$$

when $\lambda \leq \varepsilon$. Note that by the magnetic Hardy inequality these estimates provide the resolvent estimate

$$\int \frac{|u|^2}{|x|^2} \leq C \int |x|^2 |f|^2. \tag{4.0.6}$$

for all $\lambda \in \mathbb{R}$ and $\varepsilon > 0$.

Our second main result shows that for $\lambda \geq \lambda_0 > 0$ the solution $u$ satisfies the radiation condition

$$\sup_{R \geq 1} R \int_{|x| \geq R} |\nabla_A (e^{-i\lambda^{1/2}|x|} u)|^2 \leq C \int (1 + |x|)|x|^2 |f|^2 \tag{4.0.7}$$

where $C = C(\lambda_0) > 0$.

Having disposed of these results, the remainder of the chapter will be devoted to the study of some spectral properties of the magnetic Schrödinger operator $H_A$. For this purpose, we will first give the limiting absorption principle for the equation (4.0.2) with singular potentials of the type $\frac{1}{|x|^2}$ at the origin. Indeed, we will construct the solution $u_\pm = u(\lambda \pm i0, f)$ of the equation $(H_A + \lambda)u_\pm = f$ as the limit

$$u(\lambda \pm i0, f) = \lim_{\varepsilon \to 0} u(\lambda \pm i\varepsilon, f), \tag{4.0.8}$$

where $u_{\pm\varepsilon} = u(\lambda \pm i\varepsilon, f)$ is the unique solution of the equation

$$(H_A + \lambda \pm i\varepsilon)u_{\pm\varepsilon} = f. \tag{4.0.9}$$

Once the limiting absorption method is applicable, the absolute continuity of $H_A$ on $(0, \infty)$ readily follows.

Let $R(z) = (H_A + z)^{-1}$ denote the resolvent of $H_A$ so that $R(z)f = u(z, f)$. On the one hand, let us recall the formula that relates the spectral measure of $H_A$ with its resolvent. Let $\Delta = (\lambda_1, \lambda_2)$ where $0 < \lambda_1 < \lambda_2 < \infty$ and $E(\Delta)$ the spectral measure associated with $H_A$ where $\Delta$ varies over all Borel sets of the reals. Then we have the well known formula (see section 1.5)

$$(E(\Delta)f, f) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \lim_{\nu \to 0} \int_{\lambda_1 - \nu}^{\lambda_2 + \nu} \left( \Im \int f \bar{u} dx \right) d\lambda. \tag{4.0.10}$$

From this, we will prove that $H_A$ is an absolute continuous operator on $(0, \infty)$. On the other hand, we will be able to give the spectral representation of $H_A$ through the cross section of the solution $u_+ = R(\lambda + i0)f$ of the equation (4.0.2).

Let us denote $r = |x|$, $\omega = \frac{x}{|x|}$ and $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$. As in the classical case, by the asymptotic expansion of the free Helmholtz equation, if we denote

$$(\mathcal{F}(\lambda, r)f)(\omega) = C(\lambda) r^{\frac{d-1}{2}} e^{-i\lambda^{1/2} r} u(r\omega), \qquad \omega \in S^{d-1}, \tag{4.0.11}$$

where $C(\lambda) = e^{\frac{(d-3)\pi i}{4}} \pi^{-1/2} \lambda^{1/4}$, then the far field pattern of $u$ is the limit

$$g_\lambda(\omega) = \lim_{r\to\infty} r^{\frac{d-1}{2}} e^{-i\lambda^{1/2} r} u(r\omega) \qquad \text{in} \qquad L^2(S^{d-1}). \tag{4.0.12}$$

and the cross-section its absolute value. More concretely, the cross-section is given by

$$|g_\lambda(\omega)| = \lim_{r\to\infty} |(\mathcal{F}(\lambda, r) f)(\omega)| \qquad \text{in} \qquad L^2(S^{d-1}). \tag{4.0.13}$$

The existence of the limit (4.0.13) together with a suitable Sommerfeld condition for the solution $u$ permits to show the identity

$$\int_{|x|=1} |g_\lambda(\omega)|^2 = \Im \int f\bar{u}. \tag{4.0.14}$$

Consequently, a combination of (4.0.14) and (4.0.10) will provide some spectral properties of $H_A$.

There are several works in which the spectral representation for Schrödinger operators is obtained from the limiting absorption principle, by considering the following limit

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} e^{-iK(x,\lambda)} R(\lambda + i0) f(r\omega) \qquad \text{in} \qquad L^2(S^{d-1}). \tag{4.0.15}$$

Here $K(x, \lambda)$ is a real-valued function which behaves like $\lambda^{1/2}|x|$ at infinity and $R(\lambda + i0)$ denotes the boundary value of the resolvent of the corresponding operator on the upper side of the positive real axis.

For $H = -\Delta + V$, Agmon [A] obtained the spectral representation in the case of short range potential $V$ i.e., $V(x) = O((1+|x|)^{-1-\delta})$, $\delta > 0$, constructing the so called generalized eigenfunctions. In this case, $K(x, \lambda) = \lambda^{1/2}|x|$. For long range potentials, there have been many investigations since [Ik]. The best reference for us is due to Isozaki [Is], where it is given the spectral representation for $H$ by considering the limit (4.0.15) with $K(x, \lambda)$ as an approximate solution of the eikonal equation

$$|\nabla K(x, \lambda)|^2 + V(x) = \lambda. \tag{4.0.16}$$

Here $V(x)$ is a real-valued $C^3(\mathbb{R}^d)$ function such that for some $\mu > 0$

$$D_x^\alpha V(x) = O(|x|^{-|\alpha|-\mu}) \quad \text{as} \quad |x| \to \infty \quad (0 \le |\alpha| \le 3),$$

where $D_x = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right)$ and $\alpha$ is a multi-index.

The magnetic case has been studied by Iwatsuka [Iw], taking $K(x, \lambda)$ of the form $\lambda^{1/2}|x| - m(x)$, where $m(x)$ is a certain function depending on the magnetic potential $A(x)$ and related to the tangential component of the magnetic field $B$. More concretely, the function $m(x)$ is constructed as

$$m(x) = \sum_{k=1}^{d} x_k \int_0^1 A_k(tx)dt, \qquad (x \in \mathbb{R}^d). \tag{4.0.17}$$

As a consequence, it may be concluded that

$$\frac{\partial m}{\partial x_j} = A_j(x) + \int_0^1 \sum_{k=1}^{d} x_k B_{kj}(tx)dt. \tag{4.0.18}$$

Observe that the function $m(x)$ is related to the trapping component of the magnetic field $B$. See section 1.6 for more details. Then the spectral representation theorem for $H_A$ is obtained in the case of short range $V$ and for magnetic potentials such that $A_j \in C^2(\mathbb{R}^d)$ with $|B_{jk}| \leq C_0(1+|x|)^{-\frac{3}{2}-\delta}$ and $\left|\frac{\partial B_{jk}}{\partial x_j}\right| \leq C_0(1+|x|)^{-2-\delta}$, where $B_{jk} = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial A_k}$ $(j, k = 1, \ldots, d)$, $C_0, \delta > 0$.

In our case, even though we consider long range perturbations, we will take $K(x, \lambda) = \lambda^{1/2}|x|$ and we do not need to construct nor the solution of the eikonal equation as in [Is] neither the function $m(x)$ mentioned above (see [Iw] for more details). Therefore, following [Ik] and [Iw], we will prove that the limit (4.0.13) exists and the identity (4.0.14) holds. As a consequence, denoting $P_{ac} = E(0, \infty)$ the projection onto the absolute continuous subspace for $H_A$, we will deduce

$$(P_{ac}f, f) = \int_0^\infty \|g_\lambda(\omega)\|^2_{L^2(S^{d-1})}\, d\lambda. \tag{4.0.19}$$

## 4.1 Main results

Let us pass now to present the main results of this chapter. To this end, we first introduce the framework in which we work in the sequel. According to section 1.2 of this dissertation, the self-adjointness of $H_A$, that is fundamental for deriving all theorems, can be concluded under some local integrability assumptions on the potentials. More concretely, we have already proved in section 1.2 that assuming

$$\int V|u|^2 < 1^- \int |\nabla u|^2, \tag{4.1.1}$$

$$A_j \in L^2_{loc}, \qquad\qquad V \in L^1_{loc}, \tag{4.1.2}$$

then $H_A$ is self-adjoint operator in $L^2(\mathbb{R}^d)$ with form domain

$$D(H_A) := \{\phi \in L^2(\mathbb{R}^d) : \int |\nabla_A \phi|^2 - \int V|\phi|^2 < \infty\}. \tag{4.1.3}$$

In order to get the estimate (4.0.4), we will make the following assumption:

**(H1)**

$$\int (\partial_r(rV))_-|u|^2 < A_V \int |\nabla u|^2,$$

$$\left(\int |x|^2|B_\tau|^2|u|^2\right)^{1/2} < A_B \left(\int |\nabla u|^2\right)^{1/2},$$

where

$$A_V + 2A_B < 1. \tag{4.1.4}$$

**Remark 4.1.1.** *Note that from* (4.1.1) *it follows that*

$$\int |x|V|u|^2 < \int |x||\nabla u|^2. \tag{4.1.5}$$

*We just only need to define $w = |x|^{1/2}u$. Then, by (4.1.1) and integration by parts, we get*

$$\begin{aligned}\int |x|V|u|^2 &= \int V|w|^2 < \int |\nabla w|^2 \\ &= \int |x||\nabla u|^2 + \frac{1}{4}\int \frac{|u|^2}{|x|} + \Re \int \frac{x}{|x|}\cdot \nabla u \bar{u} \\ &= \int |x||\nabla u|^2 - \frac{(2d-3)}{4}\int \frac{|u|^2}{|x|} \\ &< \int |x||\nabla u|^2.\end{aligned}$$

**Remark 4.1.2.** *Observe that we are considering potentials that are singular at the origin and that decay at infinity. In both cases, we require some smallness on them.*

**Remark 4.1.3.** *In this chapter it would be enough to assume*

$$\int V|u|^2 < 1^- \int |\nabla_A u|^2, \tag{4.1.6}$$

$$\int (\partial_r(rV))_-|u|^2 < A_V \int |\nabla_A u|^2, \tag{4.1.7}$$

$$\left(\int |x|^2|B_\tau|^2|u|^2\right)^{1/2} < A_B \left(\int |\nabla_A u|^2\right)^{1/2}, \tag{4.1.8}$$

*which are weaker conditions than (4.1.1) and* **(H1)***. Note that from assumptions (4.1.1) and* **(H1)** *together with the diamagnetic inequality (1.1.4) we can conclude the above ones.*

Now we are ready to state the first theorem.

**Theorem 4.1.4.** *Let $d \geq 3$, $\varepsilon > 0$, $f$ such that $\||x|f\|_{L^2} < \infty$ and assume that (4.1.1),* **(H1)** *hold. Then, there exists $C > 0$ independent of $\lambda, \varepsilon$ such that any solution $u \in H^1_A(\mathbb{R}^d)$ of the electromagnetic Helmholtz equation (4.0.3) satisfies*

*(i) For $0 < \varepsilon < \lambda$,*

$$\int |\nabla_A(e^{-i\lambda^{1/2}|x|}u)|^2 \leq C \int |x|^2|f|^2. \tag{4.1.9}$$

*(iii) If $\lambda \leq \varepsilon$, then*

$$\int |\nabla_A u|^2 \leq C \int |x|^2|f|^2. \tag{4.1.10}$$

**Remark 4.1.5.** *Note that from the above result, by the magnetic Hardy inequality (1.4.8) we deduce the uniform resolvent estimate*

$$\int \frac{|u|^2}{|x|^2} \leq C \int |x|^2|f|^2, \tag{4.1.11}$$

*for all $d \geq 3$ and for any $\lambda \in \mathbb{R}$.*

One of the main significances of this theorem is that it allows one to deduce the resolvent estimate (4.1.11). We emphasize that this weighted $L^2$ estimate for the resolvent of $H_A$ plays a fundamental role for proving dispersive estimates on the time dependent Schrödinger operator. Moreover, it generalizes the corresponding result of Kato and Yajima [KaYa] where the operator in question is restricted to the Laplace operator in $\mathbb{R}^d$ ($d \geq 3$) and it recovers a more recent results proved by Burq, Planchom, Stalker and Tahvilder-Zadeh [BPST1], [BPST2] in their study of the Strichartz estimates for Schrödinger operator $-\Delta+V$. Regarding to the magnetic Schrödinger operator, there are several works related to this issue. Firstly, we should mention the papers by Fanelli and Vega ([FV]) and by D'Ancona, Fanelli, Vega and Visciglia [DFVV] where they show magnetic virial identities and Strichartz estimates for the Schrödinger equation with electromagnetic potential, respectively. Very recently, Fanelli, Felli, Fontelos and Primo [FFFP] study the dispersive property of the Schrödinger equation with singular electromagnetic potentials. Furthermore, Mochizuki [M3] proves the estimate (4.1.11) assuming the following smallness conditions on the potentials

$$\max\{|B(x)|, |V(x)|\} \leq \frac{\varepsilon_0}{|x|^2} \qquad \text{in} \qquad \mathbb{R}^d \tag{4.1.12}$$

where $0 < \varepsilon_0 < \frac{1}{4\sqrt{2}}$ ($d = 3$) or $< \left(\frac{(d-1)(d-3)}{8}\right)^{\frac{1}{2}}$ ($d \geq 4$). In our case, the constant $\varepsilon_0$ is determined by the one that appears in the standard Hardy inequality, as we can see in the following example:

**Example 4.1.6.** Let us take

$$(\partial_r(rV))_- \leq \frac{\nu_1}{|x|^2}, \qquad |B_\tau| \leq \frac{\nu_2}{|x|^2}, \tag{4.1.13}$$

where

$$\frac{4\nu_1}{(d-2)^2} + \frac{2\nu_2}{(d-2)} < 1.$$

In particular,

$$V = \frac{\nu_1}{|x|^2} \quad \text{with} \quad 0 < \nu_1 < \frac{(d-2)^2}{4}. \tag{4.1.14}$$

Then, one can easily check that (4.1.1) and (**H1**) are held.

**Example 4.1.7.** One can also work with Coulomb type electric potential $V$ and long range magnetic potential $A$ such that $B_\tau = 0$. In fact, if we take

$$V(x) = \frac{V_\infty\left(\frac{x}{|x|}\right)}{|x|} \quad \text{with} \quad V_\infty < 0,$$

then $\partial_r(rV) = 0$ and our assumptions are satisfied.

We emphasize that under the assumptions of Theorem 4.1.4 above, it may be concluded the uniqueness result for the equation (4.0.2). As a consequence, we deduce the limiting absorption principle for the Helmholtz equation with potentials $V$ and $A$ that can have sharp singularities at the origin. See section 4.4.1 below.

For the Sommerfeld radiation condition (4.0.7) one needs to put some further restrictions on the potentials. In fact, one can preserve the same kind of singularity at the origin, but needs to require more decay at infinity. We make the following assumption.

(**H2**)

$$|B_\tau| + |V| \leq \begin{cases} \frac{c}{|x|^2} & \text{if } |x| \leq 1 \\ \frac{c}{|x|^{5/2+\alpha}} & \text{if } |x| \geq 1. \end{cases}$$

for some $c > 0$, $\alpha > 0$.

We can now state the second result.

**Theorem 4.1.8.** *For $d \geq 3$, let $\lambda_0 > 0$. Under the hypotheses of Theroem 4.1.4, if moreover $\||x|^{3/2}f\|_{L^2} < \infty$ and* (**H2**) *holds, then for any $\lambda \geq \lambda_0$ solutions $u \in H^1_A(\mathbb{R}^d)$ of the equation (4.0.3) satisfies*

$$\sup_{R\geq 1} R \int_{|x|\geq R} |\nabla_A(e^{-i\sqrt{\lambda}|x|}u)|^2 \leq C \int (1+|x|)|x|^2|f|^2, \tag{4.1.15}$$

*where $C = C(\lambda_0) > 0$.*

The radiation condition (4.1.15) extends the Sommerfeld condition given by

$$\int_{|x|\geq 1} |x|^{\alpha}|\nabla_A(e^{-i\lambda^{1/2}|x|}u)|^2 < \infty \tag{4.1.16}$$

for any $0 < \alpha < 1$ that has been proved in Chapter 2. In order to get (4.1.15), we will first prove the a-priori estimate

$$\lambda|||u|||_1 \leq C\int |x|^2|f|^2 \tag{4.1.17}$$

for $\lambda > \varepsilon > 0$, which can be easily obtained from the estimate (4.1.9) of Theorem 4.1.4. In addition, we point out that estimate (4.1.15) is crucial to show that $|\mathcal{F}(\lambda, r)f| \in H^1(S^{d-1})$, a fundamental property for proving the existence of the cross-section.

Under the hypotheses of Theorem 4.1.8, from (4.1.15) and (4.1.17) it may be concluded that there exists a sequence $\{r_n\}_{n\in\mathbb{N}}$ tending to infinity such that

$$\lim_{n\to\infty}\left|r_n^{\frac{d-1}{2}}e^{-i\lambda^{1/2}r_n}u(r_n\omega)\right| = |g_\lambda(\omega)| \qquad \text{in} \qquad L^2(S^{d-1}). \tag{4.1.18}$$

However, the existence of the limit (4.1.18) for a certain sequence does not ensure uniqueness of a cross-section.

Following Iwatsuka [Iw] and assuming that the potentials satisfy (4.1.1), **(H1)**, by the estimates (4.1.9) and (4.1.11) we will prove that for a given $\phi \in C^\infty(S^{d-1})$ the following limit exists

$$\lim_{r\to\infty}\int_{|x|=1}|\mathcal{F}(\lambda, r)f(\omega)|^2\phi(\omega). \tag{4.1.19}$$

It is worth pointing out that a combination of (4.1.18), (4.1.19) ensures the existence and uniqueness of cross-section for the magnetic case. In particular, we do it for singular magnetic potentials $A$ such that $B_\tau = 0$. These potentials are not included in [Iw] and as far as we know, there are no results related to the far field pattern of solutions of the magnetic Schrödinger equation with singular magnetic potentials.

From (4.1.18), (4.1.19) and the limiting absorption principle we will be in a position to give the spectral representation of the magnetic Schrödinger operator $H_A$, which is established by the third result of this chapter.

**Theorem 4.1.9.** *Let the potential $V$ and the trapping component of the magnetic field $B$ satisfy the hypotheses of Theorem 4.1.8. Then:*

*(1) There exists $g = |g_\lambda|^2 \in L^1(S^{d-1})$ where*

$$|g_\lambda| = \lim_{r\to\infty}|r^{\frac{d-1}{2}}e^{-i\lambda^{1/2}r}u(r\omega)| \qquad in \qquad L^2(S^{d-1}), \tag{4.1.20}$$

*such that satisfies*

$$\int_{S^{d-1}} g(\omega)d\sigma(\omega) \leq C\int |x|^2|f|^2 \tag{4.1.21}$$

*for some $C > 0$.*

*(2) Let $P_{ac} = E(0,\infty)$ be the projection onto the absolute continuous subspace for $H_A$. Then*

$$(P_{ac}f, f) = \int_0^\infty \|g(\omega)\|_{L^1(S^{d-1})}\, d\lambda. \tag{4.1.22}$$

The rest of the chapter is devoted to the proofs of the above results. The next section deals with the proof of the uniform resolvent estimates that has been stated in Theorem 4.1.4. Section 4.3 establishes the key Sommerfeld radiation condition 4.1.15, proving Theorem 4.1.8. Finally, in section 4.4 we present some applications of the first two results, which will imply the proof of Theorem 4.1.9.

## 4.2 Proof of Theorem 4.1.4

The proof will be divided into two parts depending on the relation between $\varepsilon$ and $\lambda$.

We begin with the case when $0 < \varepsilon < \lambda$.

**Proposition 4.2.1.** *Let $d \geq 3$, $0 < \varepsilon < \lambda$, $f$ such that $\||x|f\|_{L^2} < \infty$. Assume that (4.1.1),* **(H1)** *hold. Then, the solution $u \in H^1_A(\mathbb{R}^d)$ of the Helmholtz equation (4.0.3) satisfies*

$$\int \left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2 + \frac{\varepsilon}{\lambda^{1/2}}\int |x|\left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2 \leq C\int |x|^2|f|^2, \tag{4.2.1}$$

*where $C > 0$ is independent of $\varepsilon, \lambda$.*

*Proof.* The proof is based on the equality (1.3.21) that has been given in Proposition 1.3.9. Let us denote $r = |x|$. We define

$$\psi(r) = \frac{r^2}{2}$$

so that $\psi'(r) = r$, $\psi''(r) = 1$ and we put it into the identity (1.3.21). Thus by (1.3.25), Remark 1.3.10 and integration by parts, it follows that

$$\begin{aligned}
&\frac{1}{2}\int \left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2 + \frac{\varepsilon}{2\lambda^{1/2}}\int |x|\left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2 \\
&= \Im\int |x|B_\tau\cdot\nabla_A u\bar{u} - \frac{1}{2}\int(\partial_r(rV))|u|^2 + \frac{\varepsilon}{2\lambda^{1/2}}\int |x|V|u|^2 \\
&+ \frac{\varepsilon(d-1)}{4\lambda^{1/2}}\int\frac{|u|^2}{|x|} - \frac{\varepsilon}{2\lambda^{1/2}}\Re\int |x|f\bar{u} - \frac{(d-1)}{2}\Re\int f\bar{u} \\
&- \Re\int |x|f\left(\overline{\nabla_A^r u} + i\lambda^{1/2}\bar{u}\right).
\end{aligned} \tag{4.2.2}$$

Let $v = e^{-i\lambda^{1/2}|x|}u$ and observe that $|v| = |u|$, $|\nabla_A v| = \left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|$. Let us estimate the right-hand side of (4.2.2). We start with the observation that

$$B_\tau \cdot \nabla_A u = B_\tau \cdot \left(\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right). \tag{4.2.3}$$

Hence by Cauchy-Schwarz inequality, **(H1)** and the diamagnetic inequality (1.1.4), we have

$$\begin{aligned}
\Im \int |x| B_\tau \cdot \nabla_A u \bar{u} &\leq \left(\int |x|^2 |B_\tau|^2 |v|^2\right)^{1/2} \left(\int |\nabla_A v|^2\right)^{1/2} \\
&< A_B \left(\int |\nabla |v||^2\right)^{1/2} \left(\int |\nabla_A v|^2\right)^{1/2} \\
&\leq A_B \int |\nabla_A v|^2.
\end{aligned} \tag{4.2.4}$$

Similarly, we get

$$\begin{aligned}
-\frac{1}{2}\int (\partial_r(rV))|u|^2 &\leq \frac{1}{2}\int (\partial_r(rV))_- |u|^2 \\
&< \frac{A_V}{2}\int |\nabla_A v|^2.
\end{aligned} \tag{4.2.5}$$

and combining **(H1)** with (4.1.5), yields

$$\frac{\varepsilon}{2\lambda^{1/2}}\int |x| V |u|^2 < \frac{\varepsilon A_V}{2\lambda^{1/2}}\int |x||\nabla_A v|^2.$$

Let us now compute the term $\frac{\varepsilon}{\lambda^{1/2}}\int \frac{|u|^2}{|x|}$. To this end, let $\delta > 0$. Then since $\varepsilon < \lambda$ and by the a-priori estimate (1.3.9), we have

$$\begin{aligned}
\frac{\varepsilon}{\lambda^{1/2}}\int \frac{|u|^2}{|x|} &= \frac{\varepsilon}{\lambda^{1/2}}\int_{|x|<\frac{\lambda^{1/2}\delta}{\varepsilon}} \frac{|u|^2}{|x|} + \frac{\varepsilon}{\lambda^{1/2}}\int_{|x|\geq \frac{\lambda^{1/2}\delta}{\varepsilon}} \int \frac{|u|^2}{|x|} \\
&\leq \delta \int \frac{|u|^2}{|x|^2} + \frac{\varepsilon}{\delta}\int |u|^2 \\
&\leq \delta \int \frac{|u|^2}{|x|^2} + \frac{1}{\delta}\int |f||u|.
\end{aligned} \tag{4.2.6}$$

Consequently, by Cauchy-Schwarz inequality and magnetic Hardy inequality (1.4.8), it follows that

$$\begin{aligned}
\frac{\varepsilon}{\lambda^{1/2}}\int \frac{|u|^2}{|x|} &\leq (\delta + \kappa)\int \frac{|u|^2}{|x|^2} + \frac{1}{4\kappa\delta^2}\int |x|^2 |f|^2 \\
&\leq \frac{4(\delta+\kappa)}{(d-2)^2}\int |\nabla_A v|^2 + C_{\kappa,\delta}\int |x|^2|f|^2,
\end{aligned} \tag{4.2.7}$$

for $\kappa > 0$. The same reasoning applying to the terms containing $f$. By $\varepsilon < \lambda$ and the a-priori estimate (1.3.9) we deduce

$$\begin{aligned}
-\frac{\varepsilon}{2\lambda^{1/2}}\Re\int |x|f\bar{u} &\leq \frac{\varepsilon^{3/2}}{4\lambda^{1/2}}\int |u|^2 + \frac{\varepsilon^{1/2}}{4\lambda^{1/2}}\int |x|^2|f|^2 \\
&\leq \frac{\varepsilon}{4}\int |u|^2 + \frac{1}{4}\int |x|^2|f|^2 \\
&\leq \kappa\int \frac{|u|^2}{|x|^2} + C_\kappa\int |x|^2|f|^2 \\
&\leq \frac{4\kappa}{(d-2)^2}\int |\nabla_A v|^2 + C_\kappa\int |x|^2|f|^2, \qquad (4.2.8)
\end{aligned}$$

$$-\frac{(d-1)\Re}{2}\int f\bar{u} \leq \frac{4\kappa}{(d-2)^2}\int |\nabla_A v|^2 + C_\kappa\int |x|^2|f|^2, \qquad (4.2.9)$$

$$-\Re\int |x|f\left(\overline{\nabla_A^r u} + i\lambda^{1/2}\bar{u}\right) \leq \kappa\int |\nabla_A v|^2 + C_\kappa\int |x|^2|f|^2, \qquad (4.2.10)$$

for arbitrary $\kappa > 0$.

Thus it may be concluded that

$$\begin{aligned}
&\frac{1}{2}\int |\nabla_A v|^2 + \frac{\varepsilon}{2\lambda^{1/2}}\int |x||\nabla_A v|^2 \\
&< \left(\frac{2A_B + A_V}{2} + \frac{(4\kappa + d - 1)(\delta + \kappa)}{(d-2)^2} + \frac{4\kappa}{(d-2)^2} + \kappa\right)\int |\nabla_A v|^2 \\
&+ \frac{\varepsilon A_V}{2\lambda^{1/2}}\int |x||\nabla_A v|^2 + C\int |x|^2|f|^2.
\end{aligned}$$

Note that since $u \in H^1_A(\mathbb{R}^d)$, by Remark 1.3.3 it is a simple matter to check that the right-hand side of the above inequality is finite. Therefore, choosing $\kappa, \delta$ small enough and using that $A_V + 2A_B < 1$, (4.2.1) is proved. $\square$

**Remark 4.2.2.** *Note that the identity (4.2.2) is true for any $d \geq 1$. However, since the classical Hardy inequality is valid only in the three and higher dimensional case, the method of proof breaks down when $d = 1$ and $d = 2$. In the two dimensional case, the main difficulty concerns with the analysis of the term $\frac{\varepsilon}{\lambda^{1/2}}\int \frac{|u|^2}{|x|}$. Observe that if $d = 1$ this term disappears, but we still need to estimate the term $\frac{\varepsilon}{\lambda^{1/2}}\Re\int |x|f\bar{u}$.*

The proof will be completed by showing that

$$\int |\nabla_A u|^2 \leq C\int |x|^2|f|^2, \qquad (4.2.11)$$

for any $\lambda \leq \varepsilon$ and for all $d \geq 3$.

**Proposition 4.2.3.** *Let $d \geq 3$, $\lambda \leq \varepsilon$ and assume that (4.1.1) holds. Then the solution $u \in H^1_A(\mathbb{R}^d)$ of the equation (4.0.3) satisfies*

$$\int |\nabla_A u|^2 \leq C \int |x|^2 |f|^2. \tag{4.2.12}$$

*Proof.* For this purpose, let us multiply equation (4.0.3) by $\bar{u}$ and integrate over $\mathbb{R}^d$. Then, taking the real part yields

$$\int |\nabla_A u|^2 = \lambda \int |u|^2 + \int V|u|^2 - \Re \int f\bar{u}. \tag{4.2.13}$$

Since $\lambda \leq \varepsilon$, by assumption (4.1.1) together with the diamagnetic inequality (1.1.4) and the a-priori estimate (1.3.9), we have

$$\begin{aligned} \int |\nabla_A u|^2 &\leq 2 \int |f||u| \\ &\leq \left( \int \frac{|u|^2}{|x|^2} \right)^{1/2} \left( \int |x|^2 |f|^2 \right)^{1/2}. \end{aligned} \tag{4.2.14}$$

This combining with the magnetic Hardy inequality gives (4.2.12), which is our claim. $\square$

**Remark 4.2.4.** *We emphasize that when $d \geq 3$, by the magnetic Hardy inequality (1.4.8) any solution $u \in H^1_A(\mathbb{R}^d)$ of the Helmholtz equation (4.0.3) satisfies for all $\lambda \in \mathbb{R}$ the resolvent estimate*

$$\int \frac{|u|^2}{|x|^2} \leq C \int |x|^2 |f|^2, \tag{4.2.15}$$

*where $C > 0$ is uniform on $\varepsilon$ and $\lambda$. Observe that this is the key inequality that is used in [BPST2] for proving the Strichartz estimates.*

## 4.3 Proof of Theorem 4.1.8

We now proceed to show the sharp Sommerfeld radiation condition (4.0.7). In order to get this inequality, we first present some preliminaries. On the one hand, we need to control the Agmon-Hörmander norm of solutions $u$ of the electromagnetic Helmholtz equation (4.0.3). On the other hand, we will give the $\alpha = 1$ version of (4.1.16) in $\mathbb{R}^d$, which consists of the following estimate

$$\int |x| \left| \nabla^r_A u - i\lambda^{1/2} u + \frac{(d-1)}{2|x|} u \right|^2 < \infty \tag{4.3.1}$$

and will be useful for our purpose. Note that in this case we loss the tangential part of the gradient and we add the term $\frac{(d-1)u}{2|x|}$ to the usual Sommerfeld term.

### 4.3.1 A priori estimate

We begin by proving the following a-priori estimate which follows from the inequality (4.2.1) above.

**Proposition 4.3.1.** *Let $d \geq 3$, $\lambda > \varepsilon > 0$ and $f$ such that, $\||x|f\|_{L^2} < \infty$. Assume that (4.1.1),* **(H1)** *hold. Then, any solution $u \in H^1_A(\mathbb{R}^d)$ of the Helmholtz equation (4.0.3) satisfies*

$$\lambda|||u|||_1 + |||\nabla_A u|||_1 \leq C \int |x|^2|f|^2 \tag{4.3.2}$$

*where $C > 0$ is independent of $\varepsilon, \lambda$.*

*Proof.* The proof follows applying similar arguments as in the proof of Proposition 2.2.12. Let us start with the observation that

$$\left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2 = |\nabla_A u|^2 + \lambda|u|^2 - 2\Im\lambda^{1/2}\frac{x}{|x|}\cdot\nabla_A u\bar{u}.$$

Let us integrate the above identity over the sphere $S_r := \{|x| = r\}$, obtaining

$$\begin{aligned}\int_{S_r}(\lambda|u|^2 + |\nabla_A u|^2)d\sigma_r = \int_{S_r}&|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u|^2 d\sigma_r \\ &+ 2\Im\lambda^{1/2}\int_{S_r}\frac{x}{|x|}\cdot\nabla_A u\bar{u}d\sigma_r.\end{aligned} \tag{4.3.3}$$

Let us multiply now equation (4.0.3) by $\bar{u}$, integrate it over the ball $B_r := \{|x| \leq r\}$ and take the imaginary part. Since $\varepsilon > 0$, it follows that

$$\Im\int_{S_r}\frac{x}{|x|}\cdot\nabla_A u\bar{u}d\sigma_r \leq \Im\int_{B_r} f\bar{u}. \tag{4.3.4}$$

Combining this with (4.3.3) yields

$$\int_{S_r}(\lambda|u|^2 + |\nabla_A u|^2)d\sigma_r \leq \int_{S_r}|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u|^2 d\sigma_r + 2\Im\lambda^{1/2}\int_{B_r} f\bar{u}. \tag{4.3.5}$$

Let $R \geq 1$. We multiply both sides of (4.3.5) by $\frac{1}{R}$ and integrate from 0 to $R$ with respect to $r$. Then from what has already proved we obtain

$$\begin{aligned}\frac{1}{R}\int_{|x|\leq R}(\lambda|u|^2 + |\nabla_A u|^2) &\leq \frac{1}{R}\int_{|x|\leq R}\left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2 \\ &+ \left(\int\frac{|u|^2}{|x|^2}\right)^{1/2}\left(\int|x|^2|f|^2\right)^{1/2} \\ &\leq C\int|x|^2|f|^2.\end{aligned} \tag{4.3.6}$$

As a consequence, taking the supremum over $R \geq 1$, we obtain (4.3.2), which is the desired conclusion.

□

### 4.3.2 $\alpha = 1$ version for the constant coefficient case

We proceed to show (4.3.1) for the constant coefficient case, i.e. $A \equiv 0$, $V = 0$. Let us consider the Helmholtz equation

$$\Delta u + \lambda u + i\varepsilon u = f. \tag{4.3.7}$$

Then we can state the following inequality.

**Lemma 4.3.2.** *Let $d \geq 1$, $\lambda > 0$, $\varepsilon > 0$ and $f$ such that $\||x|^{3/2} f\|_{L^2} < \infty$, $\||x|^2 f\|_{L^2} < \infty$. Then any solution $u \in H^1(\mathbb{R}^d)$ of the equation (4.3.7) satisfies*

$$\begin{aligned} &\frac{1}{4}\int |x| \left|\partial_r u - i\lambda^{1/2} u + \frac{d-1}{2|x|} u\right|^2 + \frac{\varepsilon}{4\lambda^{1/2}} \int |x|^2 \left|\nabla u - i\lambda^{1/2} \frac{x}{|x|} u\right|^2 \\ &\qquad \leq \frac{1}{4}\int |x|^3 |f|^2 + \frac{d}{4\lambda^{1/2}} \int |f||u| + \frac{\varepsilon}{4\lambda^{1/2}} \int |x|^2 |f||u|, \end{aligned} \tag{4.3.8}$$

*Proof.* The proof is based on the analogue identity of (1.3.21) for $A \equiv 0$, $V = 0$. Note that in this case, $\nabla_A \equiv \nabla$. Thus we have

$$\begin{aligned} &\frac{1}{2}\int \psi''(|\partial_r u|^2 + \lambda |u|^2) - \Im \lambda^{1/2} \int \psi'' \partial_r u \bar{u} + \int \left(\frac{\psi'}{|x|} - \frac{\psi''}{2}\right) |\nabla^\perp u|^2 \\ &+ \Re \frac{(d-1)}{2} \int \nabla \left(\frac{\psi'}{|x|}\right) \cdot \nabla u \bar{u} + \frac{\varepsilon}{2\lambda^{1/2}} \int \psi' \left|\nabla (e^{-i\lambda^{1/2}|x|} u)\right|^2 \\ &+ \frac{\varepsilon}{2\lambda^{1/2}} \Re \int \psi'' \partial_r u \bar{u} = -\frac{\varepsilon}{2\lambda^{1/2}} \Re \int \psi' f \bar{u} - \Re \int f \psi' (\partial_r \bar{u} + i\lambda^{1/2} \bar{u}) \\ &- \frac{(d-1)}{2} \Re \int \frac{\psi'}{|x|} f \bar{u}. \end{aligned} \tag{4.3.9}$$

Let us define

$$\psi(r) = \frac{r^3}{3}, \qquad r = |x|$$

and we put it into (4.3.9). Hence, since

$$\begin{aligned} \left|\partial_r u - i\lambda^{1/2} u + \frac{(d-1)}{2|x|} u\right|^2 &= |\partial_r u|^2 + \lambda |u|^2 + \frac{(d-1)^2}{4|x|^2} |u|^2 \\ &\quad - 2\lambda^{1/2} \Im \partial_r u \bar{u} + \Re \frac{(d-1)}{2|x|} \partial_r u \bar{u}, \end{aligned} \tag{4.3.10}$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^d}|x|\left|\partial_r u - i\lambda^{1/2}u + \frac{(d-1)}{2|x|}u\right|^2 + \frac{\varepsilon}{4\lambda^{1/2}}\int_{\mathbb{R}^d}|x|^2\left|\nabla u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2\\
&= \frac{d\varepsilon}{4\lambda^{1/2}}\int_{\mathbb{R}^d}|u|^2 - \frac{1}{2}\Re\int f|x|^2\left(\partial_r\bar{u} + i\lambda^{1/2}\bar{u} + \frac{(d-1)}{2}\bar{u}\right)\\
&- \frac{\varepsilon}{4\lambda^{1/2}}\Re\int_{\mathbb{R}^d}|x|^2 f\bar{u}, \qquad (4.3.11)
\end{aligned}$$

and the lemma follows. □

**Remark 4.3.3.** *It is not our purpose to give the corresponding estimate for the electromagnetic case. However, under suitable assumptions on the potentials, (4.3.1) may be obtained in much the same way as in the constant coefficient case, the only difference being in the analysis of the potential terms.*

### 4.3.3 Key Sommerfeld condition

We are now in a position to show the goal of this section.

**Proposition 4.3.4.** *Let $d \geq 3$, $\lambda_0, \varepsilon > 0$ with $\lambda_0 > \varepsilon$ and $f$ such that $N(f) < \infty$, $\||x|f\|_{L^2} < \infty$, $\||x|^{3/2}f\|_{L^2} < \infty$, $\||x|^2 f\|_{L^2} < \infty$. Let the potentials satisfy* **(H2)**. *Then, there exist positive constants $C_i, i = 1, 2$ (independent of $\varepsilon$ and $\lambda$) such that for any $R \geq 1$ and $\lambda \geq \lambda_0$ the solution $u \in H^1_A(\mathbb{R}^d)$ to the Helmholtz equation (4.0.3) satisfies*

$$\begin{aligned}
&\int_{|x|\leq R}|x|\left|\nabla_A^r u - i\lambda^{1/2}u + \frac{(d-1)}{2|x|}u\right|^2 + R\int_{|x|\geq 2R}|\nabla_A(e^{-i\lambda^{1/2}|x|}u)|^2\\
&+ \frac{\varepsilon}{\lambda^{1/2}}\int_{|x|\leq R}|x|^2|\nabla_A(e^{-i\lambda^{1/2}|x|}u)|^2 + \frac{\varepsilon R}{\lambda^{1/2}}\int_{|x|\geq 2R}|x||\nabla_A(e^{-i\lambda^{1/2}|x|}u)|^2\\
&\leq C_1\left[\int\left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2 + \int\frac{|u|^2}{|x|^2} + |||u|||_1^2\right]\\
&+ C_2\left[\int(1+|x|)|x|^2|f|^2 + \varepsilon\int|x|^4|f|^2\right]. \qquad (4.3.12)
\end{aligned}$$

*Proof.* Let $R \geq 1$. Without loss of generality, noting that

$$\inf_{R\geq R_1}\int_{|x|=R}|u|^2 \leq C|||u|||^2_{R_1}$$

and

$$\inf_{R\geq R_1}\int_{|x|=R}|x||u|^2 \leq \int|u|^2,$$

there exists $R_1$ such that $R \leq R_1 \leq 2R$ and satisfies

$$\int_{|x|=R_1} |u|^2 \leq C|||u|||^2_{R_1}, \tag{4.3.13}$$

$$\int_{|x|=R_1} |x||u|^2 \leq C \int |u|^2. \tag{4.3.14}$$

We will prove the estimate (4.3.12) for this $R_1$, and then, since $R_1$ and $R$ are comparable, we will deduce the result for any $R \geq 1$.

Let us define the multiplier

$$\psi'(|x|) = \begin{cases} |x|^2 & \text{if } |x| \leq 1, \\ |x| & \text{if } |x| \geq 1, \end{cases} \tag{4.3.15}$$

and set $\psi'_{R_1}(|x|) = R_1^2 \psi'\left(\frac{|x|}{R_1}\right)$. Thus we get

$$\psi'_{R_1}(|x|) = \begin{cases} |x|^2 & \text{if } |x| \leq R_1, \\ R_1|x| & \text{if } |x| \geq R_1, \end{cases} \tag{4.3.16}$$

so that in the distributional sense yields

$$\psi''_{R_1}(|x|) = \begin{cases} 2|x| & \text{if } |x| \leq R_1, \\ R_1 & \text{if } |x| \geq R_1. \end{cases} \tag{4.3.17}$$

Let us put the above multiplier into the identity (1.3.21). For simplicity, we start by considering the case when $A_j = V = 0$, $j = 1, \ldots, d$. Thus denoting $v = e^{-i\lambda^{1/2}|x|}u$ we get

$$\begin{aligned}
&\int_{|x|\leq R_1} |x| \left| \nabla_A^r u - i\lambda^{1/2} u + \frac{(d-1)}{2|x|} u \right|^2 + \frac{R_1}{2} \int_{|x|\geq R_1} \left| \nabla_A u - i\lambda^{1/2} \frac{x}{|x|} u \right|^2 \\
&+ \frac{\varepsilon}{2\lambda^{1/2}} \left[ \int_{|x|\leq R_1} |x|^2 \left|\nabla_A v\right|^2 + R_1 \int_{|x|\geq R_1} |x| \left|\nabla_A v\right|^2 \right] \\
&= \frac{d\varepsilon}{2\lambda^{1/2}} \int_{|x|\leq R_1} |u|^2 + \frac{\varepsilon(d-1)R_1}{4\lambda^{1/2}} \int_{|x|\geq R_1} \frac{|u|^2}{|x|} \\
&- \frac{\varepsilon R_1}{2\lambda^{1/2}} \Re \int_{|x|\geq R_1} |x| f\bar{u} - \Re \int_{|x|\leq R_1} f|x|^2 \left( \nabla_A^r \bar{u} + i\lambda^{1/2}\bar{u} + \frac{d-1}{2|x|}\bar{u} \right) \\
&- \Re R_1 \int_{|x|\geq R_1} |x| f(\nabla_A^r \bar{u} + i\lambda^{1/2}\bar{u}) - \frac{(d-1)R_1}{2} \Re \int_{|x|\geq R_1} f\bar{u} \\
&- \frac{\varepsilon}{2\lambda^{1/2}} \Re \int_{|x|\leq R_1} |x|^2 f\bar{u} + \frac{(d-1)}{4} \int_{|x|=R_1} |u|^2 + \frac{\varepsilon}{4\lambda^{1/2}} \int_{|x|=R_1} |x||u|^2.
\end{aligned} \tag{4.3.18}$$

Let us analyze now the right hand side of the above equality. Note that the integral terms over $\{|x| \geq R_1\}$ correspond to the $\alpha = 0$ case of (4.1.16) that has been already proved in the previous section. The terms over $\{|x| \leq R_1\}$ reduce to the $\alpha = 1$ version above. Regarding to these terms, by (4.3.8) we only need to estimate the following ones

$$\frac{1}{\lambda^{1/2}} \int |f||u| \leq \frac{1}{\lambda_0^{1/2}} \left( \frac{|u|^2}{|x|^2} \right)^{1/2} \left( \int |x|^2 |f|^2 \right)^{1/2}$$

and

$$\frac{\varepsilon}{\lambda^{1/2}} \int |x|^2 |f||u| \leq \frac{1}{\lambda_0^{1/2}} \left( \varepsilon \int |u|^2 \right)^{1/2} \left( \varepsilon \int |x|^4 |f|^2 \right)^{1/2}. \tag{4.3.19}$$

The surface integrals pose no problem. By (4.3.13) and (4.3.14) we have

$$\begin{aligned} \int_{|x|=R_1} |u|^2 + \frac{\varepsilon}{\lambda^{1/2}} \int_{|x|=R_1} |x||u|^2 &\leq C|||u|||_{R_1}^2 + \frac{C\varepsilon}{\lambda^{1/2}} \int |u|^2 \\ &\leq C|||u|||_{R_1}^2 + \frac{C}{\lambda^{1/2}} \left( \int \frac{|u|^2}{|x|^2} \right)^{1/2} \left( \int |x|^2 |f|^2 \right)^{1/2}. \end{aligned}$$

We now turn to the potential terms. After substituting the above multiplier into the identity (1.3.21), in the right-hand side of the resulting equality the integrals related to $B_\tau$ and $V$ are

$$\begin{aligned} &\Im \int_{|x|\leq R_1} |x|^2 B_\tau \cdot \overline{\nabla_A u} u - \Im R_1 \int_{|x|\geq R_1} |x| B_\tau \cdot \nabla_A u \bar{u} - \int_{|x|\leq R_1} |x| V |u|^2 \\ &- \frac{1}{2} \int_{|x|\leq R_1} |x|^2 (\partial_r V) |u|^2 - \frac{R_1}{2} \int_{|x|\geq R_1} V |u|^2 - \frac{R_1}{2} \int_{|x|\geq R_1} |x| (\partial_r V) |u|^2 \\ &+ \frac{\varepsilon}{2\lambda^{1/2}} \int_{|x|\leq R_1} |x|^2 V |u|^2 + \frac{\varepsilon R_1}{2\lambda^{1/2}} \int_{|x|\geq R_1} |x| V |u|^2. \end{aligned}$$

Let us treat the above terms. We start with the magnetic ones. Let us recall that $v =$

$e^{-i\lambda^{1/2}|x|}u$. Then, by Cauchy-Schwarz inequality, **(H2)**, (4.2.1) and (4.2.3) we get

$$
\begin{aligned}
&\Im \int_{|x|\leq R_1} |x|^2 B_\tau \cdot \overline{\nabla_A u} u + \Im R_1 \int_{|x|\geq R_1} |x| B_\tau \cdot \overline{\nabla_A u} u \\
&\quad \leq \int_{\mathbb{R}^d} |x|^2 |B_\tau| |\nabla_A v| |v| \\
&\quad \leq \left( \int_{\mathbb{R}^d} |\nabla_A v|^2 \right)^{1/2} \left( \int_{\mathbb{R}^d} |x|^4 |B_\tau|^2 |v|^2 \right)^{1/2} \\
&\quad \leq \left( \int |\nabla_A v|^2 \right)^{1/2} \left( c \int_{|x|\leq 1} \frac{|u|^2}{|x|^2} + c \int_{|x|\geq 1} \frac{|u|^2}{|x|^{1+2\alpha}} \right)^{1/2} \\
&\quad \leq C \left[ \int |\nabla_A v|^2 + \int \frac{|u|^2}{|x|^2} + |||u|||_1^2 \right].
\end{aligned}
$$

As far as the electric potential is concerned, note that after integrating by parts the terms containing $\partial_r V$, one can rewrite them as follows

$$
\begin{aligned}
&\frac{d-1}{2} \int_{|x|\leq R_1} |x| V |u|^2 + \Re \int_{|x|\leq R_1} V |x|^2 \frac{x}{|x|} \cdot \nabla_A u \bar{u} \\
&+ \frac{(d-1)R_1}{2} \int_{|x|\geq R_1} V |u|^2 + \Re R_1 \int_{|x|\geq R_1} V |x| \frac{x}{|x|} \cdot \nabla_A u \bar{u} \\
&+ \frac{\varepsilon}{2\lambda^{1/2}} \int_{|x|\leq R_1} |x|^2 V |u|^2 + \frac{\varepsilon R_1}{2\lambda^{1/2}} \int_{|x|\geq R_1} |x| V |u|^2.
\end{aligned}
$$

Then by Cauchy-Schwarz inequality, the a-priori estimate (1.3.9) and assumption **(H2)**, analysis similar to the above implies that these terms are upper bounded by

$$
\begin{aligned}
&\frac{d-1}{2} \int |x| |V| |u|^2 + \int |V| |x|^2 |\nabla_A v| |v| + \frac{\varepsilon}{2\lambda^{1/2}} \int |x|^2 |V| |u|^2 \\
&\leq C \left( \int |\nabla_A v|^2 + \int \frac{|u|^2}{|x|^2} + |||u|||_1^2 + \int |x|^2 |f|^2 \right).
\end{aligned}
$$

Putting everything together the proposition follows. $\square$

Combining this result with Proposition 4.3.1 and estimate (4.2.15), provides the following inequality which in particular proves Theorem 4.1.8.

**Corollary 4.3.5.** *Under the hypotheses of Theorem 4.1.4, let $\varepsilon > 0$ and $f$ such that also satisfies $|||x|^{3/2} f\|_{L^2} < \infty$, $|||x|^2 f\|_{L^2} < \infty$. Then, for any $R \geq 1$ the solution $u \in H^1_A(\mathbb{R}^d)$ of* *the Helmholtz equation (4.0.3) satisfies*

$$\begin{aligned}
&\int_{|x|\le\frac{R}{2}} |x| \left|\nabla_A^r u - i\lambda^{1/2}u + \frac{(d-1)}{2|x|}u\right|^2 + R\int_{|x|\ge R} |\nabla_A(e^{-i\lambda^{1/2}|x|}u)|^2 \\
&+ \frac{\varepsilon}{\lambda^{1/2}}\int_{|x|\le\frac{R}{2}} |x|^2|\nabla_A(e^{-i\lambda^{1/2}|x|}u)|^2 + \frac{\varepsilon R}{\lambda^{1/2}}\int_{|x|\ge R} |x||\nabla_A(e^{-i\lambda^{1/2}|x|}u)|^2 \\
&\le C\left[\int (1+|x|)|x|^2|f|^2 + \varepsilon\int |x|^4|f|^2\right], \qquad (4.3.20)
\end{aligned}$$

*where $C = C(\lambda_0)$ is independent of $\varepsilon$.*

**Remark 4.3.6.** *Note that taking the supremum over $R \ge 1$ in (4.3.20), we get (4.1.15) and the proof is complete.*

**Remark 4.3.7.** *This result provides an extra a-priori estimate for the surface integral. In fact, the solution $u$ of the equation (4.0.3) holds*

$$\int_{|x|=R} |u|^2 < \infty, \qquad \forall R \ge 1. \qquad (4.3.21)$$

*Note that we can rewrite (4.3.18) for any $R \ge 1$ with the boundary terms in the left hand side of the identity. Hence from (4.3.12) it is immediate that*

$$\int_{|x|=R} |u|^2 + \frac{\varepsilon}{\lambda^{1/2}}\int_{|x|=R} |x||u|^2 \le C\left(\int (1+|x|+\varepsilon|x|^2)|x|^2|f|^2\right),$$

*where $C = C(\lambda_0) > 0$ is independent of $\varepsilon$.*

## 4.4 Applications

Firstly, under the hypotheses of Theorem 4.1.4, we will prove the limiting absorption principle for the equation (4.0.2) by the same method as in the previous chapters. Note that since condition (4.1.9) for the solution of the equation (4.0.3) has been already showed, we can restrict ourselves for proving the uniqueness result for the equation (4.0.2).

Secondly, we proceed with the study of the cross-section of the electromagnetic Helmholtz equation (4.0.2), giving the proof of the first part of Theorem 4.1.9. This will follow by the radiation condition (4.1.15) and the resolvent estimates (4.1.11), (4.3.2).

Finally, the limiting absorption principle and the existence and uniqueness of the cross-section will allow us to give the spectral representation of $H_A$, which completes the proof of Theorem 4.1.9.

### 4.4.1 Limiting absorption principle

Let us first state a uniqueness theorem for the electromagnetic Schrödinger operator with potentials satisfying **(H1)** provided that

$$\liminf_{R\to\infty} \int_{|x|=R} V|u|^2 = 0. \tag{4.4.1}$$

Let us consider the homogeneous electromagnetic Helmholtz equation

$$(\nabla + iA(x))^2 u + V(x)u + \lambda u = 0. \tag{4.4.2}$$

**Theorem 4.4.1.** *Let $d \geq 1$, $\lambda_0 > 0$ and assume that* **(H1)** *and (4.4.1) hold. Let $u$ be a solution of (4.4.2) with $u, \nabla_A u \in L^2_{loc}$ such that*

$$\lim_{R\to\infty} \frac{1}{R} \int_{R\leq|x|\leq 2R} (\lambda|u|^2 + |\nabla_A u|^2) = 0. \tag{4.4.3}$$

*Then $u \equiv 0$.*

*Proof.* The proof is based on the multiplier method. By (4.4.3), without loss of generality there exists a sequence $\{R_j\}$ tending to infinity such that

$$\lim_{R_j\to\infty} \frac{1}{R_j} \int_{R_j\leq|x|\leq 2R_j} (\lambda|u|^2 + |\nabla_A u|^2) = 0. \tag{4.4.4}$$

Let us multiply the equation (4.4.2) by the combination of the symmetric and the antisymmetric multipliers

$$\nabla\psi \cdot \overline{\nabla_A u} + \frac{1}{2}\Delta\psi\bar{u} + \varphi\bar{u},$$

where $\psi, \varphi$ are a real valued functions and integrate over the ball $\{|x| < R_j\}$. Hence we have

$$\begin{aligned}
&\int_{|x|<R_j} \nabla_A u \cdot D^2\psi \cdot \overline{\nabla_A u} - \int_{|x|<R_j} \varphi|\nabla_A u|^2 + \int_{|x|<R_j} \varphi\lambda|u|^2 \\
&- \frac{1}{4}\int_{|x|<R_j} (\Delta^2\psi - 2\Delta\varphi)|u|^2 + \int_{|x|<R_j} \varphi V|u|^2 + \frac{1}{2}\int_{|x|<R_j} \nabla V \cdot \nabla\psi|u|^2 \\
&= \Im \sum_{k,m=1}^{d} \int_{|x|<R_j} \frac{\partial\psi}{\partial x_k} B_{km} u\overline{(\nabla_A)_m u} + \frac{1}{4}\int_{S_{R_j}} \nabla(\Delta\psi)\cdot\frac{x}{|x|}|u|^2 \\
&+ \frac{1}{2}\Re\int_{S_{R_j}} \frac{x}{|x|}\cdot\overline{\nabla_A u}\Delta\psi u + \frac{1}{2}\int_{S_{R_j}} \frac{x}{|x|}\cdot\nabla\varphi|u|^2 - \Re\int_{S_{R_j}} \nabla_A^r u\varphi\bar{u} \\
&+ \frac{1}{2}\int_{S_{R_j}} (\lambda + V)\frac{x}{|x|}\cdot\nabla\psi|u|^2 - \frac{1}{2}\int_{S_{R_j}} \frac{x}{|x|}\cdot\nabla\psi|\nabla_A u|^2,
\end{aligned} \tag{4.4.5}$$

being $S_{R_j} = \{|x| = R_j\}$.

Let $R$ such that $1 \leq \frac{R_j}{2} \leq R \leq R_j$ and we consider the following multipliers

$$\varphi(x) = \frac{1}{2R}, \qquad \nabla\psi(x) = \frac{x}{R}. \tag{4.4.6}$$

Let us insert them into the identity (4.4.5). Noting that the boundary terms can be upper bounded by

$$C \int_{|x|=R_j} \{|\nabla_A u|^2 + (\lambda + V)|u|^2\} d\sigma_{R_j}, \tag{4.4.7}$$

it follows that

$$\begin{aligned} \frac{1}{2R} \int_{|x|\leq R_j} (\lambda|u|^2 + |\nabla_A u|^2) \leq{}& \frac{1}{2R} \int_{|x|\leq R_j} (\partial_r(rV))_-|u|^2 \\ &+ \frac{1}{R} \int_{|x|\leq R_j} |x||B_\tau||\nabla_A u||u| \\ &+ C \int_{|x|=R_j} (|\nabla_A u|^2 + (\lambda + V)|u|^2). \end{aligned} \tag{4.4.8}$$

Let $\theta(r) \in C^\infty(\mathbb{R})$ be a cut-off function such that $0 \leq \theta \leq 1$, given by

$$\theta(r) = \begin{cases} 1 & \text{if } r \leq 1 \\ 0 & \text{if } r \geq 2 \end{cases} \tag{4.4.9}$$

and set $\theta_R = \theta\left(\frac{|x|}{R}\right)$. Hence, by **(H1)** and the diamagnetic inequality (1.1.4) we have

$$\begin{aligned} \frac{1}{2R} \int_{|x|\leq R_j} (\partial_r(rV))_-|u|^2 \leq{}& \frac{1}{2R} \int (\partial_r(rV))_-|\theta_{R_j} u|^2 \\ \leq{}& \frac{A_V}{2R} \int_{|x|\leq R_j} |\nabla_A u|^2 + \frac{C}{R_j^2} \int_{R_j\leq|x|\leq 2R_j} |u|^2 \\ &+ \frac{A_V}{2R} \int_{R_j\leq|x|\leq 2R_j} |\nabla_A u|^2. \end{aligned} \tag{4.4.10}$$

Similarly, we obtain

$$\begin{aligned} \frac{1}{R} \int_{|x|\leq R_j} |B_\tau||\nabla_A u||u| \leq{}& \left(\frac{1}{R} \int_{|x|\leq R_j} |\nabla_A u|^2\right)^{\frac{1}{2}} \left(\frac{1}{R} \int |\nabla_A(\theta_{R_j} u)|^2\right)^{\frac{1}{2}} \\ \leq{}& \frac{(A_B + \kappa)}{R} \int_{|x|\leq R_j} |\nabla_A u|^2 + \frac{C}{R_j^2} \int_{R_j\leq|x|\leq 2R_j} |u|^2 \\ &+ \frac{C}{R} \int_{R_j\leq|x|\leq 2R_j} |\nabla_A u|^2. \end{aligned} \tag{4.4.11}$$

As a consequence, since $\{|x| \le R\} \subset \{|x| \le R_j\}$, it may be concluded that

$$\sup_{R \le R_j} \frac{1}{R} \int_{|x| \le R} \lambda |u|^2 \le \frac{C}{R_j} \int_{R_j \le |x| \le 2R_j} (|\nabla_A u|^2 + \lambda |u|^2) + C \int_{|x|=R_j} ((\lambda + V)|u|^2 + |\nabla_A u|^2). \tag{4.4.12}$$

Then taking the lim inf in $j$, by (4.4.1) and (4.4.4) the theorem follows. □

**Remark 4.4.2.** *Note that in this case it is not needed the unique continuation property for $H_A$. The smallness conditions on the potentials allows us to prove uniqueness just by multiplier method and integration by parts.*

Combining the uniqueness of a solution of the equation (4.0.2) with the estimate (4.1.9) for $\lambda \ge \lambda_0 > 0$, the same method as in section 2.2.4 imply the limiting absorption principle for the electromagnetic Helmholtz equation assuming sharp singularities on $V$ and $B_\tau$.

**Theorem 4.4.3.** *(LAP) Let $\lambda_0 > 0$. Under the hypotheses of Theorem 4.1.4, if moreover $V$ holds (4.4.1), then there exists a unique solution $u$ of the equation* (4.0.2) *such that for any $\lambda \ge \lambda_0 > 0$ satisfies the radiation condition*

$$\int |\nabla_A (e^{-i\lambda^{1/2}|x|} u)|^2 \le C \int |x|^2 |f|^2 \tag{4.4.13}$$

*and the a-priori estimate*

$$\int \frac{|u|^2}{|x|^2} \le C \int |x|^2 |f|^2, \tag{4.4.14}$$

*where $C > 0$ is independent of $\lambda$.*

**Remark 4.4.4.** *It is worth pointing out that uniqueness result also follows by assuming some local conditions on the potentials. In fact, given $\lambda > 0$ if we require that there exists $R_0 = R_0(\lambda) > 0$ such that*

*(***H1a)**

$$\int_{|x| \le R_0} (\partial_r (rV))_- |u|^2 < A_V \Lambda_{R_0} + \int_{|x| \le R_0} (\partial_r (rV))_+ |u|^2,$$

$$\left( \int_{|x| \le R_0} |x|^2 |B_\tau|^2 |u|^2 \right)^{1/2} < A_B \Lambda_{R_0}^{1/2},$$

*with*

$$\Lambda_{R_0} = \int_{|x| \le R_0} |\nabla u|^2 + \sup_{R \ge R_0} \frac{(d-1)}{2R} \int_{|x|=R} |u|^2,$$

*where* $A_V + 2A_B < 1$.

(**H1b**)

$$\frac{1}{R_0}\int_{|x|\geq R_0}[(\partial_r V)_-]\,|u|^2 < A''_V\lambda|||u|||^2_{R_0} + \int_{|x|\geq R_0}\frac{1}{|x|}(\partial_r V)_+|u|^2$$

$$\frac{1}{R_0}\int_{|x|\geq R_0}|x|^2|B_\tau|^2|u|^2 < A''_B\lambda|||u|||^2_{R_0}.$$

*where*

$$1 - A''_V - 2A''_B > 0, \tag{4.4.15}$$

*then Theorem 4.4.1 follows. In this case condition (4.4.3) can be replaced with a weaker one,*

$$\liminf\int_{|x|=R}(\lambda|u|^2 + |\nabla_A u|^2) \to 0 \qquad as \qquad R\to\infty. \tag{4.4.16}$$

*Furthermore, combining these assumptions with the condition (4.1.1), it may be concluded the Morrey-Campanato type estimate*

$$\lambda|||u|||^2_{R_0} + |||\nabla_A u|||^2_{R_0} \leq C(1+\varepsilon)(N_{R_0}(f))^2, \tag{4.4.17}$$

*for all* $\lambda \geq 0$ *being* $C$ *independent of* $\lambda$, $\varepsilon$ *and the Sommerfeld radiation condition*

$$\int_{|x|\geq 1}|\nabla_A(e^{-i\lambda^{1/2}|x|}u)|^2 \leq C\int|x|^2|f|^2 + (1+\varepsilon)(N_{R_0}(f))^2, \tag{4.4.18}$$

*for the solution* $u$ *of the equation (4.0.3).*
*As a consequence, we deduce the limiting absorption principle for the equation (4.0.2) with potentials satisfying (4.1.1),* **(H1a)**, **(H1b)** *such that the solution holds the a-priori estimate (4.4.17). Note that the potentials given in Examples 4.1.6 and 4.1.7 also hold the above assumptions. In fact, potentials that have the sharp singularity at the origin and that decay as a short-range potential at infinity are included.*

### 4.4.2 Cross-section

Our next goal is to prove existence and uniqueness of the cross-section of the solution $u$ of the equation (4.0.2), which proves Theorem 4.1.9 (i).

For this purpose, from (4.3.2) and (4.1.15) we first prove the existence of the strong limit

$$\lim_{r_n\to\infty}\left|r_n^{\frac{d-1}{2}}e^{-i\lambda^{1/2}r_n}u(r_n\omega)\right| \qquad \text{in} \qquad L^2(S^{d-1}) \tag{4.4.19}$$

for certain specified sequence $\{r_n\}_{n\in\mathbb{N}}$ tending to infinity as $n\to\infty$.

Furthermore, by the resolvent estimate (4.1.11) it may be concluded that there exists the limit

$$\lim_{r\to\infty}\int_{|x|=1}|\mathcal{F}(\lambda,r)f(\omega)|^2\,\phi(\omega)\qquad\forall\phi\in C^\infty(S^{d-1}),\tag{4.4.20}$$

being

$$(\mathcal{F}(\lambda,r)f)(\omega)=C(\lambda)r^{\frac{d-1}{2}}e^{-i\lambda^{1/2}r}u(r\omega),\qquad\omega\in S^{d-1}.\tag{4.4.21}$$

Consequently, we deduce the existence of the limit (4.1.20) and the first part of Theorem 4.1.9 follows.

### Existence

Let us start proving the existence of the limit (4.4.19) which asserts existence of a cross.section. This will follow under the hypotheses of Theorem 4.1.8.

**Proposition 4.4.5.** *There exists a sequence $\{r_n\}_{n\in\mathbb{N}}$ tending to infinity such that $|\mathcal{F}(\lambda,r_n)f|$ converges strongly to $|g_\lambda|$ in $L^2(S^{d-1})$ as $n\to\infty$. $|g_\lambda|\in L^2(S^{d-1})$ is called scattering cross section of $u$ and satisfies*

$$\int_{S^{d-1}}|g_\lambda(\omega)|^2\le C\int|x|^2|f|^2.\tag{4.4.22}$$

*Proof.* Our proof starts with the observation that from (4.1.15) and (4.3.2) there exists a sequence $\{r_n\}_{n\in\mathbb{N}}$ that tends to infinity such that for each $r_n$

$$r_n^2\int_{|x|=r_n}\left|\nabla_A u-i\lambda^{1/2}\frac{x}{|x|}u\right|^2d\sigma_r<+\infty.\tag{4.4.23}$$

and

$$\lambda\int_{|x|=r_n}|u|^2d\sigma_r<+\infty,\tag{4.4.24}$$

respectively.

We write $x=r_n\omega$, where $\omega\in S^{d-1}$ and we take

$$h_n(\omega)=\left(\mathcal{F}(\lambda,r_n)f\right)(\omega).\tag{4.4.25}$$

Thus we have

$$\begin{aligned}\int_{S^{d-1}}|h_n(\omega)|^2d\sigma(\omega)&=r_n^{d-1}\int_{S^{d-1}}|u(r_n\omega)|^2d\sigma(\omega)\\&=\int_{|x|=r_n}|u(x)|^2d\sigma_r.\end{aligned}\tag{4.4.26}$$

Furthermore, by the diamagnetic inequality (1.1.4) we get

$$\begin{aligned}\int_{S^{d-1}} |\nabla_\omega |h_n(\omega)||^2 d\sigma(\omega) &= r_n^{d+1}\int_{S^{d-1}} \left|\nabla^\tau \left|e^{-i\lambda^{1/2}r_n}u(r_n\omega)\right|\right|^2 d\sigma(\omega)\\ &\leq r_n^2 \int_{|x|=r_n} \left|\nabla \left|e^{-i\lambda^{1/2}r_n}u\right|\right|^2 d\sigma_r\\ &\leq r_n^2 \int_{|x|=r_n} \left|\nabla_A \left(e^{-i\lambda^{1/2}r_n}u\right)\right|^2 d\sigma_r. \end{aligned} \tag{4.4.27}$$

Hence it may be concluded that $|h_n| \in H^1(S^{d-1})$ and by the Rellich theorem we deduce that $\exists!|g_\lambda| \in L^2(S^{d-1})$ such that $|h_n| \to |g_\lambda|$ in $L^2(S^{d-1})$. This $|g_\lambda|$ is the cross-section of the solution $u$. In addition, writing the solution $u$ as

$$u(r_n\omega) = \frac{h_n(\omega)e^{i\lambda^{1/2}r_n}}{r_n^{\frac{d-1}{2}}}, \tag{4.4.28}$$

it follows that

$$\lim_{R\to+\infty}\inf \int_{|x|=R} \left| |u| - \frac{|g_\lambda|}{|x|^{\frac{d-1}{2}}}\right|^2 d\sigma_R = 0. \tag{4.4.29}$$

Let us prove now (4.4.22). Let us multiply the equation (4.0.2) by $\bar{u}$ and integrate over the the ball $\{|x| \leq R\}$ for some $R \geq 1$. Let us take the imaginary part, obtaining

$$\Im \int_{|x|=R} \frac{x}{|x|}\cdot \nabla_A u\bar{u} = \Im \int_{|x|\leq R} f\bar{u}. \tag{4.4.30}$$

Now observe that the left hand-side of the above equality can be written as

$$\Im \int_{|x|=R} \frac{x}{|x|}\cdot \left(\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right)\bar{u} + \lambda^{1/2}\int_{|x|=R} |u|^2. \tag{4.4.31}$$

Computing the lim inf as $R \to \infty$ on the both sides of the identity, since from (4.1.15) yields

$$\lim_{R\to\infty}\inf \int_{|x|=R} \left|\nabla_A u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2 d\sigma = 0, \tag{4.4.32}$$

we conclude that

$$\lambda^{1/2}\lim_{R\to\infty}\inf \int_{|x|=R} |u|^2 = \Im \int_{\mathbb{R}^d} f\bar{u}. \tag{4.4.33}$$

On the other hand, from (4.4.29) it is easily seen that

$$\int_{|x|=1} |g_\lambda(\omega)|^2\, d\sigma(\omega) = \lim_{R\to\infty}\inf \int_{|x|=R} |u|^2. \tag{4.4.34}$$

Hence, by (4.4.33) and (4.4.34) we obtain

$$\int_{|x|=1} |g_\lambda(\omega)|^2 \, d\sigma(\omega) = \lambda^{1/2} \Im \int_{\mathbb{R}^d} f\bar{u}, \tag{4.4.35}$$

which combining with (4.3.2) gives the desired estimate and the proof is complete. □

**Remark 4.4.6.** *Observe that from the strong convergence*

$$|\mathcal{F}(\lambda, r_n) f| \quad \to \quad |g_\lambda| \quad in \quad L^2(S^{d-1}) \quad as \quad n \to \infty, \tag{4.4.36}$$

*we deduce*

$$|\mathcal{F}(\lambda, r_n) f|^2 \quad \to \quad |g_\lambda|^2 \quad in \quad L^1(S^{d-1}) \quad as \quad n \to \infty. \tag{4.4.37}$$

*We denote* $g(\omega) = |g_\lambda(\omega)|^2$, $\omega \in S^{d-1}$.

**Remark 4.4.7.** *Note that under the assumptions that have been mentioned in Remark 4.4.4, from the identity (4.4.35) and by the a-priori estimate (4.4.17) it follows that*

$$\int_{|\omega|=1} g(\omega) \le C(N_{R_0}(f))^2. \tag{4.4.38}$$

The existence of the limit (4.4.19) for a certain sequence $\{r_n\}_{n=1}^{\infty}$ diverging to infinity does not ensure uniqueness of the cross-section. Indeed, if we take another sequence $\{r_m\}_{m\in\mathbb{N}}$ such that $r_m \to \infty$ as $m \to \infty$, we get that there exists $g_1 \in L^1(S^{d-1})$ satisfying

$$\lim_{m\to\infty} \left| r_m^{\frac{d-1}{2}} e^{-i\lambda^{1/2} r_m} u(r_m \omega) \right|^2 = g_1(\omega) \quad \text{in} \quad L^1(S^{d-1}). \tag{4.4.39}$$

### Uniqueness

Basing on the ideas developed by Isozaki [Is] and Iwatsuka [Iw] and adapting them to our case, we prove the existence of the limit (4.4.20). This issue together with Proposition 4.4.5 proves uniqueness of a cross-section.

To this end, some properties related to the surface integral involving the solution $u$ of the equation (4.0.2) will be needed. Thus we will divide the proof into a sequence of lemmas.

Let us denote

$$\mathcal{D}_r = \nabla_A^r - i\lambda^{1/2} + \frac{(d-1)}{2r}. \tag{4.4.40}$$

**Remark 4.4.8.** *Note that from (4.1.9) and (4.1.11) it follows that*

$$\int |\mathcal{D}_r u|^2 \le C \int |x|^2 |f|^2, \tag{4.4.41}$$

*for some* $C > 0$.

Unless otherwise stated, in this paragraph we will work under the assumptions of Theorem 4.1.4.

**Lemma 4.4.9.** *Let $f$ such that $\||x|f\|_{L^2}$ is finite, $u = R(\lambda+i0)f$. Let $v$ such that $\lambda|||v|||^2 < \infty$ and $\int \left|\nabla_A(e^{-i\lambda^{1/2}|x|}v)\right|^2 < +\infty$. Then*

$$\frac{d}{dr}\int_{|x|=r}(\mathcal{D}_r u)\bar{v}\,d\sigma_r = -2i\lambda^{1/2}\int_{|x|=r}(\mathcal{D}_r u)\bar{v}\,d\sigma_r + F(r) \tag{4.4.42}$$

*where $\int_1^\infty |F(r)|dr < \infty$.*

*Proof.* By a straightforward calculation we have

$$\frac{d}{dr}\int_{|x|=r}(\mathcal{D}_r u)\bar{v} = \int_{|x|=r}\mathcal{D}_r(\mathcal{D}_r u)\bar{v} + \int_{|x|=r}(\mathcal{D}_r u)(\overline{\mathcal{D}_r v}). \tag{4.4.43}$$

By our second assumption on $v$ and by Remark 4.4.8, the second term of the right hand side of (4.4.43) belongs to $L^1((1,\infty))$. On the other hand, first note that

$$\mathcal{D}_r(\mathcal{D}_r) = \nabla_A^r(\nabla_A^r) + \frac{(d-1)}{r}\nabla_A^r + \lambda - \frac{(d-1)(d-3)}{4r^2} - 2i\lambda^{1/2}\mathcal{D}_r. \tag{4.4.44}$$

Moreover, we have

$$\int_{|x|=r}\left[\nabla_A^r(\nabla_A^r u)\bar{v} + \frac{(d-1)\nabla_A^r u\bar{v}}{r}\right] = \int_{|x|=r}\nabla_A^2 u\bar{v} + \int_{|x|=r}\nabla_A^\perp u\overline{\nabla_A^\perp v}. \tag{4.4.45}$$

This is obtained by differentiating the Green's formula

$$\int_{|x|<r}\nabla_A^2 u\bar{v} = -\int_{|x|<r}\nabla_A u\cdot\overline{\nabla_A v} + \int_{|x|=r}\nabla_A^r u\bar{v}, \tag{4.4.46}$$

with respect to $r$ and by noting that $\nabla_A = \nabla_A^r + \nabla_A^\perp$ is an orthogonal sum decomposition.
Hence, combining (4.4.44) with (4.4.45) and by the equation (4.0.2) we get

$$\int_{|x|=r}\left[\nabla_A^r(\nabla_A^r u)\bar{v} + \frac{(d-1)}{r}\nabla_A^r + \lambda u\bar{v}\right] = F_1(r), \tag{4.4.47}$$

where

$$F_1(r) = \int_{|x|=r}\nabla_A^\perp u\overline{\nabla_A^\perp v} - \int_{|x|=r}Vu\bar{v} + \int_{|x|=r}f\bar{v}. \tag{4.4.48}$$

In addition, by (4.3.2), (4.1.9), assumptions on $v$ and **(H3)** it is very easy to check that $F_1(r) \in L^1((1,\infty))$. Furthermore, by (4.1.11) we also have

$$\int_{|x|=r}\frac{(d-1)(d-3)}{4r^2}u\bar{v} \in L^1((1,\infty)). \tag{4.4.49}$$

As a consequence, by (4.4.43)-(4.4.49) we deduce

$$\frac{d}{dr}\int_{|x|=r}(\mathcal{D}_r u)\bar{v} = -2i\lambda^{1/2}\int_{|x|=r}(\mathcal{D}_r u)\bar{v} + F(r) \tag{4.4.50}$$

where $F(r) = \int_{|x|=r}(\mathcal{D}_r u)(\overline{\mathcal{D}_r v}) + F_1(r) + \int_{|x|=r}\frac{(d-1)(d-3)}{4r^2}u\bar{v} \in L^1((1,\infty))$, which completes the proof of the lemma. □

**Lemma 4.4.10.** *Under the same assumption as in Lemma 4.4.9 it follows that*

$$\int_{|x|=r}(\mathcal{D}_r u)\bar{v}d\sigma_r \quad \to \quad 0 \qquad as \qquad r \to \infty. \tag{4.4.51}$$

*Proof.* Let us put $\phi(r) = \int_{|x|=r}(\mathcal{D}_r u)\bar{v}d\sigma_r$. Then by Lemma 4.4.9 we have

$$\frac{d}{dr}\phi(r) = -2i\sqrt{\lambda}\phi(r) + F(r), \qquad \int_1^\infty |F(r)|dr < +\infty. \tag{4.4.52}$$

Letting $\psi(r) = e^{2i\sqrt{\lambda}r}\phi(r)$ we get

$$\frac{d}{dr}\psi(r) = e^{2i\sqrt{\lambda}r}F(r), \tag{4.4.53}$$

which implies

$$\psi(r) = \psi(1) + \int_1^r e^{2i\sqrt{\lambda}s}F(s)ds. \tag{4.4.54}$$

Since $F(s) \in L^1((1,\infty))$, we have that there exists a limit

$$\lim_{r\to\infty}\psi(r). \tag{4.4.55}$$

On the other hand, since $\int\frac{|v|^2}{|x|^2} < \infty$ and Remark 4.4.8, by Cauchy-Schwarz inequality we obtain

$$\int_1^\infty r^{-1}|\phi(r)|dr < \infty. \tag{4.4.56}$$

Thus we conclude that

$$\liminf_{r\to\infty}|\psi(r)| = 0,$$

which combining with the existence of the limit (4.4.55), gives

$$|\psi(r)| = |\phi(r)| \to 0 \qquad \text{as} \qquad r \to \infty.$$

□

**Lemma 4.4.11.** *Let $f$ such that $\||x|f\|_{L^2} < \infty$, $u = R(\lambda + i0)f$. Then it satisfies*

$$(R(\lambda - i0)f - R(\lambda + i0)f, f) = \lim_{r\to\infty} 2i\sqrt{\lambda} \int_{|x|=r} |u|^2. \tag{4.4.57}$$

*Proof.* Since $\nabla_A^2 u + Vu + \lambda u = f$, by using the following Green's formula

$$\int_{|x|<r} f\nabla_A^2 g = -\int_{|x|<r} \nabla_A f \cdot \nabla_A g + \int_{|x|=r} f\frac{x}{|x|} \cdot \nabla_A g, \tag{4.4.58}$$

yields

$$\begin{aligned}\int_{|x|<r}(u\bar{f} - f\bar{u})dx &= \int_{|x|<r}(u\overline{\nabla_A^2 u} - \nabla_A^2 u\bar{u}) \\ &= \int_{|x|=r}(\overline{\nabla_A^r u}u - \nabla_A^r u\bar{u}) \\ &= \int_{|x|=r}\left[(\overline{\mathcal{D}_r u})u - (\mathcal{D}_r u)\bar{u}\right] - 2i\sqrt{\lambda}\int_{|x|=r}|u|^2 d\sigma_r.\end{aligned}$$

By Lemma 4.4.10 and the fact that the left hand side of the above equality tends to $(R(\lambda + i0)f - R(\lambda - i0)f, f)$ as $r \to \infty$, the lemma follows. □

In view of Lemma 4.4.11, since

$$\int_{|x|=1} |(\mathcal{F}(\lambda, r)f)(\omega)|^2 \, d\sigma = \int_{|x|=r} |u(r\omega)|^2 d\sigma_r, \tag{4.4.59}$$

for $f$ such that $\||x|f\|_{L^2} < \infty$ yields

$$\frac{1}{2\pi i}\left(R(\lambda - i0)f - R(\lambda + i0)f, f\right) = \lim_{r\to\infty} \|\mathcal{F}(\lambda, r)f\|^2_{L^2(S^{d-1})}. \tag{4.4.60}$$

Now we are in a position to prove the existence of the limit of $|\mathcal{F}(\lambda, r)f|^2$.

**Lemma 4.4.12.** *Let $f$ such that $\||x|f\|_{L^2} < \infty$. Given $\phi \in C^\infty(S^{d-1})$ there exists the limit*

$$\lim_{r\to\infty} \int_{|x|=1} |\mathcal{F}(\lambda, r)f(\omega)|^2 \, \phi(\omega). \tag{4.4.61}$$

*Proof.* Let $\phi(\omega) \in C^\infty(S^{d-1})$ and we define

$$v = \rho(r)u(r\omega)\phi(\omega) \qquad \left(r = |x|, \omega = \frac{x}{r}\right), \tag{4.4.62}$$

where $\rho(r) \in C^\infty(\mathbb{R}^+)$ such that $\rho(r) = 0$ if $r < 1$ and $\rho(r) = 1$ if $r > 2$. Note that we have $\int \frac{|v|^2}{|x|^2} < \infty$ and $\int |\nabla_A(e^{-i\lambda^{1/2}|x|}v)|^2 < \infty$.

Let

$$g = (\nabla_A^2 + V + \lambda)v. \tag{4.4.63}$$

Then, by a straightforward calculation we have

$$g = f\rho\phi + \frac{2\rho}{|x|}\nabla_A^\perp u \cdot \nabla_\omega \phi + \frac{\rho}{|x|^2}\Lambda\phi u + 2\rho'\nabla_A^r u\phi + \rho'' u\phi + \frac{\rho'(d-1)}{|x|}u\phi \tag{4.4.64}$$

where $\Lambda$ is the Laplace Beltrami operator on $S^{d-1}$. Therefore, from the a-priori estimates (4.1.11) and (4.3.2) it follows that

$$\begin{aligned}\int |x|^2|g|^2 &\le C\left[\int |x|^2|f|^2 + \int |\nabla_A^\perp u|^2 + |||\nabla_A u|||_1^2 + \int \frac{|u|^2}{|x|^2}\right]\\ &\le C\int |x|^2|f|^2.\end{aligned}$$

Now letting $u = R(\lambda + i0)f$, by Green's formula (4.4.58) we see that

$$\int_{|x|<r}\left\{(\nabla_A^2 v)\bar{u} - v(\overline{\nabla_A^2 u})\right\} = \int_{|x|=r}\{(\mathcal{D}_r v)\bar{u} - v(\overline{\mathcal{D}_r u})\} + 2i\sqrt{\lambda}\int_{|x|=r} v\bar{u}. \tag{4.4.65}$$

Note that letting $r \to \infty$, the left hand side of the above identity tends to

$$(g, R(\lambda + i0)f) - (v, f).$$

Since $\mathcal{D}_r v = (\mathcal{D}_r u)\rho\phi$ if $r > 2$, by Lemma 4.4.10 the first term of the right hand side of (4.4.65) tends to 0 as $r \to \infty$. Moreover, by the definitions of $v$ and $\mathcal{F}(\lambda, r)f$ it follows that

$$\begin{aligned}\lim_{r\to\infty} 2i\sqrt{\lambda}\int_{|x|=r} v\bar{u} &= \lim_{r\to\infty} 2i\sqrt{\lambda}\int_{|x|=r}|u|^2\phi\\ &= \lim_{r\to\infty} C(\lambda)\int_{|x|=1}|\mathcal{F}(\lambda, r)f|^2\phi.\end{aligned}$$

Therefore, we have shown that

$$(g, R(\lambda + i0)f) - (v, f) = \lim_{r\to\infty} C(\lambda)\int_{|x|=1}|\mathcal{F}(\lambda, r)f|^2\phi, \tag{4.4.66}$$

which proves the lemma. □

**Proof of Theorem 4.1.9 (i)**

From the existence of the limit (4.4.61), together with the existence of the limit

$$\lim_{r_n\to\infty}\int_{|x|=1}|\mathcal{F}(\lambda,r_n)f(\omega)|^2=g(\omega)\qquad g\in L^1(S^{d-1}) \tag{4.4.67}$$

proved in Proposition 4.4.5, it may be concluded that

$$\lim_{r\to\infty}\int_{|x|=1}|\mathcal{F}(\lambda,r)f(\omega)|^2\phi(\omega)=\int_{|x|=1}g(\omega)\phi(\omega)\qquad\forall\phi\in C^\infty(S^{d-1}). \tag{4.4.68}$$

In particular, if we take $\phi(\omega)=1$ we obtain

$$\lim_{r\to\infty}\int_{|x|=1}|\mathcal{F}(\lambda,r)f(\omega)|^2=\int_{|x|=1}g(\omega). \tag{4.4.69}$$

As a consequence, since $g(\omega)=|g_\lambda(\omega)|^2$, from (4.4.22) we deduce (4.1.21) and the proof is complete.

**Remark 4.4.13.** *Observe that from Proposition 4.4.5 and Remark 4.4.6, denoting $F=\{f:\||x|f\|_{L^2}<\infty,\||x|^{3/2}f\|_{L^2}<\infty\}$, we have defined*

$$\begin{aligned} T_\lambda : F &\quad\to\quad L^2(S^{d-1})\\ f &\quad\to\quad T_\lambda f=|g_\lambda| \end{aligned} \tag{4.4.70}$$

*such that*

$$\int_{|x|=1}|T_\lambda f|^2\ \le C\int|x|^2|f|^2. \tag{4.4.71}$$

*Moreover, from Remark 4.4.4 we have*

$$\int_{|x|=1}|T_\lambda f|^2\ \le C(N(f))^2. \tag{4.4.72}$$

*Thus one may extend the operator $T_\lambda$ for any $f$ such that $N(f)<\infty$. In fact, if we denote $B=\{f:N(f)<\infty\}$, then it may be concluded that $T_\lambda:B\to L^2(S^{d-1})$ is a continuous operator.*

**Remark 4.4.14.** *Adding some extra assumption on the magnetic potential $A$, one could prove existence and uniqueness of the far field pattern. In fact, if we require $|A|\le\frac{C}{|x|}$, then it may be concluded that $\mathcal{F}(\lambda,r)f(\omega)\in H^1(S^{d-1})$. Hence, proceeding as in Proposition 4.4.5, the existence of the far field pattern would follow.*
*In addition, if we put more restriction on the magnetic potential $A$, i.e. if we consider*

$|A| \leq \frac{C}{(1+|x|)^{\frac{3}{2}+\mu}}$ *with* $\mu > 0$*, then following Iwatsuka [Iw] we could also prove uniqueness of the far field pattern. The only difference regarding to this part is in Lemma 4.4.12. In this case, one needs to define* $v$ *by*

$$v = \rho(r) r^{-\frac{(d-1)}{2}} e^{i\lambda^{1/2} r} \phi(\omega).$$

*Observe that if* $|A(x)| \leq \frac{C}{(1+|x|)^{\gamma}}$ *with* $\gamma > \frac{1}{2}$*, then* $\int \left|\nabla_A (e^{-i\lambda^{1/2}|x|} v)\right|^2 < \infty$*. On the other hand, by a straightforward calculation we have that if* $r > 2$ *yields*

$$\begin{aligned} e^{-i\lambda^{1/2} r} g &= C_d r^{-\frac{(d-1)}{2}-2} \phi(\omega) - \frac{(d-1)}{2} r^{-\frac{(d-1)}{2}-1} \frac{x}{|x|} \cdot A(x)\phi(\omega) \\ &+ i\lambda^{1/2} r^{-\frac{(d-1)}{2}} \frac{x}{|x|} \cdot A(x)\phi(\omega) - r^{-\frac{(d-1)}{2}} \left[A(x)\cdot A(x) + V(x)\right]\phi(\omega) \\ &+ r^{-\frac{(d-1)}{2}-1} A \cdot \frac{\partial \phi}{\partial \omega} + r^{-\frac{(d-1)}{2}-2} \Lambda \phi(\omega). \end{aligned}$$

*Hence, for* $A$ *and* $V$ *decaying like* $(1+|x|)^{-\frac{3}{2}+\mu}$ *with* $\mu > 0$*, we get* $\int |x|^2 |g|^2 < \infty$*. The rest of the proof runs as before.*

### 4.4.3 Spectral representation

Finally, our purpose is to study some spectral properties of the electromagnetic Schrödinger operator

$$H_A = \nabla_A^2 + V$$

in $\mathbb{R}^d$, $d \geq 3$ with potentials satisfying the same assumptions as in Theorem 4.1.8. In particular, we will prove the second part of Theorem 4.1.9.

Let us first recall that $H_A$ is a self-adjoint operator with form domain

$$D(H_A) = \{u \in L^2(\mathbb{R}^d) : \int |\nabla_A u|^2 - \int V|u|^2 < \infty\}. \tag{4.4.73}$$

Let $E(B)$ the spectral measure associated with $H_A$, where $B$ varies over all Borel sets of the reals. We shall first show that $E((0,\infty))H_A$ is an absolutely continuous operator.

Let $R(z) = (H_A + z)^{-1}$ denote the resolvent of $H_A$ and recall that for $z = \lambda + i\varepsilon$ with $\lambda \neq 0$, $\varepsilon > 0$ and $f$ such that $\int |x|^2|f|^2 < \infty$, a unique solution

$$u(z,f) = R(z)f(x)$$

of the equation $(H_A + z)u = f$ satisfying (4.3.2) can be determined.

Let $\lambda > 0$. From Theorem 4.4.3 it follows that for any $f$ such that $\|(1+|x|)f\|_{L^2} < \infty$ a unique solution $u_\pm(\lambda, f)$ of the equation

$$(H_A + \lambda)u_\pm = f$$

satisfying the corresponding Sommerfeld radiation condition

$$\int |\nabla_A(e^{\mp i\lambda^{1/2}|x|}u_\pm)|^2 < \infty$$

can be constructed as the limit

$$u_\pm(\lambda, f) = \lim_{\varepsilon\to 0} u(\lambda \pm i\varepsilon, f) \tag{4.4.74}$$

in $(H^1_A)_{loc}$. Let $\Delta = (\lambda_1, \lambda_2)$ where $0 < \lambda_1 < \lambda_2 < \infty$. Then, employing the following well-known formula stated in section 1.5

$$(E(\Delta)f, f) = \lim_{\varepsilon\to 0}\lim_{\nu\to 0}\frac{1}{2\pi i}\int_{\lambda_1-\nu}^{\lambda_2+\nu}(R(\lambda - i\varepsilon)f - R(\lambda + i\varepsilon)f, f)\, d\lambda \qquad (f \in L^2),$$

by (4.4.74), the fact that

$$(u(\lambda - i\varepsilon, f) - u(\lambda + i\varepsilon, f), f)$$

is uniformly bounded for $(\lambda, \varepsilon) \in [\lambda_1, \lambda_2] \times [0, 1]$, together with the Lebesgue dominated convergence theorem, one obtains

$$(E(\Delta)f, f) = \frac{1}{2\pi i}\int_\Delta (u_-(\lambda, f) - u_+(\lambda, f), f)\, d\lambda. \tag{4.4.75}$$

Thus noting that the space $L^2(1 + |x|) := \{f : \int(1 + |x|)^2|f|^2 < \infty\}$ is dense in $L^2(\mathbb{R}^d)$ and $(u_-(\lambda, f) - u_+(\lambda, f), f)$ is a continuous function of $\lambda \in (0,\infty)$, we state the following result.

**Theorem 4.4.15.** *Under the hypotheses of Theorem 4.4.3, $E((0,\infty))H_A$ is an absolutely continuous operator. Moreover, it satisfies*

$$(E(0,\infty)f, f) = \frac{1}{2\pi i}\int_0^\infty (u_-(\lambda, f) - u_+(\lambda, f), f)\, d\lambda. \tag{4.4.76}$$

As a consequence, since $(u_-(\lambda, f) - u_+(\lambda, f), f) = 2\Im\int f\bar{u}$ and

$$\Im\int f\bar{u} = C_\lambda\int_{|x|=1} g(\omega),$$

we obtain (4.1.22) and Theorem 4.1.9 follows.

**Remark 4.4.16.** *By Remark 4.4.4, replacing the assumption* **(H1)** *on the potentials by the conditions* **(H1a)**, **(H1b)**, *the same results hold for $f$ such that $N(f) < \infty$.*

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