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The impact of liberalizing cost-sharing on basic models of network formation *

By Norma Olaizola[†] and Federico Valenciano[‡]

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Abstract

This paper studies the impact of "liberalizing" the cost-sharing of links on some basic models of network formation. This is done in a setting where both doubly supported and singly supported links are possible, and which includes the two seminal models of network formation by Jackson and Wolinsky and Bala and Goyal as extreme cases. In this setting, the notion of pairwise stability is extended and it is proved that liberalizing cost-sharing for doubly supported links widens the range of values of the parameters where the efficient networks formed by such type of links are pairwise stable, while the range of values of the parameters where the efficient networks formed by singly supported links are pairwise stable shrinks, but the region where the latter are efficient and pairwise stable remains the same.

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Key words: Network formation, Unilateral link-formation, Bilateral link-formation, Cost-sharing, Efficiency, Stability

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1 Introduction

Jackson and Wolinsky's (1996) connections model is perhaps the most influential model of strategic formation of networks[1]. In this seminal model the formation of links is based on bilateral agreements, and it is assumed that the cost of a link is equally shared by the two players involved. In return for their investments in links, players benefit from the information received through the network. In this setting[2], Jackson and Wolinsky characterize efficient structures (i.e. those that maximize the aggregate payoff) and establish the range of values of the parameters for which each of them is stable.

The initial motivation of this paper is a question that arises naturally when examining Jackson and Wolinsky's (1996) connections model: if any two players can coordinate to form a link, can coordination not be extended to the way in which its cost is shared? If so, what is the impact on stability in general and on that of efficient structures in particular? This paper seeks to provide an answer to these questions, but by addressing them in a more general setting that we outline briefly[3].

A model introduced by Olaizola and Valenciano (2015b) merges and integrates as extreme cases Jackson and Wolinsky's (1996) connections model and the strictly non-cooperative version provided by Bala and Goyal's (2000) two-way flow model, where links can be created unilaterally. The merger is achieved by assuming that two types of links can be formed: strong links and weak ones. *Strong links* work better and must be supported by the two players involved, their cost is twice that of weak links and each player must pay half that cost. The flow through them suffers some decay, i.e. only $\delta \in (0,1)$ out of a unit of information at one node reaches the other. *Weak links* are those supported by only one player who pays for the cost, and they work worse: only $\alpha \in (0,\delta)$ out of a unit of information at one node reaches the other. Parameters δ and α are referred to as the *flow-level* through strong links and weak links respectively. This link-formation model "bridges the gap" between the two benchmark models in the following sense: if $\alpha = 0$ only strong links work, which is equivalent to Jackson and Wolinsky's (1996) connections model, and when $\alpha = \delta$ both strong links and weak links work equally well, which is equivalent to Bala and Goyal's (2000) two-way flow model.

This bridge-model can be further specified in two ways. One is by assuming a *strictly non-cooperative* environment where *coordination is not possible*. This does not preclude the formation of strong links if it is assumed that a doubly supported link necessarily becomes strong. The appropiate stability notion in this setting is the *Nash equilibrium*, as in Bala and Goyal's (2000) setting. Another possible scenario is to allow for *pairwise coordination* for the formation of strong links. In this case, *pairwise*

[1] Goyal (2007), Jackson (2008) and Vega-Redondo (2007) are excellent monographs on social and economic networks.

[2] The model is described in more detail later.

[3] Meléndez-Jiménez (2008) considers a network formation model where the cost-shares are decided through a bargaining process, but the approach, the setting and the issues addressed are completely different. This is discussed in more detail in the final section.

stability (conveniently adapted) is the appropriate stability notion, as in Jackson and Wolinsky (1996). In Olaizola and Valenciano (2015b) the efficient networks for the bridge-model, i.e. those that maximize the aggregate payoff, which are the same in both scenarios, are characterized and their stability is studied from both points of view. In this paper we only consider the second scenario, where pairwise coordination is possible, given that our purpose is to study the possibility of *extending pairwise coordination further* to cost-sharing of strong links. To that end this bridge-model is further specified so as to allow a precise, complete specification of all admissible deviations w.r.t. which stability is defined. Consistently with Jackson and Wolinsky's (1996) results, efficient structures formed by strong links turn out to be pairwise stable only within a subset of the region where they are efficient[4].

The rest of the paper is organized as follows. Section 2 reviews the reference model, Olaizola and Valenciano (2015b), as outlined above, and its results. Section 3 refines the specification of the bridge model and studies the impact of liberalizing cost-sharing for strong links. Finally, Section 4 contains some concluding comments on the results presented in the paper, points out some related literature and suggests possible further research.

2 A bridge model

We first briefly review the model introduced by Olaizola and Valenciano (2015b), where costs of strong links must be equally shared, as outlined in the introduction. Individuals may invest in *links* with other individuals, thus creating a network which can be described by a graph. Each node $i \in N$ represents an individual referred to as *player*[5] i. A map $g_i : N \backslash \{i\} \rightarrow \{0,1\}$ specifies the links in which player i invests. Denote $g_{ij} := g_i(j)$, and $g_{ij} = 1$ ($g_{ij} = 0$) means that i invests (does not invest) in a link with j. Thus, vector $g_i = (g_{ij})_{j \in N \backslash \{i\}} \in \{0,1\}^{N \backslash \{i\}}$ specifies the links in which i invests. $G_i := \{0,1\}^{N \backslash \{i\}}$ denotes the set of i's possible link-investments and $G_N = G_1 \times G_2 \times ... \times G_n$ the set of *link-investment profiles.* Each $g \in G_N$ univocally determines a graph or *network* (N, Γ_g) of links invested in, where $\Gamma_g := \{(i,j) \in N \times N : g_{ij} = 1\}$. If $g_{ij} = 1$ we equivalently write $ij \in g$, and if $g_{ij} = g_{ji} = 1$ write $\overline{ij} \in g$ and say that i and j are connected by a *strong link*, while when only one of them, i or j, invests in it they are said to be connected by a *weak link.* If $g_{ij} = 1$ in a graph g, $g - ij$ denotes the graph that results from replacing $g_{ij} = 1$ by $g_{ij} = 0$ in g; and if $g_{ij} = 0$, $g + ij$ denotes the graph that results from replacing $g_{ij} = 0$ by $g_{ij} = 1$. Similarly, if $g_{ij} = g_{ji} = 1$, $g - \overline{ij} = (g - ij) - ji$, and if $g_{ij} = g_{ji} = 0$,

[4] In Jackson and Wolinsky's (1996) model the only efficient structure which is pairwise stable for the whole range of values of the parameters for which it is efficient is the complete network of strong links, while the star of strong links is pairwise stable only within a subset of the region where it is efficient.

[5] A player can be an individual or a multi-agent entity. Because of this and in order to avoid biased language, we often refer to players by the more neutral term "nodes".

$g + \overline{ij} = (g + ij) + ji$. Given $g \in G_N$, a *path* of *length* $k \geq 1$ *from* i *to* j in g is a sequence of $k+1$ distinct nodes where i is the first and j the last, s.t. i and j, and any two consecutive nodes are connected by a link (weak or strong). $N(i; g)$ denotes the set of players connected with i by a path.

An *all-encompassing star* is a graph where one node is involved in links with all other nodes, and there are no other links. An *all-encompassing star* of weak links is said to be *periphery-sponsored* (*center-sponsored*) if the center supports no links (all links). A *complete* (*weak-complete*, *strong-complete*) graph is one where any two nodes are involved in a link (weak link, strong link). The *empty* network is the trivial network where no two players are connected by a link.

If g represents the links invested in by every player, the following is assumed:

1. Investment by a player in a link with another entails a *cost* $c > 0$.

2. Each player has a particular type of information or other good[6] of *value* 1 for any player who receives it complete.

3. Flow through links is *not* perfect (though it is better through strong links), so nobody else receives this information intact. Let δ $(0 < \delta < 1)$ be the fraction of the value of information at one node that reaches the other node through a *strong* link, and let α $(0 \leq \alpha \leq \delta < 1)$ be the fraction of the value of information at one node that reaches the other through a *weak* link. For a pair of nodes $i \neq j$, let $\mathcal{P}_{ij}(g)$ denote the set of paths in g from i to j. For each $p \in \mathcal{P}_{ij}(g)$, let $\ell(p)$ denote the length of p and $\omega(p)$ the number of weak links in p. Then i's valuation of the unit of information originating from j that arrives via p is

$$I_i(p) = \delta^{\ell(p)-\omega(p)}\alpha^{\omega(p)}.$$

If information at j reaches i via the best possible route from j to i, then i's valuation of information originating from j is

$$I_{ij}(g) = \max_{p \in \mathcal{P}_{ij}(g)} I_i(p),$$

and i's overall information is

$$I_i(g) = \sum_{j \in N(i;g)} I_{ij}(g).$$

Thus player i's payoff in g is:

$$\Pi_i(g) = I_i(g) - c\mu_i^d(g) = \sum_{j \in N(i;g)} \max_{p \in \mathcal{P}_{ij}(g)} \delta^{\ell(p)-\omega(p)}\alpha^{\omega(p)} - c\mu_i^d(g), \tag{1}$$

where $\mu_i^d(g)$ is the number of links (weak or strong) in which i invests.

[6] Although other interpretations are possible, in general, we give preference to the interpretation in terms of information.

Remark: Note that if $\alpha = 0$, this model is equivalent to Jackson and Wolinsky's (1996) connections model, where only strong links are feasible, and when $\alpha = \delta$ it is equivalent to Bala and Goyal's (2000) two-way flow model, where strong links are unstable and inefficient.

2.1 Efficiency

A network is said to be *efficient* if it maximizes the aggregate payoff. The following result characterizes the only efficient architectures and establishes the rank of values of the parameters for which each of them is efficient:

Proposition 1 (*Proposition* 3*, Olaizola and Valenciano* (2015*b*)) *If the payoff function is given by* (1) *with* $0 \leq \alpha \leq \delta < 1$*, then the only efficient networks are*:
(*i*) *The strong-complete graph if* $c < \min\{\delta - \delta^2, 2(\delta - \alpha)\}$.
(*ii*) *The weak-complete graph if*

$$2(\delta - \alpha) < c < 2\left(\alpha - \alpha^2\right)$$

and $c(n-4) < 2n\alpha - 4\delta - 2(n-2)\delta^2$.
(*iii*) *All-encompassing stars of strong links if*

$$\delta - \delta^2 < c < \min\{2(\delta - \alpha) + (n-2)\left(\delta^2 - \alpha^2\right), \delta + (n-2)\delta^2/2\}, \tag{2}$$

and

$$c(n-4) > 2n\alpha - 4\delta - 2(n-2)\delta^2. \tag{3}$$

(*iv*) *All-encompassing stars of weak links if*

$$\max\{2(\delta - \alpha) + (n-2)\left(\delta^2 - \alpha^2\right), 2\left(\alpha - \alpha^2\right)\} < c < 2\alpha + (n-2)\alpha^2. \tag{4}$$

(*v*) *The empty network if*

$$c > \max\{2\alpha + (n-2)\alpha^2, \delta + (n-2)\delta^2/2\}.$$

Remark: As a corollary, making $\alpha = 0$ in Proposition 1 yields the characterizing result of the efficient networks established in Jackson and Wolinsky's (1996) connections model; while by making $\alpha = \delta$ it yields the efficiency results in Bala and Goyal's (2000) two-way flow model.

2.2 Stability

In Jackson and Wolinsky's (1996) setting only strong links make sense and actually form, thus their notion of *pairwise stability* consists of two requirements: (i) no player has an incentive to sever a link, and (ii) no two players not linked have an incentive to create a strong link. Severing a link is the only unilateral option of a player in Jackson and Wolinsky's model, but in the current setting weak links can be created unilaterally and so can even strong links by making an existing weak link double. Thus, in this setting the notion of pairwise stability must be revised. We consider the following extension of pairwise stability, to which we refer in the same terms (though to avoid confusion with the original notion we add an asterisk) allowing for a player to invest in a new link, withdraw support for a link, or switch support from a link to another:

Definition 1 (*Olaizola and Valenciano,* 2015*b*) *A network g is pairwise* stable if for all i, j $(i \neq j)$:*
(*i*) *if $ij \in g$, then $\Pi_i(g - ij) \leq \Pi_i(g)$,*
(*ii*) *if $ij \notin g$, then $\Pi_i(g + ij) \leq \Pi_i(g)$,*
(*iii*) *if $ij \in g$, $ij' \notin g$, then $\Pi_i((g - ij) + ij') \leq \Pi_i(g)$, and*
(*iv*) *if $ij \notin g$, $ji \notin g$, and $\Pi_i(g + \overline{ij}) > \Pi_i(g)$, then $\Pi_j(g + \overline{ij}) < \Pi_j(g)$.*

The following result establishes the range of values of the parameters for pairwise* stability of each of the efficient networks:

Proposition 2 (*Proposition* 6*, Olaizola and Valenciano* (2015*b*)) *If the payoff function is given by* (1) *with $0 \leq \alpha \leq \delta < 1$, we have:*
(*i*) *If $0 < c < \min\{\delta - \delta^2, \delta - \alpha\}$, then the strong-complete graph is the unique pairwise* stable network.*
(*ii*) *If $\delta - \alpha < c < \alpha - \alpha^2$ and $\delta < 2\alpha/(1+\alpha)$, then weak-complete graphs are the unique pairwise* stable networks.*
(*iii*) *If $\delta - \delta^2 < c < \delta - \alpha$, then all-encompassing stars of strong links are pairwise* stable.*
(*iv*) *If $\delta - \alpha^2 < c < \alpha + (n-2)\alpha^2$, then all-encompassing periphery-sponsored stars of weak links are pairwise* stable.*
(*v*) *If $\max\{(\delta - \alpha)(1 + (n-2)\alpha), \delta - \alpha^2\} < c < \alpha$, then all all-encompassing stars of weak links are pairwise* stable.*
(*vi*) *If $c > \delta$, then the empty network is pairwise* stable.*

Remarks:

(i) Again, as a corollary, making $\alpha = 0$ in Proposition 2 yields the range of the parameters where each of the efficient structures are pairwise stable in Jackson and Wolinsky's (1996) connections model.

(ii) For none of the five structures do the region where each one is efficient and the region where it is pairwise* stable coincide. Those with strong links (strong-complete and all-encompassing stars of strong links) are pairwise* stable only within a subset of the region where they are efficient. For instance, all-encompassing stars of strong links are pairwise* stable only if

$$\delta - \delta^2 < c < \delta - \alpha, \tag{5}$$

while they are efficient in the much wider region where (2) and (3) hold. As to efficient structures formed by weak links, the regions where they are efficient and those where they are pairwise* stable are different[7]. Note that these results are obtained under the assumption that the cost of a strong link is to be shared equally by the two players supporting it and the cost of a weak link is paid by the player who supports it. As stated in the introduction, the point of this paper is to study the effect of relaxing this assumption by allowing players to freely agree on how they share the cost of strong links.

3 The impact of liberalizing cost-paying

Thus we consider a variation of the bridge model described in Section 2 where the *cost-shares of each strong link can be freely agreed upon* by the two players who form it.

In Jackson and Wolinsky's model and in the bridge model a network is completely specified by $g \in G_N$, which implies the unique link-investment that yields such a network: the cost of each weak link is paid for by the only player supporting it, and that of strong links is shared equally by the two players involved, and the assumed flow-level through each type of link follows from the assumptions of the model. However, if the way of *sharing the cost of each strong link can be freely settled* by the two players who form it, then for an analysis of stability a complete description of the network in this scenario is needed. This requires the investment of each player in each link in which he/she is involved to be specified by introducing a *matrix of cost-shares*, $\mathbf{c} = (c_{ij})_{i,j \in N}$, where $c_{ij} \geq 0$ (with $c_{ii} = 0$) is the investment of player i in the link connecting i and j. Moreover, the flow-level of each link in the resulting network must be specified. Let $\delta_{ij}(\mathbf{c}) := (\delta_{ij}(c_{ij}, c_{ji}))_{i,j \in N}$ be the flow-level through link ij in which i invests c_{ij} and j invests c_{ji}. Then assume:

$$\delta_{ij}(c_{ij}, c_{ji}) := \begin{cases} \delta, \text{ if } c_{ij} + c_{ji} \geq 2c, \\ \alpha, \text{ if } c_{ij} + c_{ji} < 2c \ \& \ \max\{c_{ij}, c_{ji}\} \geq c, \\ 0, \text{ if } \max\{c_{ij}, c_{ji}\} < c. \end{cases} \tag{6}$$

Remarks: (i) Notice that (6) defines a *matrix of flow-levels* $\delta(\mathbf{c})$ which univocally specifies a network of strong and weak links $g(\mathbf{c}) \in G_N$ *for any conceivable*[8] *link-investment of the players* $\mathbf{c}$. Namely, $g(\mathbf{c}) = g$ s.t. $g_{ij} = g_{ji} = 1$ (strong link) in the first case ($c_{ij} + c_{ji} \geq 2c$), $g_{ij} = 1$ and $g_{ji} = 0$ (weak link) in the second case if i is the only player which invests at least c, and $g_{ij} = g_{ji} = 0$ (no link) in the third case ($\max\{c_{ij}, c_{ji}\} < c$). Note that *different* $\mathbf{c}$'s *may yield the same* $g(\mathbf{c})$. If $g \in G_N$, and $g = g(\mathbf{c})$, $\mathbf{c}$ is said to be *consistent* for g.

[7] In the case of non periphery-sponsored stars of weak links, the intersection of these regions is empty for certain values of the parameters.

[8] This is required in order to avoid undefined situations once we specify, as we do presently, the admissible deviations w.r.t. which stability is to be defined.

(ii) Now the payoff function (1) can be rewritten in terms of $g = g(\mathbf{c})$ as

$$\Pi_i(g, \mathbf{c}) = \sum_{j \in N(i;g)} \max_{p \in \mathcal{P}_{ij}(g)} \delta^{\ell(p)-\omega(p)} \alpha^{\omega(p)} - \sum_{j:ij \in g} c_{ij}. \tag{7}$$

(iii) Note that the efficiency of a network $(g, \mathbf{c})$ depends on the costs of links (weak and strong) but *not* on how the cost of each link is paid for. In other words, the results relative to efficiency in Proposition 1 apply to this scenario, i.e. the efficient structures are the same for the same ranges of values of the parameters[9]. However, the possibility of freely sharing the cost of strong links may affect stability. In fact, the extension of Jackson and Wolinsky's notion of pairwise stability provided in Definition 1 *must be further revised* in this setting. To begin with, the actions allowed for players, given a network $g(\mathbf{c})$, w.r.t. which a notion of stability is to be formulated must be specified. In a strictly non-cooperative context each player i would freely choose[10] $(c_{ij})_{j \in N \setminus i}$, while the pairwise stability notion, in both Jackson and Wolinsky (1996) and its extension in Definition 1, allows coordination to create strong links but *restricts* the admissible unilateral moves of players (w.r.t. Nash equilibrium). Consistently, we assume the following *admissible unilateral moves*:

- *Any player* can modify (increase or decrease) the investment in *one* link.

- *Any player* can modify the investment in *two* links by transferring part of the investment in one of them to the other[11].

We also assume *feasible bilateral moves*:

- *Any two players* not connected by a strong link, (i.e. disconnected or connected by a weak one) can create a strong link *and* share freely its cost.

The following definition formalizes the idea of stability w.r.t. these actions extending Definition 1.

Definition 2 *A network $g \in G_N$ admits a pairwise stable cost-share allocation (CSA), if there exists a matrix of cost-shares $\mathbf{c}$ consistent for g s.t. no player has an incentive to make any unilateral move and no pair of players has an incentive to create a new strong link. It is said then that $\mathbf{c}$ is a pairwise stable cost-share allocation (CSA) for g.*

Depending on the network g and the values of the parameters, a pairwise stable cost-share allocation may not exist for g, and when such allocation do exist, they are generally *not* unique. But when a network admits a pairwise stable cost-share allocation, it represents a feasible outcome *stable w.r.t. admissible unilateral and bilateral actions* which adapts pairwise stability to the current scenario. We now address the

[9] Just observe that if $g(\mathbf{c}) = g \in G_N$, the cost of $g(\mathbf{c}) = g$ cannot be lower than that of g in the context of the bridge model.

[10] Which could properly be called i's strategy, as along with (7) it specifies a non-cooperative game.

[11] This move is the natural extension to this setting of the possibility of switching support stated in condition (iii) in Defiinition 1.

question of the existence of pairwise stable CSAs for the efficient structures characterized by Proposition 1.

In order to simplify the proofs the following easy lemma, whose proof is omitted, stating that a necessary condition for CSA-pairwise stability is that no player wastes money, will be of use:

Lemma 1 *If $g \in G_N$ and $g = g(\mathbf{c})$, for $\mathbf{c}$ to be a pairwise stable cost-share allocation for g the following are necessary conditions*:
(i) If $ij \in g$ and $ji \notin g : c_{ij} = c$ and $c_{ij} = 0$.
(ii) If $\overline{ij} \in g : c_{ij} + c_{ji} = 2c$.
(iii) If $ij \notin g$ and $ji \notin g : c_{ij} = c_{ji} = 0$.

Proof. Assume $g \in G_N$ and $g = g(\mathbf{c})$. It is immediate to check that if any of the three conditions does not hold for a pair of players then at least one of the two players has an incentive to make an admissible move. ■

The following result establishes the impact of assuming free-sharing of costs on the stability of the five efficient structures characterized by Proposition 1, by establishing conditions under which each of them admits a pairwise stable cost-share allocation. A comparison with the conditions for pairwise* stability in Proposition 2 is discussed later.

Proposition 3 *If the payoff function is given by* (7) :
(i) A pairwise stable cost-share allocation exists for the strong-complete network whenever the following condition holds

$$c \leq \min\{\delta - \delta^2, \delta - \left(\delta^2 + \alpha\right)/2, 2\left(\delta - \alpha\right)\}.$$

(ii) A pairwise stable cost-share allocation exists for a weak-complete network whenever the following conditions hold

$$2\left(\delta - \alpha\right) \leq c \leq \alpha - \alpha^2.$$

(iii) A pairwise stable cost-share allocation exists for an all-encompassing star of strong links whenever the following conditions hold

$$\delta - \delta^2 \leq c \leq \min\{(\delta - \alpha)(2 + (n-2)\delta), \delta\left(1 + (n-2)\delta/2\right) - \alpha/2\}.$$

(iv) A pairwise stable cost-share allocation exists for a periphery-sponsored all-encompassing star of weak links whenever the following conditions hold

$$\max\{\delta - \alpha^2, (\delta - \alpha)\left(2 + (n-2)\,\alpha\right)\} \leq c \leq \alpha + (n-2)\,\alpha^2.$$

(v) A pairwise stable cost-share allocation exists for an all-encompassing star of weak links whenever the following conditions hold

$$\max\{\delta - \alpha^2, (\delta - \alpha)\left(2 + (n-2)\,\alpha\right)\} \leq c \leq \alpha.$$

(vi) The trivial allocation of costs ($c_{ij} = 0$ for all i, j) is a pairwise stable cost-share allocation for the empty network whenever the following condition holds

$$c \geq \delta.$$

Proof. (i) Let g be the *strong-complete* network. First, observe that if the cost of each link in g is shared equally by the two players forming it, i.e. $\mathbf{c} = (c_{ij})_{i,j \in N}$ with $c_{ij} = c_{ji} = c$ for all $i \neq j$, then $\mathbf{c}$ is a pairwise stable CSA for g in the whole region where g is pairwise* stable (i.e. $0 < c \leq \min\{\delta - \delta^2, \delta - \alpha\}$ according to Proposition 2-(i)[12]). Just note that in this case the feasible moves w.r.t. which CSA-pairwise stability is defined do not actually extend the options of those w.r.t. which pairwise* stability is defined.

Now observe that allowing for *non*-egalitarian cost-sharing increases the range of values of the parameters where the strong-complete network admits a pairwise stable CSA. Assume $\mathbf{c} = (c_{ij})_{i,j \in N}$ with $c_{ij} \neq c_{ji}$ for a strong link $\overline{ij}$, and assume w.l.o.g. $c_{ij} > c_{ji}$. In view of Lemma 1, assume $c_{ij} + c_{ji} = 2c$. Thus, the only possibly improving admissible moves for player j entail decreasing investment in the link, but among them the optimal one is withdrawing support to the link, for which j has no incentive if $\delta - c_{ji} \geq \max\{\delta^2, \alpha\}$, that is if

$$c_{ji} \leq \min\{\delta - \delta^2, \delta - \alpha\}. \tag{8}$$

There are two possibly optimal admissible moves for player i. First, to withdraw support from $\overline{ij}$, for which i has no incentive if $\delta - c_{ij} \geq \delta^2$, that is if

$$c_{ij} \leq \delta - \delta^2; \tag{9}$$

and second, to lower the investment in the link to c, for which i has no incentive if $\delta - c_{ij} \geq \max\{\delta^2, \alpha\} - c$, that is if

$$c_{ij} \leq \min\{\delta - \delta^2, \delta - \alpha\} + c. \tag{10}$$

The three conditions along with $c_{ij} + c_{ji} = 2c$ are compatible if

$$2c \leq \delta - \delta^2 + \min\{\delta - \delta^2, \delta - \alpha\} \text{ and}$$

$$2c \leq c + 2\min\{\delta - \delta^2, \delta - \alpha\}.$$

That is, if

$$c \leq \min\{\delta - \delta^2, \delta - \left(\delta^2 + \alpha\right)/2\} \text{ and}$$

$$c \leq 2\min\{\delta - \delta^2, \delta - \alpha\}.$$

[12] Note that in Proposition 2-(i) both inequalities are strict to ensure uniqueness, but pairwise* stabilty is also guaranteed at the boundary where $c = \delta - \delta^2$.

That is, if

$$c \le \min\{\delta - \delta^2, \delta - \left(\delta^2 + \alpha\right)/2, 2\left(\delta - \alpha\right)\}. \tag{11}$$

Finally, note that, this condition is weaker than the one for pairwise* stability. Just note that the strong-complete network is pairwise* stable if $c \le \min\{\delta - \delta^2, \delta - \alpha\}$, and $c = \delta - \alpha$, $c = \delta - \delta^2$ and $c = \delta - \left(\delta^2 + \alpha\right)/2$ intersect at $\delta = \sqrt{\alpha}$, and for $0 < \delta \le \sqrt{\alpha}$ we have

$$\left(\delta - \left(\delta^2 + \alpha\right)/2\right) - \left(\delta - \alpha\right) \ge \alpha - \left(\alpha + \alpha\right)/2 = 0.$$

(*ii*) Let g be a *weak-complete* network. In view of Lemma 1, assume that each link is supported by only one player, who pays c for it, i.e. $\mathbf{c} = (c_{ij})_{i,j \in N}$ with $\max\{c_{ij}, c_{ji}\} = c$ and $\min\{c_{ij}, c_{ji}\} = 0$ for all $i \neq j$. The only action not included in the repertoire considered for pairwise* stability but possibly improving is to form a strong link and *freely share its cost.* Therefore within the region where this structure is pairwise* stable, i.e. $\delta - \alpha < c < \alpha - \alpha^2$ and $\delta < 2\alpha/\left(1+\alpha\right)$ (see Proposition 2-(*ii*)), no action can improve a player's payoff, *except perhaps forming a strong link.* Let us see how this new option actually further *restricts* this region. In Olaizola and Valenciano's (2015b) model the only way in which this could be done is by one player "doubling" a weak link, for which there is no incentive if $c \ge \delta - \alpha$. But now players forming a strong link can freely share its cost. Assume that the weak link ij is supported by i, and i and j envisage forming a strong link and paying c_{ij} and c_{ji} for it. Both players would have incentives to do so if $\alpha - c < \delta - c_{ij}$ and $\alpha < \delta - c_{ji}$, which are compatible with the necessary condition $c_{ij} + c_{ji} = 2c$ (Lemma 1) if $c < 2\left(\delta - \alpha\right)$. In other words, no pair of players has an incentive to make a strong link if $c \ge 2\left(\delta - \alpha\right)$. Therefore, the weaker condition $c \ge \delta - \alpha$ for pairwise* stability (Proposition 2-(*ii*)) must be replaced by this *stronger* one for the existence of a pairwise stable CSA for g. Moreover, this condition along with $c < \alpha - \alpha^2$ implies $\delta < 2\alpha/\left(1+\alpha\right)$. Therefore g admits a pairwise stable CSA if

$$2\left(\delta - \alpha\right) \le c \le \alpha - \alpha^2.$$

(*iii*) Let g be an *all-encompassing star of strong links.* Let i_o be its center and j any peripheral node. Let c_{i_oj} and c_{ji_o} be the shares of the cost to be paid by each of them. In view of Lemma 1 assume $c_{i_oj} + c_{ji_o} = 2c$.

It seems natural to expect a wider range of feasible agreements when the center does not pay more than a peripheral player for the link connecting them[13], thus assume: $c_{i_oj} \le c_{ji_o}$ for some j. If $c_{i_oj} = c_{ji_o}$ for all j, then $\mathbf{c}$ is pairwise stable CSA for g in the same region where according to Proposition 2-(*iii*) the star of strong link is pairwise* stable, as in this case the feasible moves w.r.t. which CSA-pairwise stability is defined do not actually widen the options of those w.r.t. which pairwise* stability is defined. If $c_{i_oj} < c_{ji_o}$ for all j, then for $\mathbf{c}$ to be a pairwise stable CSA for g the following conditions must hold. For the center to have an incentive to pay c_{i_oj}, $\delta - c_{i_oj} \ge \alpha$ must hold. That is

$$c_{i_oj} \le \delta - \alpha. \tag{12}$$

[13] In fact, there also exist pairwise stable CSAs with $c_{i_oj} > c_{ji_o}$, but within a smaller region.

For a peripheral player to have an incentive to pay c_{ji_o}, two conditions must hold. First,

$$c_{ji_o} \leq \delta(1 + (n-2)\delta), \tag{13}$$

otherwise it would be profitable for j to withdraw support for the link. Second,

$$\delta(1 + (n-2)\delta) - c_{ji_o} \geq \alpha(1 + (n-2)\delta) - c$$

i.e.

$$c_{ji_o} \leq (\delta - \alpha)(1 + (n-2)\delta) + c, \tag{14}$$

otherwise player j's payoff could be improved by lowering the investment in the link to c. On the other hand, no two peripheral players are interested in creating a strong link if $\delta - c \leq \delta^2$, i.e.

$$c \geq \delta - \delta^2, \tag{15}$$

because in that case no share of the cost can improve the payoff of one player without decreasing that of the other. Thus all three conditions (12), (13), and (14) are necessary for there to be a margin for sharing the cost, in other words, for $\mathbf{c}$ to be a pairwise stable CSA for g. Again by Lemma 1, summing up (12) and (13), and (12) and (14), yields two upper bounds for $2c$. That is

$$c_{i_oj} + c_{ji_o} = 2c \leq \min\{(\delta - \alpha)(2 + (n-2)\delta) + c, \delta\,(2 + (n-2)\delta) - \alpha\},$$

which along with (15) yields the range of values of the parameters where an all-encompassing star of strong links admits a pairwise stable CSA:

$$\delta - \delta^2 \leq c \leq \min\{(\delta - \alpha)(2 + (n-2)\delta), \delta\,(1 + (n-2)\delta/2) - \alpha/2\}. \tag{16}$$

(*iv*)-(*v*) Assume g is an *all-encompassing star of weak links*. Under the conditions for which such structure is pairwise* stable, that is, Proposition 2-(*iv*) or (*v*) if it is periphery-sponsored, and (*v*) otherwise, no admissible unilateral action can improve the payoff of a player or form a new strong link between two peripheral players. All that remains is to study the possibility of creating a new strong link between the center and a peripheral player and *freely agreeing how to share its cost.*

Let i_o be the center and j any peripheral node. By Lemma 1, assume that each link is supported by only one player who pays c for it, i.e. $\mathbf{c} = (c_{ij})_{i,j \in N}$ with $\max\{c_{i_oj}, c_{ji_o}\} = c$ and $\min\{c_{i_oj}, c_{ji_o}\} = 0$ for all j. Assume that a peripheral node, j, supports the weak link with the center i_o. Player j has an incentive to form a strong link with i_o and pay for it c_{ji_o} if $\alpha + (n-2)\,\alpha^2 - c < \delta + (n-2)\,\alpha\delta - c_{ji_o}$, and i_o has an incentive to pay for it c_{i_oj} if $\alpha < \delta - c_{i_oj}$. Assuming by Lemma 1 $c_{i_oj} + c_{ji_o} = 2c$, there is no room for both conditions if

$$c \geq (\delta - \alpha)\,(2 + (n-2)\,\alpha)\,.$$

The same condition is obtained if the weak link is supported by the center. Therefore, adding this condition to those in Proposition 2-(iv) and (v) shows that a periphery-sponsored all-encompassing star of weak links admits a pairwise stable CSA if:

$$\max\{\delta - \alpha^2, (\delta - \alpha)(2 + (n-2)\alpha)\} \leq c \leq \alpha + (n-2)\alpha^2,$$

and *any* all-encompassing star of weak links admits a pairwise stable CSA if:

$$\max\{\delta - \alpha^2, (\delta - \alpha)(2 + (n-2)\alpha)\} \leq c \leq \alpha.$$

A comparison of these conditions with those established in Proposition 2-(iv) and (v) clearly shows that the conditions for CSA-pairwise stability are stronger than those for pairwise* stability.

(vi) Let g be the empty network, with respect to pairwise* stability the new option of forming strong links and sharing their cost in any way offers no chance of improving any two players payoffs if $c \geq \delta$. ■

Remarks:

(i) In view of Proposition 3-(i), the strong-complete network admits a pairwise stable CSA whenever (11) holds. As shown in the proof of Proposition 3-(i), these conditions are *weaker* than those under which the strong-complete network is pairwise* stable (Proposition 2-(i)). Therefore, interestingly enough, the possibility of asymmetry in the way of sharing the cost of links in a strong-complete network *extends* the region where this entirely symmetric structure can be stabilized. Counterintuitive as it may seem at first sight, the reason is clear: in comparison with the situation of equal sharing of the cost of strong links, the player paying less for a link has less incentive to withdraw support for it. As to the player paying more, withdrawing support for the link would make it disappear altogether, which makes him willing to pay more than c for it.

Comparing Proposition 1-(i), Proposition 2-(i) and Proposition 3-(i) reveals the following: the region where the strong-complete network is *efficient and not pairwise* stable, but admits a pairwise stable CSA* is

$$\delta - \alpha < c < \min\{2(\delta - \alpha), \delta - \left(\delta^2 + \alpha\right)/2\}.$$

Note however that this is so *only if the costs of all* links are shared asymmetrically and conditions (8), (9), (10) and $c_{ij} + c_{ji} = 2c$ hold for all i, j. While in the region

$$c < \min\{\delta - \delta^2, \delta - \alpha\},$$

where the strong-complete network is *efficient and pairwise* stable*, a pairwise stable CSA may include *both* symmetrically and asymmetrically cost-shared links, as far as (8), (9), (10) and $c_{ij} + c_{ji} = 2c$ for all i, j hold for the asymmetrically cost-shared links.

Figure 1 represents the region where the strong-complete network is efficient and admits a pairwise stable CSA for $n = 20$ and $\alpha = 0.2$ (Figure 1-a) and $\alpha = 0.6$ (Figure

1-b) bounded by continuous thick lines, while the region where it is pairwise* stable is the *smaller* region below the dashed line.

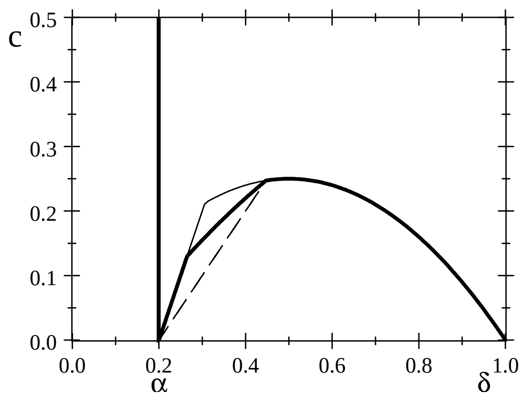

Figure 1-a: Strong-complete
$\alpha = 0.2$

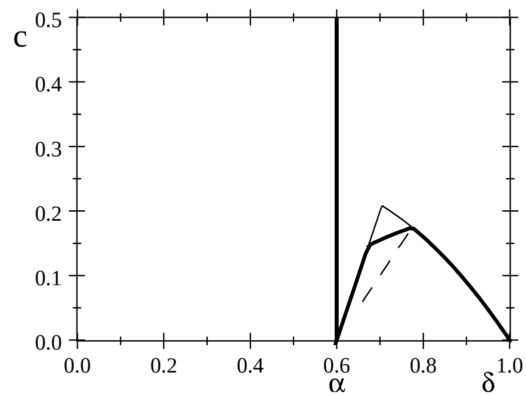

Figure 1-b: Strong-complete
$\alpha = 0.6$

(ii) A comparison of Proposition 2-(*ii*) and Proposition 3-(*ii*) shows clearly that the region where weak-complete networks are stable *shrinks* when costs are freely-shared. But observe that the region where a weak-complete network is stable in either sense, i.e. is pairwise* stable or admits a pairwise stable CSA, *and efficient is the same.*

Figure 2 represents this region for $n = 20$ and $\alpha = 0.2$ (Figure 2-a) and $\alpha = 0.6$ (Figure 2-b) bounded by continuous thick lines, while the region where it is pairwise* stable is the *greater* region that results by expanding it rightwards up to the dashed line. The region above, bounded by a thin continuous line, is where this structure is efficient but is not stable in either sense.

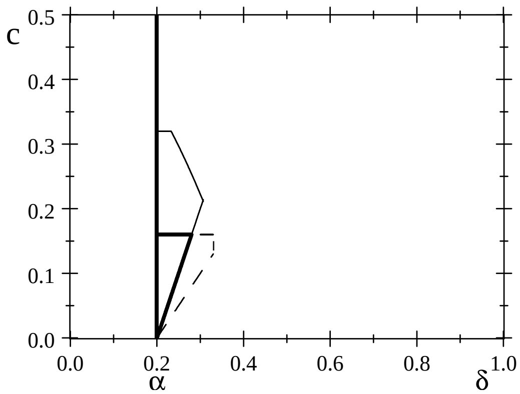

Figure 2-a: Weak-complete
$\alpha = 0.2; n = 20$

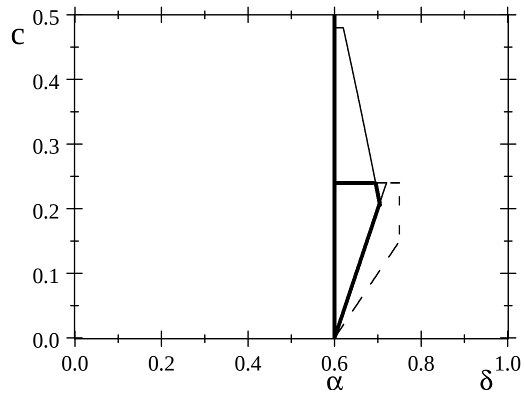

Figure 2-b: Weak-complete
$\alpha = 0.6; n = 20$

(*iii*) Note that pairwise* stability conditions for a star of strong links (5) require $c \leq \delta - \alpha$, while if free-sharing of the cost of strong links is assumed a pairwise stable CSA is admitted for a considerably *wider* set of values of the parameters. It also follows from (16) that the number of players contributes to the widening of the range of values of the other parameters where a star of strong links admits a pairwise stable CSA, while the greater α is the more stringent this condition becomes.

Figure 3 represents this region for $n = 20$ and $\alpha = 0.2$ (Figure 3-a) and $\alpha = 0.6$ (Figure 3-b) enclosed by continuous thick lines, which is the intersection of two regions bounded by thin continuous lines, where the star of strong links is efficient and admits a pairwise stable CSA, while the region where it is pairwise* stable is the considerably *smaller* subset of this region below the dashed line.

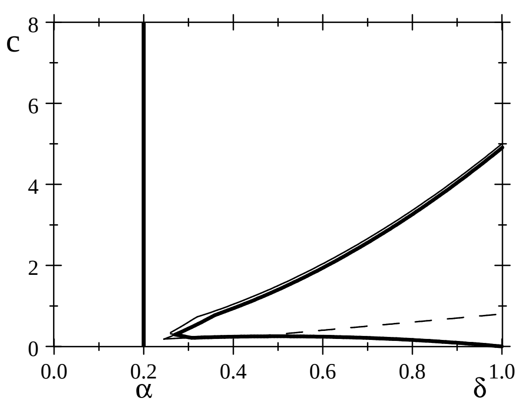

Figure 3-c: Strong-stars
$\alpha = 0.2; n = 10$

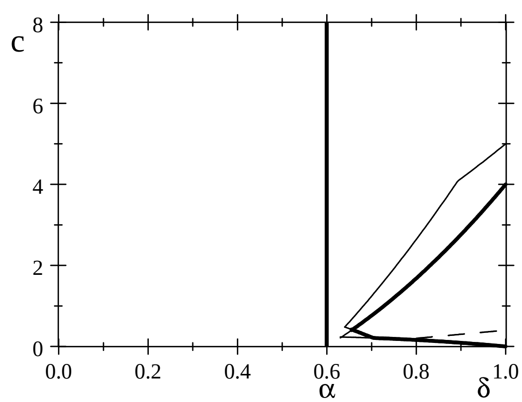

Figure 3-d: Strong-stars
$\alpha = 0.6; n = 10$

(iv) Again, as with weak-complete networks, free-sharing of the cost of strong links *reduces* the region of stability of stars of weak links. Among the stars of weak links, periphery-sponsored ones are also stable in this sense in a wider region. But, as with weak-complete networks, the intersection of the region where an all-encompassing star (periphery-sponsored or not) of weak links is efficient and that where it is pairwise* stable is the same to the intersection with the region where it admits a pairwise stable CSA.

For the case of a periphery-sponsored all-encompassing star of weak links, Figure 4 represents this region for $n = 20$ and $\alpha = 0.2$ (Figure 4-a) and $\alpha = 0.6$ (Figure 4-b) bounded by continuous thick lines, while the region where it is pairwise* stable is the *greater* region that results by expanding rightwards up to the dashed line. The region above, bounded by a thin continuous line, is where this structure is efficient but is not stable in either sense.

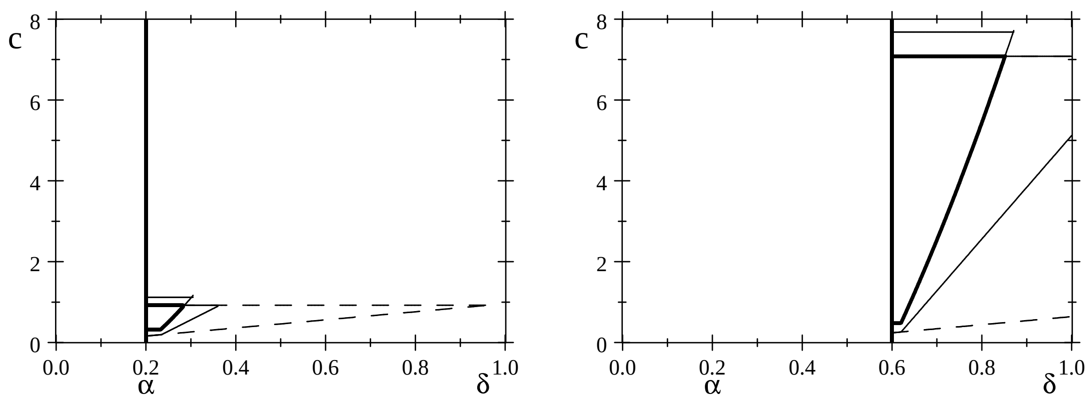

Figure 4-a: Weak-stars
$\alpha = 0.2; n = 20$

Figure 4-b: Weak-stars
$\alpha = 0.6; n = 20$

(v) Finally, the empty network admits a pairwise stable CSA in the same region where it is pairwise* stable, that is, for $c \geq \delta$.

(vi) The extreme case $\alpha = 0$ corresponds to Jackson and Wolinsky's (1996) model. In this case condition (11) in Proposition 3-(i) becomes $c \leq \delta - \delta^2$ which is the region where the complete network in their model is efficient and pairwise stable (Propositions 1 and 2, Jackson and Wolinsky, 1996), and consequently free-sharing of costs does *not* extend the stability region of the complete network, which is pairwise stable in their model.

As for all-encompassing stars of strong links, condition (16) becomes

$$\delta - \delta^2 \leq c \leq \delta + (n-2)\delta^2/2$$

when $\alpha = 0$, which is the region where such structures are efficient (Proposition 1, Jackson and Wolinsky, 1996). This means that in Jackson and Wolinsky's (1996) model all-encompassing stars can be stabilized by cost-share equilibrium allocations *within the whole region where they are efficient.* Thus the following corollary arises for Jackson and Wolinsky's model:

Corollary 1 *If the payoff function is given by* (7) *with* $\alpha = 0$*, i.e. in Jackson and Wolinsky's* (1996) *model*:
(*i*) *A pairwise stable cost-share allocation exists for the strong-complete network if and only if it is efficient.*
(*ii*) *A pairwise stable cost-share allocation exists for an all-encompassing star of strong links if and only if it is efficient.*

4 Concluding remarks

We explored the impact of extending the possibility of coordination to form strong links to how their cost is shared within the model introduced in Olaizola and Valenciano (2015b). In a context where any two players can coordinate to create such links, it seems natural to extend the possibility of pairwise coordination to cost-sharing. We show that this possibility favors the formation and stability of strong links to the detriment of weak ones. Specifically, it extends the range of values where the strong-complete network and all-encompassing stars of strong links are stable. Moreover, the same is true for the region where such structures are stable *and* efficient. By contrast, this possibility reduces the range of values where weak-complete networks and all-encompassing stars of weak links are stable, but *not* the region where such structures are stable *and* efficient, which remains the same. In other words, liberalizing cost-sharing never negatively affects the stability of efficient structures.

These conclusions apply in their strongest terms to Jackson and Wolinsky's (1996) model, which corresponds to the particular case $\alpha = 0$ in ours. In this case, the non-empty[14] efficient structures admit a pairwise stable cost-share allocation *whenever they are efficient* and then alone.

4.1 Related literature

As mentioned in the introduction, Meléndez-Jiménez (2008) (see also Meléndez-Jiménez (2007)) considers a network formation model where cost shares are not exogenously fixed. But this is the only point in common with the model considered here. In his model, links are a direct source of revenue for the players who form them, determined by a stag-hunt coordination game between them, who bargain the resulting benefit. He analyzes the model in both static and dynamic settings, showing that while the static

[14]Note that the empty network admits a pairwise stable cost-share allocation in the same region where it is pairwise* stable.

game has multiple equilibria only one is stochastically stable. The differences between this model and the one presented in this paper are clear. First, our setting is an extension of Jackson and Wolinsky's (1996) connections model where the information that runs through the network is *the only source of revenue*, not links themselves as in Meléndez-Jiménez (2008), where the resulting network is only a by-product of bilateral negotiations on benefits accessible to every pair of players. Second, in our setting, negotiation between the players who form each link can implicitly be interpreted as the source of cost-shares, but we pass over this bargaining process and concentrate on the stability of possible outcomes of those processes, while explicit assumptions about the formation of links in Meléndez-Jiménez (2008) allow him to specify such bargaining problems. The stochastic dynamic analysis is the most important ingredient in the approach in his model, while in ours dynamics is completely missing, and is mentioned later as a line of further research.

A more closely related paper is Olaizola-Valenciano (2015b), which addresses a similar study to the one undertaken here, but based on a simpler model that also allows weak and strong links introduced in Olaizola-Valenciano (2015a). This is a first step toward Olaizola-Valenciano (2015c), which is the starting point here.

4.2 Further work

The model studied here suggests other variations, perhaps extending the possibility of liberalizing cost-sharing to links of two or more levels of strength and cost. On the other hand, in the model presented here we have not addressed the question of how the network forms, i.e. we have not provided a dynamic model of network formation that gives support for pairwise stable cost-share allocations, which suggests an interesting line for extending the model.

References

[1] Bala, V., and S. Goyal, 2000, A non-cooperative model of network formation, *Econometrica* 68, 1181-1229.

[2] Goyal, S., 2007, *Connections. An Introduction to the Economics of Networks*, Princeton University Press. Princeton.

[3] Jackson, M., 2008, *Social and Economic Networks*, Princeton University Press. Princeton.

[4] Jackson, M., and A. Wolinsky, 1996, A strategic model of social and economic networks, *Journal of Economic Theory* 71, 44-74.

[5] Meléndez-Jiménez, M.A., 2007, A bargaining approach to coordination in networks,. *IVIE, WP-AD* 2002-27.

[6] Meléndez-Jiménez, M.A., 2008, A bargaining approach to coordination in networks, *International Journal of Game Theory* 37, 439-456.

[7] Olaizola, N., and F. Valenciano, 2015a, Unilateral vs. Bilateral link-formation: A transition without decay, *Mathematical Social Sciences* 74, 13-28.

[8] Olaizola, N., and F. Valenciano, 2015b, A unifying model of network formation *WP Ikerlanak* IL 85/15

[9] Olaizola, N., and F. Valenciano, 2015c, Efficiency vs. stability in a mixed network formation model. *WP Ikerlanak* IL 87/15

[10] Vega-Redondo, F., 2007, *Complex Social Networks*, Econometric Society Monographs, Cambridge University Press.