THE MINIMAL OVERLAP COST SHARING RULE

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Abstract. In this paper we introduce a new cost sharing rule-the minimal overlap cost sharing rule-which is associated with the minimal overlap rule for claims problems defined by O'Neill (1982). An axiomatic characterization is given by employing a unique axiom: demand separability. Variations of this axiom enable the serial cost sharing rule (Moulin and Shenker, 1992) and the rules of a family (Albizuri, 2010) that generalize the serial cost sharing rule to be characterized. Finally, a family that includes the minimal overlap cost sharing rule is defined and obtained by means of an axiomatic characterization.

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1. Introduction

One of the most interesting applications of cooperative game theory is in solving cost allocation problems. In this paper we consider cost sharing problems in which a group of agents shares a joint process to produce a certain private good. Each agent demands a quantity q_i of the good. If the cost function is denoted by C, a cost sharing rule allocates the total production cost, i.e. $C\left(\sum_{i\in N} q_i\right)$ among the agents. A new cost sharing rule is introduced and an axiomatic characterization is given. The set of additive cost sharing rules with constant returns is linearly isomorphic (Moulin (2002)) to that of monotonic division rules for claims problems. Our rule is associated by that isomorphism with a well known division rule for claims problems, the minimal overlap rule, which is an extension by O'Neill (1982) of the classical Ibn Ezra's rule. That is why it is called the minimal overlap cost sharing rule.

The new cost sharing rule is characterized by means of a unique axiom, which takes into account that given two agents the demand of the agent who asks for more comprised what the other agent asks for plus the demand difference. The axiom requires the agent with the higher demand to pay what the other agent pays plus

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an allocation corresponding to the remaining demand. This axiom is called demand separability. We also show that a variation of demand separability enable the serial cost sharing rule (Moulin and Shenker (1992)) to be characterized. In addition, all the cost sharing rules of a family introduced by Albizuri (2010), which includes the serial cost sharing rule, can be characterized by variations of demand separability.

Finally, a family of cost sharing rules that includes the minimal overlap cost sharing rule is considered. They are called generalized monotonic Ibn Ezra's cost sharing rules. As mentioned above, the minimal overlap rule for claims problems is a monotonic extension of Ibn Ezra's rule. If all the possible monotonic extensions of Ibn Ezra's rule are considered, the associated isomorphic family of cost sharing rules is precisely the family introduced in this paper. An axiomatic characterization that gives the entire family is proposed.

The rest of the paper is organized as follows. Section 2 introduces the necessary preliminaries. Section 3 defines and characterizes the minimal overlap cost sharing rule, the serial cost sharing rule and the α-serial cost sharing rules. In Section 4 the family formed by the generalized monotonic Ibn Ezra's cost sharing rules is defined and axiomatically characterized. The paper ends with a list of references.

2. Preliminaries

Let U denote a set of potential *agents*. Given a non-empty finite subset N of U, $\mathbb{R}^N$ denotes the $|N|$-dimensional Euclidean space whose axes are labeled with the members of N, $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_i \geq 0\}$ and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. Given $q \in \mathbb{R}^N_+$, if $N = \{1, 2, \dots, n\}$ and $q_1 \leq q_2 \leq \cdots \leq q_n$, then $q_0 = 0$, $q^0 = 0$ and $q^j = (n-j+1)\, q_j + q_{j-1} + ... + q_1$ for every $j \in N$. If $q \in \mathbb{R}^N_+$ and $S \subseteq N$, $q(S) = \sum_{i \in S} q_i$, and $q_S \in \mathbb{R}^S_+$ satisfies $(q_S)_i = q_i$ for all $i \in S$. If $q \in \mathbb{R}^N_+$ and $x \in \mathbb{R}_+$, $q^{-x} \in \mathbb{R}^N_+$ satisfies $(q^{-x})_i = \max\{q_i - x, 0\}$ for all $i \in N$.

A triple (N, q, C) is called a *cost sharing problem*, if N is a non-empty finite subset of U (the set of agents involved in the problem), $q \in \mathbb{R}^N_+$ (the demand profile of the cost sharing problem) and C is a nondecreasing function defined on $\mathbb{R}_+$ such that $C(0) = 0$ (the cost function of the cost sharing problem).

The set of all cost sharing problems with the foregoing properties is denoted by Γ^U, and Γ^N denotes the subset of Γ^U formed by the cost sharing problems with a set of agents N.

A *cost sharing rule* φ on a subset Γ of Γ^U associates a vector $\varphi(N, q, C) \in \mathbb{R}^N_+$ with each $(N, q, C) \in \Gamma$ satisfying

$$\sum_{i \in N} \varphi_i(N, q, C) = C\,(q\,(N)) \qquad \text{(efficiency)}.$$

Hence, a cost sharing rule allocates total cost among the n agents.

On the other hand, a *claims problem* (or bankruptcy or rationing problem) with a set of claimants $N \subseteq U$ is an ordered pair (q, E) where $q \in \mathbb{R}_+^N$ specifies a claim $q_i \in \mathbb{R}_+$ for each agent i, and $0 \leq E \leq \sum_{i \in N} q_i$ represents the amount to be divided.

The space of all claims problems is denoted by C^U, and C^N denotes the set of all claims problems with set of claimants N.

A *division rule* (or bankruptcy rule) is a function that associates a vector $r(q, E) \in \mathbb{R}_+^N$ specifying an award for each agent i with each claims problem $(q, E) \in C^N$ such that $0 \leqslant r(q, E) \leqslant q$ and $\sum_{i \in N} r_i(q, E) = E$.

The minimal overlap rule is a division rule that provides each agent with the sum of the partial awards from the various units to which he/she has laid claim, where for each unit equal division among all agents claiming it prevails and claims are arranged on specific parts of the amount available, called units, so that the number of units claimed by exactly one claimant is maximized, and for each $k = 2, ..., n-1$ successively, the number of units claimed by exactly k claimants is maximized provided that the $k-1$ maximization exercises have been solved.

The minimal overlap rule, introduced by O'Neill (1982), is an extension of Ibn Ezra's rule, which is defined only when the endowment is less than or equal to the largest claim. Following Chun and Thomson (2005), Alcalde et al. (2008) formalize the minimal overlap rule, denoted r^{mo}, as follows:

For each (q, E) and each $i \in N$, assume with no loss of generality that $N = \{1, 2, \ldots, n\}$ and $q_1 \leq \cdots \leq q_n$.

(a) If $E \geq q_n$, then

$$r_i^{mo}(q, E) = \sum_{j=1}^{i} \frac{\min\{q_j, t\} - \min\{q_{j-1}, t\}}{n - j + 1} + \max\{q_i - t, 0\},$$

where t is the unique solution for the equation

$$\sum_{k=1}^{n} \max\{q_k - t, 0\} = E - t.$$

(b) If $E \leq q_n$, then[1]

$$r_i^{mo}(q, E) = \sum_{j=1}^{i} \frac{\min\{q_j, E\} - \min\{q_{j-1}, E\}}{n - j + 1}.$$

Alcalde et al. (2008) prove the following proposition, where r^{IE} denotes Ibn Ezra's rule.

[1]This is the Ibn Ezra's rule.

Proposition 1 (Alcalde et al. (2008))**.** *Let* $(q, E) \in C^N$ *such that* $N = \{1, 2, \ldots, n\}$ *and* $q_1 \leq \cdots \leq q_n$*. Then,*

$$r_i^{mo}(q, E) = r_i^{IE}(q, E') + r_i^{UL}(q - r^{IE}(q, E'), E - E'),$$

where $E' = \min\{E, q_n\}$ *and* r^{UL} *is the uniform losses rule, i.e., for each claims problem* $(\widehat{q}, \widehat{E})$ *and agent* i

$$r_i^{UL}(\widehat{q}, \widehat{E}) = \max\{0, \widehat{q}_i - \lambda\},$$

with λ *satisfying*

$$\sum_{i=1}^{n} \max\{0, \widehat{q}_i - \lambda\} = \widehat{E}.$$

Following the survey by Moulin (2002), it is shown by Moulin and Shenker (1994) that the set of monotonic division rules is linearly isomorphic to that of additive cost sharing rules with constant returns. Fix the agents set N.

A division rule r is monotonic if: $E \leq E' \leq \sum_{i\in N} q_i \Longrightarrow r(E, q) \leq r(E', q)$.

A cost sharing rule φ on Γ is additive if

$$\varphi(N, q, C^1) + \varphi(N, q, C^2) = \varphi(N, q, C^1 + C^2) \text{ for all } q \in \mathbb{R}_+^N \text{ and } C^1, C^2 \in \Gamma,$$

and φ has constant returns if for all $\lambda \in \mathbb{R}_+$ and all $q \in \mathbb{R}_+^N$,

$$C(x) = \lambda x \text{ for all } \lambda \in \mathbb{R}_+ \Longrightarrow \varphi(N, q, C) = \lambda q.$$

Linear isomorphism holds if absolutely continuous cost functions are taken. And it is given by

$$r \to \varphi : \varphi(N, q, C) = \int_0^{q(N)} C'(t)\, dr(q, t). \tag{1}$$

3. The minimal overlap cost sharing rule

Expression (1) allows a cost sharing rule associated with each monotonic division rule for claims problems to be defined. In this section we take the minimal overlap rule and obtain its associated solution.

First, Proposition 1 gives the following alternative definition for the minimal overlap rule, where $G(x) = \min(x, E)$ for all $x \in \mathbb{R}_+$.

$$r_i^{mo}(q, E) = \sum_{j=1}^{i} \frac{(G(q_j) - G(q_{j-1})) + \left(G(\overline{q}_{j-1}) - G(\overline{q}_j)\right)}{n - j + 1},$$

where $\overline{q}_j = \sum_{k=j+1}^{n} q_k - (n - j - 1)\, q_j$, and $\overline{q}_n = \overline{q}_{n-1}$.

Then, considering an absolutely continuous $(N,q,C)\in\Gamma^N$, the associated cost sharing rule with r^{mo} by means of expression (1) satisfies

$$\begin{aligned}\varphi_i(N,q,C) &= \int_0^{q(N)} C'(t)\,dr_i^{mo}(c,t)\\ &= \sum_{j=1}^{i}\left(\int_{q_{j-1}}^{q_j}\frac{C'(t)}{n-j+1}dt+\int_{\overline{q}_j}^{\overline{q}_{j-1}}\frac{C'(t)}{n-j+1}dt\right)\\ &= \sum_{j=1}^{i}\frac{\left(C(q_j)-C(q_{j-1})\right)+\left(C\left(\overline{q}_{j-1}\right)-C\left(\overline{q}_j\right)\right)}{n-j+1}.\end{aligned}$$

This last expression makes sense if we consider Γ^N as the domain of the cost sharing rule. Thus, we give the following definition.

Definition 1. *The minimal overlap cost sharing rule, denoted by φ^{mo}, is defined for each $(N,q,C)\in\Gamma^N$ by*

$$\varphi_i^{mo}(N,q,C)=\sum_{j=1}^{i}\frac{\left(C(q_j)-C(q_{j-1})\right)+\left(C\left(\overline{q}_{j-1}\right)-C\left(\overline{q}_j\right)\right)}{n-j+1}.$$

The minimal overlap cost sharing rule is characterized by means of one axiom.

To state it, given (N,q,C) and $\lambda\in\mathbb{R}_+$, the cost sharing problem $\left(N,q,C^{-\lambda}\right)$ denotes the associated cost sharing problem defined by

$$C^{-\lambda}(x)=C(x+\lambda)-C(\lambda). \tag{2}$$

As can be seen in Fig. 1, $C^{-\lambda}(x)$ measures the cost of x units to be produced when λ units have already been produced.

Fig. 1

To characterize the minimal overlap cost sharing rule we consider the following property.

Demand separability. Let $(N,q,C)\in\Gamma^N$ and $i\in N$. Then, for all $j\in N$ such that $q_j\geq q_i$ it holds

$$\varphi_j(N,q,C)=\varphi_i(N,q,C)+\varphi_j(N,q^{-q_i},C^{-q_i}).$$

According to this axiom the allocation of an agent, say j, whose demand is greater than or equal to that of another, say i, is the sum of the allocation of the latter agent plus the allocation of the former in a residual cost sharing problem. Notice that the demand of agent j is composed of the demand of agent i, i.e. q_i, plus a further quantity. This axiom is a way to reflect this decomposition. It requires agent j to pay what agent i pays plus an allocation corresponding to the remaining demand of agent j. Since the remaining demand of agent j is $q_j - q_i$, it is given by the demand profile q^{-q_i}. The remaining demands of all agents whose demand is greater than or equal to q_i are also considered, and that those agents also pay the allocation of agent i. Therefore, we consider the new demand profile q^{-q_i} in the remaining cost sharing problem. Moreover, since q_i units have already been paid, we consider the cost function C^{-q_i}. Both the new demand profile and the new cost function determine the new cost sharing problem (N, q^{-q_i}, C^{-q_i}), and the corresponding cost shares are again paid by the agents.

Proposition 2. *The minimal overlap cost sharing rule satisfies demand separability.*

Proof. Let $(N, q, C) \in \Gamma^N$ and $i, j \in N$ such that $N = \{1, 2, \ldots, n\}$, $q_1 \leq \cdots \leq q_n$ and $q_j \geq q_i$. If $q_j = q_i$, then by definition

$$\varphi_j^{mo}(N, q^{-q_i}, C^{-q_i}) = 0 = \varphi_j^{mo}(N, q, C) - \varphi_i^{mo}(N, q, C).$$

Assume that $q_j > q_i$. By definition,

$$\begin{aligned} &\varphi_j^{mo}(N, q^{-q_i}, C^{-q_i}) \\ &= \sum_{k=1}^{j} \frac{C^{-q_i}\left(\left(q^{-q_i}\right)_k\right) - C^{-q_i}\left(\left(q^{-q_i}\right)_{k-1}\right)}{n-k+1} \\ &+ \sum_{k=1}^{j} \frac{C^{-q_i}\left(\overline{\left(q^{-q_i}\right)}_{k-1}\right) - C^{-q_i}\left(\overline{\left(q^{-q_i}\right)}_k\right)}{n-k+1} \end{aligned} \tag{3}$$

And since $\left(q^{-q_i}\right)_k = 0$ for $k = 1, ..., i$,

$$\begin{aligned} &\sum_{k=1}^{j} \frac{C^{-q_i}\left(\left(q^{-q_i}\right)_k\right) - C^{-q_i}\left(\left(q^{-q_i}\right)_{k-1}\right)}{n-k+1} \\ &= \sum_{k=i+1}^{j} \frac{C^{-q_i}\left(\left(q^{-q_i}\right)_k\right) - C^{-q_i}\left(\left(q^{-q_i}\right)_{k-1}\right)}{n-k+1}, \end{aligned}$$

and taking into account that

$$C^{-q_i}\left(\left(q^{-q_i}\right)_k\right) = C\left(\left(q^{-q_i}\right)_k + q_i\right) - C\left(q_i\right) = C\left(q_k\right) - C\left(q_i\right)$$

for $k = i, ..., j$, then

$$\sum_{k=1}^{j} \frac{C^{-q_i}\left(\left(q^{-q_i}\right)_k\right) - C^{-q_i}\left(\left(q^{-q_i}\right)_{k-1}\right)}{n-k+1} = \sum_{k=i+1}^{j} \frac{C\left(q_k\right) - C\left(q_{k-1}\right)}{n-k+1}. \tag{4}$$

Let us prove

$$\overline{\left(q^{-q_i}\right)}_k = \overline{q}_k - q_i \tag{5}$$

for $k = i, ..., n$. Distinguish two cases. If $k < n$, by definition,

$$\overline{(q^{-q_i})}_k = \sum_{k'=k+1}^{n} \left(q^{-q_i}\right)_{k'} - (n-k-1)\left(q^{-q_i}\right)_k$$
$$= \sum_{k'=k+1}^{n} (q_{k'} - q_i) - (n-k-1)(q_k - q_i)$$
$$= \sum_{k'=k+1}^{n} q_{k'} - (n-k-1)\, q_k - q_i$$
$$= \overline{q}_k - q_i.$$

And if $k = n$, then also by definition, $\overline{(q^{-q_i})}_n = \overline{(q^{-q_i})}_{n-1} = \overline{q}_{n-1} - q_i$.

On the other hand, if $k = 0, ..., i$, then

$$\overline{(q^{-q_i})}_k = \sum_{k'=i+1}^{n} (q_{k'} - q_i) = \overline{q}_i - q_i. \tag{6}$$

Hence, by (6) it holds that $\overline{(q^{-q_i})}_k = \overline{(q^{-q_i})}_{k'}$ if $k, k' \in \{0, ..., i\}$, and therefore

$$\sum_{k=1}^{j} \frac{C^{-q_i}\left(\overline{(q^{-q_i})}_{k-1}\right) - C^{-q_i}\left(\overline{(q^{-q_i})}_k\right)}{n-k+1}$$
$$= \sum_{k=i+1}^{j} \frac{C^{-q_i}\left(\overline{(q^{-q_i})}_{k-1}\right) - C^{-q_i}\left(\overline{(q^{-q_i})}_k\right)}{n-k+1}.$$

Applying (5) and the definition of C^{-q_i}, the last expression equals

$$= \sum_{k=i+1}^{j} \frac{C\left(\overline{q}_{k-1}\right) - C\left(\overline{q}_k\right)}{n-k+1}. \tag{7}$$

Then, taking into account (3), (4) and (7),

$$\varphi_j^{mo}(N, q^{-q_i}, C^{-q_i})$$
$$= \sum_{k=i+1}^{j} \frac{C(q_k) - C(q_{k-1})}{n-k+1} + \sum_{k=i+1}^{j} \frac{C\left(\overline{q}_{k-1}\right) - C\left(\overline{q}_k\right)}{n-k+1}.$$

Since by definition of φ^{mo},

$$\varphi_j^{mo}(N, q, C) - \varphi_i^{mo}(N, q, C)$$
$$= \sum_{k=i+1}^{j} \frac{\left(C(q_j) - C(q_{j-1})\right) + \left(C\left(\overline{q}_{j-1}\right) - C\left(\overline{q}_j\right)\right)}{n-k+1},$$

then

$$\varphi_j^{mo}(N, q, C) - \varphi_i^{mo}(N, q, C) = \varphi_j^{mo}(N, q^{-q_i}, C^{-q_i}).$$

□

Theorem 1. *The minimal overlap cost sharing rule is the only cost sharing rule on Γ^N that satisfies demand separability.*

Proof. The minimal overlap cost sharing rule satisfies demands separability by the previous Proposition. We prove unicity by induction on the number α of non null demands. Let φ be a cost sharing rule on Γ^N. We assume without loss of generality that $N = \{1, 2, \ldots, n\}$. If $\alpha = 0$, then since φ is a cost sharing rule $\varphi_i(N, q, C) = 0$ for all $i \in N$ and all $(N, q, C) \in \Gamma^N$. We suppose that $\varphi(N, q, C)$ is determined when $\alpha < m$, and let us prove that it is determined when $\alpha = m$. Assume that $q_1 \leq \cdots \leq q_n$. Let $j = n - m = |\{k \in N : q_k = 0\}|$. It holds $j \neq n$. If $j \neq 0$, let $k \in \{1, ..., j\}$. Demands separability implies (notice that $q_k = 0$)

$$\varphi_n(N, q, C) = \varphi_k(N, q, C) + \varphi_n(N, q^{-q_k}, C^{-q_k}) = \varphi_k(N, q, C) + \varphi_n(N, q, C),$$

and therefore

$$\varphi_k(N, q, C) = 0. \tag{8}$$

If $j \neq 0$ or $j = 0$ let us determine $\varphi_k(N, q, C)$ for $k = j + 1, ..., n$. If $j = n - 1$, then $k = n$ and $\varphi_n(N, q, C)$ is determined since φ is a cost sharing rule and (8) is satisfied. If $j < n - 1$, applying demands separability,

$$\varphi_k(N, q, C) = \varphi_{j+1}(N, q, C) + \varphi_k(N, q^{-q_{j+1}}, C^{-q_{j+1}}) \tag{9}$$

for $k = j + 2, ..., n$. Therefore,

$$\sum_{k=j+2}^{n} \varphi_k(N, q, C) = (n - j - 1)\, \varphi_{j+1}(N, q, C) + \sum_{k=j+2}^{n} \varphi_k(N, q^{-q_{j+1}}, C^{-q_{j+1}}),$$

and taking into account (8) and that φ is a cost sharing rule,

$$C\left(q(N)\right) - \varphi_{j+1}(N, q, C)$$

$$= (n - j - 1)\, \varphi_{j+1}(N, q, C) + \sum_{k=j+2}^{n} \varphi_k(N, q^{-q_{j+1}}, C^{-q_{j+1}}).$$

Moreover, since $\left(q^{-q_{j+1}}\right)_k = 0$ for $k = 1, ..., j + 1$ and φ is a cost sharing rule, it holds that

$$\sum_{k=j+2}^{n} \varphi_k(N, q^{-q_{j+1}}, C^{-q_{j+1}}) = C\left(q(N) - (n - j - 1)\, q_{j+1}\right) - C\left(q_{j+1}\right),$$

which substituting in the above expression implies

$$\varphi_{j+1}(N, q, C) = \frac{C\left(q(N)\right) - \left(C\left(q(N) - (n - j - 1)\, q_{j+1}\right) - C\left(q_{j+1}\right)\right)}{n - j},$$

and by (9) and the induction hypothesis, $\varphi_k(N, q, C)$ is determined for $k = j + 1, ..., n$. $\square$

The well known serial cost sharing rule, introduced by Moulin and Shenker (1992), can be characterized by means of an axiom similar to demands separability. With no loss of generality we assume that $N = \{1, 2, \ldots, n\}$ and $q_1 \leq \cdots \leq q_n$. First recall the definition of this rule.

Definition 2 (Moulin and Shenker (1992)). *The serial cost sharing rule, denoted by φ^s, is defined for each $(N, q, C) \in \Gamma^N$ by*

$$\varphi_i^s(N, q, C) = \sum_{j=1}^{i} \frac{C\left(q^j\right) - C\left(q^{j-1}\right)}{n - j + 1}.$$

Given (N, q, C), we take in (2) the value $\lambda = q^i$, that is, we consider C^{-q^i}. Substituting C^{-q_i} by C^{-q^i} in demand separability we obtain this axiom.

Demand separability*. Let $(N, q, C) \in \Gamma^N$ and $i \in N$. Then for all $j \in N$ such that $q_j \geq q_i$ it holds that

$$\varphi_j(N, q, C) = \varphi_i(N, q, C) + \varphi_j(N, q^{-q_i}, C^{-q^i}).$$

As in demand separability, agent j is required to pay what agent i pays plus an allocation corresponding to the remaining demand of agent j. The remaining demands are also given by q^{-q_i}, that is, the new demand profile is q^{-q_i}. But now we take into account the total demand received already by the agents, that is, q^i. And therefore the resulting cost function is C^{-q^i}, which results in a new cost sharing problem (N, q^{-q_i}, C^{-q^i}).

Theorem 2. *The serial cost sharing rule is the only cost sharing rule on Γ^N that satisfies demand separability*.*

Instead of showing Theorem 2, we show a more general one that characterizes α-cost sharing rules (Albizuri (2010)). These rules are also serial like rules, but instead of sharing the increments in the cost of quantities q^j, they share the increments in the cost of portions αq^j, and the cost associated with the rest, i.e. $(1 - \alpha) q^j$ is measured with respect to $q(N)$. Formally, assume $\alpha \in [0, 1]$.

Definition 3 (Albizuri (2010)). *The α-serial cost sharing rule, denoted by φ^α, is defined for each $(N, q, C) \in \Gamma^N$ by*

$$\varphi_i^\alpha(N, q, C) = \sum_{j=1}^{i} \frac{C\left(\alpha q^j\right) - C\left(\alpha q^{j-1}\right)}{n - j + 1}$$

$$+ \sum_{j=1}^{i} \frac{C\left(q(N) - (1 - \alpha) q^{j-1}\right) - C\left(q(N) - (1 - \alpha) q^j\right)}{n - j + 1}.$$

When $\alpha = 1$ the α-serial cost sharing rule coincides with the serial cost sharing rule and when $\alpha = 0$ with the dual serial cost sharing rule (Albizuri and Zarzuelo (2007)).

We generalize demands separability* by considering $C^{-\alpha q^i}$ instead of C^{-q^i}. That is, we assume that the cost associated with q^i is given by αq^i units at the beginning of the production cost and by $(1-\alpha)\, q^i$ units at the end of the production cost.

α-Demand separability*. Let $(N, q, C) \in \Gamma^N$ and $i \in N$. Then, for all $j \in N$ such that $q_j \geq q_i$ it holds that

$$\varphi_j(N, q, C) = \varphi_i(N, q, C) + \varphi_j(N, q^{-q_i}, C^{-\alpha q^i}).$$

Theorem 3. *The α-serial cost sharing rule is the only cost sharing rule on Γ^N that satisfies α-demand separability*.*

Proof. First we prove that φ^α satisfies α-demand separability*. Let $(N, q, C) \in \Gamma^N$ and $i, j \in N$ such that $N = \{1, 2, \ldots, n\}$, $q_1 \leq \cdots \leq q_n$ and $q_j \geq q_i$. If $q_j = q_i$, then $\varphi_j^\alpha(N, q^{-q_i}, C^{-\alpha q^i}) = 0 = \varphi_j^\alpha(N, q, C) - \varphi_i^\alpha(N, q, C)$. Assume that $q_j > q_i$. Taking into account the definition,

$$\begin{aligned}
&\varphi_j^\alpha(N, q^{-q_i}, C^{-\alpha q^i}) \\
&= \sum_{k=i+1}^{j} \frac{C^{-\alpha q^i}\left(\alpha \left(q^{-q_i}\right)^k\right) - C^{-\alpha q^i}\left(\alpha \left(q^{-q_i}\right)^{k-1}\right)}{n-k+1} \\
&\quad + \sum_{k=i+1}^{j} \left[\frac{C^{-\alpha q^i}\left(q^{-q_i}(N) - (1-\alpha)\left(q^{-q_i}\right)^{k-1}\right)}{n-k+1} \right. \\
&\qquad \left. - \frac{C^{-\alpha q^i}\left(q^{-q_i}(N) - (1-\alpha)\left(q^{-q_i}\right)^{k}\right)}{n-k+1} \right],
\end{aligned} \tag{10}$$

where we have also taken into account that $\left(q^{-q_i}\right)^k = 0$ for $k = 1, ..., i$. Moreover,

$$C^{-\alpha q^i}\left(\alpha \left(q^{-q_i}\right)^k\right) = C\left(\alpha \left(q^{-q_i}\right)^k + \alpha q^i\right) - C\left(\alpha q^i\right) = C\left(\alpha q^k\right) - C\left(\alpha q^i\right)$$

and

$$\begin{aligned}
&C^{-\alpha q^i}\left(q^{-q_i}(N) - (1-\alpha)\left(q^{-q_i}\right)^k\right) \\
&= C\left(q(N) - q^i - (1-\alpha)\left(q^k - q^i\right) + \alpha q^i\right) - C\left(\alpha q^i\right) \\
&= C\left(q(N) - (1-\alpha)\, q^k\right) - C\left(\alpha q^i\right)
\end{aligned}$$

for $k = i, ..., j$. Therefore, then (10) turns into

$$\sum_{k=i+1}^{j} \frac{C\left(\alpha q^k\right) - C\left(\alpha q^{k-1}\right)}{n-k+1}$$

$$+ \sum_{k=i+1}^{j} \frac{C\left(q\left(N\right)-\left(1-\alpha\right)q^{k-1}\right)-C\left(q\left(N\right)-\left(1-\alpha\right)q^{k}\right)}{n-k+1}$$

$$= \varphi_j^{\alpha}(N,q,C) - \varphi_i^{\alpha}(N,q,C).$$

Unicity can be proven in a similar way as in Theorem 1, so we omit it. □

4. Generalized monotonic Ibn Ezra's cost sharing rule

In this section we consider monotonic division rules that extend Ibn Ezra's rule and the cost sharing rules associated with them. The family of those cost sharing rules is characterized by four axioms.

We say that a monotonic division rule r extends Ibn Ezra's rule if $r(q,E) = r^{IE}(q,E)$ when $E \leq q_n$. The monotonic extensions of Ibn Ezra's rule are denoted by r^{GIE}. The associated additive cost sharing rule is defined on $\Gamma_{ac}^N \subset \Gamma^N$, where Γ_{ac}^N denotes the subset of absolutely continuous functions, and gives the following for each $(N,q,C) \in \Gamma_{ac}^N$

$$\varphi^{GIE}\left(N,q,C\right) = \int_0^{q(N)} C'\left(t\right)dr^{GIE}\left(q,t\right).$$

We call φ^{GIE} the generalized monotonic Ibn Ezra's cost sharing rule. The axioms that characterize the family are additivity, a requirement that the rules have constant returns, and the following two axioms. We assume that $N = \{1,2,\ldots,n\}$ and $q_1 \leq \cdots \leq q_n$.

Bounded cost. If for some $i \in N$, $C\left(t\right) = C\left(\min\left(t,q_i\right)\right)$ for all $t \in \mathbb{R}_+$, then

$$\varphi_j\left(N,q,C\right) = \varphi_i\left(N,q,C\right).$$

if $j > i$.

The cost function is bounded by the cost of the demand of agent i. The axiom requires agents with higher demands than i to pay the same as i. They do not pay more since the cost does not increase with such higher demands.

Null agent. If $C\left(t\right) = C\left(q_n\right)$ for $t \geq q_n$ and for some $i \in N$, $C(t) = 0$ for $t \leq q_i$, then $\varphi_i\left(N,q,C\right) = 0$.

The null agent property is applied when the cost function is bounded by the cost of the demand of agent n. It requires an agent to pay nothing if the cost vanishes for his/her demand.

Theorem 4. *A cost sharing rule on Γ_{ac}^N is a generalized monotonic Ibn Ezra's cost sharing rule if and only if it has constant returns and satisfies additivity, bounded cost and null agent.*

Proof. Let φ^{GIE} be a generalized monotonic Ibn Ezra's cost sharing rule. By the isomorphism, it has constant returns and satisfies additivity. We now show that it satisfies bounded cost. If for some $i \in N$, $C(t) = C(\min(t, q_i))$ for all $t \in \mathbb{R}_+$, then $C'(t) = 0$ when $t > q_i$. Moreover, $dr_i^{GIE}(q,t) = dr_j^{GIE}(q,t)$ when $t < q_i$ and $j > i$. Hence,

$$\varphi_i^{GIE}(N,q,C) = \int_0^{q_i} C'(t)\, dr_i^{GIE}(q,t) = \varphi_j^{GIE}(N,q,C).$$

if $j > i$. To show null agent take into account that $dr_i^{GIE}(q,t) = 0$ when $q_i < t < q_n$. Moreover, if for some $i \in N$, $C(t) = 0$ for $t \le q_i$ and $C(t) = C(q_n)$ for $t \ge q_n$, then $C'(t) = 0$ when $t < q_i$ or $t > q_n$. Therefore, $\varphi_i^{GIE}(N,q,C) = 0$.

Conversely, let φ be a cost sharing rule that has constant returns and satisfies additivity, bounded cost and null agent. We prove that the associated division rule r is a monotonic extension of Ibn Ezra's rule. It is monotonic since φ has constant returns and satisfies additivity. Now consider $\left(N,q,C^E\right)$, where $E \le q_n$ and $C^E(t) = \min\{t, E\}$, and prove that $\varphi\left(N,q,C^E\right) = r^{IE}(q,E)$. This is done by induction on $i \in N$.

Some notation is needed. Given $q \in \mathbb{R}_+^N$, $E \in \mathbb{R}_+$, a cost function C and $i \in N$, we define these cost functions:

$$C_{1,E}^i(t) = \begin{cases} C(t) & \text{if } t \le \min\{E, q_i\} \\ C(\min\{E, q_i\}) & \text{otherwise,} \end{cases}$$

and

$$C_{2,E}^i(t) = \begin{cases} 0 & \text{if } t \le \min\{E, q_i\} \\ C(t) - C(\min\{E, q_i\}) & \text{otherwise.} \end{cases}$$

Let $i = 1$. By additivity,

$$\varphi_1\left(N,q,C^E\right) = \varphi_1\left(N,q,\left(C^E\right)_{1,E}^1\right) + \varphi_1\left(N,q,\left(C^E\right)_{2,E}^1\right),$$

and taking into account null agent,

$$\varphi_1\left(N,q,\left(C^E\right)\right) = \varphi_1\left(N,q,\left(C^E\right)_{1,E}^1\right).$$

Moreover, bounded cost implies $\varphi_1\left(N,q,\left(C^E\right)_{1,E}^1\right) = \varphi_i\left(N,q,\left(C^E\right)_{1,E}^1\right)$ for $i = 2,...,n$. Therefore, since φ is a cost sharing rule,

$$\varphi_1\left(N,q,\left(C^E\right)_{1,E}^1\right) = \frac{\min\{q_1, E\}}{n} = r_1^{IE}(q,E).$$

Assume that $\varphi_i\left(N,q,C^E\right) = r_i^{IE}(q,E)$ for all $E \le q_n$ when $i < j$ and prove that $\varphi_j\left(N,q,C^E\right) = r_j^{IE}(q,E)$ for all $E \le q_n$.

By additivity and null agent,

$$\varphi_j\left(N, q, C^E\right) = \varphi_j\left(N, q, \left(C^E\right)^j_{1,E}\right).$$

Since $\left(C^E\right)^j_{1,E} = C^{\min\{E,q_j\}}$, the induction hypothesis implies

$$\varphi_i\left(N, q, \left(C^E\right)^j_{1,E}\right) = r_i^{IE}(q, \min\{E, q_j\})$$

when $i < j$, and bounded cost implies

$$\varphi_j\left(N, q, \left(C^E\right)^j_{1,E}\right) = \varphi_k\left(N, q, \left(C^E\right)^j_{1,E}\right)$$

when $k \geq j$. Since φ is a cost sharing rule,

$$\varphi_j\left(N, q, C^{\min\{E,q_j\}}\right) = r_j^{IE}(q, \min\{E, q_j\}) = r_j^{IE}(q, E),$$

and the result is obtained. □

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