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UNILATERAL VS. BILATERAL LINK-FORMATION: BRIDGING THE GAP

by

Norma Olaizola and Federico Valenciano

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UPV EHU

University of the Basque Country

Unilateral vs. Bilateral link-formation: Bridging the gap*

By Norma Olaizola† and Federico Valenciano‡

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Abstract

We provide a model that bridges the gap between two benchmark models of strategic network formation: Jackson and Wolinsky's model based on bilateral formation of links, and Bala and Goyal's two-way flow model, where links can be unilaterally formed. In the model introduced and studied here a link can be created unilaterally. When it is only supported by one of the two players the flow through the link suffers a certain decay, but when it is supported by both the flow runs without friction. When the decay in links supported by only one player is maximal (i.e. there is no flow) we have Jackson and Wolinsky's connections model without decay, while when flow in such links is perfect we have Bala and Goyal's two-way flow model. We study Nash, strict Nash and pairwise stability for the intermediate models. Efficiency and dynamics are also examined.

JEL Classification Numbers: A14, C72, D20, J00

Key words: Network formation, Unilateral link-formation, Bilateral link-formation, Stability, Efficiency, Dynamics.

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†*BRiDGE* group (http://www.bridgebilbao.es), Departamento de Fundamentos del Análisis Económico I, Universidad del País Vasco, Avenida Lehendakari Aguirre 83, 48015 Bilbao, Spain; norma.olaizola@ehu.es.

‡*BRiDGE* group (http://www.bridgebilbao.es), Departamento de Economía Aplicada IV, Universidad del País Vasco, Avenida Lehendakari Aguirre 83, 48015 Bilbao, Spain; federico.valenciano@ehu.es.

1 Introduction

Two basic models of strategic network formation in economic literature are those of Jackson and Wolinsky (1996), where the formation of a link between two players requires the agreement of both, and Bala and Goyal (2000), where a link can be formed unilaterally by any player. The first model has two variants: the connections model and the coauthors model. Bala and Goyal's model has also two variants: the one-way flow model, where the flow through a link runs toward a player only if he/she supports it, and the two-way flow model, where flow runs in both directions regardless of which player supports it. These models have been extended in different directions, all of which branch out from one of these stems[1]. In this paper we thicken the trunk instead, by providing a model that bridges the gap between two of these basic benchmark models of strategic network formation. In a previous paper (Olaizola and Valenciano, 2014) we provide a model that integrates Bala and Goyal's one-way and two-way flow models of network formation as particular extreme cases of a more general one that we call the "asymmetric flow" model, and characterize Nash and strict Nash structures for the whole range of intermediate models. This paper is a contribution to the theory of strategic network formation which takes this unification a step further. More precisely, we provide a model which has Jackson and Wolinsky's connections model without decay and Bala and Goyal's two-way flow model as extreme cases. In the model introduced and studied here, a link can be created unilaterally, but when it is only supported by one of the two players (such a link is referred to as a "weak" link) the flow through the link suffers a certain decay (the same in both directions). However, when a link is supported by both players (a "strong" link) the flow runs without friction in both directions. When the decay in weak links is maximal (i.e. there is no flow) we have Jackson and Wolinsky's connections model without decay, while when flow in such links is perfect we have Bala and Goyal's two-way flow model. In contrast to these two extreme cases, it seems reasonable to consider intermediate situations, where strong doubly-supported links "work" better than weak singly-supported ones, but both types of link are feasible. Moreover, this provides an extension of both models and allows for a study of the "transition" from one to the other, providing a "neighborhood" of each model which provides a point of view for testing the robustness of the results for each of the extreme cases. We study Nash, strict Nash and pairwise stability for the intermediate models. We also discuss efficiency and dynamics.

A noteworthy outcome of this exercise is the emergence (for certain ranges of the parameters, i.e. cost and decay through weak links) in this transition of a particularly simple type of core-periphery structure[2] in equilibrium, namely networks with a "core"

[1] Given the (to our knowledge) completely different direction of the joint extension studied here, we omit an unavoidable, too long or incomplete list of extensions orthogonal to this one. Excellent monographs on social and economic networks are Goyal (2007), Jackson (2008) and Vega-Redondo (2007).

[2] Core-periphery networks arise in different contexts: geographical, biological, social, economic, financial, industrial research, etc. See, for instance, Csermely et al. (2013), and the literature cited

consisting of a tree of strong links whose terminal nodes are involved in one or more weak links with "peripheral" players, each of whom supports a single weak link with the core. Moreover, this arises in a context of ex-ante homogeneity[3], and interestingly enough in a model "close" to -in fact, intermediate between- the two best known economic models of network formation.

The rest of the paper is organized as follows. Section 2 introduces notation and terminology relative to graphs. Section 3 reviews the strategic models of network formation of Jackson and Wolinsky (1996) and Bala and Goyal (2000). In Section 4, a model that bridges the gap between these two is presented and Nash stable, Nash strictly stable and pairwise stable structures are studied for the intermediate models. Section 5 addresses the question of efficiency, and Section 6 is devoted to dynamics. Finally, Section 7 summarizes the main conclusions and points out some lines of further research.

2 Graphs: notation and terminology

A *directed* N*-graph* is a pair (N, Γ), where $N = \{1, 2, ..., n\}$ is a finite set with $n \geq 3$ whose elements are called *nodes*, and Γ is a subset of $N \times N$, whose elements are called links. When both (i, j), and (j, i) are in Γ, we say that i and j are connected by a *strong link*, while if only one of them is there we say that they are connected by a *weak link*. If $M \subseteq N$, the M*-subgraph* of (N, Γ) is the M-graph $(M, \Gamma|_M)$ with

$$\Gamma|_M := \{(i, j) \in M \times M : (i, j) \in \Gamma\}.$$

Alternatively, a graph Γ can be specified by a map $g_\Gamma : N \times N \to \{0, 1\}$,

$$g_\Gamma(i, j) := \begin{cases} 1, & \text{if } (i, j) \in \Gamma \\ 0, & \text{if } (i, j) \notin \Gamma. \end{cases}$$

When we specify a graph Γ in this way by a map g, we denote $g_{ij} := g(i, j)$, and if $g_{ij} = 1$ link (i, j) is referred to as "link ij in g", and we write $ij \in g$. Note that for $M \subseteq N$, subgraph $\Gamma|_M$ is specified by $g|_{M \times M}$ but, abusing the notation, this subgraph is denoted by $g|_M$. The *empty graph* is denoted by g^e (i.e. $g^e(i, j) = 0$, for all i, j).

If $g_{ij} = 1$ in a graph g, $g - ij$ denotes the graph that results from replacing $g_{ij} = 1$ by $g_{ij} = 0$ in g; and if $g_{ij} = 0$, $g + ij$ denotes the graph that results from replacing $g_{ij} = 0$ by $g_{ij} = 1$. Similarly, if $g_{ij} = g_{ji} = 1$, $g - \overline{ij} = (g - ij) - ji$, and if $g_{ij} = g_{ji} = 0$, $g + \overline{ij} = (g + ij) + ji$. An *isolated* node in a graph g is a node that is not involved in

therein.

[3] Given their presence in economic contexts, models of formation of core-periphery networks have attracted considerable attention. In some models, such structures arise via the assumption of some form of heterogeneity among the players (e.g. van der Leij et al., 2014), while in others they originate in spite of ex ante homogeneity (e.g. Galeotti and Goyal, 2010).

any link, that is, a node i s.t. for all $j \neq i$, $g_{ij} = g_{ji} = 0$. A node is *peripheral* in a graph g if it is involved in a single link (weak or strong).

Given a graph g, a *path* of *length* k *from* j *to* i in g is a sequence of $k+1$ distinct nodes $j_0, j_1, ..., j_k$, s.t. $j = j_0$, $i = j_k$, and for all $l = 1, ..., k$, $g_{j_{l-1}j_l} = 1$ or $g_{j_l j_{l-1}} = 1$; if for all $l = 1, ..., k$, $g_{j_l j_{l-1}} = 1$, the path is said to be *weakly i-oriented*, and *i-oriented* if for all $l = 1, ..., k$, $g_{j_l j_{l-1}} = 1$ and $g_{j_{l-1}j_l} = 0$. A graph g is *acyclic* or *contains no cycles* if there is no sequence of k ($k > 2$) distinct nodes, $i_1, ..., i_k$, s.t. for all $l = 1, ..., k-1$, $g_{i_l i_{l+1}} = 1$ or $g_{i_{l+1} i_l} = 1$, and $g_{i_1 i_k} = 1$ or $g_{i_k i_1} = 1$.

Definition 1 *Given a graph g, and $C \subseteq N$, the subgraph $g\mid_C$ is said to be*:
(i) A weak component of g if for any two nodes $i, j \in C$ ($i \neq j$) there is a path from j to i in g, and no subset of N strictly containing C meets this condition.
(ii) A strong component if for any two nodes $i, j \in C$ ($i \neq j$) there is a path of strong links from j to i in g, and no subset of N strictly containing C meets this condition.

When a component in either sense consists of a single node we say that it is a *trivial component*. In both senses, an *isolated* node, i.e. a node that is not involved in any link, is a trivial component. The *size* of a component is the number of nodes from which it is formed. For instance, the three graphs[4] in Figure 1 have a unique weak component, (a) has four strong componentes (two of them trivial), (b) has nine strong componentes (all of them trivial), and (c) has only one strong component.

Based on these definitions we have two different notions of *connectedness*. We say that a graph g is *weakly* (*strongly*) *connected*[5] if g is the unique weak (strong) component of g. Note that strong connectedness implies weak connectedness. The three graphs in Figure 1 are weakly connected, but only (c) is strongly connected.

A weak (strong) component $g\mid_C$ of a graph g is *minimal* if for all $i, j \in C$ s.t. $g_{ij} = 1$, the number of weak (strong) components of g is smaller than the number of weak (strong) components of $g - ij$.

A graph is *minimally* weakly (strongly) connected if it is weakly (strongly) connected and minimal. In both cases, a minimally connected graph is a *tree* (of weak links in one case, of strong links in the other), but, in principle, any node in such trees can be seen as the *root*, i.e. a reference node from which there is a unique path connecting it with any other. Note that a weakly connected graph *with no cycles* is a tree in general formed by weak and strong links, and in general neither minimally weakly nor strongly connected. A minimally weakly connected graph is said to be an

[4]Throughout the text, a strong link between two nodes is represented by a thick segment connecting them, while a weak link is represented by a thin segment between them only touching the node that supports it.

[5]Note that the sense in which the term "strongly connected" is used here differs from its usual meaning in the literature, where a directed network or digraph is said to be strongly connected when for any two distinct nodes, i, j there is a weakly i-oriented path from j to i. In our context, a clear distinction between weak and strong links invites the use of the term in the sense in which we use it here.

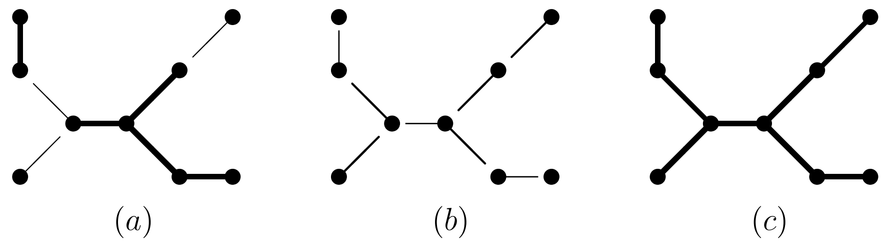


Figure 1: Weak and strong connectedness

i-oriented tree if for any node $j \neq i$, the only path from j to i is i-oriented. Graph (b) in Figure 1 is minimally weakly connected, and graph (c) is minimally strongly connected.

Given a graph g, the following notation is also used:

$N^d(i;g) := \{j \in N : g_{ij} = 1\}$ (i.e. set of nodes with which i supports a link),
$N^e(i;g) := \{j \in N : g_{ji} = 1\}$ (i.e. set of nodes which support a link with i),
$N^o(i;g) := N^d(i;g) \cup N^e(i;g)$ (i.e. set of nodes involved in a link with i).

The set of nodes connected with i by a path is denoted by $N(i;g)$. Note that none of these sets contains i. Their cardinalities are denoted by $\mu_i^d(g) := \#N^d(i;g)$, $\mu_i^e(g) := \#N^e(i;g)$, $\mu_i^o(g) := \#N^o(i;g)$, and $\mu_i(g) := \#N(i;g)$.

We consider two measures of distance between nodes in a graph g based on two different notions of the length of a path. When there is no path connecting two nodes the distance between them for both the two notions is said to be ∞. The *distance* between two nodes i, j $(i \neq j)$, denoted by $d(i,j;g)$, is the length of the shortest path connecting them. In Section 4 we consider a situation where the flow through weak links has some friction or decay, in contrast with strong links, through which flow is without friction. This motivates the following notion. The *discounting length* of a path from j to i in g is the length of the path *minus* the number of strong links in that path, i.e. the number of weak links in it. The *discounting distance* between j and i $(i \neq j)$ in g, denoted by $\lambda(i,j;g)$, is defined as the discounting length of the path from j to i with the shortest discounting length. Note that $d(i,j;g) \geq \lambda(i,j;g)$. The following example illustrates the two notions of distance.

Example 1: Consider the 6-node graph g formed by links: 12, 23, 32, 43, 45, 56.

Then, $d(1,6;g) = d(6,1;g) = 5$, $\lambda(1,6;g) = \lambda(6,1;g) = 4$, $d(2,3;g) = d(3,2;g) = 1$, $\lambda(2,3;g) = \lambda(3,2;g) = 0$.

3 Strategic models of network formation

We consider situations where individuals may initiate or support *links* with other individuals under certain assumptions, thus creating a network formalized as a graph. We assume that at each node $i \in N$ there is an agent identified by label i and referred to as *player*[6] i. Each player i may initiate or, more generally, *intend* to initiate links with other players as an intended link may or may not, depending on the assumptions, actually be formed[7]. A map $g_i : N\backslash\{i\} \to \{0,1\}$ describes the links intended by i. We denote $g_{ij} := g_i(j)$, and $g_{ij} = 1$ ($g_{ij} = 0$) means that i *intends* (does not intend) to form a link with j. Thus, vector $g_i = (g_{ij})_{j\in N\backslash\{i\}} \in \{0,1\}^{N\backslash\{i\}}$ specifies the links intended by i and is referred to as a *strategy* of player i. $G_i := \{0,1\}^{N\backslash\{i\}}$ denotes the set of i's strategies and $G_N = G_1 \times G_2 \times ... \times G_n$ the set of *strategy profiles.* A strategy profile $g \in G_N$ univocally determines a *graph* (N, Γ_g) of intended links, where $\Gamma_g := \{(i,j) \in N \times N : g_{ij} = 1\}$. Given a strategy profile $g \in G_N$ and $i \in N$, g_{-i} denotes the $N\backslash\{i\}$ strategy profile that results by eliminating g_i in g, i.e. all links intended by i, and (g_{-i}, g'_i), where $g'_i \in G_i$, denotes the strategy profile that results by replacing g_i by g'_i in g.

Let g be a strategy profile representing players' intended links. We denote by g^* the associated graph representing the *actual network* that results from g. We consider several models under different assumptions, but the following are generally assumed:

1. Whether it actually forms or not, an intended link of player i with player j means a *cost* $c_{ij} > 0$ for all $j \neq i$.

2. The player at node j has a particular type of information or other good[8] of *value* v_{ij} for player i.

3. If $\mathbf{v} = (v_{ij})_{i,j\in N}$ is the matrix of values, $\mathbf{c} = (c_{ij})_{i,j\in N}$ is the matrix of cost (assuming[9] $c_{ii} = v_{ii} = 0$), and g is the strategy profile and g^* the resulting network, the payoff of a player is given by a function

$$\Pi_i(g) = I_i(g^*, \mathbf{v}) - c_i(g, \mathbf{c}), \tag{1}$$

where $I_i(g^*, \mathbf{v})$ is the *information* received by i through the actual network g^*, and $c_i(g, \mathbf{c}) = \sum_{j\in N^d(i;g)} c_{ij}$ the *cost* incurred by i.

Under different assumptions, different models specify g^* and I_i differently. In all cases a game in strategic form is specified: $(G_N, \{\Pi_i\}_{i\in N})$, and we consider three forms

[6] However, so as not to overcomplicate our writing style due to efforts to avoid a biased language, we often refer to players by the more neutral term "nodes".

[7] This is similar to Myerson's (1977) model, where all players simultaneously announce the set of players with whom they wish form links. But while in Myerson's model links are formed if and only if they were proposed by both, we consider different scenarios here.

[8] Although other interpretations are possible, in general, we give preference to the interpretation in terms of information.

[9] Only to make it possible to call $\mathbf{c}$ and $\mathbf{v}$ matrices. Nevertheless, in practice c_{ii} and v_{ii} play no role. Note also that by definition g_{ii} remains undefined for any strategy.

of stability: *Nash* equilibrium and two different refinements of it: *strict Nash* equilibrium and *pairwise* stability (Jackson and Wolinsky, 1996).

Definition 2 *A strategy profile g is:*
(i) A Nash equilibrium if $\Pi_i(g_{-i}, g_i') \leq \Pi_i(g)$, *for all i and all* $g_i' \in G_i$.
(ii) A strict Nash equilibrium if $\Pi_i(g_{-i}, g_i') < \Pi_i(g)$, *for all i and all* $g_i' \in G_i$ $(g_i' \neq g_i)$.
(iii) Pairwise stable if it is a Nash equilibrium and for any pair of players i, j $(i \neq j)$ *s.t.* $g_{ij} = g_{ji} = 0$, *if* $\Pi_i(g + \overline{ij}) > \Pi_i(g)$ *then* $\Pi_j(g + \overline{ij}) < \Pi_j(g)$.

We consider two basic models relating g^* to g:

$$(i)\quad g_{ij}^* := g_{ij}^{\min} = \min\{g_{ij}, g_{ji}\}. \tag{2}$$

$$(ii)\quad g_{ij}^* := g_{ij}^{\max} = \max\{g_{ij}, g_{ji}\}. \tag{3}$$

Under assumption (2) only links intended by *both* players actually form. This is Jackson and Wolinsky's (1996) model of network formation, where establishing a link requires that both players intend it. Under assumption (3) a link forms between two players as soon as at least one of them intends it. Thus, in this case a player can create a strong link unilaterally. This is Bala and Goyal's (2000) *two-way flow* network formation model.

If every node receives the value of the players with whom it is connected by a path in g^* without friction, then, according to each of these specifications of the resulting actual network, i.e., whether g^* is given by (2) or (3), the payoff of a player i given by (1) becomes

$$\Pi_i^{\min}(g) = \sum_{j \in N(i; g^{\min})} v_{ij} - \sum_{j \in N^d(i;g)} c_{ij}, \tag{4}$$

$$\Pi_i^{\max}(g) = \sum_{j \in N(i; g^{\max})} v_{ij} - \sum_{j \in N^d(i;g)} c_{ij}. \tag{5}$$

The model specified by (2) and (4) is Jackson and Wolinsky's connections model *without decay* ("J&W model" often in what follows), that is, assuming that the flow through a link of the actual network is perfect or without loss. Similarly, (3) and (5) specify Bala and Goyal's two-way flow model without decay ("B&G model" often in what follows).

4 Between the J&W and B&G models

In both the J&W and B&G models, a level of friction in the flow through a link can be assumed, so that only a fraction δ $(0 \leq \delta \leq 1)$, referred to as the *level of decay*, of the information at one node reaches the other through that link[10]. In both cases, whether

[10] In fact, this is the only situation considered by Jackson and Wolinsky (1996), while Bala and Goyal (2000) also consider the case with no decay.

perfect flow or a level of decay is assumed, the flow is assumed to be *homogeneous* (i.e. the same through all actual links) in both models. In order to bridge the gap between these models, making a transition from one to the other possible, we introduce a very simple form of *endogenous* heterogeneity[11] relative to the level of decay. We consider a model where information flows *through strong links without friction in both directions*, while flow *through weak links is the same in both directions but with a certain decay.*

This can be formalized as follows. In both benchmark models the actual network g^* specifies the decay $\delta_{ij}^{g^*}$ through each link $ij \in g^*$, namely, $\delta_{ij}^{g^*} = g_{ij}^*$ if there is no decay, and $\delta_{ij}^{g^*} := \delta g_{ij}^*$ if the level of decay is δ. Note that the *decay matrix* $(\delta_{ij}^{g^*})_{i,j\in N}$ (assuming $\delta_{ii}^{g^*} = 0$) encapsulates all the relevant information (along with costs and values).

Assume now that when players' strategy profile is g the resulting decay matrix of the actual network, $\delta^g = (\delta_{ij}^g)_{i,j\in N}$, is given by

$$\delta_{ij}^g := \alpha g_{ij}^{\max} + (1-\alpha) g_{ij}^{\min}, \tag{6}$$

for all $i, j \in N$, with $\alpha \in [0, 1]$. In this model, for $0 < \alpha < 1$, when a link is supported by *both* players ($g_{ij} = g_{ji} = 1$) we have $g_{ij}^{\max} = g_{ji}^{\max} = g_{ij}^{\min} = g_{ji}^{\min} = 1$, so that $\delta_{ij}^g = \delta_{ji}^g = 1$, i.e. *information flows without friction in both directions*, while when *one and only one* player supports it ($g_{ij} = 1$ and $g_{ji} = 0$, or $g_{ij} = 0$ and $g_{ji} = 1$) we have $g_{ij}^{\max} = g_{ji}^{\max} = 1$ and $g_{ij}^{\min} = g_{ji}^{\min} = 0$, i.e. *flow through is the same in both directions but with a certain decay* ($\delta_{ij}^g = \delta_{ji}^g = \alpha$). Note this model's similarity to and difference from the two-way flow model *with decay*. Weak links, i.e. links supported by only one player, work as in that model, while the flow through strong links, i.e. links supported by *both* players, is perfect. This important difference enriches the setting of the benchmark models with the possibility of different treatment of links with strong support[12].

Now the point is to study the stable networks in this model for the different values of the parameter α ($0 \le \alpha \le 1$) assuming *homogeneity in costs and values across players*, that is, we assume throughout the paper

$$v_{ij} = 1 \text{ and } c_{ij} = c, \text{ where } 0 < c < 1 \text{ and } i \neq j;$$

[11] See Bloch and Dutta (2009) for a model with endogenous heterogeneity where players may invest their endowments across links.

[12] Moreover, if a level of decay μ is added to the model a new one emerges, whose decay matrix is

$$\delta_{ij}^{\prime g} := \mu \delta_{ij}^g = \mu \alpha g_{ij}^{\max} + \mu (1-\alpha) g_{ij}^{\min},$$

that is, when a link is supported by both players the flow through it is the same in both directions ($\delta_{ij}^{\prime g} = \delta_{ji}^{\prime g} = \mu$), while when only one player supports it ($g_{ij} = 1$ and $g_{ji} = 0$) the flow through it is the same in both directions but with a certain *greater* decay ($\delta_{ij}^{\prime g} = \delta_{ji}^{\prime g} = \mu\alpha$). That is, with and without decay, doubly supported links are treated differently, which may be a reasonable assumption in certain contexts. Again, when $\alpha = 1$ this is B&G two-way flow model *with decay* μ, while when $\alpha = 0$ this is J&W's connections model *with decay* μ.

so that, for all values of the parameters, the cost for a player i in a profile g is given by

$$c_i(g) = c\mu_i^d(g).$$

Let us first consider the extreme cases $\alpha = 0$ and $\alpha = 1$. When $\alpha = 0$ we have Jackson and Wolinsky (1996) connections model *without decay*[13]: a link is formed if and only if both players intend it and in this case the flow through it is perfect in both directions. Thus (4) becomes

$$\Pi_i^{\min}(g) = \mu_i(g^{\min}) - c\mu_i^d(g). \tag{7}$$

Proposition 1 *If the decay matrix* δ^g *is given by* (6) *with* $\alpha = 0$ *and payoffs by* (7) :
(i) *The Nash and strict Nash profiles are those where all links are strong and all strong components are minimal.*
(ii) *The pairwise stable profiles are those minimally strongly connected.*

Proof. (i) As $\delta_{ij}^g = 1$ if and only if $g_{ij} = g_{ji} = 1$, and $\delta_{ij}^g = 0$ otherwise, in equilibrium only links supported by both players may exist. As flow is perfect within each strong component, no redundant link may exist in equilibrium. Thus, all strong components of g must be minimal. In these conditions no player has incentives to intend new links which would actually not form, nor to sever current ones because $c < 1$. Therefore such architectures of g are the only ones which are Nash equilibrium. Moreover, as $c < 1$, severing a link in any of them means a strict loss. Therefore, they are also strict Nash profiles. Note that the empty network trivially meets the conditions for Nash and strict Nash stability.

(ii) Now assume that there is more than one minimal strong component in an equilibrium profile. Then for any two players in different strong components it is profitable to form a link. Thus assuming that bilateral agreements are feasible, only those profiles with a unique minimal strong component, i.e. those minimally strongly connected, are pairwise stable. ■

Thus, in equilibrium, for any two players either there is no path that connects them or there is a unique path formed by strong links, but note that a Nash network can be non-connected, given that a player cannot form an actual link unilaterally.

As to the case $\alpha = 1$ we have Bala and Goyal (2000) two-way flow model (without decay), where a link can be unilaterally formed by any player, and (5) becomes

$$\Pi_i^{\max}(g) = \mu_i(g^{\max}) - c\mu_i^d(g). \tag{8}$$

In this model pairwise stability does not refine Nash equilibrium because bilateral agreements add nothing in this context. As to noncooperative stability, we have Bala and Goyal's well-known result.

[13] As pointed out in footnote 10, unlike Bala and Goyal (2000), this case is not considered in Jackson and Wolinsky (1996), where a decay of $\delta < 1$ is always assumed.

Proposition 2 (*Bala and Goyal,* 2000) *If the decay matrix* δ^g *is given by* (6) *with* $\alpha = 1$ *and payoffs by* (8) :
(*i*) *The Nash profiles are those minimally weakly connected.*
(*ii*) *The strict Nash profiles are center-sponsored stars, where a player supports weak links with all other players.*

Let us consider now the intermediate situations between these two extreme cases represented by the benchmark models and see how the transition occurs. That is, assume that the decay matrix of the actual network is given by (6) with $0 < \alpha < 1$. Then the information received by player i is given by

$$I_i(g) = \sum_{j \in N(i;g)} \alpha^{\lambda(i,j;g)},$$

where $\lambda(\cdot, \cdot, g)$ is the discounting distance introduced in Section 2, and the payoff function (1) becomes

$$\Pi_i(g) = \sum_{j \in N(i;g)} \alpha^{\lambda(i,j;g)} - c\mu_i^d(g). \tag{9}$$

Example 2: Consider the strategy profile given by the 6-node graph in Example 1. Player 3 receives perfectly from player 2, and receives from players 1 and 4 with a decay

α, from player 5 with a decay α^2, and from player 6 with a decay α^3; and pays only for link 32. Thus player 3's payoff is $\Pi_3(g) = 1 + 2\alpha + \alpha^2 + \alpha^3 - c$. Similarly, $\Pi_4(g) = 3\alpha + 2\alpha^2 - 2c$, and $\Pi_6(g) = \alpha + \alpha^2 + 2\alpha^3 + \alpha^4$.

We thus have a model with two parameters, α and c, both ranging from 0 to 1. In what follows we study stability for different configurations of values (α, c) of these parameters within the open square $(0,1) \times (0,1)$ (see Figure 2).

4.1 Case $\mathbf{c \leq 1 - \alpha}$: only strong links

As the following lemma shows, in this region only strong links form in equilibrium.

Lemma 1 *If the decay matrix* δ^g *is given by* (6) *and payoffs by* (9), *with* $0 < \alpha < 1$ *and* $0 < c < 1 - \alpha$, *in equilibrium only strong links occur.*

Proof. Assume g is a Nash profile where $g_{ij} = 1$ and $g_{ji} = 0$. First note that then there is no path of strong links connecting i and j, otherwise ij would be superfluous for i. Therefore the contribution of i's value to j's payoff is α, while if j makes this link strong by paying c it would be $1 - c$. Thus, as $c < 1 - \alpha$, j's payoff would improve by doing this, and g would not be a Nash profile. Thus, in a Nash profile within this range of values of α and c all links are strong. ■

The following proposition provides a characterization of Nash, strict Nash and pairwise stable profiles within the half square $(0,1)\times(0,1)$ *below the line* $c = 1-\alpha$, and shows how the results for the J&W model (Proposition 1) extend smoothly, establishing the range of values of the parameters within which the profiles described in Proposition 1 remain Nash, strict Nash or pairwise stable within this region.

Proposition 3 *If the decay matrix* δ^g *is given by* (6) *and payoffs by* (9)*, with* $0 < \alpha < 1$ *and* $0 < c < 1 - \alpha$*, then:*
(*i*) *If* $c \geq \alpha$*, the Nash* (*strict Nash*) *profiles are those minimally strongly connected and those not minimally strongly connected where all links are strong, all strong components are minimal and the maximal size of a strong component is smaller than or equal to* (*strictly smaller than*) $\frac{c}{\alpha}$.
(*ii*) *If* $c < \alpha$*, the Nash profiles are all those minimally strongly connected, which are also strict Nash.*
(*iii*) *For the whole range of values, the pairwise stable profiles are those minimally strongly connected.*

Proof. (*i*) Assume g is a Nash profile. By Lemma 1, within this range of values of α and c all links are strong in equilibrium and, as no superfluous link would be supported in equilibrium, all strong components must be minimal. If g is minimally strongly connected no player has an incentive to intend or sever a link. Otherwise, let s (integer s.t. $1 \leq s < n$) be the size of a strong component of g, and i a node that does not belong to that component. By paying for a weak link with any node in that component i would receive $\alpha s - c$, and if $\alpha s - c > 0$, i.e. if $s > \frac{c}{\alpha}$ this would mean a strict improvement in i's payoff. Therefore, for g to be a Nash profile no strong component of g may be larger than $\frac{c}{\alpha}$. Reciprocally, if these conditions hold no node has a best response that improves its payoff. As to strict Nash stability, this condition must hold strictly.

(*ii*) If $c < \alpha$, as in (*i*) it is easy to conclude that in a Nash profile all links are strong and all strong components are minimal. But now, as $c < \alpha$, it is strictly profitable to initiate a weak link with an isolated player. Therefore, a Nash profile must have a single strong component which must be minimal. Reciprocally, in any minimally strongly connected profile no node has a best response that improves its payoff. Moreover, all these profiles are strict Nash as any unilateral change of strategy would cause a loss.

(*iii*) Once bilateral agreements are feasible, a profile which is not strongly connected cannot be pairwise stable since for any two players in different strong components of a Nash network it would be profitable to form a strong link. Thus, whatever the values of c and α within the range considered, only minimally strongly connected profiles remain pairwise stable. ■

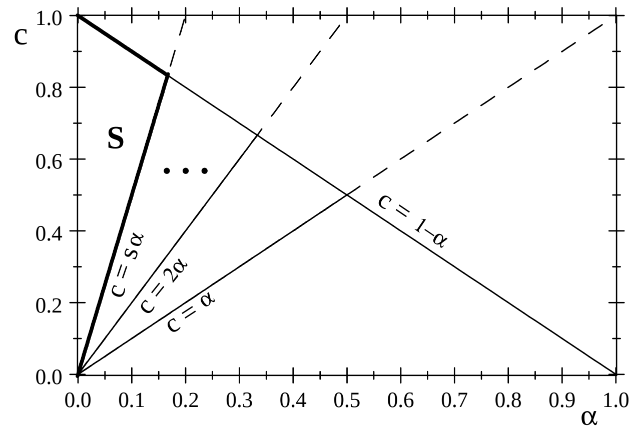


Figure 2: Stability for $c < 1 - \alpha$

Remarks:

(i) Figure 2 illustrates the situation described by Proposition 3. The left-hand side of the rectangle, i.e. $\alpha = 0$, corresponds to the J&W connections model without decay, where Nash and strict Nash profiles are those where all links are strong and all strong components of g are minimal, and those minimally strongly connected are pairwise stable (Proposition 1). Proposition 3 *characterizes Nash, strict Nash and pairwise stable profiles within the triangle below the straight line* $c = 1 - \alpha$. Within this triangle, where $c < 1 - \alpha$, as one moves rightwards from the side where $\alpha = 0$, *all* the structures characterized in Proposition 1-(i) as Nash and strict Nash when $\alpha = 0$ remain strict Nash as far as $n - 1 < \frac{c}{\alpha}$, while at $n - 1 = \frac{c}{\alpha}$ the only isolated individual in a profile where the rest of the players form a minimal strong component is indifferent as regards paying for a weak link with any individual, but when $n-1 > \frac{c}{\alpha}$ this player has an incentive to pay. In this way, as $\frac{c}{\alpha}$ decreases, smaller maximal sizes of a strong component suffice to make it profitable for any player that does not belong to that component to pay for a weak link with any player belonging to it. Thus, in region S, where $c > s\alpha$, all minimally strongly connected profiles and those profiles described in Proposition 3-(i) where the size of the largest strong component is smaller than or equal to (strictly smaller than) s are Nash (strict Nash) stable. When $\frac{c}{\alpha} > 1$ but its value is very close to 1, apart from minimally strongly connected profiles only the empty network, where all strong components are singletons, remains strict Nash. Beyond this point, i.e. when $c < \alpha$ and $c < 1 - \alpha$, the only Nash and the only strict Nash stable profiles are those minimally strongly connected.

(ii) Only minimally strongly connected profiles are pairwise stable. But in view of Proposition 3-(ii), within the range of values considered, below the line $c = \alpha$ pairwise stability adds nothing to (i.e. does not refine) Nash stability, given that in this case bilateral coordination is irrelevant because it does not really offer any new chances to the players.

As to strategy profiles on the line $c = 1 - \alpha$, we have the following conclusion relative to the Nash profiles emerging from propositions 1 and 3:

Proposition 4 *If the decay matrix* δ^g *is given by* (6) *and payoffs by* (9)*, with* $0 < \alpha < 1$ *and* $c = 1 - \alpha$: (i) *The profiles characterized as strict Nash for certain ranges of values of* $\frac{c}{\alpha}$ *in Proposition* 3 *are Nash but not strict Nash for the same ranges of values of* $\frac{c}{\alpha}$*, with the sole exception of the empty network, which remains strict Nash if* $c > \alpha$. (ii) *Among them, only minimally strongly connected profiles are pairwise stable.*

Proof. (i) The profiles characterized as strict Nash in Proposition 3, consist of one or more minimal strong components and all their links are strong. When $c = 1 - \alpha$ such profiles continue to be Nash for the same range of values of $\frac{c}{\alpha}$, but now, as $c = 1 - \alpha$, a node supporting a strong link with a peripheral node is indifferent between supporting it or not. This leaves only the empty network profile as possible strict Nash. In this case, if $c = \alpha$ a node is indifferent between supporting a weak link with any other or not, while if $c < \alpha$ for any node it is profitable to initiate a weak link with an isolated player. Finally, if $c > \alpha$ for a node initiating a weak link with an isolated player means a loss. Thus in this case the empty network is strict Nash.

(ii) From (i) a minimally strongly connected profile is a Nash network and obviously no two players have incentives to form a new strong link. If the profile consists of more than one strong component and all links are strong, any two players in different components have incentives to create a strong link. ■

As shown below, there are some other stable profiles when $c = 1 - \alpha$.

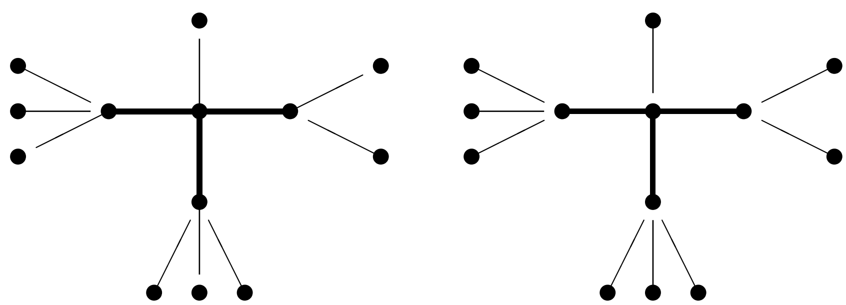
4.2 Case $\mathbf{c \geq 1 - \alpha}$: peripheral players weakly-linked

We now address the question of stability within the half square *above* the line $c = 1 - \alpha$. We first show that above this line, that is, when $c > 1 - \alpha$, *in equilibrium peripheral players are involved only in weak links.* Moreover, in this region, if $c > \alpha$, in equilibrium peripheral players *must support* their weak links.

Lemma 2 *If the decay matrix* δ^g *is given by* (6) *and payoffs by* (9)*, with* $c > 1 - \alpha$*, and* g *is a Nash profile, then:* (i) *All peripheral players are connected by a weak link.* (ii) *If in addition* $c > \alpha$*, then peripheral players support their weak links.*

Proof. Assume that g is a Nash profile and $\#N^o(i; g) = 1$, i.e. there is a unique node j s.t. $g_{ij} = 1$ or $g_{ji} = 1$. (i) If $g_{ij} = 1$ and $g_{ji} = 1$, then, if $c > 1 - \alpha$, $\Pi_j(g - ji) - \Pi_j(g) = \alpha - (1 - c) > 0$, which contradicts the fact that g is a Nash profile. Therefore, necessarily $g_{ij} = 0$ or $g_{ji} = 0$. (ii) Now assume $c > \alpha$. If $g_{ij} = 0$, then $g_{ji} = 1$, but then $\Pi_j(g - ji) - \Pi_j(g) = c - \alpha > 0$, which contradicts the fact that g is a Nash profile. Thus, necessarily $g_{ij} = 1$ and $g_{ji} = 0$. ■

Therefore *none* of the profiles characterized as Nash profiles in Proposition 3, where peripheral players are involved in strong links, continues to be Nash above the line $c = 1 - \alpha$. This raises the question of whether the profiles that result if all strong links connecting peripheral players in a profile which is strict Nash below $c = 1 - \alpha$ (as characterized by Proposition 3) are replaced by weak links (supported by the peripheral players if $c > \alpha$) are stable above the line $c = 1 - \alpha$. Note that such architectures can be precisely described in core-periphery terms as comprising one or more weak components, each consisting of a minimally strongly connected part (i.e. a *tree* of strong links), and a set of peripheral players, each of them connected by weak links with nodes in that "core"[14] (see Figure 3).



(a) (b)

Figure 3: Tree-core-periphery profiles

Formally we have the following.

Definition 3 *A profile g is said to be "tree-core-periphery" if N is partitioned into two nonempty sets, $N = P \cup Q$ with $P \cap Q = \varnothing$, such that (i) $g\mid_Q$ is a strong component of g which is minimal, (ii) each node in P is peripheral and is connected by a weak link with a node in Q, and (iii) P contains all peripheral nodes in g. We refer to Q as the "core" of g and to P as its "periphery".*

A comparison with the usual features of a core-periphery structure mentioned in footnote 14 is pertinent here. In a tree-core-periphery profile the core is a tree of strong links, i.e. (strong) connectedness is *minimal*, unlike the usual high connectedness in the core, but, given the perfect transmission through strong links, the *communication is perfect between nodes in the core*, making further links between them unnecessary. As to peripheral nodes, one single weak link connects each of them to the core and there is no interconnection between them. Thus, the core-periphery term seems adequate here.

[14] A variety of formal definitions of core-periphery graphs can be found in the literature, but the basic ingredients are a *core* consisting of a set of nodes which are highly (or completely) interconnected, and a *periphery* consisting of a set of nodes which are hardly (if at all) interconnected with one another but are connected with some (one/at least one/all) nodes in the core.

A tree-core-periphery profile where all peripheral players support their weak links with the core is said to be *periphery-sponsored* (see Figure 3-(b)). When the core is trivial, i.e. Q is a singleton, the structure is a star of weak links.

The following three lemmas help characterize stable profiles *with no cycles* within the region considered. Lemma 3 shows that in the region where $c > 1-\alpha$ the existence of peripheral players in Nash profiles with no cycles implies weak connectedness, which explains why we only consider weakly connected structures of this type. Lemma 4 is instrumental in proving Lemma 5, which establishes that a Nash profile with no cycles contains at most one non-trivial strong component.

Lemma 3 *Let the decay matrix δ^g be given by* (6) *and payoffs by* (9)*. Assume $c > 1-\alpha$, and let g be a Nash profile; then either no player is peripheral in g, or g is weakly connected.*

Proof. Let g be a Nash profile. Assume that there is a peripheral player i in g, and $N^o(i;g) = \{j\}$. In view of Lemma 2, if $c > \alpha$, it must hold that $g_{ij} = 1$ and $g_{ji} = 0$. If g is not weakly connected, take any node k in another weak component of g. Then $\Pi_k(g+kj) = \Pi_k(g)+\Pi_i(g)+\alpha^2 > \Pi_k(g)$, that is, k can improve its payoff by initiating a weak link with j, which contradicts the fact that g is a Nash profile. Assume now that $c \leq \alpha$. In this case it may be $g_{ij} = 1$ or $g_{ji} = 1$. If $g_{ij} = 1$, proceed as before, and if $g_{ji} = 1$ then k can also improve its payoff by initiating a weak link with i or j. ■

The next two lemmas establish some facts relative to Nash profiles ($\neq g^e$) with *no cycles*. Observe that such a profile must consist of a weakly connected (Lemma 3) tree-structure, in general formed by weak and strong links.

Lemma 4 *Let the decay matrix δ^g be given by* (6) *and payoffs by* (9)*, with $c > 1-\alpha$. If g is a Nash profile with no cycles, then a node connected by a weak link with a non-trivial strong component must support it.*

Proof. Let i be a node connected by a weak link with a node j in a non-trivial strong component of a Nash profile with no cycles. Assume that the weak link between i and j is supported by j. Node j must support at least one strong link $\overline{jj'}$. Given the tree-structure of g, node j does not have any incentive to withdraw its support from that strong link if

$$\alpha(I_{j'}(g-\overline{jj'})+1) \leq I_{j'}(g-\overline{jj'})+1-c \quad \text{i.e.} \quad c \leq (I_{j'}(g-\overline{jj'})+1)(1-\alpha).$$

But then it is worthwhile for i to double its link with j. To see this, note that a necessary condition for i not to have an incentive to double its link with j is

$$\alpha(I_{j'}(g-\overline{jj'})+2) \geq I_{j'}(g-\overline{jj'})+2-c \quad \text{i.e.} \quad c \geq (I_{j'}(g-\overline{jj'})+2)(1-\alpha).$$

Therefore the weak link between i and j must be supported by i. ■

The following Lemma proves that in equilibrium acyclic profiles cannot have more than one non-trivial strong component, and that when a non-trivial strong component does exist players in that component, and they alone, receive the maximal amount of information.

Lemma 5 *Let the decay matrix δ^g be given by (6) and payoffs by (9), with $c > 1 - \alpha$. If $g \neq g^e$ is a Nash profile with no cycles, then: (i) g is a weakly connected profile with at most one non-trivial strong component. (ii) If g has a unique non-trivial strong component it contains the nodes that receive the maximal amount of information in g.*

Proof. (i) A Nash profile $g \neq g^e$ with no cycles must necessarily contain peripheral players. Then, by Lemma 3, g must be weakly connected. Therefore either g is minimally weakly connected or there is at least one strong link, in other words at least one strong component with more than one node. We now prove that in this case g has *only one non-trivial strong component.* Assume that there are two strong components C and C' with more than one node. By Lemma 4, a weak link that connects a node with a non-trivial strong component must be supported by that node. This implies that in equilibrium the path connecting the two non-trivial strong components, C and C', must contain at least two weak links, say kk' and ll', with $k' \in C$ and $l' \in C'$. Now let i' (j') be a/the node in C (C') furthest away from k' (l') in C (C'). Thus i' (j') is a peripheral player in $g\mid_C$ $(g\mid_{C'})$, which, by Lemma 2, cannot be peripheral in g. That is, two nodes, i and j, must exist which *support* (by Lemma 4) *weak* links ii' and jj'. Let $\lambda(i,j;g)$ be the discounting distance between i and j (note that $\lambda(i,j;g) \geq 4$). Let t_i (t_j) denote the weak component containing i (j) in the graph resulting by deleting link ii' (jj') in g. And let $I_i^{ii'}(g - jj')$ $(I_j^{jj'}(g - ii'))$ denote the information received by i (j) in the graph that results from g by deleting jj' (ii') through link ii' (jj'), that is, the information that each of them receives from the part of the network between them via their weak link with the corresponding strong component. Assume w.l.o.g. $I_i^{ii'}(g - jj') \leq I_j^{jj'}(g - ii')$. Then we have

$$I_i(g) = I_i(t_i) + I_i^{ii'}(g - jj') + \alpha^{\lambda(i,j;g)}(I_j(t_j) + 1),$$

while if i replaces its link with i' by a link with j', i will receive

$$I_i(g - ii' + ij') = I_i(t_i) + I_j^{jj'}(g - ii') + \alpha^2(I_j(t_j) + 1).$$

But then, as $\lambda(i,j;g) > 2$, we have

$$I_i(g - ii' + ij') - I_i(g) = I_j^{jj'}(g - ii') + \alpha^2(I_j(t_j) + 1) - I_i^{ii'}(g - jj') - \alpha^{\lambda(i,j;g)}(I_j(t_j) + 1)$$

$$= I_j^{jj'}(g - ii') - I_i^{ii'}(g - jj') + (\alpha^2 - \alpha^{\lambda(i,j;g)})(I_j(t_j) + 1) > 0.$$

That is, i can improve its payoff by replacing link ii' by ij', which contradicts the fact that g is a Nash profile. Thus, a Nash profile has at most one non-trivial strong component.

(ii) Let g be a Nash profile with no cycles and a unique non-trivial strong component $g\mid_C$. Evidently, all nodes in C receive the same amount of information. Assume that $i_0 \notin C$ receives the maximal amount of information in g. As g has no cycles, there must be at least two peripheral nodes, say i and j, in $g\mid_C$. By Lemma 2-(i), every peripheral player in g must be connected by a weak link, so both i and j must be connected by a weak link with another node each, say with i' and j' respectively. By Lemma 3, such links must be supported by i' and j'. Then either link $i'i$ is critical for i_0 to receive i' or $j'j$ is critical for i_0 to receive j' (or both). Assume that $i'i$ is critical for i_0 to receive i'. Then i' has an incentive to replace its weak link with i by a weak link with i_0, which contradicts the fact of g being a Nash profile. Thus, the nodes that receive the maximal amount of information in g are those in the non-trivial strong component $g\mid_C$ and those alone. ■

The following result, based on the preceding lemmas, shows that when $c > 1 - \alpha$ *acyclic* Nash profiles must be minimally weakly connected or tree-core-periphery profiles.

Proposition 5 *Let the decay matrix δ^g be given by* (6) *and payoffs by* (9)*, with $c > 1 - \alpha$. If $g(\neq g^e)$ is a Nash profile with no cycles, then:* (i) *If $c > \alpha$ then g is necessarily a periphery-sponsored tree-core-periphery profile.* (ii) *If $c \leq \alpha$ then g is either a minimally weakly connected profile or a periphery-sponsored tree-core-periphery profile.*

Proof. Let $g \neq g^e$ be a Nash profile with no cycles. Then, by Lemma 5, g must be a weakly connected profile with at most one non-trivial strong component.

(i) Consider first the case $c > \alpha$. In this case, by Lemma 2-(ii), in equilibrium peripheral players support their weak links. Assume there are two peripheral players, i and j, at discounting distance $\lambda(i, j; g) > 2$. Assume w.l.o.g. that $I_i(g) \leq I_j(g)$, then it is advantageous for i to replace its weak link by a link with the same player linked by j, which contradicts the fact that g is a Nash profile. Therefore the discounting distance between any two peripheral players must be exactly 2, which entails the following. If g does not contain a non-trivial strong component, then g is a periphery-sponsored star. If g does contain a (necessarily unique by Lemma 5-(i)) non-trivial strong component, that component must form a tree of strong links where, by Lemma 2-(i), none of its nodes is peripheral in g, and this together with distance 2 between peripheral players implies that g must be a periphery-sponsored tree-core-periphery profile. In both cases the consequence is that g must be a periphery-sponsored tree-core-periphery profile.

(ii) Consider now the case $c \leq \alpha$. As g is a weakly connected profile without cycles, if it has no non-trivial strong component then g must be *minimally* weakly connected. Assume now that g has a (unique by Lemma 5-(i)) non-trivial strong component consisting of a tree of strong links. Then any node that does not belong to that strong component must be connected with it by a weak link or a path of weak links. We prove first that g must then be a tree-core-periphery profile. Assume node i is at discounting distance 2 from the non-trivial strong component. Then

$\lambda(i,k;g) = 2$ where k is the node in the non-trivial strong component closest (w.r.t. ordinary distance) to i. Thus, there is a node j connected with i and k by weak links. By Lemma 4, the weak link between j and k must be supported by j. Now it can be seen that the weak link connecting i and j must also be supported by j: By Lemma 5-(ii), it is the nodes in the non-trivial strong component of g that receive the maximal amount of information. Thus, if the weak link connecting i and j were supported by i, then i would strictly improve its payoff by replacing its weak link with j by a link with k. Therefore j supports both weak links, with i and k, and k is involved in at least one strong link, say with k', since k belongs to the strong component. Now we see that either k withdraws its support from its strong link with k' or i doubles its link with j. For k not to be interested in withdrawing its support from its strong link with k' the information received by k through link $\overline{kk'}$ in g (i.e. $1+I_{k'}(g-\overline{kk'})$) must be such that

$$1 + I_{k'}(g - \overline{kk'}) - c \geq \alpha(1 + I_{k'}(g - \overline{kk'})).$$

That is, $c \leq M(1-\alpha)$, where $M = 1 + I_{k'}(g - \overline{kk'})$. On the other hand, for i not to be interested in doubling its link with j $c \geq (1+\alpha+\alpha M)(1-\alpha)$ is a necessary condition. Therefore for g to be a Nash profile it must hold that

$$(1+\alpha+\alpha M)(1-\alpha) \leq c \leq M(1-\alpha).$$

There is room for c if $(1+\alpha+\alpha M) \leq M$, that is if $M \geq \frac{1+\alpha}{1-\alpha}$. But since $0 \leq c \leq 1$, the lower bound for c must be smaller than 1. That is, $(1+\alpha+\alpha M)(1-\alpha) \leq 1$, which entails $M \leq \frac{\alpha}{1-\alpha}$. This leads to the impossible requirement

$$\frac{1+\alpha}{1-\alpha} \leq M \leq \frac{\alpha}{1-\alpha}.$$

Therefore the discounting distance from any node not in the unique non-trivial strong component to it must be exactly 1. Thus, g is a tree-core-periphery profile. But then, by Lemma 4, peripheral players must support their links. In other words, g must be a periphery-sponsored tree-core-periphery profile. ■

The following proposition shows that each *periphery-sponsored* tree-core-periphery architecture *is* actually a Nash network for a range of values of c and α within the region considered, and it establishes that range. Note that in view of Proposition 5 such architectures are the only ones without cycles which are Nash stable for $c > \alpha$.

Proposition 6 *Let the decay matrix* δ^g *be given by* (6) *and payoffs by* (9), *with* $c > 1-\alpha$. *Then a periphery-sponsored tree-core-periphery profile* g, *with core* Q *and periphery* $P = N \backslash Q$ *is:*
(i) *A Nash profile if and only if*

$$c \leq q\alpha + (n-q-1)\alpha^2, \tag{10}$$

where $p = \#P$ *and* $q = \#Q$, *and*

$$c \leq (1 + p_{\min}\alpha)(1 - \alpha), \tag{11}$$

where $p_{\min}$ *is the number of peripheral nodes supporting a weak link with the same peripheral node of the tree* $g\,|_Q$ *(which forms the core) for which this number is minimal.*
(ii) *A strict Nash profile if and only if it is a periphery-sponsored star and*

$$c < \alpha(1 + (n - 2)\alpha).$$

(iii) *A pairwise stable profile if and only if it is a Nash profile, i.e.* (10) *and* (11) *hold, and*

$$c \geq 1 - \alpha^2. \tag{12}$$

Proof. (i) *Necessity* $(\Rightarrow)$. Let g be a periphery-sponsored tree-core-periphery profile, with core Q and periphery $P = N \backslash Q$, which is a Nash profile. The payoff of a peripheral node in g is $\alpha(q + (p - 1)\alpha) - c$, and this number must be nonnegative, otherwise a peripheral node has an incentive to withdraw support from its weak link with a node in the core. That is, it must hold that

$$\alpha(q + (p - 1)\alpha) \geq c.$$

Then, as $p = n - q$, this condition can be rewritten as (10) and means that no peripheral node has incentive to sever its weak link with a node in the minimally strongly connected subgraph $g\,|_Q$ formed by the nodes in the core Q. As to strong links connecting nodes in the core, the difference between the payoff of a node $i \in Q$ and its payoff after withdrawing its support from a strong link with $j \in Q$ should be nonnegative. That is, it must hold that

$$q_{ij} + p_{ij}\alpha - c - \alpha(q_{ij} + p_{ij}\alpha) \geq 0,$$

where p_{ij} and q_{ij} are the number of peripheral and non-peripheral nodes connected with i through link $\overline{ij}$ in g. Thus, if

$$c \leq (q_{ij} + p_{ij}\alpha)(1 - \alpha) \tag{13}$$

for all $i, j \in Q$ s.t. $g_{ij} = 1$, no node in the core has incentive to withdraw support from any of its strong links. But note that the nodes for which this condition is most demanding are those that support a strong link with a peripheral node *of the tree* $g\,|_Q$, and for any such node $q_{ij} = 1$. Therefore, the node/s for which this condition is most demanding is/are the node/s in the core that support a strong link with a peripheral node of the tree $g\,|_Q$ with which a minimal number of peripheral nodes are supporting weak links. If this number[15] is $p_{\min}$, condition (13) becomes (11). Therefore conditions (10) and (11) are necessary for the architecture described to be that of a Nash profile.

[15] For instance, in the example represented in Figure 2-(b), this number is 2.

Sufficiency ($\Leftarrow$). On the other hand, when both conditions hold, given the perfect flow within the core, no node can improve its payoff by supporting a different set of links. Note first that a peripheral node of the tree $g\,|_Q$ has no incentive to double any link with a peripheral player because $c > 1 - \alpha$. Note also that, as $c > 1 - \alpha$ entails $c > \alpha - \alpha^2$, a peripheral node has no incentive to support a weak link with another peripheral node.

(*ii*) As to strict stability, note that in a periphery-sponsored tree-core-periphery profile all peripheral nodes have the same payoff whatever the node in the core with which each of them chooses to link. This means that they are *not* strict Nash with one exception: when $\#Q = 1$. In this case, if inequality (10) holds strictly it becomes $c < \alpha + (n-2)\alpha^2$, while (11) becomes vacuous. Therefore the periphery-sponsored star (i.e. all peripheral players supporting a link with a single player) is the only architecture of this type that is strict Nash.

(*iii*) It is easy to check that no pair of players in the core of a tree-core-periphery profile has an incentive to establish any new strong link. As to peripheral players, no pair of them would gain by establishing a new strong link if $c \geq 1 - \alpha^2$. ■

Remarks:

(i) Inequality (10) just requires *the core to be big enough* to make it worth for peripheral players to pay for their weak links. Note that for a fixed n, the right hand side of (10) depends exclusively on q, the cardinal of the core, and *the greater the number of players in the core is, the less demanding condition* (10) *is.* More precisely, the boundary of the region where condition (10) holds is an increasing convex function of α (i.e. $c_q(\alpha) = q\alpha + (n-q-1)\alpha^2$) in the interval $\alpha \in [0,1]$, such that at $\alpha = 0$ its value is 0, and at $\alpha = 1$ its value is $n-1$, and the greater q is, the higher is the upper bound that this condition sets for c.

Condition (11) requires the cost of a link to be low enough to make it worth supporting the less rewarding strong link with a peripheral player of the core $g\,|_Q$. This condition can be rewritten like this:

$$c \leq 1 + (p_{\min} - 1)\alpha - p_{\min}\alpha^2.$$

which shows that the boundary of this region is a concave function[16] of α (i.e. $c_{p_{\min}}(\alpha) = 1 + (p_{\min} - 1)\alpha - p_{\min}\alpha^2$) in the interval $\alpha \in [0,1]$, which at $\alpha = 0$ its value is 1 and at $\alpha = 1$ its value is 0, and *the greater $p_{\min}$ the weaker condition* (11) *is.* It is then easy to check that *the region where both conditions hold is never empty.* In other words, *each periphery-sponsored tree-core-periphery profile is Nash for a certain range of values of the parameters.* Figure 4 shows the region S where both conditions hold for $n = 20$, $q = 8$ and $p_{\min} = 2$.

(ii) The maximal value of q is $n-2$, which corresponds to a core consisting of $n-2$ nodes forming a line of strongly-linked nodes, and two peripheral players linking one of

[16] In fact, it is a parabola which intersects the α-axis at $\alpha = 1$ and $\alpha = -1/p_{\min}$, and whose axis is $\alpha = (p_{\min} - 1)/2p_{\min}$.

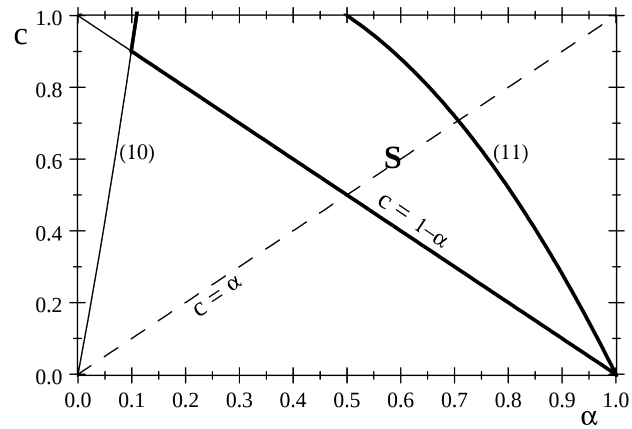


Figure 4: Tree-core-periphery Nash stability ($n = 20, q = 8, p_{\min} = 2$)

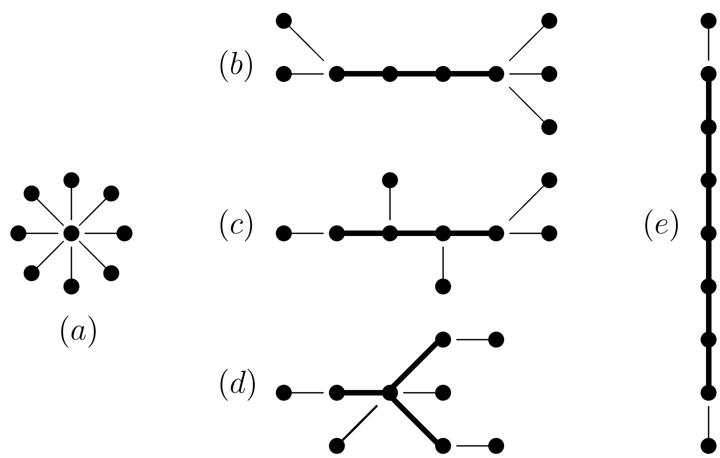


Figure 5: Periphery-sponsored tree-core-periphery profiles

the two extremes each (see Figure 5 (e)). In this case $p_{\min} = 1$ is minimal, so condition (11) is the strongest that it can be, while condition (10) is the weakest that it can be. Figure 6 shows the region S where this structure is Nash stable for $n = 20$ (thus, $q = 18$, $p_{\min} = 1$).

(iii) For a given value of q, condition (11) depends on the particular arrangement of the tree-core and of the $n-q$ weak links supported by the peripheral players. The most stable architecture for a given q (i.e. the one for which condition (11) is least demanding so that it is stable for a wider range of values of the parameters) corresponds to a core arranged as a line, all peripheral nodes supporting weak links with the extremes, and a minimal difference (0 or 1) between the number of peripheral nodes connected with each extreme, so as to make condition (11) as weak as possible. In other words, the smaller the number of peripheral nodes *in the core*, i.e. in $g\mid_Q$, and the more egalitarian

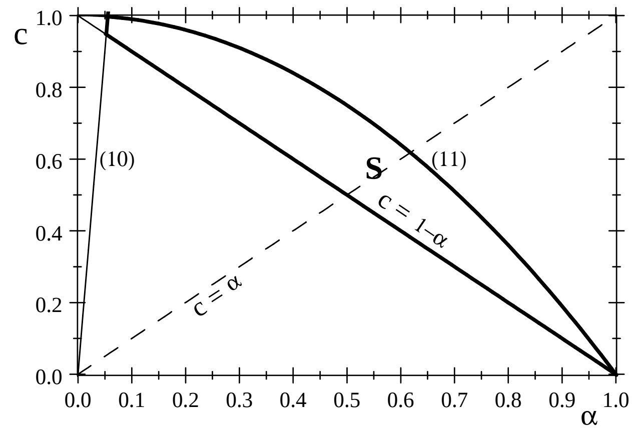


Figure 6: Tree-core-periphery Nash stability ($n = 20, q = 18, p_{\min} = 1$)

the distribution of peripheral nodes connected (exclusively) to peripheral nodes in the core is, the wider the region of stability is. Graphs (b), (c) and (d) in Figure 5 represent three different periphery-sponsored tree-core-periphery profiles for $n = 9$ and $q = 4$, of which (b) has the highest stability ($p_{\min} = 2$), and (c) and (d) the lowest since in both $p_{\min} = 1$.

(iv) The minimal value of q is 1, which corresponds to the center of a periphery-sponsored star, so condition (10) is the strongest that it can be, while condition (11) vanishes (see Figure 5 (a)). Figure 7 shows the region S where the periphery-sponsored star is Nash[17] and strict Nash for $n = 20$.

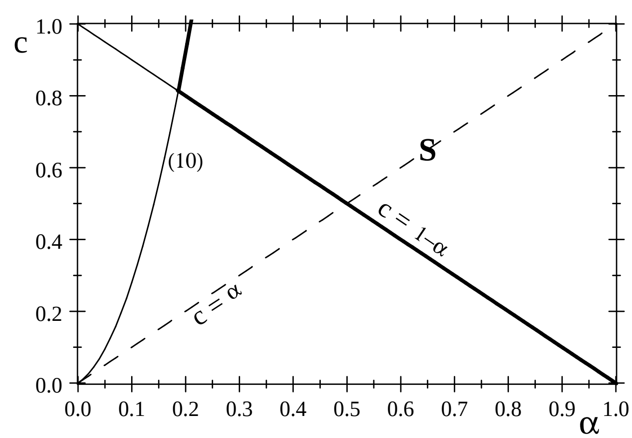


Figure 7: Periphery-sponsored star Nash stability ($n = 20, q = 1, p_{\min} = 19$)

[17] This is the only stable structure in Hojman and Szeidl's (2008) model, where benefits from connections exhibit decreasing returns and decay with network distance.

(v) In view of Proposition 6-(iii), pairwise stability sets a lower bound (condition (12)) in addition to the two upper bounds required for Nash equilibrium. Figure 8 shows the region S where the tree-core-periphery profile for $n = 20$, $q = 10$ and $p_{\min} = 3$ is pairwise stable.

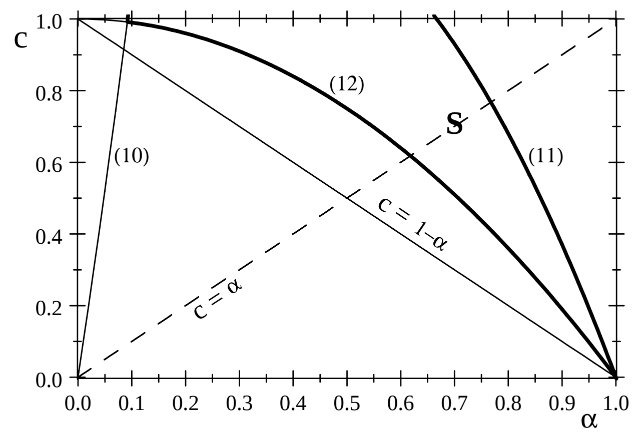


Figure 8: Tree-core-periphery pairwise stability ($n = 20, q = 10, p_{\min} = 3$)

As seen above, the peripheral nodes in a tree-core-periphery profile with a non-trivial core must support their links in equilibrium. Given that in the region where $c > 1-\alpha$ and in addition $c < \alpha$ it is worth initiating a single link with an isolated player, it is reasonable to wonder whether a non-periphery-sponsored tree-core-periphery structure with trivial core, that is, a *non*-periphery-sponsored star of weak links (i.e. a star of weak links where the center supports at least one link) can be stable. The following proposition shows the conditions under which this is so.

Proposition 7 *Let the decay matrix* δ^g *be given by* (6) *and payoffs by* (9), *with* $c > 1 - \alpha$ *and* $c < \alpha$, *and let* g *be a non-periphery-sponsored star of weak links, then:*
(i) *g is a Nash profile if and only if*

$$c \geq 1 + (n-3)\alpha - (n-2)\alpha^2. \tag{14}$$

(ii) *g is a strict Nash profile if and only if* (14) *holds strictly.*
(iii) *g is a pairwise stable profile if and only if* (14) *holds.*

Proof. (i) Assume that g is a Nash non-periphery-sponsored star of weak links. As $c < \alpha$, no player supporting a weak link has an incentive to sever it. On the other hand, the greatest incentive to double a weak link is for a peripheral player. For a peripheral player not to have an incentive to double its link it is necessary for the following to hold:

$$(1 + (n-2)\alpha)\alpha \geq 1 + (n-2)\alpha - c.$$

Which sets a lower bound for c:

$$c \geq (1-\alpha)(1+(n-2)\alpha) = 1+(n-3)\alpha-(n-2)\alpha^2.$$

(ii) It is straightforward.

(iii) Note that (14) implies that $c \geq 1-\alpha^2$. Therefore no pair of peripheral nodes has incentives to establish a new strong link. ∎

Remark: Therefore, when the core consists of more than one node the only Nash stable tree-core-periphery profiles are those which are periphery-sponsored, while the only structure of this type where some peripheral nodes are core-sponsored are stars, including center-sponsored stars. But condition (14) confines the stability of these stars to a region close to the side of the square where $\alpha = 1$ (i.e. the B&G two-way flow model, where the only strict Nash profiles are center-sponsored stars), which narrows as the number of players increases. For instance, for $n = 20$ the limiting curve is $c = 1 + 17\alpha - 18\alpha^2$, which is represented in region S of Figure 9[18]. In general, the limiting curve of (14) intersects $c = 1$ at $\alpha = \frac{n-3}{n-2}$. In other words, the only Nash stable tree-core-periphery profiles where some peripheral nodes are core-sponsored are stars and they occur in a narrow region close to $\alpha = 1$.

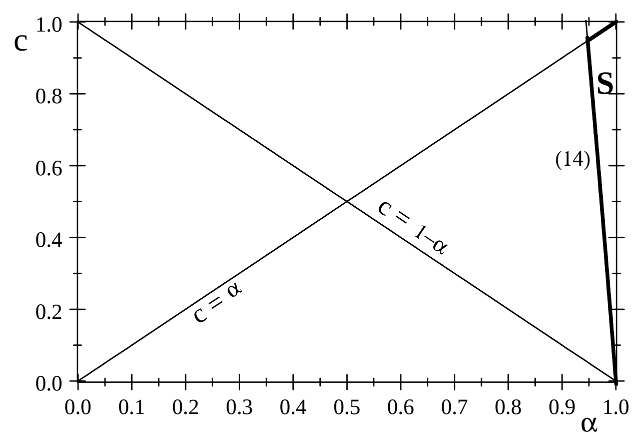


Figure 9: Non-periphery-sponsored star stability $(n = 20, q = 1, p_{\min} = 19)$

Thus stars, which can be seen as degenerated core-periphery structures with a singleton as their core, are Nash-stable for certain ranges of the parameters. Are there other Nash profiles which are *minimally* weakly connected? An immediate corollary of Proposition 5 gives the answer for $c > \alpha$.

Corollary 1 *Let the decay matrix δ^g be given by* (6) *and payoffs by* (9) *, with $c > 1-\alpha$. If $c > \alpha$ the only minimally weakly connected Nash profiles are periphery-sponsored stars.*

[18] Compare with the region where the periphery-sponsored star is Nash, represented in Figure 7.

The region where $c > 1 - \alpha$ and $c < \alpha$ remains to be explored. The following result specifies the structure of minimally weakly connected Nash profiles required, but some notation and terminology are necessary. A minimally weakly connected graph is a tree of weak links. Let i_0 be a reference node chosen as the root of that tree, so that for any other node $i \neq i_0$, there is a unique path from i to i_0, and we denote by *subtree* t_i the weak component which contains i in the graph that results from deleting in g the link containing i in this path.

Proposition 8 *Let the decay matrix δ^g be given by* (6) *and payoffs by* (9)*, with $c > 1 - \alpha$. A minimally weakly connected Nash profile g necessarily has the following structure: a node i_0 that receives most information in g is the root of a tree of weak links such that:* (*i*) *for all $i \in N^o(i_0, g)$, t_i is i-oriented;* (*ii*) *for all $ij \in g$, j is a player in the weak component of $g - ij$ containing j that receives most information from that component.*

Proof. Let g be a Nash profile that is minimally weakly connected and let i_0 be a node that receives the maximal information in g and choose it as the root of g.

(*i*) Let $i \in N^o(i_0, g)$. Consider the tree t_i and assume that it is not i-oriented, i.e. a node j in this tree supports a link with a node k closer to i (and consequently to i_0). From the choice of i_0, we have $I_{i_0}(g) \geq I_k(g)$, then we show that then j can improve its payoff by replacing jk by ji_0, that is, $I_j(g - jk + ji_0) > I_j(g)$. Note that if $d(i_0, j; g) = r$ $(r \geq 2)$, and $g - t_j$ denotes the tree that results from cutting subtree t_j off g, we have the following:

$$I_{i_0}(g) = \alpha^r (I_j(t_j) + 1) + I_{i_0}(g - t_j),$$
$$I_k(g) = \alpha (I_j(t_j) + 1) + I_k(g - t_j).$$

As $I_{i_0}(g) \geq I_k(g)$ and $\alpha^r (I_j(t_j) + 1) < \alpha (I_j(t_j) + 1)$, it follows from these two equalities, that $I_{i_0}(g - t_j) > I_k(g - t_j)$, and consequently

$$I_j(g - jk + ji_0) - I_j(g) = \alpha (I_{i_0}(g - t_j) - I_k(g - t_j)) > 0,$$

that is $I_j(g - jk + ji_0) > I_j(g)$. Therefore the link connecting j and k must be supported by k. Thus, subtree t_i is i-oriented.

(*ii*) Assume $ij \in g$. If j is not a player that receives most information from the weak component of $g - ij$ containing j, then i can improve its payoff by replacing the link with j by a link with any player in that component who receives more information than j. ■

Note that Proposition 8 establishes *necessary* conditions for a minimally weakly connected profile to be Nash. Figure 10 shows an architecture that meets them. Note that stars are a particular case of such architectures.

Remarks:

(i) Observe that condition (*ii*) in Proposition 8 is very demanding, as it is not true in general for structures that meet condition (*i*). In particular, this condition entails

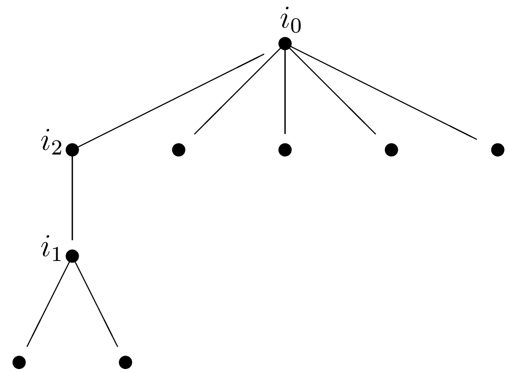


Figure 10: A minimally weakly connected Nash profile

that the i_0-rooted tree embodies and expresses a hierarchy of levels of information: the best informed player is at the root, and as one moves away from it by any path the level of information received by players decreases.

(ii) *Not* every structure of the type described by Proposition 8 is a Nash profile for some range of values of the parameters. Additional conditions are required for them to be Nash profiles: no player must have incentive to double any link and/or to initiate new ones. Note that both conditions amount to setting *lower* bounds for c. The first condition is of the form $c > K(\alpha)(1-\alpha)$ (where $K(\alpha)$ is a polynomial with nonnegative coefficients, consequently increasing with α), which requires a delicate trade-off when applied to the terminal nodes of any structure of this type, as the closer α to 1 one gets, the smaller $1-\alpha$ is, but the greater $K(\alpha)$ is. Thus a high value of $c/(1-\alpha)$ is required. In fact, as was the case with non-periphery-sponsored stars, these conditions confine the stability of minimally weakly connected Nash profiles to a region close to the side of the square where $\alpha = 1$.

Example 3: Let g be the profile in Figure 10, where node i_0 supports links with 4 players, and node i_1 supports links with 2 players, and node i_2 supports links with i_0 and i_1. It is easy to check that i_0 is the player that receives most information in g and this profile fits conditions (i) and (ii) in Proposition 8. The player with the greatest incentive to double a link is any of those with whom i_0 supports a link. For this not to be so the following must hold:

$$c \geq (1-\alpha)\left(1+4\alpha+\alpha^2+2\alpha^3\right).$$

The player with the greatest incentive to initiate a new link (with i_0 in fact) is any of those with whom i_1 supports a link. For this not to be so the following must hold:

$$c \geq \left(\alpha-\alpha^3\right)(1+4\alpha).$$

As to changing links, both i_0 or i_1 can sever one of their links and double their link with i_2, but none of these responses results in an improvement for α sufficiently large

($\alpha \geq 0.73898$ for i_0 and $\alpha \geq 0.818$ for i_1), also i_1 can sever one of its links and initiate a link with i_0, but this is not profitable for $\alpha \geq 0.75$. It can easily be checked that these inequalities are implied by either of the two preceding conditions. Figure 11 shows the region S where these conditions hold and g is a Nash profile.

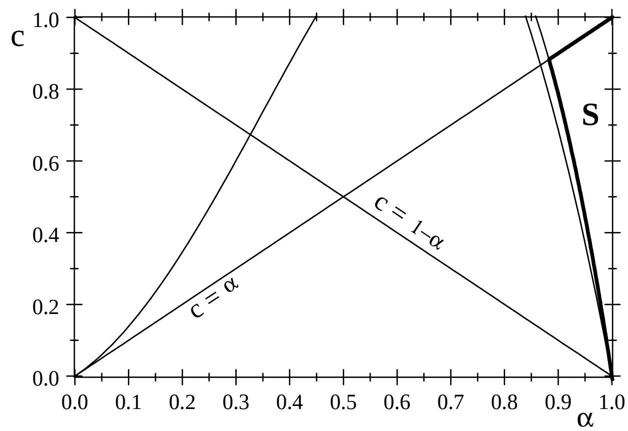


Figure 11: Stability of Example 3

As to strategy profiles on the line $c = 1 - \alpha$, the following conclusion is reached relative to the stable profiles which complements Proposition 4:

Proposition 9 *If the decay matrix* δ^g *is given by* (6) *and payoffs by* (9), *with* $0 < \alpha < 1$ *and* $c = 1 - \alpha$: (i) *Periphery-sponsored tree-core-periphery profiles satisfying conditions* (10) *and* (11) *are Nash but not strict Nash;* (ii) *Non-periphery-sponsored tree-core-periphery profiles are not Nash;* (iii) *No tree-core-periphery profile is pairwise stable.*

Proof. (i) Note that, under conditions (10) and (11), when $c = 1 - \alpha$ every player is playing a best response in a periphery-sponsored tree-core-periphery profile. Among them only the periphery-sponsored star is strict Nash when $c > 1 - \alpha$, but when $c = 1 - \alpha$ the center is indifferent between doubling links or not with peripheral players.

(ii) Since $c = 1 - \alpha$, a peripheral node involved in a weak link supported by a node in the core has an incentive to double it.

(iii) As seen in Proposition 7-(iii), $c \geq 1 - \alpha^2$ is a necessary condition for a tree-core-periphery profile to be pairwise stable, but this implies $c > 1 - \alpha$. ■

It remains to be seen whether other structures are possible for a Nash profile when $c > 1 - \alpha$. The core of the core-periphery structures considered so far, whenever it is not trivial, consists of strong links which are in fact two-node wheels. This raises the question of whether any sort of wheel-structure with three or more nodes can be the core of a Nash profile[19]. More generally, we are driven to the question of whether profiles with cycles can appear in equilibrium. We have not been able to answer this in full generality. Nevertheless, the following result shows that "wheel-core-periphery" profiles are ruled out in equilibrium or, more precisely, no structure containing a *unique* cycle with three or more nodes is Nash stable.

Proposition 10 *Let the decay matrix δ^g be given by* (6) *and payoffs by* (9)*, with $c > 1 - \alpha$. Then a profile with a weak component containing a unique cycle where* 6 *or more of its links are weak cannot be a Nash profile.*[20]

Proof. Let g be a profile and $M \subseteq N$, such that: (i) $g\mid_M$ is a weak component of g; (ii) $g\mid_M$ contains a cycle containing 6 or more weak links; and (iii) that cycle is the only one in $g\mid_M$. That is, there is a sequence of 6 or more nodes, say w.l.o.g. $Q = \{i_0, i_1, i_2, ..., i_{q-1}\} \subseteq M$, s.t. any pair $i_p, i_{p+_q 1}$ (where $p +_q 1$ means $(p+1)$ *mod* q) is connected by a weak or a strong link, but the number of weak ones is 6 or more and this is the only cycle. Let $Q_0, Q_1, ..., Q_{k-1}$ be a partition of Q into k sets where any two nodes in each Q_r $(0 \leq r \leq k-1)$ are connected by a path of strong links, and every such set contains a complete sequence of nodes in the cycle involved in consecutive strong links (or a single node involved in two weak links). Note that each Q_r, along with any other nodes connected with any node in Q_r by a path of strong links, forms a strong component of $g\mid_M$. In each sequence Q_r a player is involved in a *weak* link with a player in $Q_{r+_k 1}$. Thus k is the number of weak links in the cycle. All nodes in the same strong component must receive the same amount of information. Let w_r be the information received by any node in the strong component containing Q_r *from outside the cycle*. That is, w_r is the information received by any node in the component containing Q_r from the network that results by deleting the two weak links connecting that component to the cycle. We refer to w_r as the *weight* of the r-component. Take any pair of nodes i, j in consecutive strong components connected by a weak link, i.e. $i \in Q_r$, $j \in Q_{r+_k 1}$, and assume w.l.o.g. $g_{ij} = 1$. Let $Q_{r+_k 2}$ be the set of nodes that form the next sequence of strong links in the cycle connected with $Q_{r+_k 1}$ by a weak link. If $w_{r+_k 2} \geq w_{r+_k 1}$ and $k \geq 6$ it would be profitable for node i to replace its link with j by a link with a node in $Q_{r+_k 2}$. This is because i interchanges the discounting distances to the strong component containing $Q_{r+_k 1}$ and to that containing $Q_{r+_k 2}$ without loss and, as $k \geq 6$, *shortens* the discounting distance to one of the furthest strong components away from i. Therefore, for g to be a Nash network, it is necessary that $w_{r+_k 2} < w_{r+_k 1}$. This has two consequences for a Nash profile if $k \geq 6$. First, the number of strong components in the cycle (and therefore weak links, i.e. k) must be

[19] In Olaizola and Valenciano (2014), where intermediate models between the one-way flow and two-way flow models of Bala and Goyal (2000) are studied, "wheels of trees", structures intermediate between the oriented wheel and root-oriented trees (or stars in particular), appear as strict Nash equilibrium for values of the parameters close to the two-way flow model.

[20] In fact, the result remains valid for a smaller number of weak links in the cycle. But each number below 6 requires a specific proof, which is rather involved for cases 4 and 5, and which we omit here.

even (otherwise this condition cannot be satisfied). Second, the direction of weak links in the cycle must *alternate* so that weak links connecting players in different strong components are paid for by the player in the component whose weight is lesser. Then, if $k \geq 6$, we can assume w.l.o.g. that $w_{k-1} < w_0 > w_1 < w_2 > ... > w_{k-1}$ and k even. Moreover, now we show that in that case $w_0 = w_2 = ... = w_{k-2}$, i.e. the weight of all the "heavy" strong components is the same. Assume that $w_0 \neq w_2$, and w.l.o.g. $w_0 < w_2$. Let i be the node in Q_{k-1} that supports the only link connecting Q_{k-1} and Q_0. Then if $k \geq 6$ it is advantageous for i to replace this link by a link with any player in Q_2. Thus, when $k \geq 6$, we are left with the case where (i) the cycle connects an even number of strong components by an equal number of weak links with alternating directions, always supported by the player belonging to the strong component with the least weight; (ii) the weight of all the "heavy" strong components is the same. But now it can be seen that if $k \geq 8$ such an architecture cannot be that of a Nash profile or a weak component of a Nash profile. Let i be the node in Q_{k-1} that supports the only link connecting Q_{k-1} and Q_0. Then if $k \geq 8$ it is advantageous for i to replace this link by a link with any player in Q_2. The discounting distances from i to Q_0 and Q_2 interchange, one of the furthest nodes approaches and the rest remain at the same distance. As $w_0 = w_2$, this improves i's payoff. This concludes the proof for $k \geq 8$. As seen above, k must be even, so $k = 7$ is ruled out. Finally, if $k = 6$, we have three big strong components of equal size and three smaller ones. And within each Q_r of the small ones there are two players (one if $\#Q_r = 1$) that support a link with a big strong component each. Assume $w_0 > w_1 < w_2 > w_3 < w_4 > w_5 < w_0$, with $w_0 = w_2 = w_4$. It is thus easy to check that either the player in Q_0 that receives a weak link from a player in Q_1 has an incentive to double it, or the player in Q_4 that supports the strong link with the player in Q_4 that receives a link from a player in Q_3 has an incentive to delete it. Therefore, a profile with a weak component containing a unique cycle with 6 or more weak links cannot be a Nash profile. ■

Remark:

A close examination of the proof shows that the conclusion is more general than as stated in Proposition 10. The proof consists of showing the incompatibility of equilibrium and the existence of a unique cycle, but the uniqueness is not crucial. The proof is easily extended if the network contains a cycle (not necessarily unique) where some of the sequences of strong links and/or some of the nodes involved in two weak links in the cycle are connected with pairwise *disjoint* parts of the network. This condition is guaranteed if the cycle is unique, but in general whether such parts of the network contain other cycles or not is immaterial as long as they are disjoint.

5 Efficiency

Given the diversity of stable structures within each region of values of the parameters, comparisons in terms of efficiency (in the sense of aggregate utility and assuming a given number of players) should be made for each particular configuration of values of the parameters. First consider the case of structures which are stable when $c < 1 - \alpha$ (Proposition 3). That is, minimally strongly connected profiles and those not minimally strongly connected where all links in g are strong, all strong components are minimal and the maximal size of a strong component is smaller than or equal to $\frac{c}{\alpha}$.

Proposition 11 *Minimally strongly connected profiles are more efficient than those not minimally strongly connected where all links in g are strong and all strong components are minimal.*

Proof. The aggregated payoff of any minimally strongly connected profile g is:

$$\sum_{i \in N} \Pi_i(g) = n(n-1) - 2c(n-1) = (n-1)(n-2c).$$

This value is increasing with n and decreasing with c. It suffices to prove that this is greater than the aggregated payoff of a profile g' of the type described which consists of two minimal strong components of sizes n_1 and n_2, with $n_1 + n_2 \leq n$:

$$\sum_{i \in N} \Pi_i(g') = (n_1 - 1)(n_1 - 2c) + (n_2 - 1)(n_2 - 2c)$$

$$= n_1^2 + n_2^2 - (1 + 2c)(n_1 + n_2) + 4c.$$

Therefore

$$\sum_{i \in N} \Pi_i(g) - \sum_{i \in N} \Pi_i(g') \geq 2n_1 n_2 - 2c = 2(n_1 n_2 - c) > 0.$$

It follows easily from here that a minimally strongly connected profile is more efficient than those not minimally strongly connected where all links in g are strong and all strong components are minimal. ■

Therefore the most efficient of the architectures described in Proposition 3 is that of minimally strongly connected profiles, i.e. trees of strong links containing all nodes, which are Nash, strict Nash and pairwise stable in the whole region. Other equilibria in this area are not efficient.

Now consider the periphery-sponsored tree-core periphery profiles, stable for certain configurations of values of the parameters when $c > 1 - \alpha$ (Propositions 5 and 6).

Proposition 12 *The greater the core of a tree-core periphery profile, the more efficient it is.*

Proof. Let g be a tree-core periphery profile with n nodes, q nodes in the core ($1 \leq q \leq n-2$) and $n - q$ peripheral nodes. Denote by $\Sigma(g) := \sum_{i \in N} \Pi_i(g)$ the aggregated payoff, given by

$$\Sigma(g) = q(q - 1 + \alpha(n-q)) - (q-1)2c + (n-q)(q\alpha + (n-q-1)\alpha^2) - (n-q)c$$

$$= (1-\alpha)^2 q^2 + (\alpha^2 - 2n\alpha^2 - c + 2n\alpha - 1)q + 2c - cn - n\alpha^2 + n^2\alpha^2.$$

This function, increasing in α and decreasing in c, is increasing in q for $q > 0$, as it is easy to check that

$$\frac{\partial \Sigma(g)}{\partial q} = 2(1-\alpha)^2 q + 2n(\alpha - \alpha^2) + \alpha^2 - c - 1 > 0.$$

■

Remarks:

(i) Figure 12 shows this q-quadratic curve for $n = 10$, $\alpha = 0.4, 0.5$ and 0.6, and $c = 0.7$. The aggregated utility increases with q, the number of players in the core, and the curve is higher with a higher α. Thus, the most efficient tree-core periphery profile occurs for $q = n - 2$, i.e. $n - 2$ nodes arranged in a line of strong links and the other two players each supporting a weak link with each of the two extremes.

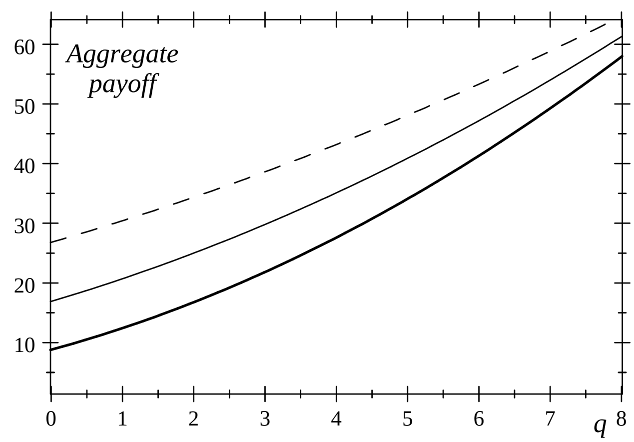


Figure 12: Aggregate payoff for $n = 10$; $c = 0.7$; $\alpha = 0.4/0.5/0.6$

(ii) Note that, unlike the case $c < 1 - \alpha$, when $c > 1 - \alpha$ efficiency and stability do not go hand in hand. As the number of nodes in the core of a tree-core periphery profile increases efficiency increases too, but condition (11) becomes in general more demanding. Moreover, the only strict Nash architecture in this region, the periphery-sponsored star, is the least efficient periphery-sponsored tree-core-periphery profile.

(iii) As to other stable architectures, as minimally weakly connected profiles, the most efficient among them are the stars of weak links, all equally efficient, but among them the periphery-sponsored star is the one which is stable for the widest range of values of the parameters.

6 Dynamics

Our next goal is to explore Bala and Goyal's dynamic model in the current context. More specifically, we consider *sequential* best response dynamics: in every period a *single* player chosen at random plays a best response (or randomizes on them if there are more than one) while all others keep their links unchanged. In this way a Markov chain on the state space of all networks is defined. Bala and Goyal (2000a) prove that *simultaneous*[21] best response dynamics converges to the center-sponsored star for the two-way flow model. Note that convergence for sequential dynamics implies convergence for simultaneous dynamics.

Given that strict Nash networks have been fully characterized only within the region $c < 1 - \alpha$, we address the convergence of dynamics only in this case. We obtain the following result:

Proposition 13 *Let the decay matrix* δ^g *be given by* (6) *and payoffs by* (9)*, with* $0 < \alpha < 1$ *and* $c < 1 - \alpha$*, then sequential best response dynamics converge to a strict Nash network with probability* 1*.*

In order to prove this, we use two lemmas and an algorithm to produce a sequence of best responses that yields the desired outcome.

When $c < 1 - \alpha$, a double link will not be severed by any of the two players if and only if there is no path of strong links connecting them other than the one consisting of that link (Lemma 1). Based on this we have the following:

Lemma 6 *Assume* $c < 1 - \alpha$*, and let* g *be an arbitrary profile and* g' *the resulting profile after a player* i *plays a best response, then:* (i) *The set of nodes in the strong component of* g' *containing* i *contains the set of nodes in the strong component of* g *containing* i, (ii) *Any strong link in which* i *is involved in* g' *will never be broken by any sequence of further best responses,* (iii) *Any weak link supported by* i *in* g' *belongs to a different strong component.*

Proof. (i) As $c < 1-\alpha$, a player i will break a strong link with another player only if a path of strong links (not containing that link) connecting them exists, and i will double a weak link unless a path of strong links connecting them exists. In consequence, the strong connection between nodes in the strong component to which i belongs will never be broken by a best response of i. Note that the size of the strong component to which i belongs may increase if new strong links are created by i doubling some weak links.

(ii) For the same reasons, for any strong link supported (i.e. doubled or not severed) by a player after a best response, neither of the two players involved will ever have an incentive to sever it after any sequence of best responses.

(iii) Otherwise, it would be superfluous. ■

[21] In the simultaneous model, at every period every player independently exhibits inertia (i.e. does nothing) with a certain probability, and otherwise plays a best response.

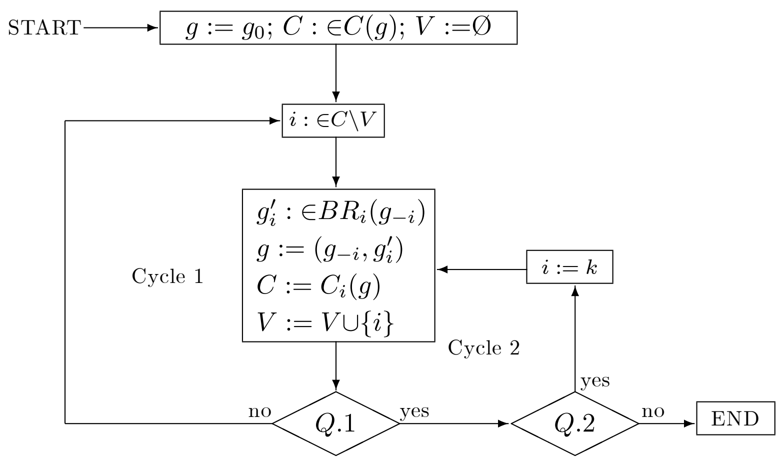


Figure 13: Procedure 1

In the following procedure, which takes as its input a profile and one of its strong components and applies a sequence of best responses, g denotes the current profile, C a strong component of g and $V \subseteq C$ a set of nodes. A sequence of best responses yields a profile with an isolated minimal strong component.

Procedure 1 (Figure 13): Initialization: $g := g_0$, where g_0 is the initial profile; C is a strong component of g_0; and $V := \varnothing$.

Step 1: Select a node $i \in C \backslash V$.

Step 2: Let i play a best response g'_i and update $g := (g_{-i}, g'_i)$, $C := C_i(g)$ (i.e. the strong component of *new* g which contains i), and $V := V \cup \{i\}$.

Q.1: $V = C$?,

- If *No*: go to Step 1, *otherwise* go to Q.2.

Q.2: Does any node in C support a weak link in g with a player in a different strong component?

- If *Yes*: Let i be one such player in a different strong component and go to Step 2.
- If *No*: END.

Claim 1 *Procedure 1 necessarily ends after a finite sequence of best responses, and at the end C is an isolated minimal strong component of a new profile g (i.e. none of its nodes is involved in any weak links with nodes in others strong components).*

Proof. By Lemma 6, after every iteration of Cycle 1, set C either remains unchanged or incorporates some new players. Thus N is an obvious "upper" bound for set C. After each iteration of Cycle 1, set $V \subseteq C$ incorporates a new player. After each iteration of Cycle 2 the sizes of both C and V increase. Therefore, condition $V = C$ is bound to hold after no more that n iterations. As to its output once $V = C$ holds, note that, by Lemma 2, no player in $V = C$ supports superfluous strong links, but perhaps some of them support some weak links with nodes in other strong components. In this

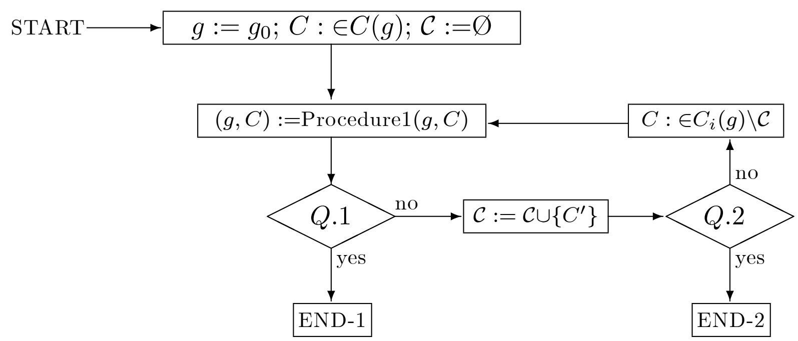


Figure 14: Algorithm 1

case, no two nodes in C support weak links with the same strong component. Note that no weak link supported by a player in a different strong component can survive this iteration, because it would have been doubled by the player receiving it in the strong component when playing a best response. As to Q.2, if the answer is *Yes* a best response of the selected player would surely include doubling the link received from a player in C, so that in the resulting new profile a strong component contains C and the selected player. Thus, this new strong component is bigger. Thus at a certain moment the answer to Q.2 must necessarily be *No*, and in that case C is a minimal strong component none of the nodes of which is involved in a weak link. ■

We now describe an algorithm that takes as its input any strategy profile g_0 and generates a sequence of best responses that gives as its output a strict Nash profile or a profile with a minimal strong component such that a sequence of best responses will lead to a minimally strongly connected profile. The variables are: a strategy profile g, a subset $C \subseteq N$ (a strong component in fact) and a collection $\mathcal{C}$ of disjoint subsets of N. The idea is to start from a strong component and, by reiterating Procedure 1, generate by best responses a sequence $(\mathcal{C})$ of *minimal* strong components which will either be interrupted if one bigger than or equal in size to $\frac{c}{\alpha}$ is generated or, otherwise, form a strict Nash profile. As shown below, in the first case a further sequence of best responses yields a minimally strongly connected profile (which is therefore strict Nash).

Algorithm 1 (Figure 14): Initialization: $g := g_0$, choose a strong component C of g, and $\mathcal{C} := \emptyset$.

Step 1: Apply Procedure 1 to profile g and its component C, and update g and C to be the output of Procedure 1.

Q.1: $\#C \geq \frac{c}{\alpha}$?

-If *Yes*, END-1, *otherwise* make : $\mathcal{C} := \mathcal{C} \cup \{C\}$ and go to Q.2.

Q.2: $N = \bigcup_{C \in \mathcal{C}} C$?

-If *Yes*, END-2; *otherwise* select a strong component C of g s.t. $C \notin \mathcal{C}$ and go to Step 1.

Claim 2 *Algorithm* 1 *ends in finite time and yields either a strict Nash profile* (*END*-2) *or a profile with a minimal strong component of size* $\geq \frac{c}{\alpha}$ (*END*-1)*.*

Proof. In each iteration of the unique cycle of the algorithm (apart from the two within Procedure 1), a new isolated strong component is added to $\mathcal{C}$. Therefore, it must end in no more than n rounds, and it may end in two ways. At END-1, the algorithm yields a profile with an isolated minimal strong component of size $\geq \frac{c}{\alpha}$. Otherwise, whenever the answer to Q.1 is *No*, current C is an isolated minimal strong component, not big enough to make it worthwhile for any player outside C to initiate a weak link with a player in C. Then C is added to $\mathcal{C}$ and the process continues. At END-2 the algorithm yields a profile where all links are strong and which consists of several minimal strong components (collected in $\mathcal{C}$) of size smaller than $\frac{c}{\alpha}$. By Proposition 3, such a profile is strict Nash. ■

It only remains to be shown that from any profile with an isolated minimal strong component of size $\geq \frac{c}{\alpha}$ (END-1) there is a sequence of best responses which yields a strict Nash profile[22].

Lemma 7 *For any strategy profile with an isolated minimal strong component of size* $\geq \frac{c}{\alpha}$*, there is a sequence of best responses that yields a minimally strongly connected profile.*

Proof. Let C be an isolated minimal strong component of a profile g s.t. $\#C \geq \frac{c}{\alpha}$. If $C = N$ we are done. Otherwise, take $i \notin C$. As $\#C \geq \frac{c}{\alpha}$, there is a best response of i in which i supports a weak link with one player $j \in C$ (if $\#C = \frac{c}{\alpha}$, player i is indifferent between supporting it or not, otherwise one such link is certainly part of any best response of i). Let i play that best response and let g' be the new profile. Now let j play a best response and let g'' be the resulting profile. Then we are sure to have $g''_{ij} = g''_{ij} = 1$, so that if C'' is the strong component of g'' containing C and i, obviously $\#C'' > \#C \geq \frac{c}{\alpha}$. Now, by applying Procedure 1 to g'' and C'' a profile with an isolated minimal strong component of size greater than or equal to that of C'', and consequently strictly greater than C, is generated. It is clear that by reiterating the argument a sequence of profiles is generated each of which has a minimal strong component greater than the one in the preceding one. Such a sequence is bound to reach a minimally strongly connected profile. ■

By Proposition 3, a minimally strongly connected profile is strict Nash whenever $c < 1 - \alpha$. Consequently, based on Algorithm 1 and Lemma 7, we have the result stated in Proposition 13.

[22] In fact, it is easy to prove that such a sequence exists for any profile with a strong component of size $\geq \frac{c}{\alpha}$.

7 Concluding remarks

This paper seeks to make a contribution to the theory of network formation. We introduce a model which incorporates as extreme cases two benchmark models of network formation: Jackson and Wolinsky's (1996) connections model and Bala and Goyal's (2000) two-way flow model. We study stability in three senses (Nash, strict Nash and pairwise), efficiency and dynamics.

Proposition 3 fully characterizes Nash, strict Nash and pairwise stable architectures for the whole range of values of the parameters within the region $c < 1 - \alpha$. As to the region where $c > 1 - \alpha$, the results in Section 4.2 do not provide a full characterization since the following issues are still open: (i) are multiple cycles possible in equilibrium?, (ii) a full characterization of weakly connected architectures for $c < \alpha$ has not been yet achieved, only necessary conditions which are, however, very demanding.

If cycles are ruled out in equilibrium, Propositions 5 and 6 would provide a complete characterization of stable architectures for $c > 1 - \alpha$ and $c > \alpha$. Finally, in the region where $c > 1 - \alpha$ and $c < \alpha$ we conjecture that, apart from the periphery-sponsored star which is stable in the whole region, only certain minimally weakly connected architectures can be stable and only for high values of $c/(1 - \alpha)$, that is in a narrow region very close to $\alpha = 1$, as is the case with non-periphery-sponsored stars.

The point of view provided by this continuum of models bridging the gap between the two benchmark models permits a comparison of the ways in which the results for each of them expand. The results for Jackson and Wolinsky's model without decay expand smoothly below the line $c = 1 - \alpha$ (i.e. when $c < 1 - \alpha$). In this region, as parameter α increases, stable architectures where all links are strong and all components are minimal, remain stable until the size of the largest component makes it worth initiating a weak link with it, while minimally strongly connected profiles are the only pairwise stable ones in the whole region. Above the line $c = 1-\alpha$ (i.e. when $c > 1-\alpha$), the expansion of Bala and Goyal's two-way flow model is, by contrast, more complex. The only stable architecture in Bala and Goyal's model, the center-sponsored star (as any non-periphery-sponsored star), remains stable only in a region where α is very close to 1, that is, a situation very close to the two-way flow model[23]. As α increases, periphery-sponsored tree-core-periphery profiles emerge in equilibrium.

There are several lines of further research. The most obvious is the unsettled question as to the possibility of there being multiple cycles in equilibrium. Also a similar intermediate model between Jackson and Wolinsky's (1996) connections model and Bala and Goyal's (2000) one-way flow model could be studied. The effects of introducing decay in the model as pointed out in footnote 12 could also be explored.

[23] A similar situation is observed in the model which bridges the gap between the two Bala and Goyal's models studied in Olaizola and Valenciano (2014), where the extreme case represented by the two-way flow model appears as a singularity. A small asymmetry (i.e. a small degree of decay in the reverse direction of a link) makes all root-oriented trees strictly stable, but when asymmetry disappears only the center-sponsored star remains stable.

This would mean a model intermediate between the two benchmark models *with decay*.

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