

IKERLANAK

**EFFICIENCY VS. STABILITY
IN A MIXED NETWORK
FORMATION MODEL**

by

Norma Olaizola and Federico Valenciano

2015

Working Paper Series: IL. 87/15

Departamento de Fundamentos del Análisis Económico I

Ekonomi Analisiaren Oinarriak I Saila



University of the Basque Country

Efficiency vs. stability in a mixed network formation model*

By Norma Olaizola[†] and Federico Valenciano[‡]

March 26, 2015

Abstract

The purpose of this paper is twofold. First, the incomplete results relative to efficiency in a transitional model introduced in a previous paper, distinguishing two types of links, strong or doubly-supported and weak or singly-supported, are completed with a full-characterization. Second, as it turns out, efficient structures are stable only for a small range of values of the parameters within the much wider range where they are efficient. This motivates the study of the impact on stability of allowing players to negotiate bilaterally the shares of the cost of doubly-supported links.

JEL Classification Numbers: A14, C72, D20, J00

Key words: Network formation, Unilateral link-formation, Bilateral link-formation, Stability, Efficiency, Cost share

*This research is supported by the Spanish Ministerio de Economía y Competitividad under projects ECO2012-31626 and ECO2012-31346. Both authors also benefit from the Basque Government Departamento de Educación, Política Lingüística y Cultura funding for Grupos Consolidados IT869-13 and IT568-13.

[†]*BRiDGE* group (<http://www.bridgebilbao.es>), Departamento de Fundamentos del Análisis Económico I, Universidad del País Vasco, Avenida Lehendakari Aguirre 83, 48015 Bilbao, Spain; norma.olaizola@ehu.es.

[‡]*BRiDGE* group (<http://www.bridgebilbao.es>), Departamento de Economía Aplicada IV, Universidad del País Vasco, Avenida Lehendakari Aguirre 83, 48015 Bilbao, Spain; federico.valenciano@ehu.es.

1 Introduction

A previous paper, Olaizola and Valenciano (2015a), proposes a “transitional” or mixed model between a no-decay version of Jackson and Wolinsky’s (1996) connections model and Bala and Goyal’s (2000) two-way flow model¹. The transition is achieved by distinguishing between “weak” links, supported by only one player, and “strong” links, supported by both players. The flow through strong links is assumed to be perfect, while it suffers some decay through weak links. That paper studies stability, efficiency and dynamics in that model. However, the results regarding efficiency are incomplete as only one type of efficient structures is identified and its efficiency is established only within a small region of values of the parameters. In contrast with this, a full-characterization of efficient structures is achieved in a more complex setting studied in Olaizola and Valenciano (2015b), where a model integrating Jackson and Wolinsky (1996) and Bala and Goyal (2000) as extreme cases, both with decay, is introduced. Using a similar strategy to the one used in the proofs in that paper, we here refine the results relative to efficiency in Olaizola and Valenciano (2015a) providing a full-characterization of efficient structures. It is proved that there are only two types of non-empty efficient structures -trees of strong links and stars of weak links- and the region where each of them is efficient is determined.

A second contribution of this paper is as follows. Olaizola and Valenciano (2015a) establish the stability (in the sense of Nash equilibrium and in the sense of pairwise stability) of some structures, including the only two proved here to be efficient. As it turns out, the trees of strong links are stable only within a small part of the region where they are efficient. This lack of stability of efficient structures motivates the study of the impact of “liberalizing” the cost-paying of strong links. In Olaizola and Valenciano (2015a) it is assumed that the cost of a strong link must be equally shared by the two players who form it, but in a context where any pair of players can coordinate to form a link, it seems natural to assume that they can also negotiate the shares of its cost. We study the impact of assuming that the players who form a strong link can bargain how the cost is to be shared.

The rest of the paper is organized as follows. Section 2 briefly reviews the model introduced in Olaizola and Valenciano (2015a). Section 3 addresses the question of efficiency. Section 4 studies the impact of assuming pairwise negotiable costs and establishes the conditions for the existence of cost share equilibrium allocations.

¹These two seminal papers are the basic references of economic models of network formation. They have been extended in different directions. See Goyal (2007), Jackson (2008) and Vega-Redondo (2007) and references therein.

2 The model

We consider the following situation. *Players* or *nodes*² within a set N may form links through which information runs. It is assumed that each node contains an information of value 1 to whoever receives it intact. A link created unilaterally and supported by only one player has a cost $c > 0$ ³ and allows information to flow in both directions. When a link is singly-supported there is *decay* in the flow of information, i.e. only $\alpha \in (0, 1)$ out of a unit of information at one node reaches the other. When a link is supported by the two players it connects, with $2c$ being the total cost of it, the link allows information to flow in both directions *without friction*. We refer to links supported by only one player as *weak* links and to those supported by both as *strong* links. A strong link is not necessarily the result of coordination and agreement, though the possibility of coordination obviously affects stability.

Formally⁴, a map $g_i : N \setminus \{i\} \rightarrow \{0, 1\}$ specifies the links supported by player i . We write $g_{ij} := g_i(j)$, and $g_{ij} = 1$ or $ij \in g$ ($g_{ij} = 0$ or $ij \notin g$) means that i *supports* (*does not support*) a link with j . $\overline{ij} \in g$ means that i and j are connected by a strong link, i.e. $ij, ji \in g$. If $ij \in g$ ($\overline{ij} \in g$) the resulting network by eliminating link ij (\overline{ij}) in g is denoted by $g - ij$ ($g - \overline{ij}$). Thus, vector $g_i = (g_{ij})_{j \in N \setminus \{i\}} \in \{0, 1\}^{N \setminus \{i\}}$ specifies the links supported by i and is referred to as a *strategy* of player i . $G_i := \{0, 1\}^{N \setminus \{i\}}$ denotes the set of i 's strategies and $G_N = G_1 \times G_2 \times \dots \times G_n$ the set of *strategy profiles*. Thus, a strategy profile $g \in G_N$ determines an N -*network*. It is assumed that the fraction of a unit of information at node j that reaches node i through a link between them when players' strategy profile is g , denoted by δ_{ij}^g , is given by

$$\delta_{ij}^g := \alpha g_{ij}^{\max} + (1 - \alpha) g_{ij}^{\min}, \quad (1)$$

where $g_{ij}^{\max} = \max\{g_{ij}, g_{ji}\}$, and $g_{ij}^{\min} = \min\{g_{ij}, g_{ji}\}$. Thus, $\delta_{ij}^g = \delta_{ji}^g = 1$ if $g_{ij} = g_{ji} = 1$, and $\delta_{ij}^g = \delta_{ji}^g = \alpha$ if $g_{ij} = 1$ and $g_{ji} = 0$. Note that when $\alpha = 0$ (1) yields a no-decay version of Jackson and Wolinsky's (1996) bilateral link-formation model, and for $\alpha = 1$ it yields Bala and Goyal's (2000) unilateral two-way flow model without decay.

In terms of δ_{ij}^g , the payoff of player i in a network resulting from g is the information he/she receives minus the cost of the links he/she pays for. Let the *discounting length* of a path from j to i in g be the number of weak links in it. And let the *discounting distance* between j and i ($i \neq j$) in g , denoted by $\lambda(i, j; g)$, be the discounting length of the path from j to i with the shortest discounting length. The information received

²To avoid a biased language we often prefer the term "node".

³In Olaizola and Valenciano (2015a) it is assumed that $c < 1$ in order to simplify the presentation, but this assumption is withdrawn here.

⁴To avoid a repetition of preliminaries entirely similar to those in Section 2 in Olaizola and Valenciano (2015a), we assume the reader to be familiar with the notation and terminology relative to networks, and we restrict our attention to the basic notation and a few peculiarities of the model to make the paper basically self-contained.

by player i is given by

$$I_i(g) = \sum_{j \in N(i;g)} \alpha^{\lambda(i,j;g)},$$

where $N(i;g)$ denotes the set of nodes connected with i by a path in g . Thus the payoff function becomes

$$\Pi_i(g) = \sum_{j \in N(i;g)} \alpha^{\lambda(i,j;g)} - c\mu_i^d(g), \quad (2)$$

where $\mu_i^d(g)$ denotes the number of links in g supported by i .

3 Efficiency

The aggregate payoff of a network g is referred to as the *value* of the network and denoted by $v(g)$. A network g is said to *dominate* another g' if $v(g) \geq v(g')$. A network is *efficient* if it dominates any other for a particular configuration of values of the parameters. We make use of the following notions.

Definition 1 Given network g , and $K \subseteq N$, K is said to be⁵ :

- (i) A *weak component* of g if for any two nodes $i, j \in K$ ($i \neq j$) there is a path from j to i in g , and no subset of N strictly containing K meets this condition.
- (ii) A *strong component* of g if for any two nodes $i, j \in K$ ($i \neq j$) there is a path of strong links from j to i in g , and no subset of N strictly containing K meets this condition.

A *trivial component* is a component in either sense that consists of a single node. We say that g is *weakly (strongly) connected* if g is the unique weak (strong) component of g . A weak (strong) component K of a network g is *minimal* if for all $i, j \in K$ s.t. $g_{ij} = 1$, the number of weak (strong) components of g is smaller than the number of weak (strong) components in $g - ij$.

A graph is *minimally weakly (strongly) connected* if it is weakly (strongly) connected and minimal. In both cases, a minimally connected graph is a *tree* (of weak links in one case, of strong links in the other).

The following structures are going to play a role in what follows. A *strong-core network* is a weakly connected network g with a *core* consisting of a strong component, which is the only one and minimal if it is non-trivial, and each of the remaining nodes (if any) are peripheral⁶ and are connected with the core by a weak link. In other words, a strong-core network consists of a tree of strong links with which some nodes are connected by weak links⁷. A strong-core network with k_s strong links and k_w weak

⁵If $g|_K$ denotes the restriction of g to K , it is clear that $g|_K$ specifies a K -network.

⁶Peripheral players are those involved in only one link (weak or strong).

⁷Tree-core-periphery structures, introduced in Olaizola and Valenciano (2015a), are strong-core networks whose core contains no peripheral nodes.

links (consequently with $k_s + k_w + 1$ nodes and a core with $k_s + 1$ nodes) is denoted by S_{k_s, k_w} . Observe that minimally strongly connected networks ($k_w = 0$) and *stars* of weak links ($k_s = 0$) are extreme cases of strong-core networks.

Lemma 1 *If the payoff function is given by (2) with $0 \leq \alpha < 1$, then the maximal value of a weak component containing m nodes and $m - 1$ or more strong links is only reached by a minimally strongly connected component with $m - 1$ strong links.*

Proof. Let K be a weak component containing m nodes and $k_s \geq m - 1$ strong links. The maximal amount of aggregate information of a component with m nodes is $m(m - 1)$, and K 's total cost is at least that of $m - 1$ strong links, i.e. $2c(m - 1)$. Therefore the aggregate payoff of K is not greater than $m(m - 1) - 2c(m - 1) = (m - 1)(m - 2c)$, which is the aggregate payoff for any minimally strongly connected component with m nodes. Moreover, it can be immediate seen that only a component with such a structure yields that aggregate payoff. ■

Lemma 2 *If the payoff function is given by (2) with $0 \leq \alpha < 1$ and $c > 2(\alpha - \alpha^2)$, then a weak component containing m nodes and fewer than $m - 1$ strong links is dominated by a strong-core component with the same number of strong links.*

Proof. Let K be a weak component containing m nodes and $k_s < m - 1$ strong links and $k_w \geq m - 1 - k_s > 0$ weak links. Without loss of generality, it can be assumed that no link is superfluous. We discuss separately two cases.

Case 1: $c \geq 2\alpha$.

Then,

$$v(K) = k_s(2 - 2c) + k_w(2\alpha - c) + p(\alpha),$$

where $p(\alpha)$ is a polynomial on α with integer positive coefficients (summing up to $\max\{m(m - 1) - 2(k_s + k_w), 0\}$) multiplying monomials of the form α^q with $q \geq 0$. As $\alpha < 1$, we have:

$$\begin{aligned} v(K) &\leq k_s(2 - 2c) + k_w(2\alpha - c) + k_s(k_s - 1) \\ &+ k_s(m - 1 - k_s)2\alpha + (m - 1 - k_s)(m - 2 - k_s)\alpha^2, \end{aligned}$$

while the value of a strong-core component with k_s strong links and $m - 1 - k_s$ nodes connected with it by weak links is

$$\begin{aligned} v(S_{k_s, m-1-k_s}) &= k_s(2 - 2c) + (m - 1 - k_s)(2\alpha - c) \\ &+ k_s(k_s - 1) + k_s(m - 1 - k_s)2\alpha + (m - 1 - k_s)(m - 2 - k_s)\alpha^2. \end{aligned}$$

Thus, the difference is

$$v(S_{k_s, m-1-k_s}) - v(K) = (c - 2\alpha)(k_s + k_w - (m - 1)) \geq 0,$$

given that $k_s + k_w \geq m - 1$ and $c \geq 2\alpha$.

Case 2: $2(\alpha - \alpha^2) < c < 2\alpha$.

Thus we have

$$v(K) \leq k_s(2 - 2c) + k_w(2\alpha - c) + A + B\alpha + C\alpha^2,$$

where

$$A = \min\{k_s(k_s - 1), m(m - 1) - 2k_s - 2k_w\}.$$

There are two cases depending on which of these numbers is smaller:

Case 2.1: $A = m(m - 1) - 2k_s - 2k_w$. In this case $B = C = 0$, and we have

$$v(K) \leq k_s(2 - 2c) + k_w(2\alpha - c) + (m(m - 1) - 2k_s - 2k_w),$$

while the value of a strong-core component with k_s strong links and $m - 1 - k_s$ weak links is

$$\begin{aligned} v(S_{k_s, m-1-k_s}) &= k_s(2 - 2c) + (m - 1 - k_s)(2\alpha - c) \\ &+ k_s(k_s - 1) + k_s(m - 1 - k_s)2\alpha + (m - 1 - k_s)(m - 2 - k_s)\alpha^2. \end{aligned}$$

Thus, the difference is

$$\begin{aligned} &v(S_{k_s, m-1-k_s}) - v(K) \\ &\geq (m - 1 - k_s - k_w)(2\alpha - c) + (k_s(k_s - 1) - m(m - 1) + 2k_s + 2k_w) \\ &\quad + k_s(m - 1 - k_s)2\alpha + (m - 1 - k_s)(m - 2 - k_s)\alpha^2 \\ &= a(2\alpha - c) + b + d\alpha + e\alpha^2, \end{aligned}$$

where a, b, d and e denote the coefficients in the last expression. Note that $a \leq 0$, while b, d and e are ≥ 0 (d and e obviously, and b because we are assuming $k_s(k_s - 1) \geq A = m(m - 1) - 2k_s - 2k_w$). As $2\alpha - c < 2\alpha^2$, by replacing $2\alpha - c$ by $2\alpha^2$ in the last expression and taking into account that $\alpha < 1$ we have

$$\begin{aligned} v(S_{k_s, m-1-k_s}) - v(K) &\geq a2\alpha^2 + b + d\alpha + e\alpha^2 \\ &\geq a2\alpha^2 + b\alpha^2 + d\alpha^2 + e\alpha^2 = (2a + b + d + e)\alpha^2. \end{aligned}$$

Therefore, if $2a + b + d + e \geq 0$ the proof is concluded in case 2.1, and summing up these coefficients we have $2a + b + d + e = 0$.

Case 2.2: $A = k_s(k_s - 1)$. In this case $k_s(k_s - 1)/2$ is the maximal number of non-directly linked pairs that can receive 1 from each other. Now

$$B = \min\{2k_s(m - 1 - k_s), m(m - 1) - 2k_s - 2k_w - k_s(k_s - 1)\}.$$

Thus, we again have two cases:

Case 2.2.1: $B = m(m-1) - 2k_s - 2k_w - k_s(k_s - 1)$. In this case $C = 0$, and

$$v(K) \leq k_s(2-2c) + k_w(2\alpha - c) + k_s(k_s - 1) + (m(m-1) - 2k_s - 2k_w - k_s(k_s - 1))\alpha.$$

Thus, subtracting this value from that of a strong-core component with k_s strong links and $m-1-k_s$ weak links, the difference is

$$\begin{aligned} v(S_{k_s, m-1-k_s}) - v(K) &\geq (m-1-k_s-k_w)(2\alpha - c) \\ &+ (2k_s(m-1-k_s) - (m(m-1) - 2k_s - 2k_w - k_s(k_s - 1)))\alpha \\ &+ (m-1-k_s)(m-2-k_s)\alpha^2 = a(2\alpha - c) + b\alpha + d\alpha^2, \end{aligned}$$

and proceeding just as in the first case we similarly conclude that $v(S_{k_s, m-1-k_s}) - v(K) \geq 0$.

Case 2.2.2: $B = 2k_s(m-1-k_s)$. In this case

$$C = m(m-1) - 2k_s - 2k_w - k_s(k_s - 1) - 2k_s(m-1-k_s), \text{ and}$$

$$\begin{aligned} v(S_{k_s, m-1-k_s}) - v(K) &\geq (m-1-k_s-k_w)(2\alpha - c) \\ &+ (k_s + k_w - (m-1))2\alpha^2 = a(2\alpha - c) + b\alpha^2, \end{aligned}$$

and proceeding again as before we conclude that $v(S_{k_s, m-1-k_s}) - v(K) \geq 0$. ■

Lemma 3 (*Proposition 11, Olaizola and Valenciano, 2015a*)⁸ *If the payoff function is given by (2) with $0 \leq \alpha < 1$ and $c < 2(1-\alpha)$, then the only non-empty efficient networks are those minimally strongly connected.*

Lemmas 1, 2 and 3 establish that, for different configurations of values of the parameters (see Figures 1 and 2), any component is dominated by a strong-core component. The following lemma shows that strong-core component are always dominated by one of the two extreme types of strong-core component, either by minimally strongly connected networks (all-encompassing core) or by stars of weak links (trivial core).

Lemma 4 *If the payoff function is given by (2) with $0 \leq \alpha < 1$, a strong-core component containing both strong and weak links is strictly dominated either by a minimally strongly connected one with the same number of links or by a star with the same number of links all of which are weak.*

Proof. The value of a strong-core component S_{k_s, k_w} connecting m nodes is given by

$$v(S_{k_s, k_w}) = k_s(2-2c) + k_w(2\alpha - c) + k_s(k_s - 1) + 2k_s k_w \alpha + k_w(k_w - 1)\alpha^2.$$

⁸This is the only result relative to efficiency established in Olaizola and Valenciano (2015a).

By making double a weak link, S_{k_s+1,k_w-1} results, and

$$v(S_{k_s+1,k_w-1}) = (k_s + 1)(2 - 2c) + (k_w - 1)(2\alpha - c) \\ + (k_s + 1)k_s + 2(k_s + 1)(k_w - 1)\alpha + (k_w - 1)(k_w - 2)\alpha^2.$$

Thus, as $k_w = m - 1 - k_s$, $v(S_{k_s+1,k_w-1}) - v(S_{k_s,k_w}) =$

$$(2 - 2c) - (2\alpha - c) + 2(m - 2)\alpha(1 - \alpha) + 2k_s(1 - \alpha)^2. \quad (3)$$

Note that if this number is > 0 , the greater k_s is the greater this number will be, and consequently the value of a minimally strongly connected component of m nodes is greater than that of S_{k_s,k_w} .

Consider now a strong-core component with one strong link less and one weak link more, i.e. S_{k_s-1,k_w+1} , whose value is

$$v(S_{k_s-1,k_w+1}) = (k_s - 1)(2 - 2c) + (k_w + 1)(2\alpha - c) \\ + (k_s - 1)(k_s - 2) + 2(k_s - 1)(k_w + 1)\alpha + (k_w + 1)k_w\alpha^2.$$

Thus, as $k_w = m - 1 - k_s$, $v(S_{k_s-1,k_w+1}) - v(S_{k_s,k_w}) =$

$$-(2 - 2c) + (2\alpha - c) + 2(1 - m\alpha + (m - 1)\alpha^2) - 2k_s(1 - \alpha)^2. \quad (4)$$

If this number is positive, the smaller k_s is the greater this number will be and consequently the value of a star of $k_s + k_w$ weak links is greater than that of S_{k_s,k_w} .

It only remains to show that S_{k_s,k_w} is strictly dominated either by S_{k_s+1,k_w-1} or by S_{k_s-1,k_w+1} ; that is, either (3) or (4) is greater than 0. Write $X = (2 - 2c) - (2\alpha - c)$, $Y = 2(m - 2)\alpha(1 - \alpha) + 2k_s(1 - \alpha)^2$ and $Y' = 2(1 - m\alpha + (m - 1)\alpha^2) - 2k_s(1 - \alpha)^2$. Thus we prove that necessarily either

$$v(S_{k_s+1,k_w-1}) - v(S_{k_s,k_w}) = X + Y > 0 \quad \text{or} \quad v(S_{k_s-1,k_w+1}) - v(S_{k_s,k_w}) = -X + Y' > 0.$$

Assume $X + Y \leq 0$, i.e. $X \leq -Y$, then we prove that $-X + Y' > 0$, i.e. $X < Y'$. For this it suffices to show that $-Y < Y'$, i.e. $Y + Y' > 0$. In fact we have $Y + Y' = 2(1 - \alpha)^2 > 0$. ■

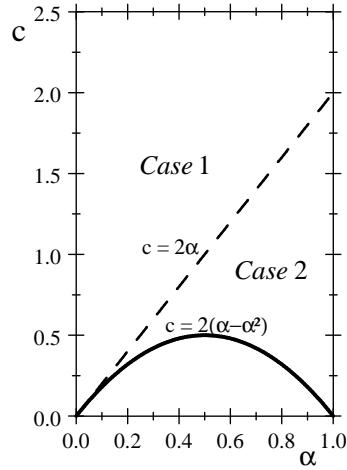


Figure 1: Lemma 2

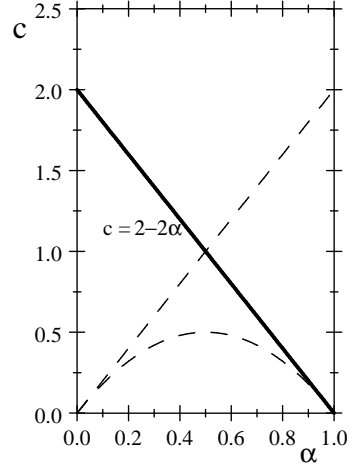


Figure 2: Lemma 3

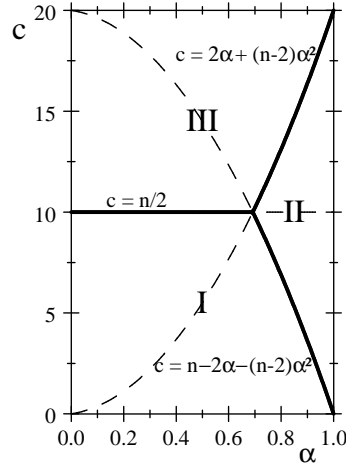


Figure 3: Proposition 1
($n = 20$)

The preceding lemmas nearly show the domination of the extreme types of strong-core networks. The next proposition completes the proof and the characterization by establishing the precise configurations of values of the parameters where such structures are efficient.

Proposition 1 *If the payoff function is given by (2) with $0 \leq \alpha < 1$, then the unique efficient networks are:*

(i) *The minimally strongly connected ones if*

$$c < \min\{n/2, n - 2\alpha - (n - 2)\alpha^2\}. \quad (\text{Region I in Figure 3})$$

(ii) *The all-encompassing stars of weak links if*

$$n - 2\alpha - (n - 2)\alpha^2 < c < 2\alpha + (n - 2)\alpha^2. \quad (\text{Region II in Figure 3})$$

(iii) *The empty network if*

$$c > \max\{n/2, 2\alpha + (n - 2)\alpha^2\}. \quad (\text{Region III in Figure 3})$$

Proof. In view of the preceding lemmas any non-empty component of an efficient network must be either minimally strongly connected or a star of weak links. As the value of a component of an efficient network must be non-negative, it can be immediately seen that the value of a minimally strongly connected component (a star of weak links) with $m_1 + m_2$ nodes is greater than the sum of the values of two minimally strongly connected components (two stars of weak links) with m_1 and m_2 nodes each. Thus, a non-empty efficient network must be weakly connected.

(i) First note that a minimally strongly connected network yields a positive aggregate payoff only if $(n - 1)(n - 2c) > 0$, i.e. only if $c < n/2$, and that payoff is greater than that of an all-encompassing star of weak links if and only if

$$(n - 1)(n - 2c) > (n - 1)\alpha + (n - 1)(\alpha + (n - 2)\alpha^2) - (n - 1)c$$

i.e. if $c < n - 2\alpha - (n - 2)\alpha^2$.

(ii) When $c > n - 2\alpha - (n - 2)\alpha^2$, an all-encompassing star of weak links beats a minimally strongly connected network, but it yields a positive payoff only if

$$(n - 1)(2\alpha + (n - 2)\alpha^2) - c > 0,$$

i.e. if $c < 2\alpha + (n - 2)\alpha^2$.

(iii) When c is above $n/2$ and above $2\alpha + (n - 2)\alpha^2$ neither of the two dominant structures yields a positive payoff and consequently the only efficient network is the empty one. ■

4 Cost share equilibrium allocations

Thus there are only two types of non-empty efficient networks, those minimally strongly connected and all-encompassing stars of weak links. In Olaizola and Valenciano (2015a) the stability of both structures is established assuming $c < 1$. Minimally strongly connected networks are Nash, strict Nash and pairwise stable only if $c < 1 - \alpha$ (Proposition 3, 2015a), which is a small subset of the set of values of the parameters where these structures are efficient, as is established in Proposition 1-(i). The reason why this is so

is very simple: it is assumed in that model that the cost of a strong link, $2c$, is equally shared by the two players who form it. Thus, in a minimally strongly connected network, as soon as $c > 1 - \alpha$ players supporting a strong link with a peripheral node cease to have incentives to pay c for it. Nevertheless, even for c close to $n/2$ the value of the network is positive if α is not too close to 1. In fact, this assumption is made in the original connections model of Jackson and Wolinsky (1996), where the only feasible links are doubly supported. But in a context in which players can pairwise coordinate to form links, it seems only natural to assume that they can also pairwise coordinate, i.e. negotiate and agree upon the way of sharing the cost of strong links⁹. This modification of the setting and its impact on the stability of minimally connected networks is the second goal of this paper.

Let g be a minimally strongly connected profile and assume that players can negotiate bilaterally the share of the cost of each strong link. A first necessary condition for network g to be worth forming is that it yields a *positive* aggregate payoff. This is so if and only if $n(n - 1) - (n - 1)2c > 0$, i.e. if

$$2c < n. \quad (5)$$

Denote by c_i^j i 's share of the cost of a strong link $\overline{ij} \in g$. For an *allocation of costs* $(c_{ij})_{\overline{ij} \in g}$ to be *feasible* it must be

$$c_i^j + c_j^i = 2c \quad (\text{for all } i, j \in N, \text{ s.t. } \overline{ij} \in g). \quad (6)$$

For each $\overline{ij} \in g$, let K_i^j be the set of nodes in the component of $g - \overline{ij}$ containing i , and k_j^i its number. As an obvious condition of *individual rationality*, player i will not pay for link $\overline{ij} \in g$ more than what he/she receives through it; that is, it must be

$$c_i^j \leq k_j^i \quad (\text{for all } i, j \in N, \text{ s.t. } \overline{ij} \in g). \quad (7)$$

On the other hand, in case of disagreement, player i has the option of paying c for a weak link with j and receive αk_j^i if the players in K_j^i remain in a strong component and player j refuses to support link ji . Therefore, it must be $k_j^i - c_i^j \geq \alpha k_j^i - c$. Thus, the following condition of *outside-option-proof* must hold:

$$c_i^j \leq c + (1 - \alpha)k_j^i \quad (\text{for all } i, j \in N, \text{ s.t. } \overline{ij} \in g). \quad (8)$$

Definition 2 *We say that $(c_{ij})_{\overline{ij} \in g}$ is a cost share equilibrium allocation for g if it satisfies conditions (6), (7) and (8).*

⁹This approach, consistent with a model where cooperation is restricted to pairs of players, is different from the cooperative game-theoretic one considered in Jackson and Wolinsky (1996).

The following proposition establishes necessary and sufficient conditions for the existence of cost share equilibrium allocations for a minimally strongly connected network.

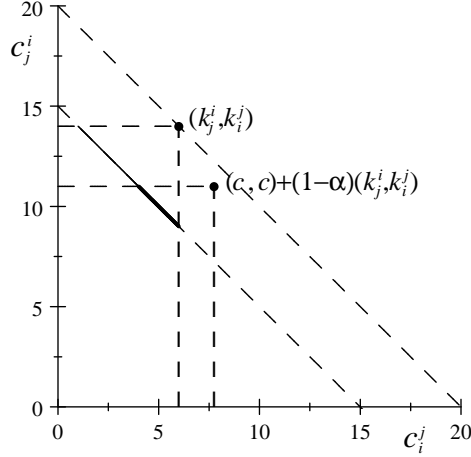


Figure 4: Cost share
($n = 20, 2c = 15$)

Proposition 2 *If the payoff function is given by (2) with $0 \leq \alpha < 1$, for any minimally strongly connected network, a cost share equilibrium allocation exists if and only if condition (5) and the following condition hold:*

$$c \leq n - \alpha(n - 1). \quad (9)$$

Proof. Let g be a minimally strongly connected profile and assume (5), which is an obvious necessary condition. We then prove that there exist $(c_i^j)_{\overline{ij} \in g}$, s.t. (6), (7) and (8) hold if and only if (9) holds. Note first that, as $k_i^j + k_j^i = n$, conditions (6) and (7) are compatible if and only if (5) holds. Similarly, conditions (6) and (8) are compatible too, because

$$2c + (1 - \alpha)(k_i^j + k_j^i) = 2c + (1 - \alpha)n > 2c.$$

Remains to be seen that all three conditions are compatible. For each $\overline{ij} \in g$, condition (7) and condition (8) specify a segment within the straight line $c_i^j + c_j^i = 2c$ each (see Figure 4). We show that the intersection of these two segments is not empty if and only if (9) holds. Without loss of generality, assume $k_i^j \geq k_j^i$. Then the intersection is not empty if and only if

$$2c - k_j^i \leq c + (1 - \alpha)k_i^j,$$

or, equivalently, if $c \leq k_j^i + (1 - \alpha)k_i^j = n - \alpha k_i^j$. But this condition is most stringent when $k_i^j = n - 1$, which corresponds to the case when j is a peripheral node, that is, $c \leq n - \alpha(n - 1)$. ■

Remarks:

1. Thus condition (5) ensures a positive aggregate payoff, which is an obvious necessary condition and makes feasibility and individual rationality compatible. Condition (9) guarantees that feasible agreements both individually rational and outside-option-proof by ensuring that there is room for negotiating the payment of links with peripheral nodes, those for which this condition is most demanding. Thus, when the two conditions hold, cost share equilibrium allocations exist. Consistent with intuition, the number of players plays in favor of stability, while the value of α goes against.

2. Thus, in contrast with the results in Olaizola and Valenciano (2015a), where minimally strongly connected profiles are unstable as soon as $c > 1 - \alpha$ due to the assumption that the cost of strong links should be shared equally by the players involved, when these shares can be negotiated such profiles can be stabilized within a much wider region by cost share equilibrium allocations. Figure 5 represents these regions for $n = 20$. Under the assumption that costs must be shared equally, minimally strongly connected networks are stable only within the triangle below $c = 1 - \alpha$ (line (1) in the figure), while assuming cost negotiable equilibrium allocations exist below lines $c = n/2$ and $c = n - \alpha(n - 1)$ (line (2) in the figure), a much wider area. Observe that it covers a great part of the region where such structures are efficient, even a small part of where they are not, as they are dominated by stars of weak links above $c = n - 2\alpha - (n - 2)\alpha^2$ (line (3) in the figure).

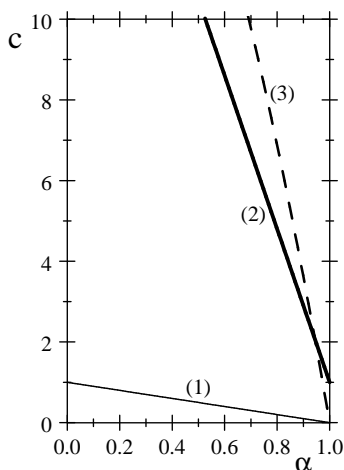


Figure 5: Stability vs. efficiency example ($n = 20$)

3. Proposition 2 establishes necessary and sufficient conditions for cost share equilibrium allocations to exist, but, in general, they are not unique. Assuming expected utility preferences, any bargaining solution would yield a specific allocation of costs.

For instance, if utilities are linear in payoffs the middle point of the segment whose nonemptiness has been established in the proof corresponds to the Nash bargaining solution. For example, assume $n = 7$, $c = 3$ and $\alpha = 7/12$. Obviously conditions (5) and (9) hold. If the seven nodes form a star of strong links each of the six peripheral players would pay 5.25 and the center 0.75 for each of the six links. If they form a line, i.e. six consecutive strong links, the two extreme links would be paid for 5.25 by the peripheral and 0.75 by the other player; the shares for the two next links at each side would be 1.5 and 4.5; and in the two central links the split would be 3.5 and 2.5. In all cases the player occupying a more central position would pay less.

References

- [1] Bala, V., and S. Goyal, 2000, A noncooperative model of network formation, *Econometrica* 68, 1181-1229.
- [2] Goyal, S., 2007, *Connections. An Introduction to the Economics of Networks*, Princeton University Press. Princeton.
- [3] Jackson, M., 2008, *Social and Economic Networks*, Princeton University Press. Princeton.
- [4] Jackson, M., and A. Wolinsky, 1996, A strategic model of social and economic networks, *Journal of Economic Theory* 71, 44-74.
- [5] Olaizola, N., and F. Valenciano, 2015a, Unilateral vs. bilateral link-formation: A transition without decay. *Mathematical Social Sciences* 74, 13-28.
- [6] Olaizola, N., and F. Valenciano, 2015b, A unifying model of strategic network formation, *Ikerlanak Working Paper Series: IL 85/15*, Department of Foundations of the Economic Analysis I, University of the Basque Country.
- [7] Nash, N., 1950, The bargaining problem. *Econometrica* 18, 155-162.
- [8] Vega-Redondo, F., 2007, *Complex Social Networks*, Econometric Society Monographs, Cambridge University Press.