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Zitterbewegung physics in a single trapped ion

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1 Introduction

Relativistic quantum mechanics (RQM) is the formulation of quantum mechanics that incorporates special relativity. This theory is applicable to particles that propagate with velocities that are near the speed of light. If this particles are spin-1/2 fermions, the equation that governs its dynamics is the Dirac equation. The motion that this equation predicts for a free particle has an oscillating term; this oscillating movement is named *zitterbewegung* ("trembling motion" in German). Although the *zitterbewegung* of a free relativistic particle has never been observed, the physics related with the Dirac equation have been simulated using an ion-trap quantum simulator.
A quantum simulator is a controllable quantum system that intends to imitate another different quantum system (the simulated system). For that, the quantum simulator must obey a physical model that is equivalent (mathematically) to the model associated to the simulated system. The Dirac model and the models used in quantum optics, the theory that studies the interaction between light and matter, have the same mathematical components. This leads us to believe that, using quantum optics based technology -like trapped ions-, it is possible to create a controllable quantum system whose Hamiltonian model is equivalent to the Dirac Hamiltonian. In 2010, the quantum simulation of the Dirac equation and the reproduction of properties like the *zitterbewegung* was performed in trapped ions, a non-relativistic quantum platform [1].

This Hamiltonian model equivalence has open a gate between quantum optics and relativistic quantum mechanics; the objective of this work is to use that gate to export concepts that are originally from quantum optics to the field of RQM and viceversa; then, explore and analyse the different results that we could obtain and their meaning in each of these two fields of physics.

The work is divided in two parts; chapters 2 and 3 contain the basic theoretical concepts of RQM and trapped ions respectively, while chapter 4 explains how the model of the $1+1$ dimensional Dirac equation is implemented in a single trapped ion. The chapters 5 and 6 contain the work that I have developed during the last months. More precisely, chapter 5 introduces a feasible method to implement, in a single trapped ion, the Foldy-Wouthuysen transformation, which takes us to a representation where the Dirac Hamiltonian is diagonal and the *zitterbewegung* disappears. In chapter 6 we do an analysis about how typical states from quantum optics evolve under the $1+1$ dimensional Dirac Hamiltonian; we also study and try to understand the behaviour of the *zitterbewegung* produced by such states.

2 Relativistic quantum mechanics

Relativistic quantum mechanics (RQM) [2] is the mathematical formulation of quantum mechanics applied in the context of special relativity; just like non-relativistic quantum mechanics is formulated in the context of what is called Galilean relativity. RQM has achieved the successful explanation of topics like antimatter or spin 1/2 particles. The main equation from which this success emerges is the Dirac equation.
The subsections 2.1, 2.2 and 2.3 intend to do a theoretical approach to RQM from the non-relativistic formulation and in subsection 2.4 we introduce one of the most surprising results of the Dirac equation, the *zitterbewegung*.

2.1 Non-relativistic quantum mechanics

In non-relativistic quantum mechanics, to obtain the dynamical equation for a free particle of mass m, we consider the classical energy momentum relation

$$E = \frac{p^2}{2m} \tag{1}$$

and substitute E and p by their respective differential operators

$$E \to i\hbar\frac{\partial}{\partial t}, \qquad p \to -i\hbar\nabla. \tag{2}$$

The resulting operator acts on a complex wavefunction $\psi(\mathbf{x}, t)$,

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\psi = 0 \tag{3}$$

where we interpret $\rho = |\psi|^2$ as the probability distribution. Now we will calculate the density flux of a beam of particles, $\mathbf{j}$. From the conservation of probability law, the rate at which the number of particles in a given volume decreases is equal to the total flux of particles coming out of that volume, that is,

$$-\frac{\partial}{\partial t}\int_V \rho \; dV = \int_S \mathbf{j}\cdot\hat{\mathbf{n}} \; dS = \int_V \nabla\cdot\mathbf{j} \; dV \tag{4}$$

where the last equality is Gauss's theorem. According to Eq. (4), the probability and the flux densities are related by the "continuity" equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0. \tag{5}$$

To determine the flux, we start from the following expressions:

$$\begin{aligned} -i\psi^*(i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\psi = 0) \\ -i\psi(i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\psi = 0)^*. \end{aligned} \tag{6}$$

Combining this two equations and remembering that $\partial\rho/\partial t = \psi^*\partial\psi/\partial t + \psi\partial\psi^*/\partial t$, we obtain

$$\frac{\partial\rho}{\partial t} - \frac{i\hbar}{2m}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) = 0. \tag{7}$$

Comparing this with Eq. (5) we identify the probability flux density as

$$\mathbf{j} = -\frac{i\hbar}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*) = 0. \tag{8}$$

If we take the following solution for Eq. (3)

$$\psi = Ne^{i\mathbf{p}\cdot\mathbf{x}-iEt}, \tag{9}$$

which describes a free particle of energy E and momentum $\mathbf{p}$, the probability density and a flux density are

$$\rho = |N|^2, \qquad \mathbf{j} = \frac{\mathbf{p}}{m}|N|^2. \tag{10}$$

2.2 The Klein-Gordon equation

In relativistic quantum mechanics the energy momentum relation for a free particle is given by

$$E^2 = p^2 + m^2 \tag{11}$$

where m is the rest mass of the particle. Making the operator substitutions introduced in Eq. (2), we obtain

$$-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi = m^2\phi. \tag{12}$$

This equation is known as the Klein-Gordon equation. Multiplying the Klein-Gordon equation by $-i\phi^*$ and the complex conjugate equation by $-i\phi$, and subtracting them, we obtain the relativistic analogue of Eq. (7)

$$\frac{\partial}{\partial t}\left[i(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t})\right] + \nabla\cdot[-i(\phi^*\nabla\phi - \phi\nabla\phi^*)] = 0. \tag{13}$$

Comparing it with Eq. (5) we easily identify the probability and the flux densities

$$\begin{aligned} \rho &= i(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}) \\ \mathbf{j} &= -i(\phi^* \nabla \phi - \phi \nabla \phi^*). \end{aligned} \tag{14}$$

For example, for a free particle of energy E and momentum $\mathbf{p}$ described by the following solution of the Klein-Gordon equation

$$\phi = N e^{i\mathbf{p}\cdot\mathbf{x} - iEt}, \tag{15}$$

we find from Eq. (14) that

$$\rho = 2E|N|^2, \qquad \mathbf{j} = 2\mathbf{p}|N|^2. \tag{16}$$

Substitution of Eq. (15) into Eq. (12) gives the following eigenvalues for the Klein-Gordon equation

$$E = \pm(\mathbf{p}^2 + m^2)^{1/2} \tag{17}$$

We clearly have a problem here. $E < 0$ solutions are associated with a negative probability density in Eq. (16).

2.3 The 1+1 dimensional Dirac equation

Dirac wanted to write an equation which was linear in $\partial/\partial t$. In order to be covariant, it must then also be linear in $\partial/\partial x$ and has therefore the general form

$$H\psi = (\alpha p_x + \beta m)\,\psi. \tag{18}$$

The two coefficients α and β are determined by the requirement that a free particle must satisfy the relativistic energy-momentum relation (11),

$$H^2\psi = \left(p_x^2 + m^2\right)\psi, \tag{19}$$

and from Eq. (18), we have that

$$H^2\psi = \left(\alpha^2 p_x^2 + (\alpha\beta + \beta\alpha)p_x m + \beta^2 m^2\right)\psi. \tag{20}$$

Comparing with Eq. (19), we see that α and β must anticommute. Also $\alpha^2 = 1$ and $\beta^2 = 1$. Since the coefficients do not commute, they cannot be simply numbers, and we are

led to considering matrices operating on a wave vector Ψ, which is now a two component column vector. Let us consider

$$\alpha = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \beta = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{21}$$

Multiplying Dirac equation(18) by β from the left, we obtain

$$i\sigma_z \partial_t \Psi = -i\sigma_z \sigma_x \partial_x \Psi + m\Psi. \tag{22}$$

Now, we are going to focus in obtaining a continuity equation of this matrix equation. For that we consider the Hermitian conjugate of Eq. (18) ,

$$-i\partial_t \Psi^\dagger \sigma_z^\dagger = i\partial_x \Psi^\dagger (\sigma_z \sigma_x)^\dagger + m\Psi^\dagger \tag{23}$$

We can easily prove that $\sigma_z^\dagger = \sigma_z$ and $\sigma_z^\dagger = \sigma_x$, and thus that they must anticommute, $(\sigma_z \sigma_x)^\dagger = \sigma_x \sigma_z = -\sigma_z \sigma_x$. Substituting these results we can rewrite the Eq. (23) to obtain

$$i\partial_t \Psi^\dagger \sigma_z = i\partial_x \Psi^\dagger \sigma_z \sigma_x - m\Psi^\dagger \tag{24}$$

Multiplying Eq. (22) from the left by $\Psi^\dagger$ and (24) from the right by Ψ and adding them, we find that the two ∂_x terms kill each other. Therefore, we need to correct our method, by multiplying Eq. (22) from the left by $\Psi^\dagger \sigma_z$ and (24) from the right by $\sigma_z \Psi$, and adding them. Now, we get the following equation

$$\partial_t \Psi^\dagger \Psi + \Psi^\dagger \partial_t \Psi + \partial_x \Psi^\dagger \sigma_x \Psi + \Psi^\dagger \sigma_x \partial_x \Psi = 0. \tag{25}$$

We may note that this equation can be rewritten as

$$\partial_t (\Psi^\dagger \Psi) + \partial_x (\Psi^\dagger \sigma_x \Psi) = 0, \tag{26}$$

which suggests that we should identify $\Psi^\dagger \Psi$ and $\Psi^\dagger \alpha \Psi$ as the probability density and the flux, ρ and j_x respectively. Now, the probability density

$$\rho \equiv \Psi^\dagger \Psi = \begin{pmatrix} \psi_1^* & \psi_2^* \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = |\psi_1|^2 + |\psi_2|^2 \tag{27}$$

is positive definite. Avoiding the main drawback of the Klein-Gordon equation and providing a physically reasonable framework to describe relativistic quantum mechanics. This final result was the one that motivated the work of Dirac.

2.4 *Zitterbewegung*

Zitterbewegung [3] is one of the most counterintuitive results of the Dirac equation. The basic idea is that the expectation value of the standard position operator x that is predicted by the Dirac Hamiltonian for a free particle has an oscillating term, a "trembling motion", that is called *Zitterbewegung*. In principle we do not expect from a free particle to have such an oscillatory motion, because in non-relativistic quantum and classical mechanics, oscillating dynamics are related to forces; and not to free particles.
In the Heisenberg picture, states do not depend on time, but operators do. The evolution of an operator A at time t in the Heisenberg picture is given by ($\hbar = 1$)

$$A(t) = e^{iHt} A e^{-iHt} \tag{28}$$

where H is the Hamiltonian. In our case, the Hamiltonian is the $1+1$ dimensional Dirac Hamiltonian, which is

$$H_{\mathrm{D}} = cp\sigma_x + mc^2\sigma_z, \tag{29}$$

and according to Eq. (28), the evolution of the position operator is given by

$$x(t) = e^{iH_{\mathrm{D}}t} x e^{-iH_{\mathrm{D}}t}. \tag{30}$$

In the Heisenberg picture any operator A obeys this relation

$$\frac{dA}{dt}(t) = i[H, A(t)], \tag{31}$$

so in our case the position operators time derivative at $t = 0$ is given by

$$\frac{dx}{dt}(0) = i[H_{\mathrm{D}}, x] = c\sigma_x \tag{32}$$

and its time evolution is given by

$$\frac{dx}{dt}(t) = c\sigma_x(t) \tag{33}$$

where $\sigma_x(t) = e^{iH_{\mathrm{D}}t}\sigma_x e^{-iH_{\mathrm{D}}t}$. Equation (33) shows that $c\sigma_x(t)$ can be understood as the velocity operator, so we will call it the *standard velocity operator*.
We will call $c^2 p H_{\mathrm{D}}^{-1}$ the *classical velocity operator*, because of its similarity with the common expression for the velocity in the non-relativistic limit, $v = c^2 p/E$.
What we want is to find an analytical expression for the evolution of the position operator. Let us define an operator F in the following way

$$F \equiv c\sigma_x - c^2 p H_{\mathrm{D}}^{-1} \tag{34}$$

the time evolution of this operator is given by

$$F(t) = e^{iH_{\mathrm{D}}t} F e^{-iH_{\mathrm{D}}t} \tag{35}$$

but, using that $\{H_{\mathrm{D}}, \sigma_x\} = cp\{\sigma_x, \sigma_x\} = 2cp$ is easy to verify that F anticommutes with H_{D} ($H_{\mathrm{D}}F = -FH_{\mathrm{D}}$) and therefore Eq. (35) can be written as

$$F(t) = e^{2iH_{\mathrm{D}}t} F. \tag{36}$$

So, according to Eq. (34), the evolution of the *standard velocity operator* is

$$c\sigma_x(t) = F(t) + c^2 p H_{\mathrm{D}}^{-1}, \tag{37}$$

note that since $[H_{\mathrm{D}}, c^2 p H_{\mathrm{D}}^{-1}] = 0$ the *classical velocity operator* is constant in time. Now that we have the expression for the evolution of the *standard velocity operator*, we can conclude that it has an oscillating part $F(t)$ and a constant part $c^2 p H_{\mathrm{D}}^{-1}$; according to this, it seems like the velocity oscillates around the expected value of the "classical velocity". This phenomena is called *zitterbewegung*.
The time evolution of the position operator is achieved by integrating $c\sigma_x(t)$ in time between 0 and t

$$x(t) = x + \int_0^t d\tau c\sigma_x(\tau). \tag{38}$$

If we calculate the integral we arrive to the expression

$$x(t) = x + c^2 p H_{\mathrm{D}}^{-1} t + \frac{1}{2iH_{\mathrm{D}}}(e^{2iH_{\mathrm{D}}t} - 1)F. \tag{39}$$

Looking at Eq. (39) we can see that there is a "classical part" of the motion that evolves linearly in time, and another part that oscillates, which is responsible for the "trembling motion". Note also that in the limit of $m \to \infty$ the position operator goes to its initial expression ($x(t) = x$).

3 Single trapped ion

Trapped-ion systems [4] are one of the most promising technologies for quantum simulations. It consist in trap the ion using for example a Paul trap -that uses time-dependent electromagnetic fields- and cooling its vibrational degree of freedom to a quantum regime. Then, lasers provide an interaction between the motional degree of freedom and the internal electronic degree of freedom.

3.1 Introduction to single trapped ion physics

3.1.1 The two-level approximation

The internal electronic structure of an ion confined in a linear Paul trap can be approximated by a two-level system, being $|g\rangle$ and $|e\rangle$ the ground and excited states respectively, and the energy difference $\hbar\omega_0 = \hbar(\omega_e - \omega_g)$ [4]. According to that the two-level Hamiltonian is

$$H^e = \hbar\omega_g |g\rangle\langle g| + \hbar\omega_e |e\rangle\langle e|. \tag{40}$$

This can be written also as

$$H^e = \frac{\hbar\omega_0}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + \hbar\frac{\omega_e + \omega_g}{2}(|e\rangle\langle e| + |g\rangle\langle g|). \tag{41}$$

Any operator connected to a two-level system can be expressed using the spin-1/2 algebra, since the three Pauli matrices and I, the 2×2 identity matrix, span the full vector space of 2×2 Hermitian matrices. In the particular case at hand the mapping is

$$\begin{aligned} |e\rangle\langle e| + |g\rangle\langle g| \longrightarrow \mathbb{I}_2^\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad |e\rangle\langle g| \longrightarrow \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ |g\rangle\langle e| \longrightarrow \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad |e\rangle\langle e| - |g\rangle\langle g| \longrightarrow \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{42}$$

where,

$$|e\rangle \;\rightarrow\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |g\rangle \;\rightarrow\; \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{43}$$

With this mapping the two-level Hamiltonian is reexpressed as

$$H^e = \frac{\hbar\omega_0}{2}\sigma_z, \tag{44}$$

where the energy is rescaled by $-\hbar(\omega_e + \omega_g)/2$ to suppress the state-independent energy contribution in Eq. (41).

3.1.2 Total Hamiltonian and interaction Hamiltonian

The fundamental Hamiltonian describing the interaction of a two-level trapped ion, cooled down to the motional quantum regime, is given by

$$H = \frac{\hbar\omega_0}{2}\sigma_z + \hbar\nu a^\dagger a + \hbar\Omega(\sigma_+ + \sigma_-)(e^{i(kx-\omega t+\phi)} + e^{-i(kx-\omega t+\phi)}) \tag{45}$$

where $\sigma_\pm$ and σ_z are Pauli operators associated to the two internal levels of the ion, $a(a^\dagger)$ is the annihilation(creation) operator of the motional degree of freedom, ω_0 is the frequency of the internal electronic transition, ν is the trap frequency, ω is the frequency of the driving laser field, k is the laser wave vector, ϕ is the laser phase, and Ω is the Rabi frequency associated with the atom-laser coupling strength.

We have a Hamiltonian, such as

$$H = H_0 + V(t) \tag{46}$$

where we assume that energy eigenvalues E_n and eigenkets $|n\rangle$ of H_0

$$H_0|n\rangle = E_n|n\rangle \tag{47}$$

are completely known.
The interaction ket is defined, as

$$|\psi\rangle_I(t) \equiv e^{\frac{iH_0t}{\hbar}}|\psi\rangle(t) \tag{48}$$

and it's time evolution is given by the equation

$$i\hbar\partial_t|\psi\rangle_I(t) = H^I|\psi\rangle_I(t) \tag{49}$$

where the interaction Hamiltonian $H^I = e^{\frac{iH_0t}{\hbar}}V(t)e^{\frac{-iH_0t}{\hbar}}$.

In our case, the transformed interaction Hamiltonian is

$$H^I = e^{i(\frac{\omega_0}{2}\sigma_z+\nu a^\dagger a)t}\hbar\Omega(\sigma_+ + \sigma_-)(e^{i(kx-\omega_0 t+\phi)} + e^{-i(kx-\omega_0 t+\phi)})e^{-i(\frac{\omega_0}{2}\sigma_z+\nu a^\dagger a)t}. \tag{50}$$

In order to simplify the expression obtained for the interaction Hamiltonian, we may remove the operators in the exponentials. Let us start with the two-level operators. Combining the very well known formula

$$e^{iG\lambda}Ae^{-iG\lambda} = A + i\lambda[G,A] + \frac{(i\lambda)^2}{2!}[G,[G,A]] + ... \tag{51}$$

with the commutation relations $[\sigma_+, \sigma_z] = -2\sigma_+$ and $[\sigma_-, \sigma_z] = 2\sigma_-$, we can easily prove the following equality

$$e^{i\frac{\omega_0 t}{2}\sigma_z}(\sigma_+ + \sigma_-)e^{-i\frac{\omega_0 t}{2}\sigma_z} = \sigma_+ e^{i\omega_0 t} + \sigma_- e^{-i\omega_0 t}. \tag{52}$$

Using this, our Hamiltonian takes the form

$$H^I = e^{i\nu a^\dagger a t}\hbar\Omega(\sigma_+ e^{i\omega_0 t} + \sigma_- e^{-i\omega_0 t})(e^{i(kx-\omega t+\phi)} + e^{-i(kx-\omega t+\phi)})e^{-i\nu a^\dagger a t} \tag{53}$$

Multiplying the time-dependent factors in the above expression leads to terms like $e^{\pm i(\omega\pm\omega_0)t}$. Two terms are rapidly oscillating because ω and ω_0 add up, while the two other terms oscillate with frequency $\delta = \omega - \omega_0 \ll \omega_0$. It can be proved, if also the contition $\Omega \ll \omega_0$ is fulfilled, that the contribution of the rapidly oscillating terms in the time evolution, are neglectable. Therefore if the commented two conditions are fulfilled, we can neglect the rapidly oscilating terms; doing so is called the *optical rotating-wave approximation*, and leads to the following Hamiltonian,

$$H^I = e^{i\nu a^\dagger a t}\hbar\Omega(\sigma_+ e^{i(kx-\delta t+\phi)} + \sigma_- e^{-i(kx-\delta t+\phi)})e^{-i\nu a^\dagger a t}. \tag{54}$$

Let us focus now on the motional operators. First we shall remember that x, the position operator of the ion, is written in term of a and $a^\dagger$ as

$$x = \sqrt{\frac{\hbar}{2m\nu}}(a + a^\dagger)\cdot \tag{55}$$

Using again the formula (51) now with the commutation relations $[a^\dagger a, a^\dagger] = a^\dagger$ and $[a^\dagger a, a] = -a$, we can prove that

$$e^{i\nu t a^\dagger a} x e^{-i\nu t a^\dagger a} = \sqrt{\frac{\hbar}{2m\nu}}(a^\dagger e^{i\nu t} + a e^{-i\nu t}). \tag{56}$$

We can continue calculating the same for x^n. Here, it is useful to remaind that $e^{-A}e^{A} = \mathbb{I}$ if A is a Hilbert space operator. Using the expression above, we can proceed

$$\begin{aligned} e^{i\nu t a^\dagger a} x^n e^{-i\nu t a^\dagger a} &= e^{i\nu t a^\dagger a} x \cdot x \cdot x ... e^{-i\nu t a^\dagger a} \\ = e^{i\nu t a^\dagger a} x e^{-i\nu t a^\dagger a} e^{i\nu t a^\dagger a} x e^{-i\nu t a^\dagger a} e^{i\nu t a^\dagger a} x ... e^{-i\nu t a^\dagger a} \\ &= (\sqrt{\frac{\hbar}{2m\nu}})^n (a^\dagger e^{i\nu t} + a e^{-i\nu t})^n. \end{aligned} \tag{57}$$

In our Hamiltonian, the terms that concern us, are multiplied by $e^{\pm ikx}$. However, the already achieved expressions will be quite useful if we expand the exponential

$$\begin{aligned} e^{i\nu t a^\dagger a} e^{\pm ikx} e^{-i\nu t a^\dagger a} = e^{i\nu t a^\dagger a} \sum_{m=0}^{\infty} \frac{(\pm ikx)^m}{m!} e^{-i\nu t a^\dagger a} \\ = \sum_{m=0}^{\infty} \frac{(\pm ik)^m}{m!} e^{i\nu t a^\dagger a} x^m e^{-i\nu t a^\dagger a} = \sum_{m=0}^{\infty} \frac{(\pm ik)^m}{m!} (\sqrt{\frac{\hbar}{2m\nu}})^m (a^\dagger e^{i\nu t} + a e^{-i\nu t})^m \end{aligned} \tag{58}$$

Introducing the Lamb-Dicke parameter $\eta \equiv k\sqrt{\frac{\hbar}{2m\nu}}$, we finally get

$$e^{i\nu t a^\dagger a} e^{\pm ikx} e^{-i\nu t a^\dagger a} = e^{\pm i\eta(e^{i\nu t}a^\dagger + e^{-i\nu t}a)} \tag{59}$$

Using this last result, the interaction Hamiltonian takes the form

$$H^I = \hbar\Omega\sigma_+ e^{i(\eta(e^{i\nu t}a^\dagger + e^{-i\nu t}a) - \delta t + \phi)} + \text{H.c.} \tag{60}$$

where H.c. means the Hermitian Conjugate.

3.2 Lamb-Dicke regime: carrier, first blue, and red sidebands

The interaction Hamiltonian in Eq. (60) can be simplified if we consider the ion confined on the Lamb-Dicke regime, where the condition $\eta\sqrt{\langle(a+a^\dagger)^2\rangle} \ll 1$ must hold for all times. Once we have this condition we can expand the exponential to the lowest order in η,

$$H_{\text{LD}} = \hbar\Omega\sigma_+\{1 + i\eta(e^{i\nu t}a^\dagger + e^{-i\nu t}a)\}e^{i(-\delta t+\phi)} + \text{H.c.}. \tag{61}$$

3.2.1 The carrier interaction

Let us consider $\delta = 0$. The Hamiltonian splits in three parts,

$$H = \hbar\Omega\{(\sigma_+ e^{i\phi} + \sigma_- e^{-i\phi}) + i\eta(a^\dagger\sigma_+ e^{i\phi} + a\sigma_- e^{-i\phi})e^{i\nu t} + i\eta(a\sigma_+ e^{i\phi} + a^\dagger\sigma_- e^{-i\phi})e^{-i\nu t}\} \tag{62}$$

where two of the parts oscillate with ν and $-\nu$ respectively. However, the first part does not have time dependence. When a part of the interaction Hamiltonian does not depend on time, it is in resonance condition, and under a second *Rotating wave approximation* (RWA),called *vibrational RWA*, we can neglect the two oscillating terms. So, the first resonance for $\delta = 0$ is called the *carrier resonance* and its effective interaction Hamiltonian is

$$H_{\text{car}} = \hbar\Omega(\sigma_+ e^{i\phi} + \sigma_- e^{-i\phi}). \tag{63}$$

Let me go a little further here. If instead of $\delta = 0$ we assume that $\delta = -\xi$, were $|\xi| \ll |\nu|$, applying the *vibrational RWA*, the same two terms can be neglected. It is called *vibrational* because the terms that are neglected are oscillating with frequency ν, that is the natural frequency of the motional degree of freedom. This case were $\delta = -\xi$ is called the *detuned carrier* interaction and its effective hamiltonian is

$$H_{\text{d/car}} = \hbar\Omega(\sigma_+ e^{i\phi} e^{i\xi t} + \sigma_- e^{-i\phi} e^{-i\xi t}). \tag{64}$$

Anyway, returning to the resonant *carrier* interaction, suppose that the initial state is, for example $|n, g\rangle$ -where $|n\rangle$ is a Fock state and $|g\rangle$ is the ground state of the two-level electronic structure. It can be proved that the evolution of this state is

$$|n, g\rangle \ \rightarrow \ \cos\left(\frac{\Omega t}{2}\right)|n, g\rangle + e^{i\varphi}\sin\left(\frac{\Omega t}{2}\right)|n, e\rangle \tag{65}$$

Thus, we realize that the *resonant carrier* Hamiltonian give rise to transitions of the type $|n, g\rangle \leftrightarrow |n, e\rangle$ with a frequency frequency Ω, that is called Rabi frequency. But the *carrier* does not affect to the motional state; to obtain an interaction that generates give transitions simultaneously in the electronic and the motional state, we must excite other sidebands.

3.2.2 First red sideband: Jaynes-Cummings (JC) model

If we bring back the Lamb-Dicke regime Hamiltonian of Eq. (61), and if, instead of $\delta = 0$, we choose $\delta = -\nu$, the resonant part is called *first red sideband* and the effective Hamiltonian has the form

$$H_{\text{rsb}} = \hbar\Omega i\eta(a\sigma_+ e^{i\phi} - a^\dagger\sigma_- e^{-i\phi}). \tag{66}$$

In this case, together with the two-level operators, we have the motional creation and annihilation operators. This Hamiltonian, is known as the Jaynes-Cummings Hamiltonian, and is one of the main models in quantum optics.

If we have the initial state $|g, n\rangle$,it can be prove that the evolution is given by,

$$|n, g\rangle \ \rightarrow \ \cos\left(\frac{\Omega t}{2}\sqrt{n}\right)|n, g\rangle + e^{i\phi}\sin\left(\frac{\Omega t}{2}\sqrt{n}\right)|n-1, e\rangle \tag{67}$$

Looking to this evolution we can conclude that the *first red sideband* Hamiltonian gives rise to transitions of the type $|n, g\rangle \leftrightarrow |n-1, e\rangle$, with Rabi frequency $\Omega_{n,n-1} = \eta\Omega\sqrt{n}$. So, if we measure the internal state of the ion, and if we get that it is on the excited state $|e\rangle$ we would know also that the motional state is $|n-1\rangle$. In other words, the motional state is entangled with the ions internal state. This entanglement property of the JC Hamiltonian makes the model one of the most appropriate to study the basic properties of the quantum correlations (entanglement).

Finally, as we have done with the *carrier* interaction, here we can also choose $\delta = -\nu - \xi$, where $|\xi| \ll |\nu|$, and the effective Hamiltonian is the *detuned red sideband*

$$H_{\text{d/rsb}} = \hbar\Omega i\eta(a\sigma_+ e^{i\phi}e^{i\xi t} - a^\dagger\sigma_- e^{-i\phi'}e^{-i\xi t}). \tag{68}$$

3.2.3 First blue sideband: anti-Jaynes-Cummings (anti-JC) model

In the *first red sideband* we have taken $\delta = -\nu$. In contrast, for the *first blue sideband* we take $\delta = \nu$. This, produces a different resonance on the Lamb-Dicke Hamiltonian (61), the resultant effective Hamiltonian is

$$H_{\text{bsb}} = \hbar\Omega i\eta(a^\dagger\sigma_+ e^{i\phi} - a\sigma_- e^{-i\phi}). \tag{69}$$

This Hamiltonian is known as the anti-James-Cummings Hamiltonian (anti-JC). While the JC Hamiltonian produces transitions of the type $|n, g\rangle \leftrightarrow |n-1, e\rangle$, the anti-JC Hamiltonian produces transitions of the type $|n, g\rangle \leftrightarrow |e, n+1\rangle$. The exact expression for the evolution of the initial state $|n, g\rangle$ is given by

$$|n, g\rangle \;\rightarrow\; \cos\left(\frac{\Omega t}{2}\sqrt{n+1}\right)|n, g\rangle + e^{i\phi}\sin\left(\frac{\Omega t}{2}\sqrt{n+1}\right)|n+1, e\rangle \tag{70}$$

where the Rabi frequency is $\Omega_{n,n+1} = \eta\Omega\sqrt{n+1}$. The anti-JC, of course also produces entanglement between the motional and electronic states.

The *detuned* case of the anti-JC has the form

$$H_{\text{d/bsb}} = \hbar\Omega i\eta(a^\dagger\sigma_+ e^{i\phi}e^{i\xi t} - a\sigma_- e^{-i\phi}e^{-i\xi t}). \tag{71}$$

4 Quantum simulation of the $1+1$ dimensional Dirac equation in trapped ions

In subsection 3.2, we have learned the theoretical treatment of the generation of the Jaynes-Cummings and the anti-Jaynes-Cummings Hamiltonians in trapped ions. Now we can start thinking on how to generate a Hamiltonian equivalent to the $1+1$ dimensional Dirac Hamiltonian. Actually to generate the $1+1$ Dirac Hamiltonian we need at least two lasers, so we can excite more than one mode simultaneously.

4.1 Two lasers with a single ion

The theoretical approach for the problem is identical to the one of a single laser. Now, the total Hamiltonian has one more term, that corresponds to the coupling between the ion and the second laser. The treatment of the second coupling term, is strictly the same as that of the first coupling term, as we are going to see. We start with the following Hamiltonian

$$\begin{aligned} H = \frac{\hbar\omega_0}{2}\sigma_z + \hbar\nu a^\dagger a + \hbar\Omega_1(\sigma_+ + \sigma_-)[e^{i(k_1x-\omega_1t+\phi_1)} + e^{-i(k_1x-\omega_1t+\phi_1)}] \\ +\hbar\Omega_2(\sigma_+ + \sigma_-)[e^{i(k_2x-\omega_2t+\phi_2)} + e^{-i(k_2x-\omega_2t+\phi_2)}] \end{aligned} \tag{72}$$

In this case the interaction Hamiltonian is

$$\begin{aligned} H^I = e^{i(\frac{\omega_0}{2}\sigma_z+\nu a^\dagger a)t}\hbar\Omega_1(\sigma_+ + \sigma_-)[e^{i(k_1x-\omega_1t+\phi_1)} + e^{-i(k_1x-\omega_1t+\phi_1)}]e^{-i(\frac{\omega_0}{2}\sigma_z+\nu a^\dagger a)t} \\ +e^{i(\frac{\omega_0}{2}\sigma_z+\nu a^\dagger a)t}\hbar\Omega_2(\sigma_+ + \sigma_-)[e^{i(k_2x-\omega_2t+\phi_2)} + e^{-i(k_2x-\omega_2t+\phi_2)}]e^{-i(\frac{\omega_0}{2}\sigma_z+\nu a^\dagger a)t}. \end{aligned} \tag{73}$$

To simplify this expression, we just have to follow the same steps as for the single laser case. Using the useful equality (52), we can rewrite the interaction Hamiltonian

$$\begin{aligned} H^I = e^{i\nu a^\dagger at}\hbar\Omega_1(\sigma_+e^{i\omega_0t} + \sigma_-e^{-i\omega_0t})[e^{i(k_1x-\omega_1t+\phi_1)} + e^{-i(k_1x-\omega_1t+\phi_1)}]e^{-i\nu a^\dagger at} \\ +e^{i\nu a^\dagger at}\hbar\Omega_2(\sigma_+e^{i\omega_0t} + \sigma_-e^{-i\omega_0t})[e^{i(k_2x-\omega_2t+\phi_2)} + e^{-i(k_2x-\omega_2t+\phi_2)}]e^{-i\nu a^\dagger at} \end{aligned} \tag{74}$$

The *optical rotating-wave approximation* can be applied (independently and in the same way) in the two terms, giving rise to the following expression

$$\begin{aligned} H^I = e^{i\nu a^\dagger at}\hbar\Omega_1(\sigma_+e^{i(k_1x-\delta_1t+\phi_1)} + \sigma_-e^{-i(k_1x-\delta_1t+\phi_1)})e^{-i\nu a^\dagger at} \\ +e^{i\nu a^\dagger at}\hbar\Omega_2(\sigma_+e^{i(k_2x-\delta_2t+\phi_2)} + \sigma_-e^{-i(k_2x-\delta_2t+\phi_2)})e^{-i\nu a^\dagger at}. \end{aligned} \tag{75}$$

where $\delta_1 = \omega_1 - \omega_0 \ll \omega_0$ and $\delta_2 = \omega_2 - \omega_0 \ll \omega_0$. Introducing the Lamb-Dicke parameters $\eta_1 \equiv k_1\sqrt{\frac{\hbar}{2m\nu}}$ and $\eta_2 \equiv k_2\sqrt{\frac{\hbar}{2m\nu}}$, and using the already proved equality (59), the interaction Hamiltonian takes the form

$$\begin{aligned} H^I = \hbar\Omega_1(\sigma_+ e^{i(\eta_1(e^{i\nu t}a^\dagger + e^{-i\nu t}a) - \delta_1 t + \phi_1)} + \sigma_- e^{-i(\eta_1(e^{i\nu t}a^\dagger + e^{-i\nu t}a) - \delta_1 t + \phi_1)}) \\ + \hbar\Omega_2(\sigma_+ e^{i(\eta_2(e^{i\nu t}a^\dagger + e^{-i\nu t}a) - \delta_2 t + \phi_2)} + \sigma_- e^{-i(\eta_2(e^{i\nu t}a^\dagger + e^{-i\nu t}a) - \delta_2 t + \phi_2)}) \end{aligned} \tag{76}$$

If the ion is confined to the Lamb-Dicke regime the expression above simplifies into this one

$$\begin{aligned} H_{\mathrm{LD}} = \hbar\Omega_1\sigma_+\{1 + i\eta_1(e^{i\nu t}a^\dagger + e^{-i\nu t}a)\}e^{i(-\delta_1 t + \phi_1)} \\ + \hbar\Omega_2\sigma_+\{1 + i\eta_2(e^{i\nu t}a^\dagger + e^{-i\nu t}a)\}e^{i(-\delta_2 t + \phi_2)} \\ + \mathrm{H.c.} \end{aligned} \tag{77}$$

At this point we can clearly see what the second laser provides us: the power to excite simultaneously two sidebands instead of one.

4.2 Simultaneous JC and anti-JC interactions

In chapter 4.1 we have obtained the expression for the interaction Hamiltonian of a single ion in the Lamb-Dicke regime excited by two lasers, Eq. (77). If we choose $\delta_1 = -\nu - \xi$ and $\delta_2 = \nu - \xi$, and suppose that $\Omega_1 = \Omega_2$ and $\eta_1 = \eta_2$, we would excite simultaneously the first blue and red sidebands with a detuning ξ ($-\nu + \delta_1 = \xi = \nu - \delta_2$). If we suppose that $|\xi| \ll |\nu|$, we can apply the *vibrational rotating wave approximation*, and in such a way the Hamiltonian that we get is the following one

$$H_{\mathrm{d/rb}} = \hbar\Omega i\eta\sigma_+(ae^{i\phi_1} + a^\dagger e^{i\phi_2})e^{i\xi t} + \mathrm{H.c.} \tag{78}$$

that can be rewritten in a more comfortable way like this

$$H_{\mathrm{d/rb}} = \hbar\Omega\eta\sigma_+ e^{i(\phi_1 + \frac{\pi}{2})}(a + a^\dagger e^{i\varphi})e^{i\xi t} + \mathrm{H.c.}, \tag{79}$$

where $\varphi = \phi_2 - \phi_1$. Now, if we choose $\phi_1 = -\pi$ and $\varphi = \pi$, the resulting Hamiltonian is

$$H_{\mathrm{d/rb}} = \hbar\Omega\eta(\sigma_+ e^{i\xi t} + \sigma_- e^{-i\xi t})\frac{a - a^\dagger}{i}. \tag{80}$$

The Hamiltonian (79) has some similarity with the $1+1$ dimensional Dirac Hamiltonian, except for the detuning ξ, but, actually this detuning is crucial, because it has a close

relation with the mass term ($mc^2\sigma_z$) of the Dirac Hamiltonian.
It can be probed without much work, that any Hamiltonian of the type

$$H = \hbar\chi_1(\sigma_+ e^{i\chi_2 t} + \sigma_- e^{-i\chi_2 t}) \tag{81}$$

where $[\chi_1, \sigma_i] = 0 = [\chi_2, \sigma_i]$ for $i = x, y, z$. If we go to another picture defined by

$$|\psi'\rangle = e^{-i\frac{\chi_2}{2}\sigma_z t}|\psi\rangle, \tag{82}$$

then, the Hamiltonian in the new picture is

$$H' = \hbar\chi_1(\sigma_+ + \sigma_-) + \frac{\hbar\chi_2}{2}\sigma_z. \tag{83}$$

In our problem, this means that if we go to the picture defined by

$$|\psi\rangle_{II} \equiv e^{-i\frac{\xi}{2}\sigma_z t}|\psi\rangle_I \tag{84}$$

the Hamiltonian (80) will become, in the new picture

$$H^{II} = \hbar\Omega\eta(\sigma_+ + \sigma_-)\frac{a - a^\dagger}{i} + \frac{\hbar\xi}{2}\sigma_z. \tag{85}$$

This Hamiltonian already looks quite similar to the $1+1$ dimensional Dirac Hamiltonian, but still we need to define the momentum and the position operators for the simulated particle. We do it in the following way

$$x = \Delta_x(a + a^\dagger) \qquad p = \frac{\hbar}{2\Delta_x}\frac{a - a^\dagger}{i}, \tag{86}$$

which fulfill the commutation relation $[x, p] = i\hbar$.
Now, replacing $\sigma_+ + \sigma_- = \sigma_x$, we can rewrite the Hamiltonian in the following form

$$H^{II} = 2\eta\Delta_x\Omega(\sigma_+ + \sigma_-)p + \frac{\hbar\xi}{2}\sigma_z \tag{87}$$

and compare it with the $1+1$ dimensional Dirac Hamiltonian

$$H_{\mathrm{D}} = cp\sigma_x + mc^2\sigma_z. \tag{88}$$

We can the the two coefficients of Eq. (87) with the two of Eq. (88)

$$mc^2 \equiv \frac{\hbar\xi}{2} \qquad c \equiv 2\eta\Delta_x\Omega \tag{89}$$

as the particles rests energy and the speed of light respectively. The evolution of $|\psi\rangle_{II}$ is given by

$$i\hbar\partial_t|\psi\rangle_{II}(t) = H^{II}|\psi\rangle_{II}(t) \tag{90}$$

which is actually a $1+1$ dimensional Dirac equation for a certain mass m and speed of light c. So far, we have engineered the quantum simulation of the Dirac equation in trapped ions, making it behave as if it was a relativistic particle. This has already been implemented in the laboratory [1].

5 Implementation of the Foldy-Wouthuysen transformation in trapped ions

In chapter 2.3 we have already pointed out that one of the new concepts that the Dirac equation brings us is the existence of negative energy eigenvalues, that is, particles with negative energy. According to that, the Hilbert space of the Dirac Hamiltonian can be seen as the orthogonal direct sum of two subspaces: the positive energies subspace and the negative energies subspace. In fact the *zitterbewegung* is a consequence of the "interference" between this two subspaces; so if we take a state from one of this two subspaces, this state does not show *zitterbewegung*.
Using what is called the Foldy-Wouthuysen transformation we are able to diagonalize the Dirac Hamiltonian. Diagonalizing the Hamiltonian the "mixed" terms between the positive energy and negative energy subspaces vanish, and the transformed states do not show *zitterbewegung*.

The objective of this chapter is the study of the implementation of this transformation in trapped ions.

5.1 The Foldy-Wouthuysen transformation in 1+1 dimensions

The Foldy-Wouthuysen transformation in $1+1$ dimensions is a unitary transformation of the form

$$|\psi\rangle \to |\psi\rangle' = W_{\mathrm{FW}}|\psi\rangle \tag{91}$$

where

$$W_{\mathrm{FW}} = e^{i\sigma_y\theta} = \cos\theta + i\sigma_y \sin\theta. \tag{92}$$

If the transformation is applied tho the Dirac Hamiltonian,

$$H'_{\mathrm{D}} \equiv W_{\mathrm{FW}} H_{\mathrm{D}} W_{\mathrm{FW}}^{-1} \tag{93}$$

the transformed Hamiltonian has the form

$$H'_{\mathrm{D}} = (\cos\theta + i\sigma_y \sin\theta)(cp\sigma_x + mc^2\sigma_z)(\cos\theta - i\sigma_y \sin\theta). \tag{94}$$

Using the anticommutation relation $\{\sigma_i, \sigma_j\} = 2\mathbb{I}_2^{\sigma}\delta_{ij}$ and imposing that θ must commute with p and $\sigma_{x,y,z}$ Eq. (94) is rewritten as

$$H'_{\mathrm{D}} = (cp\sigma_x + mc^2\sigma_z)(\cos\theta - i\sigma_y \sin\theta)^2 \tag{95}$$

and using

$$(\cos\theta - i\sigma_y \sin\theta)^2 = (e^{-i\sigma_y\theta})^2 = e^{-i\sigma_y 2\theta} = \cos 2\theta - i\sigma_y \sin 2\theta \tag{96}$$

the transformed Hamiltonian can be rewritten as

$$H'_{\rm D} = \sigma_x[cp\cos 2\theta - mc^2 \sin 2\theta] + \sigma_z[mc^2 \cos 2\theta + cp \sin 2\theta]. \tag{97}$$

We can obtain a diagonalized $H'_{\rm D}$, by choosing θ such that the σ_x term in (97) vanishes; that corresponds to

$$\tan 2\theta \equiv p/m. \tag{98}$$

With that particular choice, the transformed Hamiltonian becomes

$$H'_{\rm D} = \sqrt{p^2c^2 + m^2c^4}\,\sigma_z \tag{99}$$

which is diagonal in the basis of eigenfunctions of σ_z

$$H'_{\rm D} = \begin{pmatrix} \sqrt{p^2c^2 + m^2c^4} & 0 \\ 0 & -\sqrt{p^2c^2 + m^2c^4} \end{pmatrix}. \tag{100}$$

The Foldy-Wouthuysen transformation is the continuous transformation defined by (91), for any θ. The specific choice in (98), takes us to the Newton-Wigner (NW) representation.

As we have done in subsection 2.4, we want to obtain the velocity operator, but now we are in the Newton-Wigner representation. So the velocity operator is defined as

$$v_{\rm NW} \equiv \frac{dx}{dt} = i[H'_{\rm D}, x] = i\sigma_z[h_{\rm D}, x] \tag{101}$$

where $h_{\rm D} \equiv \sqrt{p^2c^2 + m^2c^4}$. Using that $[h_{\rm D}^2, x] = [p^2, x]c^2 = -2ipc^2$, we arrive at

$$[h_{\rm D}^2, x] = h_{\rm D}[h_{\rm D}, x] + [h_{\rm D}, x]h_{\rm D} = -2ipc^2. \tag{102}$$

We need an expression for $[h_{\rm D}, x]$ that satisfies (102). It is straightforward to verify that the following,

$$[h_{\rm D}, x] = -i\frac{pc^2}{h_{\rm D}} \tag{103}$$

satisfies this relation. So, we obtain that the velocity operator in the Newton-Wigner representation is

$$v_{\rm NW} = \frac{pc^2}{h_{\rm D}}\sigma_z = c^2 p (H'_{\rm D})^{-1}. \tag{104}$$

The evolution of the operator is given by

$$v_{\mathrm{NW}}(t) = e^{iH'_{\mathrm{D}}t} v_{\mathrm{NW}} e^{-iH'_{\mathrm{D}}t} \tag{105}$$

and given that $[H'_{\mathrm{D}}, p\sigma_z/h_{\mathrm{D}}] = 0$, we conclude that the operator remains constant in time. As it is obvious, the Newton-Wigner velocity operator does not have any oscillating term, which in Eq. (37) was the origin of the *zitterbewegung*. Integrating the velocity operator we obtain the position operator at time t:

$$x_{\mathrm{NW}}(t) = \int_0^t d\tau v_{\mathrm{NW}}(\tau) = v_{\mathrm{NW}} \int_0^t d\tau = x_{\mathrm{NW}} + c^2 p (H'_{\mathrm{D}})^{-1} t. \tag{106}$$

This operator, unlike the ordinary position operator in Eq. (39) does not show *zitterbewegung*.

5.2 Implementation in trapped ions

What we have to do is to find a way to generate the unitary transformation W_{FW} in trapped ions,which has the form

$$W_{\mathrm{FW}} = e^{i\sigma_y \theta} = e^{i\sigma_y \frac{1}{2} \arctan \frac{p}{mc}}. \tag{107}$$

Suppose for a moment that we knew how to engineer, in trapped ions, the dynamics that corresponds to a Hamiltonian of the type

$$H = -\hbar g \sigma_y \frac{1}{2} \arctan \frac{p}{mc}. \tag{108}$$

In this case, the time evolution operator would be given by

$$U(t) = e^{-i\frac{Ht}{\hbar}} = e^{i\sigma_y \frac{1}{2} \arctan \frac{p}{mc} gt}. \tag{109}$$

Letting the initial state evolve until $t = \frac{1}{g}$ or doing our Foldy-Wouthuysen transformation is mathematically equivalent. Unfortunately we do not know how to implement a dynamic that corresponds to the Hamiltonian (108). But we can make some approximations.
Suppose that $\langle p \rangle \ll mc$, in this case it is rightful to make a Taylor expansion of the arctangent function and neglect all terms except the first one,

$$\arctan \frac{p}{mc} \simeq \frac{p}{mc}. \tag{110}$$

So in the case in which the condition $\langle p \rangle \ll mc$ is fulfilled, our Foldy-Wouthuysen transformation can be written as (we will call it the approximate Foldy-Wouthuysen transformation)

$$W^{ap}_{\mathrm{FW}} = e^{i\sigma_y \frac{p}{2mc}}. \tag{111}$$

This expression looks more familiar because the dynamics that corresponds to "$\sigma_y p$ kind" Hamiltonians are feasible in the trapped ions platforms. As we have learned in subsection 4.2, if we excite simultaneously the first red and blue sidebands, we combine a James-Cummings Hamiltonian with an anti-Jaynes-Cummings Hamiltonian. But, unlike in subsection 4.2, this time we are going to choose the *resonant* ($\xi = 0$) JC and anti-JC interactions, with laser phases $\phi_1 = \pi/2$ and $\phi_2 = -\pi/2$. Making those substitutions in the Lamb-Dicke Hamiltonian (77), we get

$$H_{\sigma_y p} = -2\hbar\Omega\eta\Delta_x\sigma_y p. \tag{112}$$

According tho this, to generate the unitary operation

$$U_{\sigma_y p}(t) = e^{i\frac{2\sigma_y p\Omega\eta\Delta_x t}{\hbar}} \tag{113}$$

what we have to do is to generate the dynamics that corresponds to Hamiltonian (112) until a time t and immediately stop dynamic. The question now is, what has to be the value of t in order for this transformation to be equivalent to the proposed Foldy-Wouthuysen transformation. The answer is given by this equation

$$2\Omega\eta\Delta_x t = \frac{1}{2mc}, \tag{114}$$

where m and c are determined by the relations in Eq. (89). Substituting those relations on the equation (115) we get

$$t_{\mathrm{FW}} = \frac{1}{\xi}. \tag{115}$$

The discussion now may be if those times are feasible in trapped ions. On the one hand, the ion has a quantum coherence time, and on the other hand we have the lasers, that they need a minimum time to produce the excitation of the sidebands. Since the detuning ξ must be comparable with the coupling constant of the interaction between the ion and the laser; these frequencies are typically of $10 - 100kHz$. That means that $t_{\mathrm{FW}} \sim 10 - 100\mu s$. They have been able to generate JC type dynamics in time scales of nanoseconds [5], and the internal state coherence time for ions is $\sim ms$ [6]. So we can conclude that the generation of such a dynamic during time t_{FW} is feasible in trapped ions platform.
So, in the case that $\langle p\rangle \ll mc$,

$$U_{\sigma_y p}(t = \frac{1}{\xi}) = W^{ap}_{\mathrm{FW}}. \tag{116}$$

We also know that the measurement of the expected value of the position of the simulated Dirac particle has been done [1]. Now, before doing the measurement we would have to

apply the evolution that corresponds to the Hamiltonian (112) and stop it at $t = \frac{1}{\xi}$. What we would get it would be

$$|\psi\rangle'(t_0) = U_{\sigma_y p}(t = \frac{1}{\xi})|\psi\rangle(t_0) = W_{\mathrm{FW}}^{ap}|\psi\rangle(t_0) \tag{117}$$

and now, if we measure the expectation value of this state we are actually measuring

$$\langle\psi|(W_{\mathrm{FW}}^{ap})^{-1} x W_{\mathrm{FW}}^{ap}|\psi\rangle(t_0) \tag{118}$$

which is nothing but the expected value of the approximate Newton-Wigner position operator at time t_0

$$\langle x_{\mathrm{NW}}^{ap}\rangle(t_0) = \langle\psi|x_{\mathrm{NW}}^{ap}|\psi\rangle(t_0). \tag{119}$$

If we plot this quantity as a function of time, it should not show oscillations when $\langle p\rangle \ll mc$. In Figure 1, we have plot the expectation values of the position operator, the theoretical Newton-Wigner position operator and the approximate Newton-Wigner position operator for different masses as a function of time. As initial state we have chosen $|0\rangle \otimes |+\rangle_x$, where $|0\rangle$ is the $n = 0$ Fock state, and $|+\rangle_x$ is the eigenket of σ_x with positive eigenvalue. Instead of the mass, we have define a variable R which is proportional to the mass but has no dimensions. R is formally defined as

$$R \equiv \frac{mc}{p_0} \tag{120}$$

where $p = p_0 \frac{(a-a^\dagger)}{i}$, is the momentum of the simulated particle that we have defined in Eq. (86). We have also considered instead of time the dimensionless parameter $\eta\Omega t$.

We have plotted the theoretical expectation value of x_{NW} only to compare it with our approximation, and to see how bad it gets when the mass gets getting smaller. At $R = 6$ and $R = 3$ for example, the expectation value of the approximate Newton-Wigner position operator is consistent with the theoretical curve; at $R = 1$ it gets a little worse, but still is consistent with the theoretical curve; however at $R = 0.5$ the approximation is not good anymore.
Another useful magnitude to evaluate the validity of the approximation is to define the fidelity at a certain time t_0 in the following way

$$Fi = \langle\psi_{\mathrm{FW}}^{ap}|\psi_{\mathrm{FW}}\rangle(t_0) = \langle\psi|(W_{\mathrm{FW}}^{ap})^{-1} W_{\mathrm{FW}}|\psi\rangle(t_0). \tag{121}$$

The fidelity should get closer to 1 when the approximation gets better.

In figure 2, we have ploted the fidelity at a certain time, and we can clearly see clearly that for $R = 1$ the fidelity is close to one. Also we can say that at $R = 0.5$ the approximation is not good anymore. The curve shows a logarithmic behavior, that justifies the approximation for any value of R greater than one.

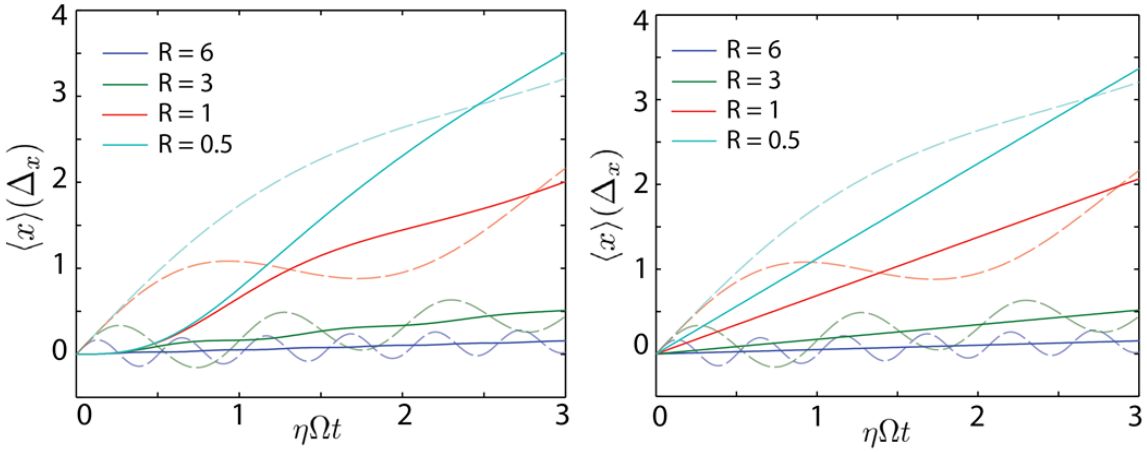

Figure 1: **Left:** Solid lines represent the expectation value of the approximate Newton-Wigner position operator for the initial state $|0\rangle \otimes |+\rangle_x$, as a function of time. **Right:** Solid lines represent the expectation value of the theoretical Newton-Wigner operator for the initial state $|0\rangle \otimes |+\rangle_x$, as afunction of time. The dashed lines, in **left** and **right**, represent the expectation value of the ordinary position operator as a function of time, for the same initial state.

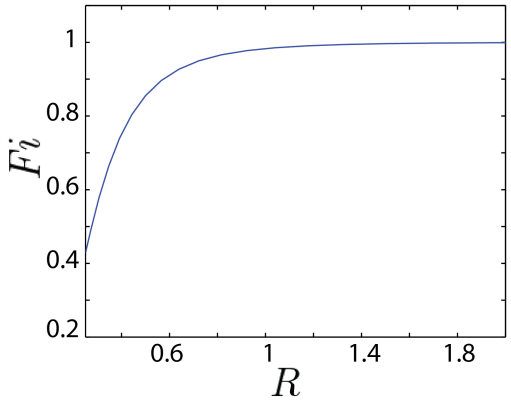

Figure 2: Fidelity as a function of R at time $\eta\Omega t = 1.85$, with the initial state $|0, +\rangle_x$.

6 *Zitterbewegung:* initial state and evolution

In this chapter, we will analyze the behaviour of the expected value of the standard position operator for different initial states. The framework in which we are, invites us to use typical states from a quantum optics scenario and study their behavior with a Hamiltonian typical from RQM. We will attempt to clarify what is the role of the motional part of the initial wave function, independently of the spin part, and viceversa. Finally, we also consider initial states where the motional and the spinorial degrees of freedom are entangled.

6.1 Fock states

Let us start analyzing the behaviour of $\langle x\rangle(t)$ for different spinorial states. We have chosen the eigenvalues of $\sigma_z(|\pm\rangle)$, $\sigma_x(|\pm\rangle_x)$ and of $\sigma_y(|\pm\rangle_y)$, together with a Fock state ($|n\rangle$).
To obtain the plots of figure 3 what we have to compute numerically is the quantity

$$\langle x\rangle(t) = \langle\psi_0|e^{iH_{\rm D}t}xe^{-iH_{\rm D}t}|\psi_0\rangle \tag{122}$$

where $H_{\rm D}$ is the $1+1$ Dirac Hamiltonian and $\hbar = 1$.

At a first view, we clearly see the oscillations associated with the *zitterbewegung*. We also observe that, in consistency with the theoretical expression in Eq. (39), when $m \to \infty$, the expectation value, like the average velocity, goes to zero. We call average velocity to the slope of the linear contribution of the expectation value of $x(t)$ (39), around which the position oscillates.
To give a deeper insight to figure 3 we need to extract information from the operator $x(t)$; we have already the theoretical expression in Eq. (39), but this expression does not help us very much. In general, it is not easy to simplify the expression for the operator $x(t) = e^{iH_{\rm D}t}xe^{-iH_{\rm D}t}$, since the expression of $[(H_{\rm D})^n, x]$ becomes more complicated as n increases. But, still we can extract some interesting information, and predict or explain some of the behaviours shown in figure 3. First of all we shall remember that the Dirac Hamiltonian fulfils the following ($c = 1$)

$${H_{\rm D}}^2 = (p^2 + m^2) \otimes \mathbb{I}_2^\sigma. \tag{123}$$

That means that the even powers of $H_{\rm D}$ do not affect on the spin state. If we define $h_{\rm D}$ as

$$h_{\rm D} = \sqrt{p^2 + m^2} \tag{124}$$

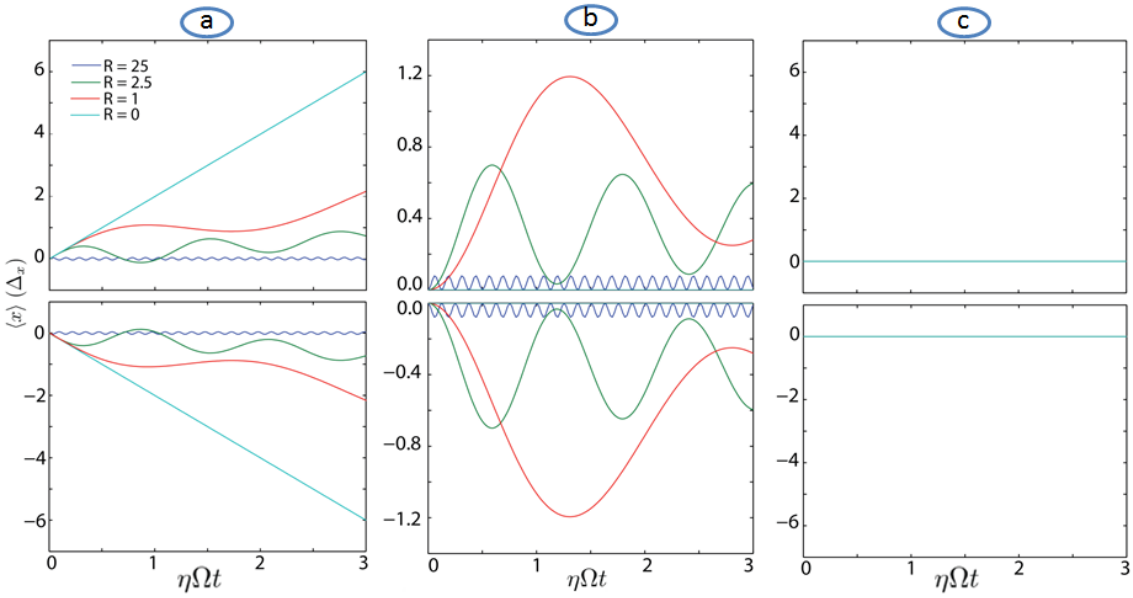

Figure 3: **a:** It is shown the expectation value of the standard position operator $\langle x \rangle$ versus time, for the states $|0,+\rangle_x$ (**up**) and $|0,-\rangle_x$ (**down**), and for different masses. **b:**$\langle x \rangle$ versus time, for the states $|0,-\rangle_y$ (**up**) and $|0,+\rangle_y$ (**down**), and for different masses. **c:**$\langle x \rangle$ versus time, for the states $|0,+\rangle$ (**up**) and $|0,-\rangle$ (**down**), and for different masses.

then the following is true for all integer n:

$$H_{\mathrm{D}}{}^{2n} = h_{\mathrm{D}}{}^{2n} \otimes \mathbb{I}_2^{\sigma}. \tag{125}$$

Using the binomial theorem we can easily reach the expression for any even power of h_{D} and in consequence of H_{D}

$$h_{\mathrm{D}}{}^{2n} = (p^2 + m^2)^n = \sum_{k=0}^{n} \binom{n}{k} m^{2k} p^{2(n-k)}. \tag{126}$$

Now let us consider the operator $e^{-iH_D t}$. We can separate in the following way the odd and the even powers of the expansion

$$e^{-iH_{\mathrm{D}}t} = \sum_{k=0}^{\infty} \frac{(-1)^n (t)^{2n}}{(2n)!} H_{\mathrm{D}}{}^{2n} - i \sum_{k=0}^{\infty} \frac{(-1)^n (t)^{2n+1}}{(2n+1)!} H_{\mathrm{D}}{}^{2n+1}, \tag{127}$$

which is equivalent to

$$e^{-iH_{\mathrm{D}}t} = \sum_{k=0}^{\infty} \frac{(-1)^n (t)^{2n}}{(2n)!} h_{\mathrm{D}}{}^{2n} \otimes \mathbb{I}_2^{\sigma} - i(\sum_{k=0}^{\infty} \frac{(-1)^n (t)^{2n+1}}{(2n+1)!} h_{\mathrm{D}}{}^{2n} \otimes \mathbb{I}_2^{\sigma}) H_{\mathrm{D}}, \tag{128}$$

and defining Γ_1 and Γ_2 as

$$\begin{aligned} \Gamma_1(p^2, t) &\equiv \sum_{k=0}^{\infty} \frac{(-1)^n (t)^{2n}}{(2n)!} h_{\mathrm{D}}{}^{2n} = \cos h_{\mathrm{D}} t \\ \Gamma_2(p^2, t) &\equiv -i \sum_{k=0}^{\infty} \frac{(-1)^n (t)^{2n+1}}{(2n+1)!} h_{\mathrm{D}}{}^{2n} = -\frac{i}{h_{\mathrm{D}}} \sin h_{\mathrm{D}} t \end{aligned} \tag{129}$$

and rewriting the expression with them, we have

$$e^{-iH_{\mathrm{D}}t} = \Gamma_1 \otimes \mathbb{I}_2^{\sigma} + \Gamma_2 p \otimes \sigma_x + \Gamma_2 m \otimes \sigma_z. \tag{130}$$

Of course Γ_1 and Γ_2 have nothing to do with spin, and are even functions in p (are invariant under the transformation $p \to -p$). It can easily be checked that $\Gamma_1^{\dagger} = \Gamma_1$, $\Gamma_2^{\dagger} = -\Gamma_2$ and highlight this relation:

$$\partial_t \Gamma_2 = -i\Gamma_1. \tag{131}$$

With this new expression of the evolution operator, where the motional and spinorial parts can be treated separately, we will attempt to explain why in figure 3 the initial states with

the spin part $|\pm\rangle$ give $\langle x\rangle(t) = 0$ for all times.
For the motional part, we choose a Fock state. The probability density $|\langle n|p\rangle|^2 = |\psi_n(p)|^2$ is always symmetric; remember that in this case the expectation value of an operator A odd in p, is zero:

$$\langle\psi|A|\psi\rangle = 0 \qquad \text{if} \qquad \langle \mathrm{n}|-\mathrm{p}\rangle = \pm\langle \mathrm{n}|\mathrm{p}\rangle,\ \langle -\mathrm{p}|\mathrm{A}|-\mathrm{p}\rangle = -\langle \mathrm{p}|\mathrm{A}|\mathrm{p}\rangle. \tag{132}$$

To calculate the evolution of the initial state $|n\rangle\otimes|+\rangle$, we must remember that $\sigma_z|+\rangle = |+\rangle$ and $\sigma_x|+\rangle = |-\rangle$,

$$|n,+\rangle \ \rightarrow \ \ e^{-iH_D t}|n,+\rangle = (\Gamma_1 + m\Gamma_2)|n,+\rangle + p\Gamma_2|n,-\rangle \tag{133}$$

and so, as $\langle+|-\rangle = 0$, the expectation value consists in evaluating this two terms

$$\langle n|(\Gamma_1 - m\Gamma_2)x(\Gamma_1 + m\Gamma_2)|n\rangle\ ,\ \langle n|p\Gamma_2 x\Gamma_2 p)|n\rangle, \tag{134}$$

that turn out to be zero both, because of this particular combination of odd and even functions ($x = -i\partial_p$ is an odd function in p). In the case of $|n\rangle\otimes|-\rangle$, we recover the same. Furthermore, we can say that this happens not only for Fock states, but for any motional state that has a symmetric probability density in p.

To have a more general vision of the dependence with respect to the spin state, we shall calculate the expression for $x(t) = e^{iH_D t}xe^{-iH_D t}$, although the expression in essence is complicated:

$$x(t) = (\Gamma_1\otimes\mathbb{I}_2^\sigma - \Gamma_2 p\otimes\sigma_x - \Gamma_2 m\otimes\sigma_z)x(\Gamma_1\otimes\mathbb{I}_2^\sigma + \Gamma_2 p\otimes\sigma_x + \Gamma_2 m\otimes\sigma_z) \tag{135}$$

using $\sigma_x\sigma_z = -i\sigma_y$, we can say that the expression splits into this four spin parts,

$$\begin{aligned} x(t) = (\Gamma_1 x\Gamma_1 - m^2\Gamma_2 x\Gamma_2 - p\Gamma_2 xp\Gamma_2)\mathbb{I}_2^\sigma \\ +m(\Gamma_1 x\Gamma_2 - \Gamma_2 x\Gamma_1)\sigma_z + (\Gamma_1 xp\Gamma_2 - \Gamma_2 px\Gamma_1)\sigma_x + m(\Gamma_2)^2\sigma_y \end{aligned} \tag{136}$$

For the commented symmetric motional states, the parts that go with $\mathbb{I}_2^\sigma$ and with σ_z are zero, since $\Gamma_1 x\Gamma_1$, $\Gamma_2 x\Gamma_2$, $p\Gamma_2 xp\Gamma_2$, $\Gamma_1 x\Gamma_2$ and $\Gamma_2 x\Gamma_1$ are odd functions in p. The ones that go with σ_x and σ_y, are even functions, so they have a value, depending of course on time, mass and in general on the form of the wavefunction $\psi(p)$. Thus, we can see more clearly why the initial states that were eigenfunctions of σ_x and the ones that were eigenfunctions of σ_y did not gave null expected values. Moreover, the analytical expression for this two functions are given by:

$$\begin{aligned} \langle x\rangle_x(t) = \langle n|\Gamma_1 xp\Gamma_2 - \Gamma_2 px\Gamma_1|n\rangle \\ \langle x\rangle_y(t) = m\langle n|\Gamma_2^2|n\rangle \end{aligned} \tag{137}$$

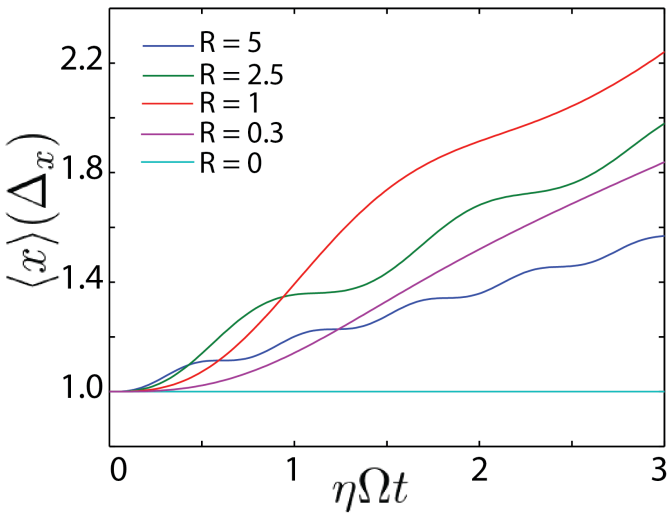

Figure 4: The expectation value of the standard position operator versus time, for different masses and for the initial state $|\alpha_0\rangle \otimes |+\rangle$ (where $\alpha_0 = 0.5 + i$)

6.2 Coherent states

A coherent state $|\alpha\rangle$ is an eigenstate of the annihilation operator a

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{138}$$

with eigenvalue α. The eigenvalue will be in general complex $\alpha = |\alpha|e^{i\theta}$, since the annihilation operator is not Hermitian. The coherent state can be written like a combination of infinite Fock states:

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{139}$$

In fact, a coherent state can be understood as a $|0\rangle$ Fock state displaced in the phase space. While the $|0\rangle$ state is centred in the centre of the phase space $(x, p) = (0, 0)$, the coherent state $|\alpha\rangle$ is centred in in the point $(x, p) = (|\alpha| \cos\theta, |\alpha| \sin\theta)$.

The wavefunction of a coherent state, in general, has no symmetry with respect to p, therefore we can not use the argument of symmetry like in the previous section; not at least for a single coherent state. To probe that, in figure 4 we have plotted the expectation value of the position for a certain coherent initial state with spin state $|+\rangle$.

As we expected, the function is not null for all times. This is because the wavefunction

has no symmetry with respect to p anymore. But if we look at the behaviour of the curves, there is another remarkable thing: in both limits $m \to 0$ and $m \to \infty$, the average velocity goes to zero; that means that at certain mass ($m = m_{max}$), the average velocity module is maximum. That is not what happens in the case of $|\pm\rangle_x$; in this case the behaviour is similar in both coherent or Fock state cases; in this case the maximum value of the average velocity module (c) is reached at $m = 0$.
The result of m_{max} is very interesting because what it means in terms of a Dirac particle, is that if we have a particle with such an initial state, the mass at which it is going to obtain the maximum average velocity is not zero. Remember that the momentum is determined by the initial state, so it would be logical that the fastest particle is the one that has the smallest mass.
Let us write the analytical expression (136) for each spinorial initial state with a coherent motional state:

$$\begin{aligned} \langle x\rangle_z(t) &= f(t) + m\langle\alpha|\Gamma_1 x\Gamma_2 - \Gamma_2 x\Gamma_1|\alpha\rangle \\ \langle x\rangle_x(t) &= f(t) + \langle\alpha|\Gamma_1 xp\Gamma_2 - \Gamma_2 px\Gamma_1|\alpha\rangle \\ \langle x\rangle_y(t) &= f(t) + m\langle\alpha|\Gamma_2^2|\alpha\rangle \end{aligned} \tag{140}$$

where $f(t) = \langle\alpha|\Gamma_1 x\Gamma_1 - m^2\Gamma_2 x\Gamma_2 - p\Gamma_2 xp\Gamma_2|\alpha\rangle$. The parts that go with σ_z, σ_x and σ_y seem to be imperative in the determination of the behaviour of the initial state, because $f(t)$ is an evolution function shared by all, and the three cases are quite different. The two cases that show the maximum average velocity in ($m = m_{max}$), that are in fact $\langle x\rangle_z(t)$ and $\langle x\rangle_y(t)$, have both an m outside of the integral. This, is useful of course to understand why the two cases give null expectation value when $m \to 0$, apart from $m \to \infty$.

Those analytic expressions however are not trivial, and we do not see how to find m_{max} in a analytic way.

6.3 Entangled states

If a quantum system has more than one degree of freedom -like in our case, motional and spinorial-, sometimes it is possible to write the whole state like the tensor product of two states, each one associated only with one of the degrees of freedom, and therefore belonging to different subspaces,

$$|\psi\rangle = |\psi_x\rangle \otimes |\theta\rangle_\sigma. \tag{141}$$

In $|\psi_x\rangle$, is all the information about the motional state and the information about the spin is held entirely in $|\theta\rangle_\sigma$. It is said that a state is entangled, if it is not possible to express it as a tensor product of two different vectors of different subspaces. All states that we have consider were tensor products of motional and spin state; for example, $|0\rangle \otimes |+\rangle$ or $|1\rangle \otimes |-\rangle$, thus are not entangled states. But the state

$$|\psi\rangle = |0\rangle|+\rangle + |1\rangle|-\rangle \tag{142}$$

is an entangled state, because we can not write it as a single tensor product of the motional and the spin parts. This kind of states will have new contributions in the expectation value of $x(t)$, because apart from the already discussed contributions $|0\rangle|+\rangle$ and $|1\rangle|-\rangle$, we have some new terms such as

$$|\psi\rangle = \langle 0,+|x(t)|1,-\rangle. \tag{143}$$

If we plot $\langle x\rangle(t)$ for the initial state

$$|\psi\rangle = |0,+\rangle + |1,-\rangle \tag{144}$$

what we obtain is that the expected value is zero for all times. Let us write all the contributions :

$$\begin{aligned}\langle x(t)\rangle = \langle 0|\Gamma_1 x\Gamma_1 - m^2\Gamma_2 x\Gamma_2 - p\Gamma_2 xp\Gamma_2 + m(\Gamma_1 x\Gamma_2 - \Gamma_2 x\Gamma_1)|0\rangle \\ + \langle 1|\Gamma_1 x\Gamma_1 - m^2\Gamma_2 x\Gamma_2 - p\Gamma_2 xp\Gamma_2 - m(\Gamma_1 x\Gamma_2 - \Gamma_2 x\Gamma_1)|1\rangle \\ + \langle 1|\Gamma_1 xp\Gamma_2 - \Gamma_2 px\Gamma_1 + im(\Gamma_2)^2|0\rangle \\ + \langle 0|\Gamma_1 xp\Gamma_2 - \Gamma_2 px\Gamma_1 - im(\Gamma_2)^2|1\rangle\end{aligned} \tag{145}$$

From the first section, we already know that the first two terms are zero because of the symmetry of the wavefuntion. The analysis of the other two terms follows a similar argument.
The wavefunction of the Fock state $|0\rangle$ is even in the p representation, while $|1\rangle$ is odd. Although the operators are even functions in p, the presence of the odd state turns the integrand odd and therefore the integral null.
This is not the case if we choose $|0\rangle$ and $|2\rangle$, or $|1\rangle$ and $|3\rangle$, instead of $|0\rangle$ and $|1\rangle$. Actually, if we do not want this contribution to be zero, we have to choose two different symmetric states or two different antisymmetric states.

In figure 5 we have plotted the expectation value of position operator for the initial state

$$|\psi\rangle = |0,+\rangle + |2,-\rangle. \tag{146}$$

Something similar happens for the initial state $|1,+\rangle + |3,-\rangle$.

Now let us do something similar with coherent states. Figure 6 contains the plot of the

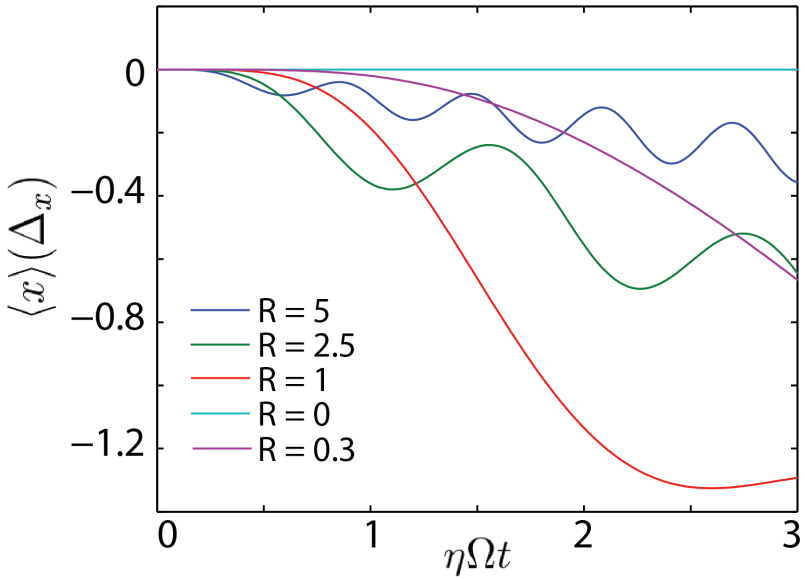

Figure 5: The expectation value of the standard position operator versus time, for different masses and for the initial state $|0,+\rangle + |2,-\rangle$.

expectation value of position for $|\alpha_1\rangle|+\rangle$ ($\alpha_1 = \frac{1}{\sqrt{2}} + i$) and for $|\alpha_2\rangle|+\rangle$ ($\alpha_2 = -\frac{1}{\sqrt{2}} + i$).

Looks like the sum of the two curves should give zero. However if we consider an initial state consisting in the superposition of these two states,

$$|\psi\rangle = |\alpha_1,+\rangle + |\alpha_2,-\rangle \tag{147}$$

the expectation value of the position operator consists in four terms,

$$\begin{aligned} x(t) = &\langle\alpha_1|\Gamma_1 x\Gamma_1 - m^2\Gamma_2 x\Gamma_2 - p\Gamma_2 xp\Gamma_2 + m(\Gamma_1 x\Gamma_2 - \Gamma_2 x\Gamma_1)|\alpha_1\rangle \\ &+ \langle\alpha_2|\Gamma_1 x\Gamma_1 - m^2\Gamma_2 x\Gamma_2 - p\Gamma_2 xp\Gamma_2 - m(\Gamma_1 x\Gamma_2 - \Gamma_2 x\Gamma_1)|\alpha_2\rangle \\ &+ \langle\alpha_2|\Gamma_1 xp\Gamma_2 - \Gamma_2 px\Gamma_1 + im(\Gamma_2)^2|\alpha_1\rangle \\ &+ \langle\alpha_1|\Gamma_1 xp\Gamma_2 - \Gamma_2 px\Gamma_1 - im(\Gamma_2)^2|\alpha_2\rangle \end{aligned} \tag{148}$$

The first two terms will kill each other, but there are still two other terms. In figure 7 we can see a 3D plot of the function (148) versus time and R. It is obvious that the expectation value of the position operator depends strongly in R. At $R = 4$ the function has the typical behaviour that we have seen in figure 5, with a negative classical velocity in this case with more attenuated oscillations. On the contrary, while $R \to 0$ the curve tends asymptotically to a curve which has a classical velocity which is positive and which has no oscillations. That the classical velocity asymptotically tends to a constat value is something familiar,

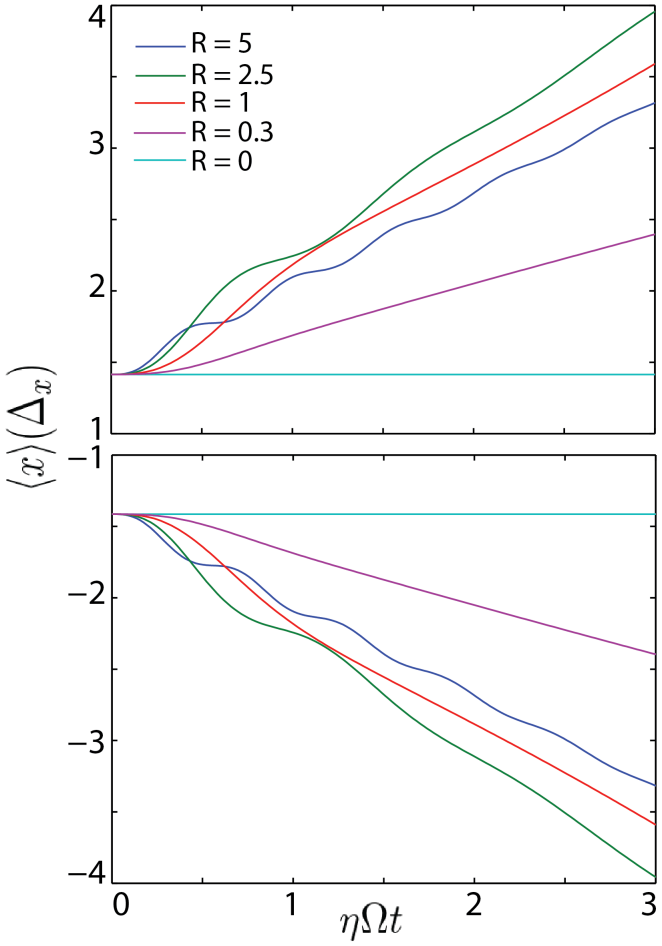

Figure 6: The expectation value of the standard position operator versus time, for different masses and for the initial states $|\alpha_1\rangle|+\rangle$ ($\alpha_1 = \frac{1}{\sqrt{2}} + i$) (**up**) and for $|\alpha_2\rangle|+\rangle$ ($\alpha_2 = -\frac{1}{\sqrt{2}} + i$) (**down**).

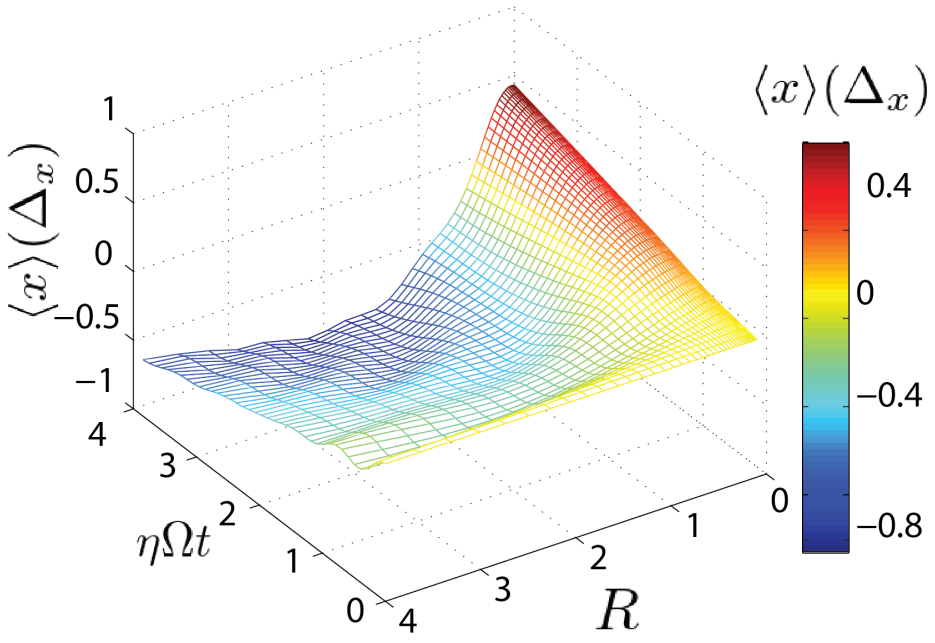

Figure 7: The expectation value of the position operator versus $\eta\Omega t$ and R with the initial state $|\alpha_1, +\rangle + |\alpha_2, -\rangle$ (where $\alpha_1 = \frac{1}{\sqrt{2}} + i$ and $\alpha_2 = -\frac{1}{\sqrt{2}} + i$.)

but in this case this constant value is not (c).
If the average velocity at $R = 0$ is positive and for $R = 4$ is negative, there must be an R_0 for which the average velocity is zero. That would mean that at this certain mass, the particle would stay oscillating around the initial point ($\langle x \rangle(0) = 0$ in this case); if instead R_0 we choose a little bigger $R = R_0 + \epsilon$ the particle will acquire an average velocity in negative direction. The acquired average velocity would be in the opposite direction if $R = R_0 - \epsilon$.
The plot in figure 7 is got if the initial state is the one of Eq. (147). If we change the initial state the behavior of function (148) would be different.

7 Conclusions

The quantum simulation of relativistic quantum mechanics has created a framework where concepts of quantum optics and from RQM coexists. The main objective of this work was to use this framework to exchange concepts of both theories.

In chapter 5 we have proposed a way to implement the Foldy-Wouthuysen transformation in trapped ions. We have concluded that the transformation is feasible with state-of-the-art trapped ions technology, under certain approximations. Moveover, we have provided numerical simulation to validate our proposal, and we have conclude that in a wide regime of masses, the approximation holds and therefore that the *zitterbewegung* disappears.

In chapter 6 we have done a detailed analysis of the evolution and the *zitterbewegung* of different initial states under the $1+1$ dimensional Dirac Hamiltonian. We have selected initial states from quantum optics -like coherent or entangled states-, which have been poorly studied in the framework of RQM. We have realized that the mass in the Dirac equation plays an important role in the evolution of this initial states. We have found that the smallest mass particle is no always the fastest one. Indeed, we have found that certain initial states will evolve in one direction or in the opposite depending on the value of the mass. We have also provided a framework based in the symmetry of the wavefunctions of initial states to predict some features of their evolution.

This works represent only an small step in the field opened by the quantum simulations of RQM. Farther development in this line of research could included, for example, an analysis of the feasibility of the implementation of the ideas developed in chapter 6. It could also be interesting to obtain the exact solution for what we have called m_{max}; the value of the mass for which the classical velocity is maximum for a given initial state.

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