Selected topics in Financial engineering: First-exit times and dependence structures of Marshall–Olkin kind

Lexuri Fernández Loroño

Dissertation presented for the degree of Philosophiæ Doctor

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Universidad del País Vasco
Euskal Herriko Unibertsitatea

under the supervision of

Matthias Scherer and Luis Ángel Seco

Foundations of Economic Analysis II
Faculty of Economics and Business Studies
University of the Basque Country
Bilbao, 2015

To my parents,
for each day we spent far away from each other,
and to my friend Pablo,
for his genuine friendship.

Abstract

First-exit time problems in different settings are investigated. An accurate and efficient Monte Carlo simulation technique to estimate first-exit time probabilities of a jump diffusion process with two constant barriers is implemented. This technique gives rise to applications such as the pricing of financial derivatives whose value depend on (a) barrier(s) hitting event and in structural credit risk models with *early payment* and *default* events. Furthermore, the probability distribution of the mean of default times, which are dependent under the Marshall–Olkin law, is computed. The Marshall–Olkin distribution is a core probability law in reliability and life-testing applications. Exact expressions for the distribution of the mean of default times are derived in the general bivariate case and for low dimensions in the exchangeable one. When the dimension tends to infinity, we prove that the mean of these dependent default times converges to the exponential functional of a Lévy subordinator. Finally, different simulation techniques to simulate Lévy-frailty copulas, that are built from an α-stable Lévy subordinator, are analysed in terms of computational speed. The possibility to simulate these copulas in an efficient way allows us to numerically compute, with a low computational cost, the exponential functional of a Lévy subordinator.

Acknowledgements

First and foremost I would like to thank my supervisors Matthias Scherer and Luis Seco. Their exceptional guidance and support made my PhD experience productive and memorable. I would like to express my special appreciation to Matthias for his contributions with ideas, insights, and time. His experience and enthusiasm for maths encouraged me to work hard and allowed me to grow as a scientific researcher. I am also very grateful to Luis. His insightful advices and inestimable help have been priceless on my career. Furthermore, I would like to thank José María Usategui for his unconditional support.

I thank my co-authors Peter Hieber and Jan-Frederik Mai. It has been a valuable experience to work together. The astute discussions have been very fruitful.

I gratefully acknowledge all the members of the committee of my Ph.D. defense.

I would like to thank the University of the Basque Country and specially the Department of Foundations of Economic Analysis II for providing me with the FPI-UPV/EHU grant. Special thanks go to Miguel Ángel Martínez and Ines García. Furthermore, I would like to thank the Technical University of Munich and specially Rudi Zagst for hosting me during my research visits and for the financial support. The teaching experience and the opportunity to participate in several scientific conferences have been exceptionally useful on my research career.

I would like to express special gratitude to my friend and officemate Asma. Sharing mathematical and non-mathematical concerns, laughs and cries made this trip pleasant and fruitful. My great thanks go to my friends and *chicas súper bien* Ángela and Dani. Their friendship have been priceless in defeat and successful moments. I will forever be thankful to my friends Alejandro, Germán, Pablo and Pedro. They taught me that there is always a positive to every negative. My time at the Technical University of Munich was made enjoyable in large part due to many friends at the Chair of Mathematical Finance.

Last but not least, my heartfelt thanks go to my parents Arantza and Vidal for raising me with the love of science. I am grateful to my brother Aitor for being my support in the moments when there was no one to answer my queries. Words can not express how thankful I am to my grandmother María Luisa for her faithful support in all my goals.

I would like to thank all the people who contributed in the success of this Ph.D. thesis and express my apology because I could not personally mention everyone of them here.

Lexuri Fernández
Munich, June 2015

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1 Introduction

> ``Talent is luck. The important thing in life is courage.''
> Manhattan (1979).

The efficient management of various risk sources (market risk, default risk, etc.) is a crucial concern for all agents acting on financial markets. An inappropriate treatment of the risks that can be faced within financial activities repeatedly lead to economic catastrophes in history. Examples of financial disasters with a significant influence on society can be found in the stock market crash which took place in 1987 or the global financial crisis generated in 2008. Consequently there exists a necessity to develop sophisticated tools that provide security against financial crisis.

Besides the mentioned tools that protect financial markets against negative events, the well understanding of markets is also essential to avoid downturns. One example of an "unrealistic" assumption is to consider the elements in financial markets to be independent. The joint behaviour of assets, and therefore the joint behaviour of portfolios in which these assets are contained, is a statistical fact that should not be neglected when implementing stochastic models. These comovement effects claim an appropriate treatment of the dependent multivariate stochastic models which is fundamental in asset pricing and risk management. These are reasons why dependence structures achieve a significant importance.

The problems considered in this thesis are divided into two parts: in the first one first-exit time probabilities are analysed. They provide efficient tools to price certain options and are thus relevant to understand market risk. Moreover, they apper in the context of credit risk via the framework of structural default models. The second part aims at investigating the dependence structures of Marshall–Olkin type.

One way to reduce the exposure to risk is to apply hedging strategies, i.e. insuring oneself from negative events. Barrier options can be used to construct an appropriate hedge in different situations. As a consequence an accurate method of pricing them will ensure an efficient protection against the risk of losses. However, a common situation in mathematical finance is the fact that it is just possible to observe quantities of interest at a finite number of moments, while reality is happening in continuous time. Therefore the information between these observation moments is missing. This is the reason for the necessity of procedures that help understanding what occurs in these "blind" periods. The mentioned circumstance is obvious in applications where the event of reaching a threshold or an objective is crucial such as in pricing barrier options. Here resides one of the reasons to analyse first-exit times. In the same direction first-exit time probabilities are useful for estimating default probabilities in structural credit-risk approaches.

Diffusion processes have been used in the literature over the years to model stock prices, see e.g., [Black and Scholes, 1973], [Barndorff-Nielsen and Shephard, 2001], [Merton, 1976], [Duffie et al., 2000]. Nevertheless, none of these models simultaneously covers the properties present in real applications such as leptokurtic stock returns or jumps in stock prices. And the models that cover some of these characteristics possess the disadvantage of analytical complexity to get closed-form solutions of first-passage time probabilities. A possibility to circumvent this disadvantage is to use the jump-diffusion process proposed by [Kou, 2002], [Scott, 1997].

One of the applications motivating this research is the computation of first-passage time probabilities of a jump-diffusion process. Analytical solutions exist when the jump size follows the double exponential distribution: an explicit solution of the Laplace transform of the first-exit time is obtained in [Kou and Wang, 2003] for a single barrier and in [Sepp, 2004] for the double barrier case. However when one aims at extending this problem to models with arbitrary jump-size distributions or non-constant barriers, the question becomes analytically challenging and in most cases numerical schemes have to be considered.

Different numerical techniques have been treated in the literature for this purpose, e.g., [Boyarchenko and Levendorskiĭ, 2002], [Boyarchenko and Levendorskiĭ, 2012], [Cont and Voltchkova, 2005]. We rely on Monte Carlo simulations. Monte Carlo methods have become a key tool in the treatment of different problems in financial engineering. These simulations possess the property of being simple to understand and work with. The basic idea behind this numerical method is to simulate trajectories over a temporal path and to compute the average of the final values of the sample paths.

The standard Monte Carlo technique, when applied to estimate first-passage time probabilities, has the disadvantages of a sampling and discretization error. The sampling error is caused by the fact of just using a finite number of trajectories to estimate the expectation. The magnitude of this error can be estimated by the central limit theorem and the law of large numbers proves that it can be reduced if one simulates an infinite amount of trajectories (see, e.g., [Glasserman, 2004], [Hull, 2008]). The discretization error appears when aiming at analysing continuous-time models. It is a direct consequence of observing the trajectories just at finite instants during their lifetime. One natural way of improving this situation is considering a finer grid. Nevertheless, the fact of increasing the number of simulated trajectories and sharpening the grid implies a higher computational cost. These handicaps motivated researchers to implement renewal Monte Carlo methods, e.g., [Boyle et al., 1997], [Caflisch, 1998], [Ribeiro and Webber, 2006], [Figueroa-Lopez and Tankov, 2014].

In the current work, we focus on the Brownian-bridge technique, e.g., [Gobet, 2009], [Metwally and Atiya, 2002]. The Brownian-bridge technique presents the advantages of being unbiased and significantly faster than the standard Monte Carlo method. It has different applications in finance, see e.g., [Hieber and Scherer, 2010], [Ruf and Scherer, 2011], [Henriksen, 2011]. The contribution of this thesis is the design of fast and reliable Brownian-bridge algorithms to price exotic double-barrier products that consider different events depending on which barrier has been reached first and to evaluate products that depend on first-passage times in structural credit-risk models.

Moreover, first-exit times constitute the main element in the construction of an interesting subfamily of the Marshall–Olkin law. This multivariate exponential distribution, which has key impact in reliability theory and life testing among others (see, e.g., [Vesely, 1977], [Klein et al., 1989], [Rao, 2009]), was first introduced by [Marshall and Olkin, 1967b]. Since the canonical construction of this probability distribution is based on a fatal-shock model, it allows to describe events of default which are interesting in credit-risk modelling, e.g., [Giesecke, 2003], [Mai, 2010]. The Marshall–Olkin law has interesting statistical properties such as: its parameters possess an intuitive interpretation, it allows the extension of the lack-of-memory property to the multidimensional case, and it belongs to the class of extreme-value distributions, e.g., [Mai and Scherer, 2011], [Mai and Scherer, 2012]. Other important features are the asymmetric tail distribution that fits in most financial applications where there is often a stronger dependence among big losses than among big gains (see, e.g., [Li, 2008]) and the existence of the singular component i.e. there is a positive probability that all components take the same value (see, e.g., [Mai and Scherer, 2011]).

The existence of dependence structures in different real-world applications, such as insurance, e.g., [Müller, 1997] or finance, e.g., [Embrechts et al., 2002], motivates the study of how to "build" dependence. Copulas have become a basic tool to construct multivariate distributions, see e.g., [Embrechts et al., 2003], [Durante and Sempi, 2010], [Mai and Scherer, 2012].

Within the research of dependence structures, we first analyse the average lifetime of the Marshall–Olkin law. The distribution of a sum of random variables has been treated considerably in the literature, e.g., [Wüthrich, 2003], [Cossette and Marceau, 2000], [Arbenz et al., 2011], [Puccetti and Rüschendorf, 2013]. For mathematical tractability and computational convenience, the individual random variables $(X_1, \ldots, X_d)$ are often considered to be independent, see e.g., [Bennett, 1962]. But this assumption is unrealistic in a financial context. The distribution of the sum of dependent variables is known when the variables are elliptically distributed (see, e.g., [Fang et al., 1990]), a stability result that (at least partially) explains the popularity of elliptical distributions. However it could be that it does not hold in all existing applications. In [de Acosta, 1985] upper bounds for the sum of exchangeable vectors of conditionally independent and identically distributed (CIID) variables are provided. In [Wüthrich, 2003] the asymptotic quantile behaviour of a sum of dependent variables, where the dependence structure is given by an Archimedean copula, is analysed. The contribution of the present thesis regarding the sum of dependent components is the explicit implementation of the distribution of the sum of dependent random variables when the dependence is given by the Marshall–Olkin law. Closed-form solutions are derived for the survival and density function as well as for the Laplace transform in the general bivariate case. The expression for the survival function is obtained for low dimensional cases in the exchangeable Marshall–Olkin subfamily and a guide on how to extend these results to higher dimensions is provided. In addition, the asymptotic distribution in the infinite case within the extendible subclass, i.e. the subfamily of Marshall–Olkin distribution CIID components, is identified. And this way we cover the behaviour of the probability distribution of the average lifetime of the Marshall–Olkin law for large dimensions.

Studying the extendible subclass of Marshall–Olkin copulas more deeply one identifies the family of Lévy-frailty copulas. These copulas were discovered in 2009 by [Mai and Scherer, 2009]. Their dynamic structure and the possibility to simultaneously model default events make them interesting in credit portfolio models. In addition, they serve as a starting point to develop more sophisticated models in the field of credit risk, see e.g., [Bernhart et al., 2013]. They possess several interesting properties such as the link they provide between completely monotone sequences and multivariate distribution functions or their connection with widely investigated Archimedean copulas, see e.g., [Mai and Scherer, 2009], [Kimberling, 1974]. Lévy-frailty copulas constitute the extendible subfamily of the Marshall–Olkin law (see, e.g., [Mai, 2010]). They make initially independent exponential random variables dependent via the first-passage times of Lévy subordinators. Regarding possible applications, Lévy-frailty copulas can be used to efficiently simulate the exponential functional of a Lévy subordinator as it is analysed within this research. Distributional properties of such type of random numbers make them interesting in several applications, e.g., [Gjessing and Paulsen, 1997], [Carmona et al., 2001], [Bertoin et al., 2004], [Bertoin and Yor, 2005], [Kuznetsov and Pardo, 2010], [Rivero, 2009].

As it is mentioned previously, the necessity to design models based on dependence structures motivates the investigation of copulas and therefore of algorithms to sample from them. The simulation of copulas have been treated in the literature over the last years: [Mai and Scherer, 2012] provides different simulation techniques for the efficient sampling of Archimedean, Marshall–Olkin and elliptical copulas, [Jäckel and Bubley, 2002] explains how to simulate some specific copulas using Monte Carlo simulations or [Hofert, 2010] provides algorithms to efficiently simulate nested Archimedean copulas. Marshall–Olkin copulas, in their general version, present the disadvantage to be arduous to simulate in high dimensions. Nevertheless, due to the existence of a stochastic model behind the construction of the extendible subfamily of the Marshall–Olkin law, Lévy-frailty copulas have advantages in simulation with respect to more general Marshall–Olkin copulas (see [Mai and Scherer, 2012], Chapter 3).

The present thesis investigates different simulation techniques for Lévy-frailty copulas that are built from α-stable Lévy subordinators. One the one hand, these copulas are based on first-exit times of the α-stable Lévy subordinator. α-stable Lévy subordinators are a special case of stable Lévy processes and due to the heavy-tailed property, as well as their asymmetric behaviour, they become suitable in several applications in different financial areas, e.g., [Janicki and Weron, 1993], [Samoradnitsky and Taqqu, 1994]. On the other hand, Lévy-frailty copulas are characterized by the Laplace exponent of the respective Lévy subordinator and the α-stable subordinator possess a convenient form of this exponent (see, e.g., [Applebaum, 2009]). In addition, the stable subordinator has a close link to the Pareto distribution which allows to approximate it by a compound Poisson process with Pareto distributed big jumps. The Pareto distribution plays a key role in extreme-value theory being significantly useful in insurance and finance (see, e.g., [Embrechts et al., 1999]). Therefore, we provide an approximation of the α-stable Lévy subordinator by a compound Poisson process that allows to sample these Lévy-frailty copulas in high dimensions with low computational effort.

As mentioned in the beginning of this section, this dissertation is divided in two main parts: the simulation of first-exit times of jump-diffusion processes on the one hand, and the analysis of the dependence structures of the Marshall–Olkin kind, on the other hand. The thesis is constituted of six chapters despite of the present introduction. Chapter 2 introduces basic notions on the concepts mentioned through this thesis. Chapter 3 describes an efficient and unbiased Monte Carlo simulation to obtain estimates for double-barrier first-passage time probabilities of a jump-diffusion process. Chapters 4 and 5 are concerned with dependence structures of Marshall–Olkin kind. Chapter 4 computes the probability distribution of the average lifetime of the Marshall–Olkin law while Chapter 5 compares different numerical techniques to simulate Lévy-frailty copulas built from the α-stable Lévy subordinator. The pseudocodes of the algorithms described in Chapters 3 and 5 are given in Chapter 6. Finally Chapter 7 summarizes the results.

2 Mathematical preliminaries

“Life has a malicious way of dealing with great potential.”
Melinda & Melinda (2004).

In this chapter we introduce the required mathematical notion. Section 2.2 is dedicated to basic definitions on stochastic processes: we introduce Brownian motion, the Poisson process and the compound Poisson process, which belong to the family of Lévy processes. In Section 2.3 we introduce Lévy subordinators, almost sure non decreasing Lévy processes. In Section 2.4 basics on first-passage times and the Brownian-bridge construction are explained. Section 2.6 introduces copulas and finally in Section 2.7 we explain standards on the Marshall–Olkin distribution.

2.1 Notation

Let us first clarify some notation we use through this work.

- Sets:
  We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. $\mathbb{R}^d$ is the set of d-dimensional vectors of real numbers. $\mathbb{N}_0$ represents the set of positive integers and 0. We interpret the variable $t \geq 0$ as time.

- Probability spaces:
  We denote probability spaces using $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω denotes the set of all possible outcomes. $\mathcal{F}$ is the σ-algebra, a collection of measurable subsets of Ω such that $\Omega \in \mathcal{F}, \emptyset \in \mathcal{F}$, if $A \in \mathcal{F}, A^c = \Omega/A \in \mathcal{F}$, and $\forall A_i \in \mathcal{F}, \cup_{i=1}^{\infty} A_i \in \mathcal{F}$, $i \in \mathbb{N}$. We will interpret $\mathcal{F}_t$ as the information accumulated until time t. One important σ-algebra is the Borel σ-algebra, $\mathcal{B}(\mathbb{R}^d)$, the smallest σ-algebra containing all open sets in $\mathbb{R}^d$. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is an increasing and right continuous family of subsets of $\mathcal{F}$ satisfying $\mathcal{F}_s \subseteq \mathcal{F}_t \; \forall \, s \leq t$. Finally $\mathbb{P}$ is the probability measure, a set function $\mathbb{P} : \mathcal{F} \to [0,1]$ such that $\mathbb{P}(\Omega) = 1$, $\mathbb{P}(A_i) \geq 0$, and $\mathbb{P}\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$, $\forall A_i \cap A_j = \emptyset \in \mathcal{F}, i \neq j$, $i, j \in \mathbb{N}$. $\mathbb{P}$ assigns probability $\mathbb{P}(A_i)$ to all possible events

$A_i \in \mathcal{F}$, $i \in \mathbb{N}$. The expectation on this probability space is denoted by $\mathbb{E}$ and random variables (mapping from Ω to $\mathbb{R}$) by capital letters, e.g., X, U, E.

- Stochastic processes:
  The stochastic processes we work with in this dissertation are mostly denoted by X_t, we introduce the generic notation W_t for the standard Brownian motion and B_t for the Brownian motion with drift $\mu \in \mathbb{R}$ and volatility $\sigma > 0$, $B_t = \mu\, t + \sigma\, W_t$. Λ_t denotes Lévy subordinators.

- Distribution functions, densities and related objects:
  We use capital letters, F, G, for cumulative distribution functions and $\bar{F}$, $\bar{G}$ for survival functions. We use small letters, f, g, for probability density functions. Copulas are denoted by C and survival copulas by $\hat{C}$. Variables for probability and density functions are x_1, $x_2, \ldots$, y_1, $y_2, \ldots$ and for copulas u_1, $u_2, \ldots$

- Some probability distributions:

  - **Poisson distribution** with mean $\beta > 0$: $\mathcal{P}(\beta)$. The probability (discrete) density function of $X \sim \mathcal{P}(\beta)$ is given by

    $$\mathbb{P}(X = k) = \beta^k \frac{e^{-\beta}}{k!}, \quad k \in \{0, 1, 2, \ldots\}.$$

  - **Gaussian distribution**: $\mathcal{N}(\mu, \sigma^2)$, where $\mu \in \mathbb{R}$ is the mean and $\sigma > 0$ the standard deviation. The probability density function and the probability distribution function are given by

    $$\begin{aligned} f(x; \mu, \sigma) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}, \\ F(x; \mu, \sigma) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy, \quad x \in \mathbb{R}. \end{aligned}$$

  - **Continuous uniform distribution** on $-\infty < a < b < \infty$: $\mathcal{U}([a, b])$, where its probability density function and its distribution function are

    $$\begin{aligned} f(x; a, b) &= \begin{cases} \frac{1}{b-a}, & x \in [a, b] \\ 0, & \text{otherwise} \end{cases}, \\ F(x; a, b) &= \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & x \in [a, b] \\ 1, & x > b \end{cases}. \end{aligned}$$

  - **Exponential distribution**: $\text{Exp}(\lambda)$, where λ is a positive real number. The density function and the distribution function are given by

    $$\begin{aligned} f(x; \lambda) &= \lambda e^{-\lambda x}, \quad x \geq 0, \\ F(x; \lambda) &= 1 - e^{-\lambda x}, \quad x \geq 0. \end{aligned}$$

- **Erlang distribution**: continuous distribution with shape parameter $k \in \mathbb{N}$ and rate parameter $\lambda > 0$ (real). Its density and distribution functions are given by

$$\begin{aligned} f(x;\lambda,k) &= \frac{x^{k-1}\lambda^k e^{-\lambda x}}{(k-1)!}, \quad x \in [0,\infty), \\ F(x;\lambda,k) &= 1 - e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}, \quad x \in [0,\infty). \end{aligned}$$

- **Pareto distribution**: power law probability distribution whose parameters are $\gamma > 0$ (real), shape parameter, and $\kappa > 0$ (real), scale parameter. The probability density function and distribution function are given by

$$\begin{aligned} f(x;\gamma,\kappa) &= \frac{\gamma\kappa^\gamma}{x^{\gamma+1}}, \quad x \geq \kappa, \\ F(x;\gamma,\kappa) &= 1 - \left(\frac{1}{x}\right)^\gamma, \quad x \geq \kappa. \end{aligned}$$

- **α-stable distribution**: $\mathcal{S}(\alpha)$, $\alpha \in (0,1)$. The probability density function and the probability distribution function are not known in closed-form and thus it is characterized by its Laplace transform, i.e. for $X \sim \mathcal{S}(\alpha)$ we have

$$\mathbb{E}[e^{-xX}] = e^{-x^\alpha}, \quad x > 0.$$

- Some convergence laws of sequences of random variables:

  On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables with cumulative distribution function $X_n \sim F_{X_n}$ and X a random variable with cumulative distribution function F_X.

  - **Convergence in distribution:**
    X_n converges in distribution to X, $X_n \xrightarrow[n\to\infty]{\mathcal{L}} X$, if,

    $$\lim_{n\to\infty} F_{X_n}(x) = F_X(x),$$

    for all $x \in \mathbb{R}$ at which F_X is continuous.

  - **Almost sure convergence:**
    X_n converges to X $\mathbb{P}$-almost surely, $X_n \xrightarrow[n\to\infty]{a.s.} X$, if

    $$\mathbb{P}\left(\left\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1.$$

Note that *almost sure convergence* implies *convergence in distribution*, but not the other way round (see, e.g., [Meyer, 2002], Chapter 1, [Brémaud, 1988], Chapter 5).

2.2 Jump-diffusion processes

The model we consider when working with jump-diffusion processes is the sum of a compound Poisson process, $\{X_t\}_{t\geq 0}$, and a Brownian motion $\{B_t\}_{t\geq 0}$. In the sequel we present the Brownian motion, the Poisson process, and the compound Poisson process.

Definition 2.2.1 (*(one-dimensional)* Brownian motion)
Let $\{W_t\}_{0\leq t<\infty}$ be a continuous, adapted stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. W_t is a one-dimensional standard Brownian motion if it satisfies the following conditions:

(1) $W_0 = 0$ a.s.

(2) The increment $W_t - W_s$ is independent of $\mathcal{F}_s$, $\forall\ 0 \leq s < t$.

(3) $W_t - W_s \sim \mathcal{N}(0, t-s)$, $\forall\ 0 \leq s < t$,

where $\{\mathcal{F}_t\}_{0\leq s\leq t}$ is the natural filtration of the process $\{W_s\}_{0\leq s\leq t}$, i.e. $\mathcal{F}_t = \sigma(\{W_s\}_{0\leq s\leq t})$ is the smallest σ-algebra with respect to which W_s is measurable for all $0 \leq s \leq t$.

Definition 2.2.2 (Poisson process)
Let $N_t = \{N_t\}_{t\geq 0}$ be the standard Poisson process with intensity $\beta > 0$. The standard Poisson process is a Lévy process[1] with increasing piecewise constant paths that is represented in the following way:

$$N_t := \sum_{n=1}^{\infty} \mathbb{1}_{\{\tau_1+\ldots+\tau_n\leq t\}}, \qquad t > 0, \tag{2.1}$$

where $\{\tau_n\}_{n\in\mathbb{N}}$ are i.i.d. random variables exponentially distributed with parameter $\beta > 0$.

Definition 2.2.3 (Compound Poisson process)
Considering a sequence of independent random variables $\{Y_k\}_{k\geq 1}$ with a given distribution $\mathcal{L}(Y_k)$, $\forall k \geq 1$, and a standard Poisson process $\{N_t\}_{t\geq 0}$ with parameter β independent of $\{Y_k\}_{k\geq 1}$, the compound Poisson process, $\{X_t\}_{t\geq 0}$, is defined as:

$$X_t = \sum_{k=1}^{N_t} Y_k. \tag{2.2}$$

Compound Poisson processes belong to the class of Lévy processes which can be interpreted as random walks in continuous time. For more information on Lévy processes we refer to [Bertoin, 1998], [Sato, 1999], [Cont and Tankov, 2004].

[1] A Lévy process $\{X_t\}_{t\geq 0}$ is a *càdlàg* (right continuous with left limits) process that satisfies stochastic continuity, i.e. $\lim_{h\downarrow 0} \mathbb{P}(|X_{t+h} - X_t| \geq \epsilon) = 0$, $\forall t \geq 0$, $\forall \epsilon > 0$, has independent and stationary increments, and $X_0 = 0$ holds $\mathbb{P}$-almost surely.

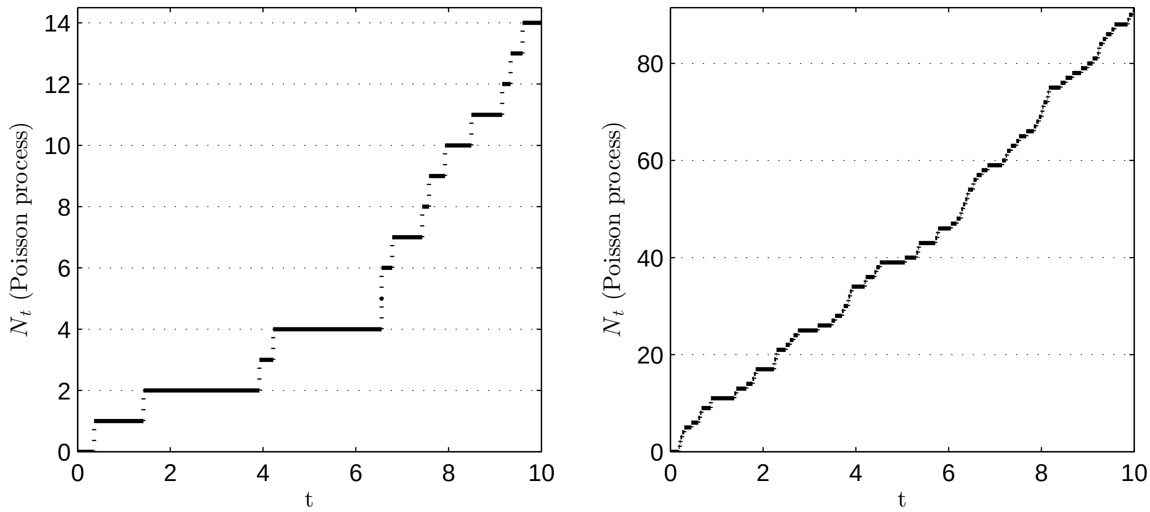


Figure 2.1: Simulated path of a Poisson process with parameter $\beta = 1$ (on the left) and $\beta = 8$ (on the right). The number of jumps in $[0, T]$ is Poisson-distributed with parameter $\beta \cdot T$.

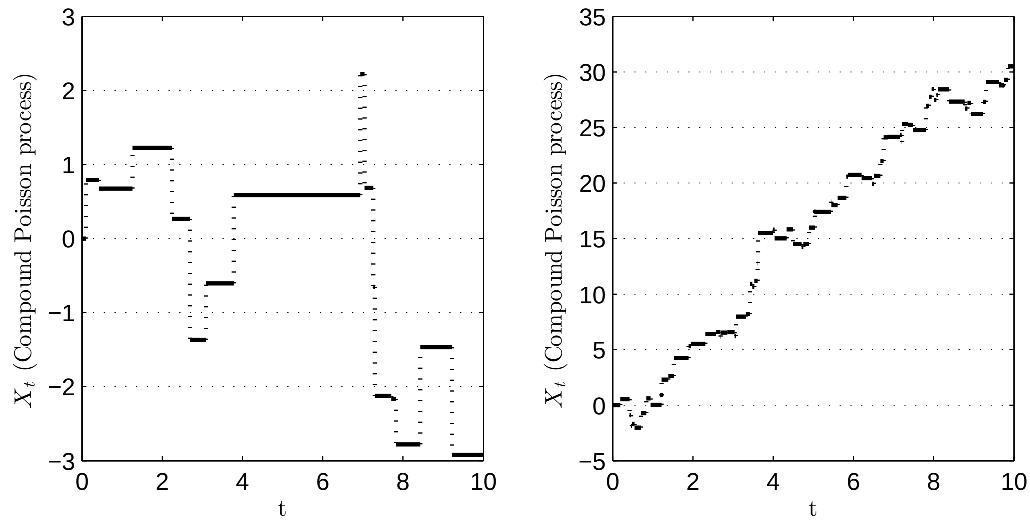


Figure 2.2: Simulated path of a compound Poisson process with $\beta = 1$ (on the left) and $\beta = 8$ (on the right). The distribution of the jumps is the standard Gaussian distribution. The compound Poisson process jumps at the same time as the Poisson process N_t.

2.3 Basics in Lévy subordinators

Lévy subordinators are a special case of Lévy processes, namely, the set of non-decreasing ones. The increments of Lévy processes are independent and stationary, i.e. do not depend on the past values of the process, not even on the present value.

Definition 2.3.1 (Lévy subordinator)
On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *let* $\Lambda = \{\Lambda_t\}_{t\geq 0}$ *be a positive real-valued stochastic process.* Λ *is a Lévy subordinator if it is almost surely non decreasing Lévy process, has càdlàg*[2] *paths. More precisely, it satisfies the following conditions:*

(1) Λ *is stochastically continuous, i.e.*

$$\lim_{h\downarrow 0} \mathbb{P}(|\Lambda_{t+h} - \Lambda_t| \geq \epsilon) = 0, \qquad \forall t \geq 0, \quad \forall \epsilon > 0.$$

(2) The increments of Λ, $\Lambda_{t_1} - \Lambda_0, \ldots, \Lambda_{t_n} - \Lambda_{t_{n-1}}$, $0 = t_0 < t_1 < t_2 < \ldots < t_n$, *are stochastically independent.*

(3) The increments of Λ *are stationary, i.e.*

$$\Lambda_{t+h} - \Lambda_t \stackrel{d}{=} \Lambda_h, \qquad \forall t \geq 0, \quad \forall h \geq 0.$$

(4) $t \to \Lambda_t$ *is almost surely non-decreasing.*

The characteristic function of Lévy processes gives an easy way to work with these stochastic processes. Since Lévy subordinators only take non-negative values, it is natural to consider the Laplace transform.

Definition 2.3.2 (Laplace transform of Lévy subordinators)
On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *we define the Laplace exponent of a Lévy subordinator* Λ *as* $\Psi : [0, \infty) \to [0, \infty)$. *So that the Laplace transform of* Λ *is given by*

$$\mathbb{E}[e^{-x\Lambda_t}] = e^{-t\Psi(x)}, \qquad x \geq 0, \quad t \geq 0. \tag{2.3}$$

The Lévy–Khintchine formula gives a characterisation of Lévy processes (resp. subordinators) through their characteristic function (resp. Laplace transform) (see, e.g., [Bertoin, 1998], [Sato, 1999], [Protter, 2004]).

Theorem 2.3.1 (Lévy–Khintchine formula for Lévy subordinators)
Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* Λ *a Lévy subordinator. Then there exists a unique non-negative drift* $\mu \geq 0$ *and a unique positive measure* ν *which satisfies*

$$\int_{(0,1]} t\nu(dt) < \infty, \quad \nu((\epsilon, \infty]) < \infty, \qquad \forall \epsilon > 0, \tag{2.4}$$

so that the Laplace exponent of Λ *can be written as*

$$\Psi(x) = \mu x + \int_{(0,\infty]} (1 - e^{-tx})\nu(dx), \qquad x \geq 0, \quad t \geq 0. \tag{2.5}$$

[2] *càdlàg* refers to the abbreviation of the French sentence *"continues à droite, limites à gauche"*, right continuous with left limits.

Proof. This theorem was first established by Paul Lévy [Lévy, 1954] and A. Khintchine [Khintchine, 1937, Khintchine, 1938]. A sketch of a proof can be found in [Applebaum, 2009] and [Bertoin, 1998]. □

The positive measure ν, mentioned above, is called *Lévy measure*. More details about the Lévy measure are given in Chapter 5. The distribution of Lévy subordinators is completely characterized by the Laplace transform of Λ, i.e. subordinators with the same Laplace transform have the same probability law.

Example 1 (Poisson process)
Let $N_t = \{N_t\}_{t\geq 0}$ be the standard Poisson process:

$$N_t := \sum_{n=1}^{\infty} \mathbb{1}_{\{\tau_1+\ldots+\tau_n\leq t\}}, \qquad t > 0, \tag{2.6}$$

where $\{\tau_n\}_{n\in\mathbb{N}}$ are i.i.d. random variables exponentially distributed with parameter $\beta > 0$.
The Laplace transform of N_t is computed as

$$\mathbb{E}[e^{-xN_t}] = \sum_{k=0}^{\infty} e^{-xk}\frac{(\beta t)^k}{k!}e^{-\beta t} = e^{-t\beta(1-e^{-x})}, \qquad x \geq 0. \tag{2.7}$$

For a more detailed background on Lévy subordinators we encourage the reader to study [Bertoin, 1998], [Bertoin, 2000], [Cont and Tankov, 2004], [Applebaum, 2009].

2.4 First-exit times and Brownian-bridge construction

First-exit times:
There exist situations where we are interested in the exact moment a phenomenon takes place for the first time. Since the occurrence of this event can change the course of other phenomenons, we need to focus our attention on investigating when, for the first time, these events happen.

We can find a formal definition as well as standards on *stopping times* in [Karatzas and Shreve, 1991].

Definition 2.4.1 (Stopping times)
On a measurable space $(\Omega, \mathcal{F})$ we define a random time T as a first stopping time of a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if the event $\{T \leq t\}$ belongs to the σ-field $\mathcal{F}_t$, for all $t \geq 0$.

In terms of stochastic processes, first-passage times are defined as first moments where a given stochastic process reaches a threshold. There are situations where the process is influenced by just one threshold and situations with two reachable barriers.

On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let $X = \{X_t : t \geq 0, X_0 = 0\}$ be a stochastic process. For a single constant upper (respectively lower) threshold, $a > 0$ (resp. $b < 0$), the first-passage time is given by:

$$T_a := \inf\{t > 0 : X_t \geq a\}, \quad T_b := \inf\{t > 0 : X_t \leq b\}. \tag{2.8}$$

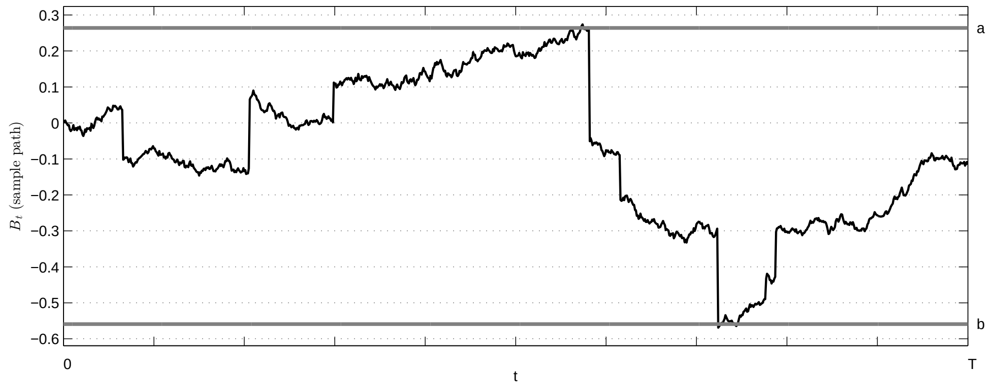


Figure 2.3: Simulated stochastic process within two barriers $b < 0 < a$.

In a double barrier case, let $b < 0 < a$ be two constant barriers:

$$T_{ab} := \inf\{t > 0 : X_t \notin (b, a)\}. \tag{2.9}$$

There are situations in the double barrier case where the order which barrier has been reached first matters. We denote the first-exit time when the upper (resp. lower) barrier has been hit first as T^+_{ab} (resp. T^-_{ab}):

$$\begin{aligned} T^+_{ab} &:= \inf\{t > 0 : X_t \geq a \,|\, X_s > b, \forall s < t\}, \\ T^-_{ab} &:= \inf\{t > 0 : X_t \leq b \,|\, X_s < a, \forall s < t\}. \end{aligned} \tag{2.10}$$

Brownian-bridge construction:
When simulating stochastic processes on a discrete time grid, one can not observe the behaviour of the simulated path between two given observation points, t_{i-1} and t_i, but we know the value that the simulated process takes on these nodes, $X_{t_{i-1}}$ and X_{t_i}. Through this work we apply the Brownian-bridge technique to compute the probability of reaching a barrier between two observation points.

The construction of the Brownian-bridge is based on conditioning the one-dimensional Brownian motion on its start- and endpoints. Let $W = \{W_t\}_{0 \leq t \leq T}$ be a one-dimensional Brownian motion. And let us assume that at time points $t = 0, t = 1, \ldots, t = T$, the values W takes are known: $W_0, W_1, \ldots, W_T$.

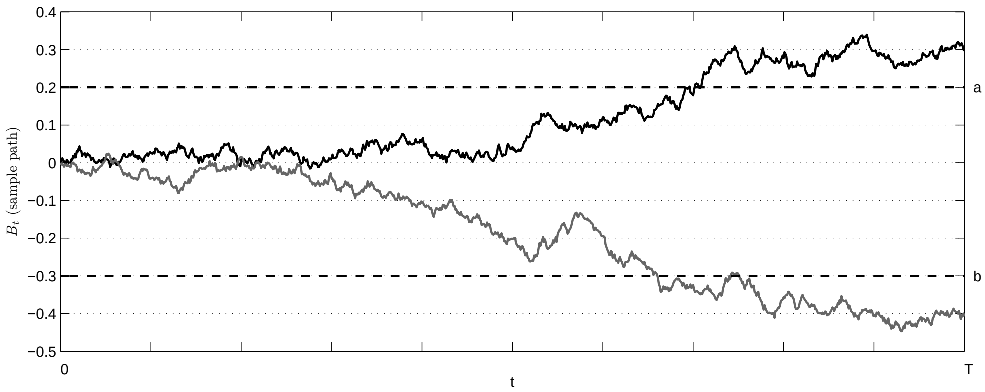
Using the conditioning formula[3] and the independence of the Brownian motion incre-

[3]**Conditioning formula**: Assuming that the partitioned vector $X := (X_{[1]}, X_{[2]})$, into subvectors $X_{[1]}$ and $X_{[2]}$, follows the multivariate normal distribution,

$$\begin{pmatrix} X_{[1]} \\ X_{[2]} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_{[1]} \\ \mu_{[2]} \end{pmatrix}, \begin{pmatrix} \Sigma_{[11]} & \Sigma_{[12]} \\ \Sigma_{[21]} & \Sigma_{[22]} \end{pmatrix} \right),$$

such that $\Sigma^{-1}_{[22]}$ exists. Then,

$$(X_{[1]} | X_{[2]} = x) \sim \mathcal{N}\left(\mu_{[1]} + \Sigma_{[12]}\Sigma^{-1}_{[22]}(x - \mu_{[2]}), \Sigma_{[11]} - \Sigma_{[12]}\Sigma^{-1}_{[22]}\Sigma_{[21]} \right).$$



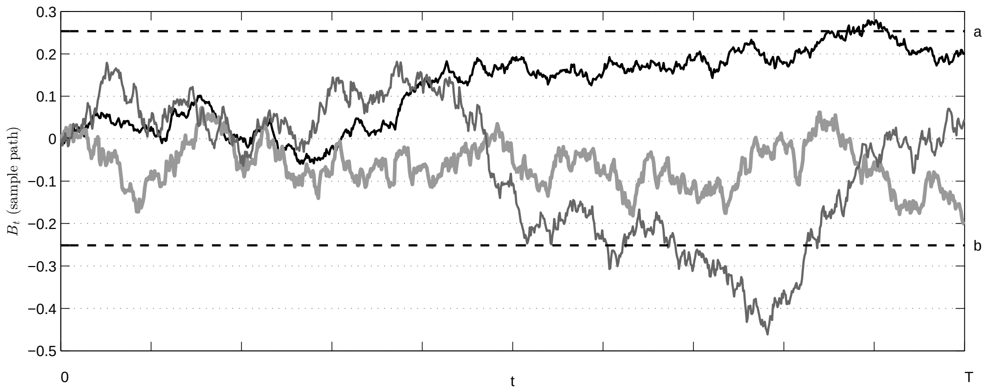


Figure 2.4: Possible situations while simulating stochastic processes within two barriers. In the plot above the barrier hitting event is clear, since the end-points of the paths are on a different side of the barrier than the start-point. In the plot below, however, the start and end-points are on the same side of the barriers but the barriers have been reached.

ments, it is possible to show that for a given $t \in [0, T]$,

$$\begin{aligned}&\big(W_t \,|\, W_{t_{i-1}} = x_{i-1}, W_{t_i} = x_i\big) \\ &\quad = \mathcal{N}\left(\frac{x_{i-1}(t_i - t) + x_i(t - t_{i-1})}{t_i - t_{i-1}}, \frac{(t_i - t)(t - t_{i-1})}{t_i - t_{i-1}}\right),\end{aligned} \tag{2.11}$$

such that t_{i-1} and t_i are the closest points to t where the values $W_{t_{i-1}}$ and W_{t_i} are known (see, e.g., [Glasserman, 2004] (Chapter 3), [Korn et al., 2010] (Chapter 4)).

Since $W_0, W_{t_1}, \ldots, W_T$ are known, the sampling of the Brownian-bridge at one specific intermediate point $t \in (t_{i-1}, t_i)$ is achieved via the conditional distribution in (2.11) by setting

$$W_t = \frac{x_{i-1}(t_i - t) + x_i(t - t_{i-1})}{t_i - t_{i-1}} + \sqrt{\frac{(t_i - t)(t - t_{i-1})}{t_i - t_{i-1}}}\, Z,$$

such that $t_{i-1} < t < t_i$. Z is a random variable $\mathcal{N}(0,1)$-distributed independent of $W_0, W_{t_1}, \ldots, W_T$. Repeating this procedure with all known values one can sample all the intermediate values in $[W_0, W_{t_1}], [W_{t_1}, W_{t_2}], \ldots, [W_{t_{n-1}}, W_T]$.

2.5 Black–Scholes model

In this section we introduce the Black–Scholes model and briefly explain the pricing of options in this framework (for a more detailed background on pricing options under the Black–Scholes model, see e.g., [Wilmott, 2006], [Hull, 2008]).

The simplest financial options, named *plain vanilla options*, are contracts that give the holder of the option the right, but not the obligation, to buy (or sell) the underlying asset of the option at the fixed maturity date of the contract at a price (strike price) which has been previously set. On the other hand, the counterpart has the obligation to trade the asset when the holder exercises the right to buy (or sell) it. When the option is exercised the holder will get the payoff $f(S_T)$, which varies depending on an option, at time $t = T$ (maturity of the contract). Through this study we evaluate barrier options which are options whose payoff depends on whether the price of the underlying asset of the contract approaches a given threshold during the lifetime of the option.

[Black and Scholes, 1973] introduce a simple model to evaluate options which consist of a risky and a riskless asset. The risky asset is interpreted as a stock price of the option and its dynamics are described as

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t, \quad t \geq 0, \tag{2.12}$$

where $\mu \in \mathbb{R}$ is the drift, $\sigma > 0$ the volatility, and W the standard Wiener process. Note that $\mu\, dt$ is the deterministic term of the process and $\sigma\, dW_t$ the stochastic one. The process that solves Equation (2.12) is called geometric Brownian motion. Applying Itô's

Lemma yields the solution for the stochastic differential equation (SDE) in (2.12) (see, e.g., [Karatzas and Shreve, 1991] Chapter 3)

$$S_t = S_0\, e^{(\mu - 1/2\, \sigma^2)\, t + \sigma\, W_t}, \tag{2.13}$$

S_0 being the initial value (at time $t = 0$) of the process.

The riskless asset grows in time deterministically at a constant rate $r \in \mathbb{R}$:

$$\frac{d\beta(t)}{\beta(t)} = r\, dt, \quad \beta(0) = 1, \tag{2.14}$$

whose unique solution is $\beta(t) = e^{\int_0^t r ds}$.

Risk-neutral measure:

Before we formally explain the risk-neutral measure let us first introduce the concept of a *martingale.*

Definition 2.5.1 (Martingale)
On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *let* $\{X_t\}_{t\geq 0}$ *be a real-valued process, adapted to a filtration* $\{\mathcal{F}_t\}_{t\geq 0}$. *Let us assume that* $\mathbb{E}[|X_t|] < \infty$ *for all* $t \geq 0$. *Then* X_t *is a martingale, if for every* $0 \leq s < t$. $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$. *In case* $\mathbb{E}[X_t|\mathcal{F}_s] \leq X_s$, X_t *is called submartingale and if* $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$ *a supermartingale.*

Starting from the fact that participants in financial markets are risk averse, it seems natural to consider the existence of attempts to profit with no risk. This procedure, which is called *arbitrage*, can be done by simultaneously entering into transactions in different –two or more– markets. Nevertheless, the big amount of riskless profiting attempts has the consequence of a decrease in arbitrage opportunities.

In a framework with no arbitrage opportunities there exists a discount factor $\tilde{\beta}(t)$ such that $\beta(t)/\tilde{\beta}(t)$ is a positive martingale with initial value $\beta(0)/\tilde{\beta}(0) = 1$. Using this positive martingale it is possible to define a change of probability measure (see, e.g., [Glasserman, 2004], Chapter 1)

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\beta(t)}{\tilde{\beta}(t)}, \quad 0 \leq t \leq T.$$

The measure $\mathbb{Q}$ is called a risk-neutral measure, and it is equivalent to $\mathbb{P}$, i.e. $\mathbb{Q}(A) = 0 \Leftrightarrow \mathbb{P}(A) = 0$, $\forall A \subseteq \mathcal{F}$.

The risk-neutral measure is an equivalent martingale measure under which $S_t/\beta(t)$ is a martingale. For pricing purposes *risk-neutrality* is a crucial tool. More precisely, the risk-neutral pricing formula (see, e.g., [Bingham and Kiesel, 2004], p. 120) states that in an arbitrage-free complete market[4] every financial derivative can be priced computing the present value of its expected payoff, so

[4] A complete market is a market where all financial assets can be replicated using some self-financing strategy. For a more detailed background in complete markets see [Bingham and Kiesel, 2004], Chapter 4.3

$$V(S_t, t) = \frac{\beta(t)}{\beta(T)} \mathbb{E}_{\mathbb{Q}} \left[f(S_t) | \mathcal{F}_t \right].$$

Coming back to the dynamics of the assets in the Black–Scholes model, in (2.14) we define the dynamics of the riskless asset. r, which is interpreted as the riskless interest rate, can be assumed to be the drift of $\beta(t)$. Let us further consider the following relation:

$$d\tilde{W}_t = dW_t + \tilde{\mu}_t dt, \tag{2.15}$$

for $\tilde{\mu} \equiv \frac{\mu - r}{\sigma}$. It follows from Girsanov's Theorem (given in Appendix A) that $\tilde{W}$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$.

Now substituting (2.15) in (2.12), we get the dynamics of the risky asset in terms of the riskless interest rate:

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t = \mu\, dt + \sigma \left(d\tilde{W}_t - \frac{\mu - r}{\sigma} dt \right) = r\, dt + \sigma\, d\tilde{W}_t,$$

where r is the riskless interest rate and in this case

$$S_t = S_0 e^{\left(r - \frac{\sigma^2}{2} \right) t + \sigma \tilde{W}_t}.$$

Note that since $\tilde{W}$ is the standard Brownian motion under the risk-neutral measure $\mathbb{Q}$, the discounted price process, $S_t/\beta(t) = e^{-rt} S_t$ is a martingale under the risk-neutral measure.

Now if we aim at computing the present price (at time $t = 0$) of *plain vanilla options* with initial value of the underlying asset S_0, strike price K, and maturity T, in the Black–Scholes model, we need to compute

$$\text{call option}: \quad C(S_0, 0) = \frac{1}{\beta(T)} \mathbb{E}_{\mathbb{Q}} \left[(S_T - K)^+ \right]; \; f(S_T) = (S_T - K)^+$$
$$\text{put option}: \quad P(S_0, 0) = \frac{1}{\beta(T)} \mathbb{E}_{\mathbb{Q}} \left[(K - S_T)^+ \right]; \; f(S_T) = (K - S_T)^+.$$

2.6 Basics in copula theory

In order to describe the probability distribution of a random vector, the knowledge of all marginal laws as well as how these univariate marginals are related is needed. We express the information about how the marginals are related via copulas. We graphically describe copulas using a two-dimensional scatterplot, i.e. a graphical set of bivariate data (two variables) which gives a visual picture of the relationship between the two variables.

Definition 2.6.1 (Copula)
Let $C : [0,1]^d \to [0,1]$ be the multivariate distribution function of a random vector $(U_1,\ldots,U_d)$ with uniform marginals, $U_i \sim \mathcal{U}[0,1]$, $i = 1,\ldots,d$, so C satisfies

$$C(u_1,\ldots,u_d) := \mathbb{P}\left(U_1 \le u_1,\ldots,U_d \le u_d\right), \quad u_1,\ldots,u_d \in [0,1]. \tag{2.16}$$

Sklar's Theorem [Sklar, 1959] states that the multivariate distribution of each random vector $(X_1,\ldots,X_d)$ can be built combining univariate marginals and a copula.

Theorem 2.6.1 (Sklar's Theorem)
Let $(X_1,\ldots,X_d)$ be a random vector with d-dimensional distribution function F and $F_1,\ldots,F_d$ the univariate marginal functions of F. Then there exist a copula C such that,

$$F(x_1,\ldots,x_d) = C(F_1(x_1),\ldots,F_d(x_d)), \quad (x_1,\ldots,x_d) \in \mathbb{R}^d. \tag{2.17}$$

In addition, if $F_1,\ldots,F_d$ are continuous, the copula C is unique.

Proof. This theorem was originally stated in [Sklar, 1959]. The proof can be found in [Rüschendorf, 2009], [Mai and Scherer, 2012] (Chapter 1). □

Through this investigation the variables $(X_1,\ldots,X_d)$ are interpreted as lifetimes and in this case it is more convenient to describe the dependence between them using survival probability functions. The analogous version of Sklar's Theorem [Sklar, 1959] states that a multivariate survival function $\bar{F}(x_1,\ldots,x_d) := \mathbb{P}(X_1 > x_1,\ldots,X_d > x_d)$ can be split into its marginal functions and a survival copula $\hat{C}$:

$$\bar{F}(x_1,\ldots,x_d) = \hat{C}(\bar{F}_1(x_1),\ldots,\bar{F}_d(x_d)), \quad (x_1,\ldots,x_d) \in \mathbb{R}^d. \tag{2.18}$$

The relation between a copula C and the respective survival copula $\hat{C}$ is given by

$$\hat{C}(u_1,\ldots,u_d) = 1 + \sum_{k=1}^{d}(-1)^k \sum_{1\le j_1\le\ldots\le j_k\le d} C_{j_1,\ldots,j_k}(1-u_{j_1},\ldots,1-u_{j_k}), \tag{2.19}$$

such that $u_1,\ldots,u_d \in (0,1)$.

The proof of this result can be found in [Mai and Scherer, 2012] (Chapter 1).

Example 2 (Basic copulas)
In this example we analyse two simple copulas. Remember that from Definition 2.6.1 $C(u_1,\ldots,u_d) := \mathbb{P}(U_1 \le u_1,\ldots,U_d \le u_d)$, $U_i \sim \mathcal{U}([0,1])$, $(U_1,\ldots,U_d) \sim C$, and $u_1,\ldots,u_d \in [0,1]$.

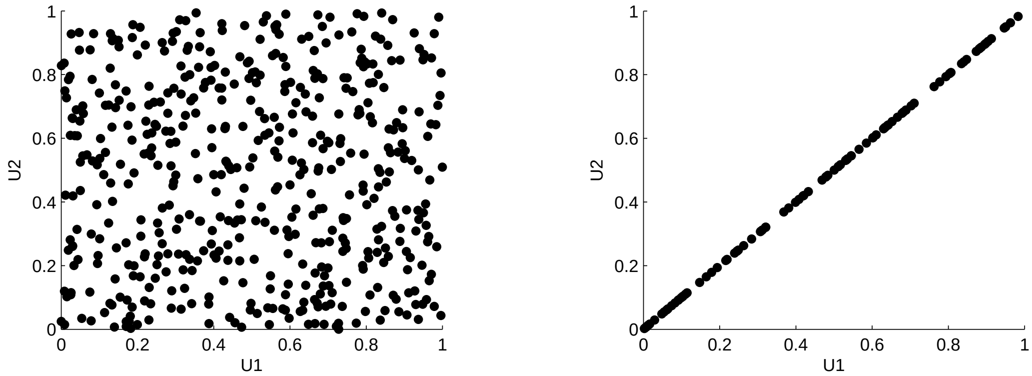
(1) Independence copula: Let $U_1,\ldots,U_d$ be uniformly distributed i.i.d. random variables, $U_i \sim \mathcal{U}([0,1])$. Then the independence copula is given by,

$$\begin{aligned} C : [0,1]^d &\to [0,1], \\ C(u_1,\ldots,u_d) &:= \prod_{i=1}^{d} u_i. \end{aligned}$$

(2) Comonotonicity copula: Let $U \sim \mathcal{U}([0,1])$ be a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In addition, let us consider $(U, \ldots, U) \in [0,1]^d$ a random vector with $\mathcal{U}([0,1])$ distributed marginal functions and joint distribution function,

$$\begin{aligned} C : [0,1]^d &\rightarrow [0,1], \\ C(u_1, \ldots, u_d) &:= \min\{u_1, \ldots, u_d\}. \end{aligned}$$

In this case, the distribution function C defines the comonotonicity copula.



(a) Independence copula.

(b) Comonotonicity copula.

Figure 2.5: Scatterplots of 500 samples from the bivariate independence copula and 100 samples from the bivariate comonotonicity copula. (a) We sample $U_1 \sim \mathcal{U}([0,1])$ and $U_2 \sim \mathcal{U}([0,1])$ independent of U_1. (b) We sample $U_1 \sim \mathcal{U}([0,1])$ and set $U_2 = U_1$.

2.7 Marshall–Olkin law

The Marshall–Olkin distribution is a d-dimensional exponential distribution characterized by lifting the univariate *lack of memory property* to higher dimensions. It was first introduced by [Marshall and Olkin, 1967b].

The exponential distribution possess some statistically attractive properties (in the univariate and multivariate cases) such as the *lack of memory property* or the *min-stability* property.

***Lack of memory* property:**
Let X be a variable supported on $[0, \infty)$. The univariate *lack of memory property* is defined as:

$$\mathbb{P}(X > x + y | X > y) = \mathbb{P}(X > x), \quad \forall x, y > 0. \tag{2.20}$$

If X has support $[0, \infty)$ and satisfies the univariate *lack of memory property*, then X is exponentially distributed. Generalizing to higher dimensions, if $(X_1, \ldots, X_d)$ has support

$[0,\infty)^d$ and all possible subvectors $(X_{i_1},\ldots,X_{i_k})$, where $1 \le i_1 < \ldots < i_k \le d$, satisfy the multidimensional *lack of memory property*,

$$\mathbb{P}(X_{i_1} > x_{i_1}+y,\ldots,X_{i_k} > x_{i_k}+y | X_{i_1} > y,\ldots,X_{i_k} > y) \\ = \mathbb{P}(X_{i_1} > x_{i_1},\ldots,X_{i_k} > x_{i_k}), \tag{2.21}$$

where $x_{i_1},\ldots,x_{i_k},y > 0$, it is shown in [Marshall and Olkin, 1967b] that the only distribution with support $[0,\infty)^d$ satisfying condition (2.21) is characterized by the survival function of Marshall–Olkin kind.

Example 3 (Increments of the Poisson and compound Poisson processes)
In case of Poisson and compound Poisson processes mentioned in Section 2.2, the waiting times between the jump times $\{\tau_n\}_{n\in\mathbb{N}}$, follow the exponential distribution with parameter β. Due to the lack of memory property satisfied by the exponential distribution, we can ensure that the increments of the Poisson and compound Poisson processes are independent and identically distributed.

□

Definition 2.7.1 (Marshall–Olkin distribution)
Let $(X_1,\ldots,X_d)$ represent a system of residual lifetimes with support $[0,\infty)^d$. Assume that the remaining components in this vector have a joint distribution that is independent of the age of the system, i.e. $(X_1,\ldots,X_d)$ satisfies the multidimensional lack of memory property (2.21). *Then*

$$\begin{aligned}\bar{F}(x_1,\ldots,x_d) &:= \mathbb{P}(X_1 > x_1,\ldots,X_d > x_d) \\ &= \exp\left(-\sum_{\emptyset\neq I\subset\{1,\ldots,d\}} \lambda_I \max_{i\in I}\{x_i\}\right), \quad x_1,\ldots,x_d \ge 0,\end{aligned} \tag{2.22}$$

for certain parameters $\lambda_I \ge 0$, $\emptyset \neq I \subset \{1,\ldots,d\}$, and $\sum_{I:k\in I}\lambda_I > 0$, $k = 1,\ldots,d$. This multivariate law is called Marshall–Olkin distribution.

Min-stability of the Marshall–Olkin distribution:
Note that an interesting property of the exponential distribution is the min-stability, i.e. if $X_1,\ldots,X_d$ are independent variables exponentially distributed with parameters $\lambda_1.\ldots,\lambda_d$, $\{\lambda_i > 0 : i = 1,\ldots,d\}$, then the minimum between these variables is also exponentially distributed:

$$\min\{X_1,\ldots,X_d\} \sim \mathrm{Exp}(\lambda_1+\ldots+\lambda_2). \tag{2.23}$$

The proof of this result is given in Appendix B.

The Marshall–Olkin distribution belongs to the class of min-stable multivariate exponential distributions, i.e. if $(X_1,\ldots,X_d)$ and $(Y_1,\ldots,Y_d)$ are two independent random vectors following the Marshall–Olkin distribution with parameters $\lambda_I \geq 0$ and $\eta_I \geq 0$ respectively, then

$$(\min\{X_1, Y_1\}, \ldots, \min\{X_d, Y_d\}),$$

follows the Marshall–Olkin distribution with parameter $\lambda_I + \eta_I$, $\emptyset \neq I \subset \{1, \ldots, d\}$.

The proof of this result can be found in [Mai and Scherer, 2012] (page 138). □

Canonical construction:
The canonical construction of the Marshall–Olkin distribution is based on the following *fatal-shock* model, see [Marshall and Olkin, 1967a, Downton, 1970].

Let E_I, $\emptyset \neq I \subset \{1, \ldots, d\}$, be exponentially distributed random variables with parameters $\lambda_I \geq 0$. We assume all E_I to be independent and interpret them as the arrival times of exogenous shocks to the respective components in I and define

$$X_k := \min\{E_I | \emptyset \neq I \subset \{1, \ldots, d\}, k \in I\} \in [0, \infty), \quad k = 1, \ldots, d, \tag{2.24}$$

where the variable X_k is the first time a shock hits component k. The parameters $\lambda_I \geq 0$ represent the intensities of the exogenous shocks. Some of these can be 0, in which case $E_I \equiv \infty$. We require $\sum_{\emptyset \neq I : k \in I} \lambda_I > 0$, so for each $k = 1, \ldots, d$ there is at least one subset $I \subset \{1, \ldots, d\}$, containing k, such that $\lambda_I > 0$. Therefore, (2.24) is well-defined. The random vector $(X_1, \ldots, X_d)$ as defined in Equation (2.24) follows the Marshall–Olkin distribution.

Singular component:
Marshall–Olkin multivariate distributions are not absolutely continuous, i.e. there is a positive probability that several components take the same value. These probabilities are computed in the following way:

$$\mathbb{P}(X_1 = \ldots = X_d) = \frac{\lambda_{\{1,\ldots,d\}}}{\sum_{\emptyset \neq I \subset \{1,\ldots,d\}} \lambda_I} \geq 0. \tag{2.25}$$

The proof of this results can be found in [Mai and Scherer, 2012], page 120.

Lemma 2.7.1 (Marshall–Olkin survival copula)
Let $(X_1, \ldots, X_d)$ be a random vector built as in (2.24). Then the survival copula $\hat{C}$ of the Marshall–Olkin distribution is defined as

$$\hat{C}(u_1, \ldots, u_d) = \prod_{\emptyset \neq I \subset \{1,\ldots,d\}} \min_{k \in I} \left\{ u_k^{\frac{\lambda_I}{\sum_{J : k \in J} \lambda_J}} \right\}, \quad u_1, \ldots, u_d \in [0, 1]. \tag{2.26}$$

We can distinguish different subfamilies with additional properties within the Marshall–Olkin distribution. This subfamilies are introduced in Chapters 4 and 5.

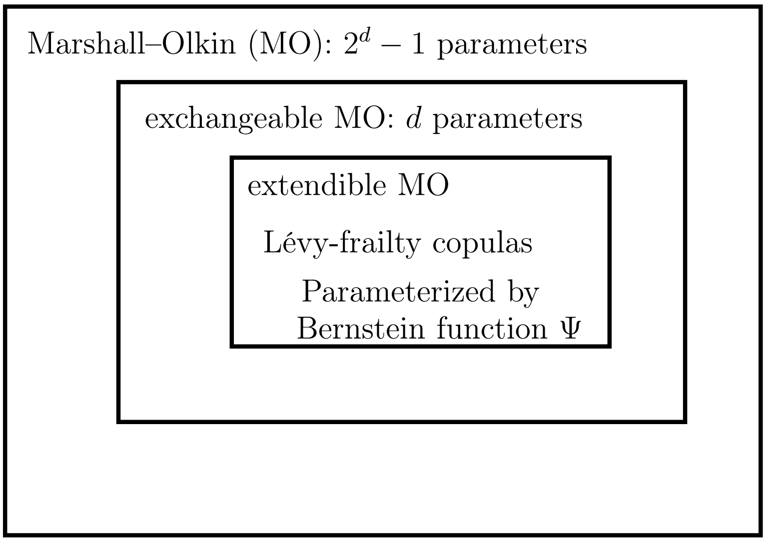


Figure 2.6: Subfamilies of the Marshall–Olkin law.

3 Double-barrier first-passage times of jump-diffusion processes

> "You know not everything in the world is sinister... just practically everything."
> Scoop (2006).

First-passage time problems have become of interest in a wide range of applications e.g., finance (see, e.g., [Kou and Wang, 2003], [Alili and Kyprianou, 2005], [Hieber and Scherer, 2012]), engineering (see, e.g., [Crandall et al., 1966], [Vanmarcke, 1975], [Peters and Barenbrug, 2002]) and physics (see, e.g., [Siegert, 1951], [Montroll, 1969]), over the past decades. Due to the lack of analytical solutions in some cases, fast and accurate numerical techniques have been developed to compute the distribution of first-exit times. We analyse an efficient and unbiased Monte-Carlo simulation to obtain double-barrier first-passage time probabilities of a jump-diffusion process with arbitrary jump size distribution. The pricing of exotic derivatives, e.g., *corridor bonus certificates* or *digital first-touch options*, that depend on whether or not the underlying asset price exceeds certain threshold levels are some of the applications of the double-barrier first-passage time in mathematical finance. They also have become relevant in structural credit-risk models if one considers two exit events, e.g., default and early repayment.

This chapter is based on the paper "Double-barrier first-passage times of a jump-diffusion processes", written by Fernández L., Hieber P. and Scherer M., and published in Monte Carlo Methods Appl. 19 (2013), $107-141$*, DOI 10.1515/mcma-2013-0005.*

3.1 Brownian-bridge and first-exit times probabilities

We aim at analysing first-passage times probabilities of Brownian motion on two constant barriers $a, b \in \mathbb{R}$. On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let us consider the jump-diffusion process $\{X_t\}_{t\geq 0}$, which is the sum of a Brownian motion and a compound Poisson process,

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i,\ t \geq 0, \tag{3.1}$$

with drift $\mu \in \mathbb{R}$, volatility $\sigma > 0$, and initial value $X_0 = 0$, where $\{W_t\}_{t\geq 0}$ is a Wiener process. $N = \{N_t\}_{t\geq 0}$, the counting process, is a Poisson process with intensity $\lambda \geq 0$, and the jumps $Y = \{Y_i\}_{i\in\mathbb{N}}$ are i.i.d. with distribution $\mathbb{P}_Y$. All random elements are mutually independent. Removing the jumps from Equation (3.1) we get the Brownian motion with drift μ and volatility σ, i.e.

$$B_t = \mu t + \sigma W_t. \tag{3.2}$$

We define in the following the Brownian-bridge probabilities for two constant barriers $b < 0 = B_0 < a$, $a, b \in \mathbb{R}$ i.e. the probability that a Brownian motion reaches one of the thresholds conditioning on the start- and endpoint of the stochastic process.

Definition 3.1.1 (Brownian-bridge probabilities)
Let $\{B_t\}_{t\geq 0}$ be a Brownian motion as defined in Equation (3.2). Let us denote its start- and endpoint in the time interval (t_{i-1}, t_i) by $x_{i-1} := B_{t_{i-1}} \in (b, a)$, respectively $x_i := B_{t_i} \in \mathbb{R}$. Assume that x_{i-1} and x_i are known, so we define the Brownian-bridge probabilities as

$$BB^+_{ab}(t_{i-1}, t_i, x_{i-1}, x_i) := \mathbb{P}\big(t_{i-1} < T^+_{ab} < t_i \,|\, B_{t_{i-1}} = x_{i-1},\ B_{t_i} = x_i\big),$$
$$BB^-_{ab}(t_{i-1}, t_i, x_{i-1}, x_i) := \mathbb{P}\big(t_{i-1} < T^-_{ab} < t_i \,|\, B_{t_{i-1}} = x_{i-1},\ B_{t_i} = x_i\big).$$

[Jeanblanc et al., 2009] provide the Brownian-bridge probabilities when there is just a single barrier.

$$BB^+_{a-\infty} = \lim_{b\to-\infty} BB^+_{ab}(t_{i-1}, t_i, x_{i-1}, x_i) = \begin{cases} \exp\Big(-\frac{2(a-x_{i-1})(a-x_i)}{\sigma^2(t_i-t_{i-1})}\Big), & \max(x_i, x_{i-1}) < a, \\ 1, & \text{else.} \end{cases}$$
$$BB^-_{\infty b} = \lim_{a\to\infty} BB^-_{ab}(t_{i-1}, t_i, x_{i-1}, x_i) = \begin{cases} \exp\Big(-\frac{2(x_{i-1}-b)(x_i-b)}{\sigma^2(t_i-t_{i-1})}\Big), & \min(x_i, x_{i-1}) > b, \\ 1, & \text{else.} \end{cases}$$

In Lemma 3.1.1 we display the closed-form solutions for Brownian-bridge probabilities BB^+_{ab} and BB^-_{ab} in Definition 3.1.1. These probabilities are computed depending on the location of the endpoint of the Brownian motion.

Lemma 3.1.1 (Brownian-bridge probabilities)
Let $\{B_t\}_{t\geq 0}$ be a Brownian motion with volatility $\sigma > 0$. Assuming that $x_{i-1} := B_{t_{i-1}} \in (b, a)$ and $0 \leq t_{i-1} < t_i < \infty$.

(i) If $x_i := B_{t_i} \in (b, a)$,

$$BB_{ab}(t_{i-1}, t_i, x_{i-1}, x_i) := BB^+_{ab}(t_{i-1}, t_i, x_{i-1}, x_i) + BB^-_{ab}(t_{i-1}, t_i, x_{i-1}, x_i)$$

$$= \sum_{n=-\infty}^{\infty} \Bigg[\exp\Bigg(-\frac{2n(a-b)}{\sigma^2(t_i - t_{i-1})}\big(x_{i-1} - x_i + n(a-b)\big)\Bigg) + \exp\Bigg(-\frac{2\big(x_i - na + (n-1)b\big)\big(x_{i-1} - na + (n-1)b\big)}{\sigma^2(t_i - t_{i-1})}\Bigg)\Bigg] - 1.$$

(ii) *If* $x_i := B_{t_i} \notin (b, a)$, $BB_{ab}(t_{i-1}, t_i, x_{i-1}, x_i) = 1$.

(iii) *If* $x_i := B_{t_i} \in (-\infty, a)$,

$$\begin{aligned} &BB^+_{ab}(t_{i-1}, t_i, x_{i-1}, x_i) \\ &\quad = \sum_{n=1}^{\infty} \Bigg[\exp\Bigg(-\frac{2(x_{i-1} - na + (n-1)b)\big(x_i - na + (n-1)b\big)}{\sigma^2(t_i - t_{i-1})}\Bigg) \\ &\qquad - \exp\Bigg(-\frac{2n(a-b)}{\sigma^2(t_i - t_{i-1})}\big(x_{i-1} - x_i + n(a-b)\big)\Bigg)\Bigg], \end{aligned} \tag{3.3}$$

(iv) *If* $x_i := B_{t_i} \in (b, \infty)$,

$$BB^-_{ab}(t_{i-1}, t_i, x_{i-1}, x_i) = BB^+_{-b\,-a}(t_{i-1}, t_i, -x_{i-1}, -x_i).$$

(v) *If* $x_i > a$, *the probability of hitting the level* a *first is* $BB^+_{ab}(t_{i-1}, t_i, x_{i-1}, x_i) = 1 - BB^-_{ab}(t_{i-1}, t_i, x_{i-1}, x_i)$.

(vi) *If* $b > x_i$, *the probability of hitting the level* b *first is* $BB^-_{ab}(t_{i-1}, t_i, x_{i-1}, x_i) = 1 - BB^+_{ab}(t_{i-1}, t_i, x_{i-1}, x_i)$.

(vii) *If* $x_{i-1} := B_{t_{i-1}} \notin (b, a)$, *we set* $BB^{\pm}_{ab}(t_{i-1}, t_i, x_{i-1}, x_i) = 0$.

Proof. The proof can be found in [Anderson, 1960] or in [Novikov et al., 1999], Remark 2. For the proof of $BB_{ab}(t_{i-1}, t_i, x_{i-1}, x_i)$, see, e.g., [Geman and Yor, 1996]. □

Let us analyse in the following the convergence of the infinite series in Lemma 3.1.1 above. [Schröder, 2000] introduces a way to measure the distance of the process to the barriers, i.e.

$$\kappa := \sigma\sqrt{(t_i - t_{i-1})/2}\,\exp(b - a). \tag{3.4}$$

If we aim at numerically computing the infinite series, we need to truncate them. It is possible to observe the accuracy of the truncation in terms of κ (Equation (3.4)). We show in Figure 3.1 the logarithmic absolute error produced when we truncate the series in Equation (3.3) after N summands.

If we analyse the plot in Figure 3.1, we can conclude that the truncated series rapidly converges to the infinite one. So a low number of terms in the sum in Equation (3.3), e.g., $N = 6$, is sufficient for applications.

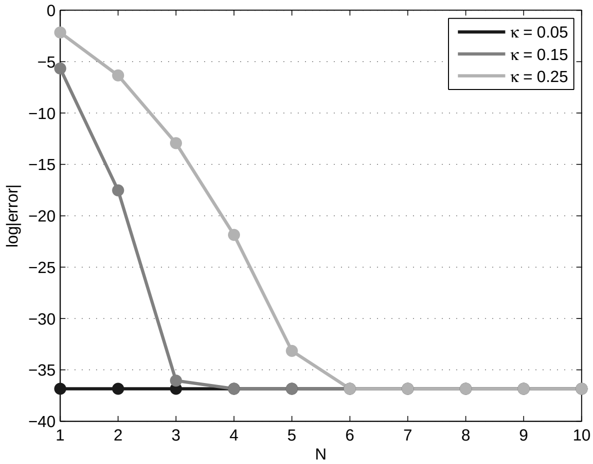


Figure 3.1: Logarithmic absolute error of the infinite series in Lemma 3.1.1, Equation (3.4) after truncating it for different values of κ. We choose $x_{i-1} = 0$, $x_i = 0.1$, $t_i - t_{i-1} = 1$, $a = \log(1.2)$, and $b = \log(0.8)$. The parameter σ is computed in terms of κ, $\sigma = \kappa / \left(\sqrt{(t_i - t_{i-1})/2} \exp(b - a) \right)$.

There are cases where the information whether a barrier has been reached or not is enough but there exist applications where we need to know in addition which barrier has been hit first. This information is obtained by the first-passage time probabilities. In Lemma 3.1.2 these probabilities for two constant barriers are computed and we display them using two different representations.

Lemma 3.1.2 (First-passage time probabilities)
Let $\{B_t\}_{t \geq 0}$ be a Brownian motion with drift $\mu \in \mathbb{R}$ and volatility $\sigma > 0$. Assuming that $x_{i-1} := B_{t_{i-1}} \in (b, a)$ and $0 \leq t_{i-1} < t_i < \infty$, and setting $BM^+_{ab}(t_{i-1}, t_i, x_{i-1}) := \mathbb{P}(t_{i-1} < T^+_{ab} < t_i)$:

(i) First representation,

$$
\begin{aligned}
&BM^+_{ab}(t_{i-1}, t_i, x_{i-1}) \qquad (3.5)\\
&\quad = \sum_{n=1}^{\infty} \Bigg[\exp\left(-\frac{2\mu(x_{i-1} - (na - (n-1)b))}{\sigma^2} \right) \times \\
&\qquad F\left(\frac{x_{i-1} - \mu(t_i - t_{i-1}) - ((2n-1)a - 2(n-1)b)}{\sigma\sqrt{t_i - t_{i-1}}} \right) - \exp\left(\frac{2n(a-b)\mu}{\sigma^2} \right) \times \\
&\qquad F\left(\frac{-x_{i-1} - \mu(t_i - t_{i-1}) - ((2n-1)a - 2nb)}{\sigma\sqrt{t_i - t_{i-1}}} \right) \Bigg] \\
&\qquad + \sum_{n=1}^{\infty} \Bigg[\exp\left(-\frac{2\mu\left((n-1)(a-b)\right)}{\sigma^2} \right) \times \\
&\qquad F\left(\frac{x_{i-1} + \mu(t_i - t_{i-1}) - ((2n-1)a - 2(n-1)b)}{\sigma\sqrt{t_i - t_{i-1}}} \right)
\end{aligned}
$$

$$- \exp\left(-\frac{2\mu(x_{i-1}+(n-1)a-nb)}{\sigma^2}\right)\times$$
$$F\left(\frac{-x_{i-1}+\mu(t_i-t_{i-1})-((2n-1)a-2nb)}{\sigma\sqrt{t_i-t_{i-1}}}\right)\Bigg],$$

where $F(\cdot)$ represents, in this case, the cumulative distribution function of a standard normally distributed random variable.

(ii) The second representation is given by:

if $\mu \neq 0$

$$BM_{ab}^{+}(t_{i-1},t_i,x_{i-1}) \tag{3.6}$$
$$= \frac{\exp\left(-\frac{2\mu(b-x_{i-1})}{\sigma^2}\right)-1}{\exp\left(-\frac{2\mu(b-x_{i-1})}{\sigma^2}\right)-\exp\left(-\frac{2\mu(a-x_{i-1})}{\sigma^2}\right)}$$
$$+\frac{\sigma^2\pi}{(a-b)^2}\exp\left(\frac{\mu(a-x_{i-1})}{\sigma^2}\right)\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{\frac{\mu^2}{2\sigma^2}+\frac{\sigma^2n^2\pi^2}{2(a-b)^2}}\times$$
$$\exp\left(-\Big(\frac{\mu^2}{2\sigma^2}+\frac{\sigma^2n^2\pi^2}{2(a-b)^2}\Big)(t_i-t_{i-1})\right)\sin\left(\frac{n\pi(b-x_{i-1})}{a-b}\right),$$

and if $\mu = 0$

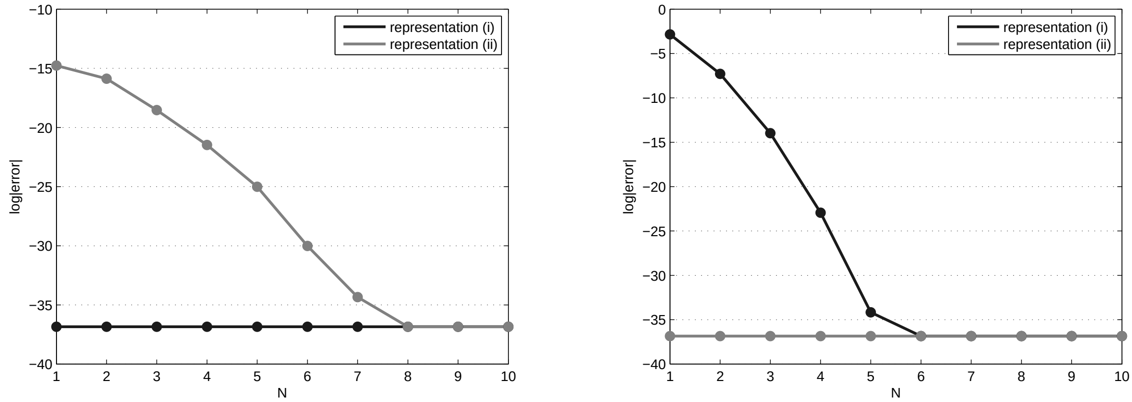
$$BM_{ab}^{+}(t_{i-1},t_i,x_{i-1})$$
$$= \frac{x_{i-1}-b}{a-b}+\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\exp\left(-\frac{\sigma^2n^2\pi^2}{2(a-b)^2}(t_i-t_{i-1})\right)\times \tag{3.7}$$
$$\sin\left(\frac{n\pi(b-x_{i-1})}{a-b}\right).$$

Note that,

$$\mathbb{P}(t_{i-1} < T_{ab}^{-} < t_i) = BM_{-b\,-a}^{+}(t_{i-1},t_i,-x_{i-1}),$$
$$\mathbb{P}(t_{i-1} < T_{ab} < t_i) = \mathbb{P}(t_{i-1} < T_{ab}^{+} < t_i)+\mathbb{P}(t_{i-1} < T_{ab}^{-} < t_i).$$

Proof. The proof of this lemma is given in Appendix A. □

Figure 3.2 shows the logarithmic absolute error generated when the infinite series in Equations (3.5) and (3.6) are truncated after N terms. We measure this error in terms of the parameter κ. One can observe that for small parameters of κ the first representation



(a) $\kappa = 0.05$ (b) $\kappa = 0.25$

Figure 3.2: Logarithmic absolute error of the truncation of the infinite series in Lemma 3.1.2, Equation (3.5) (Representation (i)) and Equation (3.6) (Representation (ii)) for different values of κ. We choose $\mu = 0.1$, $x_{i-1} = 0$, $x_i = 0.1$, $t_i - t_{i-1} = 1$, $a = \log(1.2)$, and $b = \log(0.8)$. The parameter σ is computed in terms of κ, $\sigma = \kappa / \left(\sqrt{(t_i - t_{i-1})/2} \exp(b - a) \right)$.

works better than the second one. However, for bigger values of κ it is more suitable to choose the second representation. If we analyse the convergence, we can conclude that in both representations the series rapidly converges.

There exist two different methods, which provide closed-form solutions in terms of infinite series, to compute these probabilities: renewal-type arguments together with Fourier inversion (see, e.g., [Darling and Siegert, 1953]) and risk-neutral valuation (see, e.g., [Kunitomo and Ikeda, 1992], [Lin, 1998])

In some applications we are also interested in the time $t_{i-1} < t < t_i$ this first-exit event happens. Definition 3.1.3 introduces first-passage time intensities. Due to the fact that there is a non-zero probability that the upper, respectively lower, barrier is never hit and thus $\int_0^\infty f_{ab}^{\pm}(t, x_{i-1})\, dt \leq 1$, the term "intensity" instead of "density" is used.

Definition 3.1.2 (First-passage time intensities)
Let $\{B_t\}_{t \geq 0}$ be a Brownian motion with drift. Let us consider two constant barriers $b < 0 < a$ and assume that $x_{i-1} = B_{t_{i-1}} \in (b, a)$. First-passage time intensities are defined as

$$f_{ab}^{+}(t, x_{i-1}) := \mathbb{P}\big(T_{ab}^{+} \in dt \mid B_{t_{i-1}} = x_{i-1}\big),$$
$$f_{ab}^{-}(t, x_{i-1}) := \mathbb{P}\big(T_{ab}^{-} \in dt \mid B_{t_{i-1}} = x_{i-1}\big).$$

We display in Lemma 3.1.3 the analytical expressions for $f_{ab}^{\pm}(t, x_{i-1})$.

Lemma 3.1.3 (First-passage time intensities)
Let $\{B_t\}_{t\geq 0}$ *be a Brownian motion with volatility* $\sigma > 0$ *and* $x_{i-1} := B_{t_{i-1}} \in (b,a)$. *Considering* $t_{i-1} < t < t_i < \infty$,

$$f_{ab}^{+}(t, x_{i-1}) = \frac{\sigma^2\pi}{(a-b)^2} \exp\Big(\frac{\mu(a-x_{i-1})}{\sigma^2}\Big) \sum_{n=1}^{\infty}(-1)^n n \sin\Big(\frac{\pi n(b-x_{i-1})}{a-b}\Big)\times$$
$$\exp\left(-\Big(\frac{\mu^2}{2\sigma^2} + \frac{\pi^2 n^2\sigma^2}{2(a-b)^2}\Big)(t-t_{i-1})\right),$$

$$f_{ab}^{-}(t, x_{i-1}) = \frac{\sigma^2\pi}{(a-b)^2} \exp\Big(\frac{\mu(b-x_{i-1})}{\sigma^2}\Big) \sum_{n=1}^{\infty}(-1)^n n \sin\Big(\frac{\pi n(-a+x_{i-1})}{a-b}\Big)\times$$
$$\exp\left(-\Big(\frac{\mu^2}{2\sigma^2} + \frac{\pi^2 n^2\sigma^2}{2(a-b)^2}\Big)(t-t_{i-1})\right).$$

Proof. These intensities are obtained from the second representation in Lemma 3.1.2 (see Appendix A, proof of Lemma 3.1.2). Note that, when $a \to \infty$, $f_{ab}^{+}(t, x_{i-1})$, converges to an inverse Gaussian density and we obtain Equation (11) in [Metwally and Atiya, 2002]. □

In a similar way we introduce the Brownian-bridge first-passage time intensities in the sequel, i.e. Lemma 3.1.4.

Definition 3.1.3 (Brownian-bridge first-passage time intensities)
Let $\{B_t\}_{t\geq 0}$ *be a Brownian motion with drift and fix start- and endpoint* $x_{i-1} := B_{t_{i-1}} \in (b,a)$, *respectively* $x_i := B_{t_i} \in \mathbb{R}$, *considered on the interval* $[t_{i-1}, t_i]$. *We define the Brownian-bridge first-passage time intensities as*

$$g_{ab}^{+}(t, x_{i-1}, x_i) := \mathbb{P}\big(T_{ab}^{+} \in dt \mid B_{t_{i-1}} = x_{i-1},\ B_{t_i} = x_i\big),$$
$$g_{ab}^{-}(t, x_{i-1}, x_i) := \mathbb{P}\big(T_{ab}^{-} \in dt \mid B_{t_{i-1}} = x_{i-1},\ B_{t_i} = x_i\big).$$

Closed-form expressions for the Brownian-bridge first-passage time intensities are provided in Lemma 3.1.4

Lemma 3.1.4 (Brownian-bridge first-passage time intensities)
Let $\{B_t\}_{t\geq 0}$ *be a Brownian motion with volatility* $\sigma > 0$. *Assume that* $x_{i-1} := B_{t_{i-1}} \in (b,a)$. *Considering* $x_i := B_{t_i}$ *and* $t_{i-1} < t < t_i < \infty$, *the first-passage time intensities*

$$g_{ab}^{+}(t, x_{i-1}, x_i) = \frac{\sigma^2\pi}{(a-b)^2}\frac{\sqrt{t_i - t_{i-1}}}{\sqrt{t_i - t}} \exp\left(\frac{(x_i - x_{i-1})^2}{2\sigma^2(t_i - t_{i-1})} - \frac{(x_i - a)^2}{2\sigma^2(t_i - t)}\right)\times \tag{3.8}$$
$$\sum_{n=1}^{\infty}(-1)^n n \exp\left(-\frac{\pi^2 n^2\sigma^2}{2(a-b)^2}(t-t_{i-1})\right)\times$$
$$\sin\Big(\frac{\pi n(b-x_{i-1})}{a-b}\Big),$$
$$g_{ab}^{-}(t, x_{i-1}, x_i) = g_{-b\,-a}^{+}(t, -x_{i-1}, -x_i).$$

Proof. We now aim at getting the expressions for the Brownian-bridge first-passage time intensities (Definition 3.1.3, Chapter 3). The idea for this work can be found in [Feller, 1966] and [Metwally and Atiya, 2002]. We compute g_{ab}^+, however note that since g_{ab}^+ and g_{ab}^- are symmetric, g_{ab}^- is implemented in the same way.

$$
\begin{aligned}
g_{ab}^+(t, x_{i-1}, x_i) &:= \mathbb{P}\big(T_{ab}^+ \in dt \,|\, B_{t_{i-1}} = x_{i-1},\, B_{t_i} = x_i\big) \\
&= \frac{\mathbb{P}\big(T_{ab}^+ \in dt, x_i \in dx \,|\, B_{t_{i-1}} = x_{i-1}\big)}{\mathbb{P}\big(x_i \in dx \,|\, B_{t_{i-1}} = x_{i-1}\big)} \\
&= \frac{\mathbb{P}\big(T_{ab}^+ \in dt \,|\, B_{t_{i-1}} = x_{i-1}\big) \cdot \mathbb{P}\big(x_i \in dx \,|\, t = T_{ab}^+, B_{t_{i-1}} = x_{i-1}\big)}{\mathbb{P}\big(x_i \in dx \,|\, B_{t_{i-1}} = x_{i-1}\big)} \\
&= \frac{\mathbb{P}\big(T_{ab}^+ \in dt \,|\, B_{t_{i-1}} = x_{i-1}\big) \cdot \mathbb{P}\big(x_i \in dx \,|\, t = T_{ab}^+, B_t = a\big)}{\mathbb{P}\big(x_i \in dx \,|\, B_{t_{i-1}} = x_{i-1}\big)} \\
&= \frac{f_{ab}^+(t, x_{i-1}) \cdot f_{\{a+\mu(t_i-t), \sigma\sqrt{t_i-t}\}}\big(x_i\big)}{f_{\{x_{i-1}+\mu(t_i-t_{i-1}), \sigma\sqrt{t_i-t_{i-1}}\}}\big(x_i\big)},
\end{aligned}
$$

where $f_{\{\mu,\sigma\}}(\cdot)$ is the density function of a Gaussian variable with mean μ and variance σ^2,

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{\sigma^2\pi(-1)^n n}{(a-b)^2} \exp\Big(-\Big(\frac{\mu^2}{2\sigma^2} + \frac{\pi^2 n^2 \sigma^2}{2(a-b)^2}\Big)(t - t_{i-1})\Big) \sin\Big(\frac{\pi n(b - x_{i-1})}{a-b}\Big) \times \\
&\qquad \exp\left(\frac{\mu(a - x_{i-1})}{\sigma^2}\right) \frac{\frac{1}{\sqrt{2\pi(t_i-t)\sigma}} \exp\left(-\frac{(x_i-a-\mu(t_i-t))^2}{2\sigma^2(t_i-t)}\right)}{\frac{1}{\sqrt{2\pi(t_i-t_{i-1})\sigma}} \exp\left(-\frac{(x_i-x_{i-1}-\mu(t_i-t_{i-1}))^2}{2\sigma^2(t_i-t_{i-1})}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sigma^2\pi}{(a-b)^2} \sum_{n=1}^{\infty}(-1)^n n \exp\Big(-\frac{\pi^2 n^2\sigma^2}{2(a-b)^2}(t - t_{i-1})\Big) \sin\Big(\frac{\pi n(b - x_{i-1})}{a-b}\Big) \times \\
&\qquad \frac{\sqrt{t_i - t_{i-1}}}{\sqrt{t_i - t}} \exp\left(-\frac{\mu^2}{2\sigma^2}(t - t_{i-1})\right) \exp\left(\frac{\mu(a - x_{i-1})}{\sigma^2}\right) \times \\
&\qquad \exp\left(\frac{(x_i - x_{i-1} - \mu(t_i - t_{i-1}))^2}{2\sigma^2(t_i - t_{i-1})} - \frac{(x_i - a - \mu(t_i - t))^2}{2\sigma^2(t_i - t)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sigma^2\pi}{(a-b)^2} \frac{\sqrt{t_i - t_{i-1}}}{\sqrt{t_i - t}} \exp\left(\frac{(x_i - x_{i-1})^2}{2\sigma^2(t_i - t_{i-1})} - \frac{(x_i - a)^2}{2\sigma^2(t_i - t)}\right) \times \\
&\qquad \sum_{n=1}^{\infty}(-1)^n n \exp\left(-\frac{\pi^2 n^2 \sigma^2}{2(a-b)^2}(t - t_{i-1})\right) \sin\left(\frac{\pi n(b - x_{i-1})}{a-b}\right).
\end{aligned}
$$

□

3.2 Simulation of first-exit times. Algorithms

In this section we describe the idea behind the algorithms to simulate the first-passage times for a jump-diffusion process within two constant barriers using the Brownian-bridge technique. The first algorithm computes the probabilities of first reaching one of the barriers (Algorithm 6.2.1). The second algorithm (Algorithm 6.2.4) calculates the probabilities of first hitting any of the barriers and in addition, the expectations of the payoff of the product we aim at pricing $\mathbb{E}[w(T_{ab}, B_T, \varepsilon)]$, i.e. a function that depends on a first-exit time T_{ab}, the final value of the asset path B_T, and the barrier hitting event ε. The pseudocodes for these algorithms are provided in Chapter 6.

In a nutshell the idea consists of first sampling the jump-times which are the observation points through the temporal path $[0, T]$. Then we simulate the jump-diffusion process between two successive jump-times and check whether the barrier hitting event took place between these two observation nodes. Two possibilities should be taken into account:

(i) The start- and endpoint of the stochastic process are not on the same side of the barrier. In this case, the barrier has been reached.

(ii) The start- and endpoint of the simulated process are on the same side of the barrier. In this case it could be that the barrier has been reached between the observation points (start- and endpoint of the process) or it could be that the barrier has not been reached. We compute the Brownian-bridge probabilities to answer how likely the two cases are.

One also has to take into consideration that the stochastic process could cross any of the barriers due to the jumps. In Figure 4.4 there is a graphical explanation of the idea behind the algorithms.

Note that the barrier hitting probabilities after the first time step need to be cumulative, e.g., the probability that the process reaches one of the barriers in the second-time step could only happen in case it survived in the first time step.

Let us introduce the cumulative probabilities.

- The probability of hitting first the upper or lower barrier from $t = t_{i-1}$ to $t = t_i$,

$$P^+_{i-1,i} = BB^+_{ab}(t_{i-1}, t_i, B_{t_{i-1}}, B_{t_i}) \quad \text{or} \quad P^-_{i-1,i} = BB^-_{ab}(t_{i-1}, t_i, B_{t_{i-1}}, B_{t_i}).$$

- The probability of surviving during the interval (t_{i-1}, t_i),

$$P_{i-1,i} = 1 - P^+_{i-1,i} - P^-_{i-1,i}.$$

So, after the first-time interval (t_0, t_1):

- The probability that a trajectory reaches the upper or the lower barrier in $(t_{i-1}, t_i), i \geq 2$ (see Figure 3.2),

$$\prod_{j=1}^{i-1} P_{j-1,j} \cdot P^+_{i-1,i}, \quad \text{or} \quad \prod_{j=1}^{i-1} P_{j-1,j} \cdot P^-_{i-1,i}, \quad i \geq 2.$$

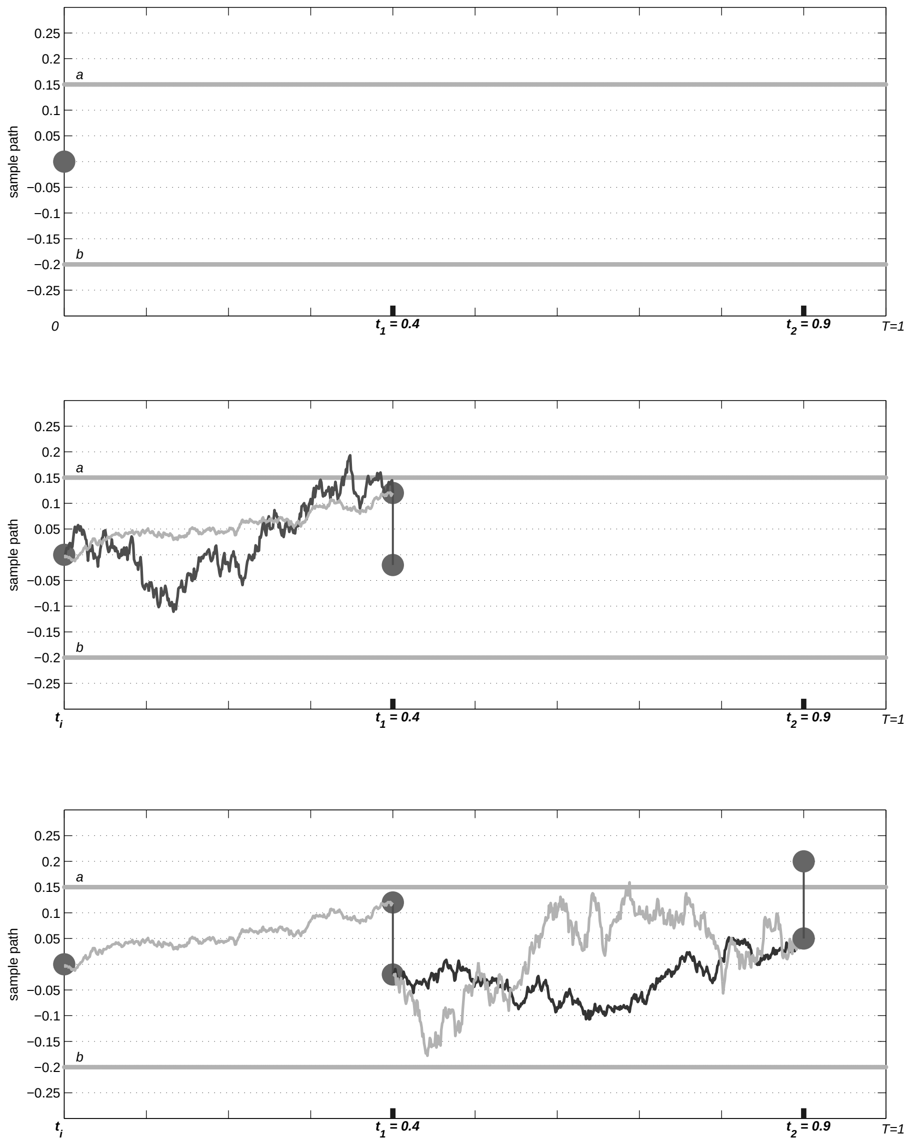


Figure 3.3: Brownian-bridge algorithm: the graph (top) shows the simulated number of jumps, $N_T = 2$, and the simulated jump-times, $t_1 = 0.4$ and $t_2 = 0.9$. The graph (middle) shows possible simulated paths of the process from t_0 till t_1. It could happen that a path reaches a barrier (dark path) or that it does not reach any barrier (light path) between two observation points. In the lower graph, possible paths are simulated between two successive jump instants. Note that a barrier crossing event could also happen due to a jump.

- The probability that the process hits any of the barries due to a jump in t_i, $i \geq 1$,

$$\prod_{j=1}^{i-1} P_{j-1,j}.$$

In general:

- The probability of reaching the upper barrier first in $[t_0, t_i)$,

$$\sum_{j=1}^{i-1} P_{j-1,j} \cdot P_{i-1,i}^{+}.$$

- The probability of crossing the lower barrier first in $[t_0, t_i)$,

$$\sum_{j=1}^{i-1} P_{j-1,j} \cdot P_{i-1,i}^{-}.$$

- The probability of surviving within $[t_0, t_i)$

$$\prod_{j=1}^{i-1} P_{j-1,j}.$$

3.3 Applications

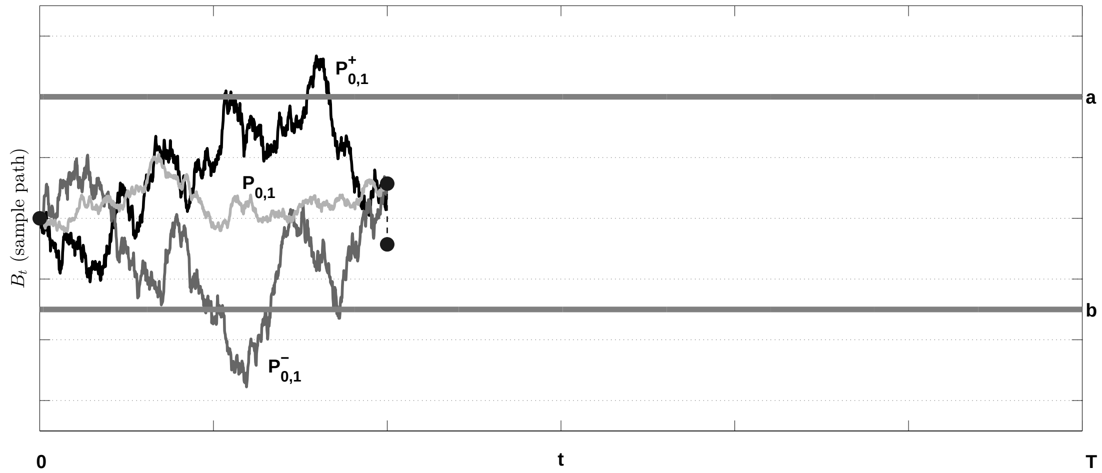
In this section we present possible applications for the algorithms presented in the previous section. We display numerical results for the pricing of some financial products and we explain a possible application in credit-risk management.

3.3.1 Pricing financial products

For the pricing of financial derivatives we consider the generalization of the Black–Scholes model given in [Merton, 1976] (for the Black–Scholes model see Chapter 2, Section 2.5). We define the stock price process as the exponential value of the jump-diffusion process in (3.1),

$$S_t := S_0 \exp(X_t) = S_0 \exp\Bigg(\underbrace{\left(r - \frac{1}{2}\sigma^2 - \delta\right) t + \sigma W_t + \sum_{i=1}^{N_t} Y_i}_{X_t}\Bigg), \quad t \geq 0, \tag{3.9}$$

where r is the risk-free interest rate and $\delta = \lambda\left(\mathbb{E}[\exp(Y_1)] - 1\right)$ the drift adjustments due to the jumps. Since we assume the exponential of the jump-diffusion process to describe the stock price process, we also need to modify the barriers according to it. In



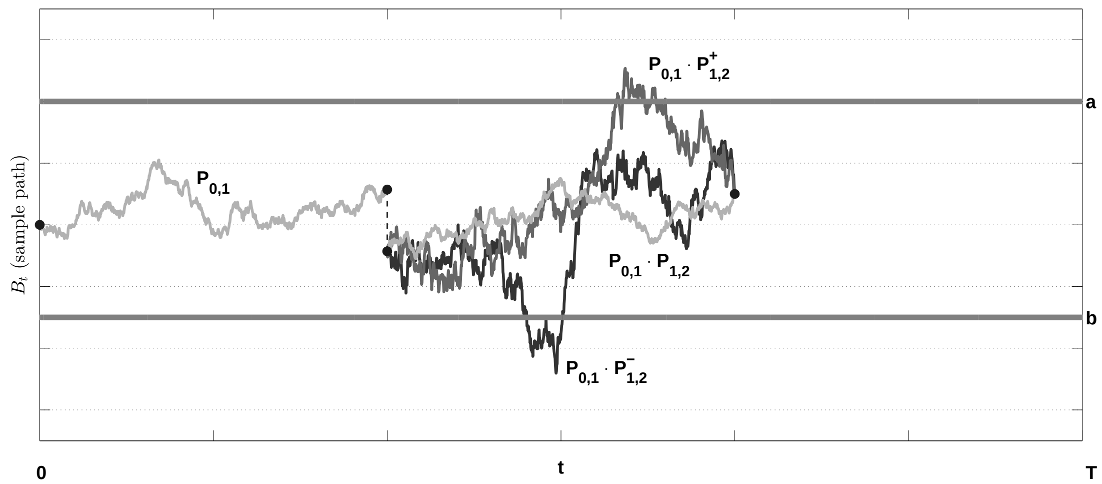


Figure 3.4: Different situations and respective probabilities of barrier reaching events. The plot (top) shows all possible situations in the first time step: (i) the upper or lower barrier has been reach with probability $P^+_{0,1}$ or $P^-_{0,1}$ respectively, (ii) non of the barriers have been hit, the probability of this event is $P_{0,1}$. The plot (down) shows all possible situations in the next time step knowing that non of the barriers have been reached in the previous step: (i) the upper barrier is hit, the probability of this event is $P_{0,1} \cdot P^+_{1,2}$, (ii) the lower barrier is hit, the probability of this event is $P_{0,1} \cdot P^-_{1,2}$, (iii) non of the barriers have been reached, this event happens with probability $P_{0,1} \cdot P_{1,2}$.

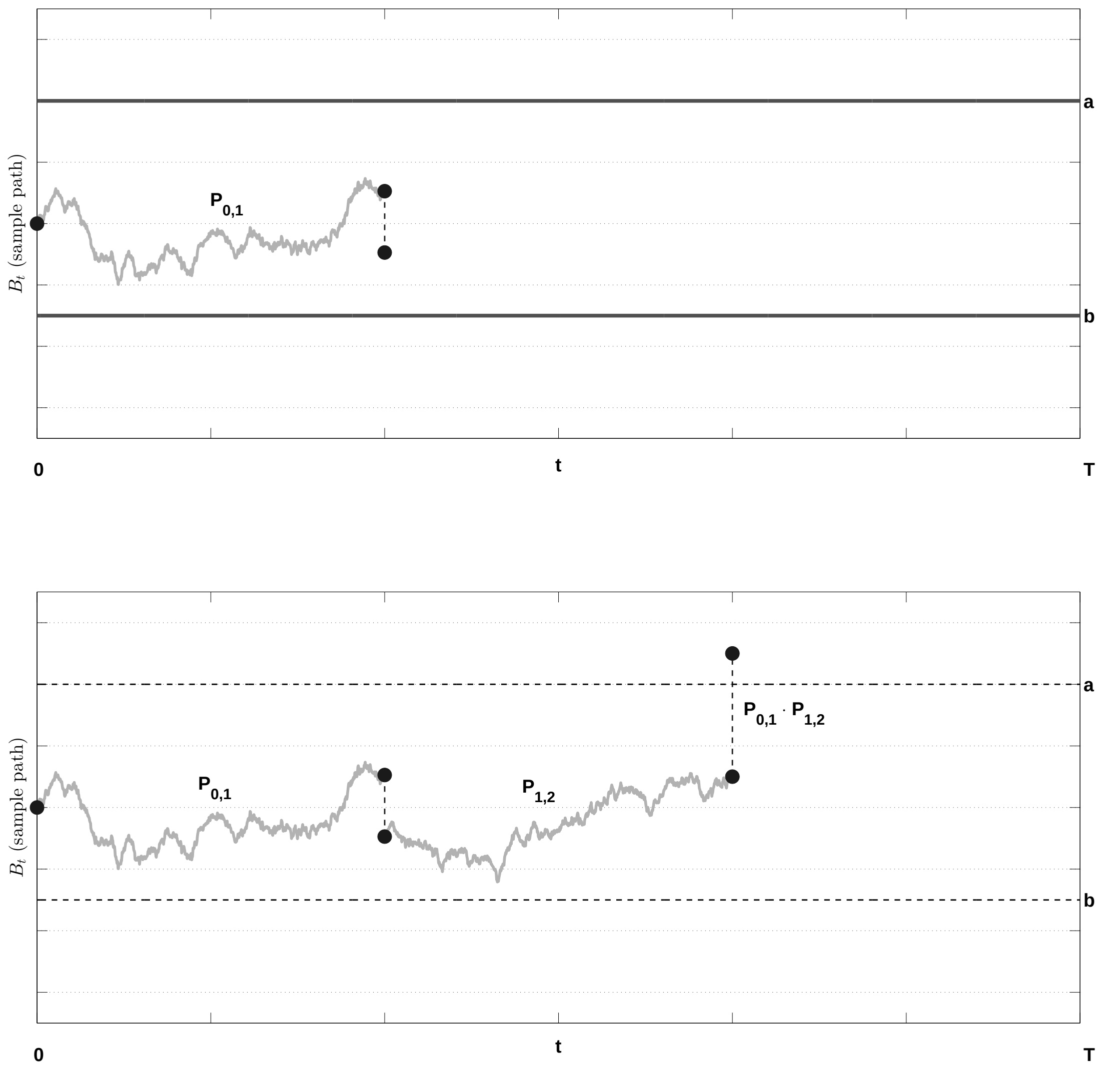


Figure 3.5: Situation where a barrier is reached due to a jump. Non of the barriers have been reached in the first two steps and in the third step the upper barrier is reached due to an upper jump. The probability of this event is $P_{0,1} \cdot P_{1,2}$. Note that the only possibility to hit a barrier in $t_i, i \geq 1$ is that the process survives in the previous time-steps.

this new setting we define the upper barrier as $\tilde{a} = S_0 \exp(a)$ and the lower barrier as $\tilde{b} = S_0 \exp(b)$. Therefore the barrier hitting probabilities can be written as

$$\begin{aligned}\mathbb{P}(T_{ab} > T) &= \mathbb{P}(X_t \in (b,a), \forall t \in [0,T]) \\ &= \mathbb{P}\Big(X_t \in (\log(\tilde{b}/S_0), \log(\tilde{a}/S_0)), \forall t \in [0,T]\Big) = \mathbb{P}(S_t \in (\tilde{b}, \tilde{a}), \forall t \in [0,T]).\end{aligned}$$

The jumps, in this case, follow the double exponential distribution, which could be interpreted as two exponential distributions, one on each side of a given boundary ($x = 0$ in this case), together added (see, e.g., [Kou and Wang, 2003]):

$$\mathbb{P}_Y(dx) = p\,\lambda_\oplus e^{-\lambda_\oplus x}\mathbb{1}_{\{x\geq 0\}}dx + (1-p)\,\lambda_\ominus e^{\lambda_\ominus x}\mathbb{1}_{\{x<0\}}dx,$$

where $0 \leq p \leq 1$ is the probability for a positive jump. Positive jumps are exponentially distributed with parameter $\lambda_\oplus > 1$[1], and negative jumps with parameter $\lambda_\ominus > 0$. The drift adjustment in this case is given by $\delta := \lambda\big(\mathbb{E}_\mathbb{Q}[\exp(Y_1)] - 1\big) = \lambda\big(p\lambda_\oplus/(\lambda_\oplus - 1) + (1 - p)\lambda_\ominus/(\lambda_\ominus + 1) - 1\big)$.

The double exponential jump-diffusion process has the advantage that one can compare the results with the Laplace transforms of the first-passage times as presented in, e.g., [Kou and Wang, 2003], [Sepp, 2004]. Nevertheless, the Brownian-bridge algorithm allows to change the jump-size distribution.

Digital first-touch options

A digital first touch option is a financial contract that pays \$1 at maturity T if the underlying process, during the lifetime of the derivative, reaches one or both barriers in case of a *knock-in* option, and if it does not cross any threshold in case of a *knock out* option. These options are the most liquid and actively traded exotic options on FX markets (see, e.g., [Carr and Crosby, 2010]) and they can be used as a tool to construct more complex derivatives (see, e.g., [Boyarchenko and Levendorskiĭ, 2002]).

Through this work we price *knock-in* options. The owner of an *up-and-in* option receives \$1 if the underlying process crosses first the upper barrier. In contrast the owner of a *down-and-in* gets the payment of \$1 if the stock-price process reaches first the lower barrier. The owner does not get anything if the underlying does not reach any of the barriers. Note that in case of *knock-out* options, the owner receives the payment in case the underlying remains within both barriers until the expiration date of the contract or hits the upper (resp. lower) barrier in case of *down-and-out* (resp. *up-and-out*) options.

Under the risk-neutral measure $\mathbb{Q}$, the *up-and-in* option can be priced as

$$X^+(0) := e^{-rT}\,\mathbb{E}_\mathbb{Q}\big[\mathbb{1}_{\{T^+_{ab}\leq T\}}\big] = e^{-rT}\,\mathbb{Q}(T^+_{ab} \leq T),$$

where $a := \ln(\tilde{a}/S_0)$, $b := \ln(\tilde{b}/S_0)$.

[1] The condition $\lambda_\oplus > 1$ is required to guarantee the existence of the first moment of S_t in (3.9).

Similarly, the price of a *down-and-in* option is given by

$$X^{-}(0) := e^{-rT}\,\mathbb{E}_{\mathbb{Q}}\big[\mathbb{1}_{\{T_{ab}^{-}\le T\}}\big] = e^{-rT}\,\mathbb{Q}(T_{ab}^{-}\le T). \tag{3.10}$$

Table 3.1 compares the prices of *(knock-in) digital first-touch options* in a double exponential jump-diffusion model obtained by the standard Monte-Carlo simulation on a discrete grid using the Brownian-bridge algorithm, Algorithm 6.2.1. The parameters considered in the simulation are: $r = 5\%$, $\sigma = 20\%$, $p = 0.5$, $\alpha_{\oplus} = \alpha_{\ominus} = 5$, and $T = 1$ (expiration date of the contract). Furthermore, we set $\tilde{b} = 80$, $\tilde{a} = 120$, and $S_0 = 100$.

In the standard Monte-Carlo algorithm, we use 250, respectively 1000, discretization steps. According to the expected number of jumps per year λ. We consider different scenarios for different intensities of jumps: "Black–Scholes" ($\lambda = 0$), "Low" ($\lambda = 0.5$), "Middle" ($\lambda = 2$), and "High" ($\lambda = 8$).

In Table 3.1, we can see that the Brownian-bridge algorithm is significantly faster than the brute–force Monte-Carlo simulation on a discrete grid. In addition the Brownian-bridge algorithm computes unbiased results so we can conclude that the values got using the Brownian-bridge technique are close to the exact prices by [Sepp, 2004].

Once that we conclude that it is more convenient, regarding the simulation accuracy and simulation speed, to use the Brownian-bridge algorithm to simulate first-passage times problems, we apply the Brownian-bridge technique (Algorithm 6.2.1) to price *corridor bonus certificates.*

Corridor bonus certificates

Corridor bonus certificates are financial products whose payoff depends on a barrier hitting event and in addition the amount that the owner of the certificate gets varies depending on which barrier has been reached first. These certificates are emitted by (major) banks, e.g., *Société Générale*, and there is a variety of these financial products depending on the underlying assets.

We again consider a scenario with two constant barriers, $0 < \tilde{b} < \tilde{a}$, $\tilde{a}, \tilde{b} \in \mathbb{R}$, and the stock-price process with initial value $\tilde{b} < S_0 < \tilde{a}$. We assume that the expiration time of the certificate is T. The payoff of these certificates depends on the value of the underlying at time T, a prior committed value F and the first-passage time events during the lifetime of the certificate:

(i) If the stock-price remains between the two barriers, during the lifetime of the certificate, then the owner gets the fixed amount F at time T.

(ii) If the stock-price hits the lower barrier first, then the payoff is the minimum between the value of the stock price at time T, S_T, and the fixed amount F.

(iii) If the upper barrier has been reached first, then the payment the owner of the certificate receives is $\max\{\min\{2\,S_0 - S_T, F\}, 0\}$.

		Black–Scholes	Jump-diffusion		
		$\lambda = 0$	Low ($\lambda = 0.5$)	Middle ($\lambda = 2$)	High ($\lambda = 8$)
StdMC (250)	$\hat{X}^+(0)$	0.3734 ± 0.0008	0.3765 ± 0.0008	0.3836 ± 0.0008	0.3815 ± 0.0008
	relative bias	4.6%	3.8%	1.7%	0.3%
	runtime	47.2s	46.1s	45.0s	41.7s
StdMC (1000)	$\hat{X}^+(0)$	0.3820 ± 0.0003	0.3838 ± 0.0003	0.3880 ± 0.0008	0.3825 ± 0.0004
	relative bias	2.2%	1.8%	1.3%	0.2%
	runtime	185.6s	183.8s	179.4s	161.7s
Brownian-bridge	$\hat{X}^+(0)$	0.3907 ± 0.0002	0.3915 ± 0.0002	0.3928 ± 0.0003	0.3821 ± 0.0003
	relative bias	0.0%	0.0%	0.0%	0.0%
	runtime	18.1s	19.3s	14.3s	30.1s
Exact price	$X^+(0)$	0.3908	0.3913	0.3928	0.3822

Table 3.1: Estimated prices for $X^+(0)$ and confidence intervals at the confidence level $\alpha = 90\%$ of *(upper barrier) digital first-touch options* for different jump intensities λ. The prices are estimated using the standard Monte-Carlo technique, with 250 and 1000 times-steps respectively and the Brownian-bridge algorithm (Algorithm 6.2.1) using $K = 10^6$ simulation runs. The exact value of the option was estimated by inverting the Laplace transforms presented by, e.g., [Sepp, 2004]. We also display the bias and runtime for each algorithm. The relative bias is the relative difference between the simulated value $\hat{X}^+(0)$ divided by the true value $X^+(0)$ of the option. The computation time was calculated using Matlab 2012a on a 3.1 GHz PC.

	Black-Scholes	Jump-diffusion		
	$\lambda = 0$	Low ($\lambda = 0.5$)	Middle ($\lambda = 2$)	High ($\lambda = 8$)
$\lambda_\oplus = \lambda_\ominus = 10$	118.75 ± 0.00	116.88 ± 0.02	109.84 ± 0.04	82.46 ± 0.07
$\lambda_\oplus = 2\,\lambda_\ominus = 20$	118.75 ± 0.00	118.05 ± 0.01	114.01 ± 0.03	89.68 ± 0.07
$\lambda_\oplus = \lambda_\ominus/2 = 10$	118.75 ± 0.00	117.28 ± 0.02	112.47 ± 0.04	91.06 ± 0.07

Table 3.2: Estimated prices for $CB^+(0)$ using Algorithm 6.2.1 for different values of $\lambda_\oplus$ and $\lambda_\ominus$, and for different scenarios depending on jump-size intensities, λ. The parameters we use are $S_0 = 100$, $\tilde{a} = 140$, $\tilde{b} = 60$, $r = 1\%$, $T = 1$, $\sigma = 10\%$, and $F = 120$. The numbers of simulated trajectories is $K = 10^6$.

So,

$$\text{payoff}(\{S_t\}_{0\leq t\leq T}) = \begin{cases} F, & T_{ab} > T, \\ \min(S_T, F), & T_{ab}^- \leq T, \\ \max\left(\min(2\,S_0 - S_T, F), 0\right), & T_{ab}^+ \leq T, \end{cases} \tag{3.11}$$

remember that a and b are the barriers in the logarithmic setting: $a = \log(\tilde{a}/S_0)$ and $b = \log(\tilde{b}/S_0)$.

The price of these certificates under the Black–Scholes framework, using the risk-neutral measure $\mathbb{Q}$, conditional on $\mathbb{P}(T_{ab}^- \leq T)$, $\mathbb{P}(T_{ab}^+ \leq T)$, and S_T is computed in the following way:

$$\begin{aligned} CB^+(0) :=& \, e^{-rT}\,\mathbb{E}_{\mathbb{Q}}\big[\text{payoff}(\{S_t\}_{0\leq t\leq T})\big] \\ =& \, e^{-rT}\,\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}\big[\text{payoff}(\{S_t\}_{0\leq t\leq T})\,\big|\,T_{ab}^-,\,T_{ab}^+,\,S_T\big]\Big] \\ =& \, e^{-rT}\,\mathbb{E}_{\mathbb{Q}}\Big[F,\big(1 - \mathbb{1}_{\{T_{ab}^+\leq T\}} - \mathbb{1}_{\{T_{ab}^-\leq T\}}\big) \\ & + \mathbb{1}_{\{T_{ab}^+\leq T\}}\max(\min(2S_0 - S_T, F), 0) + \mathbb{1}_{\{T_{ab}^-\leq T\}}\min(S_T, F)\Big]. \end{aligned}$$

Table 3.2 displays the prices of *corridor bonus certificates* simulated with Algorithm 6.2.1 for different values of $\lambda_\oplus$ and $\lambda_\ominus$ in four different scenarios depending on the intensity of the jumps, λ: Black–Scholes (no jumps, $\lambda = 0$), "low", "middle" and "high".

3.3.2 Credit-risk

Credit-risk can broadly be defined as the risk of failing on a payment of a debt. In structural models this failure, named *default*, is a consequence of not having sufficient assets, i.e. the number of assets of a company reaches a significantly low level comparing to the liabilities of the company (standards in credit-risk and structured credit-risk can be found in e.g., [Shimko, 2004], [Duffie and Singleton, 2012]). Therefore, bonds can be priced as an option on firm's assets. Let us consider, as an illustrative example, a company funded with shares (equity) and bonds (debt). Let us assume that the company makes the compromise to pay the debt to the bondholder at an agreed time T. If at time T the total value of the firm's assets, S, is higher than the value of the debt, D, then the company pays to the bondholder the value of the debt. However, if the total value of the assets is lower than the value of the debt, the bondholder exercises the right to liquidate the company and gets the amount obtained from the liquidation, L. In this case D is the lower barrier, $\tilde{b} = D$. But there are situations where the company would like to repay the bond before maturity T, for example due to its healthy condition to improve the credit rating. This case is described as having an upper threshold $\tilde{a}$ (see, e.g., [Downing et al., 2005], [Gabaix et al., 2007], [Dobránszky and Schoutens, 2008]) and when this barrier is reached first the bondholder gets the same amount as in the case with no default.

Let $B(0)$ be the present value (at time $t = 0$) of a bond with nominal value 1 which pays continuous interest at a rate d. The riskless interest rate is denoted by r. Assume that the maturity of the bond is T. In addition the holder of the bond will get an amount $L \in [0, 1]$, recovery rate, in case the default happens and he exercises the right to get his debt paid. Then B is computed in the following way:

$$
\begin{aligned}
B(0) &= e^{-rT}\mathbb{Q}(T_{ab} > T) + \mathbb{E}_{\mathbb{Q}}\left[\int_0^T de^{-rt}\,\mathbb{1}_{\{T_{ab}>T\}}dt\right] + B^+(0) + B^-(0) \\
&= e^{-rT}\mathbb{Q}(T_{ab} > T) + \frac{d}{r}\left(1 - e^{-rT}\right)\mathbb{Q}(T_{ab} > T) + B^+(0) + B^-(0) \\
&= \left[\frac{d}{r} + \left(1 - \frac{d}{r}\right) e^{-rT}\right] \mathbb{Q}(T_{ab} > T) + B^+(0) + B^-(0),
\end{aligned}
$$

where

$$
\begin{aligned}
B^+(0) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-rT_{ab}^+} + \int_0^{T_{ab}^+} d\,e^{-rt}dt\,\mathbb{1}_{\{T_{ab}^+\leq T\}}\right], \\
B^-(0) &= \mathbb{E}_{\mathbb{Q}}\left[L\,e^{-rT_{ab}^-} + \int_0^{T_{ab}^-} d\,e^{-rt}dt\,\mathbb{1}_{\{T_{ab}^-\leq T\}}\right],
\end{aligned}
$$

such that $a := \log(\tilde{a}/S_0)$ and $b := \log(\tilde{b}/S_0)$. $B^+(0)$ describes the prepayment case and $B^-(0)$ the situation where the firm defaults.

Note that to price these bonds we need the first-exit time events as well as the moment when exactly these events occurs. Therefore the value of these bonds can be estimated using Algorithm 6.2.4.

4

The mean of Marshall–Olkin dependent exponential random variables

"People are afraid to face how great a part of life is dependent on luck.
It's scary to think so much is out of one's control."
Match Point (2005).

In this chapter we investigate the probability distribution of $S_d := X_1 + \ldots + X_d$, where the vector $(X_1, \ldots, X_d)$ is distributed according to the Marshall–Olkin law. Closed-form solutions are derived in the general bivariate case and for $d \in \{2, 3, 4\}$ in the exchangeable subfamily. Our computations can be extended to higher dimensions, which, however, become cumbersome due to the large number of involved parameters. We consider the extendible subfamily of Marshall–Olkin distributions to identify the limiting distribution of S_d/d when d tends to infinity. This result might serve as a convenient approximation in high-dimensional situations. Possible fields of application for the presented results are reliability theory, insurance, and credit-risk modelling.

This chapter is based on the paper "The mean of Marshall–Olkin dependent exponential random variables", written by Fernández L., Mai J.-F. and Scherer M., and published in Marshall–Olkin Distributions - Advances in Theory and Practice, Cherubini U., Mulinacci S., Durante F. (eds.), Springer Proceedings in Mathematics & Statistics. Springer (2015).

4.1 General bivariate case

In this section we derive the probability distribution of $S_2 = aX_1 + bX_2$, where a, b are positive constants. Knowing that the dimension d represents the number of components in the system, the quantity $S_2/2$ $(a = b = 1/2)$ is precisely the average lifetime of the two components in the system.

Remark 4.1.1 (Simplify notation)
To derive the results in this chapter we will use the concepts introduced in Section 2.7, Chapter 2. To simplify notation, regarding parameters λ_I*, instead of using* $\lambda_{\{1\}}$, $\lambda_{\{2\}}$, $\lambda_{\{12\}} \dots$ *we use* λ_1, λ_2, $\lambda_{12} \dots$ *and referring to variables* E_I *instead of denoting them by* $E_{\{1\}}, E_{\{2\}}, E_{\{12\}} \dots$ *we use the notation* $E_1, E_2, E_{12} \dots$.

Lemma 4.1.1 (The weighted sum of two lifetimes)
On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *let* (X_1, X_2) *be a random vector constructed as in* (2.24) *and let* a, b *be positive constants. The survival function of the weighted sum of* X_1 *and* X_2 *is computed as*

$$\begin{aligned} \mathbb{P}(aX_1 + bX_2 > x) \quad = \quad & \frac{\lambda_1}{\lambda_1 - (\lambda_2 + \lambda_{12})\frac{a}{b}} e^{-(\lambda_2+\lambda_{12})\frac{x}{b}} \left(1 - e^{-(\lambda_1 - (\lambda_2+\lambda_{12})\frac{a}{b})\frac{x}{a+b}}\right) \\ & + \frac{\lambda_2}{\lambda_2 - (\lambda_1 + \lambda_{12})\frac{b}{a}} e^{-(\lambda_1+\lambda_{12})\frac{x}{a}} \left(1 - e^{-(\lambda_2 - (\lambda_1+\lambda_{12})\frac{b}{a})\frac{x}{a+b}}\right) \\ & + e^{-(\lambda_1+\lambda_2+\lambda_{12})\frac{x}{a+b}}. \end{aligned} \tag{4.1}$$

Proof.

We need to compute the following probability:

$$\begin{aligned} \mathbb{P}(aX_1 + bX_2 > x) \quad = \quad & \mathbb{P}(aX_1 + bX_2 > x, X_1 < X_2) + \mathbb{P}(aX_1 + bX_2 > x, X_2 < X_1) \\ & + \mathbb{P}(aX_1 + bX_2 > x, X_1 = X_2). \end{aligned}$$

Observe that,

$$\begin{aligned} & X_1 < X_2 \Leftrightarrow E_1 < X_2, \\ & X_2 < X_1 \Leftrightarrow E_2 < X_1, \\ & X_1 = X_2 \Leftrightarrow E_{12} < \min\{E_1, E_2\}, \end{aligned}$$

and by the so called min-stability of the exponential distribution (Equation (2.23)),

$$\min\{E_1, E_2\} \sim \mathrm{Exp}(\lambda_1 + \lambda_2).$$

Then,

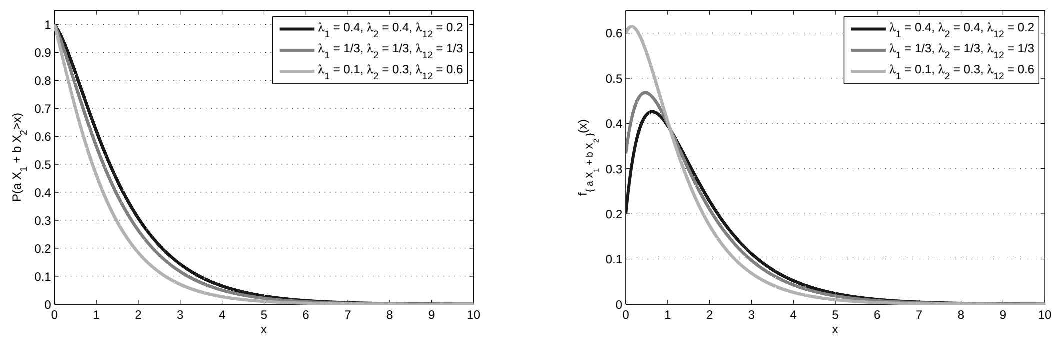
$$\begin{aligned} & \mathbb{P}(aX_1 + bX_2 > x, X_1 < X_2) \\ & \quad = \mathbb{P}(aX_1 + bX_2 > x, E_1 < X_2) \overset{(*)}{=} \\ & X_2 > E_1 \Leftrightarrow \min\{E_2, E_{12}\} > E_1 \Rightarrow E_{12} > E_2 > E_1 \quad \text{or} \quad E_2 > E_{12} > E_1. \\ & \text{So,} \quad E_{12} > E_1, \quad \text{and therefore,} \quad X_1 = \min\{E_1, E_{12}\} = E_1 \\ & \quad \overset{(*)}{=} \mathbb{P}\left(X_2 > E_1 > \frac{x - bX_2}{a}\right) \\ & \quad = \mathbb{E}\left[\mathbb{P}\left(X_2 > E_1 > \frac{x - bX_2}{a} | E_1\right)\right] \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \mathbb{P}\left(X_2 > y_1 > \frac{x - bX_2}{a}\right) f_{E_1}(y_1) dy_1 \\
&= \int_0^\infty \mathbb{P}\left(X_2 > \max\left\{y_1, \frac{x - aX_1}{b}\right\}\right) f_{E_1}(y_1) dy_1 \\
&= \int_0^{\frac{x}{a+b}} \mathbb{P}\left(X_2 > \frac{x - ay_1}{b}\right) f_{E_1}(y_1) dy_1 + \int_{\frac{x}{a+b}}^\infty \mathbb{P}(X_2 > y_1) f_{E_1}(y_1) dy_1 \\
&= \frac{\lambda_1}{\lambda_1 - (\lambda_2 + \lambda_{12})\frac{a}{b}} e^{-(\lambda_2+\lambda_{12})\frac{x}{b}} \left(1 - e^{-(\lambda_1 - (\lambda_2+\lambda_{12})\frac{a}{b})\frac{x}{a+b}}\right) \\
&\quad + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_{12}} e^{-(\lambda_1+\lambda_2+\lambda_{12})\frac{x}{a+b}}.
\end{aligned}
$$

$\mathbb{P}(aX_1 + bX_2 > x, X_2 < X_1)$ and $\mathbb{P}(aX_1 + bX_2 > x, X_1 = X_2)$ are computed in the same way so,

$$
\begin{aligned}
\mathbb{P}(aX_1 + bX_2 > x) = &\frac{\lambda_1}{\lambda_1 - (\lambda_2 + \lambda_{12})\frac{a}{b}} e^{-(\lambda_2+\lambda_{12})\frac{x}{b}} \left(1 - e^{-(\lambda_1 - (\lambda_2+\lambda_{12})\frac{a}{b})\frac{x}{a+b}}\right) \\
&+ \frac{\lambda_2}{\lambda_2 - (\lambda_1 + \lambda_{12})\frac{b}{a}} e^{-(\lambda_1+\lambda_{12})\frac{x}{a}} \left(1 - e^{-(\lambda_2 - (\lambda_1+\lambda_{12})\frac{b}{a})\frac{x}{a+b}}\right) \\
&+ e^{-(\lambda_1+\lambda_2+\lambda_{12})\frac{x}{a+b}}.
\end{aligned}
$$

□



(a) Survival function. (b) Density function.

Figure 4.1: The survival and density function of $S_2 = aX_1 + bX_2$, where $a = 30\%$ and $b = 70\%$.

Once the survival function of S_2 is known, one can further compute the density and the Laplace transform of S_2.

Corollary 4.1.1 (Probability density function)
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, (X_1, X_2) a random vector constructed as in (2.24), and let a, b be positive constants. The probability density function of $S_2 = aX_1 + bX_2$ is given by,

$$f_{S_2}(x) = \frac{\lambda_1(\lambda_2+\lambda_{12})}{\lambda_1 b - (\lambda_2+\lambda_{12})a} e^{-(\lambda_2+\lambda_{12})\frac{x}{b}} \left(1 - e^{-\left(\lambda_1 - (\lambda_2+\lambda_{12})\frac{a}{b}\right)\frac{x}{a+b}}\right) \tag{4.2}$$
$$+ \frac{\lambda_2(\lambda_1+\lambda_{12})}{\lambda_2 a - (\lambda_1+\lambda_{12})b} e^{-(\lambda_1+\lambda_{12})\frac{x}{a}} \left(1 - e^{-\left(\lambda_2 - (\lambda_1+\lambda_{12})\frac{b}{a}\right)\frac{x}{a+b}}\right)$$
$$+ \frac{\lambda_{12}}{a+b} e^{-(\lambda_1+\lambda_2+\lambda_{12})\frac{x}{a+b}}.$$

Proof. The density function is derived using the survival function in Equation (4.1), i.e.

$$f_{S_2}(x) = \frac{d}{dx}\left(1 - \mathbb{P}(aX_1 + bX_2 > x)\right).$$

□

Remark 4.1.2 (Conflictive points)
Note that in Equations (4.1) and (4.2) there is a non-defined situation when $\lambda_1 = (\lambda_2 + \lambda_{12})a/b$ or $\lambda_2 = (\lambda_1 + \lambda_{12})b/a$. However, these parameter constellations can be treated computing the existing limits when λ_1 and λ_2 approach these conflictive points.

$$\lim_{\lambda_1 \to (\lambda_2+\lambda_{12})a/b} \mathbb{P}(aX_1 + bX_2 > x) = \left(\frac{(\lambda_2+\lambda_{12})ax}{b(a+b)} + 1\right) e^{-(\lambda_2+\lambda_{12})\frac{x}{b}}$$
$$- \frac{\lambda_2 a}{\lambda_{12}(a+b)} e^{-\left(\lambda_2\frac{a}{b} + \lambda_{12}\frac{a+b}{b}\right)\frac{x}{a}} \left(1 - e^{-\lambda_{12}\frac{x}{a}}\right),$$
$$\lim_{\lambda_2 \to (\lambda_1+\lambda_{12})b/a} \mathbb{P}(aX_1 + bX_2 > x) = \left(\frac{(\lambda_1+\lambda_{12})bx}{a(a+b)} + 1\right) e^{-(\lambda_1+\lambda_{12})\frac{x}{a}}$$
$$- \frac{\lambda_1 b}{\lambda_{12}(a+b)} e^{-\left(\lambda_1\frac{b}{a} + \lambda_{12}\frac{a+b}{a}\right)\frac{x}{b}} \left(1 - e^{-\lambda_{12}\frac{x}{b}}\right),$$
$$\lim_{\lambda_1 \to (\lambda_2+\lambda_{12})a/b} f_{S_2}(x) = \left(\frac{(\lambda_2+\lambda_{12})^2 ax}{b^2(a+b)} + \frac{\lambda_2(\lambda_2 a + \lambda_{12}(a+b))}{b\lambda_{12}(a+b)} + \frac{\lambda_{12}}{a+b}\right) e^{-(\lambda_2+\lambda_{12})\frac{x}{b}}$$
$$- \frac{\lambda_2(\lambda_2 a + \lambda_{12}(a+b))}{\lambda_{12} b(a+b)} e^{-(\lambda_2 a + \lambda_{12}(a+b))\frac{x}{ab}}, \tag{4.3}$$
$$\lim_{\lambda_2 \to (\lambda_1+\lambda_{12})b/a} f_{S_2}(x) = \left(\frac{(\lambda_1+\lambda_{12})^2 bx}{a^2(a+b)} + \frac{\lambda_1(\lambda_1 b + \lambda_{12}(a+b))}{a\lambda_{12}(a+b)} + \frac{\lambda_{12}}{a+b}\right) e^{-(\lambda_1+\lambda_{12})\frac{x}{a}}$$
$$- \frac{\lambda_1(\lambda_1 b + \lambda_{12}(a+b))}{\lambda_{12} a(a+b)} e^{-(\lambda_1 b + \lambda_{12}(a+b))\frac{x}{ab}}. \tag{4.4}$$

Proof. Remember that,

$$\mathbb{P}(aX_1 + bX_2 > x) = \frac{\lambda_1}{\lambda_1 - (\lambda_2+\lambda_{12})\frac{a}{b}} e^{-(\lambda_2+\lambda_{12})\frac{x}{b}} \left(1 - e^{-(\lambda_1 - (\lambda_2+\lambda_{12})\frac{a}{b})\frac{x}{a+b}}\right)$$

$$
\begin{aligned}
&+ \frac{\lambda_2}{\lambda_2 - (\lambda_1+\lambda_{12})\frac{b}{a}} e^{-(\lambda_1+\lambda_{12})\frac{x}{a}} \left(1 - e^{-(\lambda_2-(\lambda_1+\lambda_{12})\frac{b}{a})\frac{x}{a+b}}\right) \\
&+ e^{-(\lambda_1+\lambda_2+\lambda_{12})\frac{x}{a+b}}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\lim_{\lambda_1 \to (\lambda_2+\lambda_{12})a/b} \mathbb{P}(aX_1 + bX_2 > x) \\
&= \frac{0}{0} - \frac{\lambda_2 a}{\lambda_{12}(a+b)} e^{-\left(\lambda_2 \frac{a}{b} + \lambda_{12}\frac{a+b}{b}\right)\frac{x}{a}} \left(1 - e^{-\lambda_{12}\frac{x}{a}}\right) + e^{-(\lambda_2+\lambda_{12})\frac{x}{b}},
\end{aligned}
$$

we solve the "$\frac{0}{0}$" problem using l'Hôpital's rule:

$$
\begin{aligned}
&\lim_{\lambda_1 \to (\lambda_2+\lambda_{12})a/b} \frac{\lambda_1}{\lambda_1 - (\lambda_2+\lambda_{12})\frac{a}{b}} e^{-(\lambda_2+\lambda_{12})\frac{x}{b}} \left(1 - e^{-\left(\lambda_1-(\lambda_2+\lambda_{12})\frac{a}{b}\right)\frac{x}{a+b}}\right) = \text{“}\frac{0}{0}\text{”} \\
&\Leftrightarrow e^{-(\lambda_2+\lambda_{12})\frac{x}{b}} \lim_{\lambda_1 \to (\lambda_2+\lambda_{12})a/b} \frac{1 - e^{-\left(\lambda_1-(\lambda_2+\lambda_{12})\frac{a}{b}\right)\frac{x}{a+b}} + \lambda_1 \frac{x}{a+b} e^{-\left(\lambda_1-(\lambda_2+\lambda_{12})\frac{a}{b}\right)\frac{x}{a+b}}}{1} \\
&= \frac{\lambda_1 x}{a+b} e^{-(\lambda_2+\lambda_{12})\frac{x}{b}} \\
&= \frac{(\lambda_2+\lambda_{12})ax}{b(a+b)} e^{-(\lambda_2+\lambda_{12})\frac{x}{b}},
\end{aligned}
$$

For the probability density function,

$$
\begin{aligned}
f_{S_2}(x) = &\frac{\lambda_1(\lambda_2+\lambda_{12})}{\lambda_1 b - (\lambda_2+\lambda_{12}a)} e^{-\left((\lambda_2+\lambda_{12})\frac{x}{b}\right)} \left(1 - e^{-\left(\lambda_1-(\lambda_2+\lambda_{12})\frac{a}{b}\right)\frac{x}{a+b}}\right) \\
&+ \frac{\lambda_2(\lambda_1+\lambda_{12})}{\lambda_2 a - (\lambda_1+\lambda_{12}b)} e^{-\left((\lambda_1+\lambda_{12})\frac{x}{a}\right)} \left(1 - e^{-\left(\lambda_2-(\lambda_1+\lambda_{12})\frac{b}{a}\right)\frac{x}{a+b}}\right) \\
&+ \frac{\lambda_{12}}{a+b} e^{-(\lambda_1+\lambda_2+\lambda_{12})\frac{x}{a+b}}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\lambda_1 \to (\lambda_2+\lambda_{12})a/b} f_{S_2}(x) = &\text{“}\frac{0}{0}\text{”} - \frac{\lambda_2(\lambda_2 a + \lambda_{12}(a+b))}{\lambda_{12} b(a+b)} e^{-(\lambda_2 a + \lambda_{12}(a+b))\frac{x}{ab}} \\
&+ \frac{\lambda_2(\lambda_2 a + \lambda_{12}(a+b))}{\lambda_{12} b(a+b)} e^{-(\lambda_2+\lambda_{12})\frac{x}{b}} + \frac{\lambda_{12}}{a+b} e^{-(\lambda_2+\lambda_{12})\frac{x}{b}},
\end{aligned}
$$

we deal with the "$\frac{0}{0}$" term applying l'Hôpital's rule:

$$
\begin{aligned}
&\lim_{\lambda_1 \to (\lambda_2+\lambda_{12})a/b} \frac{\lambda_1(\lambda_2+\lambda_{12})}{b\left(\lambda_1 - (\lambda_2+\lambda_{12})\frac{a}{b}\right)} e^{-(\lambda_2+\lambda_{12})\frac{x}{b}} \left(1 - e^{-\left(\lambda_1-(\lambda_2+\lambda_{12})\frac{a}{b}\right)\frac{x}{a+b}}\right) = \text{“}\frac{0}{0}\text{”} \\
&\Leftrightarrow (\lambda_2+\lambda_{12}) e^{-(\lambda_2+\lambda_{12})\frac{x}{b}} \lim_{\lambda_1 \to (\lambda_2+\lambda_{12})a/b} \frac{1 - e^{-\left(\lambda_1-(\lambda_2+\lambda_{12})\frac{a}{b}\right)\frac{x}{a+b}} + \frac{\lambda_1}{a+b} e^{-\left(\lambda_1-(\lambda_2+\lambda_{12})\frac{a}{b}\right)\frac{x}{a+b}}}{b}
\end{aligned}
$$

$$= \frac{(\lambda_2 + \lambda_{12})^2 ax}{b^2(a+b)} e^{-(\lambda_2+\lambda_{12})\frac{x}{b}}.$$

For the survival function and the density function, the procedure when $\lambda_2 = (\lambda_1 + \lambda_{12})b/a$ is exactly the same. □

Corollary 4.1.2 (Laplace transform)
On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let us consider (X_1, X_2) a random vector constructed as in (2.24) and let a, b be positive constants. Then the Laplace transform of $S_2 = aX_1 + bX_2$ is given by

$$\begin{aligned}
\psi_{S_2}(t) &= \mathbb{E}\left[e^{-tS_2}\right] \\
&= \frac{\lambda_1(\lambda_2+\lambda_{12})b}{(\lambda_1 b - (\lambda_2+\lambda_{12})a)(\lambda_2+\lambda_{12}+bt)} \\
&\quad + \frac{\lambda_2(\lambda_1+\lambda_{12})a}{(\lambda_2 a - (\lambda_1+\lambda_{12})b)(\lambda_1+\lambda_{12}+at)} \\
&\quad - \left(\frac{\lambda_1(\lambda_2+\lambda_{12})}{\lambda_1 b - (\lambda_2+\lambda_{12})a} + \frac{\lambda_2(\lambda_1+\lambda_{12})}{\lambda_2 a - (\lambda_1+\lambda_{12})b}\right) \times \\
&\quad \frac{a+b}{\lambda_1+\lambda_2+\lambda_{12}+(a+b)t} \\
&\quad + \frac{\lambda_{12}}{\lambda_1+\lambda_2+\lambda_{12}+(a+b)t}.
\end{aligned} \tag{4.5}$$

Proof. The Laplace transform is computed by evaluating the integral

$$\begin{aligned}
\psi_{S_2}(t) &= \int_0^\infty e^{-tx} f_{S_2}(x)dx \\
&= \frac{\lambda_1(\lambda_2+\lambda_{12})}{\lambda_1 b - (\lambda_2+\lambda_{12})a} \int_0^\infty e^{-(\lambda_2+\lambda_{12}+tb)\frac{x}{b}} dx \\
&\quad - \frac{\lambda_1(\lambda_2+\lambda_{12})}{\lambda_1 b - (\lambda_2+\lambda_{12})a} \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12}+t(a+b))\frac{x}{a+b}} dx \\
&\quad + \frac{\lambda_2(\lambda_1+\lambda_{12})}{\lambda_2 a - (\lambda_1+\lambda_{12})b} \int_0^\infty e^{-(\lambda_1+\lambda_{12}+ta)\frac{x}{a}} dx \\
&\quad - \frac{\lambda_2(\lambda_1+\lambda_{12})}{\lambda_2 a - (\lambda_1+\lambda_{12})b} \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12}+t(a+b))\frac{x}{a+b}} dx \\
&\quad + \frac{\lambda_{12}}{a+b} \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12}+t(a+b))\frac{x}{a+b}} dx.
\end{aligned}$$

□

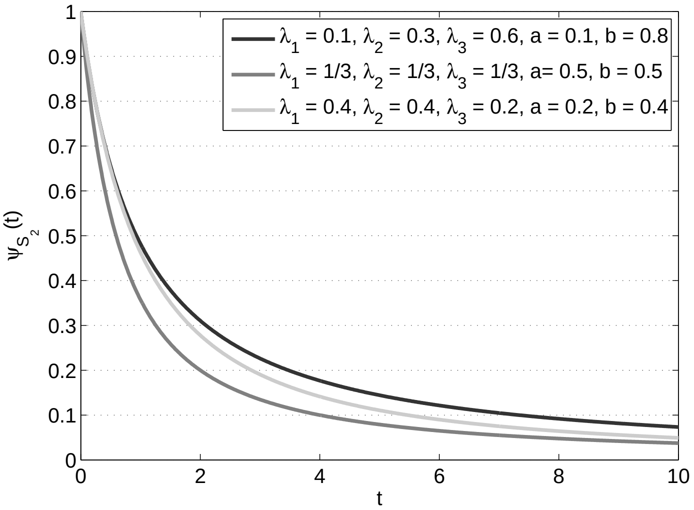
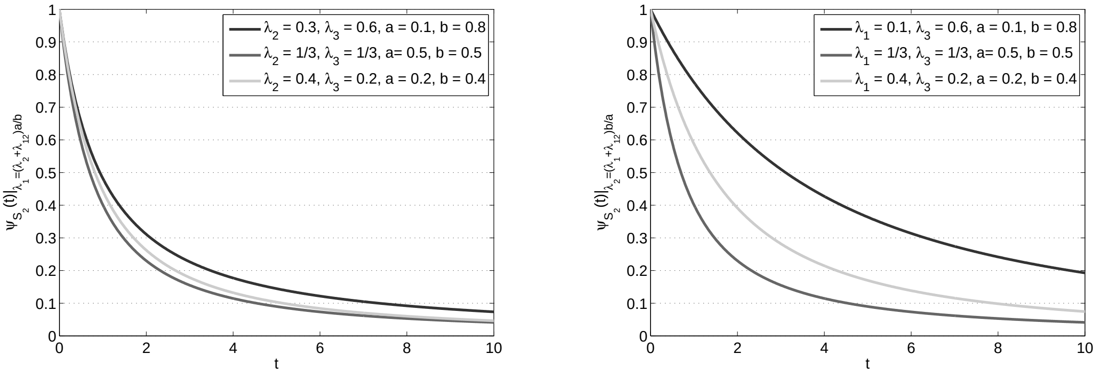


Figure 4.2: Laplace transform for different values of parameters λ_1, λ_2, λ_{12} and weights a and b.

Remark 4.1.3 (Conflictive points in the Laplace transform)
When $\lambda_1 = (\lambda_2 + \lambda_{12})a/b$ or $\lambda_2 = (\lambda_1 + \lambda_{12})b/a$, the Laplace transform has to be computed evaluating the integral $\int_0^\infty e^{-tx} f_{S_2}(x)dx$ using the expressions for $f_{S_2}(x)$ in Equations (4.3) *and* (4.4)*. This yields:*

$$\begin{aligned}
\psi_{S_2}(t)|_{\lambda_1=(\lambda_2+\lambda_{12})a/b} &= \frac{(\lambda_2+\lambda_{12})^2 a}{(a+b)(\lambda_2+\lambda_{12}+bt)^2} \\
&- \frac{\lambda_2(\lambda_2 a+\lambda_{12}(a+b))a}{\lambda_{12}(a+b)(\lambda_2 a+\lambda_{12}(a+b)+abt)} \\
&+ \frac{\lambda_2(\lambda_2 a+\lambda_{12}(a+b))}{\lambda_{12}(a+b)(\lambda_2+\lambda_{12}+bt)} \\
&+ \frac{\lambda_{12} b}{(a+b)(\lambda_2+\lambda_{12}+bt)}, \\
\psi_{S_2}(t)|_{\lambda_2=(\lambda_1+\lambda_{12})b/a} &= \frac{(\lambda_1+\lambda_{12})^2 b}{(a+b)(\lambda_1+\lambda_{12}+at)^2} \\
&- \frac{\lambda_1(\lambda_1 b+\lambda_{12}(a+b))b}{\lambda_{12}(a+b)(\lambda_1 b+\lambda_{12}(a+b)+abt)} \\
&+ \frac{\lambda_1(\lambda_1 b+\lambda_{12}(a+b))}{\lambda_{12}(a+b)(\lambda_1+\lambda_{12}+at)} \\
&+ \frac{\lambda_{12} a}{(a+b)(\lambda_1+\lambda_{12}+at)}.
\end{aligned}$$

If one aims at generalizing these results to higher dimensions, one notices that the number of involved shocks and parameters, i.e. $2^d - 1$ in dimension d, renders this problem extremely intractable already for moderate dimensions d. A subclass with fewer



(a) $\Psi(t)|_{\lambda_1=(\lambda_2+\lambda_{12})a/b}$.

(b) $\Psi(t)|_{\lambda_2=(\lambda_1+\lambda_{12})b/a}$.

Figure 4.3: Laplace transform when $\lambda_1 = (\lambda_2 + \lambda_{12})a/b$ (left) and $\lambda_2 = (\lambda_1 + \lambda_{12})b/a$ (right) for different values of parameters λ_1, λ_2, λ_{12} and weights a and b.

parameters is obtained by considering the Marshall–Olkin law with exchangeable components. This yields a parametric family with d parameters in dimension d, allowing us to derive the distribution of S_d in higher dimensions.

4.2 The exchangeable Marshall–Olkin law

The aim of this section is to derive the survival function of S_d in the exchangeable case. We introduce the subfamily of exchangeable Marshall–Olkin laws in order to deal with the problem of overparameterization. For a deeper background on exchangeable Marshall–Olkin laws see [Mai and Scherer, 2011], [Mai and Scherer, 2012] (Chapter 3, Section 3.2).

Definition 4.2.1 (Exchangeable random vector)
A random vector $(X_1, \ldots, X_d)$ is said to be exchangeable if for all permutations π on $\{1, \ldots, d\}$ it satisfies

$$\mathbb{P}(X_1 > x_1, \ldots, X_d > x_d) = \mathbb{P}(X_1 > x_{\pi(1)}, \ldots, X_d > x_{\pi(d)}), \quad x_1, \ldots, x_d \in \mathbb{R}.$$

Alternatively in the Marshall–Olkin context, if it satisfies the exchangeability condition:

$$|I| = |J| \Rightarrow \lambda_I = \lambda_J. \tag{4.6}$$

The proof that (4.6) is equivalent to $(X_1, \ldots, X_d)$ being exchangeable can be found in [Mai and Scherer, 2012], page 124. Condition (4.6) intuitively means that two shocks affecting subsets with identical cardinalities have the same intensity λ_I. Hence, in this section we denote by λ_1 the intensity of all shocks affecting precisely one component, by λ_2 all shocks affecting two components, and so on. In addition, this condition allows to identify the parameters of each E_I in a simple way as we illustrate it in Example 4 below.

Example 4 (Influence of the exchangeability condition)
Let us check the influence of the exchangeability condition in our problem comparing the non-exchangeable and the exchangeable cases when $d = 3$:

(i) Non-exchangeable case:

$$\begin{aligned}
&E_1 \sim Exp(\lambda_1), \quad E_2 \sim Exp(\lambda_2), \quad E_3 \sim Exp(\lambda_3),\\
&E_{12} \sim Exp(\lambda_{12}), \quad E_{13} \sim Exp(\lambda_{13}), \quad E_{23} \sim Exp(\lambda_{23}),\\
&E_{123} \sim Exp(\lambda_{123}),\\
&X_1 = \min\{E_1, E_{12}, E_{13}, E_{123}\} \sim Exp(\lambda_1 + \lambda_{12} + \lambda_{13} + \lambda_{123}),\\
&X_2 = \min\{E_2, E_{12}, E_{23}, E_{123}\} \sim Exp(\lambda_2 + \lambda_{12} + \lambda_{23} + \lambda_{123}),\\
&X_3 = \min\{E_3, E_{13}, E_{23}, E_{123}\} \sim Exp(\lambda_3 + \lambda_{13} + \lambda_{23} + \lambda_{123}).
\end{aligned}$$

(ii) Exchangeable case:

$$\begin{aligned}
&E_1, E_2, E_3 \sim Exp(\lambda_1), \quad E_{12}, E_{13}, E_{23} \sim Exp(\lambda_2), \quad E_{123} \sim Exp(\lambda_3),\\
&X_1 = \min\{E_1, E_{12}, E_{13}, E_{123}\} \sim Exp(\lambda_1 + 2\lambda_2 + \lambda_3),\\
&X_2 = \min\{E_2, E_{12}, E_{23}, E_{123}\} \sim Exp(\lambda_1 + 2\lambda_2 + \lambda_3),\\
&X_3 = \min\{E_3, E_{13}, E_{23}, E_{123}\} \sim Exp(\lambda_1 + 2\lambda_2 + \lambda_3).
\end{aligned}$$

Observe that in the exchangeable case instead of dealing with $2^d - 1$ parameters λ_I we just have to work with d parameters $\lambda_1, \ldots, \lambda_d$, which simplifies the process of computing the required probabilities.

Definition 4.2.2 (Exchangeable Marshall–Olkin distribution)
Let $(X_1, \ldots, X_d)$ be a random vector following the Marshall–Olkin distribution, defined as in Equation (2.24). Then the survival function of the exchangeable Marshall–Olkin law is given by

$$\bar{F}(x_1, \ldots, x_d) = \exp\left(- \sum_{k=1}^{d} x_{(d+1-k)} \sum_{i=0}^{d-k} \binom{d-k}{i} \lambda_{i+1} \right), \quad x_1, \ldots, x_d \geq 0,$$

$x_{(1)} \leq \ldots \leq x_{(d)}$ being the ordered list of $x_1, \ldots, x_d$.

In the following, we present the survival function of the sum of components of Marshall–Olkin random vectors in low dimensional exchangeable cases (2, 3, and 4-dimensional).

Lemma 4.2.1 (The sum of $d \in \{2, 3, 4\}$ lifetimes)
On the probability space $(\Omega, \mathcal{F}, \mathbb{P}) \ldots$

1. *…let (X_1, X_2) be a 2-dimensional exchangeable Marshall–Olkin random vector. Then,*

$$\mathbb{P}(X_1 + X_2 > x) \;=\; \frac{2\lambda_1 e^{-(\lambda_1+\lambda_2)x}}{\lambda_2} \left(e^{\lambda_2 \frac{x}{2}} - 1\right) + e^{-(2\lambda_1+\lambda_2)\frac{x}{2}}, \quad x \geq 0. \quad (4.7)$$

2. *... let (X_1, X_2, X_3) be a 3-dimensional exchangeable Marshall–Olkin random vector. Then,*

$$\begin{aligned}\mathbb{P}(X_1+X_2+X_3>x) = e^{-(3\lambda_1+3\lambda_2+\lambda_3)\frac{x}{3}} \\ &+\frac{6\lambda_1(2\lambda_1+3\lambda_2+\lambda_3)}{(3\lambda_2+\lambda_3)(\lambda_2+\lambda_3)}\times \\ &e^{-(2\lambda_1+3\lambda_2+\lambda_3)\frac{x}{2}}\Big(e^{\left(\frac{3\lambda_2+\lambda_3}{2}\right)\frac{x}{3}}-1\Big) \\ &+\frac{3\lambda_2(\lambda_2+\lambda_3)-6\lambda_1(\lambda_1+\lambda_2)}{(\lambda_2+\lambda_3)(3\lambda_2+2\lambda_3)}\times \\ &e^{-(\lambda_1+2\lambda_2+\lambda_3)x}\Big(e^{(3\lambda_2+2\lambda_3)\frac{x}{3}}-1\Big), \quad x\geq 0.\end{aligned} \tag{4.8}$$

3. *... let (X_1, X_2, X_3, X_4) be a 4-dimensional exchangeable Marshall–Olkin random vector. Then,*

$$\begin{aligned}\mathbb{P}(X_1+X_2+X_3+X_4>x) &= 24\cdot P_1+12\cdot P_2+12\cdot +12\cdot P_4 \\ &\quad +4\cdot P_5+4\cdot P_6+6\cdot P_7+P_8, \quad x\geq 0,\end{aligned} \tag{4.9}$$

where,

$$\begin{aligned}
P_1 &= \mathbb{P}(X_1+X_2+X_3+X_4>x \,|\, X_1<X_2<X_3<X_4) \\
&= \lambda_1(\lambda_1+\lambda_2)f_{11}\Big(\frac{32f_{10}}{f_1f_2f_4f_5}e^{-f_1\frac{x}{4}}-\frac{27f_{10}}{f_2f_3f_7f_8}e^{-f_3\frac{x}{3}}+\frac{4f_{10}}{f_4f_6f_7f_9}e^{-f_9\frac{x}{2}} \\
&\quad -\frac{1}{f_5f_6f_8}e^{-f_{10}x}\Big), \\
P_2 &= \mathbb{P}(X_1+X_2+X_3+X_4>x \,|\, X_1<X_2<X_3=X_4) \\
&= \lambda_1(\lambda_1+\lambda_2)f_6\Big(\frac{8}{f_1f_2f_4}e^{-f_1\frac{x}{4}}-\frac{9}{f_2f_3f_7}e^{-f_3\frac{x}{3}}+\frac{2}{f_4f_7f_9}e^{-f_9\frac{x}{2}}\Big), \\
P_3 &= \mathbb{P}(X_1+X_2+X_3+X_4>x \,|\, X_1<X_2=X_3<X_4) \\
&= \lambda_1(\lambda_2+\lambda_3)\Big[\frac{f_{10}}{f_2}\Big(\frac{16}{f_1f_5}e^{-f_1\frac{x}{4}}-\frac{9}{f_3f_8}e^{-f_3\frac{x}{3}}\Big)+\frac{1}{f_5f_8}e^{-f_{10}x}\Big], \\
P_4 &= \mathbb{P}(X_1+X_2+X_3+X_4>x \,|\, X_1=X_2<X_3<X_4) \\
&= \lambda_2f_{11}\Big[\Big(\frac{2f_{10}}{f_4f_6f_9}-\frac{1}{f_5f_6}+\frac{1}{f_1f_9}\Big)e^{-f_1\frac{x}{4}}-\frac{2f_{10}}{f_4f_6f_9}e^{-f_9\frac{x}{2}}+\frac{1}{f_5f_6}e^{-f_{10}x}\Big], \\
P_5 &= \mathbb{P}(X_1+X_2+X_3+X_4>x \,|\, X_1=X_2=X_3<X_4) \\
&= \lambda_3\Big(\frac{4f_{10}}{f_1f_5}e^{-f_1\frac{x}{4}}-\frac{1}{f_5}e^{-f_{10}x}\Big), \\
P_6 &= \mathbb{P}(X_1+X_2+X_3+X_4>x \,|\, X_1<X_2=X_3=X_4) \\
&= \frac{\lambda_1}{f_2}(\lambda_3+\lambda_4)\Big(\frac{4}{f_1}e^{-f_1\frac{x}{4}}-\frac{3}{f_3}e^{-f_3\frac{x}{3}}\Big),
\end{aligned}$$

$$\begin{aligned}
P_7 &= \mathbb{P}(X_1 + X_2 + X_3 + X_4 > x \,|\, X_1 = X_2 < X_3 = X_4) \\
&= \frac{\lambda_2 f_6}{f_4}\Big(\frac{2}{f_1}e^{-f_1\frac{x}{4}} - \frac{1}{f_9}e^{-f_9\frac{x}{2}}\Big), \\
P_8 &= \mathbb{P}(X_1 + X_2 + X_3 + X_4 > x \,|\, X_1 = X_2 = X_3 = X_4) = \frac{\lambda_4}{f_1}e^{-f_1\frac{x}{4}},
\end{aligned}$$

and

$$\begin{aligned}
&f_1 = 4\lambda_1 + 6\lambda_2 + 4\lambda_3 + \lambda_4, && f_5 = 6\lambda_2 + 8\lambda_3 + 3\lambda_4, && f_9 = 2\lambda_1 + 5\lambda_2 + 4\lambda_3 + \lambda_4, \\
&f_2 = 6\lambda_2 + 4\lambda_3 + \lambda_4, && f_6 = \lambda_2 + 2\lambda_3 + \lambda_4, && f_{10} = \lambda_1 + 3\lambda_2 + 3\lambda_3 + \lambda_4, \\
&f_3 = 3\lambda_1 + 6\lambda_2 + 4\lambda_3 + \lambda_4, && f_7 = 3\lambda_2 + 4\lambda_3 + \lambda_4, && f_{11} = \lambda_1 + 2\lambda_2 + \lambda_3. \\
&f_4 = 4\lambda_2 + 4\lambda_3 + \lambda_4, && f_8 = 3\lambda_2 + 5\lambda_3 + 2\lambda_4, &&
\end{aligned}$$

Proof. We prove the case $d = 2$ in details and we give a sketch of a prove when $d = 3$ and $d = 4$. The cases $d = 3$ and $d = 4$ are in Appendix B.1.

To clarify notation, $f_Y(\cdot)$ represents the probability density function of the distribution of variable Y.

$$\begin{aligned}
\mathbb{P}(X_1 + X_2 > x) &= \mathbb{P}(X_1 + X_2 > x \,|\, X_1 < X_2)\mathbb{P}(X_1 < X_2) \\
&\quad + \mathbb{P}(X_1 + X_2 > x \,|\, X_2 < X_1)\mathbb{P}(X_2 < X_1) \\
&\quad + \mathbb{P}(X_1 + X_2 > x \,|\, X_1 = X_2)\mathbb{P}(X_1 = X_2),
\end{aligned}$$

such that $E_1, E_2 \sim \text{Exp}(\lambda_1)$ and $E_{12} \sim \text{Exp}(\lambda_2)$, and note that since we are working on the exchangeable case,

$$\mathbb{P}(X_1 + X_2 > x \,|\, X_1 > X_2)\mathbb{P}(X_1 > X_2) = \mathbb{P}(X_1 + X_2 > x \,|\, X_1 < X_2)\mathbb{P}(X_1 < X_2).$$

So,

$$\begin{aligned}
\mathbb{P}(X_1 + X_2 > x) &= 2\,\mathbb{P}(X_1 + X_2 > x \,|\, X_2 < X_1)\,\mathbb{P}(X_2 < X_1) \\
&\quad + \mathbb{P}(X_1 + X_2 > x \,|\, X_1 = X_2)\,\mathbb{P}(X_1 = X_2).
\end{aligned}$$

Taking into account that, $X_2 < X_1 \Leftrightarrow E_2 < \min\{E_1, E_{12}\}$ and $X_1 = X_2 \Leftrightarrow \min\{E_1, E_2\} > E_{12}$,

$$\begin{aligned}
&\mathbb{P}(X_1 + X_2 > x) \\
&\quad = 2\,\mathbb{P}(E_2 + \min\{E_1, E_{12}\} > x \,|\, E_2 < \min\{E_1, E_{12}\})\,\mathbb{P}(E_2 < \min\{E_1, E_{12}\}) \\
&\qquad + \mathbb{P}(E_{12} + E_{12} > x \,|\, E_{12} < \min\{E_1, E_2\})\,\mathbb{P}(E_{12} < \min\{E_1, E_2\}) \\
&\quad = 2\frac{\mathbb{P}(\min\{E_1, E_{12}\} > E_2 > x - \min\{E_1, E_{12}\})}{\mathbb{P}(E_2 < \min\{E_1, E_{12}\})}\,\mathbb{P}(E_2 < \min\{E_1, E_{12}\})
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\mathbb{P}(\min\{E_1, E_2\} > E_{12} > \frac{x}{2})}{\mathbb{P}(E_{12} < \min\{E_1, E_2\})} \mathbb{P}(E_{12} < \min\{E_1, E_2\}) \\
&= 2\mathbb{E}\left[\mathbb{P}(\min\{E_1, E_{12}\} > E_2 > x - \min\{E_1, E_{12}\} \mid E_2)\right] \\
&+ \mathbb{E}\left[\mathbb{P}\left(\min\{E_1, E_2\} > E_{12} > \frac{x}{2} \mid \min\{E_1, E_2\}\right)\right].
\end{aligned}
$$

Then, from the so-called *min-stability* of the exponential distribution (Lemma B.1),

$$
\min\{E_1, E_2\} \sim \mathrm{Exp}(2\lambda_1) \quad \text{and} \quad \min\{E_1, E_{12}\} \sim \mathrm{Exp}(\lambda_1 + \lambda_2),
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{P}(\min\{E_1, E_{12}\} > E_2 > x - \min\{E_1, E_{12}\} \mid E_2)\right] \\
&= \int_0^\infty \mathbb{P}(\min\{E_1, E_{12}\} > E_2 > x - \min\{E_1, E_{12}\}) f_{E_2}(y) dy \\
&= \frac{\lambda_1}{\lambda_2} e^{-(\lambda_1+\lambda_2)x}\left(e^{\lambda_2 \frac{x}{2}} - 1\right) \\
&\quad + \frac{\lambda_1}{2\lambda_1 + \lambda_2} e^{-(2\lambda_1+\lambda_2)\frac{x}{2}},
\end{aligned}
$$

and following the same procedure

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{P}\left(\min\{E_1, E_2\} > E_{12} > \frac{x}{2} \mid \min\{E_1, E_2\}\right)\right] \\
&= \int_0^\infty \mathbb{P}\left(y > E_{12} > \frac{x}{2}\right) f_{\min\{E_1, E_2\}}(y) dy \\
&= \frac{\lambda_2}{2\lambda_1 + \lambda_2} e^{-(2\lambda_1+\lambda_2)\frac{x}{2}}.
\end{aligned}
$$

So,

$$
\begin{aligned}
&\mathbb{P}(X_1 + X_2 > x) \\
&= 2\Big(\frac{\lambda_1}{\lambda_2} e^{-(\lambda_1+\lambda_2)x}\left(e^{\lambda_2 \frac{x}{2}} - 1\right) + \frac{\lambda_1}{2\lambda_1 + \lambda_2} e^{-(2\lambda_1+\lambda_2)\frac{x}{2}}\Big) + \frac{\lambda_2}{2\lambda_1 + \lambda_2} e^{-(2\lambda_1+\lambda 2)\frac{x}{2}} \\
&= \frac{2\lambda_1 e^{-(\lambda_1+\lambda_2)x}}{\lambda_2}\left(e^{\lambda_2 \frac{x}{2}} - 1\right) + e^{-(2\lambda_1+\lambda_2)\frac{x}{2}}.
\end{aligned}
$$

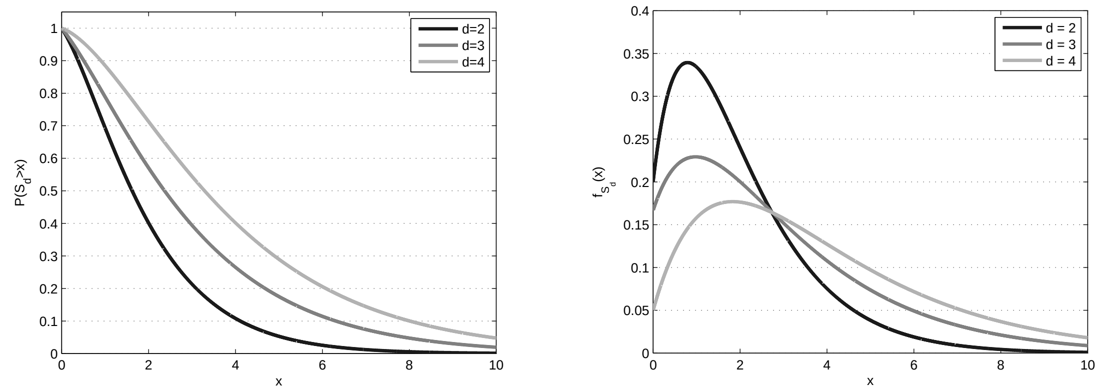
□

Remark 4.2.1 (Generalizing the results to higher dimensions)
Marshall–Olkin multivariate distributions are not absolutely continuous, i.e. there is a positive probability that several components take the same value, i.e. $\mathbb{P}(X_1 = \ldots = X_d) > 0$. *It is possible to compute the expression*

$$
\mathbb{P}(X_1 + \ldots + X_d > x, X_1 = \ldots = X_d),
$$

for all dimensions $d \in \mathbb{N}$, *by recalling Pascal's triangle (see Table 4.1).*

$$
\begin{aligned}
PM_d^d &:= \mathbb{P}(X_1 + \ldots + X_d > x, X_1 = \ldots = X_d) \qquad (4.10) \\
&= \frac{\lambda_d}{\sum_{i=0}^d \binom{d}{i}\lambda_i} e^{-\left(\sum_{i=0}^d \binom{d}{i}\lambda_i\right)\frac{x}{d}}, \quad \lambda_0 = 0.
\end{aligned}
$$



(a) Survival function. (b) Density function.

Figure 4.4: Plots of the survival and density function for S_d, $d = 2, 3, 4$, in the exchangeable case. The parameters considered are in the two-dimensional case: $\lambda_1 = 0.6, \lambda_2 = 0.4$, in the three-dimensional case: $\lambda_1 = 0.1, \lambda_2 = 0.2, \lambda_3 = 0.5$, and in the four-dimensional case: $\lambda_1 = 0.05, \lambda_2 = 0.1, \lambda_3 = 0.15, \lambda_4 = 0.2$.

						1						
					1		1					
$d = 2:$				1		**2**		1				
$d = 3:$			1		**3**		**3**		1			
$d = 4:$		1		**4**		**6**		**4**		1		
$d = 5:$	1		**5**		**10**		**10**		**5**		1	

Table 4.1: Pascal's triangle.

Observe that on the one hand from a sum of d elements in S_d we have to take into account the cases where we have k, $k \in \{0, 1, \ldots, d-1\}$, equalities in the condition of the conditional probabilities, and on the other hand we need to calculate how many times each conditional probability has to be added in the sum S_d.

The number of cases with k equalities in the condition of the conditional probabilities is given by the binomial coefficient $\binom{d-1}{k}$.

Take for example the case $d = 4$:

(i) Number of cases where $k = 0$, i.e. there is no equality in the condition: $\binom{3}{0} = 1$,

$$\mathbb{P}(X_1 + \ldots + X_4 > x, X_{i_1} < \ldots < X_{i_4}), \quad \text{where} \quad i_k \neq i_j \in \{1, 2, 3, 4\}.$$

(ii) Number of cases where there is one equality ($k = 1$) in the condition: $\binom{3}{1} = 3$,

$$\begin{aligned} &\mathbb{P}(X_1 + \ldots + X_4 > x, X_{i_1} = X_{i_2} < X_{i_3} < X_{i_4}), \\ &\mathbb{P}(X_1 + \ldots + X_4 > x, X_{i_1} < X_{i_2} = X_{i_3} < X_{i_4}), \\ &\mathbb{P}(X_1 + \ldots + X_4 > x, X_{i_1} < X_{i_2} < X_{i_3} = X_{i_4}), \end{aligned}$$

where $i_k \neq i_j \in \{1, 2, 3, 4\}$.

(iii) Number of cases where there are 2 equalities ($k = 2$) in the condition: $\binom{3}{2} = 3$,

$$\begin{aligned} &\mathbb{P}(X_1 + \ldots + X_4 > x, X_{i_1} = X_{i_2} = X_{i_3} < X_{i_4}), \\ &\mathbb{P}(X_1 + \ldots + X_4 > x, X_{i_1} < X_{i_2} = X_{i_3} = X_{i_4}), \\ &\mathbb{P}(X_1 + \ldots + X_4 > x, X_{i_1} = X_{i_2} < X_{i_3} = X_{i_4}), \end{aligned}$$

such that $i_k \neq i_j \in \{1, 2, 3, 4\}$.

And the number of times each conditional probability has to be added in the sum S_d can be computed using the *permutation of multisets*

$$PM_d^{a_1, a_2, \ldots, a_{k-1}, a_k} := \frac{d!}{a_1! \cdot a_2! \cdot \ldots \cdot a_{k-1}! \cdot a_k!}, \tag{4.11}$$

where in our case $a_1, \ldots, a_k$ represent the number of elements which are equal and how they are located in each condition. Note that $\sum_{i=1}^{k} a_i = d$. Let us illustrate this relation with the example of $d = 4$:

$$\begin{aligned} \mathbb{P}(X_1 + \ldots + X_4 > x) \;=\; & PM_4^{1,1,1,1} \cdot \mathbb{P}(X_1 + \ldots + X_4, \underbrace{X_1}_{1} < \underbrace{X_2}_{1} < \underbrace{X_3}_{1} < \underbrace{X_4}_{1}) \\ & + PM_4^{2,1,1} \cdot \mathbb{P}(X_1 + \ldots + X_4 > x, \underbrace{X_1 = X_2}_{2} < \underbrace{X_3}_{1} < \underbrace{X_4}_{1}) \\ & + PM_4^{1,2,1} \cdot \mathbb{P}(X_1 + \ldots + X_4 > x, \underbrace{X_1}_{1} < \underbrace{X_2 = X_3}_{2} < \underbrace{X_4}_{1}) \\ & + PM_4^{1,1,2} \cdot \mathbb{P}(X_1 + \ldots + X_4 > x, \underbrace{X_1}_{1} < \underbrace{X_2}_{1} < \underbrace{X_3 = X_4}_{2}) \\ & + PM_4^{2,2} \cdot \mathbb{P}(X_1 + \ldots + X_4 > x, \underbrace{X_1 = X_2}_{2} < \underbrace{X_3 = X_4}_{2}) \\ & + PM_4^{3,1} \cdot \mathbb{P}(X_1 + \ldots + X_4 > x, \underbrace{X_1 = X_2 = X_3}_{3} < \underbrace{X_4}_{1}) \\ & + PM_4^{1,3} \cdot \mathbb{P}(X_1 + \ldots + X_4 > x, \underbrace{X_1}_{1} < \underbrace{X_2 = X_3 = X_4}_{3}) \\ & + PM_4^{4}, \end{aligned}$$

the expression for PM_4^4 is given in Equation (4.10).

Example 5 (Illustrating the effect of different levels of dependence)
We analyse the effect of different levels of dependence on the survival and density function of S_d. We show this influence graphically in Figure 4.5.

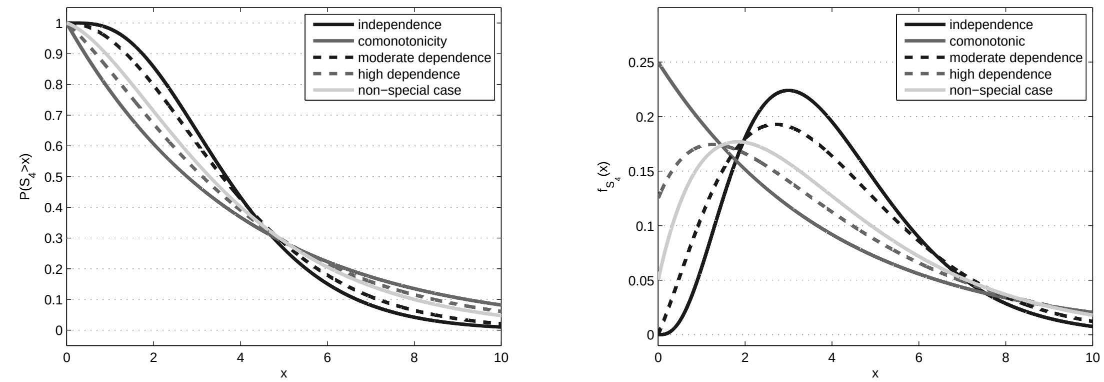
a) *Independence case: Shocks arriving to just one element are the only ones present in the system, i.e. $\lambda_1 > 0$ and $\lambda_2 = \lambda_3 = \lambda_4 = 0$. In this case the probability distribution of S_d follows the Erlang distribution with rate λ_1 and shape parameter $k = 4$.*

b) *Comonotonic case: The shock arriving to all components at the same time is the only one influencing the system, i.e. $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\lambda_4 > 0$, and the distribution of S_d is exponential with mean $4/\lambda_4$.*

c) *Moderate dependence case: In this case the shocks influencing fewer components jointly have the strongest influence, i.e. $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$.*

d) *High dependence case: Shocks arriving to most components jointly have the strongest influence, i.e. $\lambda_4 > \lambda_3 > \lambda_2 > \lambda_1 > 0$.*

e) *Non-special case: $\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$.*



(a) Survival function.

(b) Density function.

Figure 4.5: $\mathbb{P}(S_4 > x)$ and $f_{S_4}(x)$ for different assumptions concerning the dependence: a) independence, b) comonotonicity, c) moderate dependence (we consider $\lambda_4 = 0$), d) high dependence (we consider $\lambda_1 = 0$), e) non-special case. In all examples, the marginal laws are considered to be the same, X_i unit exponential random variables, $i = 1, \ldots, 4$.

One can observe in Figure 4.5 that the intersection of the survival functions is around the expected value $\mathbb{E}[S_4] = 4$. When the dependence between the components of the system is strong, the probability of the system to collapse before this intersection is lower than in the cases where the dependence is weak, but once the system survives till this intersection point, in cases with strong dependence the probability that the system will last alive longer is higher than in cases where the dependence is weak. This interpretation can be also seen *in the densities. In weak dependence cases, the mass of the probability is concentrated around the expected value, which is translated into having a strong depth in the slope of the survival function.*

While working with the exchangeable subfamily of the Marshall–Olkin distribution we simplify the number of parameters we needed to deal with in Section 4.1. However, due to the large number of cases that one has to take into account as d increases, i.e. 2^{d-1}, the analytical derivation of $\mathbb{P}(X_1 + \ldots + X_d > x)$ becomes cumbersome for $d \gg 2$.

4.3 The extendible Marshall–Olkin law

Until now we have investigated the distributional behaviour of S_d in low dimensional cases $d \in \{2, 3, 4\}$. But, because of the large number of parameters involved in the implementation of the results (Section 4.1) and the extensive number of cases that has to be taken into consideration while d increases (Section 4.2), the generalization of $\mathbb{P}(X_1 + \ldots + X_d > x)$ to all dimensions $d \in \mathbb{N}$ becomes challenging.

In this section we aim at analysing how the probability distribution of S_d/d behaves in the limit when the system grows in dimension, i.e. for $d \to \infty$. For this purpose we work with the extendible subfamily of the Marshall–Olkin law, since we must be able to extend the dimension of the vector $(X_1, \ldots, X_d)$ without destroying its distributional structure. This subfamily of the Marshall–Olkin distribution is based on a stochastic model with conditionally independent and identically distributed components.

Definition 4.3.1 (Extendible random vector)
A random vector $(X_1, \ldots, X_d)$ is called extendible if there exists an infinite exchangeable sequence $\{\tilde{X}_k\}_{k\in\mathbb{N}}$ such that

$$(X_1, \ldots, X_d) \overset{\mathcal{L}}{=} (\tilde{X}_1, \ldots, \tilde{X}_d).$$

De Finetti's Theorem (see [Finetti, 1937]) states that this is equivalent to $(\tilde{X}_1, \ldots, \tilde{X}_d)$ being conditionally i.i.d.

Definition 4.3.2 (Lévy-frailty canonical construction)
Let $\{\Lambda_t, t \geq 0\}$ be a Lévy subordinator. For extendible Marshall–Olkin laws there is a canonical construction based on these processes:

$$X_k = \inf\{t > 0 : \Lambda_t \geq \varepsilon_k\}, \quad k = 1, \ldots, d. \tag{4.12}$$

Component X_k is the first-passage time of Λ_t across ε_k and $\{\varepsilon_k\}_{k\in\mathbb{N}}$ is an i.i.d. sequence of unit exponential random variables. This construction is called the Lévy-frailty construction and it defines the subclass of extendible Marshall–Olkin distributions. Further information on these distributions can be found in [Mai and Scherer, 2009], [Mai and Scherer, 2012] (Chapter 3).

Definition 4.3.3 (Extendible Marshall–Olkin distribution)
Let $\{\Lambda_t, t \geq 0\}$ be a Lévy subordinator, $\{\Psi(k)\}_{k\in\mathbb{N}}$ a sequence derived from evaluating the

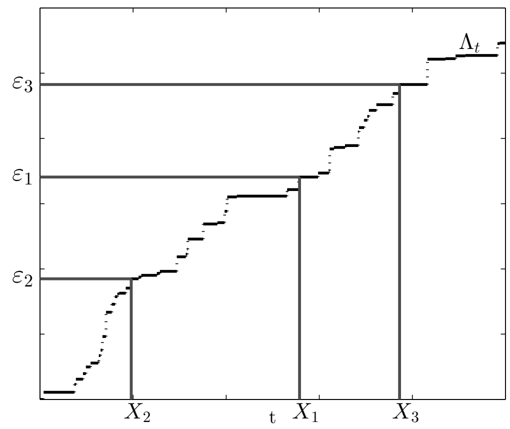


Figure 4.6: Illustration of the Lévy-frailty canonical construction for a compound Poisson process with Exp(1)-distributed jump sizes and jump-intensity $\beta = 8$ in dimension $d = 3$. ε_1, ε_2, and ε_3 follow the exponential distribution with parameter $\lambda = 1$.

Laplace exponent Ψ of Λ_t at the natural numbers, and $(X_1, \ldots, X_d)$ an extendible vector. We define the survival function of $(X_1, \ldots, X_d)$ as:

$$\mathbb{P}(X_1 > x_1, \ldots, X_d > x_d) = \exp\left(- \sum_{k=1}^{d} x_{(d-k+1)} \left(\Psi(k) - \Psi(k-1) \right) \right),$$

where $x_{(1)} \leq \ldots \leq x_{(d)}$ is the ordered list of $x_1, \ldots, x_d \geq 0$.

It is shown in [Mai, 2010] that $(X_1, \ldots, X_d)$ follows the Marshall–Olkin distribution with parameters

$$\lambda_k = \sum_{i=0}^{k-1} (-1)^i \left(\Psi(d-k+i+1) - \Psi(d-k+i) \right), \quad k = 1, \ldots, d.$$

Once we construct the vector of first-passage times of a *Lévy-subordinator*, $(X_1, \ldots, X_d)$, we can prove that when $d \to \infty$, S_d/d has the same distribution as the *exponential functional* of a Lévy-subordinator, $I_\infty = \int_0^\infty e^{-\Lambda_s} ds$, defined in the following definition.

Definition 4.3.4 (Exponential functional of a Lévy subordinator)
Let $\{\Lambda_t, t \geq 0\}$ be a Lévy subordinator. Then the exponential functional of a Lévy process, Λ_t, is defined as

$$I_t = \int_0^t e^{-\Lambda_s} ds,$$

and at its terminal value $t = \infty$ it is given by

$$I_\infty = \int_0^\infty e^{-\Lambda_s} ds.$$

For further information on exponential functionals of Lévy subordinators we refer the reader to [Bertoin and Yor, 2005] and [Carmona et al., 2001].

Lemma 4.3.1 (The sum of $d \nearrow \infty$ lifetimes)
Let $(X_1, \ldots, X_d)$ be a random vector following the extendible Marshall–Olkin distribution. Then,

$$\lim_{d \nearrow \infty} \frac{S_d}{d} \xrightarrow{\mathcal{L}} I_\infty. \tag{4.13}$$

Proof. Define from Equation (4.12) $X_k := \inf\{t > 0 : \Lambda_t \geq \varepsilon_k\}$. If we prove that $\lim_{d \nearrow \infty} \frac{S_d}{d}$ converges to I_∞ $\mathbb{P}$-almost surely, since $\mathbb{P}$-almost sure convergence implies convergence in distribution, $\lim_{d \nearrow \infty} \frac{S_d}{d}$ converges in distribution to I_∞.

$$\lim_{d \nearrow \infty} \frac{1}{d} \sum_{k=1}^{d} X_k \xrightarrow{\text{a.s.}} I_\infty = \int_0^\infty e^{-\Lambda_s} ds \Leftrightarrow \mathbb{P}\left(\left| \lim_{d \nearrow \infty} \frac{1}{d} \sum_{k=1}^{d} X_k - \int_0^\infty e^{-\Lambda_s} ds \right| = 0 \right) = 1.$$

So,

$$\mathbb{P}\left(\left| \lim_{d \nearrow \infty} \frac{1}{d} \sum_{k=1}^{d} X_k - \int_0^\infty e^{-\Lambda_s} ds \right| = 0 \right) = \mathbb{E}\left[\mathbb{P}\left(\left| \underbrace{\lim_{d \nearrow \infty} \frac{1}{d} \sum_{k=1}^{d} X_k}_{\mathbb{E}[X_1|\Lambda] \quad \text{a.s.}} - \int_0^\infty e^{-\Lambda_s} ds \right| = 0 \,\Big|\, \Lambda \right) \right]$$

Observe that,

$$\begin{aligned}
\mathbb{E}[X_1 \,|\, \Lambda] = \int_0^\infty x d\mathbb{P}(X_1 \leq x \,|\, \Lambda) &= \int_0^\infty x d\mathbb{P}(\varepsilon_1 \leq \Lambda_x \,|\, \Lambda) = \int_0^\infty x d(1 - e^{-\Lambda_x}) = \int_0^\infty -x d(e^{-\Lambda_x}) \\
&\text{applying integration by parts} \\
&= \Big[-x e^{-\Lambda_x} \Big]_{x=0}^{x=\infty} + \int_0^\infty e^{-\Lambda_x} dx \\
&= 0 + \int_0^\infty e^{-\Lambda_x} dx.
\end{aligned}$$

□

Example 6 (The limit of S_d/d in a Poisson-frailty model)
In this example we aim at numerically illustrating the result presented in Lemma 4.3.1 above. For this purpose we consider the standard Poisson process $N_t = \{N_t\}_{t \geq 0}$ with intensity $\beta > 0$, which is a Lévy subordinator (see Definition 2.2.2 in Chapter 2). We want to analyse that $\mathbb{P}(S_d/d > x)$, $d = 2, 3, 4$, $x \geq 0$ converges to the survival function of the exponential functional of the Poisson process,

$$I_\infty = \int_0^\infty e^{-N_t} dt, \tag{4.14}$$

when $d \to \infty$.

[Bertoin et al., 2004] compute the Laplace transform of the exponential functional of the standard Poisson process

$$\mathbb{E}[e^{\tilde{\lambda} I_\infty}] = \left(\prod_{j=0}^{\infty} (1 - \tilde{\lambda} e^{-j}) \right)^{-1}, \quad \tilde{\lambda} < 1.$$

Using the Gaver–Stehfest Laplace inversion technique (see [Kou and Wang, 2003], [Gaver, 1966], [Stehfest, 1970]), we numerically compute the survival function of the exponential functional of I_∞:

$$\mathbb{P}(I_\infty^{(q)} > x), \quad x \in \mathbb{R}^+.$$

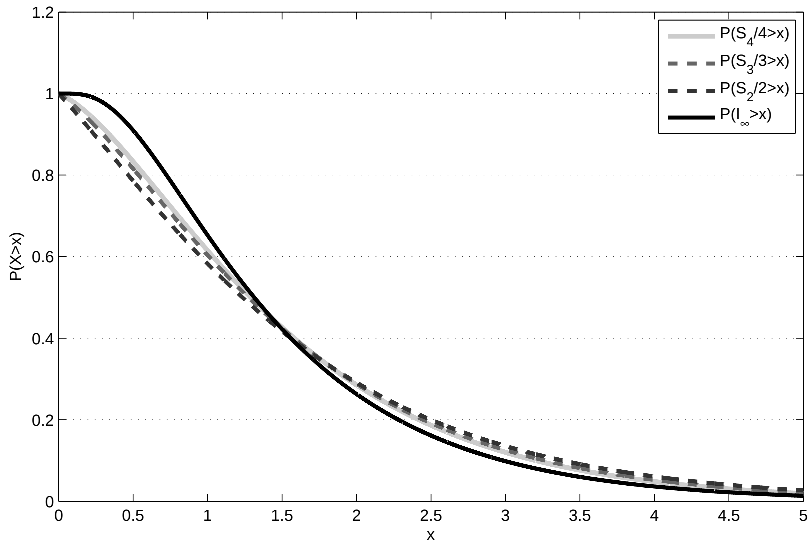


Figure 4.7: Plot of $\mathbb{P}(S_d/d > x)$, $d = 2, 3, 4$ together with $\mathbb{P}(I_\infty > x)$, $x \geq 0$, where $\beta = 1$.

With this example we visualize how $\mathbb{P}\left(S_d/d > x\right)$, $d \in \mathbb{N}$, converges to $\mathbb{P}\left(I_\infty > x\right)$ when $d \to \infty$. In this case the components of the system strongly depend on each other, i.e. $0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$.

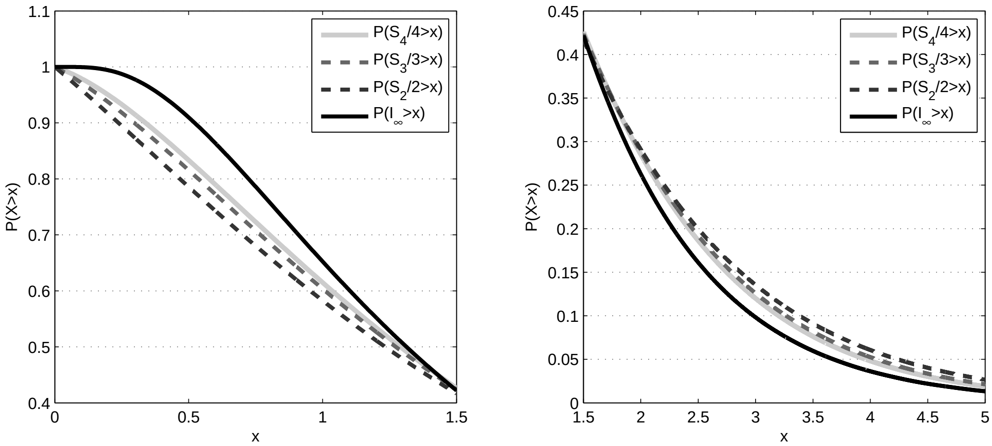


Figure 4.8: Zoom into Figure 4.7.

5

Simulating Lévy-frailty copulas built from an α-stable Lévy subordinator

> "I really doubt that it's mathematically possible for me to be in two dreams at one time."
> Whatever Works (2009)

Lévy-frailty copulas, which belong to the family of survival copulas of the Marshall–Olkin law (see Figure 2.6), were originally introduced in [Mai and Scherer, 2009]. These copulas are based on a stochastic model with conditionally independent and identically distributed components (CIID). Lévy-frailty copulas are defined in terms of the Laplace exponent of Lévy subordinators. They allow to construct dependence structures over initially independent random vectors using first-passage times of Lévy subordinators. We focus on Lévy-frailty copulas built from an α-stable Lévy subordinator. Since α-stable subordinators possess a convenient functional form of the Laplace exponent, they are attractive in different applications (see e.g. [Applebaum, 2009]). Different simulation techniques to sample Lévy-frailty copulas built from α-stable subordinators are investigated in this chapter. We measure the efficiency of these computational methods to sample these copulas in terms of computational speed. We compare a method based on the recursive formula for general exchangeable Marshall–Olkin copulas, the simulation of the involved α-stable subordinator on a fine grid, and the simulation of the approximation of the α-stable subordinator by a compound Poisson process. For this purpose we consider different values of the dimension of the copulas and index α of the subordinator.

5.1 α-stable Lévy subordinators

In this section α-stable Lévy subordinators are introduced. As it was previously mention in Chapter 2 (Section 2.3), Lévy subordinators are a special case of Lévy processes. We first introduce basic concepts about Lévy processes. Standard references on these processes are provided in, e.g., [Bertoin, 1998], [Sato, 1999], [Applebaum, 2009]. Concerning the simulation of Lévy processes and applications in finance we refer the reader to [Schoutens, 2003] and [Cont and Tankov, 2004].

Definition 5.1.1 (Jump process)
The jump process of a Lévy process $\{X_t\}_{t\geq 0}$ *is given by*

$$\{\Delta X_t := X_t - X_{t^-}\}_{t\geq 0}.$$

Definition 5.1.2 (Random jump measure)
On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *let* $\{X_t\}_{t\geq 0}$ *be a Lévy process. Then the number of jumps of* X_t *with jump size in* $A \subset \mathbb{R} \setminus \{0\}$ *in a time interval* $[0,t]$ *is measured by the so called random jump measure:*

$$m^X(\omega; [0,t] \times A) := |(s, \Delta X_s(\omega)) \in [0,t] \times A|, \quad \forall \omega \in \Omega,$$

such that, $\Delta X_s(\omega) = X_s(\omega) - X_{s^-}(\omega)$.

Note that, for a given time interval $[0,t]$ and jump size in $A \subset \mathbb{R}$, $m^X(\omega, \cdot)$ is a random measure. In case of Lévy subordinators the definition remains the same besides of $A \subset (0, \infty]$ due to the non-decreasingness of the subordinator.

Taking the average of the jump measure in a unit-time interval, it is possible to calculate the average number of jumps of X_t in this interval with jump size in $B \in \mathcal{B}(\mathbb{R})$. This measure is called Lévy measure (see [Cont and Tankov, 2004], Chapter 3).

Definition 5.1.3 (Lévy measure)
On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *let* $\{X_t\}_{t\geq 0}$ *be a Lévy process. The Lévy measure is given by,*

$$\nu(B) := \mathbb{E}\left[|\{s \in (0,1] : \Delta X_t \neq 0, \Delta X_t \in B\}|\right], \quad B \in \mathcal{B}(\mathbb{R} \setminus \{0\}).$$

The Lévy measure for a Lévy subordinator, $\{\Lambda_t\}_{t\geq 0}$, is given by:

$$\nu(B) := \mathbb{E}\left[|\{s \in (0,1] : \Delta \Lambda_t \neq 0, \Delta \Lambda_t \in B\}|\right], \quad B \in \mathcal{B}((0,\infty]).$$

As previously mentioned in Chapter 2 (Section 2.3) the distributional law of Lévy processes is characterized by their characteristic function. In case of Lévy subordinators, due to their non-negativity, this characterization is given by the Laplace transform (Definition 2.3.2). In the Lévy–Khintchine formula (Theorem 2.3.1) for Lévy subordinators we find that the Lévy measure also satisfies the following properties:

$$\int_{(0,1]} x\, \nu(dx) < \infty, \tag{5.1}$$

$$\nu((\epsilon, \infty]) < \infty, \quad \forall \epsilon > 0. \tag{5.2}$$

Let us now describe the structure of a Lévy process in terms of a continuous part and a part that can be expressed as a compensated sum of jumps. This representation is given by the Lévy–Itô decomposition.

Theorem 5.1.1 (Lévy–Itô decomposition)
Let $\{X_t\}_{t\geq 0}$ be a Lévy process with jump process $\{\Delta X_t\}_{t\geq 0}$. Then, the following holds:

$$\begin{aligned} X_t = \mu\, t + \sigma\, W_t + \int_0^t \int_{|x|\geq 1} x\, m^X(ds, dx) \\ + \lim_{\epsilon\to 0} \left(\int_0^t \int_{\epsilon<|x|<1} x\, m^X(ds, dx) - t \int_{\epsilon<|x|<1} x\, \nu(dx) \right), \end{aligned} \tag{5.3}$$

where μ and $\sigma > 0$ are real numbers and W_t is the standard Brownian motion.

Note that the third term in (5.3) represents jumps with absolute jump size bigger than one, *big jumps.* However, in the last term we consider jumps with absolute jumps size smaller than one, which are named *small jumps.*

For the proof of this theorem we refer the reader to [Sato, 1999] (Chapter 4) or [Kyprianou, 2006] (Chapter 2).

Remark 5.1.1 (Convergence of *small jumps*)
The last term in the above Equation (5.3) *is the so called compensated sum of jumps. Note that when $\epsilon \to 0$ it is not possible to ensure that*

$$\int_0^t \int_{\epsilon<|x|<1} x\, m^X(ds, dx)$$

converges. However, the difference

$$\int_0^t \int_{\epsilon<|x|<1} x\, m^X(ds, dx) - t \int_{\epsilon<|x|<1} x\, \nu(dx)$$

uniformly converges to

$$\int_0^t \int_{0<|x|<1} x\, m^X(ds, dx) - t \int_{0<|x|<1} x\, \nu(dx)$$

when $\epsilon \to 0$.

This convergence is proved in [Sato, 1999], Lemma 20.6.

Definition 5.1.4 (Lévy process with finite variation)
Let $\{X_t\}_{t\geq 0}$ be a Lévy process with Lévy measure ν. $\{X_t\}_{t\geq 0}$ is said to be a process with finite variation if it fulfills $\int_{(0,1]} x\, \nu(dx) < \infty$.

In case of finite variation the small jumps do not have to be compensated. Since Lévy subordinators, $\{\Lambda_t\}_{t\geq 0}$, have finite variation, the Lévy–Itô decomposition for $\{\Lambda_t\}_{t\geq 0}$ with jump process $\{\Delta\Lambda_t\}_{t\geq 0}$, and jump sizes $\Delta\Lambda_t = x \in [0,\infty)$ is represented as

$$\Lambda_t = \mu\, t + \int_0^t \int_{x>0} x\, m^{\Lambda}(ds, dx),$$

such that μ is a non-negative real number (see [Sato, 1999], Theorem 19.3).

In conclusion Lévy subordinators can be written as a sum of a deterministic drift process, $\{\mu\, t\}_{t\geq 0}$, small and big jumps. We introduce in the following the concept of *finite and infinite activity.*

Definition 5.1.5 (Finite/infinite activity of Lévy subordinators)
Let $\{\Lambda_t\}_{t\geq 0}$ be a Lévy subordinator with Lévy measure ν. Λ_t is said to have a finite or infinity activity if,

$$\nu((0,\epsilon)) < \infty \quad or \quad \nu((0,\epsilon)) = \infty, \quad \epsilon > 0.$$

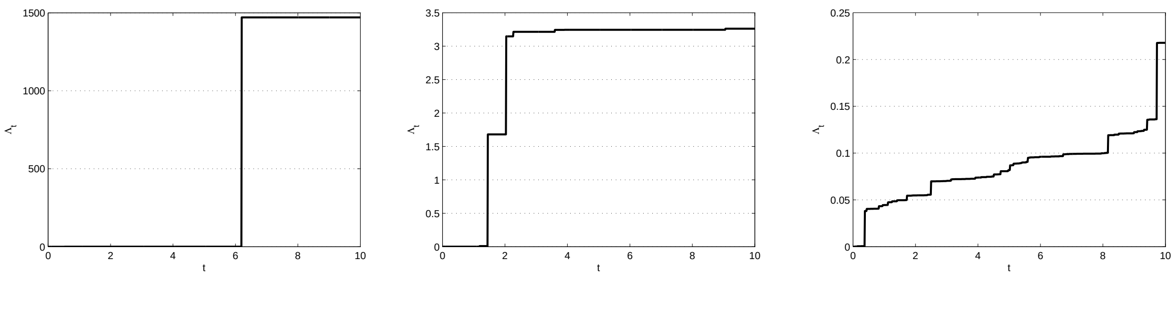
Providing an interpretation, infinite activity means that in a finite time interval the subordinator has almost surely infinite many jumps.

The α-stable Lévy subordinator belong to the set of Lévy subordinators with infinite activity.

Definition 5.1.6 (α-stable Lévy subordinator)
A Lévy subordinator Λ is said to be α-stable, $\alpha \in (0,1)$, if it has zero drift, $\mu = 0$, and Lévy measure, which is absolutely continuous with respect to the Lebesgue measure, given by

$$\nu(dx) = \frac{\alpha}{\Gamma(1-\alpha)} x^{-(1+\alpha)} \mathbb{1}_{\{x\geq 0\}} dx. \tag{5.4}$$

Figure 5.1 shows paths of the α-stable subordinator for different values of α.



(a) 0.1-stable Lévy subordinator.

(b) 0.25-stable Lévy subordinator.

(c) 0.5-stable Lévy subordinator.

Figure 5.1: Simulated paths of an α-stable subordinator with $\alpha = 0.1$, $\alpha = 0.25$, and $\alpha = 0.5$, respectively.

The α-stable subordinator has the advantage that its Laplace exponent has a convenient analytical form.

Lemma 5.1.1 (Laplace exponent of Λ)
Let $\{\Lambda_t\}_{t\geq 0}$ be the α-stable subordinator. Then the Laplace exponent of Λ is given by

$$\Psi(x) = x^\alpha, \quad x \geq 0, \quad \alpha \in (0,1).$$

Proof. From the Lévy–Khintchine formula, taking into account that $\mu = 0$ and the expression for the Lévy measure of the α-stable subordinator (Equation (5.4)), we get

$$\Psi(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-xt}) \frac{dt}{t^{1+\alpha}}.$$

Now applying the so called *Method of Sato* (see [Sato, 1999], p. 46) we write the repeated integral as a double integral and change the order of integration,

$$= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-xt})\, t^{-1-\alpha}\, dt$$

substitute $1 - e^{-xt} = \int_0^t x e^{-xy} dy$,

$$= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \left(\int_0^t x\, e^{-xy}\, dy \right) t^{-1-\alpha}\, dt$$

swap the limits of integration using Fubini's Theorem,

$$\begin{aligned} &= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \left(\int_y^\infty t^{-(1+\alpha)}\, dt \right) x\, e^{-xy}\, dy \\ &= \frac{\alpha}{\Gamma(1-\alpha)} \frac{x}{\alpha} \int_0^\infty e^{-xy} y^{-\alpha}\, dy \end{aligned}$$

apply the change of variable: $xy = u$, $dy = du/x$,

$$\begin{aligned} &= \frac{x^\alpha}{\Gamma(1-\alpha)} \int_0^\infty u^{-\alpha}\, e^{-u}\, du \\ &= \frac{x^\alpha}{\Gamma(1-\alpha)} \Gamma(1-\alpha) \\ &= x^\alpha, \end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function.

This proof is also provided in [Applebaum, 2009] (Chapter 1, Appendix). □

5.2 Lévy-frailty copulas

We introduce in the sequel Lévy-frailty copulas. [Mai and Scherer, 2009] display different examples of these copulas while [Mai, 2010] (Chapter 6) provides applications of these copulas in portfolio-credit risk.

Definition 5.2.1 (Lévy-frailty copula)
Let Ψ be a Laplace exponent of the Lévy-subordinator Λ satisfying $\Psi(1) = 1$. The Lévy-frailty copula is defined as

$$C_\Psi(u_1, \ldots, u_d) := \prod_{i=1}^{d} u_{(i)}^{\Psi(i)-\Psi(i-1)},$$

where $u_{(1)} \leq \ldots \leq u_{(d)}$ is the ordered list of $u_1, \ldots, u_d \in [0, 1]$.

Since the extendible subfamily of Marshall–Olkin distribution is part of the exchangeable one (see Figure 2.6), they possess a close link with exchangeable Marshall–Olkin copulas (previously introduced in Chapter 4, Definition 4.2.1) which we explain in the sequel.

Definition 5.2.2 (d-monotone sequence)
Let $\{\theta_1, \theta_2, \ldots, \theta_d\}$ be a finite sequence of real numbers and Δ^j, $j \in \mathbb{N}$, the difference operator, such that $\Delta^0\theta_k = \theta_k$, $\Delta^1\theta_k = \theta_{k+1} - \theta_k$, $\Delta^2\theta_k = \Delta\theta_{k+1} - \Delta\theta_k$, etc. Then $\{\theta_1, \theta_2, \ldots, \theta_d\}$ is said to be d-monotone if,

$$(-1)^j \Delta^j \theta_k \geq 0, \quad k = 1, \ldots, d, \quad j = 1, \ldots, d - k.$$

Exchangeable Marshall–Olkin copulas can be defined in terms of d-monotone sequences (see [Mai and Scherer, 2011], [Mai and Scherer, 2012], Lemma 3.7).

Definition 5.2.3 (Exchangeable Marshall–Olkin copula)
Exchangeable Marshall–Olkin copulas can be reparameterized using a d-monotone sequence $\{\theta_k\}_{k\in\mathbb{N}}$:

$$eMO = \left\{ \prod_{i=1}^{d} u_{(i)}^{\theta_{i-1}} \,|(\theta_0, \ldots, \theta_{d-1}) \text{ d-monotone}, \theta_0 = 1 \right\},$$

where $u_{(1)} \leq u_{(2)} \leq \ldots \leq u_{(d)}$ is the ordered list of $u_1, \ldots, u_d \in [0, 1]$.

If it is possible to extend a finite d-monotone sequence to an infinite sequence $\{\theta_d\}_{d\in\mathbb{N}}$, then we get a *completely monotone* sequence.

Definition 5.2.4 (Completely monotone sequence)
$\{\theta_d\}_{d\in\mathbb{N}}$ is a completely monotone sequence if,

$$(-1)^j \Delta^j \theta_d \geq 0, \quad d, j \in \mathbb{N}.$$

When a d-monotone sequence can be extended to a completely monotone sequence, then the extendible subfamily of the Marshall–Olkin distribution can be derived. As we mentioned in Chapter 4, within this subfamily we can extend the dimension of a random vector without modifying its distributional structure.

Lévy-frailty copulas built the dependence over initially independent and unit exponentially distributed random variables. Applying the Lévy-frailty canonical construction (see Definition 4.3.2), we can sample first-exit times of a Lévy subordinator:

$$X_k := \inf\{t > 0 : \Lambda_t \geq \varepsilon_k\}, \quad k \in \{1, \dots, d\}, \tag{5.5}$$

being $\{\varepsilon_k\}_{k\in\mathbb{N}}$ a sequence of i.i.d. unit exponential random variables and Λ an independent non-zero Lévy subordinator.

The survival copula for the random vector $(X_1, \dots, X_d)$, built using Equation (5.5) above, is precisely the exchangeable Marshall–Olkin copula which is parameterized using the d-monotone sequence (Definition 5.2.3). For the interested readers in the proof of this statement we refer to Theorem 3.2 in [Mai and Scherer, 2012].

Theorem 5.2.1 (Hausdorff, 1921)
On the probability space $(\Omega, \mathcal{F}, \mathbb{F})$ let $X : \Omega \to [0, 1]$ be a random variable and $\{\theta_k\}_{k\in\mathbb{N}_0}$ a sequence of real numbers. Then $\{\theta_k\}_{k\in\mathbb{N}_0}$ is completely monotone if and only if $\theta_k = \mathbb{E}[X^k]$, for all $k \in \mathbb{N}_0$.

Proof. This theorem was originally stated in [Hausdorff, 1921]. The sketch of a proof is given in [Mai and Scherer, 2012], p. 141. For a full proof we refer the reader to [Feller, 1966], p. 225. □

Theorem 5.2.2 (A bijection)
On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let $\mu > 0$ be the drift, ν the Lévy measure, and Ψ the Laplace exponent of the Lévy subordinator Λ, such that $\Psi(1) = 1$. Then there exists the following bijection between the set of all probability measures on $[0, 1]$ and (μ, ν):

(i) $\mu := \mathbb{P}(X = 1), \qquad \nu(A) := \mathbb{E}\left[\frac{1}{1-X}\mathbb{1}_{\{-ln(X)\in A\}}\right], \quad A \in \mathcal{B}((0, \infty]).$

(ii) $\mathbb{P}(x \in B) := \mu\,\mathbb{1}_{\{1\in A\}} + \int_{\{-ln(a)\,|\,a\in A-\{1\}\}}(1 - e^{-t})\nu(dt).$

Proof. We refer the reader to [Mai and Scherer, 2012], p. 144. □

On the one side using Theorem 5.2.1 it is possible to get a *one-to-one* correspondence between measures on $[0, 1]$ and completely monotone sequences. On the other side in Theorem 5.2.2 a *one-to-one* correspondence between measures on $[0, 1]$ and characteristics of a Lévy subordinator, Λ, is set. In addition, the following relation between measures on [0,1] and Laplace exponent of Lévy subordinators is true (see [Mai and Scherer, 2012], p. 145):

$$\Psi(k + 1) - \Psi(k)$$

$$
\begin{aligned}
&= \mu + \int_{(0,\infty]} e^{-kt}\,(1-e^{-t})\nu(dt) \\
&= \mathbb{P}(X=1) + \int_{(0,\infty]} e^{-kt}\,(1-e^{-t})\,(1-e^{-t})^{-1}\,\mathbb{P}(-\log X \in dt) \\
&= \mathbb{E}\left[X^k\,\mathbb{1}_{\{X=1\}}\right] + \mathbb{E}\left[X^k\,\mathbb{1}_{\{X\in[0,1)\}}\right] \\
&= \mathbb{E}[X^k] \\
&= \theta_k, \quad k \in \mathbb{N}_0.
\end{aligned}
$$

So, for each completely monotone sequence $\{\theta_k\}_{k\in\mathbb{N}_0}$ with $\theta_0 = 1$ there exists a unique Lévy subordinator satisfying $\Psi(1) = 1$.

In conclusion, the link between Lévy-frailty copulas and exchangeable Marshall–Olkin copulas is given by:

$$
\prod_{i=1}^{d} u_{(i)}^{\Psi(i)-\Psi(i-1)} \leftrightarrow \prod_{i=1}^{d} u_{(i)}^{\theta_{i-1}},
$$

where $u_{(1)} \leq u_{(2)} \leq \ldots \leq u_{(d)}$ is the ordered list of $u_1, \ldots, u_d \in [0,1]$.

So, Lévy-frailty copulas are survival copulas of random vectors of first-passage times of Lévy subordinators.

Figure 5.2 shows scatterplots of 500 samples of the Lévy-frailty copula built from the α-stable subordinator in dimension $d = 2$. We compare the dependence in terms of the index α of the subordinator and conclude that the dependence is lighter for bigger values of α.

In the following section we approximate the α-stable Lévy subordinator by a compound Poisson process (CPP).

5.3 Approximation of the α-stable subordinator by a CPP

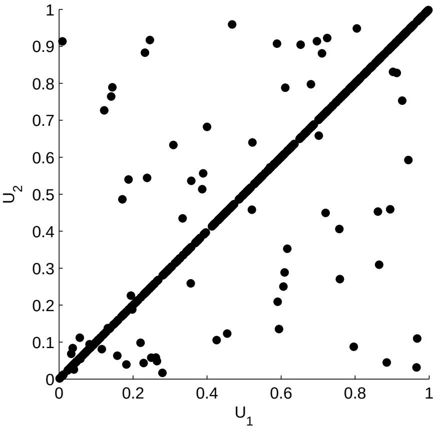
One of the techniques to simulate Lévy-frailty copulas built from an α-stable Lévy subordinator is based on simulating an approximated process of the α-stable subordinator by the compound Poisson process and sampling the first-exit times applying the Lévy-frailty canonical construction.

From the Lévy–Itô decomposition we know that Lévy subordinators can be written via,

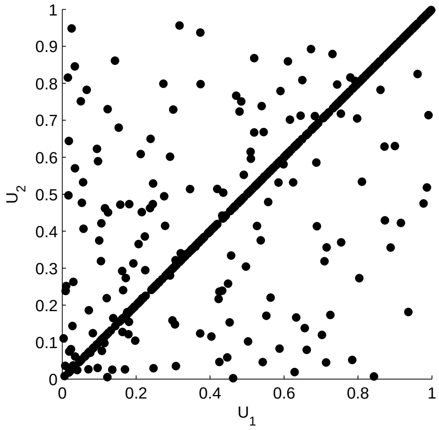
$$
\Lambda_t = \mu\,t + \int_0^t \int_{[0,\infty)} x\,m^{\Lambda}(ds,dx) \tag{5.6}
$$

being x the jump size, $x = \Delta\Lambda_t \in [0,\infty)$.

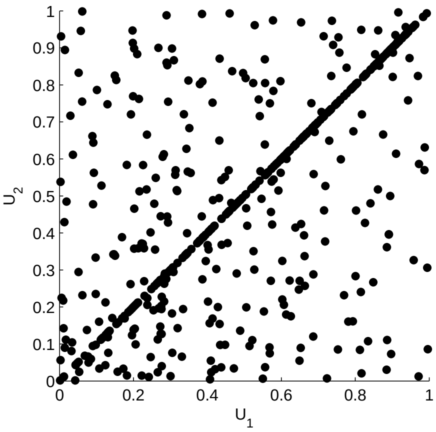
Since subordinators are finite variation processes, it is not necessary to compensate the small jumps, in this case it is enough if one considers the expectation (see [Cont and



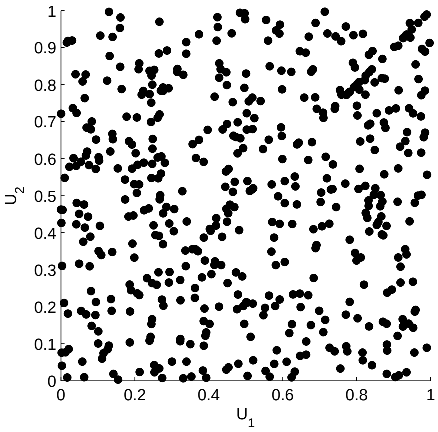
(a) Lévy-frailty copula built from an 0.1-stable Lévy subordinator.



(b) Lévy-frailty copula built from an 0.25-stable Lévy subordinator.



(c) Lévy-frailty copula built from an 0.5-stable Lévy subordinator.



(d) Lévy-frailty copula built from an 0.9-stable Lévy subordinator.

Figure 5.2: Scatterplots of 500 samples of a two-dimensional Lévy-frailty copula built from an α-stable Lévy subordinator with indexes $\alpha = 0.1$, $\alpha = 0.25$, $\alpha = 0.5$, and $\alpha = 0.9$. We can observe that for bigger values of α lighter is the dependence.

Tankov, 2004], p. 185). Taking into consideration that one of the properties of the α-stable subordinators is that $\mu = 0$, the subordinator in Equation (5.6) can be approximated in the following way:

$$\Lambda_t^\epsilon = \sum_{s<t} \Delta\Lambda_s \, \mathbb{1}_{\{\Delta\Lambda_s \geq \epsilon\}} + \mathbb{E}\left[\sum_{s<t} \Delta\Lambda_s \, \mathbb{1}_{\{0<\Delta\Lambda_s<\epsilon\}}\right]. \tag{5.7}$$

We can now compute the expectation of the small jumps:

$$\begin{aligned} \mathbb{E}\left[\sum_{s\leq t} \Delta X_s \mathbb{1}_{\{0<\Delta X_s<\epsilon\}}\right] &= \int_0^t \int_{[0,\infty)} x \mathbb{1}_{\{0<\Delta X_s<\epsilon\}} \nu(dx) ds \\ &= \int_0^t \int_{[0,\infty)} \frac{\alpha}{\Gamma(1-\alpha)} x x^{-(1+\alpha)} \mathbb{1}_{\{0<x<\epsilon\}} dx ds \\ &= \int_0^t \int_0^\epsilon \frac{\alpha}{\Gamma(1-\alpha)} x x^{-(1+\alpha)} dx ds \\ &= \frac{\alpha \cdot \epsilon^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} t. \end{aligned} \tag{5.8}$$

In conclusion, substituting this expectation in (5.7), the approximation of the α-stable subordinator by a compound Poisson process is given by:

$$\Lambda_t^\epsilon = \sum_{s<t} \Delta\Lambda_s \, \mathbb{1}_{\{\Delta\Lambda_s \geq \epsilon\}} + \frac{\alpha \cdot \epsilon^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} t. \tag{5.9}$$

Note that the term $\frac{\alpha \cdot \epsilon^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}$ now serves as the drift μ.

Let us now compute the Lévy measure and the Laplace exponent of the approximated process in (5.9).

The Lévy measure gives the information of the average number of jumps in a unit time interval, i.e. the intensity and magnitude of the jumps. So, in this case the intensity of the jumps:

$$\int_\epsilon^\infty \frac{\alpha}{\Gamma(1-\alpha)} x^{-(1+\alpha)} dx = \frac{1}{\Gamma(1-\alpha)} \left(\epsilon^{-\alpha} - \lim_{x\to\infty} x^{-\alpha}\right) = \frac{\epsilon^{-\alpha}}{\Gamma(1-\alpha)}. \tag{5.10}$$

Recall that the Lévy measure of the α-stable distribution is

$$\nu(dx) = \frac{1}{\Gamma(1-\alpha)} \frac{\alpha}{x^{1+\alpha}} \mathbb{1}_{\{x\geq 0\}} \, dx,$$

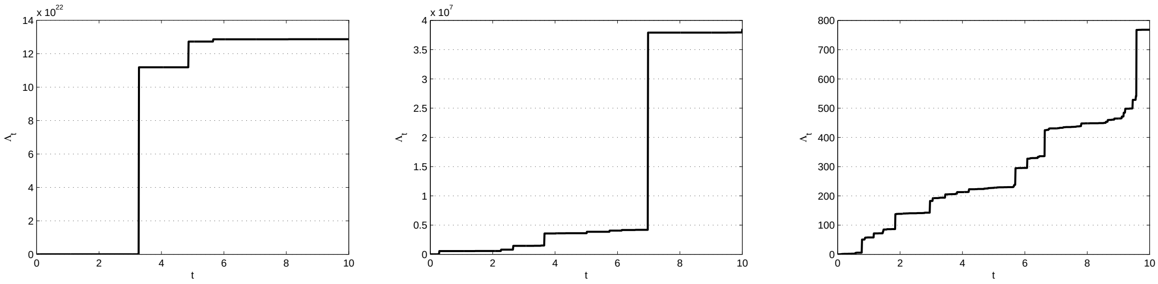
multiplying and dividing it by ϵ^α and taking into consideration that the small jumps have been already truncated, we get that

$$\nu_\epsilon(dx) = \frac{\epsilon^{-\alpha}}{\Gamma(1-\alpha)} \frac{\epsilon^\alpha \, \alpha}{x^{1+\alpha}} \mathbb{1}_{\{x\geq\epsilon\}} \, dx = \frac{\alpha}{\Gamma(1-\alpha) \, x^{1+\alpha}} \mathbb{1}_{\{x\geq\epsilon\}} \, dx, \tag{5.11}$$

being $\frac{\epsilon^{-\alpha}}{\Gamma(1-\alpha)}$ the intensity of jumps computed in Equation (5.10) and $\frac{\epsilon^{\alpha}\,\alpha}{x^{1+\alpha}}\mathbb{1}_{\{x\geq\epsilon\}}dx$ the density function of the Pareto distribution (see Chapter 2, Section 2.1).

Therefore, the approximated process of the α-stable subordinator by the compound Poisson process is based on a sum of big jumps following the Pareto distribution and the expected value of the small jumps used as drift.

The following Figure 5.3 displays the paths of the approximated process of the α-stable subordinator by a compound Poisson process for different values of parameter α.



(a) $\Lambda^{\epsilon}_t\,|\,\alpha = 0.1$ (b) $\Lambda^{\epsilon}_t\,|\,\alpha = 0.25$ (c) $\Lambda^{\epsilon}_t\,|\,\alpha = 0.5$

Figure 5.3: Simulated paths of the α-stable subordinator approximated by the compound Poisson process for $\alpha = 0.1$, $\alpha = 0.25$, $\alpha = 0.5$, and $\epsilon = 10^{-3}$.

Lemma 5.3.1 (Laplace exponent of Λ^{ϵ})
Let Λ_t be an α-stable subordinator with Laplace exponent $\Psi(x) = x^{\alpha}$, $x \geq 0$, $\alpha \in (0,1)$. Let Λ^{ϵ} be the approximated α-stable subordinator by a compound Poisson process. Then the Laplace exponent, Ψ_{ϵ}, of Λ^{ϵ} is given by,

$$\Psi_{\epsilon}(x) = \frac{(1-\alpha)\,\Gamma(1-\alpha,\epsilon\,x)\,x^{\alpha} + \alpha\,\epsilon^{1-\alpha}\,x + (1-\alpha)\,\epsilon^{-\alpha}\,(1-e^{-\epsilon x})}{\alpha\,\epsilon^{1-\alpha} + (1-\alpha)\,\epsilon^{-\alpha}\,(1-e^{-\epsilon}) + (1-\alpha)\,\Gamma(1-\alpha,\epsilon)}. \tag{5.12}$$

such that $\Gamma(\cdot)$ is the Gamma function and $\Gamma(\cdot,\cdot)$ the upper incomplete Gamma function, which is defined as $Gamma(b,s) = \int_s^{\infty} u^{b-1}\,e^{-u}du$, that satisfies $\Gamma(b) = \Gamma(b,0)$, and being b a complex number and strictly positive integer, and s an integer number.

Proof. This result is proved in the same way as the result in Lemma 5.1.1.

$$\Lambda^{\epsilon}_t = \sum_{s<t} \Delta\Lambda_s\,\mathbb{1}_{\{\Delta\Lambda_s\geq\epsilon\}} + \frac{\alpha\cdot\epsilon^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\,t$$

where big jumps, $\sum_{s\leq t}\Delta\Lambda_s\mathbb{1}_{\{\Delta\Lambda_s\geq\epsilon\}}$, follow the Pareto distribution.

From the Lévy–Khintchine formula

$$\Psi_{\epsilon}(x) = \mu\,x + \int_{(0,\infty]} (1-e^{-tx})\nu_{\epsilon}(dt) \overset{(*)}{=}$$

and remembering that,

$$\nu_\epsilon(dt) = \frac{\epsilon^{-\alpha}}{\Gamma(1-\alpha)}\,\frac{\alpha\,\epsilon^{\alpha}}{t^{1+\alpha}}\,\mathbb{1}_{\{t>\epsilon\}}dt,$$

$$\stackrel{(*)}{=} \frac{\alpha\,\epsilon^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)}\,x + \frac{\alpha}{\Gamma(1-\alpha)}\int_{(0,\infty]}(1-e^{-t\,x})\frac{1}{t^{1+\alpha}}\,\mathbb{1}_{\{t>\epsilon\}}dt \stackrel{(**)}{=}$$

Note that, we can express $(1-e^{-t\,x})$ in the following way:

$$1-e^{-t\,x} = \int_0^t xe^{-x\,y}dy,$$

so,

$$\stackrel{(**)}{=} \frac{\alpha\,\epsilon^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)}\,x + \frac{\alpha}{\Gamma(1-\alpha)}\int_{(\epsilon,\infty]}\left[\int_0^t xe^{-x\,y}dy\right]\frac{1}{t^{1+\alpha}}dt,$$

now we change the order of the integrals,

$$\begin{aligned}
&= \frac{\alpha\,\epsilon^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)}\,x + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\epsilon}\left[\int_{\epsilon}^{\infty}\frac{1}{t^{1+\alpha}}dt\right]xe^{-x\,y}dy \\
&\quad + \frac{\alpha}{\Gamma(1-\alpha)}\int_{\epsilon}^{\infty}\left[\int_y^{\infty}\frac{1}{t^{1+\alpha}}dt\right]xe^{-x\,y}dy \\
&= \frac{\alpha\,\epsilon^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)}\,x + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\epsilon}\left(\left[\frac{t^{-\alpha}}{-\alpha}\right]_{\epsilon}^{\infty}\right)xe^{-x\,y}dy \\
&\quad + \frac{\alpha}{\Gamma(1-\alpha)}\int_{\epsilon}^{\infty}\left(\left[t^{-\alpha}\right]_y^{\infty}\right)xe^{-x\,y}dy \\
&= \frac{\alpha\,\epsilon^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)}\,x + \frac{\epsilon^{-\alpha}}{\Gamma(1-\alpha)}\int_0^{\infty}xe^{-x\,y}dy + \frac{\alpha}{\Gamma(1-\alpha)}\,\frac{x}{\alpha}\int_{(\epsilon,\infty]}y^{-\alpha}e^{-x\,y}dy,
\end{aligned}$$

choosing $x\,y = u$,

$$\begin{aligned}
&= \frac{\alpha\,\epsilon^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)}\,x + \frac{\epsilon^{-\alpha}}{\Gamma(1-\alpha)}(1-e^{-\epsilon\,x}) + \frac{x}{\Gamma(1-\alpha)}\int_{\epsilon\,x}^{\infty}\frac{u^{-\alpha}}{x^{1-\alpha}}\,e^{-u}du \\
&= \frac{\alpha\,\epsilon^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)}\,x + \frac{\epsilon^{-\alpha}}{\Gamma(1-\alpha)}(1-e^{-\epsilon\,x}) + \frac{x^{\alpha}}{\Gamma(1-\alpha)}\int_{\epsilon\,x}^{\infty}u^{-\alpha}\,e^{-u}du \\
&= \frac{\alpha\,\epsilon^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)}\,x + \frac{\epsilon^{-\alpha}}{\Gamma(1-\alpha)}(1-e^{-\epsilon\,x}) + \frac{x^{\alpha}}{\Gamma(1-\alpha)}\Gamma(1-\alpha,\epsilon\,x). \qquad (5.13)
\end{aligned}$$

Note that Ψ_ϵ has to fulfil $\Psi_\epsilon(1) = 1$, so we divide the expression in Equation (5.13) by $\Psi_\epsilon(1)$,

$$\Psi_\epsilon(1) = \frac{\alpha\,\epsilon^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)} + \frac{\epsilon^{-\alpha}}{\Gamma(1-\alpha)}(1-e^{-\epsilon}) + \frac{\Gamma(1-\alpha,\epsilon)}{\Gamma(1-\alpha)},$$

therefore,

$$\Psi_\epsilon(x) = \frac{(1-\alpha)\,\Gamma(1-\alpha,\epsilon\,x)\,x^\alpha + \alpha\,\epsilon^{1-\alpha}\,x + (1-\alpha)\,\epsilon^{-\alpha}\,(1-e^{-\epsilon x})}{\alpha\,\epsilon^{1-\alpha} + (1-\alpha)\,\epsilon^{-\alpha}\,(1-e^{-\epsilon}) + (1-\alpha)\,\Gamma(1-\alpha,\epsilon)}.$$

□

Lemma 5.3.2 (Quality of the approximation)
Let Ψ_ϵ be the Laplace exponent of the approximated α-stable Lévy subordinator by a compound Poisson process. We call C_{Ψ_ϵ} the Lévy-frailty copula parameterized in terms of Ψ_ϵ. Then the quality of the approximation between C_{Ψ_ϵ} and C_Ψ is given by,

$$\|C_{\Psi_\epsilon} - C_\Psi\|_\infty \leq \delta \Leftrightarrow \epsilon \leq \left(\frac{\delta(1-\alpha)\Gamma(1-\alpha,1)}{4(d-1)}\right)^{\frac{1}{1-\alpha}}. \tag{5.14}$$

Proof. We need to find ϵ that satisfies

$$\begin{aligned} &\|C_{\Psi_\epsilon}(u_1,\ldots,u_d) - C_\Psi(u_1,\ldots,u_d)\|_\infty \\ &= \sup_{u_1,\ldots,u_d\in[0,1]} |C_{\Psi_\epsilon}(u_1,\ldots,u_d) - C_\Psi(u_1,\ldots,u_d)| \leq \delta. \end{aligned} \tag{5.15}$$

Recall that $\theta_k = \Psi(k+1) - \Psi(k) = (k+1)^\alpha - k^\alpha \leq 1$. $\{\theta_k\}_{k\in\mathbb{N}_0}$ is a completely monotone sequence, i.e., $(-1)^k\Delta\theta_k \geq 0$, so $(k+1)^\alpha - k^\alpha$ is decreasing, for every k. In conclusion,

$$1 = \theta_0 \geq \theta_k, \quad \forall k \geq 1.$$

Let us consider the bivariate case in (5.15):

$$\sup_{u_1,u_2\in[0,1]} \left|u_{(1)}u_{(2)}^{\theta_{1,\epsilon}} - u_{(1)}u_{(2)}^{\theta_1}\right| = \sup_{u_1,u_2\in[0,1]} u_{(1)}\left|u_{(2)}^{\theta_{1,\epsilon}} - u_{(2)}^{\theta_1}\right| \leq \left|u_{(2)}^{\theta_{1,\epsilon}} - u_{(2)}^{\theta_1}\right| \leq \delta,$$

note that,

$$\begin{aligned} &\sup_{u_1,\ldots,u_d\in[0,1]} |C_{\Psi_\epsilon}(u_1,\ldots,u_d) - C_\Psi(u_1,\ldots,u_d)| \\ &\quad= \sup_{u_1,\ldots,u_d\in[0,1]} \left|u_{(1)}u_{(2)}^{\theta_{1,\epsilon}}\cdot\ldots\cdot u_{(d)}^{\theta_{d-1,\epsilon}} - u_{(1)}u_{(2)}^{\theta_1}\cdot\ldots\cdot u_{(d)}^{\theta_{d-1}}\right| \\ &\quad\leq \left|u_{(d)}^{\theta_{d-1,\epsilon}} - u_{(d)}^{\theta_{d-1}}\right| + \left|u_{(d-1)}^{\theta_{d-2,\epsilon}} - u_{(d-1)}^{\theta_{d-2}}\right| + \ldots + \left|u_{(2)}^{\theta_{1,\epsilon}} - u_{(2)}^{\theta_1}\right|. \end{aligned}$$

Let us now work with one of the terms on the sum above: $\left|u_{(k+1)}^{\theta_{k,\epsilon}} - u_{(k+1)}^{\theta_k}\right|$.

We apply now the *Mean Value Theorem* (e.g. [Larson and Edwards, 2013], p. 172). Let us consider $f(x,t) = x^t$. Then,

$$|f(x,a) - f(x,b)| \leq \left|\sup_{t\in[a,b]}\frac{\partial}{\partial t}f(x,t)(a-b)\right| = \left|x^t\log(x)(a-b)\right|, \quad a > b \in \mathbb{R}.$$

Since $x \in [0,1]$,

$$|f(x,a) - f(x,b)| \leq |x\log(x)|\,|a-b| \leq |a-b|\,.$$

Now choosing $a = \theta_{k,\epsilon}$ and $b = \theta_k$:

$$\begin{aligned}
&\left|u_{(k+1)}^{\theta_{k,\epsilon}} - u_{(k+1)}^{\theta_k}\right| \\
&\quad \leq |a_{k,\epsilon} - a_k| \\
&\quad = \left|\frac{\alpha\epsilon^{1-\alpha} + (1-\alpha)\epsilon^{-\alpha}\left(e^{-\epsilon\cdot k}(1-e^{-\epsilon})\right)}{\alpha\epsilon^{1-\alpha} + (1-\alpha)\epsilon^{-\alpha}(1-e^{-\epsilon}) + (1-\alpha)\Gamma\left(1-\alpha,\epsilon\right)}\right. \\
&\qquad \left. + \frac{(1-\alpha)\left[\Gamma\left(1-\alpha,\epsilon(k+1)\right)(k+1)^{\alpha} - \Gamma\left(1-\alpha,\epsilon k\right)k^{\alpha}\right]}{\alpha\epsilon^{1-\alpha} + (1-\alpha)\epsilon^{-\alpha}(1-e^{-\epsilon}) + (1-\alpha)\Gamma\left(1-\alpha,\epsilon\right)} - \left[(k+1)^{\alpha} - k^{\alpha}\right]\right|,
\end{aligned}$$

note that $\Gamma\left(1-\alpha,\epsilon(k+1)\right) \leq \Gamma\left(1-\alpha,\epsilon k\right)$,

$$\begin{aligned}
&\leq \left|\frac{\alpha\epsilon^{1-\alpha} + (1-\alpha)\epsilon^{-\alpha}\left(e^{-\epsilon\cdot k}(1-e^{-\epsilon})\right) + (1-\alpha)\Gamma\left(1-\alpha,\epsilon k\right)\left[(k+1)^{\alpha} - k^{\alpha}\right]}{\alpha\epsilon^{1-\alpha} + (1-\alpha)\epsilon^{-\alpha}(1-e^{-\epsilon}) + (1-\alpha)\Gamma\left(1-\alpha,\epsilon\right)}\right. \\
&\qquad \left. - \left((k+1)^{\alpha} - k^{\alpha}\right)\right|,
\end{aligned}$$

since $\Gamma(1-\alpha,\epsilon k) \leq \Gamma(1-\alpha,\epsilon)$ and $e^{-\epsilon k} \leq 1$,

$$\begin{aligned}
&\leq \left|\frac{\alpha\epsilon^{1-\alpha} + (1-\alpha)\epsilon^{-\alpha}(1-e^{-\epsilon}) + (1-\alpha)\Gamma\left(1-\alpha,\epsilon\right)\left[(k+1)^{\alpha} - k^{\alpha}\right]}{\alpha\epsilon^{1-\alpha} + (1-\alpha)\epsilon^{-\alpha}(1-e^{-\epsilon}) + (1-\alpha)\Gamma\left(1-\alpha,\epsilon\right)} - \left((k+1)^{\alpha} - k^{\alpha}\right)\right| \\
&= \left|\frac{\alpha\epsilon^{1-\alpha} + (1-\alpha)\epsilon^{-\alpha}(1-e^{-\epsilon}) + \left[(k+1)^{\alpha} - k^{\alpha}\right]\left[-\alpha\epsilon^{1-\alpha} - (1-\alpha)\epsilon^{-\alpha}(1-e^{-\epsilon})\right]}{\alpha\epsilon^{1-\alpha} + (1-\alpha)\epsilon^{-\alpha}(1-e^{-\epsilon}) + (1-\alpha)\Gamma\left(1-\alpha,\epsilon\right)}\right|,
\end{aligned}$$

we apply now the triangular inequality,

$$\leq \frac{|\alpha\epsilon^{1-\alpha}| + |(1-\alpha)\epsilon^{-\alpha}(1-e^{-\epsilon})| + |\left[(k+1)^{\alpha} - k^{\alpha}\right]\left[-\alpha\epsilon^{1-\alpha} - (1-\alpha)\epsilon^{-\alpha}(1-e^{-\epsilon})\right]|}{|\alpha\epsilon^{1-\alpha} + (1-\alpha)\epsilon^{-\alpha}(1-e^{-\epsilon}) + (1-\alpha)\Gamma\left(1-\alpha,\epsilon\right)|},$$

recall that $(k+1)^{\alpha} - k^{\alpha} \leq 1$,

$$\leq \frac{2\alpha\epsilon^{1-\alpha} + 2(1-\alpha)\epsilon^{-\alpha}(1-e^{-\epsilon})}{\alpha\epsilon^{1-\alpha} + (1-\alpha)\epsilon^{-\alpha}(1-e^{-\epsilon}) + (1-\alpha)\Gamma(1-\alpha,\epsilon)},$$

due to $\alpha\epsilon^{1-\alpha} \to 0$ and $(1-\alpha)\epsilon^{-\alpha}(1-e^{-\epsilon}) \to 0$ when $\epsilon \searrow 0$,

$$\leq \frac{2\alpha\epsilon^{1-\alpha} + 2(1-\alpha)\epsilon^{-\alpha}(1-e^{-\epsilon})}{(1-\alpha)\Gamma(1-\alpha,\epsilon)},$$

since $\alpha \in (0,1)$ and $\Gamma(1-\alpha,\epsilon) \geq \Gamma(1-\alpha,1)$,

$$\leq \frac{2\epsilon^{1-\alpha} + 2\epsilon^{-\alpha}(1-e^{-\epsilon})}{(1-\alpha)\Gamma(1-\alpha,\epsilon)},$$

w.l.o.g. we can consider $\epsilon \in [0,1]$, so $(1-e^{-\epsilon}) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\epsilon^n}{n!} \leq \epsilon$,

$$\leq \frac{4\epsilon^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha,1)}.$$

So, if we get the value of ϵ such that $\frac{4\epsilon^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha,1)} \leq \frac{\delta}{d-1}, \quad \delta \in \mathbb{R}$, then

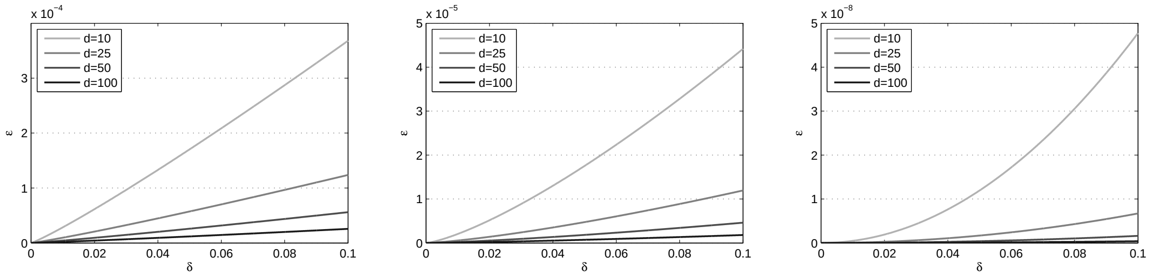
$$\begin{aligned}
&\sup_{u_1,\ldots,u_d \in [0,1]} |C_{\Psi_\epsilon}(u_1,\ldots,u_d) - C_{\Psi}(u_1,\ldots,u_d)| \\
&\leq \left|u_{(d)}^{\theta_{d-1,\epsilon}} - u_{(d)}^{\theta_{d-1}}\right| + \left|u_{(d-1)}^{\theta_{d-2,\epsilon}} - u_{(d-1)}^{\theta_{d-2}}\right| + \ldots + \left|u_{(2)}^{\theta_{1,\epsilon}} - u_{(2)}^{\theta_{1}}\right| \\
&\leq (d-1)\frac{4\epsilon^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha,1)} \leq \delta,
\end{aligned}$$

In conclusion,

$$\frac{4\epsilon^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha,1)} \leq \frac{\delta}{d-1} \Leftrightarrow \epsilon \leq \left(\frac{\delta(1-\alpha)\Gamma(1-\alpha,1)}{4(d-1)}\right)^{\frac{1}{1-\alpha}}.$$

□

Figure 5.4 describes how the parameter δ performs depending on the values of the index α of the stable subordinator and dimension d of the copula.



(a) $\alpha = 0.1$.

(b) $\alpha = 0.25$.

(c) $\alpha = 0.5$.

Figure 5.4: Behaviour of the parameter ϵ in terms of δ in Equation (5.14) for different values of the parameter α and dimension of a copula, d.

5.4 Simulating Lévy-frailty copulas: Algorithms

In this section we aim at explaining the numerical techniques we consider to simulate Lévy-frailty copulas built from α-stable subordinators. The pseudocodes of these algorithms are provided in Chapter 6.

Algorithm 5.4.1 (Simulate eMO copulas)
In Section 5.2 we explained that there exists a link between Lévy-frailty copulas and exchangeable Marshall–Olkin copulas, therefore the dependence structure built either using the exchangeable Marshall–Olkin copula or the Lévy-frailty copula is the same. Algorithm 5.4.1 recursively simulates the exchangeable Marshall–Olkin copula.

Intuitively, let us consider a system with d components in it. This system is influenced by external shocks that kill the components in the system. It could happen that a shock arriving in the system kills just one component or several at once.

The idea behind this algorithm is based on counting the amount of components in the system destroyed at each step and on measuring the time needed to destroy them.

The number of components H destroyed at each time step follows a discrete probability distribution given by (see [Mai and Scherer, 2012], p. 135):

$$\mathbb{P}(H = k) = \frac{\binom{d}{k} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \theta_{d-k+j}}{\sum_{j=0}^{d-1} \theta_j}, \quad 1 \leq k \leq d \in \mathbb{N}. \tag{5.16}$$

Note that at each time at least one component is killed.

Since within the simulation it is not possible to know exactly which components are annihilated at each step, we randomly permute the killing times at the final step. Remember that these first-exit times $(X_1, \dots, X_d)$ follow the Marshall–Olkin distribution, however using the Probability Integral Transform (e.g. [Roussas, 2014], Chapter 11) we get the normalization to uniform variables $(U_1, \dots, U_d)$:

$$U_k = \exp(-X_k) \sim \mathcal{U}([0, 1]), \quad X_k \sim Exp(1), \quad k = 1, \dots, d.$$

Algorithm 5.4.2 (Simulate the α-stable Lévy subordinator)
In this case we simulate the α-stable Lévy subordinator on a fine grid. Due to the infinite activity property of these subordinators there exists a discretization bias so the finer we choose the grid the more accurate are the results. The simulation of the paths of the α-stable subordinator is achieved via the cumulative sum of

$$(dt)^{\frac{1}{\alpha}} \mathcal{S}(\alpha),$$

being $\mathcal{S}(\alpha)$, α-stable random variables, $\alpha \in (0, 1)$, and dt the time step in the discretization of the temporal path. Time steps are equidistant through the time interval.

The first-exit times are estimated using the canonical construction in the Lévy-frailty environment

$$X_k = \inf\{t > 0 : \Lambda_t \geq \varepsilon_k\}, \tag{5.17}$$

where $\{\varepsilon_k\}_{k=1,\dots,d}$ are unit exponential i.i.d. random variables. And applying the Probability Integral Transform we normalize the first-exit times to uniform distribution obtaining $U_1, \dots, U_d$.

While simulating Lévy-frailty copulas applying this technique there is a practical suggestion one should take into consideration. As we mentioned above the idea behind this *algorithm is to simulate the stable subordinator on a grid and to check whether the thresholds given by $\varepsilon_1, \ldots, \varepsilon_d$ have been reached. One "natural" way of proceeding could be checking the canonical condition in Equation* (5.17) *in every node of the grid for each X_i, $i = 1, \ldots, d$, i.e. at each time step. However there exists the possibility to first sample the vector of $(\varepsilon_1, \ldots, \varepsilon_d)$ and sort it afterwards, $\varepsilon_{(1)} < \ldots < \varepsilon_{(d)}$. This way the canonical construction does not have to be checked more than once through the whole temporal path, i.e. as soon as the condition in* (5.17) *is satisfied for a given $\varepsilon_{(i)}$ we obtain $X_{(i)}$, and we continue computing $X_{(i+1)}$. Once we get the vector $(X_{(1)}, \ldots, X_{(d)})$ we apply the order statistics of $(\varepsilon_1, \ldots, \varepsilon_d)$ to sort back $(X_{(1)}, \ldots, X_{(d)})$ and get $(X_1, \ldots, X_d)$. Figure 5.5 graphically illustrates this procedure.*

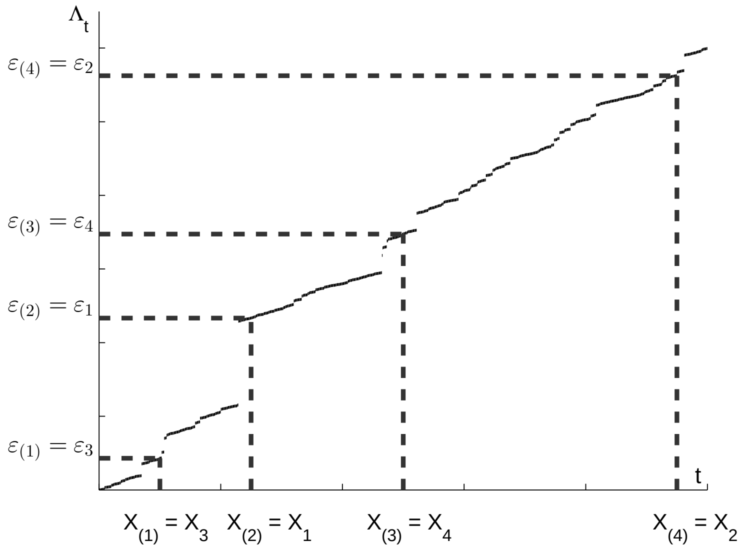
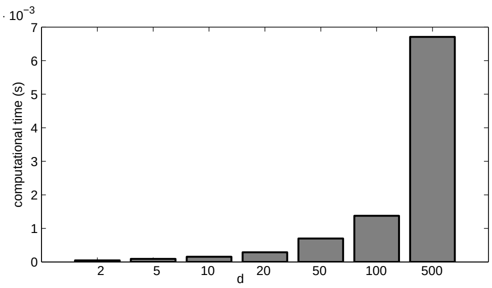


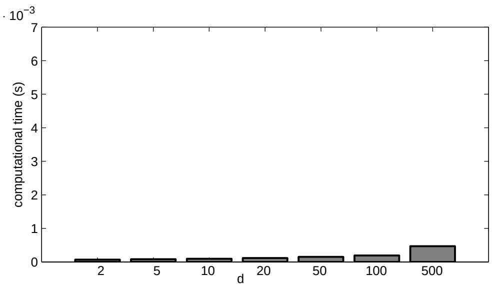
Figure 5.5: Possible situation while checking the canonical construction $X_k = \inf\{t > 0 : \Lambda_t \geq \varepsilon_k\}$ after sorting the vector $(\varepsilon_1, \ldots, \varepsilon_d)$.

The advantage of sorting the vector $(\varepsilon_1, \ldots, \varepsilon_d)$ can be observed for big values of the dimension of the copula as it is shown in the bar charts in Figure 5.6. For small values of d the differences regarding the computational effort are not significant. This is due to the complexity of the algorithm: in case the vector $(\varepsilon_1, \ldots, \varepsilon_d)$ is not sorted, the condition $\Lambda_t > \varepsilon_k$ has to be checked in all nodes N of the grid for each X_k, $k = 1, \ldots, d$ so the cost is $N \cdot d$. However when $(\varepsilon_1, \ldots, \varepsilon_d)$ is sorted, the grid has to be run just once so the complexity resides just in sorting the vector and therefore the cost is $N + d \cdot \ln(d)$.

Algorithm 5.4.3 (Approximate Λ_t by a compound Poisson process)
In this case we approximate the α-stable Lévy subordinator by a compound Poisson subordinator. We consider the big jumps to follow the Pareto distribution and we truncate the small jumps by their expected value. The discretization of the temporal path $[0, T]$ is given by the (big) jump times $\{t_j \,|\, t_j \leq T, j \in \mathbb{N}_0\}$. From Equation (5.9)*, taking into consideration that big jumps follow the Pareto distribution, we simulate:*



(a) non-sorted $(\varepsilon_1, \ldots, \varepsilon_d)$.



(b) sorted $(\varepsilon_1, \ldots, \varepsilon_d)$.

Figure 5.6: Computational effort of Algorithm 5.4.2 depending on whether the vector of the arrival times of the shocks in the system, $(\varepsilon_1, \ldots, \varepsilon_d)$, has been sorted or not. The advantage of sorting the vector is significant for big values of the dimension d.

$$\Lambda_{t_j} = \mathcal{Z}_p + \frac{\alpha \cdot \epsilon^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} t_j, \tag{5.18}$$

being $\mathcal{Z}_p$ *a random variable following the Pareto distribution.*

This algorithm is based on simulating first the arrival times of the shocks $\varepsilon_1, \ldots, \varepsilon_d$ *and a "large enough" sample of jump times and jump sizes. Once the data for shocks and jumps are simulated, the cumulative sum of the process in* (5.18) *is computed* [1]*. The first-exit times* $X_1, \ldots, X_d$ *are obtained using the canonical construction* (5.17) *and normalized to uniform distribution applying the Probability Integral Transform.*

Remark 5.4.1 (Simulating Pareto distributed random numbers)
It is possible to generate Pareto distributed (pd) random numbers from Generalised Pareto distribution (gpd) available in most simulation softwares. The relation between both distributions is given by

$$gpd(x; \mu, \sigma, \xi) = gpd\left(x; \kappa, \frac{\kappa}{\gamma}, \frac{1}{\gamma}\right) = pd(x; \gamma, \kappa),$$

being $\mu \in (-\infty, \infty)$ *the location parameter,* $\sigma \in (0, \infty)$ *the scale parameter, and* $\xi \in (-\infty, \infty)$ *the shape parameter. Standard references in Pareto distribution and Generalised Pareto distribution can be found in e.g. [Arnold, 2015], [Embrechts et al., 1997].*

5.5 Numerical results and application

We compare the numerical techniques introduced in the previous section according to the computational speed. For this purpose we simulate Lévy-frailty copulas built from the α-stable subordinator and measure the time each algorithm needs to sample these copulas.

[1] One can check whether the simulated sample is large enough computing the maximum of $\varepsilon_1, \ldots, \varepsilon_d$ and comparing with the values of the process Λ_t.

We first analyse how each algorithm performs depending on the different parameters d and α, and afterwards we proceed with the overall comparison between the three techniques.

We analyse the average value of a sample of $k = 10^6$ computational times, i.e. we simulate k times the involved copula measuring the computational effort at each simulation and compute the expected value of these computational times [2].

5.5.1 Simulate eMO copulas

These results are obtained after simulating Lévy-frailty copulas with Algorithm 5.4.1. Due to the existing link between Lévy-frailty and exchangeable Marshall–Olkin copulas (Section 5.1), the idea behind this technique is to recursively simulate these latter ones. In Figure 5.7 the scatterplot of 1000 samples for a 2-dimensional Lévy-frailty copula as well as the histograms of the marginals are plotted. The marginals follow the unit exponential distribution.

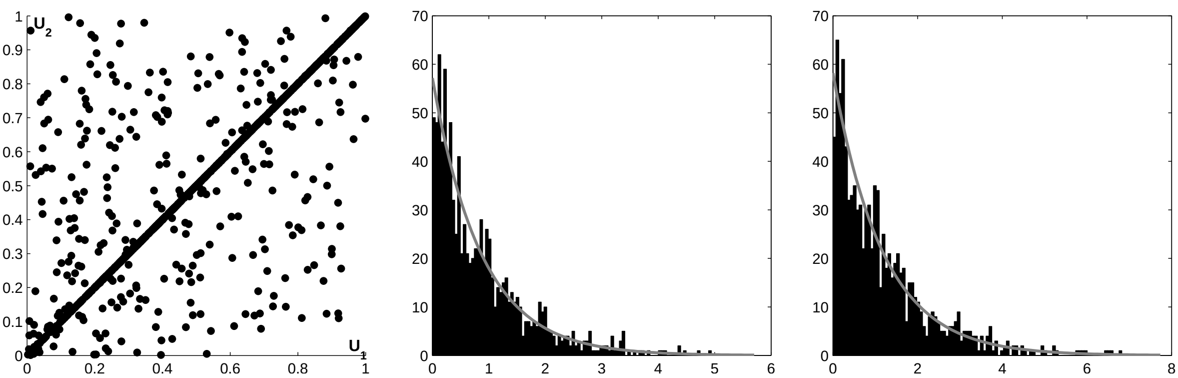


Figure 5.7: Scatterplot of 1000 samples of a 2-dimensional Lévy-frailty copula built from an 0.25-stable Lévy subordinator simulated by Algorithm 5.4.1, $(U_1, U_2) := (\exp(-X_1), \exp(-X_2))$ (left). Histograms of the marginals of the random vector (X_1, X_2), which follows the Marshall–Olkin distribution built by Equation (5.17) (middle and right). One can observe that the marginals follow the exponential distribution with parameter $\lambda = 1$.

Table 5.1 shows the computational times for this algorithm.

[2] Computational times were computed using Matlab R2014a on a 2.4 GHz PC.

	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$
d	CpTime (s)		
2	0.0003	0.0003	0.0004
5	0.0010	0.0012	0.0021
10	0.0033	0.0045	0.0104
20	0.0129	0.0193	0.0641

Table 5.1: Computational time (CpTime) measured in seconds to simulate a Lévy-frailty copula built from the α-stable Lévy subordinator using Algorithm 5.4.1. We compare the computational speed between different dimensions, d, and different values of parameter α. For bigger dimension and bigger values of α the algorithm becomes more expensive.

We can observe that for a larger dimensional copula the algorithm needs more time to simulate it. This is explained by the fact that for bigger dimensions more exit-times have to be computed. In addition one has to consider that the worst case in this algorithm is given when at each step just one element of the system is reached and therefore the algorithm is recursively called as many times as the value of the dimension, d.

It is also possible to realise that for bigger values of parameter α, i.e. for lighter dependence between elements in the system, the time required to simulate the copula is bigger. This could be due to the fact that in cases with stronger dependence the probability to annihilate several elements at once is higher and in conclusion the whole set of the components in the system is reached by the shocks in a shorter time.

5.5.2 Simulate the α-stable Lévy subordinator

In this case the Lévy-frailty copulas are obtained using Algorithm 5.4.2: simulating first the α-stable subordinator on a fine grid and computing the first-exit times using the canonical construction in the Lévy-frailty environment

$$X_k = \inf\{t > 0 : \Lambda_t \geq \varepsilon_k\},$$

where Λ_t is the α-stable subordinator and $\{\varepsilon_k\}_{k=1,\ldots,d}$ the sequence of i.i.d. unit exponential random variables.

The time steps are equidistant, i.e. $dt = 1/n$, $n \geq 10^3 \in \mathbb{N}$. Remember that the stable subordinator is simulated via the cumulative sum of $(dt)^{\frac{1}{\alpha}}\,\mathcal{S}(\alpha)$, such that $\mathcal{S}(\alpha)$ is a random variable following the stable distribution, independent for each time step.

Figure 5.8 displays the scatterplot in dimension 2 of the Lévy-frailty copula built from the 0.5-stable subordinator and the marginals that follow the exponential distribution with parameter $\lambda = 1$.

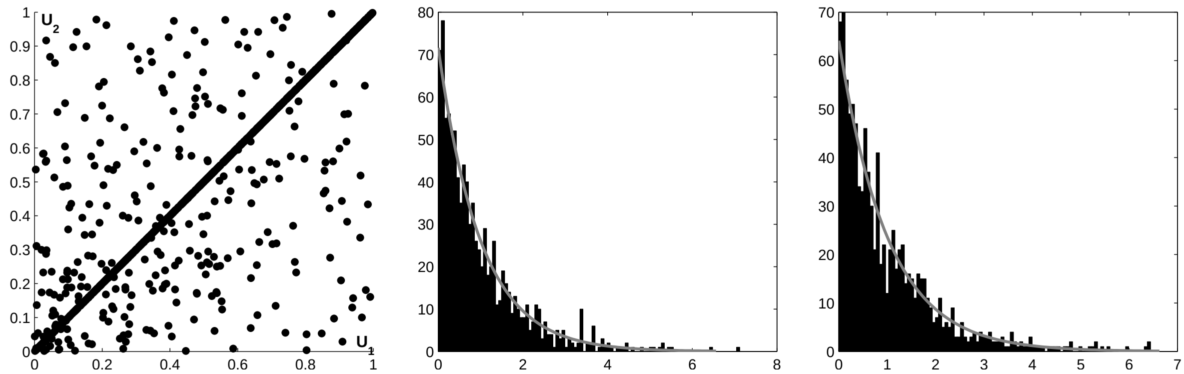


Figure 5.8: Scatterplot of 1000 samples of a 2-dimensional Lévy-frailty copula built from an 0.25-stable Lévy subordinator using Algorithm 5.4.2, $(U_1, U_2) := (\exp(-X_1), \exp(-X_2))$ (left). And the histograms of the marginals of the random vector (X_1, X_2) which follows the Marshall–Olkin distribution (middle and right). Both marginals follow the unit exponential distribution.

In the following Table 5.2 the computational effort, depending on the index α and the dimension d, to simulate the copulas using Algorithm 5.4.2 is displayed. We compare the results for different time-steps, $\Delta_t = 10^{-3}$ and $\Delta_t = 10^{-4}$.

There are no significant differences concerning the dimension d of the copula. Concerning the parameter α, the algorithm needs similar time to sample the copula. We can observe that the algorithm works "a little" faster when $\alpha = 0.5$. This is due to the effort Matlab makes to generate random variables following the stable distribution. Nevertheless, the computational time increases (~ 8 times) when the grid becomes finer ($\Delta_t = 10^{-4}$).

The fact that it is not possible to determine the accuracy of this simulation technique is a disadvantage that has to be taken into consideration. Due to this downside it is not possible to establish a suitable size of the time-step on the discretization of the temporal path.

5.5.3 Approximation by a compound Poisson process

We analyse in the following the computational effort to simulate the Lévy-frailty copulas built from the stable subordinator using Algorithm 5.4.3. Remember that in this case, the idea is based on simulating a process of a compound Poisson type

$$\Lambda_{t_j} = \mathcal{Z}_p + \frac{\alpha\, \epsilon^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)}\, t_j, \quad \mathcal{Z}_p \sim pd(x; \gamma, \kappa), \quad \epsilon > 0,$$

and computing the first-passage times via the canonical construction

$$X_k = \inf\{t > 0 : \Lambda_t \geq \varepsilon_k\}, \quad \varepsilon_k \sim \text{Exp}(1)\,(\text{i.i.d}), \quad k = 1, \dots, d.$$

Figure 5.9 shows the scatterplot of 1000 samples of a 2-dimensional Lévy-frailty copula built from the 0.25-stable subordinator using Algorithm 5.4.3 as well as the histograms

	CpTime (s) ($\Delta_t = 10^{-3}$)			CpTime (s) ($\Delta_t = 10^{-4}$)		
d	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$
2	0.0017	0.0016	0.0012	0.0135	0.0134	0.0105
5	0.0017	0.0016	0.0013	0.0136	0.0137	0.0106
10	0.0017	0.0016	0.0013	0.0136	0.0141	0.0106
20	0.0017	0.0016	0.0013	0.0136	0.0141	0.0107

Table 5.2: Computational effort to simulate the α-stable Lévy subordinator and to mimic the canonical construction in (5.5) in order to simulate the required Lévy-frailty copulas. We consider different dimensions d, indexes α, and time steps Δ_t. There are no significant differences regarding the dimension and the index α of the copula. However, the computational time increases when we consider a finer grid.

of the marginals of the random vector (X_1, X_2) which follows the Marshall–Olkin distribution.

In Table 5.3 the computational times to simulate the Lévy-frailty copulas built from the stable subordinator using Algorithm 5.4.3 are displayed. The parameters for the Pareto distribution in order to sample big jumps are chosen according to Equation (5.11): $\gamma = \alpha$ and $\kappa = \epsilon$.

If we analyse the results, we can conclude that there are no significant differences regarding the dimension d and index α when the dependence is strong, i.e. the algorithm works in a similar way for different values of d and parameter $\alpha \in (0, 0.25)$. However, for lower dependence levels and bigger dimensions of the copula, the computational effort increases. Due to the efficiency this algorithm shows when simulating Lévy-frailty copulas in dimension $d = 20$, we analyse the simulation times for bigger dimensions of the copula (see Table 5.4).

Analysing the results in Table 5.4 we can conclude that when α takes small values, the algorithm shows low computational times. Nevertheless, when α takes bigger values ($\alpha = 0.5$) the simulation effort considerably increases.

5.5.4 Overall comparison

Within this section we collect the results we obtained above and we compare them and analyse which algorithm is the most efficient one with respect to its computational time.

We can observe that, besides of very small dimensions, the first algorithm, *simulate eMO copulas*, is the most expensive whatever the value parameter α takes. This is due to the possibility that at each time just one element is reached so the algorithm is potentially called as many times as the value of the dimension, d. However, note that this simulation

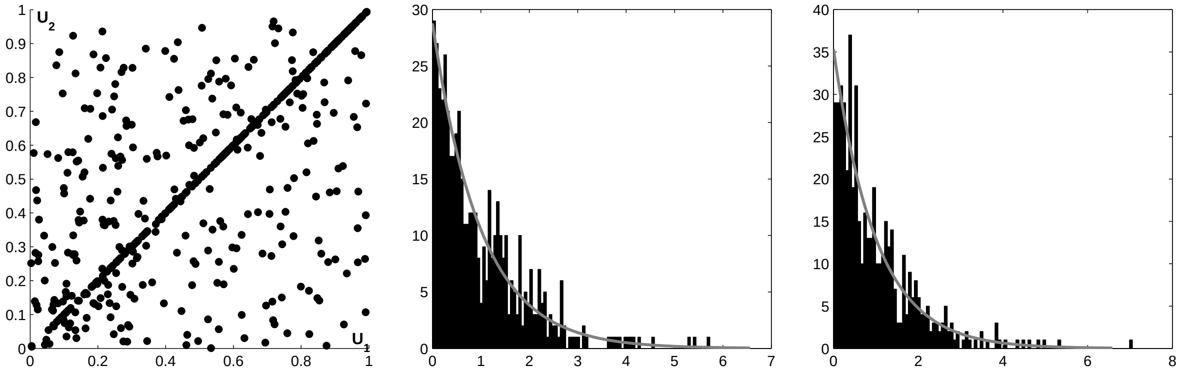


Figure 5.9: Scatterplot of 1000 samples of a 2-dimensional Lévy-frailty copula built from an 0.25-stable Lévy subordinator using Algorithm 5.4.3, $(U_1, U_2) := (\exp(-X_1), \exp(-X_2))$ (left), and histograms of marginals of the random vector (X_1, X_2) which follows the Marshall–Olkin law built using the canonical construction in Equation (5.17) (middle and right).

	$\alpha = 0.1$			$\alpha = 0.25$			$\alpha = 0.5$		
d	CpTime (s)	ϵ	δ	CpTime (s)	ϵ	δ	CpTime (s)	ϵ	δ
2	0.0002	10^{-3}	0.0218	0.0002	10^{-4}	0.0175	0.0003	10^{-6}	0.0132
5	0.0002	10^{-4}	0.0110	0.0002	10^{-5}	0.0080	0.0008	10^{-7}	0.0167
10	0.0002	10^{-4}	0.0247	0.0002	10^{-5}	0.0181	0.0011	10^{-7}	0.0375
20	0.0002	10^{-4}	0.0522	0.0002	10^{-5}	0.0381	0.0044	10^{-8}	0.0251

Table 5.3: Computational time needed to simulate Lévy-frailty copulas built from the α-stable Lévy subordinator approximating the subordinator by a compound Poisson process. We display the results for different values of the copula dimension, d, and parameter α. We consider different values of ϵ in order to get accurate results regarding δ. Regarding the dimension of the copula, when α takes small values, the differences are not significant. However, when the dependence becomes weaker, i.e. α takes bigger values, the algorithm becomes more expensive for bigger dimensions. Concerning parameter α, for stronger dependence, the algorithm is more efficient.

	$\alpha = 0.1$			$\alpha = 0.25$			$\alpha = 0.5$		
d	CpTime (s)	ϵ	δ	CpTime (s)	ϵ	δ	CpTime (s)	ϵ	δ
50	0.0002	10^{-5}	0.0169	0.0003	10^{-6}	0.0175	0.0354	10^{-9}	0.0204
100	0.0003	10^{-5}	0.0349	0.0003	10^{-6}	0.0353	0.0528	10^{-9}	0.0413

Table 5.4: Computational times for the simulation of Lévy-frailty copulas in high dimensions using Algorithm 5.4.3. The computational cost is higher for bigger copulas when the index α takes values close to 0.5. The differences are not significant for small values of α. If we compare the computational times regarding the parameter α, we can conclude that for lower dependence levels the algorithm is slower.

technique is exact, so that it ensures that the results are precise.

Algorithm 5.4.2, *simulate α-stable Lévy subordinator*, when $\Delta_t \leq 10^{-3}$ shows smaller simulation times than Algorithm 5.4.1 for bigger dimensions. Nevertheless, it is not possible to determine how accurate the results, computed using this simulation method, are. If we modify the size of the time step in order to get a finer grid and in conclusion more accurate results, the computational effort increases considerably.

The third algorithm, *Approximate by a compound Poisson process*, shows an efficient behaviour when simulating these Lévy-frailty copulas for either small or big dimensions. Therefore, it provides a simulation technique to sample these copulas in bigger dimensions than the previous techniques with a low computational cost. In addition, it allows to measure the accuracy of the results.

5.5.5 Application

We consider in the following an example where the simulation of Lévy-frailty copulas built from an α-stable subordinator in large dimensions is applied. In Chapter 4, Lemma 4.3.1 it is proved that an exponential functional of a Lévy subordinator in its terminal value converges in distribution to the arithmetic mean of a sum of dependent variables following the Marshall–Olkin distribution:

$$\lim_{d \nearrow \infty} \frac{X_1 + \ldots + X_d}{d} \stackrel{\mathcal{L}}{=} I_\infty,$$

where the random variable $I_\infty = \int_0^\infty e^{-\Lambda_t} dt$ is the exponential functional of a Lévy subordinator Λ.

Therefore these algorithms, most precisely Algorithm 5.4.3, provide a tool to estimate the distribution of the exponential functional of the α-stable subordinator and this way they allow to compute the average lifetime of the Marshall–Olkin law in the infinite case.

6
Algorithms

> "They took a bite out of crime."
> Small Time Crooks (2000).

6.1 Notation

Let us explain first the notation used in the pseudo codes for a better understanding of the algorithms.

- Random numbers:
  - `poissonrnd(`β`)`: Poisson random numbers with parameter β.
  - `uniformrnd(a,b)`: (continuous) uniformly distributed random numbers on the interval $[a, b]$.
  - `normalrnd(`μ`,`σ^2`)`: normally distributed random numbers with mean parameter μ and variance σ^2.
  - `exprnd(`λ`)`: exponentially distributed random numbers with parameter λ.
  - `paretornd(`γ`,`k`)`: standard Pareto distributed random numbers with parameters γ and k.
- Functions:
  - `@NAME_ OF_ FUNCTION`: calls to an external function called `‘‘NAME_ OF_ FUNCTION’’`
- Given commands:
  - `mean`: computes the arithmetic mean of a given vector.
  - `sort`: sorts the elements of a given array in increasing order.

- `length`: computes the length of a given vector.
- `diff`: returns (in a vector) the differences between adjacent elements of a vector.

6.2 Brownian-bridge techniques

Algorithm 6.2.1 (Brownian-bridge technique 1)
This algorithm samples the first-passage time probabilities $\mathbb{P}(T_{ab}^{+} \leq T)$ *and* $\mathbb{P}(T_{ab} \leq T)$*. To this end, it requires as input variables the number of simulation runs,* K*, the drift and volatility coefficients,* μ *and* σ*, respectively, the barriers* a *and* b*, the parameters for the jumps (in our case the jumps follow the double exponential distribution with parameters:* λ*,* $\lambda_{\oplus}$*,* $\lambda_{\ominus}$*, and* p*), and the truncation number of the sum in Equation* (3.3) *in Lemma 3.1.1. As an output, the algorithm generates a* $K \times 3$ *matrix whose columns contain for each simulation run:*

1. *the realised final path value,* B_T*,*
2. *the conditional probability of hitting the upper barrier,* P_{ab}^{+}*,*
3. *the conditional probability that the path stays within the corridor* (b, a)*,* P_{ab}*.*

FUNCTION $[B_T, P_{ab}^{+}, P_{ab}] =$ *sample_ firstPassage*$(K, \mu, \sigma, a, b, \lambda, \lambda_{\oplus}, \lambda_{\ominus}, p, N)$

(1) Simulate the paths and compute the conditional barrier crossing probabilities

FOR $k = 1, \ldots, K$

(A) Simulate the number of jumps within $[0, T]$

$N_T :=$ *poissonrnd*(λT);

(B) Simulate the jump times. Conditional on N_T*, these jumps are distributed as order statistics of i.i.d.* $\mathcal{U}([0, T])$ *random variables, see [Sato, 1999], p. 17.*

FOR $i = 1, \ldots, N_T$
$\tilde{t}[i] :=$ *uniformrnd(0,T)*;
END

Order the random numbers such as $0 < t_1 < \ldots < t_{N_T} < T$

$t :=$ *sort*$(\tilde{t})$;

Compute the time-steps

$\Delta t :=$ *diff(t)* ;

(C) Generate two independent series of random variables independent of N_T:

```
FOR  i = 1, ..., N_T
  b[i] := normalrnd(μ Δt[i], σ² Δt[i]);
  y[i] := @DoubleExpRN(λ, λ_⊕, λ_⊖, p, 1)¹;
END
```

(D) Simulate the asset path on the grid of the jump times (set $B_{t_0} = 0$ and $t_{N_T+1} = T$):

```
Set  B̃[0] := 0;
FOR  i = 1, ..., N_T + 1
  B̃[i] := B[i-1] + b[i];       (before the jumps)
  B[i] := B̃[i] + y[i];        (after the jumps)
END
B := B[1:end-1];        (there are no jumps in t = 0 and t = T)
```

(E) Compute the conditional barrier crossing probabilities between the grid points.

```
Set  P[0] := 1;
     P+[0] := 1;

FOR  i = 1, ..., N_T + 1
```

(a) Compute the probabilities

```
P+_{i-1,i} := @COMPUTE_ BBup(a,b,B[i-1],B̃[i],σ,Δt,N);
P-_{i-1,i} := @COMPUTE_ BBup(-b,-a,-B[i-1],−B̃[i],σ,Δt,N);
P_{i-1,i} := 1-P+_{i-1,i} − P-_{i-1,i};
P[i] := P[i-1] · P_{i-1,i};
P+[i] := P+[i-1] + P[i-1] · P+_{i-1,i};
```

(b) Check the location of the paths before the jumps

```
IF  B̃[i] ≤ b  OR  B̃[i] ≥ a,   THEN
  BB+[k] := P+[i];
  BB := 0;
  i := N_T + 1;       (go to (F))
ELSEIF  B̃[i] > b  AND  B̃[i] < a,   THEN
```

[1] Simulate the jump size.

(c) *Check the endpoint of the path*

```
IF   i = N_T + 1,   THEN
     BB+ [k] := P+ [N_T + 1];
     BB[k] := P[N_T + 1];
```

(d) *Check whether the barrier crossing occurs due to a jump*

```
ELSE
     IF   B[i] > a,   THEN
          BB+ [k] := P[i] + P+ [i];
          BB[k] := 0;
          i := N_t + 1;    (go to (F))
     ELSEIF   B[i] < b,   THEN
          BB[k] := 0;
          i := N_t + 1;    (go to (F))
     END
END
  END
END
```

(F) *Compute the realised final path value,*

Set $B_T[k] := \tilde{B}[N_T + 1]$;

END

(2) *Estimate the unconditional quantities in question via the sample mean of all conditional quantities over all runs*

$P^{+}_{ab} := mean(BB^{+})$;

$P_{ab} := mean(BB)$;

Algorithm 6.2.2 (Brownian-bridge probabilities)

Algorithm 6.2.2 computes the Brownian-bridge probabilities BB^{+}_{ab} in Lemma 3.1.1. The input variables for this algorithm are the upper and lower barriers, a and b, respectively, the start- and endpoint of the path in a given time interval, B_0 and B_1, respectively, volatility σ, time step Δt, and the truncation number of the infinite sum N (Equation (3.3)*).*

$FUNCTION\ BB^{+}_{ab} = COMPUTE\_\ BBup(a,b,B_0,B_1,\sigma,\Delta t,N)$

$sum := 0;$

$FOR\ \ n = 1,\dots,N$

$$s := exp\left\{\frac{2\left(-B_0 + n\cdot a\text{-}(n\text{-}1)\cdot b\right)\left(B_1 - n\cdot a\text{+}(n\text{-}1)\cdot b\right)}{\sigma^2\Delta t}\right\}$$

$$- \ exp\left\{\frac{2\cdot n\cdot (a-b)\cdot (B_1 - B_0 - n\cdot (a-b))}{\sigma\Delta t}\right\};$$

$sum := sum + s;$

END

$BB^{+}_{ab} := sum;$

Algorithm 6.2.3 (Generate double exponential random numbers)
This algorithms generates random numbers following the double exponential distribution needed to sample the jump sizes in Algorithm 6.2.1 and Algorithm 6.2.4. As input variables parameters λ, $\lambda_{\oplus}$, $\lambda_{\ominus}$, p, and the size of the sample of this random numbers N are required.

$FUNCTION\ Y = DaubleExpRN(\lambda,\lambda_{\oplus},\lambda_{\ominus},p,N)$

(1) *Simulate a sample of standard uniform random number $\mathcal{U}([0,1])$ and two samples of exponentially distributed random numbers with parameter $\lambda_{\oplus}$ and $\lambda_{\ominus}$ respectively*

$FOR\ \ n = 1,\dots,N$

$U[n] := uniformrnd(0,1);$

$E_1[n] := exprnd(\lambda_{\oplus});$

$E_2[n] := -exprnd(\lambda_{\ominus});$

END

(2) *Differentiate the positive and negative jumps comparing the values of $\boldsymbol{U}$ with p and safe the indices in two vectors:* ***positiveJ*** *and* ***negativeJ***

$positiveJ := \quad index\ of \quad U \leq p;$

$negativeJ := \quad index\ of \quad U > p;$

(3) *Distribute the values of vectors $\boldsymbol{E_1}$ and $\boldsymbol{E_2}$ in $\boldsymbol{Y}$ according to the indices in vectors* ***positiveJ*** *and* ***negativeJ***

$Y(positiveJ) := E_1;$

$Y(negativeJ) := E_2;$

Algorithm 6.2.4 (Brownian-bridge technique 2)
This algorithm evaluates expectations of the form $X(0) := \mathbb{E}[w(\hat{T}_{ab}, B_T, \mathcal{E})]$, *where* $\mathcal{E} \in \{\oplus, \ominus, \emptyset\}$, *that depend on the first-passage times* T_{ab}, T^+_{ab}, *and* T^-_{ab} *and the final path value* B_T. *The inter-jump periods* (t_{i-1}, t_i) *are considered sequentially. The first Steps (1)(A-D) are implemented in the same way as in Algorithm 6.2.1. Since the only difference from Algorithm 6.2.1 resides in the evaluation of expectation* $X(0) := \mathbb{E}[w(\hat{T}_{ab}, B_T, \mathcal{E})]$ *the input variables are the same for both algorithms.*

FUNCTION $[B_T, X] =$ sample_ firstPassage2($K, \mu, \sigma, a, b, \lambda, \lambda_\oplus, \lambda_\ominus, p$)

(1) *Repeat Steps (A)–(E) for each simulation*

FOR $k = 1, \ldots, K$

(A)–(D) *The same as in Algorithm 6.2.1.*

(E) *Check whether a barrier crossing occurs continuously.*

FOR $i = 1, \ldots, N_T + 1$

(a) *Sample a standard uniform random variable* $\boldsymbol{U} \sim \mathcal{U}([0,1])$ *which will determine whether a barrier has been reached*

$U :=$ uniformrnd(0,1)

(b) *Calculate the barrier crossing probabilities*

$V :=$ @COMPUTE_ BBup($-b, -a, -B[i-1], -\tilde{B}[i], \sigma, \Delta t, N$);
$W :=$ @COMPUTE_ BBup($a, b, B[i-1], \tilde{B}[i], \sigma, \Delta t, N$);
IF $U < V$, THEN (*the lower barrier has been hit*)
$E[k] := \ominus$;
$T_{ab}[k] := t[i-1] + (t[i]-t[i-1])\dfrac{U}{V}$;
$g^-_{ab} :=$ @gUp($-b, -a, -B[i-1], -\tilde{B}[i], \sigma, T_{ab}[k], t[i-1], t[i], N$);
$p[k] := g^-_{ab}\dfrac{t[i]-t[i-1]}{V}$;
and return to Step (1)
ELSEIF $U > 1-W$, THEN (*the upper barrier has been hit*)
$E[k] := \oplus$;
$T_{ab}[k] := t[i-1] + (t[i]-t[i-1])\dfrac{(1-U)}{W}$;
$g^+_{ab} :=$ @gUp($a, b, B[i-1], \tilde{B}[i], \sigma, T_{ab}[k], t[i-1], t[i], N$);
$p[k] := g^+_{ab}\dfrac{t[i]-t[i-1]}{W}$;
and return to Step (1)
ELSEIF $i = N_T + 1$, THEN

```
            E[k] := ∅;
            T_ab[k] := T;
            p[k] := 1;
        return to Step (1)
        ELSE
        continue with Step (c)
        END
```

(c) Check whether a barrier crossing occurs due to a jump

```
        IF  B_{t_i} > a,   THEN
            E[k] := ⊕;
            T_ab[k] := t[i];
            p[k] := 1;
        return to Step (1)
        ELSEIF  B_{t_i} < b,   THEN
            E[k] := ⊖;
            T_ab[k] := t[i];
            p[k] := 1;
        return to Step (1)
        ELSEIF  (B_{t_i} > b)  AND  (B_{t_i} < a),   THEN
        non of the barriers have been reached so return to Step (E)
        END
      END
    END
```

(2) Compute expectation[2] $X(0) := \mathbb{E}[w(T_{ab}, B_T, \mathcal{E})]$

```
    Set  B_T[k] := B̃[N_T + 1];
    Compute  X := mean(p · w);
```

Note that if the lower (resp. upper) barrier has been crossed (Step (1)(E)(b)), the first-exit time is taken uniform in (t_{i-1}, t_i) *and it is weighted according to its actual density, p. Recalling that,*

$$\int_{t_{i-1}}^{t_i} \frac{g_{ab}^{-}\big(t, B_{t_{i-1}}, B_{t_i -}\big)}{v}\, dt = 1 = \int_{t_{i-1}}^{t_i} \frac{1}{t_i - t_{i-1}}\, p(t)\, dt,$$

and

$$\int_{t_{i-1}}^{t_i} \frac{g_{ab}^{+}\big(t, B_{t_{i-1}}, B_{t_i -}\big)}{w}\, dt = 1 = \int_{t_{i-1}}^{t_i} \frac{1}{t_i - t_{i-1}}\, p(t)\, dt.$$

[2] $X(0) \cong \frac{1}{K}\sum_{k=1}^{K} p(k, \hat{T}_{ab}(k))\ w\big(\hat{T}_{ab}(k), B_T(k), \mathcal{E}(k)\big)$ where $w\big(\hat{T}_{ab}(k), B_T(k), \mathcal{E}(k)\big)$ is the quantity that needs to be estimated conditional on the sampled quantities.

The two densities in the latter expressions coincide, thus we can conclude that $p = g^{-}_{ab}\big(\hat{T}^{-}_{ab}, B_{t_{i-1}}, B_{t_i-}\big)\,(t_i - t_{i-1})/v$ *and* $p = g^{+}_{ab}\big(\hat{T}^{+}_{ab}, B_{t_{i-1}}, B_{t_i-}\big)\,(t_i - t_{i-1})/w$*. We note that the importance sampling weight is on average 1, i.e.* $\mathbb{E}\big[p\big] = 1$*. If the barrier-hitting event has happened due to a jump, then the weight of the path is* $p = 1$ *and the first-exit time* t_i*.*

Algorithm 6.2.5 (Brownian-bridge first-passage time intensities)
This algorithm calculates the Brownian-bridge first-passage time intensity (Lemma 3.1.4, Equation (3.8)*). The input variables for this algorithms are the barriers, a and b, the start- and endpoint of the process B,* B_0*, and* B_1 *respectively, the volatility* σ*, the time where the barrier hitting event has ocurred* t*, the jump times* t_0 *and* t_1 *such that* $t_0 < t < t_1$*, and the truncation number of the infinite sum in* (3.8)*.*

$$\mathtt{FUNCTION}\quad g^{+}_{ab} = gUp\,(a, b, B_0, B_1, \sigma, t, t_0, t_1, N)$$

$sum := 0;$

$\mathtt{FOR}\;\; n = 1, \ldots, N$

$$s := (-1)^n\, n\, exp\left\{-\frac{\pi^2 n^2 \sigma^2}{2(a\text{-}b)^2}\right\}\; sin\left(\frac{\pi\, n\, (b - B_0)}{a\text{-}b}\right);$$

$\quad sum := sum + s;$

$\mathtt{END}$

$$g^{+}_{ab} := \frac{\sigma^2 \pi}{(a\text{-}b)^2}\,\frac{\sqrt{t_1 - t_0}}{\sqrt{t_1 - t}}\; exp\left\{\frac{(B_1 - B_0)^2}{2\sigma^2(t_1 - t_0)} - \frac{B_1 - a}{2\sigma^2(t_1 - t)}\right\} \cdot sum;$$

6.3 Simulating Lévy-frailty copulas

Algorithm 6.3.1 (Simulate eMO copulas)
Algorithm 6.3.1 recursively simulates the exchangeable Marshall–Olkin copula. The idea is based on counting the amount of components in the system destroyed at each time step and on measuring the time needed to destroy them, first-exit times. Since within the simulation it is not possible to know exactly which components are annihilated at each time, we randomly permute the first-exit times in the final step. This algorithm requires as input variables the dimension of the copula, d, the d-monotone vector, $\boldsymbol{\theta} := (\theta_0, \ldots, \theta_{d-1})$*, and the vector where the first-exit times are stored, initialized as the zero vector,* $\boldsymbol{X} := (0, \overset{(d)}{\ldots}, 0)$*.*

$$\mathtt{FUNCTION}X = \mathtt{SIMULATE\_}\;\; eMO(d, \theta, X)$$

(1) Set the number of components that are "still alive" and the number of "already destroyed" ones

$alive := length(a);$

$destroyed := d - alive;$

(2) *Fix the time that all "remaining" components survived*

```
IF   destroyed = 0,
THEN
   t_0 := 0;
ELSE
   t_0 := max{X[1], ..., X[destroyed]};
END
```

(3) *Compute the intensity of the next extinction time*

$\lambda_{next} :=$ `a[1]` $+ \ldots +$ `a[alive]`;

(4) *Simulate the time until the next shock arises*

$t_{next} := t_0 +$ *exprnd* (λ_{next});

(5) *Simulate the number of destroyed components*

$h :=$ @DISCRETE_ RN;

(6) *Extend the vector of first-exit times by the number of destroyed elements, h*

```
FOR  j = 1, ..., h
   X[j + destroyed] := t_next;
END
```

(7) *Check whether all the components were already destroyed*

```
IF   alive > h,    THEN
   θ̃ = (θ[1], ..., θ[alive-h]);
   @SIMULATE_ eMO (d, θ̃, X);    (recursively call the function)
ELSE
   x := exp(−X);        (normalization to   U([0,1]))
   X := @PERMUTE_ VECTOR (x);
END
```

This pseudocode with more detailed description of each step and runtime estimations can be found in [Mai and Scherer, 2012] (p. 131-138).

Algorithm 6.3.2 (Random numbers from discrete distribution)
As we already mentioned in Chapter 5, Section 5.4, the number of elements destroyed at each time step is given by a discrete variable $|Y|$ such that

$$|Y| \sim \mathbb{P}(|Y| = k) = p_k = \frac{\binom{alive}{k} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} a[alive - k + j]}{\sum_{j=0}^{alive-1} a[j]}.$$

This random variables are computed using

$$Y = \min \left\{ k \in \mathbb{N} : \sum_{i=1}^{k} p_i \geq u \right\}, \quad u \sim \mathcal{U}\left([0,1]\right). \tag{6.1}$$

```
FUNCTION Y = DISCRETE_ RN
```

(1) Generate a random number from continuous uniform distribution

```
U := uniformrnd(0, 1);
```

(2) Compute the random numbers using Equation (6.1)

```
k := 1;
sum := 0;
WHILE  sum < U,   THEN
   sum := 0;
   FOR  i = 1, . . . , k
      dp := 3@SAMPLE_ PROB;
      sum := sum + dp;
   END
   k := k + 1;
END
Y := k − 1;
```

Algorithm 6.3.3 (Random permutation of a vector)
Algorithm 6.3.3 randomly permutes the elements of an array using the order statistics of a uniformly distributed random vector.

[3] SAMPLE_ PROB computes the discrete probabilities in (6.1)

```
FUNCTION ω = PERMUTE_ VECTOR(x)
```

(1) *Simulate n independent random numbers from continuous uniform distribution*

```
n := length(x)                                                    ;
FOR i = 1,...,n
  U[i] := uniformrnd(0,1);
END
```

(2) *Order the elements of the vector* $\boldsymbol{U}$ *and store the indices of* $\boldsymbol{U}$ *in the ordered vector* $\boldsymbol{V}$

```
V := sort(U);
FOR i = 1,...,n
  index[i] := indices of vector U in V;
END
```

(3) *Permute vector* $\boldsymbol{X}$ *using indexes of vector* $\boldsymbol{V}$

```
ω := X(index);
```

Algorithm 6.3.4 (Simulate the α-stable Lévy subordinator)
Algorithm 6.3.4 simulates the α-stable Lévy subordinator in a fine grid and then it mimics the canonical construction of first-exit times in the Lévy-frailty environment $X_k = \inf\{t > 0 : \Lambda_t \geq \varepsilon_k\}$, *where* $\{\varepsilon_k\}_{k=1,\dots,d}$ *are unit exponential i.i.d. random variables. The stable subordinator is simulated by the cumulative sum of* $dT^{\frac{1}{\alpha}}\mathcal{S}(\alpha)$, *where* $\mathcal{S}(\alpha)$ *denotes the stable distribution.*

The input variables for this algorithm are the index α of the subordinator, the dimension of the Lévy-frailty copula, d, the grid, dT, and the vector where the first-exit times as well as the vector where the normalized values are stored $\boldsymbol{X} := \left(0, \overset{(d)}{\dots}, 0\right)$, $\boldsymbol{U} := \left(0, \overset{(d)}{\dots}, 0\right)$.

```
FUNCTION U = SIMULATE_ STBSUBORDINATOR(α,d,dT,X,U)
```

(1) *Simulate d independent unit exponential random variables*

```
ε := (0,.(d).,0);
FOR k = 1,...,d
  ε[k] := exprnd(1);
END
```

(2) *Find the maximum of* $\boldsymbol{\varepsilon}$

```
ε_ max := max{ε[1],...ε[d]};
```

(3) *Sort the vector* $\boldsymbol{\varepsilon}$ *and store the indices of* $\boldsymbol{\varepsilon}$ *in the sorted vector* $\tilde{\boldsymbol{\varepsilon}}$

$\tilde{\varepsilon}$:= `sort`(ε);

`index` := *indices of* $\boldsymbol{\varepsilon}$ *in vector* $\tilde{\boldsymbol{\varepsilon}}$;

(4) *Simulate a "long enough" sample of the* α*-stable Lévy subordinator*

```
n := 1;
Λt[0] := 0;
WHILE  Λt ≤ ε_ max
   S := @GENERATE_ STBRND(α);
   Λt[n] := Λt[n-1] +  (dT)^(1/α) · S;
   n = n + 1;
END
```

(5) *Set the number of components in the system that are "still alive" and the ones that have been already "destroyed"*

```
alive := d;
destroyed := d-alive;
```

(6) *Check the condition* $\Lambda_t \geq \varepsilon_k$ *of the canonical construction*

```
t := 0;
n := 1;
WHILE  alive > 0
   WHILE  Λt[n] ≥ ε̃(destroyed+1)
      t := t + dT;
      n := n+1;
   END
```

(7) *Compute how many components, h, have been destroyed at each time step and store the first-exit times in* $\boldsymbol{X}$

```
   h := length(ε̃[(ε̃ < Λt[n]) & (ε̃ > Λt[n-1])]);
   X[1+destroyed:h+destroyed] := t;
(update variables: alive, destroyed, n, t)
   alive := alive - h;
   destroyed := d - alive;
   n := n + 1;
   t := t + dT;
END
```

(8) *Sort the vector* $\boldsymbol{X}$ *using the indices stored in Step (4) and standardised them such that it follows the standard uniform distribution*

$\tilde{X}$`(index)` $:= X$;
$U := exp(-\tilde{X})$;

Algorithm 6.3.5 (Generate α-stable random numbers)
This algorithm generates random numbers following the stable distribution. One can find more details about this procedure in [Mai and Scherer, 2012], p. 246. A general background on stable processes can be found in [Samoradnitsky and Taqqu, 1994]. The input variable for this algorithm is the index $\alpha > 0$, *(*$\alpha \in (0, 1)$ *for Lévy subordinators).*

`FUNCTION` $\mathcal{S}$ = `GENERATE_ STBRND`(α)

(1) *Sample a standard uniform random number* $\boldsymbol{U} \sim \mathcal{U}([0, 1])$ *and transform it to* $\tilde{\boldsymbol{U}} \sim \tilde{\mathcal{U}}([-\pi/2, \pi/2])$

$U :=$ `uniformrnd(0,1)`;
$\tilde{U} := \pi \cdot$ `(U-1/2)`;

(2) *Sample a unit exponentially distributed random number*

$\varepsilon =$ `exprnd(1)`;

(3) *Generate stable random numbers*

$$\mathcal{S} := sin(\alpha\,(\pi/2 + U))\; cos\,(U)^{-\frac{1}{\alpha}}\; cos\,((U - \alpha\,(\pi/2 + U))/\varepsilon)^{\frac{1-\alpha}{\alpha}};$$

Algorithm 6.3.6 (Approximate by a compound Poisson subordinator)
Algorithm 6.3.6 approximates the α*-stable Lévy subordinator by a compound Poisson process (CPP). The big jumps follow the Pareto distribution and the small jumps are truncated by their expected value. We simulate the process in* (5.18) *and we compute the first-passage times by* $X_k = \inf\{t > 0 : \Lambda_t \geq \varepsilon_k\}$, *where* $\{\varepsilon_k\}_{k=1,\ldots,d}$ *are i.i.d. unit exponential random variables.*

In this algorithm the following variables are required as input variables: dimension of the Lévy-frailty copula, d, the intensity of the jumps, $\beta := \frac{\epsilon^{-\alpha}}{\Gamma(1-\alpha)}$ *(Equation* (5.10)*), the parameters of the Pareto distribution, k and* α*, the expected value of small jumps which will work as a drift,* $\mu = \frac{\alpha\epsilon^{1-\alpha}}{\Gamma(1-\alpha)(1-\alpha)}$ *(Equation* (5.8)*), and the vector where the first-passage times and copula components are stored, initialized as zero vectors,* $\boldsymbol{X} := \left(0, \overset{(d)}{\ldots}, 0\right)$, $\boldsymbol{U} := \left(0, \overset{(d)}{\ldots}, 0\right)$.

FUNCTION U = SIMULATING_ CPP

(1) *Generate d independent unit exponential random variables*

$\varepsilon := (0, \overset{(d)}{\ldots}, 0)$;

$FOR\;\; k = 1, \ldots, d$

  $\varepsilon[k] := exprnd(1)$;

end

(2) *Compute the maximum of* ε

$\varepsilon_{max} := max\{\varepsilon[1], \ldots, \varepsilon[d]\}$;

(3) *Generate and store all required jumps sizes and jump times in the matrix* $jumps := matrix[0 \quad 0]$

$\Lambda_t^{\epsilon}[0] := 0$;

$While\;\; \Lambda_t^{\epsilon}[indi] \leq \varepsilon_{max}, \quad THEN$

  $indi := indi + 1$;

  $jumpsize := paretornd(\alpha, k)$;

  $jumptime := exprnd\left(\frac{1}{\beta}\right)$;

  $\Lambda_t^{\epsilon}[indi] := \Lambda_t^{\epsilon}[indi\text{-}1] + jumpsize + jumptime \cdot \mu$;

  $jumps := [jumpsize \quad jumptime]$;

$END.$

(4) *Simulate for each 'd' the compound Poisson process and check the condition of the canonical construction in order to compute first-exit times. Normalized them to standard uniform variables*

$FOR\;\; k = 1, \ldots, d$

  $indi := 0$;

  $time := 0$;

  $WHILE\;\; \Lambda_t^{\epsilon}[indi] < \varepsilon[k], \quad THEN$

    $indi := indi + 1$;

    $time := time + jumps[indi, 2]$;

  END

  $IF\;\; ((\Lambda_t^{\epsilon}[indi] - jumps[indi, 1]) > E[k]), \quad THEN$[4]

    $X[k] := time - \frac{\Lambda_t^{\epsilon}[indi] - jumps[indi\text{-}1, 1] - \varepsilon[k]}{\mu}$;

  $ELSE$

    $X[k] := time$;

```
    END
    U[k] := exp(-X[k]);
END
```

In our case we use MATLAB R2014a to generate these Lévy-frailty copulas. MATLAB offers the possibility to sample Pareto distributed random numbers following the generalized Pareto distribution. Therefore, taking into consideration the link between ν_ϵ and the Pareto distribution explained in Equation (5.11) *and Remark 5.4.1, the parameters we used to generate these numbers are: $\xi = 1/\alpha$ (shape), $\sigma = \epsilon/\alpha$ (scale), and $\mu = \epsilon$ (location).*

A deeper description of this algorithm is in [Mai and Scherer, 2012], p. 150-153.

[4]Note that the subordinator can reach any of the thresholds set by the arrival times of the shocks $\{\varepsilon_k\}_{k=1,\ldots,d}$ between two jump times which are the nodes of the discretization of the temporal path. Therefore it is not possible to directly know when exactly has happened the barrier hitting event. This is way the first-exit time has to be adjusted. We explained in Appendix C how this drift adjustment is computed.

7 Conclusion

`"Well, the universe is everything, and if it's expanding, someday it will break apart and that would be the end of everything!"`
`Annie Hall (1977).`

The problems we dealt with through this dissertation can be classified in two main categories: firstly, the simulation of first-exit time probabilities of a jump-diffusion process and, second, the investigation of the dependence structures of Marshall–Olkin kind.

First of all in Chapter 3, we extended the existing Brownian-bridge technique, implemented by [Metwally and Atiya, 2002] for a single barrier, to two constant barriers and simulateed first-passage time probabilities of a jump-diffusion process. We used Monte-Carlo simulations to get the mentioned first-passage times probabilities. Due to its simplicity regarding implementation, the Monte-Carlo technique is a convenient tool for the pricing of different exotic double-barrier derivatives and for credit-risk management. Although standard Monte-Carlo simulations based on a discrete grid are affected by a discretization error, we proved that the Brownian-bridge technique computes unbiased results with a lower computational cost. Therefore, the Brownian-bridge method is a fast and reliable pricing technique. This technique can also be applied in credit-risk management with two possible events: early repayment and default.

Chapters 4 and 5 focus on the study of a specific dependence structure: Marshall–Olkin copulas. Recalling on the importance that dependence has on real-world applications, copulas provide a simple way to construct dependence over initially independent random variables.

In Chapter 4 we considered a sequence of dependent random variables under the Marshall–Olkin law and studied the probability distribution of the sum of these variables. We developed explicit expressions of the survival and probability density functions as well as the Laplace transform for the general bivariate case. Nevertheless, the large number of parameters involved in the problem makes it difficult to extend these results to higher dimensions. We dealt with the mentioned overparameterization drawback considering the subclass of exchangeable Marshall–Olkin law and in this way we derived closed-form solutions for the distribution of the average lifetime in dimensions $d = 2$, $d = 3$, and $d = 4$. Although we explained the procedure of how to extend these results to bigger dimensions, the extension becomes cumbersome analytically and expensive computationally. This setback is explained by the fact that adding one factor in the sum entails an increment of 2^{d-1} on the number of cases into consideration in the implementation of the probabilities. The extendible subfamily of the Marshall–Olkin law, however, provides a way to analyse the law of the mentioned average lifetime when the dimension, d, tends to infinity. We proved that in the infinite dimensional case the probability distribution of the mean of dependent Marshall–Olkin variables converges to the probability law of the exponential functional of a Lévy subordinator.

Finally in Chapter 5 we remained working with the subclass of the extendible Marshall–Olkin law and aimed at designing a fast numerical technique to simulate Lévy-frailty copulas built from an α-stable Lévy subordinator. These copulas are based on stochastic models with conditionally independent and identically distributed components. Due to this stochastic representation, Lévy-frailty copulas present advantages when simulating them, i.e. they present a lower computational effort comparing to the other Marshall–Olkin copulas. We compared three different computational techniques to simulate the mentioned copulas. And since the simulation of copulas in high dimension is often a handicap regarding the computational cost, we provided within this research a fast technique to simulate Lévy-frailty copulas built from an α-stable Lévy subordinator when the dimension of the copula is large.

8 Appendix

A Double–barrier first passage times

We will need the next two results to prove some results in this section.

Theorem A.1 (Novikov's Condition)
On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration, let us consider the standard Brownian motion $\{W_t\}_{t\geq 0}$. Let $\{\theta_s\}_{s\geq 0}$ be an adapted process satisfying the square integrability condition, i.e. $\mathbb{E}\left[\int_0^t \theta_s^2\, ds\right] < \infty$, and let us define

$$Z(t) := \exp\left(\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right), \quad t \geq 0, \tag{8.1}$$

where $\int_0^t \theta_s dW_s$ is a well-defined stochastic integral, named Itô's Integral. If,

(i) $\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \theta_s^2 ds\right)\right] < \infty, \quad 0 \leq s \leq t,$

(ii) $\mathbb{E}[Z(t)] = 1, \quad t \geq 0,$

then, the process $\{Z_t\}_{t\geq 0}$ is a positive martingale.

Proof. The proof of this result can be found in e.g., [Novikov, 1973], [Ruf, 2013]. □

Theorem A.2 (Girsanov's Theorem)
Let us assume that all conditions in Theorem A.1 are satisfied. Let $Z(t)$ be given by (8.1) and let us define

$$\tilde{\mathbb{P}}(A) = \mathbb{E}[Z(T)\mathbb{1}_A], \tag{8.2}$$

a new probability measure on $(\Omega, \mathcal{F})$. Let us, in addition, define

$$\tilde{W}_t := W_t - \int_0^t \theta_s ds, \quad t \geq 0. \tag{8.3}$$

Then, under the probability measure $\tilde{\mathbb{P}}$, the process $\{\tilde{W}_t\}_{t\geq 0}$ is a Wiener process.

Proof. The proof of this results can be found in [Revuz and Yor, 1999], [Musiela and Rutkowski, 2006]. □

Lemma A.1 (Jacobi transformation formula)
Let us consider $x \in \mathbb{R}$ and $t^* \geq 0$, then the next equality is fulfilled:

$$\frac{-2}{\sqrt{\pi t^*}} \sum_{n=-\infty}^{\infty} \frac{(x+1/2-n)}{t^*} \exp\Big(- \frac{(x+1/2-n)^2}{t^*}\Big) = 4\pi \sum_{n=1}^{\infty} (-1)^{n+1} n \exp(-\pi^2 n^2 t^*) \sin(2\pi n x).$$

Proof. For the proof of this lemma we refer the reader to [Jacobi, 1828], [Abramowitz and Stegun, 1965].

□

Proof of Lemma 3.1.2. (i) First representation:

$$\begin{aligned} BM_{ab}^{+}(t_{i-1}, t_i, x_i) &= \mathbb{P}(t_{i-1} < T_{ab}^{+} < t_i) \\ &= \mathbb{P}(a > x_i,\, t_{i-1} < T_{ab}^{+} < t_i) + \mathbb{P}(x_i > a,\, t_{i-1} < T_{ab}^{+} < t_i). \end{aligned}$$

Now we aim at changing $\mathbb{P}(x_i > a,\, t_{i-1} < T_{ab}^{+} < t_i)$ to $\mathbb{P}(a > x_i,\, t_{i-1} < T_{ab}^{+} < t_i)$ and for this purpose we apply the reflectioin principle and the change of measure using Girsanov's Theorem above.

We define

$$Z(t) = \exp\left(\frac{\mu}{\sigma^2}(x_i - x_{i-1}) + \frac{\mu^2}{2\,\sigma^2}(t_i - t_{i-1}) \right), \tag{8.4}$$

and

$$d\mathbb{P} = Z(t)\,\tilde{\mathbb{P}},$$

Since

$$x_i - x_{i-1} \sim \mathcal{N}_{\mathbb{P}}\left(0, \sigma^2(t_i - t_{i-1})\right), \quad x_i - x_{i-1} \sim \mathcal{N}_{\tilde{\mathbb{P}}}\left(-\mu\,(t_i - t_{i-1}), \sigma^2(t_i - t_{i-1})\right), \tag{8.5}$$

if we define $\bar{x}_i = x_i + \mu\,(t_i - t_{i-1})$, then,

$$\bar{x}_i - x_{i-1} \sim \mathcal{N}_{\mathbb{P}}\left(\mu\,(t_i - t_{i-1}), \sigma^2(t_i - t_{i-1})\right), \bar{x}_i - x_{i-1} \sim \mathcal{N}_{\tilde{\mathbb{P}}}\left(0, \sigma^2(t_i - t_{i-1})\right). \tag{8.6}$$

Therefore, using (8.2),

$$\begin{aligned} &\mathbb{P}(x_i > a,\, t_{i-1} < T_{ab}^{+} < t_i) \\ &\quad = \mathbb{E}_{\mathbb{P}}\left[\mathbb{1}_{\{x_i > a \,|\, t_{i-1} < T_{ab}^{+} < t_i\}} \right] \\ &\quad = \mathbb{E}_{\tilde{\mathbb{P}}}\left[\exp\left(\frac{\mu}{\sigma^2}(x_i - x_{i-1}) + \frac{\mu^2}{2\,\sigma^2}(t_i - t_{i-1}) \right) \mathbb{1}_{\{x_i > a \,|\, t_{i-1} < T_{ab}^{+} < t_i\}} \right] \\ &\quad = \mathbb{E}_{\tilde{\mathbb{P}}}\left[\exp\left(\frac{\mu}{\sigma^2}(\bar{x}_i - \mu(t_i - t_{i-1}) - x_{i-1}) + \frac{\mu^2}{2\,\sigma^2}(t_i - t_{i-1}) \right) \times \right. \\ &\qquad \left. \mathbb{1}_{\{\bar{x}_i - \mu(t_i - t_{i-1}) > a \,|\, t_{i-1} < T_{ab}^{+} < t_i\}} \right] \end{aligned}$$

recall that $\bar{x}_i - \mu(t_i - t_{i-1}) > a \Leftrightarrow \bar{x}_i > a + \mu(t_i - t_{i-1}) > a$

$$= \mathbb{E}_{\tilde{\mathbb{P}}}\left[\exp\left(\frac{\mu}{\sigma^2}(\bar{x}_i - x_{i-1}) - \frac{\mu^2}{\sigma^2}(t_i - t_{i-1}) + \frac{\mu^2}{2\,\sigma^2}(t_i - t_{i-1}) \right) \mathbb{1}_{\{\bar{x}_i > a \,|\, t_{i-1} < T_{ab}^{+} < t_i\}} \right]$$

$$= \mathbb{E}_{\tilde{\mathbb{P}}}\left[\exp\left(\frac{\mu}{\sigma^2}(\bar{x}_i - x_{i-1}) - \frac{\mu^2}{2\,\sigma^2}(t_i - t_{i-1})\right)\,\mathbb{1}_{\{\bar{x}_i > a\,|\,t_{i-1} < T_{ab}^{+} < t_i\}}\right]$$

now taking the reflection of $\bar{x}_i$ at the barrier a, $\bar{x}_i \to 2\,a - \bar{x}_i = \tilde{x}_i$,

$$\begin{aligned}
&= \mathbb{E}_{\tilde{\mathbb{P}}}\left[\exp\left(\frac{\mu}{\sigma^2}(2a - \bar{x}_i - x_{i-1}) - \frac{\mu^2}{2\,\sigma^2}(t_i - t_{i-1})\right)\,\mathbb{1}_{\{2a-\bar{x}_i > a\,|\,t_{i-1} < T_{ab}^{+} < t_i\}}\right]\\
&= \mathbb{E}_{\tilde{\mathbb{P}}}\left[\exp\left(\frac{-\mu}{\sigma^2}(-2a + \bar{x}_i + x_{i-1}) - \frac{\mu^2}{2\,\sigma^2}(t_i - t_{i-1})\right)\,\mathbb{1}_{\{a > \tilde{x}_i\,|\,t_{i-1} < T_{ab}^{+} < t_i\}}\right]\\
&= \mathbb{E}_{\tilde{\mathbb{P}}}\left[\exp\left(\frac{-\mu}{\sigma^2}(-2a + 2x_{i-1})\right)\,\exp\left(\frac{-\mu}{\sigma^2}(\bar{x}_i - x_{i-1}) - \frac{\mu^2}{2\,\sigma^2}(t_i - t_{i-1})\right)\times\right.\\
&\qquad \left.\mathbb{1}_{\{a > \tilde{x}_i\,|\,t_{i-1} < T_{ab}^{+} < t_i\}}\right]
\end{aligned}$$

changing notation: $\bar{x}_i = x_i$ and $\tilde{x}_i = x_i$,

$$\begin{aligned}
&= \exp\left(\frac{2\,\mu}{\sigma^2}(a - x_{i-1})\right)\,\mathbb{E}_{\tilde{\mathbb{P}}}\left[\exp\left(-\frac{\mu}{\sigma^2}(x_i - x_{i-1}) - \frac{\mu^2}{2\,\sigma^2}(t_i - t_{i-1})\right)\times\right.\\
&\qquad \left.\mathbb{1}_{\{a > x_i\,|\,t_{i-1} < T_{ab}^{+} < t_i\}}\right]\\
&= \exp\left(\frac{2\,\mu}{\sigma^2}(a - x_{i-1})\right)\,\mathbb{E}_{\mathbb{P}}\left[\mathbb{1}_{\{a > x_i\,|\,t_{i-1} < T_{ab}^{+} < t_i\}}\right]\\
&= \exp\left(\frac{2\,\mu}{\sigma^2}(a - x_{i-1})\right)\,\mathbb{P}\left(a > x_i,\, t_{i-1} < T_{ab}^{+} < t_i\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{P}(x_i > a,\, t_{i-1} < T_{ab}^{+} < t_i) \qquad\qquad (8.7)\\
&\quad = \exp\left(\frac{2\,\mu}{\sigma^2}(a - x_{i-1})\right)\mathbb{P}(a > x_i,\, t_{i-1} < T_{ab}^{+} < t_i)\\
&\quad = \int_{-\infty}^{a} BB_{ab}^{+}(t_{i-1}, t_i, x_{i-1}, x_i)\left[f(x_i - x_{i-1}) + \exp\left(\frac{2\mu(a - x_{i-1})}{\sigma^2}\right)\,g(x_i - x_{i-1})\right]dx_i,
\end{aligned}$$

such that $f(\cdot)$ is the probability density function of a normally distributed variable with mean $\mu\,(t_i - t_{i-1})$ and variance $\sigma^2\,(t_i - t_{i-1})$ while $g(\cdot)$ represents the density function of a normally distributed random variable with mean $-\mu\,(t_i - t_{i-1})$ and variance $\sigma^2\,(t_i - t_{i-1})$.

Recall that the expression for $BB_{ab}^{+}(t_{i-1}, t_i, x_{i-1}, x_i)$ is given in Equation (3.3) in Chapter 3. So computing the integral in Equation (8.7),

$$\begin{aligned}
&\int_{-\infty}^{a} BB_{ab}^{+}(t_{i-1}, t_i, x_{i-1}, x_i)\left[f(x_i - x_{i-1}) + \exp\left(\frac{2\mu(a - x_{i-1})}{\sigma^2}\right)\,g(x_i - x_{i-1})\right]dx_i\\
&= \frac{1}{\sqrt{2\pi\sigma^2(t_i - t_{i-1})}}\int_{-\infty}^{a}\left[\sum_{n=1}^{\infty}\exp\left(-\frac{2(x_{i-1} - na + (n-1)b)(x_i - na + (n-1)b)}{\sigma^2\,(t_i - t_{i-1})}\right)\times\right.\\
&\quad \left.\exp\left(-\frac{2n(a-b)}{\sigma^2\,(t_i - t_{i-1})}(x_{i-1} - x_i + n(a-b))\right)\right]\left[\exp\left(-\frac{(x_i - x_{i-1} - \mu(t_i - t_{i-1}))^2}{2\,\sigma^2(t_i - t_{i-1})}\right)\right.\\
&\quad \left.+ \exp\left(\frac{2\mu(a - x_{i-1})}{\sigma^2}\right)\,\exp\left(-\frac{(x_i - x_{i-1} + \mu(t_i - t_{i-1}))^2}{2\,\sigma^2(t_i - t_{i-1})}\right)\right]dx_i,
\end{aligned}$$

now completing the squares

$$= \frac{1}{\sqrt{2\pi\sigma^2(t_i - t_{i-1})}} \sum_{n=1}^{\infty} \Bigg[\exp\left(-\frac{2\mu(x_{i-1} - na - (n-1)b)}{\sigma^2}\right) \times$$

$$\int_{-\infty}^{a} \exp\left(-\frac{1}{2}\left(\frac{x_i + x_{i-1} - \mu(t_i - t_{i-1}) - 2(na - (n-1)b)}{\sigma\sqrt{t_i - t_{i-1}}}\right)^2\right) dx_i$$

$$- \exp\left(\frac{2n(a-b)\mu}{\sigma^2}\right) \int_{-\infty}^{a} \exp\left(-\frac{1}{2}\left(\frac{x_i - x_{i-1} - \mu(t_i - t_{i-1}) - 2n(a-b)}{\sigma\sqrt{t_i - t_{i-1}}}\right)^2\right) dx_i$$

$$+ \exp\left(-\frac{2\mu(n-1)(a-b)}{\sigma^2}\right) \times$$

$$\int_{-\infty}^{a} \exp\left(-\frac{1}{2}\left(\frac{x_i + x_{i-1} + \mu(t_i - t_{i-1}) - 2(na - (n-1)b)}{\sigma\sqrt{t_i - t_{i-1}}}\right)^2\right) dx_i$$

$$- \exp\left(-\frac{2\mu(x_{i-1} + (n-1)a - nb)}{\sigma^2}\right) \times$$

$$\int_{-\infty}^{a} \exp\left(-\frac{1}{2}\left(\frac{x_i - x_{i-1} + \mu(t_i - t_{i-1}) - 2n(a-b)}{\sigma\sqrt{t_i - t_{i-1}}}\right)^2\right) dx_i \Bigg]$$

$$= \sum_{n=1}^{\infty} \Bigg[\exp\left(-\frac{2\mu(x_{i-1} - (na - (n-1)b))}{\sigma^2}\right) \times \tag{8.8}$$

$$F\left(\frac{x_{i-1} - \mu(t_i - t_{i-1}) - ((2n-1)a - 2(n-1)b)}{\sigma\sqrt{t_i - t_{i-1}}}\right)$$

$$- \exp\left(\frac{2n(a-b)\mu}{\sigma^2}\right) F\left(\frac{-x_{i-1} - \mu(t_i - t_{i-1}) - ((2n-1)a - 2nb)}{\sigma\sqrt{t_i - t_{i-1}}}\right)\Bigg]$$

$$+ \sum_{n=1}^{\infty} \Bigg[\exp\left(-\frac{2\mu\left((n-1)(a-b)\right)}{\sigma^2}\right) F\left(\frac{x_{i-1} + \mu(t_i - t_{i-1}) - ((2n-1)a - 2(n-1)b)}{\sigma\sqrt{t_i - t_{i-1}}}\right)$$

$$- \exp\left(-\frac{2\mu(x_{i-1} + (n-1)a - nb)}{\sigma^2}\right) F\left(\frac{-x_{i-1} + \mu(t_i - t_{i-1}) - ((2n-1)a - 2nb)}{\sigma\sqrt{t_i - t_{i-1}}}\right)\Bigg],$$

where $F(\cdot)$ is the standard Gaussian distribution function.

(ii) Second representation:

We will consider the case $\mu > 0$. The case $\mu < 0$ is done in a similar way substituting n by $-n+1$ in Equation (8.8) and setting

$$\sum_{n=-\infty}^{0} \exp\Big(-\frac{\mu}{\sigma^2}(2x_{i-1} - k_n + 2a)\Big) - \sum_{n=1}^{\infty} \exp\left(\frac{\mu k_n}{\sigma^2}\right)$$
$$= \frac{\exp\left(-\frac{2\mu(b - x_{i-1})}{\sigma^2}\right) - 1}{\exp\left(-\frac{2\mu(b - x_{i-1})}{\sigma^2}\right) - \exp\left(-\frac{2\mu(a - x_{i-1})}{\sigma^2}\right)}.$$

After resubstituting n by $-n+1$ the same final result is obtained.

From part (i) above,

$$
\begin{aligned}
&\mathbb{P}(t_{i-1} < T_{ab}^{+} < t_i) \\
&= \sum_{n=1}^{\infty} \left[\exp\left(-\frac{2\mu(x_{i-1} - (na - (n-1)b))}{\sigma^2} \right) \times \right. \\
&\quad F\left(\frac{x_{i-1} - \mu(t_i - t_{i-1}) - ((2n-1)a - 2(n-1)b)}{\sigma\sqrt{t_i - t_{i-1}}} \right) \\
&\quad \left. - \exp\left(\frac{2n(a-b)\mu}{\sigma^2} \right) F\left(\frac{-x_{i-1} - \mu(t_i - t_{i-1}) - ((2n-1)a - 2nb)}{\sigma\sqrt{t_i - t_{i-1}}} \right) \right] \\
&\quad + \sum_{n=1}^{\infty} \left[\exp\left(-\frac{2\mu\left((n-1)(a-b)\right)}{\sigma^2} \right) \times \right. \\
&\quad F\left(\frac{x_{i-1} + \mu(t_i - t_{i-1}) - ((2n-1)a - 2(n-1)b)}{\sigma\sqrt{t_i - t_{i-1}}} \right) \\
&\quad - \exp\left(-\frac{2\mu(x_{i-1} + (n-1)a - nb)}{\sigma^2} \right) \times \\
&\quad \left. F\left(\frac{-x_{i-1} + \mu(t_i - t_{i-1}) - ((2n-1)a - 2nb)}{\sigma\sqrt{t_i - t_{i-1}}} \right) \right] \\
&= \sum_{n=1}^{\infty} \left[\exp\left(-\frac{2\mu(x_{i-1} - (na - (n-1)b))}{\sigma^2} \right) \times \right. \\
&\quad \left. F\left(\frac{x_{i-1} - \mu(t_i - t_{i-1}) - ((2n-1)a - 2(n-1)b)}{\sigma\sqrt{t_i - t_{i-1}}} \right) \right] \\
&\quad - \sum_{n=-\infty}^{0} \left[\exp\left(-\frac{2\mu(n-1)(a-b)}{\sigma^2} \right) \times \right. \\
&\quad \left. F\left(\frac{-x_{i-1} - \mu(t_i - t_{i-1}) + (2n-1)a - 2(n-1)b)}{\sigma\sqrt{t_i - t_{i-1}}} \right) \right] \\
&\quad + \sum_{n=1}^{\infty} \left[\exp\left(-\frac{2\mu(n-1)(a-b)}{\sigma^2} \right) \times \right. \\
&\quad \left. F\left(\frac{x_{i-1} + \mu(t_i - t_{i-1}) - ((2n-1)a - 2(n-1)b)}{\sigma\sqrt{t_i - t_{i-1}}} \right) \right] \\
&\quad - \sum_{n=-\infty}^{0} \left[\exp\left(-\frac{2\mu(x_{i-1} - na + (n-1)b)}{\sigma^2} \right) \times \right. \\
&\quad \left. F\left(\frac{-x_{i-1} + \mu(t_i - t_{i-1}) + (2n-1)a - 2(n-1)b}{\sigma\sqrt{t_i - t_{i-1}}} \right) \right].
\end{aligned}
$$

We apply now the following property of the standard Gaussian distribution function $F(-x) = 1 - F(x)$:

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \left[\exp\left(-\frac{2\mu(x_{i-1} - (na - (n-1)b))}{\sigma^2} \right) \times \right. \\
&\quad \left. F\left(\frac{x_{i-1} - \mu(t_i - t_{i-1}) - ((2n-1)a - 2(n-1)b)}{\sigma\sqrt{t_i - t_{i-1}}} \right) \right]
\end{aligned} \tag{8.9}
$$

$$
\begin{aligned}
&- \sum_{n=-\infty}^{0} \left[\exp\left(-\frac{2(n-1)(a-b)\mu}{\sigma^2}\right) \times \right. \\
&\left. F\left(\frac{-x_{i-1}-\mu(t_i-t_{i-1})+(2n-1)a-2(n-1)b}{\sigma\sqrt{t_i-t_{i-1}}}\right)\right] \qquad (8.10) \\
&+ \sum_{n=1}^{\infty} \left[\exp\left(-\frac{2\mu\left((n-1)(a-b)\right)}{\sigma^2}\right) \times \right. \\
&\left. \left(1 - F\left(\frac{-x_{i-1}-\mu(t_i-t_{i-1})+(2n-1)a-2(n-1)b}{\sigma\sqrt{t_i-t_{i-1}}}\right)\right)\right] \\
&- \sum_{n=-\infty}^{0} \left[\exp\left(-\frac{2\mu(x_{i-1}-na+(n-1)b)}{\sigma^2}\right) \times \right. \\
&\left. \left(1 - F\left(\frac{x_{i-1}-\mu(t_i-t_{i-1})-((2n-1)a-2(n-1)b)}{\sigma\sqrt{t_i-t_{i-1}}}\right)\right)\right] \\
= &\sum_{n=1}^{\infty} \exp\left(-\frac{2\mu\left((n-1)(a-b)\right)}{\sigma^2}\right) - \sum_{n=-\infty}^{0} \exp\left(-\frac{2\mu(x_{i-1}-na+(n-1)b)}{\sigma^2}\right) \\
&+ \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{2\mu(x_{i-1}-(na-(n-1)b))}{\sigma^2}\right) \times \right. \qquad (8.11) \\
&F\left(\frac{x_{i-1}-\mu(t_i-t_{i-1})-((2n-1)a-2(n-1)b)}{\sigma\sqrt{t_i-t_{i-1}}}\right) - \exp\left(-\frac{2(n-1)(a-b)\mu}{\sigma^2}\right) \times \\
&\left. F\left(\frac{-x_{i-1}-\mu(t_i-t_{i-1})+(2n-1)a-2(n-1)b}{\sigma\sqrt{t_i-t_{i-1}}}\right)\right].
\end{aligned}
$$

Since we are in the case where $\mu > 0$, we can apply the Geometric power series property. Let us remember that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \Leftrightarrow |x| < 1,$$

so

$$\sum_{n=1}^{\infty} x^n = \left(\sum_{n=0}^{\infty} x^n\right) - 1 = \frac{1}{1-x} - 1 = \frac{x}{1-x} \Leftrightarrow |x| < 1.$$

On one side,

$$
\begin{aligned}
\sum_{n=1}^{\infty} \exp\left(-\frac{2\mu\left((n-1)(a-b)\right)}{\sigma^2}\right) &= \sum_{n=1}^{\infty} \exp\left(-\frac{\mu}{\sigma^2}\left(2n(a-b)-2(a-b)\right)\right) \\
&= \exp\left(\frac{\mu}{\sigma^2}2(a-b)\right) \sum_{n=1}^{\infty} \left[\exp\left(-\frac{\mu}{\sigma^2}2(a-b)\right)\right]^n \overset{(*)}{=}
\end{aligned}
$$

note that $\exp\left(-\frac{\mu}{\sigma^2}2(a-b)\right) < 1$, so

$$\overset{(*)}{=} \exp\left(\frac{\mu}{\sigma^2}2(a-b)\right) \frac{\exp\left(-\frac{\mu}{\sigma^2}2(a-b)\right)}{1-\exp\left(\frac{\mu}{\sigma^2}2(a-b)\right)}$$

$$= \frac{1}{1 - \exp\left(\frac{\mu}{\sigma^2} 2(a-b)\right)}.$$

On the other side,

$$\begin{aligned}
&\sum_{n=-\infty}^{0} \exp\left(-\frac{2\mu(x_{i-1} - na + (n-1)b)}{\sigma^2}\right) \\
&= \sum_{n=0}^{\infty} \exp\left(\frac{2\mu}{\sigma^2}(-x_{i-1} - na - (-n-1)b)\right) \\
&= \exp\left(\frac{2\mu}{\sigma^2}(b - x_{i-1})\right) \sum_{n=0}^{\infty} \exp\left(-\frac{2\mu}{\sigma^2} n(a-b)\right) \\
&= \exp\left(\frac{2\mu}{\sigma^2}(b - x_{i-1})\right) \sum_{n=0}^{\infty} \left[\exp\left(-\frac{2\mu}{\sigma^2}(a-b)\right)\right]^n \\
&= \frac{\exp\left(\frac{2\mu}{\sigma^2}(b - x_{i-1})\right)}{1 - \exp\left(-\frac{2\mu}{\sigma^2}(a-b)\right)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \exp\left(-\frac{2\mu\left((n-1)(a-b)\right)}{\sigma^2}\right) - \sum_{n=-\infty}^{0} \exp\left(-\frac{2\mu(x_{i-1} - na + (n-1)b)}{\sigma^2}\right) \\
&= \frac{1}{1 - \exp\left(\frac{\mu}{\sigma^2} 2(a-b)\right)} - \frac{\exp\left(\frac{2\mu}{\sigma^2}(b - x_{i-1})\right)}{1 - \exp\left(-\frac{2\mu}{\sigma^2}(a-b)\right)} \\
&= \frac{\exp\left(-\frac{2\mu}{\sigma^2}(b - x_{i-1})\right) - 1}{\exp\left(-\frac{2\mu}{\sigma^2}(b - x_{i-1}\right) - \exp\left(-\frac{2\mu}{\sigma^2}(a - x_{i-1})\right)}.
\end{aligned}$$

So coming back to Equation (8.9),

$$\begin{aligned}
\mathbb{P}(t_{i-1} < T_{ab}^{+} < t_i) = &\frac{\exp\left(-\frac{2\mu}{\sigma^2}(b - x_{i-1})\right) - 1}{\exp\left(-\frac{2\mu}{\sigma^2}(b - x_{i-1}\right) - \exp\left(-\frac{2\mu}{\sigma^2}(a - x_{i-1})\right)} \\
&+ \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{2\mu(x_{i-1} - (na - (n-1)b))}{\sigma^2}\right) \times \right. \\
&F\left(\frac{x_{i-1} - \mu(t_i - t_{i-1}) - ((2n-1)a - 2(n-1)b)}{\sigma\sqrt{t_i - t_{i-1}}}\right) \\
&- \exp\left(-\frac{2(n-1)(a-b)\mu}{\sigma^2}\right) \times \\
&\left. F\left(\frac{-x_{i-1} - \mu(t_i - t_{i-1}) + (2n-1)a - 2(n-1)b}{\sigma\sqrt{t_i - t_{i-1}}}\right)\right].
\end{aligned}$$

Let us define in the sequel $p_n := x_{i-1} + (2n-2)b - (2n-1)a$, so that we obtain

$$\mathbb{P}(t_{i-1} < T_{ab}^{+} < t_i)$$

$$= \frac{\exp\Big(-\frac{2\mu}{\sigma^2}(b-x_{i-1})\Big)-1}{\exp\Big(-\frac{2\mu}{\sigma^2}(b-x_{i-1}\Big)-\exp\Big(-\frac{2\mu}{\sigma^2}(a-x_{i-1})\Big)}$$

$$+\exp\Big(\frac{\mu}{\sigma^2}(a-x_{i-1})\Big)\sum_{n=-\infty}^{\infty}\Bigg[\exp\Big(-\frac{\mu p_n}{\sigma^2}\Big)\,F\Big(\frac{p_n-\mu(t_i-t_{i-1})}{\sigma\sqrt{t_i-t_{i-1}}}\Big)$$

$$-\exp\Big(-\frac{2\mu p_n}{\sigma^2}\Big)\,F\Big(\frac{-p_n-\mu(t_i-t_{i-1})}{\sigma\sqrt{t_i-t_{i-1}}}\Big)\Bigg].$$

Recall that this sum is related to the distribution function of the inverse Gaussian distribution

$$G(t)=1-\Bigg(F\Big(\frac{\alpha-\mu(t_i-t_{i-1})}{\sigma\sqrt{t_i-t_{i-1}}}\Big)-\exp\Big(\frac{2\mu}{\sigma^2}\alpha\Big)F\Big(\frac{-\alpha-\mu(t_i-t_{i-1})}{\sigma\sqrt{t_i-t_{i-1}}}\Big)\Bigg).$$

The density function of the inverse Gaussian distribution is given by,

$$f(t)=\frac{\alpha}{\sqrt{2\pi}\sigma t^{\frac{3}{2}}}\exp\Bigg(-\frac{\big(\alpha-\mu(t_i-t_{i-1})\big)^2}{2\sigma^2(t_i-t_{i-1})}\Bigg).$$

So we derive now the first-passage time intensity (Definition 3.1.3, Chapter 3)

$$f^+_{ab}(t,x_{i-1}) := \lim_{\Delta t\to 0}\frac{\mathbb{P}(t-\Delta t<T^+_{ab}<t+\Delta t)}{2\Delta t}$$

$$=\exp\Big(\frac{\mu}{\sigma^2}(a-x_{i-1})\Big)\sum_{n=-\infty}^{\infty}\exp\Big(-\frac{\mu}{\sigma^2}p_n\Big)\frac{p_n\exp\Big(-\frac{\big(p_n-\mu(t-t_{i-1})\big)^2}{2\sigma^2(t-t_{i-1})}\Big)}{\sqrt{2\pi}\sigma(t-t_{i-1})^{\frac{3}{2}}}$$

$$=\exp\Big(\frac{\mu}{\sigma^2}(a-x_{i-1})\Big)\exp\Big(-\frac{\mu^2(t-t_{i-1})}{2\sigma^2}\Big)\sum_{n=-\infty}^{\infty}\frac{p_n\exp\Big(-\frac{p_n^2}{2\sigma^2(t-t_{i-1})}\Big)}{\sqrt{2\pi}\sigma(t-t_{i-1})^{\frac{3}{2}}}$$

$$=\exp\Big(-\frac{\mu^2(t-t_{i-1})}{2\sigma^2}\Big)\exp\Big(\frac{\mu(a-x_{i-1})}{\sigma^2}\Big)\times$$

$$\sum_{n=-\infty}^{\infty}\frac{x_{i-1}-2(n-1)a+(2n-3)b}{\sqrt{2\pi\sigma^2}(t-t_{i-1})^{\frac{3}{2}}}\times$$

$$\exp\Bigg(-\frac{(x_{i-1}-2(n-1)a+(2n-3)b)^2}{2\sigma^2(t-t_{i-1})}\Bigg),$$

we apply now the *Jacobi transformation formula* in Lemma A.1, choosing $x=(x_{i-1}-b)/(2(a-b))$ and $t^*=\sigma^2(t-t_{i-1})/(2(a-b)^2)$,

$$=\exp\Big(\frac{\mu(a-x_{i-1})}{\sigma^2}\Big)\frac{\sigma^2\pi}{(a-b)^2}\sum_{n=1}^{\infty}(-1)^n n\,\sin\Big(\frac{\pi n(x_{i-1}-b)}{a-b}\Big)\times$$

$$\exp\Big(-\Big(\frac{\mu^2}{2\sigma^2}+\frac{\pi^2n^2\sigma^2}{2(a-b)^2}\Big)(t-t_{i-1})\Big).$$

Integrating over t the first-passage intensity we get the expression for $\mathbb{P}(t_{i-1}<T^+_{ab}<t_i)=\int_{t_{i-1}}^{t_i}f^+_{ab}(t,x_{i-1})dt$. In the similar way one can compute $\mathbb{P}(t_{i-1}<T^-_{ab}<t_i)$ and $\mathbb{P}(t_{i-1}<T_{ab}<t_i)$

To complete the proof it is missing to verify the Equations (8.5) and (8.6). Let us consider $Z(t)$ as in Equation (8.4), note that, on one side

$$d\mathbb{P} = \exp\left(\frac{\mu}{\sigma^2}(x_i - x_{i-1}) + \frac{\mu^2}{2\,\sigma^2}(t_i - t_{i-1})\right) d\tilde{\mathbb{P}}.$$

Then,

$$\begin{aligned}
\mathbb{P}(x \in dx) &= \tilde{\mathbb{P}}(x \in dx) Z(t) \\
&= \frac{1}{\sqrt{2\pi\,\sigma^2(t_i - t_{i-1})}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(\frac{x_i - x_{i-1} + \mu(t_i - t_{i-1})}{\sigma\,\sqrt{t_i - t_{i-1}}}\right)^2\right. \\
&\left. + \frac{\mu}{\sigma^2}(x_i - x_{i-1}) + \frac{\mu^2}{2\sigma^2}(t_i - t_{i-1})\right) dx_i \\
&= \frac{1}{\sqrt{2\pi\,\sigma^2(t_i - t_{i-1})}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\frac{(x_i - x_{i-1}) + 2\mu(t_i - t_{i-1}) + \mu^2(t_i - t_{i-1})^2}{\sigma^2(t_i - t_{i-1})}\right. \\
&\left. + \frac{\mu}{\sigma^2}(x_i - x_{i-1}) + \frac{\mu^2}{2\sigma^2}(t_i - t_{i-1})\right) dx_i \\
&= \frac{1}{\sqrt{2\pi\,\sigma^2(t_i - t_{i-1})}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(\frac{x_i - x_{i-1}}{\sigma\sqrt{t_i - t_{i-1}}}\right)^2\right) dx_i,
\end{aligned}$$

therefore, $x_i - x_{i-1} \sim \mathcal{N}_{\mathbb{P}}(0, \sigma^2(t_i - t_{i-1}))$.

On the other side,

$$d\tilde{\mathbb{P}} = \exp\left(-\frac{\mu}{\sigma^2}(x_i - x_{i-1}) - \frac{\mu^2}{2\,\sigma^2}(t_i - t_{i-1})\right) d\mathbb{P}.$$

Let $x_i = \bar{x}_i - \mu(t_i - t_{i-1})$, so,

$$\begin{aligned}
d\tilde{\mathbb{P}} &= \exp\left(-\frac{\mu}{\sigma^2}(\bar{x}_i - \mu(t_i - t_{i-1}) - x_{i-1}) - \frac{\mu^2}{2\sigma^2}(t_i - t_{i-1})\right) d\mathbb{P} \\
&= \exp\left(-\frac{\mu}{\sigma^2}(\bar{x}_i - x_{i-1}) + \frac{\mu^2}{\sigma^2}(t_i - t_{i-1}) - \frac{\mu^2}{2\sigma^2}(t_i - t_{i-1})\right) d\mathbb{P} \\
&= \exp\left(-\frac{\mu}{\sigma^2}(\bar{x}_i - x_{i-1}) + \frac{\mu^2}{2\sigma^2}(t_i - t_{i-1})\right) d\mathbb{P},
\end{aligned}$$

and we denote $\bar{Z}(t) = \exp\left(-\frac{\mu}{\sigma^2}(\bar{x}_i - x_{i-1}) + \frac{\mu^2}{2\sigma^2}(t_i - t_{i-1})\right)$. Then,

$$\begin{aligned}
\tilde{\mathbb{P}}(\bar{x} \in d\bar{x}) &= \mathbb{P}(\bar{x} \in d\bar{x}) \bar{Z}(t) \\
&= \frac{1}{\sqrt{2\pi\,\sigma^2(t_i - t_{i-1})}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(\frac{(\bar{x}_i - x_{i-1} - \mu(t_i - t_{i-1})}{\sigma\sqrt{t_i - t_{i-1}}}\right)^2\right. \\
&\left. - \frac{\mu}{\sigma^2}(\bar{x}_i - x_{i-1}) + \frac{\mu^2}{2\sigma^2}(t_i - t_{i-1})\right) d\bar{x}_i
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi\,\sigma^2(t_i-t_{i-1})}} \int_{-\infty}^{\infty} \exp\Bigg(-\frac{1}{2}\,\frac{(\bar{x}_i - x_{i-1})}{\sigma^2(t_i-t_{i-1})} + \frac{\mu}{\sigma^2}(\bar{x}_i - x_{i-1}) \\
&\quad - \frac{\mu^2}{2\sigma^2}(t_i-t_{i-1}) - \frac{\mu}{\sigma^2}(\bar{x}_i - x_{i-1}) + \frac{\mu^2}{2\sigma^2}(t_i-t_{i-1}) \Bigg)\, d\bar{x}_i \\
&= \frac{1}{\sqrt{2\pi\,\sigma^2(t_i-t_{i-1})}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(\frac{\bar{x}_i - x_{i-1}}{\sigma\,\sqrt{t_i-t_{i-1}}} \right)^2 \right) d\bar{x}_i,
\end{aligned}
$$

so, $\bar{x}_i - x_{i-1} \sim \mathcal{N}_{\tilde{\mathbb{P}}}(0, \sigma^2(t_i - t_{i-1}))$.

□

B The average lifetime of the Marshall–Olkin law

Lemma B.1 (Minimum of independent exponential random variables)
Let $X_1, \ldots, X_d$ be exponentially distributed and independent random variables with parameters $\lambda_1, \ldots, \lambda_d > 0$, respectively. Then,

$$\min\{X_1, \ldots, X_d\} \sim \mathit{Exp}\Big(\sum_{i=1}^{d} \lambda_i \Big).$$

Proof of Lemma B.1.

$$\mathbb{P}(\min\{X_1, \ldots, X_d\} > x) = \mathbb{P}(\{X_1 > x\} \cap \ldots \cap \{X_d > x\}).$$

Since $X_1, \ldots, X_d$ are independent,

$$\mathbb{P}(\{X_1 > x\} \cap \ldots \cap \{X_d > x\}) = \mathbb{P}(X_1 > x) \cdot \ldots \cdot \mathbb{P}(X_d > x).$$

$X_1, \ldots, X_d$ are exponentially distributed with parameters $\lambda_1, \ldots, \lambda_d$, and so

$$\mathbb{P}(X_1 > x) \cdot \ldots \cdot \mathbb{P}(X_d > x) = e^{-\lambda_1 x} \cdot \ldots \cdot e^{-\lambda_d x} = e^{-\sum_{i=1}^{d} \lambda_i x}.$$

Therefore,

$$\min\{X_1, \ldots, X_d\} \sim \mathrm{Exp}\Big(\sum_{i=1}^{d} \lambda_i \Big)$$

□

B.1 The exchangeable Marshall–Olkin law

Proof of Lemma 4.2.1. The sum of $d \in \{2, 3, 4\}$ lifetimes

We prove the cases when $d = 3$ and $d = 4$.

- Case $d = 3$:

  In case $d = 3$ the following survival function has to be computed:

$$\begin{aligned}
&\mathbb{P}(X_1 + X_2 + X_3 > x) \\
&\quad = 6\,\mathbb{P}(X_1 + X_2 + X_3 > x \,|\, X_1 < X_2 < X_3)\,\mathbb{P}(X_1 < X_2 < X_3) \\
&\qquad + 3\,\mathbb{P}(X_1 + X_2 + X_3 > x \,|\, X_1 = X_2 < X_3)\,\mathbb{P}(X_1 = X_2 < X_3) \\
&\qquad + 3\,\mathbb{P}(X_1 + X_2 + X_3 > x \,|\, X_1 < X_2 = X_3)\,\mathbb{P}(X_1 < X_2 = X_3) \\
&\qquad + \mathbb{P}(X_1 + X_2 + X_3 > x \,|\, X_1 = X_2 = X_3)\mathbb{P}(X_1 = X_2 = X_3).
\end{aligned} \tag{8.12}$$

  Each conditional probability in the sum (8.12) is derived exactly in the same way as when $d = 2$ obtaining the next results:

$$\begin{aligned}
&\mathbb{P}(X_1 + X_2 + X_3 > x, X_1 < X_2 < X_3) \\
&\quad = \mathbb{P}(X_1 + X_2 + X_3 > x, E_1 < \min\{E_2, E_{12}\} < X_3) \\
&\quad = \mathbb{P}(X_1 + X_2 + X_3 > x, E_1 < \min\{E_2, E_{12}\} < X_3)
\end{aligned}$$

  we call $\min\{E_2, E_{12}\} = X_2^{(d=2)}$

$$\begin{aligned}
&= \mathbb{P}(X_3 > X_2^{(d=2)} > E_1 > x - X_2^{(d=2)} - X_3) \\
&= \mathbb{E}[\mathbb{P}(X_3 > X_2^{(d=2)} > E_1 > x - X_2^{(d=2)} - X_3 \,|\, E_1)] \\
&= \int_0^\infty \mathbb{P}(X_3 > X_2^{(d=2)} > y_1 > x - X_2^{(d=2)} - X_3) f_{E_1}(y_1) dy_1,
\end{aligned}$$

  where

$$\begin{aligned}
&\mathbb{P}(X_3 > X_2^{(d=2)} > y_1 > x - X_2^{(d=2)} - X_3) \\
&\quad = \mathbb{E}[\mathbb{P}(X_3 > X_2^{(d=2)} > y_1 > x > X_2^{(d=2)} - X_3 \,|\, X_2^{(d=2)})] \\
&\quad = \int_0^\infty \mathbb{P}(X_3 > y_2, y_2 > y_1, y_1 > x - y_2 - X_3) f_{X_2^{(d=2)}}(y_2) dy_2 \\
&\quad = \int_{y_1}^\infty \mathbb{P}(X_3 > \max\{y_2, x - y_1 - y_2\}) f_{X_2^{(d=2)}}(y_2) dy_2.
\end{aligned}$$

  So,

$$\begin{aligned}
&\mathbb{P}(X_1 + X_2 + X_3 > x, X_1 < X_2 < X_3) \\
&\quad = \lambda_1(\lambda_1 + \lambda_2) \left(\frac{9(\lambda_1 + 2\lambda_2 + \lambda_3) e^{-(3\lambda_1 + 3\lambda_2 + \lambda_3)x/3}}{(3\lambda_2 + \lambda_3)(3\lambda_2 + 2\lambda_3)(3\lambda_1 + 3\lambda_2 + \lambda_3)} \right. \\
&\qquad \left. - \frac{2(\lambda_1 + 2\lambda_2 + \lambda_3) e^{-(2\lambda_1 + 3\lambda_2 + \lambda_3)x/2}}{(2\lambda_1 + 3\lambda_2 + \lambda_3)(\lambda_2 + \lambda_3)\left(\frac{3\lambda_2 + \lambda_3}{2}\right)} + \frac{e^{-(\lambda_1 + 2\lambda_2 + \lambda_3)x}}{(\lambda_2 + \lambda_3)(3\lambda_2 + 2\lambda_3)} \right).
\end{aligned}$$

In the same way,

$$\begin{aligned}
&\mathbb{P}(X_1 + X_2 + X_3 > x, X_1 = X_2 < X_3) \\
&\quad = \mathbb{P}(X_1 + X_2 + X_3 > x, E_{12} < \min\{E_1 E_2, X_3\}) \\
&\quad = \mathbb{P}(E_{12} + E_{12} + X_3 > x, E_{12} < \min\{E_1 E_2, X_3\}) \\
&\quad = \mathbb{P}(2\min\{E_1, E_2, X_3\} > 2E_{12} > x - X_3) \\
&\quad = \mathbb{P}(2\min\{E_1, E_2, X_3\} > 2E_{12} > x - X_3 \mid \min\{E_1, E_2\} < X_3)\, \mathbb{P}(\min\{E_1, E_2\} < X_3) \\
&\qquad + \mathbb{P}(2\min\{E_1, E_2, X_3\} > 2E_{12} > x - X_3 \mid X_3 < \min\{E_1, E_2\})\, \mathbb{P}(X_3 < \min\{E_1, E_2\}) \\
&\quad = \underbrace{\mathbb{P}(2X_3 > 2\min\{E_1, E_2\} > 2E_{12} > x - X_3)}_{(1)} \\
&\qquad + \underbrace{\mathbb{P}(2\min\{E_1, E_2\} > 2X_3 > 2E_{12} > x - X_3)}_{(2)},
\end{aligned}$$

on one hand,

$$\begin{aligned}
(1) \quad &\mathbb{P}(2X_3 > 2\min\{E_1, E_2\} > 2E_{12} > x - X_3) \\
&= \mathbb{E}[\mathbb{P}(2X_3 > 2\min\{E_1, E_2\} > 2E_{12} > x - X_3 \mid E_{12})] \\
&= \int_0^\infty \mathbb{P}(X_3 > \min\{E_1, E_2\}, \min\{E_1, E_2\} > y, 2y > x - X_3) f_{E_{12}}(y) dy,
\end{aligned}$$

such that

$$\begin{aligned}
&\mathbb{P}(X_3 > \min\{E_1, E_2\}, \min\{E_1, E_2\} > y, 2y > x - X_3) \\
&\quad = \mathbb{E}[\mathbb{P}(X_3 > \min\{E_1, E_2\}, \min\{E_1, E_2\} > y, 2y > x - X_3 \mid \min\{E_1, E_2\})] \\
&\quad = \int_0^\infty \mathbb{P}(X_3 > y_1, , y_1 > y, X_3 > x - 2y) f_{\min\{E_1, E_2\}}(y_1) dy_1 \\
&\quad = \int_y^\infty \mathbb{P}(X_3 > \max\{y, x - 2y\}) f_{\min\{E_1, E_2\}}(y_1) dy_1.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(2) \quad &\mathbb{P}(2\min\{E_1, E_2\} > 2X_3 > 2E_{12} > x - X_3) \\
&= \mathbb{E}[\mathbb{P}(2\min\{E_1, E_2\} > 2X_3 > 2E_{12} > x - X_3 \mid E_{12})] \\
&= \int_0^\infty \mathbb{P}(2\min\{E_1, E_2\} > 2X_3 > 2y > x - X_3) f_{E_{12}}(y) dy \\
&= \int_0^\infty \mathbb{P}(\min\{E_1, E_2\} > X_3, X_3 > y, 2y > x - X_3) f_{E_{12}}(y) dy,
\end{aligned}$$

where

$$\begin{aligned}
&\mathbb{P}(\min\{E_1, E_2\} > X_3, X_3 > y, 2y > x - X_3) \\
&\quad = \mathbb{P}(\min\{E_1, E_2\} > X_3, X_3 > \max\{y, x - 2y\}) \\
&\quad = \mathbb{E}[\mathbb{P}(\min\{E_1, E_2\} > X_3, X_3 > \max\{y, x - 2y\} \mid X_3)]
\end{aligned}$$

$$= \int_0^\infty \mathbb{P}(\min\{E_1, E_2\} > y_1, y_1 > \max\{y, x-2y\}) f_{X_3}(y_1) dy_1$$
$$= \int_{\max\{y, x-2y\}}^\infty \mathbb{P}(\min\{E_1, E_2\} > y_1) f_{X_3}(y_1) dy_1.$$

Computing these integrals,

$$\mathbb{P}(X_1 + X_2 + X_3 > x, X_1 = X_2 < X_3) = \frac{3\lambda_2(\lambda_1 + 2\lambda_2 + \lambda_3)e^{-(3\lambda_1+3\lambda_2+\lambda_3)x/3}}{(3\lambda_2 + 2\lambda_3)(3\lambda_1 + 3\lambda_2 + \lambda_3)} - \frac{\lambda_2 e^{-(\lambda_1+2\lambda_2+\lambda_3)x}}{3\lambda_2 + 2\lambda_3}.$$

Similarly,

$$\mathbb{P}(X_1 + X_2 + X_3 > x, X_1 < X_2 = X_3)$$
$$= \mathbb{P}(X_1 + X_2 + X_3 > x, E_1 < \min\{E_{23}, E_{123}\} < \min\{E_2, E_3, E_{12}E_{13}\})$$

we call $\min\{E_{23}, E_{123}\} = \tilde{X}$ and $\min\{E_2, E_3, E_{12}, E_{13}\} = \bar{X}$

$$= \mathbb{P}(\bar{X} > \tilde{X} > E_1 > x - 2\tilde{X}) = \mathbb{E}[\mathbb{P}(\bar{X} > \tilde{X} > E_1 > x - 2\tilde{X} \mid E_1)]$$
$$= \int_0^\infty \mathbb{P}(\bar{X} > \tilde{X} > y > x - 2\tilde{X}) f_{E_1}(y) dy,$$

where

$$\mathbb{P}(\bar{X} > \tilde{X} > y > x - 2\tilde{X}) = \mathbb{E}[\mathbb{P}(\bar{X} > \tilde{X} > y > x - 2\tilde{X} \mid \tilde{X})]$$
$$= \int_0^\infty \mathbb{P}(\bar{X} > y_1, y_1 > y, y > x - 2y_1) f_{\tilde{X}}(y_1) dy_1$$
$$= \int_0^\infty \mathbb{P}\left(\bar{X} > y_1, y_1 > \max\left\{y, \frac{x-y}{2}\right\}\right) f_{\tilde{X}}(y_1) dy_1$$
$$= \int_{\max\left\{y, \frac{x-y}{2}\right\}}^\infty \mathbb{P}\left(\bar{X} > y_1\right) f_{\tilde{X}}(y_1) dy_1.$$

Therefore,

$$\mathbb{P}(X_1 + X_2 + X_3 > x, X_1 < X_2 = X_3)$$
$$= \frac{3\lambda_1(\lambda_2 + \lambda_3)e^{-(3\lambda_1+3\lambda_2+\lambda_3)x/3}}{2\left(\frac{3\lambda_2+\lambda_3}{2}\right)(3\lambda_1 + 3\lambda_2 + \lambda_3)} - \frac{\lambda_1(\lambda_2 + \lambda_3)e^{-(2\lambda_1+3\lambda_2+\lambda_3)x/2}}{(2\lambda_1 + 3\lambda_2 + \lambda_3)\left(\frac{3\lambda_2+\lambda_3}{2}\right)}.$$

Finally, from Equation (4.10), one can directly calculate

$$\mathbb{P}(X_1 + X_2 + X_3 > x, X_1 = X_2 = X_3) = \frac{\lambda_3}{3\lambda_1 + 3\lambda_2 + \lambda_3} e^{-(3\lambda_1+3\lambda_2+\lambda_3)x/3}.$$

- Case $d = 4$:

  In this case we need to compute the next probability:

$$
\begin{aligned}
&\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x) \qquad (8.13)\\
&\quad = 24\,\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x \,|\, X_1 < X_2 < X_3 < X_4)\,\mathbb{P}(X_1 < X_2 < X_3 < X_4)\\
&\qquad + 12\,\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x \,|\, X_1 < X_2 < X_3 = X_4)\,\mathbb{P}(X_1 < X_2 < X_3 = X_4)\\
&\qquad + 12\,\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x \,|\, X_1 < X_2 = X_3 < X_4)\,\mathbb{P}(X_1 < X_2 = X_3 < X_4)\\
&\qquad + 12\,\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x \,|\, X_1 = X_2 < X_3 < X_4)\,\mathbb{P}(X_1 = X_2 < X_3 < X_4)\\
&\qquad + 4\,\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x \,|\, X_1 = X_2 = X_3 < X_4)\,\mathbb{P}(X_1 = X_2 = X_3 < X_4)\\
&\qquad + 4\,\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x \,|\, X_1 < X_2 = X_3 = X_4)\,\mathbb{P}(X_1 < X_2 = X_3 = X_4)\\
&\qquad + 6\,\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x \,|\, X_1 = X_2 < X_3 = X_4)\,\mathbb{P}(X_1 = X_2 < X_3 = X_4)\\
&\qquad + \mathbb{P}(X_1 + X_2 + X_3 + X_4 > x \,|\, X_1 = X_2 = X_3 = X_4)\,\mathbb{P}(X_1 = X_2 = X_3 = X_4).
\end{aligned}
$$

Let us define some notation to simplify the computations. We denote:

$$
\begin{aligned}
\tilde{X} &= \min\{E_1, E_2\},\\
\tilde{X}_2 &= \min\{E_2, E_{12}\},\\
\tilde{X}_{23} &= \min\{E_{23}, E_{123}\},\\
\tilde{X}_{1234} &= \min\{E_{234}, E_{1234}\},\\
\tilde{X}_{123} &= \min\{E_2, E_3, E_{12}, E_{13}\},\\
\tilde{X}_3 &= \min\{E_3, E_{13}, E_{23}, E_{123}\},\\
\tilde{X}_{34} &= \min\{E_{34}, E_{134}, E_{234}, E_{1234}\},\\
\tilde{X}_{\{1,2,3\}} &= \min\{E_1, E_2, E_3, E_{12}, E_{13}, E_{23}\},\\
\tilde{X}_{234} &= \min\{E_3, E_4, E_{13}, E_{14}, E_{23}, E_{24}, E_{123}, E_{124}\},\\
\hat{X} &= \min\{E_2, E_3, E_4, E_{12}, E_{13}, E_{14}, E_{23}, E_{24}, E_{34}, E_{123}, E_{124}, E_{134}\}.
\end{aligned}
$$

We calculate each of the probabilities in Equation (8.13) in the following way:
Let us start with $\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x, X_1 < X_2 < X_3 < X_4)$:
note that

$$X_1 < X_2 < X_3 < X_4 \Leftrightarrow E_1 < \tilde{X}_2 < \tilde{X}_3 < X_4.$$

Then,

$$
\begin{aligned}
&\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x, X_1 < X_2 < X_3 < X_4)\\
&\quad = \mathbb{P}(X_4 > \tilde{X}_3 > \tilde{X}_2 > E_1 > x - \tilde{X}_2 - \tilde{X}_3 - X_4)\\
&\quad = \mathbb{E}[\mathbb{P}(X_4 > \tilde{X}_3 > \tilde{X}_2 > E_1 > x - \tilde{X}_2 - \tilde{X}_3 - X_4 \,|\, E_1)]\\
&\quad = \int_0^\infty \mathbb{P}(X_4 > \tilde{X}_3 > \tilde{X}_2 > y_1 > x - \tilde{X}_2 - \tilde{X}_3 - X_4) f_{E_1}(y_1) dy_1,
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathbb{P}(X_4 > \tilde{X}_3 > \tilde{X}_2 > y_1 > x - \tilde{X}_2 - \tilde{X}_3 - X_4) \\
&\quad = \mathbb{E}[\mathbb{P}(X_4 > \tilde{X}_3 > \tilde{X}_2 > y_1 > x - \tilde{X}_2 - \tilde{X}_3 - X_4 \,|\, \tilde{X}_2)] \\
&\quad = \int_0^\infty \mathbb{P}(X_4 > \tilde{X}_3 > y_2, y_2 > y_1, y_1 > x - y_2 - \tilde{X}_3 - X_4) f_{X_2}(y_2) dy_2 \\
&\quad = \int_{y_1}^\infty \mathbb{P}(X_4 > \tilde{X}_3 > y_2, y_1 > x - y_2 > \tilde{X}_3 - X_4) f_{X_2}(y_2) dy_2,
\end{aligned}
$$

in the same way

$$
\begin{aligned}
&\mathbb{P}(X_4 > \tilde{X}_3 > y_2, y_2 > y_1, y_1 > x - y_2 - \tilde{X}_3 - X_4) \\
&\quad = \mathbb{E}[\mathbb{P}(X_4 > \tilde{X}_3 > y_2, y_2 > y_1, y_1 > x - y_2 - \tilde{X}_3 - X_4 \,|\, \tilde{X}_3)] \\
&\quad = \int_0^\infty \mathbb{P}(X_4 > y_3, y_3 > y_2, y_1 > x - y_2 - y_3 - X_4) f_{\tilde{X}_3}(y_3) dy_3 \\
&\quad = \int_{y_2}^\infty \mathbb{P}(X_4 > y_3, X_4 > x - y_1 - y_2 - y_3) f_{\tilde{X}_3}(y_3) \\
&\quad = \int_{y_2}^\infty \mathbb{P}(X_4 > \max\{y_3, x - y_1 - y_2 - y_3\}) f_{\tilde{X}_3}(y_3) dy_3.
\end{aligned}
$$

Let us compute now $\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x, X_1 < X_2 < X_3 = X_4)$:
taking into consideration that

$$
X_1 < X_2 < X_3 = X_4 \Leftrightarrow E_1 < \tilde{X}_2 < \tilde{X}_{34} < \tilde{X}_{234},
$$

we get

$$
\begin{aligned}
&\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x, X_1 < X_2 < X_3 = X_4) \\
&\quad = \mathbb{P}(\tilde{X}_{234} > \tilde{X}_{34} > \tilde{X}_2 > E_1 > x - \tilde{X}_2 - 2\tilde{X}_{34}) \\
&\quad = \mathbb{E}[\mathbb{P}(\tilde{X}_{234} > \tilde{X}_{34} > \tilde{X}_2 > E_1 > x - \tilde{X}_2 - 2\tilde{X}_{34} \,|\, E_1)] \\
&\quad = \int_0^\infty \mathbb{P}(\tilde{X}_{234} > \tilde{X}_{34} > \tilde{X}_2 > y_1 > x - \tilde{X}_2 - 2\tilde{X}_{34}) f_{E_1}(y_1), dy_1,
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathbb{P}(\tilde{X}_{234} > \tilde{X}_{34} > \tilde{X}_2 > y_1 > x - \tilde{X}_2 > 2\tilde{X}_{34}) \\
&\quad = \mathbb{E}[\mathbb{P}(\tilde{X}_{234} > \tilde{X}_{34} > \tilde{X}_2 > y_1 > x - \tilde{X}_2 > 2\tilde{X}_{34} \,|\, \tilde{X}_2)] \\
&\quad = \int_0^\infty \mathbb{P}(\tilde{X}_{234} > \tilde{X}_{34} > y_2, y_2 > y_1, y_1 > x - y_2 > 2\tilde{X}_{34}) f_{\tilde{X}_2}(y_2) dy_2 \\
&\quad = \int_{y_1}^\infty \mathbb{P}(\tilde{X}_{234} > \tilde{X}_{34} > y_2, y_1 > x - y_2 - 2\tilde{X}_{34}) f_{\tilde{X}_2}(y_2) dy_2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{P}(\tilde{X}_{234} > \tilde{X}_{34} > y_2, y_1 > x - y_2 - 2\tilde{X}_{34}) \\
&\quad = \mathbb{E}[\mathbb{P}(\tilde{X}_{234} > \tilde{X}_{34} > y_2, y_1 > x - y_2 - 2\tilde{X}_{34} \,|\, \tilde{X}_{34})] \\
&\quad = \int_0^\infty \mathbb{P}\left(\tilde{X}_{234} > y_3, y_3 > y_2, y_3 > \frac{x - y_1 - y_2}{2}\right) f_{\tilde{X}_{34}}(y_3) dy_3 \\
&\quad = \int_{\max\{y_2, \frac{x-y_1-y_2}{2}\}}^\infty \mathbb{P}(\tilde{X}_{234} > y_3) f_{\tilde{X}_{34}}(y_3) dy_3
\end{aligned}
$$

We calculate in the following, $\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x, X_1 < X_2 = X_3 < X_4)$: paying attention on

$$X_1 < X_2 = X_3 < X_4 \Leftrightarrow E_1 < \tilde{X}_{23} < \min\{X_4, \tilde{X}_{123}\},$$

we obtain,

$$
\begin{aligned}
&\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x, X_1 < X_2 = X_3 < X_4) \\
&\quad = \mathbb{P}(X_1 + X_2 + X_3 + X_4 > x, E_1 < \tilde{X}_{23} < X_4 < \tilde{X}_{123}) \\
&\qquad + \mathbb{P}(X_1 + X_2 + X_3 + X_4 > x, E_1 < \tilde{X}_{23} < \tilde{X}_{123} < X_4) \\
&\quad = \mathbb{P}(\tilde{X}_{123} > X_4 > \tilde{X}_{23} > E_1 > x - 2\tilde{X}_{23} - X_4) \\
&\qquad + \mathbb{P}(X_4 > \tilde{X}_{123} > \tilde{X}_{23} > E_1 > x - 2\tilde{X}_{23} - X_4).
\end{aligned}
$$

On one hand,

$$
\begin{aligned}
&\mathbb{P}(\tilde{X}_{123} > X_4 > \tilde{X}_{23} > E_1 > x - 2\tilde{X}_{23} - X_4) \\
&\quad = \mathbb{E}[\mathbb{P}(\tilde{X}_{123} > X_4 > \tilde{X}_{23} > E_1 > x - 2\tilde{X}_{23} - X_4 \,|\, E_1)] \\
&\quad = \int_0^\infty \mathbb{P}(\tilde{X}_{123} > X_4 > \tilde{X}_{23} > y_1 > x - 2\tilde{X}_{23} - X_4) f_{E_1}(y_1) dy_1,
\end{aligned}
$$

such that

$$
\begin{aligned}
&\mathbb{P}(\tilde{X}_{123} > X_4 > \tilde{X}_{23} > y_1 > x - 2\tilde{X}_{23} - X_4) \\
&\quad = \mathbb{E}[\mathbb{P}(\tilde{X}_{123} > X_4 > \tilde{X}_{23} > y_1 > x - 2\tilde{X}_{23} - X_4 \,|\, \tilde{X}_{23})] \\
&\quad = \int_{y_2}^\infty \mathbb{P}(\tilde{X}_{123} > X_4 > y_2, y_1 > x - 2y_2 - X_4) f_{X_{23}}(y_2) dy_2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{P}(\tilde{X}_{123} > X_4 > y_2, y_1 > x - 2y_2 - X_4) \\
&\quad = \mathbb{E}[\mathbb{P}(\tilde{X}_{123} > X_4 > y_2, y_1 > x - 2y_2 - X_4 \,|\, X_4)] \\
&\quad = \int_0^\infty \mathbb{P}(\tilde{X}_{123} > y_3 >, y_3 > y_2, y_3 > x - 2y_2 - y_1) f_{X_4}(y_3) dy_3 \\
&\quad = \int_{\max\{y_2, x-y_1-2y_2\}}^\infty \mathbb{P}(\tilde{X}_{123} > y_3) f_{X_4}(y_3) dy_3
\end{aligned}
$$

On the other hand,

$$\begin{aligned}
&\mathbb{P}(X_4 > \tilde{X}_{123} > \tilde{X}_{23} > E_1 > x - 2\tilde{X}_{23} - X_4) \\
&\quad = \mathbb{E}[\mathbb{P}(X_4 > \tilde{X}_{123} > \tilde{X}_{23} > E_1 > x - 2\tilde{X}_{23} - X_4 \,|\, E_1)] \\
&\quad = \int_0^\infty \mathbb{P}(X_4 > \tilde{X}_{123} > \tilde{X}_{23} > y_1 > x - 2\tilde{X}_{23} - X_4) f_{E_1}(y_1) dy_1,
\end{aligned}$$

where

$$\begin{aligned}
&\mathbb{P}(X_4 > \tilde{X}_{123} > \tilde{X}_{23} > y_1 > x - 2\tilde{X}_{23} - X_4) \\
&\quad = \mathbb{E}[\mathbb{P}(X_4 > \tilde{X}_{123} > \tilde{X}_{23} > y_1 > x - 2\tilde{X}_{23} - X_4 \,|\, \tilde{X}_{23})] \\
&\quad = \int_{y_1}^\infty \mathbb{P}(X_4 > \tilde{X}_{123} > y_2, y_1 > x - 2y_2 - X_4) f_{\tilde{X}_{23}}(y_2) dy_2,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}(X_4 > \tilde{X}_{123} > y_2, y_1 > x - 2y_2 - X_4) f_{\tilde{X}_{23}}(y_2) \\
&\quad = \mathbb{E}[\mathbb{P}(X_4 > \tilde{X}_{123} > y_2, y_1 > x - 2y_2 - X_4 \,|\, \tilde{X}_{123})] \\
&\quad = \int_0^\infty \mathbb{P}(X_4 > y_3, y_3 > y_2, X_4 > x - 2y_2 - y_1) f_{\tilde{X}_{123}}(y_3) dy_3 \\
&\quad = \int_{y_2}^\infty \mathbb{P}(X_4 > \max\{y_3, x - 2y_2 - y_1\}) f_{\tilde{X}_{123}}(y_3) dy_3.
\end{aligned}$$

Let us now come to the computations for $\mathbb{P}(X_1+X_2+X_3+X_4 > x, X_1 = X_2 < X_3 < X_4)$: note that,

$$\begin{aligned}
X_1 = X_2 < X_3 < X_4 \Leftrightarrow \left(E_{12} < \tilde{X}_3 < X_4 < \tilde{X}\right) &\bigcup \left(E_{12} < \tilde{X}_3 < \tilde{X} < X_4\right) \\
&\bigcup \left(E_{12} < \tilde{X} < \tilde{X}_3 < X_4\right)
\end{aligned}$$

So,

$$\begin{aligned}
&\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x, X_1 = X_2 < X_3 < X_4) \\
&\quad = \mathbb{P}(E_{12} + E_{12} + \tilde{X}_3 + X_4 > x, E_{12} < \tilde{X}_3 < X_4 < \tilde{X}) \\
&\qquad + \mathbb{P}(E_{12} + E_{12} + \tilde{X}_3 + X_4 > x, E_{12} < \tilde{X}_3 < \tilde{X} < X_4).
\end{aligned}$$

We calculate first,

$$\begin{aligned}
&\mathbb{P}\left(\tilde{X} > X_4 > \tilde{X}_3 > E_{12} > \frac{x - \tilde{X}_3 - X_4}{2}\right) \\
&\quad = \mathbb{E}\left[\mathbb{P}\left(\tilde{X} > X_4 > \tilde{X}_3 > E_{12} > \frac{x - \tilde{X}_3 - X_4}{2} \,|\, E_{12}\right)\right] \\
&\quad = \int_0^\infty \mathbb{P}\left(\tilde{X} > X_4 > \tilde{X}_3 > y_1 > \frac{x - \tilde{X}_3 - X_4}{2}\right) f_{E_{12}}(y_1) dy_1,
\end{aligned}$$

such that

$$\begin{aligned}
&\mathbb{P}\left(\tilde{X} > X_4 > \tilde{X}_3 > y_1 > \frac{x - \tilde{X}_3 - X_4}{2}\right) \\
&\quad = \mathbb{E}\left[\mathbb{P}\left(\tilde{X} > X_4 > \tilde{X}_3 > y_1 > \frac{x - \tilde{X}_3 - X_4}{2} \mid \tilde{X}_3\right)\right] \\
&\quad = \int_0^\infty \mathbb{P}\left(\tilde{X} > X_4 > y_2, y_2 > y_1, y_1 > \frac{x - y_2 - X_4}{2}\right) f_{\tilde{X}_3}(y_2) dy_2 \\
&\quad = \int_{y_1}^\infty \mathbb{P}\left(\tilde{X} > X_4 > y_2, y_1 > \frac{x - y_2 - X_4}{2}\right) f_{\tilde{X}_3}(y_2) dy_2,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}\left(\tilde{X} > X_4 > y_2, y_1 > \frac{x - y_2 - X_4}{2}\right) \\
&\quad = \mathbb{E}\left[\mathbb{P}\left(\tilde{X} > X_4 > y_2, y_1 > \frac{x - y_2 - X_4}{2} \mid X_4\right)\right] \\
&\quad = \int_0^\infty \mathbb{P}\left(\tilde{X} > y_3, y_3 > y_2, y_3 > x - y_2 - 2y_1\right) f_{X_4}(y_3) dy_3.
\end{aligned}$$

Let us proceed with,

$$\begin{aligned}
&\mathbb{P}\left(X_4 > \tilde{X} > \tilde{X}_3 > E_{12} > \frac{x - \tilde{X}_3 - X_4}{2}\right) \\
&\quad = \mathbb{E}\left[\mathbb{P}\left(X_4 > \tilde{X} > \tilde{X}_3 > E_{12} > \frac{x - \tilde{X}_3 - X_4}{2} \mid E_{12}\right)\right] \\
&\quad = \int_0^\infty \mathbb{P}\left(X_4 > \tilde{X} > \tilde{X}_3 > y_1 > \frac{x - \tilde{X}_3 - X_4}{2}\right) f_{E_{12}}(y_1) dy_1,
\end{aligned}$$

from where we get,

$$\begin{aligned}
&\mathbb{P}\left(X_4 > \tilde{X} > \tilde{X}_3 > y_1 > \frac{x - \tilde{X}_3 - X_4}{2}\right) \\
&\quad = \mathbb{E}\left[\mathbb{P}\left(X_4 > \tilde{X} > \tilde{X}_3 > y_1 > \frac{x - \tilde{X}_3 - X_4}{2} \mid \tilde{X}_3\right)\right] \\
&\quad = \int_0^\infty \mathbb{P}\left(X_4 > \tilde{X} > y_2, y_2 > y_1, y_1 > \frac{x - y_2 - X_4}{2}\right) f_{\tilde{X}_3}(y_2) dy_2 \\
&\quad = \int_{y_1}^\infty \mathbb{P}\left(X_4 > \tilde{X} > y_2, y_1 > \frac{x - y_2 - X_4}{2}\right) f_{\tilde{X}_3}(y_2) dy_2,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}\left(X_4 > \tilde{X} > y_2, y_1 > \frac{x - y_2 - X_4}{2}\right) \\
&\quad = \mathbb{E}\left[\mathbb{P}\left(X_4 > \tilde{X} > y_2, y_1 > \frac{x - y_2 - X_4}{2} \mid \tilde{X}\right)\right] \\
&\quad = \int_0^\infty \mathbb{P}\left(X_4 > y_3, y_3 > y_2, X_4 > x - y_2 - 2y_1\right) f_{\tilde{X}}(y_3) dy_3 \\
&\quad = \int_{y_2}^\infty \mathbb{P}\left(X_4 > \max\{y_3, x - y_2 - 2y_1\}\right) f_{\tilde{X}}(y_3) dy_3.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\mathbb{P}\left(X_4 > \tilde{X}_3 > \tilde{X} > E_{12} > \frac{x - \tilde{X}_3 - X_4}{2}\right) \\
&\quad = \mathbb{E}\left[\mathbb{P}\left(X_4 > \tilde{X}_3 > \tilde{X} > E_{12} > \frac{x - \tilde{X}_3 - X_4}{2} \mid E_{12}\right)\right] \\
&\quad = \int_0^\infty \mathbb{P}\left(X_4 > \tilde{X}_3 > \tilde{X} > y_1 > \frac{x - \tilde{X}_3 - X_4}{2}\right) f_{E_{12}}(y_1) dy_1,
\end{aligned}$$

such that

$$\begin{aligned}
&\mathbb{P}\left(X_4 > \tilde{X}_3 > \tilde{X} > y_1 > \frac{x - \tilde{X}_3 - X_4}{2}\right) \\
&\quad = \mathbb{E}\left[\mathbb{P}\left(X_4 > \tilde{X}_3 > \tilde{X} > y_1 > \frac{x - \tilde{X}_3 - X_4}{2} \mid \tilde{X}\right)\right] \\
&\quad = \int_0^\infty \mathbb{P}\left(X_4 > \tilde{X}_3 > y_2, y_2 > y_1, y_1 > \frac{x - \tilde{X}_3 - X_4}{2}\right) f_{\tilde{X}}(y_2) dy_2 \\
&\quad = \int_{y_1}^\infty \mathbb{P}\left(X_4 > \tilde{X}_3 > y_2, y_1 > \frac{x - \tilde{X}_3 - X_4}{2}\right) f_{\tilde{X}}(y_2) dy_2,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}\left(X_4 > \tilde{X}_3 > y_2, y_1 > \frac{x - \tilde{X}_3 - X_4}{2}\right) \\
&\quad = \mathbb{E}\left[\mathbb{P}\left(X_4 > \tilde{X}_3 > y_2, y_1 > \frac{x - \tilde{X}_3 - X_4}{2} \mid \tilde{X}_3\right)\right] \\
&\quad = \int_0^\infty \mathbb{P}\left(X_4 > y_3, y_3 > y_2, X_4 > x - 2y_1 - y_3\right) f_{\tilde{X}_3}(y_3) dy_3 \\
&\quad = \int_{y_2}^\infty \mathbb{P}(X_4 > \max\{y_3, x - 2y_1 - y_3\}) f_{\tilde{X}_3}(y_3) dy_3.
\end{aligned}$$

Let us derive now $\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x, X_1 = X_2 = X_3 < X_4)$: taking into consideration that,

$$X_1 = X_2 = X_3 < X_4 \Leftrightarrow E_{123} < \min\{X_4, \tilde{X}_{\{1,2,3\}}\},$$

then,

$$\begin{aligned}
&\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x, X_1 = X_2 = X_3 < X_4) \\
&\quad = \mathbb{P}(3E_{123} + X_4 > x, E_{123} < X_4 \tilde{X}_{\{1,2,3\}}) \\
&\qquad + \mathbb{P}(3E_{123} + X_4 > x, E_{123} \tilde{X}_{\{1,2,3\}} < X_4) \\
&\quad = \mathbb{P}\left(\tilde{X}_{\{1,2,3\}} > X_4 > E_{123} > \frac{x - X_4}{3}\right) \\
&\qquad + \mathbb{P}\left(X_4 > \tilde{X}_{\{1,2,3\}} > E_{123} > \frac{x - X_4}{3}\right).
\end{aligned}$$

On one side,

$$\begin{aligned}
&\mathbb{P}\left(\tilde{X}_{\{1,2,3\}} > X_4 > E_{123} > \frac{x - X_4}{3}\right) \\
&\quad = \mathbb{E}\left[\mathbb{P}\left(\tilde{X}_{\{1,2,3\}} > X_4 > E_{123} > \frac{x - X_4}{3} \mid E_{123}\right)\right] \\
&\quad = \int_0^\infty \mathbb{P}\left(\tilde{X}_{\{1,2,3\}} > X_4 > y_1 > \frac{x - X_4}{3}\right) f_{E_{123}}(y_1) dy_1,
\end{aligned}$$

where

$$\begin{aligned}
&\mathbb{P}\left(\tilde{X}_{\{1,2,3\}} > X_4 > y_1 > \frac{x - X_4}{3}\right) \\
&\quad = \mathbb{E}\left[\mathbb{P}\left(\tilde{X}_{\{1,2,3\}} > X_4 > y_1 > \frac{x - X_4}{3} \mid X_4\right)\right] \\
&\quad = \int_0^\infty \mathbb{P}(\tilde{X}_{\{1,2,3\}} > y_2, y_2 > y_1, y_2 > x - 3y_1) f_{X_4}(y_2) dy_2 \\
&\quad = \int_{\max\{y_1, x - 3y_1\}}^\infty \mathbb{P}(\tilde{X}_{\{1,2,3\}} > y_2) f_{X_4}(y_2) dy_2.
\end{aligned}$$

On the other side,

$$\begin{aligned}
&\mathbb{P}\left(X_4 > \tilde{X}_{\{1,2,3\}} > E_{123} > \frac{x - X_4}{3}\right) \\
&\quad = \mathbb{E}\left[\mathbb{P}\left(X_4 > \tilde{X}_{\{1,2,3\}} > E_{123} > \frac{x - X_4}{3} \mid E_{123}\right)\right] \\
&\quad = \int_0^\infty \mathbb{P}\left(X_4 > \tilde{X}_{\{1,2,3\}} > y_1 > \frac{x - X_4}{3}\right) f_{E_{123}}(y_1) dy_1,
\end{aligned}$$

such that

$$\begin{aligned}
&\mathbb{P}\left(X_4 > \tilde{X}_{\{1,2,3\}} > y_1 > \frac{x - X_4}{3}\right) \\
&\quad = \mathbb{E}\left[\mathbb{P}\left(X_4 > \tilde{X}_{\{1,2,3\}} > y_1 > \frac{x - X_4}{3} \mid \tilde{X}_{\{1,2,3\}}\right)\right] \\
&\quad = \int_0^\infty \mathbb{P}\left(X_4 > y_2, y_2 > y_1, y_1 > \frac{x - X_4}{3}\right) f_{\tilde{X}_{\{1,2,3\}}}(y_2) dy_2 \\
&\quad = \int_{y_1}^\infty \mathbb{P}(X_4 > y_2, X_4 > x - 3y_1) f_{\tilde{X}_{\{1,2,3\}}}(y_2) dy_2 \\
&\quad = \int_{y_1}^\infty \mathbb{P}(X_4 > \max\{y_2, x - 3y_1\}) f_{\tilde{X}_{\{1,2,3\}}}(y_2) dy_2.
\end{aligned}$$

We now aim at computing $\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x, X_1 < X_2 = X_3 = X_4)$: we need to take into consideration that

$$X_1 < X_2 = X_3 = X_4 \Leftrightarrow E_1 < \tilde{X}_{1234} < \hat{X}.$$

Then,

$$\begin{aligned}
&\mathbb{P}(X_1 + X_2 + X_3 + X_4 > x, X_1 < X_2 = X_3 = X_4) \\
&\quad = \mathbb{P}(\hat{X} > \tilde{X}_{1234} > E_1 > x - 3\tilde{X}_{1234}) \\
&\quad = \mathbb{E}[\mathbb{P}(\hat{X} > \tilde{X}_{1234} > E_1 > x - 3\tilde{X}_{1234} \mid E_1)] \\
&\quad = \int_0^\infty \mathbb{P}(\hat{X} > \tilde{X}_{1234} > y_1 > x - 3\tilde{X}_{1234}) f_{E_1}(y_1) dy_1,
\end{aligned}$$

where

$$\begin{aligned}
&\mathbb{P}(\hat{X} > \tilde{X}_{1234} > y_1 > x - 3\tilde{X}_{1234}) \\
&\quad = \mathbb{E}[\mathbb{P}(\hat{X} > \tilde{X}_{1234} > y_1 > x - 3\tilde{X}_{1234} \mid \tilde{X}_{1234})] \\
&\quad = \int_0^\infty \mathbb{P}\left(\hat{X} > y_2, y_2 > y_1, y_2 > \frac{x - y_1}{3}\right) f_{\tilde{X}_{1234}}(y_2) dy_2 \\
&\quad = \int_{\max\left\{y_1, \frac{x-y_1}{3}\right\}}^\infty \mathbb{P}(\hat{X} > y_2) f_{\tilde{X}_{1234}}(y_2) dy_2.
\end{aligned}$$

Let us now calculate $\mathbb{P}(X_1 + \ldots + X_4 > x, X_1 = X_2 < X_3 = X_4)$: note that:

$$\begin{aligned}
X_1 = X_2 < X_3 = X_4 \Leftrightarrow &\left(E_{12} < \tilde{X} < \tilde{X}_{34} < \tilde{X}_{234}\right) \bigcup \left(E_{12} < \tilde{X}_{34} < \tilde{X} < \tilde{X}_{234}\right) \\
&\bigcup \left(E_{12} < \tilde{X}_{34} < \tilde{X}_{234} < \tilde{X}\right),
\end{aligned}$$

and we proceed as

$$\begin{aligned}
&\mathbb{P}(X_1 + \ldots + X_4 > x, X_1 = X_2 < X_3 = X_4) \\
&\quad = \mathbb{P}(X_1 + \ldots + X_4 > x, E_{12} < \tilde{X} < \tilde{X}_{34} < \tilde{X}_{234})
\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{P}(X_1+\ldots+X_4>x, E_{12}<\tilde{X}_{34}<\tilde{X}<\tilde{X}_{234}) \\
&+\mathbb{P}(X_1+\ldots+X_4>x, E_{12}<\tilde{X}_{34}<\tilde{X}_{234}<\tilde{X}).
\end{aligned}
$$

We first compute

$$
\begin{aligned}
&\mathbb{P}(X_1+\ldots+X_4>x, X_1=X_2<X_3=X_4) \\
&\quad=\mathbb{P}\left(\tilde{X}_{234}>\tilde{X}_{34}>\tilde{X}>E_{12}>\frac{x-2\tilde{X}_{34}}{2}\right) \\
&\quad=\mathbb{E}\left[\mathbb{P}\left(\tilde{X}_{234}>\tilde{X}_{34}>\tilde{X}>E_{12}>\frac{x-2\tilde{X}_{34}}{2}\mid E_{12}\right)\right] \\
&\quad=\int_0^\infty \mathbb{P}\left(\tilde{X}_{234}>\tilde{X}_{34}>\tilde{X}>y_1>\frac{x-2\tilde{X}_{34}}{2}\right) f_{E_{12}}(y_1)dy_1,
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathbb{P}\left(\tilde{X}_{234}>\tilde{X}_{34}>\tilde{X}>y_1>\frac{x-2\tilde{X}_{34}}{2}\right) \\
&\quad=\mathbb{E}\left[\mathbb{P}\left(\tilde{X}_{234}>\tilde{X}_{34}>\tilde{X}>y_1>\frac{x-2\tilde{X}_{34}}{2}\mid \tilde{X}\right)\right] \\
&\quad=\int_0^\infty \mathbb{P}\left(\tilde{X}_{234}>\tilde{X}_{34}>y_2, y_2>y_1, y_1>\frac{x-2\tilde{X}_{34}}{2}\right) f_{\tilde{X}}(y_2)dy_2 \\
&\quad=\int_{y_1}^\infty \mathbb{P}\left(\tilde{X}_{234}>\tilde{X}_{34}>y_2, y_1>\frac{x-2\tilde{X}_{34}}{2}\right) f_{\tilde{X}}(y_2)dy_2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{P}\left(\tilde{X}_{234}>\tilde{X}_{34}>y_2, y_1>\frac{x-2\tilde{X}_{34}}{2}\right) \\
&\quad=\mathbb{E}\left[\mathbb{P}\left(\tilde{X}_{234}>\tilde{X}_{34}>y_2, y_1>\frac{x-2\tilde{X}_{34}}{2}\mid \tilde{X}_{34}\right)\right] \\
&\quad=\int_0^\infty \mathbb{P}\left(\tilde{X}_{234}>y_3, y_3>\max\left\{y_2, \frac{x-2y_1}{2}\right\}\right) f_{\tilde{X}_{34}}(y_3)dy_3 \\
&\quad=\int_{\max\left\{y_2, \frac{x-2y_1}{2}\right\}}^\infty \mathbb{P}(\tilde{X}_{234}>y_3) f_{\tilde{X}_{34}}(y_3)dy_3.
\end{aligned}
$$

We proceed now with

$$
\begin{aligned}
&\mathbb{P}(X_1+\ldots+X_4>x, E_{12}<\tilde{X}_{34}<\tilde{X}<\tilde{X}_{234}) \\
&\quad=\mathbb{P}\left(\tilde{X}_{234}>\tilde{X}>\tilde{X}_{34}>E_{12}>\frac{x-2\tilde{X}_{34}}{2}\right) \\
&\quad=\mathbb{E}\left[\mathbb{P}\left(\tilde{X}_{234}>\tilde{X}>\tilde{X}_{34}>E_{12}>\frac{x-2\tilde{X}_{34}}{2}\mid E_{12}\right)\right]
\end{aligned}
$$

$$= \int_0^\infty \mathbb{P}\left(\tilde{X}_{234} > \tilde{X} > \tilde{X}_{34} > y_1 > \frac{x - 2\tilde{X}_{34}}{2}\right) f_{E_{12}}(y_1)dy_1,$$

such as

$$\begin{aligned}
&\mathbb{P}\left(\tilde{X}_{234} > \tilde{X} > \tilde{X}_{34} > y_1 > \frac{x - 2\tilde{X}_{34}}{2}\right) \\
&\quad = \mathbb{E}\left[\mathbb{P}\left(\tilde{X}_{234} > \tilde{X} > \tilde{X}_{34} > y_1 > \frac{x - 2\tilde{X}_{34}}{2} \mid \tilde{X}_{34}\right)\right] \\
&\quad = \int_0^\infty \mathbb{P}\left(\tilde{X}_{234} > \tilde{X} > y_2, y_2 > \max\left\{y_1, \frac{x - 2y_1}{2}\right\}\right) f_{\tilde{X}_{34}}(y_2)dy_2 \\
&\quad = \int_{\max\left\{y_1, \frac{x-2y_1}{2}\right\}}^\infty \mathbb{P}(\tilde{X}_{234} > \tilde{X} > y_2) f_{\tilde{X}_{34}}(y_2)dy_2,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}(\tilde{X}_{234} > \tilde{X} > y_2) \\
&\quad = \mathbb{E}\left[\mathbb{P}(\tilde{X}_{234} > \tilde{X} > y_2 \mid \tilde{X})\right] = \int_0^\infty \mathbb{P}(\tilde{X}_{234} > y_3, y_3 > y_2) f_{\tilde{X}}(y_3)dy_3 \\
&\quad = \int_{y_2}^\infty \mathbb{P}(\tilde{X}_{234} > y_3) f_{\tilde{X}}(y_3)dy_3.
\end{aligned}$$

And finally,

$$\begin{aligned}
&\mathbb{P}(X_1 + \ldots + X_4 > x, E_{12} < \tilde{X}_{34} < \tilde{X}_{234} < \tilde{X}) \\
&\quad = \mathbb{P}\left(\tilde{X} > \tilde{X}_{234} > \tilde{X}_{34} > E_{12} > \frac{x - 2\tilde{X}_{34}}{2}\right) \\
&\quad = \mathbb{E}\left[\mathbb{P}\left(\tilde{X} > \tilde{X}_{234} > \tilde{X}_{34} > E_{12} > \frac{x - 2\tilde{X}_{34}}{2} \mid E_{12}\right)\right] \\
&\quad = \int_0^\infty \mathbb{P}\left(\tilde{X} > \tilde{X}_{234} > \tilde{X}_{34} > y_1 > \frac{x - 2\tilde{X}_{34}}{2}\right) f_{E_{12}}(y_1)dy_1,
\end{aligned}$$

where

$$\begin{aligned}
&\mathbb{P}\left(\tilde{X} > \tilde{X}_{234} > \tilde{X}_{34} > y_1 > \frac{x - 2\tilde{X}_{34}}{2}\right) \\
&\quad = \mathbb{E}\left[\mathbb{P}\left(\tilde{X} > \tilde{X}_{234} > \tilde{X}_{34} > y_1 > \frac{x - 2\tilde{X}_{34}}{2} \mid \tilde{X}_{34}\right)\right] \\
&\quad = \int_0^\infty \mathbb{P}\left(\tilde{X} > \tilde{X}_{234} > y_2, y_2 > \max\left\{y_1, \frac{x - 2y_1}{2}\right\}\right) f_{\tilde{X}_{34}}(y_2)dy_2 \\
&\quad = \int_{\max\left\{y_1, \frac{x-2y_1}{2}\right\}}^\infty \mathbb{P}(\tilde{X} > \tilde{X}_{234} > y_2) f_{\tilde{X}_{34}}(y_2)dy_2,
\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(\tilde{X} > \tilde{X}_{234} > y_2) &= \mathbb{E}[\mathbb{P}(\tilde{X} > \tilde{X}_{234} > y_2 \,|\, \tilde{X}_{234})] \\ &= \int_0^\infty \mathbb{P}(\tilde{X} > y_3, y_3 > y_2) f_{\tilde{X}_{234}}(y_3) dy_3 \\ &= \int_{y_2}^\infty \mathbb{P}(\tilde{X} > y_3,) f_{\tilde{X}_{234}}(y_3) dy_3.\end{aligned}$$

We conclude the sketch of the proof calculating $\mathbb{P}(X_1+\ldots+X_4 > x, X_1 = X_2 = X_3 = X_4)$ but this can be computed directly from the generalized case (Equation (4.10)):

$$\mathbb{P}(X_1 + \ldots + X_4 > x, X_1 = X_2 = X_3 = X_4) = \frac{\lambda_4}{4\lambda_1 + 6\lambda_2 + 4\lambda_3 + \lambda_4} e^{-(4\lambda_1+6\lambda_2+4\lambda_3+\lambda_4)x/4}.$$

□

C Simulating Lévy-frailty copulas

C.1 Drift adjustment

While simulating Lévy-frailty copulas using Algorithm 5.4.3, the first-exit times are computed using the canonical construction in the Lévy-frailty environment

$$X_k = \inf\{t > 0 : \Lambda_t \geq \varepsilon_k\}, \quad \varepsilon_k \sim \text{Exp}(1)\,(\text{i.i.d.}), \quad k = 1, \ldots, d,$$

where Λ_t is a Lévy subordinator.

Note that if the subordinator reaches the threshold E_k between two jump times (as it is shown in Figure 8.1), one does not know a priory when exactly the threshold has been reached. However it is possible to compute it.

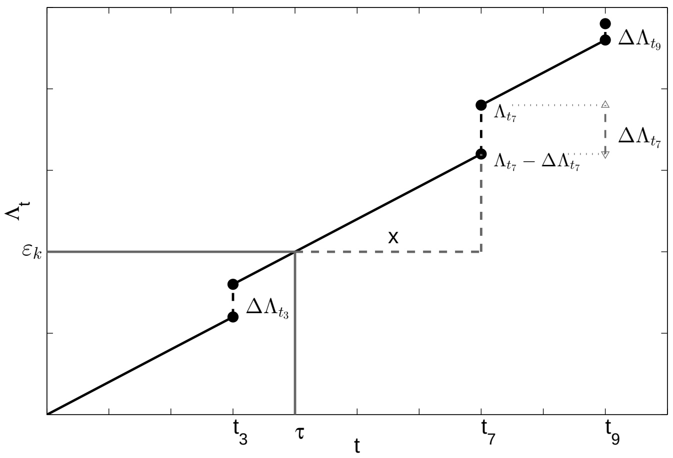


Figure 8.1: Drift adjustment in case the threshold has been reached between two jump times.

Algorithm 5.4.3 simulates a subordinator of a compound Poisson type. Compound Poisson processes are piecewise constant (see [Sato, 1999], Theorem 21.2) and since we add the drift, $\mu > 0$,

$$\Lambda_t = \mu\, t + \sum_{s \leq t} \Delta\Lambda_s \, \mathbb{1}_{\{\Delta\Lambda_s \geq \epsilon\}}, \forall \epsilon > 0,$$

it performs as in Figure 8.1.

The exact time τ when the subordinator hits the threshold ε_k can be computed using the value of the subordinator at jump time $t = t_7$, the jump size at $t = t_7$, and the drift:

$$\tau = t_7 - x, \quad \text{where,} \quad \mu = \frac{\Lambda_{t_7} - \Delta\Lambda_{t_7} - \varepsilon_k}{x} \Leftrightarrow x = \frac{\Lambda_{t_7} - \Delta\Lambda_{t_7} - \varepsilon_k}{\mu}.$$

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Bibliography

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