

Elements of Decision under Uncertainty with Applications

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1 Introduction

Risk is a central dimension of the decision-making environment and many important economic decisions involve risk. In this work the economic theory of the characterization of risk and the modeling of economic agents' responses to it are analyzed. The work is completed with several applications of decision making under risk to different economic problems. An analytically convenient unified framework to incorporate risk in economic modeling is used in this document: Expected Utility Theory.

The presentation is organized as follows: Section 2 is devoted to theory of Expected Utility. The properties of the preference relation defined on the set of risky alternatives that are required for the Expected Utility Theorem are analyzed in that section. Section 3 centers on risk aversion and its measurement. The concepts of certainty equivalent, risk premium, absolute risk aversion and relative risk aversion, and the "more risk averse than" relation are discussed in that section. The last section applies the analyses developed in sections 2 and 3 to a great variety of situations: investment in risky assets and portfolio selection, risk sharing, investment to reduce risk, insurance, taxes and income underreporting, deposit insurance and the value of information. Moreover, it includes nine Exercises. Full solutions of the exercises are provided.

2 Expected utility theory

2.1 The theory of expected utility

Consider that a decision maker faces a choice among a number of risky alternatives (or lotteries, assets or gambles). The possible outcomes of these alternatives are monetary payoffs. There are a finite number N of possible outcomes of those lotteries.¹ Let x_n be the monetary payoff associated to outcome n , with $n = 1, \dots, N$. Consider that $x_1 < x_2 < \dots < x_N$.

The outcome that will occur with each alternative is uncertain. A risky alternative is characterized by the vector of probabilities of the outcomes in that alternative. The decision maker knows the possible outcomes of each alternative and she also knows (or has a subjective opinion about) the probability of each outcome in each alternative.²

A lottery L is simple if it is given by $L = (p_1, p_2, \dots, p_N)$ with $p_n \geq 0$ for all n and $\sum_n p_n = 1$, where p_n is interpreted as the probability of outcome n occurring. A lottery is compound if some (or all) outcomes of that lottery are themselves lotteries. The lottery $(L_1, L_2, \dots, L_H; q_1, q_2, \dots, q_H)$ is a compound lottery that yields the lottery $L_h = (p_1^h, p_2^h, \dots, p_N^h)$ with probability q_h . The reduced lottery of a compound lottery $(L_1, L_2, \dots, L_H; q_1, q_2, \dots, q_H)$ is a simple lottery $L' = (p'_1, p'_2, \dots, p'_N)$ where:

$$p'_n = q_1 p_n^1 + q_2 p_n^2 + \dots + q_H p_n^H$$

for $n = 1, \dots, N$.

¹From the next section on we will allow for the possibility of an infinite number of outcomes.

²For the moment, let us include in the set of risky alternatives those (risk-free) alternatives where the probability of one of the outcomes is equal to 1.

It is assumed that for any lottery or risky alternative, only the reduced lottery is of relevance to the decision maker (note that simple lotteries are already defined in reduced form). For instance, if there are three outcomes, the reduced form of lottery $L = (L_1, L_2; \frac{1}{3}, \frac{2}{3})$, where $L_1 = (\frac{1}{2}, \frac{1}{2}, 0)$ and $L_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, is $(\frac{7}{18}, \frac{7}{18}, \frac{4}{18})$, and the reduced form of lottery $L' = (L_3, L_4; \frac{1}{2}, \frac{1}{2})$ where $L_3 = (\frac{7}{9}, 0, \frac{2}{9})$ and $L_4 = (0, \frac{7}{9}, \frac{2}{9})$ is also $(\frac{7}{18}, \frac{7}{18}, \frac{4}{18})$. Hence, any decision maker is indifferent between the lotteries L and L' .

The decision maker has a preference relation defined on the set \mathcal{L} of lotteries in reduced form. When lottery $L = (p_1, p_2, \dots, p_N)$ is at least as good as lottery $L' = (p'_1, p'_2, \dots, p'_N)$, we write $L \succsim L'$. When lottery $L = (p_1, p_2, \dots, p_N)$ is preferred (indifferent) to lottery $L' = (p'_1, p'_2, \dots, p'_N)$, we write $L \succ L'$ ($L \sim L'$). We have that $L \succ L' \Leftrightarrow L \succsim L'$ and L' is not at least as good as L . Moreover, we have that $L \sim L' \Leftrightarrow L \succsim L'$ and $L' \succsim L$.

Apart from considering that the decision maker cares only about lotteries in reduced form let us assume that the preference relation \succsim possesses the following properties:

- i) Completeness: For any $L, L' \in \mathcal{L}$, we have that $L \succsim L'$ or $L' \succsim L$ (or both),
- ii) Transitivity: For any $L, L', L'' \in \mathcal{L}$, if $L \succsim L'$ and $L' \succsim L''$, then $L \succsim L''$,
- iii) Continuity: For any $L, L', L'' \in \mathcal{L}$ such that $L \succsim L' \succsim L''$, there exists $q \in [0, 1]$ such that $L' \sim qL + (1 - q)L''$, and
- iv) Independence axiom: For any $L, L', L'' \in \mathcal{L}$ and $q \in (0, 1)$ we have:

$$L \succsim L' \text{ if and only if } qL + (1 - q)L'' \succsim qL' + (1 - q)L''.$$

When the preference relation \succsim is complete and transitive we say that it is a rational preference relation. Continuity means that small changes in the probabilities of the outcomes do not change the nature of the ordering between two risky alternatives. When the preference relation \succsim is complete, transitive and continuous there exists a continuous utility function representing \succsim (that is, a function that assigns a number to each lottery in such a way that the ordering of lotteries according to those numbers is the same as the ordering of lotteries given by \succsim).

The independence axiom states that, if we mix in the same way each of two risky alternatives with a third one, then the preference ordering of the two resulting mixtures will be *independent* of the particular third risky alternative used. In the two compound lotteries considered in the definition of the independence axiom, $(L, L''; q, 1 - q)$ and $(L', L''; q, 1 - q)$, the decision maker obtains L'' with probability $1 - q$, but in the first lottery he obtains L with probability q , while in the second lottery he obtains L' with probability q . If $L \succsim L'$ the independence axiom requires that $(L, L''; q, 1 - q) \succsim (L', L''; q, 1 - q)$ for any $L'' \in \mathcal{L}$ and $q \in (0, 1)$. Moreover, if \succsim satisfies the independence axiom, then for all $q \in (0, 1)$ and $L, L', L'' \in \mathcal{L}$ we have:

$$L \succ L' \text{ if and only if } qL + (1 - q)L'' \succ qL' + (1 - q)L'',$$

$$L \sim L' \text{ if and only if } qL + (1 - q)L'' \sim qL' + (1 - q)L''$$

and

$$L \succ L' \text{ and } L'' \succ L''' \Rightarrow qL + (1 - q)L'' \succ qL' + (1 - q)L'''.$$

It may be proved that, as a consequence of the Independence axiom, the preference relation \succsim satisfies the following property of Monotonicity: If outcome x_i is preferred to outcome x_h (if $x_h < x_i$) and lotteries L and L' only differ in the probabilities of those two outcomes then $L \succsim L'$ if and only if the probability of outcome x_i in L is greater or equal than the probability of that outcome in L' .

A utility function U over risky alternatives has an expected utility form if there is an assignment of numbers to the N outcomes ($u_1 = u(x_1), u_2 = u(x_2), \dots, u_N = u(x_N)$) such that for any risky alternative $L = (p_1, p_2, \dots, p_N)$ we have:

$$U(L) = p_1u_1 + p_2u_2 + \dots + p_Nu_N$$

(it is assumed that $u_1 \leq u_2 \leq \dots \leq u_N < \infty$). Hence, with the expected utility form the utility of a risky alternative is the expected value of the utilities u_n of the N outcomes under that alternative. If L^n denotes the lottery that yields outcome n with probability one, then $U(L^n) = u_n$.

Observe that, while the outcomes themselves are objective, their utility is subjective and may differ among decision makers.

It may be shown that U has an expected utility form if and only if it is linear, i.e., if and only if:

$$U\left(\sum_h q_h L_h\right) = \sum_h q_h U(L_h)$$

for any H lotteries $L_h \in \mathcal{L}$, $h = 1, \dots, H$, and probabilities $q_1, \dots, q_H \geq 0$ and $\sum_h q_h = 1$. In this work a utility function over risky alternatives with the expected utility form is going to be called an Expected utility function.³

A preference relation complete, transitive and continuous, that satisfies the independence axiom, is representable by a utility function with the expected utility form (this is the Expected Utility Theorem).⁴ For a utility function U with the expected utility form, that represents those preferences, it is:

$$\begin{aligned} L = (p_1, p_2, \dots, p_N) \succeq L' = (p'_1, p'_2, \dots, p'_N) \\ \Leftrightarrow U(L) = u_1 p_1 + u_2 p_2 + \dots + u_N p_N \geq u_1 p'_1 + u_2 p'_2 + \dots + u_N p'_N = U(L'). \end{aligned}$$

As a consequence of the Expected Utility Theorem a decision maker that cares only about the reduced forms of lotteries and that has preferences \succeq with properties i) to iv) will apply expected utility maximization, with some Expected utility function U that represents \succeq , to choose among lotteries.

An Expected utility function that represents the preference relation \succsim on the set of risky alternatives may be built in the following way: Define $u_1 = 0$ and $u_N = 1$. For any other outcome x_i with $1 < i < N$, calculate β_i , $0 < \beta_i < 1$, such that the decision maker is indifferent between receiving x_i for sure and a lottery where x_N is obtained with probability β_i and x_1 is obtained with probability $1 - \beta_i$. Then define $u_i = \beta_i$. Once the utility

³The first analysis of the expected utility theory is developed in von-Neumann-Morgenstern (1944). However, Bernoulli (1954, translation from 1738) was the first to suggest that a risky alternative should be valued according to the expected utility that it provides.

⁴See Mas-Colell et al. (1995, section 6.B) for a proof of the Expected Utility Theorem.

levels of the outcomes have been obtained calculate the utility of any lottery $L = (p_1, p_2, \dots, p_N)$ by using the following expected utility form:

$$U(L) = p_1 \cdot (0) + p_2 \beta_2 + \dots + p_{N-1} \beta_{N-1} + p_N \cdot (1)$$

Note from this construction of the Expected utility function that once we fix the utility levels for two outcomes or for two lotteries the utility levels of the rest of outcomes and lotteries may be uniquely determined.

If U is an Expected utility function that represents the preference relation \succsim on the set of risky alternatives, then V is another Expected utility function representing \succsim if and only if $V = \alpha + \beta U$, with $\beta > 0$ (that is, if and only if $v_i = \alpha + \beta u_i$ for all $i = 1, 2, \dots, N$). Any strictly increasing transformation Z of U will also represent the same preferences over lotteries as U . Nevertheless, if Z is a non-linear strictly increasing transformation of U then Z will not have an expected utility form. For instance, if $U(L) \geq 0$ for all L then $Z = U^2$ will represent the same preferences as U . In that case, however, $z_i = (u_i)^2$, for all $i = 1, 2, \dots, N$, and for any $L = (p_1, p_2, \dots, p_N)$ it will be $Z(L) = (U(L))^2 = (u_1 p_1 + u_2 p_2 + \dots + u_N p_N)^2 \neq (u_1)^2 p_1 + (u_2)^2 p_2 + \dots + (u_N)^2 p_N = z_1 p_1 + z_2 p_2 + \dots + z_N p_N$.

If U represents \succsim then the combination of a non-linear strictly increasing transformation of u_i , for all $i = 1, 2, \dots, N$, with the use of the expected utility form to obtain the utility of lotteries would not represent \succsim : Consider a situation where there are four possible outcomes with utilities $u_1 = 0$, $u_2 = 1$, $u_3 = 2$ and $u_4 = 3$. The utility of lottery $L = (0, 0, 1, 0)$ under U will be $U(L) = 2$ and the utility of lottery $L' = (0, \frac{1}{2}, 0, \frac{1}{2})$ will be $U(L') = 2$. Hence, the decision maker with preferences \succsim is indifferent between L and L' . Consider, instead, that $z_i = (u_i)^2$, with $i = 1, 2, 3, 4$, and that a function \bar{Z} is constructed from those z_i using the expected utility form. It follows that $\bar{Z}(L) = 0(0) + 0(1) + 1(2^2) + 0(3^2) = 4$ and $\bar{Z}(L') = 0(0) + \frac{1}{2}(1) + 0(2^2) + \frac{1}{2}(3^2) = 5$ (under \bar{Z} the decision maker would not be indifferent between L and L'). Hence, \bar{Z} does not represent the same preferences over lotteries as U .

2.2 Some criticisms of expected utility

The Theory of Expected Utility is very convenient analytically and it provides useful information as to how decision makers choose under risk. However, its plausibility has been challenged in some situations. Among the criticisms of the expected utility theory we can include the Allais Paradox, the frame dependence, the Ellsberg Paradox and ambiguity aversion, the Prospect Theory and loss aversion. We present below a numerical example of the Allais Paradox and an illustration of frame dependence. The rest of criticisms are not analyzed in this work.⁵

As a numerical example of the Allais Paradox consider the following two choices. The first choice is between lotteries L and L' where L pays 1 million € for sure and L' pays 5 million € with probability 0.9 and 0 with probability 0.1. The second choice is between lotteries L'' and L''' where L'' pays 1 million € with probability 0.1 and 0 with probability 0.9 and L''' pays 5 million € with probability 0.09 and 0 with probability 0.91. When facing these choices decision makers, in general, prefer L to L' and L''' to L'' . However, note that:

$$\begin{aligned}L'' &= (L, 0; 0.1, 0.9) \\L''' &= (L', 0; 0.1, 0.9)\end{aligned}$$

As from the independence axiom we have that $L \succ L' \Rightarrow L'' = 0.1L + 0.9(0) \succ L''' = 0.1L' + 0.9(0)$ we conclude that those usual choices are not consistent with expected utility theory.

Kreps (2004, section 15.1) proposes an example of frame dependence where decision makers are faced with two choice problems and two risky alternatives (programs) in each problem. The choice problems are the following (either death or complete recovery are the only possible outcomes):

1- “If nothing is done, the prospective flu epidemic will result in the death of 600 people. You can undertake either of two possible vaccination programs, and doing one precludes doing the other. The first will save 400 people with certainty. The second will save no one with probability 1/3 and 600 with probability 2/3. Which do you recommend?”

⁵Some references for these topics are Allais (1953), Ellsberg (1961), Kahneman and Tversky (1979) and Machina (1987).

2- “As an advisor to the staff of your country’s public health agency, you are informed that a new flu epidemic will hit your country next winter. To fight this epidemic, one of two possible vaccination programs is to be chosen, and undertaking one program precludes attempting the other. In the first program, 200 people will die with certainty. In the second, there is a 2/3 chance that no one will die, and a 1/3 chance that 600 will die. Which do you prefer?”

When facing these choice problems most decision makers choose the first program in choice problem 1 and the second program in choice problem 2. Nevertheless, those choices are inconsistent as the first program in problem 1 is the same as the first program in problem 2 and the second program in problem 1 is the same as the second program in problem 2. The inconsistency of choices is due to the different way to present the programs in the two choice problems. The choices of decision makers may thus depend on the frame used to present the risky alternatives.

3 Risk aversion and its measurement

Consider in this section, and in the following section, that there are infinite possible monetary outcomes of risky alternatives and that these outcomes are represented by a continuous variable x . Let u be the utility of the decision maker defined over possible outcomes. The function u is subjective and may differ among decision makers.

A risky alternative will be characterized by a density function $f(\cdot)$ defined on x , or by the corresponding distribution function $F(\cdot)$ ($F(x) = \int_{-\infty}^x f(t)dt$).

The utility of a risky alternative L , or the utility of the distribution function $F_L(\cdot)$ that characterizes that alternative, is, in application of the Expected Utility Theorem:

$$U(L) = U(F_L) = \int u(x)dF_L(x) = \int u(x)f_L(x)dx,$$

where $u(x)$ is the utility of outcome x for the decision maker.⁶

Consider that the utility function u is increasing and continuous and such that the (expected) utility of any risky alternative considered is finite.⁷ The strength of the expected utility approach rests on the ability to use that approach with many functional forms for u (hence, with many different subjective preferences of decision makers).

3.1 Risk aversion

A decision maker is risk averse if and only if, for every density function $f(\cdot)$:

$$\int u(x)f(x)dx \leq u\left(\int xf(x)dx\right) \quad (1)$$

with strict inequality for some $f(\cdot)$. The decision maker is strictly risk averse if and only if this inequality is strict for every $f(\cdot)$. It is risk neutral if and only if $\int u(x)f(x)dx = u\left(\int xf(x)dx\right)$ for every $f(\cdot)$ and it is strictly risk lover if and only if $\int u(x)f(x)dx > u\left(\int xf(x)dx\right)$ for every $f(\cdot)$.

As $\int xf(x)dx$ is the mean (or expected payoff) of a risky alternative with density function $f(\cdot)$, a strictly risk averse agent always prefers receiving the expected payoff of a risky alternative with certainty (and obtaining utility $u(E(x)) = u\left(\int xf(x)dx\right)$), rather than bearing the risk of that alternative (and obtaining expected utility $E(u(x)) = \int u(x)f(x)dx$).

A decision maker is risk averse, strictly risk averse, risk neutral or strictly risk lover in a particular decision problem if and only if the corresponding condition above is satisfied for any density function $f(\cdot)$ representing a risky alternative in that problem.

⁶ $U(L)$ is, thus, the mathematical expectation of the values of $u(x)$ for risky alternative L .

⁷This latter property of the utility function is a realistic property. Moreover, its plausibility may be argued with reference to the S. Petersburg-Menger Paradox (see Mas-Colell et al., 1995, section 6.c). See also the last paragraph in section 3.2 below.

If u is twice differentiable then the inequality (1) is equivalent to the concavity of u ($u''(x) \leq 0$ for every x , with $u''(x) < 0$ for some x). For a strictly risk averse decision maker, u is strictly concave ($u''(x) < 0$ for every x), for a risk neutral u is linear ($u''(x) = 0$ for every x), and for a strictly risk lover u is strictly convex ($u''(x) > 0$ for every x).

Consider the utility functions $u_1(x) = \ln x$, $u_2(x) = x^2$, $u_3(x) = 20 + 7x$, $u_4(x) = 2 - e^{-x}$ and $u_5(x) = \sqrt{x}$. All these functions are increasing functions as their first derivatives are positive for every x . The signs of the second derivatives are, for every x , $u_1''(x) = -\frac{1}{x^2} < 0$, $u_2''(x) = 2 > 0$, $u_3''(x) = 0$, $u_4''(x) = -e^{-x} < 0$ and $u_5''(x) = -\frac{1}{4}x^{-\frac{3}{2}} < 0$. Therefore, a decision maker with utility function $u_1(x)$, $u_4(x)$ or $u_5(x)$ is strictly risk averse, a decision maker with utility function $u_2(x)$ is strictly risk lover, and a decision maker with utility function $u_3(x)$ is risk neutral.

In the rest of this work it is going to be considered that the decision maker either is risk averse or strictly risk averse for the problem considered.

3.2 Certainty equivalent and risk premium

For a decision maker with utility function u the certainty equivalent of a risky alternative L with density function $f_L(\cdot)$ is $c(f_L, u)$ such that:

$$u(c(f_L, u)) = \int u(x)f_L(x)dx$$

The certainty equivalent of L is the amount of money that leaves the decision maker indifferent between receiving for sure that amount of money and playing the risky alternative L . Hence, the certainty equivalent is the minimum price the decision maker would sell L for when she already has the right to play L (she is the owner of that risky asset) or the minimum amount of money that has to be paid to the decision maker to induce her not to play L when she has the opportunity of playing that risky alternative.

The risk premium of a risky alternative L with density function $f_L(\cdot)$ is $m(f_L, u)$ such that:

$$m(f_L, u) = \int x f_L(x) dx - c(f_L, u)$$

The risk premium of L is the difference between the expected payoff of the risky alternative and the certainty equivalent, or the premium over the certainty equivalent that the risky alternative gives on average to the decision maker.

The previous definitions of certainty equivalent and risk premium are adequate when the initial wealth of the individual is 0. If the decision maker has initial wealth w , the certainty equivalent of a risky alternative L will be $c(f_L, u, w)$ such that:

$$u(w + c(f_L, u, w)) = \int u(w + x) f_L(x) dx$$

and the risk premium of that risky alternative will be $m(f_L, u, w)$ such that:

$$m(f_L, u, w) = \int x f_L(x) dx - c(f_L, u, w).$$

When the decision maker is strictly risk averse it is, for any w , $c(f, u, w) < \int x f(x) dx$ for every $f(\cdot)$ ($\Rightarrow m(f, u, w) > 0$) as a strictly risk averse decision maker prefers receiving the expected payoffs of the risky alternative rather than bearing the risk of that alternative.⁸

If the variance of a risky alternative increases while its mean remains unchanged then, for a strictly risk averse decision maker, the certainty equivalent decreases and the risk premium increases. Consider a decision maker with initial wealth equal to 0 and utility function $u(x) = \sqrt{x}$ (this decision maker is strictly risk averse). The lottery $(36, 16; \frac{1}{2}, \frac{1}{2})$ has

⁸That decision maker would be willing to sell the risky alternative L at any price greater or equal than $c(f_L, u, w)$. If $c(f_L, u, w) = 0$ then the decision maker is indifferent between playing the risky alternative L and giving it away. If $c(f_L, u, w) < 0$ then the decision maker is indifferent between playing the risky alternative L and paying $c(f_L, u, w)$ to avoid facing that alternative (hence, if $c(f_L, u, w) < 0$ the decision maker prefers not to play L rather than to play it). For a risk neutral decision maker it would be $c(f_L, u, w) = \int x f(x) dx$.

mean=26, certainty equivalent (c)=25 ($\sqrt{c} = \frac{1}{2}\sqrt{36} + \frac{1}{2}\sqrt{16} = 5$) and risk premium=26 – 25 = 1. However, the lottery $(48, 4; \frac{1}{2}, \frac{1}{2})$, with the same mean but a higher variance, has certainty equivalent=19.928 and risk premium=26 – 19.928 = 6.072.

Consider, finally, the following gamble: A coin is tossed until it comes up head. The decision maker receives 2^n where n is the number of tosses until a head comes up. The expected value of this gamble is:

$$\frac{1}{2}2 + \frac{1}{4}2^2 + \frac{1}{8}2^3 + \frac{1}{16}2^4 + \dots = 1 + 1 + 1 + 1 + \dots$$

that is, $\sum_{n=1}^{\infty} \frac{1}{2^n}2^n = \infty$. Nevertheless, most risk averse decision makers would pay less than 20 to participate in this gamble with infinite expected value (this is the St. Petersburg Paradox posed early in the eighteenth century by Bernoulli). This behavior of risk averse decision makers occurs as they take into account the (expected) utility, not the expected value of the gamble, and we consider preferences of the decision maker such that the utility of any risky alternative considered always provides a finite value.⁹ As the utility of the gamble is finite there will be a finite maximum price that the decision maker will be willing to pay to participate in the gamble.¹⁰

3.3 Measurement of risk aversion

3.3.1 Absolute risk aversion

⁹Hence, either utility is bounded (it never becomes infinite) and, as a consequence, the (expected) utility of the gamble is finite or utility is unbounded but the expected utility of the relevant gamble finite. For instance, for a decision maker with $u(x) = \ln(x)$ ($\ln(x)$ is not bounded) the utility of the gamble considered is 4 (finite). Nevertheless, the utility of that gamble would become infinite if the decision maker receives, instead, e^{2^n} when there are n tosses until a head comes up. With these latter outcomes an explanation of the behavior of the decision maker requires the consideration of a utility function that guarantees a finite value for the expected utility of the gamble and, therefore, it would not be correct to assume $u(x) = \ln(x)$.

¹⁰See Exercise C in Section 4.1.2 for an analysis of the price that a decision maker is willing to pay for a risky alternative.

Consider that z represents the final wealth of the decision maker. If the decision maker has initial wealth w and faces a risky alternative represented by a continuous variable x her final wealth would be the continuous variable $z = w + x$. Given a (twice-differentiable) utility function u the Arrow-Pratt coefficient of absolute risk aversion at z is defined as $r_A(z, u) = -\frac{u''(z)}{u'(z)}$.¹¹

The Arrow-Pratt coefficient of absolute risk aversion tries to capture the idea that the faster the marginal utility of wealth declines, the more risk averse the individual is. Hence, the degree of risk aversion of the decision maker must be related to the curvature of $u(z)$. One possible measure of the curvature of that utility function is $u''(z)$. Nevertheless, to obtain a measure invariant to positive linear transformations of the utility function it is, instead, used $-\frac{u''(z)}{u'(z)}$, with a $-$ sign to have a positive number (note that if $V = \alpha + \beta U$, with $\beta > 0$, we have $v(z) = \alpha + \beta u(z)$, $v'(z) = \beta u'(z)$, $v''(z) = \beta u''(z)$ and $-\frac{v''(z)}{v'(z)} = -\frac{u''(z)}{u'(z)}$).

Note that $r_A(z, u)$ is a local measure as it depends on the level z of final wealth. In general, $r_A(z, u)$ will change with z . We say that the utility function u exhibits constant absolute risk aversion if $r_A(z, u)$ is independent of z . A decision maker with a utility function that exhibits constant absolute risk aversion is a CARA decision maker. We say that the utility function u exhibits decreasing (increasing) absolute risk aversion if $r_A(z, u)$ is a decreasing (increasing) function of z . A decision maker with a utility function that exhibits decreasing absolute risk aversion is a DARA decision maker.

The general form of a utility function with a coefficient of absolute risk aversion equal to the constant $a > 0$, for every z , is $u(z) = \alpha - \beta e^{-az}$, where $\beta > 0$. For this utility function it is: $u'(z) = a\beta e^{-az} > 0$, $u''(z) = -a^2 \beta e^{-az} < 0$ and $r_A(z, u) = -\frac{-a^2 \beta e^{-az}}{a\beta e^{-az}} = a$. A decision maker with this utility function is a CARA decision maker with coefficient of absolute risk aversion equal to a .

The acceptance or rejection of a risky alternative by a CARA decision maker is independent of her level w of initial wealth: Consider a CARA

¹¹The measures of risk aversion studied in this section were proposed in Arrow (1963) and Pratt (1964).

decision maker with coefficient of absolute risk aversion equal to a and initial wealth w and consider a lottery L given by density function $f(\cdot)$. If the CARA decision maker rejects L her utility will be $\alpha - \beta e^{-aw}$. If she accepts L her expected utility will be:

$$\int (\alpha - \beta e^{-a(w+x)}) f(x) dx = \alpha - \beta \int e^{-a(w+x)} f(x) dx = \alpha - \beta e^{-aw} \int e^{-ax} f(x) dx$$

That CARA decision maker will accept L if:

$$\alpha - \beta e^{-aw} \int e^{-ax} f(x) dx > \alpha - \beta e^{-aw} \Leftrightarrow \int e^{-ax} f(x) dx < 1$$

and this latter condition is independent of w . Hence, the set of risky alternatives acceptable for a CARA decision maker is independent of her initial wealth w . Proceeding in the same way it may be shown that $c(f, u, w)$ and $m(f, u, w)$ do not depend on w for that decision maker. Moreover, the amount of money invested by a CARA decision maker in a risky asset is independent of w (in section 4.1.1 there is an illustration of this result).

A DARA decision maker is willing to accept more risky alternatives as her (initial) wealth w increases. Moreover, this decision maker is willing to invest a greater amount of money in a risky asset when her wealth increases. These results are illustrated in the Examples included in section 4.1.2. Hence, the following properties are equivalent:¹²

- i) The utility function u exhibits decreasing absolute risk aversion
- ii) $c(f, u, w)$ is increasing in w ($\Rightarrow m(f, u, w)$ is decreasing in w)
- iii) If $w_1 > w_2$ then $\int u(w_2 + x) f(x) dx \geq u(w_2) \Rightarrow \int u(w_1 + x) f(x) dx \geq u(w_1)$.

3.3.2 Comparisons across decision makers

Consider two decision makers 1 and 2 with, respective, utility functions $u_1(z)$ and $u_2(z)$. We say that decision maker 2 is more risk averse than decision maker 1 if, at any given initial wealth w , 1 accepts any risky

¹²See Mas-Colell et al. (1995, section 6.C) for a proof of this equivalence and for the equivalence of definitions included in section 3.3.2 below.

alternative that 2 accepts (or, equivalently, 2 rejects any risky alternative that 1 rejects). The following definitions are equivalent:

- i) Decision maker 2 is more risk averse than decision maker 1
- ii) $r_A(z, u_2) \geq r_A(z, u_1)$ for every z
- iii) $u_2(\cdot)$ is a concave transformation of $u_1(\cdot)$
- iv) $c(f, u_2) \leq c(f, u_1)$ for any $f(\cdot)$
- v) $m(f, u_2) \geq m(f, u_1)$ for any $f(\cdot)$
- vi) $\int u_2(w+x)f(x)dx \geq u_2(w+\bar{x}) \Rightarrow \int u_1(w+x)f(x)dx \geq u_1(w+\bar{x})$ for any $f(\cdot)$ and riskless alternative \bar{x} .

There is also a strict version of this equivalence of definitions where the decision maker 2 is strictly more risk averse than the decision maker 1, $u_2(\cdot)$ is a strictly concave transformation of $u_1(\cdot)$ and the inequalities in ii), iv), v) and vi) are strict. These results are illustrated in the Examples included in section 4.1.2.

The more risk averse than relation is a partial ordering of utility functions as it is not complete. Consider two decision makers 1 and 2. Decision maker 1 is CARA with coefficient of absolute risk aversion equal to a (hence, $r_a(z, u_1) = a$). Decision maker 2 has $u_2(z) = \ln(z)$. We have $u_2'(z) = \frac{1}{z}$, $u_2''(z) = -\frac{1}{z^2}$ (therefore, decision maker 2 is strictly risk averse) and $r_a(z, u_2) = \frac{1}{z}$. We have $r_a(z, u_1) > r_a(z, u_2) \Leftrightarrow z > \frac{1}{a}$ and $r_a(z, u_1) < r_a(z, u_2) \Leftrightarrow z < \frac{1}{a}$. As a consequence, we cannot compare decision makers 1 and 2 according to the relation “more risk averse than”.

If a CARA decision maker accepts a risky alternative, then any CARA decision maker with a smaller coefficient of absolute risk aversion accepts that risky alternative. Analogously, if a CARA decision maker rejects a risky alternative, then any CARA decision maker with a greater coefficient of absolute risk aversion rejects that risky alternative. Hence, a CARA decision maker with coefficient of absolute risk aversion equal to a is strictly more risk averse than any CARA decision maker with coefficient of absolute risk aversion smaller than a .

3.3.3 Relative risk aversion

Given a (twice-differentiable) utility function, the coefficient of relative risk aversion at z is defined as $r_R(z, u) = -z \frac{u''(z)}{u'(z)}$. We say that the utility function u exhibits constant relative risk aversion if $r_R(z, u)$ is independent of z . A decision maker with a utility function that exhibits constant relative risk aversion is a CRRA decision maker. We say that the utility function u exhibits decreasing (increasing) relative risk aversion if $r_R(z, u)$ is a decreasing (increasing) function of z . A decision maker with a utility function that exhibits decreasing relative risk aversion is a DRRA decision maker.

Note that: $r_R = zr_A \Rightarrow \frac{dr_R}{dz} = r_A + z \frac{dr_A}{dz}$; then $\frac{dr_R}{dz} \leq 0 \Rightarrow \frac{dr_A}{dz} < 0$, but $\frac{dr_A}{dz} < 0$ does not imply $\frac{dr_R}{dz} \leq 0$. Hence, DRRA \Rightarrow DARA and CRRA \Rightarrow DARA, but DARA does not imply DRRA or CRRA. The property of non-increasing relative risk aversion is stronger than the property of decreasing absolute risk aversion.

As a DRRA or CRRA decision maker is DARA, there will be more risky alternatives that she will consider acceptable as her wealth increases and she will, also, invest a greater amount in a risky alternative. If the decision maker had a utility function that exhibits constant relative risk aversion she would be willing to invest the same proportion of her wealth in a risky asset independently of her level of wealth (hence, she would be willing to invest a greater amount of her wealth in a risky asset as her wealth increases). Moreover, a decision maker with a utility function that exhibits decreasing relative risk aversion is willing to invest a greater proportion of her wealth in a risky asset when her wealth increases. If we consider risky projects whose outcomes are percentage gains or losses of current wealth of the decision maker, the acceptance or rejection of each risky project by a CRRA decision maker is independent of her wealth and there will be more risky projects that a DRRA decision maker will consider acceptable as her wealth changes.

The general form of a utility function with a coefficient of relative risk aversion equal to the constant ρ , where $0 < \rho \neq 1$, for every z , is $u(z) = \alpha + \beta z^{1-\rho}$. For this utility function it is: $u'(z) = (1 - \rho)\beta z^{-\rho}$,

$u''(z) = -\rho(1 - \rho) \beta z^{-\rho-1}$ and $r_R(z, u) = -z \frac{-\rho(1-\rho)\beta z^{-\rho-1}}{(1-\rho)\beta z^{-\rho}} = \rho$. Hence, for a concave increasing function we require $(1 - \rho)\beta > 0 \Rightarrow$ either $\beta > 0$ and $0 < \rho < 1$, or $\beta < 0$ and $\rho > 1$.¹³ A decision maker with this utility function is a CRRA decision maker with coefficient of relative risk aversion equal to ρ .

The general form of a utility function with a coefficient of relative risk aversion equal to 1 for every z is $u(z) = \alpha + \beta \ln z$, where $\beta > 0$. For this utility function it is: $u'(z) = \frac{\beta}{z} > 0$, $u''(z) = -\frac{\beta}{z^2} < 0$ and $r_R(z, u) = -z \frac{-\frac{\beta}{z^2}}{\frac{\beta}{z}} = 1$.¹⁴

For any pair of utility functions u_1 and u_2 , note that it is:

$$r_R(z, u_1) - r_R(z, u_2) = z(r_A(z, u_1) - r_A(z, u_2))$$

Hence, we have that $r_R(z, u_1) > r_R(z, u_2) \Leftrightarrow r_A(z, u_1) > r_A(z, u_2)$.

3.4 Types of utility functions and wealth effects

The utility function $u(z) = \alpha + \beta\sqrt{z}$, with $\beta > 0$, is used to illustrate several results and applications in this work. For this utility function it is $u'(z) = \frac{\beta}{2\sqrt{z}} > 0$, $u''(z) = -\frac{\beta z^{-\frac{3}{2}}}{4} < 0$, $r_A(z, u) = \frac{1}{2z}$ and $r_R(z, u) = \frac{1}{2}$. Hence, a decision maker with utility function $u(z) = \alpha + \beta\sqrt{z}$ has decreasing

¹³Moreover, we have:

$$\begin{aligned} r_R(z, u) = \rho &\Rightarrow -z \frac{u''(z)}{u'(z)} = \rho \Rightarrow \frac{u''(z)}{u'(z)} = -\frac{\rho}{z} \\ &\Rightarrow \ln u'(z) = -\rho \ln z + \ln k = \ln \kappa z^{-\rho} \\ &\Rightarrow u'(z) = \kappa z^{-\rho} \Rightarrow u(z) = \alpha + (1 - \rho)\kappa z^{1-\rho}; \end{aligned}$$

in general, $u(z) = \alpha + \beta z^{1-\rho}$.

¹⁴Note also that:

$$\begin{aligned} r_R(z, u) = -z \frac{u''(z)}{u'(z)} = 1 &\Rightarrow -z u''(z) = u'(z) \Rightarrow z u''(z) + u'(z) = 0 \\ &\Rightarrow z u'(z) = \beta \Rightarrow u'(z) = \frac{\beta}{z} \Rightarrow u(z) = \alpha + \beta \ln z. \end{aligned}$$

absolute risk aversion and constant relative risk aversion (that decision maker is DARA and CRRA). Therefore, for any lottery, the certainty equivalent increases with w and the risk premium decreases with w .

The utility function $u(z) = \alpha + \beta \ln z$, with $\beta > 0$, is also used below to illustrate several results and applications. For that utility function it is $r_A(z, u) = \frac{1}{z}$ and $r_R(z, u) = 1$. Hence, a decision maker with utility function $u(z) = \alpha + \beta \ln z$ has decreasing absolute risk aversion and constant relative risk aversion (that decision maker is DARA and CRRA). Therefore, for any lottery, the certainty equivalent increases with w and the risk premium decreases with w . Note that a decision maker with utility function $u(z) = \alpha + \beta \ln z$ is strictly more risk averse than a decision maker with utility function $u(z) = \alpha + \beta \sqrt{z}$, as $r_A(z, \alpha + \beta \ln z) = \frac{1}{z} > r_A(z, \alpha + \beta \sqrt{z}) = \frac{1}{2z}$ for every z .

4 Applications

This section analyzes some applications of the previous definitions and results. The applications considered are investment in risky assets and portfolio selection, risk sharing, investment to reduce risk, insurance, taxes and income underreporting, deposit insurance and the value of information. The applications are presented by means of Exercises and their solutions. In all situations considered it is assumed that the decision maker's preferences satisfy the axioms of Expected Utility.

To shorten the presentation I do not include in the solutions of the Exercises the second order conditions for the interior solutions obtained. However, it is easy to check that those second order conditions are satisfied in every interior solution.

4.1 Investment in risky assets and portfolio selection

4.1.1 Investment in a risky alternative

A risk averse decision maker with utility function $u(z)$ and initial wealth w wants to decide at time 0 on investment of her wealth. She must decide the amount s to invest in a risky alternative that pays $1 + x_1$ at time 1 with probability p per unit invested and $1 + x_2$ at time 1 with probability $1 - p$ per unit invested, where $x_1 > x_2$ (it could be $x_2 < 0$). The wealth $w - s$ not invested in the risky alternative is invested in a riskless alternative (a bond) that at time 1 pays always $1 + r$ per unit invested. Consider that there is no discounting of the future.

As the decision maker is risk averse, a necessary condition for an strictly positive investment in the risky alternative is $px_1 + (1 - p)x_2 > r$ (that is, the risky investment is actuarially favorable with respect to the riskless alternative as it implies a gain on average over the riskless alternative). Moreover, a necessary condition for an strictly positive investment in the safe, or riskless, alternative is $x_2 < r$. Hence, $x_1 > r > x_2$ is required for strictly positive investments in both alternatives.

To obtain s the decision maker solves:

$$\max_s p \cdot u(s(1 + x_1) + (w - s)(1 + r)) + (1 - p) \cdot u(s(1 + x_2) + (w - s)(1 + r)),$$

subject to $0 \leq s \leq w$. The first order condition for an interior solution is:¹⁵

$$p(x_1 - r) \cdot u'(w + sx_1 + r(w - s)) + (1 - p)(x_2 - r) \cdot u'(w + sx_2 + r(w - s)) = 0.$$

If the solution of this equation is s^* such that $0 < s^* \leq w$ then the decision maker will invest s^* in the risky asset. If the solution of that equation is a value of s such that $s > w$, then the decision maker will invest all her wealth

¹⁵The second order condition is fulfilled as:

$$p(x_1 - r)^2 \cdot u''(w + sx_1 + r(w - s)) + (1 - p)(x_2 - r)^2 \cdot u''(w + sx_2 + r(w - s)) < 0$$

in the risky asset (a corner solution). Note, however, that $s^* = 0$ cannot be a solution when $px_1 + (1 - p)x_2 > r$, as from the first order condition we have that, when $s = 0$:

$$(p(x_1 - r) + (1 - p)(x_2 - r)) \cdot u'(w + rw) > 0$$

Therefore, $px_1 + (1 - p)x_2 > r$ is also a sufficient condition for an strictly positive investment in the risky alternative. If a risky investment is actuarially favorable with respect to the riskless alternative and the decision maker may decide the amount of the risky asset to buy, then a risk averse decision maker will always buy at least a small amount of it.

It has been pointed out in Section 3.3.1 that the amount of money invested by a CARA decision maker in a risky asset is independent of her wealth. For that decision maker it is $u(z) = \alpha - \beta e^{-az}$, where $\beta > 0$, and $u'(z) = a\beta e^{-az} > 0$. For the risky and riskless alternatives that we are considering here we have from the first order condition for an interior solution that:

$$p(x_1 - r)a\beta e^{-a(w+sx_1+r(w-s))} + (1 - p)(x_2 - r)a\beta e^{-a(w+sx_2+r(w-s))} = 0$$

that is:

$$p(x_1 - r)a\beta e^{-asx_1} + (1 - p)(x_2 - r)a\beta e^{-asx_2} = 0$$

From this latter equality we have that s^* does not depend on w .

4.1.2 Exercises

Exercise A: Portfolio selection (I)

A decision maker with utility function $u(z) = \ln(z)$ and initial wealth w wants to decide at time 0 on investment of her wealth. She must decide the amount s to invest in a risky asset that pays $1 + x_1$ at time 1 with probability p per unit invested and $1 + x_2$ at time 1 with probability $1 - p$ per unit invested, where $x_1 > x_2$. The wealth $w - s$ not invested in the risky asset is invested in a government bond that at time 1 pays always $1 + r$ per unit invested. Solve for s in the general case and analyze how s changes with

p and with w (consider that there is no discounting of the future). Obtain s when $p = 0.5$, $r = 0.1$, $x_1 = 0.3$ and $x_2 = -0.06$. If the utility function were $u(z) = \sqrt{z}$, would s be greater than when the utility function is $u(z) = \ln z$? Why?

Solution

From 4.1.1 the first order condition for an interior solution is:

$$\frac{p(x_1 - r)}{w + sx_1 + r(w - s)} + \frac{(1 - p)(x_2 - r)}{w + sx_2 + r(w - s)} = 0$$

and the interior solution is $s^* = w \frac{(1+r)(px_1+(1-p)x_2-r)}{(x_1-r)(r-x_2)} > 0$. The solution will be $s^* = w \frac{(1+r)(px_1+(1-p)x_2-r)}{(x_1-r)(r-x_2)}$ if $\frac{(1+r)(px_1+(1-p)x_2-r)}{(x_1-r)(r-x_2)} \leq 1$ and $s^{**} = w$ (corner solution) if $\frac{(1+r)(px_1+(1-p)x_2-r)}{(x_1-r)(r-x_2)} > 1$.

The interior solution s^* increases with p as:

$$\frac{ds^*}{dp} = w \frac{(1+r)(x_1 - x_2)}{(x_1 - r)(r - x_2)} > 0$$

Moreover, s^* also increases with w (the utility function implies decreasing absolute risk aversion; hence, when w increases the decision maker is willing to accept more risks, i.e., in this case she is willing to invest more in the risky asset, or to keep less money in the riskless asset). However, we know that this utility function implies constant relative risk aversion (equal to 1). Therefore, the decision maker invests the same proportion of her wealth in the risky asset, independently of her level of wealth ($\frac{(1+r)(px_1+(1-p)x_2-r)}{(x_2-r)(r-x_1)}$ of her wealth in this case). Moreover, $\frac{ds^*}{dx_1} > 0$ and $\frac{ds^*}{dx_2} > 0$.

When $p = 0.5$, $r = 0.1$, $x_1 = 0.3$ and $x_2 = -0.06$ it is $s^* = 0.6875w$ (note that it is $px_1 + (1 - p)x_2 > r$ and $x_1 > r > x_2$ as required: see section 4.1.1). The decision maker invests 68.75% of her wealth in the risky asset and 31.25% of her wealth in bonds.

When the utility function is $u(z) = \sqrt{z}$, the decision maker will invest more in the risky asset because a decision maker with $u(z) = \sqrt{z}$ is less risk

averse than a decision maker with $u(z) = \ln z$, as $r_A(z, \ln z) = \frac{1}{z} > \frac{1}{2z} = r_A(z, \sqrt{z})$.

Remark: Note that if, instead, there is a probability p of winning $t_1\%$ at time 1 with the investment in the risky asset and a probability $1 - p$ of winning $t_2\%$ at time 1, where $t_1 > t_2$, we would proceed as in the solution to Exercise A, noting that now $x_1 = \frac{t_1}{100}$ and $x_2 = \frac{t_2}{100}$. \square

Exercise B: Portfolio selection (II)

A decision maker with initial wealth w and utility function $u(z) = \ln(z)$ must decide the amount s she will invest in a lottery, or risky asset, with a probability p of receiving y_1 per unit invested and a probability $1 - p$ of receiving y_2 per unit invested, with $y_1 > y_2$. The wealth not invested in the risky asset remains with the decision maker. Hence, the final wealth will be $sy_1 + w - s$ with probability p and $sy_2 + w - s$ with probability $1 - p$. Solve for s in the general case and analyze how s changes with the parameters of the problem (consider that there is no discounting of the future). Obtain s when $p = 0.5$, $y_1 = 3$ and $y_2 = 0$. Analyze how s depends on w .

Solve again the exercise considering that the utility function is, instead, $u(z) = \sqrt{z}$. Use the coefficients of absolute risk aversion to explain why the decision maker invests more in the risky asset with one utility function than with the other.

Solution

This is a particular case of the problem discussed in section 4.1.1 where $x_1 = y_1 - 1$, $x_2 = y_2 - 1$ and $r = 0$. From that section we have that $p(y_1 - 1) + (1 - p)(y_2 - 1) > 0$ is a necessary condition for an strictly positive investment in the risky asset and $y_1 - 1 > 0 > y_2 - 1$ ($\Leftrightarrow y_1 > 1 > y_2$) is a necessary condition to obtain that a strictly positive amount of wealth remains with the decision maker.

From section 4.1.1 the first order condition for an interior solution is:

$$\frac{p(y_1 - 1)}{w + s(y_1 - 1)} + \frac{(1 - p)(y_2 - 1)}{w + s(y_2 - 1)} = 0$$

and the interior solution is $s^* = \frac{py_1 + (1-p)y_2 - 1}{(y_1 - 1)(1 - y_2)}w > 0$. The solution will be $s^* = \frac{py_1 + (1-p)y_2 - 1}{(y_1 - 1)(1 - y_2)}w$ if $\frac{py_1 + (1-p)y_2 - 1}{(y_1 - 1)(1 - y_2)} \leq 1$ and $s^{**} = w$ (corner solution) if $\frac{py_1 + (1-p)y_2 - 1}{(y_1 - 1)(1 - y_2)} > 1$.

The interior solution s^* increases with p and it also increases with w (the utility function implies decreasing absolute risk aversion; hence, when w increases the decision maker is willing to accept more risks, i.e., in this case she is willing to invest more in the risky asset, or to keep less money in the riskless asset). However, we know that this utility function implies constant relative risk aversion (equal to 1). Therefore, the decision maker invests the same proportion of her wealth in the risky asset, independently of her level of wealth ($\frac{py_1 + (1-p)y_2 - 1}{(y_1 - 1)(1 - y_2)}$ of her wealth in this case). Moreover, $\frac{ds^*}{dy_1} > 0$ and $\frac{ds^*}{dy_2} > 0$.

When $p = 0.5$, $y_1 = 3$ and $y_2 = 0$, it is $s = \frac{w}{4}$. The decision maker invests $\frac{1}{4}$ of her wealth in the risky asset and she does not invest $\frac{3}{4}$ of her wealth.

When $u(z) = \sqrt{z}$, from section 4.1.1 we know that the first order condition for an interior solution is:

$$\frac{p(y_1 - 1)}{2\sqrt{w + s(y_1 - 1)}} + \frac{(1 - p)(y_2 - 1)}{2\sqrt{w + s(y_2 - 1)}} = 0$$

and the interior solution is:

$$s^* = \frac{w(py_1 + 1 - y_2 - 2p + py_2)((py_1 + (1 - p)y_2) - 1)}{(y_1 - 1)(1 - y_2)(-y_2 + 1 + 2py_2 - 2p - p^2y_2 + p^2y_1)}$$

If $p = 0.5$, $y_1 = 3$ and $y_2 = 0$, it is $s = \frac{w}{2}$. We have that s increases with w (the utility function implies decreasing absolute risk aversion; hence, when w increases the decision maker is willing to accept more risks, i.e., in this case she is willing to invest more in the risky asset). However, we know that this utility function implies constant relative risk aversion (equal to $\frac{1}{2}$); hence, the decision maker invests the same proportion of her wealth in the

risky asset, independently of her level of wealth (that proportion is $\frac{1}{2}$ in this case).

The decision maker invests less in the risky asset when $u(z) = \ln z$ than when $u(z) = \sqrt{z}$, because a decision maker with $u(z) = \ln z$ is more risk averse than a decision maker with $u(z) = \sqrt{z}$, as $r_A(z, \ln z) = \frac{1}{z} > \frac{1}{2z} = r_A(z, \sqrt{z})$. For instance, we have obtained that, if $p = 0.5$, $y_1 = 3$ and $y_2 = 0$, it is $s = \frac{w}{4}$ when $u(z) = \ln z$ and $s = \frac{w}{2}$ when $u(z) = \sqrt{z}$. \square

Exercise C: Value of risky assets. Investment in risky assets.

A decision maker has utility function $u(z) = \sqrt{z}$ and wealth $w = 500$.

i) If the decision maker accepts the risky alternative $(100, -100; p, 1 - p)$, which is the minimum value of p ?

ii) If the decision maker owns the risky asset (or lottery) $(100, -100; \frac{2}{3}, \frac{1}{3})$, what is the minimum price he will sell it for (note that this minimum price is the certainty equivalent of the risky asset)?

iii) If the decision maker does not own the risky asset, what is the maximum price that the decision maker is willing to pay for the risky asset $(100, -100; \frac{2}{3}, \frac{1}{3})$?

iv) What is the minimum amount M that has to be paid to the decision maker to induce him to accept the risky alternative $(100, -100; \frac{1}{2}, \frac{1}{2})$?

v) Which is the minimum value of H required to make the risky alternative $(H, -100; p, 1 - p)$ acceptable to the decision maker?

vi) Determine the amount s that this decision maker would invest in a risky asset that pays, per unit invested, 2 with probability $\frac{1}{2} + \gamma$ and 0 with probability $\frac{1}{2} - \gamma$, where $0 < \gamma < \frac{1}{2}$. Explain the variation of s with γ .

vii) How would a change in the level of wealth of the decision maker affect the values obtained in your answers to questions i) to vi)?

Solution

i) It must be $p\sqrt{500+100} + (1-p)\sqrt{500-100} > \sqrt{500} \Rightarrow p > \frac{\sqrt{500}-\sqrt{400}}{\sqrt{600}-\sqrt{400}} = 0.525$.

ii) The decision maker will be willing to sell the risky asset at any price S such that $\sqrt{500+S} \geq \frac{2}{3}\sqrt{500+100} + \frac{1}{3}\sqrt{500-100}$. The minimum selling price will be such that $\sqrt{500+S} = \frac{2}{3}\sqrt{600} + \frac{1}{3}\sqrt{400} \Rightarrow S = 28.84$ (at this price the decision maker is indifferent between selling and not selling the risky asset).¹⁶

iii) The decision maker will be willing to buy the risky asset at any price B such that $\sqrt{500} \leq \frac{2}{3}\sqrt{600-B} + \frac{1}{3}\sqrt{400-B}$. The maximum price he would be willing to pay for the risky asset will be such that $\sqrt{500} = \frac{2}{3}\sqrt{600-B} + \frac{1}{3}\sqrt{400-B} \Rightarrow B = 28.565$.

Note that as $28.565 < 28.84$, an owner of the risky asset with $w = 500$ and $u(z) = \sqrt{z}$ would not be able to sell it to a buyer that also has $w = 500$ and $u(z) = \sqrt{z}$.

iv) As the expected payoff of the risky alternative $(100, -100; \frac{1}{2}, \frac{1}{2})$ is 0 the risk averse decision maker will not accept that risky alternative unless it is paid for it. The solution is obtained from:

$$\frac{1}{2}\sqrt{600+M} + \frac{1}{2}\sqrt{400+M} = \sqrt{500} \Rightarrow M = 5.$$

Remark to iv): note that M is not the risk premium. The certainty equivalent of lottery $(100, -100; \frac{1}{2}, \frac{1}{2})$ is c where:

$$\sqrt{500+c} = \frac{1}{2}\sqrt{600} + \frac{1}{2}\sqrt{400} \Rightarrow c = -5.051.$$

¹⁶Hence, 28.84 is the certainty equivalent of the risky asset (or lottery) $(100, -100; \frac{2}{3}, \frac{1}{3})$ for that decision maker.

As the expected payoff of that risky alternative is 0 the risk premium is $0 - (-5.051) = 5.051$. The amount M refers to a situation where the decision maker has not initially the right to play the risky alternative (of course, that right to play is not attractive in this case).

v) The risky alternative is acceptable if $p\sqrt{w+H} + (1-p)\sqrt{w-100} > \sqrt{w}$. The minimum value of H that makes the risky alternative acceptable is such that $p\sqrt{w+H} + (1-p)\sqrt{w-100} = \sqrt{w} \Rightarrow H = -w + \frac{1}{p^2} (\sqrt{w} - (1-p)\sqrt{w-100})^2$. If, for instance, $w = 500$ and $p = 0.5$ it is $H = 111.18$. Note that, as expected, we have:

$$\frac{dH}{dp} = -\frac{2(\sqrt{w} - \sqrt{w-100})(\sqrt{w} - (1-p)\sqrt{w-100})}{p^3} < 0$$

vi) This is a particular case of the problem discussed in section 4.1.1 where $x_1 = 1$, $x_2 = -1$ and $r = 0$, and, hence, $(\frac{1}{2} + \gamma)x_1 + (\frac{1}{2} - \gamma)x_2 > r$ and $x_1 > r > x_2$, as required. From that section we know that the first order condition for an interior solution is:

$$\frac{\frac{1}{2} + \gamma}{2\sqrt{500+s}} - \frac{\frac{1}{2} - \gamma}{2\sqrt{500-s}} = 0$$

and the interior solution is $s^* = \frac{2000}{1+4\gamma^2}\gamma$.¹⁷ Note that $\gamma > 0 \Rightarrow s^* > 0$ and $\frac{ds^*}{d\gamma} = \frac{2000(1-4\gamma^2)}{(1+4\gamma^2)^2} > 0$, as $\gamma < \frac{1}{2}$ (the probability of obtaining the best outcome increases with γ and the decision maker invests more in the risky alternative as γ increases). Moreover, $\frac{ds^*}{d\gamma} > 0$ and $\gamma < \frac{1}{2} \Rightarrow s^* < \frac{2000}{1+4(\frac{1}{2})^2}(\frac{1}{2}) = 500$. Hence, for $0 < \gamma < \frac{1}{2}$ there is an interior solution ($0 < s^* < 500$) for the problem of maximization of the expected utility of the decision maker.

vii) The utility function $u(z) = \sqrt{z}$ implies decreasing absolute risk aversion (DARA). Hence, when w increases the decision maker is willing to accept more risks.¹⁸

The minimum value of p in part i) decreases with wealth (when w increases the decision maker is willing to accept more risks, i.e., to accept a

¹⁷Note that, from the first order condition we can write $s^* = 500 \frac{(\frac{0.5+\gamma}{0.5-\gamma})^2 - 1}{(\frac{0.5+\gamma}{0.5-\gamma})^2 + 1}$.

¹⁸I refer to risks that imply absolute gains and losses from current wealth.

risky alternative with smaller winning probability in this case). To accept the risky alternative it must be:

$$p\sqrt{w+100} + (1-p)\sqrt{w-100} > \sqrt{w} \Rightarrow p > \frac{\sqrt{w} - \sqrt{w-100}}{\sqrt{w+100} - \sqrt{w-100}}$$

We have that:

$$\begin{aligned} & \frac{d\left(\frac{\sqrt{w}-\sqrt{w-100}}{\sqrt{w+100}-\sqrt{w-100}}\right)}{dw} \\ &= \frac{(\sqrt{w+100}-\sqrt{w})(\sqrt{w}-\sqrt{w-100})}{2\sqrt{w}(w\sqrt{w+100}-100\sqrt{w+100}-w\sqrt{w-100}-100\sqrt{w-100})} < 0 \end{aligned}$$

as it is easy to check that:

$$\left(w\sqrt{w+100}-100\sqrt{w+100}-w\sqrt{w-100}-100\sqrt{w-100}\right) < 0.$$

The minimum selling price in part ii) increases with wealth (when w increases the decision maker is willing to accept more risks, i.e., his willingness to get rid of any risk is smaller and he only accepts to get rid of the risk for a higher selling price). For instance, if $w = 600$ the minimum selling price is such that $\sqrt{600+S} = \frac{2}{3}\sqrt{700} + \frac{1}{3}\sqrt{500} \Rightarrow S = 29.604$ and if $w = 800$ it is $\sqrt{800+S} = \frac{2}{3}\sqrt{900} + \frac{1}{3}\sqrt{700} \Rightarrow S = 30.545$.

The maximum buying price of part iii) increases with wealth (when w increases the decision maker is willing to accept more risks, i.e., in this case he is willing to pay more for the risky asset proposed). For instance, if $w = 600$ the maximum buying price is such that $\sqrt{600} = \frac{2}{3}\sqrt{700-B} + \frac{1}{3}\sqrt{500-B} \Rightarrow B = 29.408$ and if $w = 800$ it is $\sqrt{800} = \frac{2}{3}\sqrt{900-B} + \frac{1}{3}\sqrt{700-B} \Rightarrow B = 30.433$.

The value of M in iv) decreases with wealth (when w increases the decision maker is willing to accept more risks, i.e., in this case he has to be paid less to accept that risky alternative with expected value equal to 0). From iv) it is:

$$\frac{1}{2}\sqrt{w+M+100} + \left(1 - \frac{1}{2}\right)\sqrt{w+M-100} = \sqrt{w}$$

and it may be shown that $\frac{dM}{dw} < 0$.

The value of H in v) decreases with wealth (when w increases the decision maker is willing to accept more risks, i.e., in this case he is willing to accept a risky alternative that pays less in the case of good outcome). From the solution of v) we have:

$$\frac{dH}{dw} = \frac{2(1-p)(\sqrt{w}\sqrt{w-100} - w + 50)}{p^2\sqrt{w-100}\sqrt{w}} < 0$$

as $\sqrt{w}\sqrt{w-100} - (\sqrt{w-50})^2 < 0$.¹⁹

From vi) the first order condition for an interior solution is (see section 4.1.1) $\frac{\frac{1}{2}+\gamma}{2\sqrt{w+s}} - \frac{\frac{1}{2}-\gamma}{2\sqrt{w-s}} = 0$ and the interior solution is $s^* = \frac{4w}{1+4\gamma^2}\gamma$. We have that s increases with wealth (when w increases the decision maker is willing to accept more risks, i.e., in this case he is willing to invest more in the risky asset). As the utility function implies constant relative risk aversion (CRRA) the decision maker invests the same proportion ($\frac{4\gamma}{1+4\gamma^2}$) of his wealth in the risky asset, independently of the level of wealth. \square

4.2 Risk sharing

Exercise D: Risk sharing

Investor A has wealth equal to 30000 and utility function $u(z) = \sqrt{z}$. She has an idea for a project that requires an investment of 30000 and returns 0 with probability 1/2 (the initial outlay is lost) and 110000 with probability 1/2. Answer the following questions (analyze the certainty equivalent corresponding to each situation):

i) Would investor A be willing to invest all her wealth in the project?

ii) If there is another investor B with the same wealth and utility function as investor A , will investors A and B want to share evenly the project (each investor puts up 15000 and they split equally the proceeds of the investment)?

¹⁹In a concave function $u(w)$ it is $\frac{u(w-100)}{u(w-50)} < \frac{u(w-50)}{u(w)}$.

iii) If investor A is considering to share the project with investor B , which is the share γ in the project that A would prefer to offer to B , when that share implies that investor B puts up 30000γ and obtains a proportion γ of the proceeds of the investment and investor A puts up $30000(1 - \gamma)$ and obtains a proportion $1 - \gamma$ of the proceeds of the investment?

iv) Investor A may decide to sell to investor B the right to a proportion γ of the proceeds of the project at a price greater than the proportion γ of the total investment required for the project (that is, she would sell a share γ in the proceeds at a price greater than 30000γ). If A wants to be paid for that γ an amount equal to 95% of the expected value of a proportion γ of the proceedings of the project what is the share γ that B will want to buy? In this case, if there are other investors with the same wealth and utility functions as investors A and B , which is the number of investors that will be approached by investor A to share the project in that way?

Solution

i) The expected gain with the project is positive as:

$$\frac{1}{2}(0) + \frac{1}{2}(110000) - 30000 = 25000 > 0.$$

Nevertheless, the project is risky and risk averse investor A will not invest in the project as (note that $\sqrt{30000}$ is her utility without investing in the project and $\frac{1}{2}\sqrt{0} + \frac{1}{2}\sqrt{110000}$ is her expected utility if she invests in the project):

$$\sqrt{30000} = 173.21 > \frac{1}{2}\sqrt{0} + \frac{1}{2}\sqrt{110000} = 165.83.$$

The certainty equivalent of the risk faced by investor A if he undertakes the project is c_A such that $\sqrt{30000 + c_A} = \frac{1}{2}\sqrt{0} + \frac{1}{2}\sqrt{110000} \Rightarrow c_A = -2500$. Hence, $c_A < 0$ (investor A prefers not to invest in the project).

ii) Investors A and B will want to share evenly the project as (note that $\sqrt{30000}$ is the utility of each investor if they do not invest in the project and $\frac{1}{2}\sqrt{15000} + \frac{1}{2}\sqrt{15000 + 55000}$ is the expected utility of each investor if they share the project):

$$\sqrt{30000} = 173.21 < \frac{1}{2}\sqrt{15000} + \frac{1}{2}\sqrt{15000 + 55000} = 193.53$$

Risk sharing makes the project feasible. The certainty equivalent of the risk faced by each investor when they share evenly the project is c such that $\sqrt{30000 + c} = \frac{1}{2}\sqrt{15000} + \frac{1}{2}\sqrt{15000 + 55000} \Rightarrow c = 7452$. Hence, $c > 0$ for each investor (risk sharing between the two investors makes the investment desirable).

iii) When investor A offers investor B to share a proportion γ of the project the expected utility of A in this case would be $\frac{1}{2}\sqrt{30000\gamma} + \frac{1}{2}\sqrt{30000\gamma + 110000(1 - \gamma)}$ (note that 30000γ is the wealth that remains with A as she only invests $30000(1 - \gamma)$ in the project). Investor A would select γ that solves:

$$\max_{\gamma} \left(\frac{1}{2}\sqrt{30000\gamma} + \frac{1}{2}\sqrt{30000\gamma + 110000(1 - \gamma)} \right)$$

The solution to this maximization problem is $\gamma^* = 0.375$. Investor A will offer investor B a share of 37.5% in the project. Investor B will accept that share of the project as his expected utility when he accepts that share is greater than his expected utility when he rejects that share:

$$\begin{aligned} \frac{1}{2}\sqrt{30000(1 - 0.375)} + \frac{1}{2}\sqrt{30000(1 - 0.375) + 110000(0.375)} &= 190.94 \\ &> \sqrt{30000} = 173.21 \end{aligned}$$

The certainty equivalent of the risk faced by investor A when she retains a proportion $1 - 0.375$ of the project is c_A such that $\sqrt{30000 + c_A} = \frac{1}{2}\sqrt{30000(0.375)} + \frac{1}{2}\sqrt{30000(0.375) + 110000(1 - 0.375)} \Rightarrow c_A = 7813$ (greater than in case ii), as now she offers B the share in the project that she prefers). The certainty equivalent of the risk faced by investor B when he participates in the project with a share 0.375 is c_B such that $\sqrt{30000 + c_B} = \frac{1}{2}\sqrt{30000(1 - 0.375)} + \frac{1}{2}\sqrt{30000(1 - 0.375) + 110000(0.375)} \Rightarrow c_B = 6458$. Hence $c_B > 0$, and B will be willing to take a share equal to 0.375 of the project (although his certainty equivalent was greater in ii)).

iv) The expected value of a share of the project equal to γ is $(\frac{1}{2}(0) + \frac{1}{2}(110000))\gamma = 55000\gamma$. Hence, A wants to be paid $0.95(55000\gamma)$ for a share

γ of the project. At a price equal to $0.95(55000\gamma)$ investor B would solve:²⁰

$$\max_{\gamma} \left(\frac{1}{2} \sqrt{30000 - 0.95(55000\gamma)} + \frac{1}{2} \sqrt{30000 + 110000(\gamma) - 0.95(55000\gamma)} \right)$$

and the solution obtained would be $\gamma^* = 0.0547$. The certainty equivalent of the risk faced by investor B when he participates in the proceedings of the project with a share of 0.0547 and pays for that participation a price equal to $0.95(55000(0.0547)) = 2858.1$ is c_B such that $\sqrt{30000 + c_B} = \frac{1}{2} \sqrt{30000 - 2858.1} + \frac{1}{2} \sqrt{30000 + 110000(0.0547) - 2858.1} \Rightarrow c_B = 75$. As $c_B > 0$ investor B will be willing to take that participation in the project.

When $\gamma = 0.0547$ the expected utility of investor A will be $\frac{1}{2} \sqrt{2858.1} + \frac{1}{2} \sqrt{110000(1 - 0.0547) + 2858.1}$ and the certainty equivalent faced by investor A will be c_A such that $\sqrt{30000 + c_A} = \frac{1}{2} \sqrt{2858.1} + \frac{1}{2} \sqrt{110000(1 - 0.0547) + 2858.1} \Rightarrow c_A = 6162.1$ (lower than in case iii).

If there are other investors with the same wealth and utility functions as investors A and B investor A may also offer that share in the project to several of those investors. Investor A will select the number n of investors that will be invited to share the project (with a share equal to 0.0547 for each investor) solving the following problem:

$$\max_n \left(\frac{1}{2} \sqrt{n(2858.1)} + \frac{1}{2} \sqrt{n(2858.1) + 110000(1 - n(0.0547))} \right)$$

The solution to this maximization problem is $n = 16.54$. As the value of the objective function of investor A is a little bit greater when $n = 17$ than when $n = 16$ investor A will invite 17 investors to share the project. In this case investor A would retain a percentage share in the project equal to $(1 - 17(0.0547))(100) = 7.01\%$.

The certainty equivalent of the risk faced by investor A when she sells a proportion 0.0547 of the project at a price 2858.1 to each of 17 identical investors is c_A such that $\sqrt{30000 + c_A} = \frac{1}{2} \sqrt{17(2858.1)} + \frac{1}{2} \sqrt{17(2858.1) + 110000(1 - (17)0.0547)} \Rightarrow c_A = 22372$. This certainty equivalent is much bigger than the one obtained in part iii). Sharing the

²⁰Note that this objective function is concave with a positive slope at $\gamma = 0$.

project may be very profitable for investor A , even if the project would not be undertaken by investor A when there is not sharing of the project.

Remark: The results in this exercise do not require a level of investment in the project equal to the wealth of each investor. Moreover, those results would be analogous if the utility functions and levels of wealth of the investors considered were similar instead of identical. \square

4.3 Investment to reduce risk and insurance

4.3.1 Investment to reduce risk

Exercise E: Investment to reduce the probability of loss

A decision maker with utility function $u(z) = \sqrt{z}$ has wealth equal to 10000 and runs a risk of a loss of 3600. The probability of this loss is 0.2. The decision maker has the possibility of reducing the probability of loss to 0.1 by investing H in internal security against that risk. When $H = 600$ will the decision maker be willing to make that investment? If the probability of loss were $p(H)$, with $p(0) = 0.2$, $p'(H) < 0$ and $p''(H) > 0$, state the problem that the decision maker would solve to decide on the amount to invest in reducing the probability of loss.

Solution

The investment H will be made when:

$$\begin{aligned} &0.9\sqrt{10000 - H} + 0.1\sqrt{10000 - H - 3600} \\ &> 0.8\sqrt{10000} + 0.2\sqrt{10000 - 3600} = 96 \end{aligned}$$

If the required H were 600, then the investment would not be made as $0.9\sqrt{10000 - 600} + 0.1\sqrt{10000 - 600 - 3600} = 94.874 < 96$. The maximum

investment that the decision maker would be willing to make to reduce the probability of loss to 0.1 would be:

$$0.9\sqrt{10000 - H} + 0.1\sqrt{10000 - H - 3600} = 96 \Rightarrow H^* = 386.17.$$

If the probability of the loss $p(H)$ depends on the amount H invested to reduce that probability then the decision maker will solve:

$$\max_H (1 - p(H))\sqrt{10000 - H} + p(H)\sqrt{10000 - H - 3600}. \quad \square$$

4.3.2 Insurance

A risk averse decision maker with utility function $u(z)$ and wealth w faces a risky situation where he may lose L with probability p . The decision maker may buy insurance (that is, he may invest in a risk shifting contract). He has to choose the proportion α of coverage to buy, with $0 \leq \alpha \leq 1$. When $\alpha = 1$ there will be full coverage of the possible loss, when $0 < \alpha < 1$ there will be partial coverage of that loss, and when $\alpha = 0$ the decision maker does not buy insurance. The price of a level α of coverage is αm (hence, m is the price of full coverage).

The price m established by the insurance company will be such that $m \geq pL$. If $m < pL$ the insurance company would lose money on average.²¹ When $m = pL$ insurance offers fair odds (is actuarially fair).²² When $m > pL$ insurance does not offer fair odds.

With a level α of insurance coverage the final wealth of the decision maker would be $w - \alpha m$ if there were no loss and $w - \alpha m - (1 - \alpha)L$ if the loss occurred. Therefore, his expected final wealth with insurance is:

$$(1 - p)(w - \alpha m) + p(w - \alpha m - (1 - \alpha)L) = w - pL - \alpha(m - pL) \quad (2)$$

²¹We can consider that insurance companies are risk neutral as they insure many different and independent risks, and there is a very high probability that the profits they obtain are very close to the expected profits from insuring all those risks.

²²Insurance will be actuarially fair if there is perfect competition among the insurance companies that offer insurance for the risk faced by the decision maker.

With full insurance the final wealth of the decision maker will be $w - m$ for sure. Without insurance the expected final wealth of the decision maker will be $w - pL$.

To decide on the proportion of coverage to buy, the decision maker will solve:

$$\max_{\alpha} (1-p)u(w - \alpha m) + pu(w - \alpha m - (1-\alpha)L)$$

subject to $0 \leq \alpha \leq 1$. The first order condition for an interior solution is:²³

$$-m(1-p)u'(w - \alpha m) + p(L-m)u'(w - \alpha m - (1-\alpha)L) = 0 \quad (3)$$

When $m = pL$ it is $m = pL \Rightarrow m(1-p) = m - pm = pL - pm = p(L-m)$ and from (3) it follows that:

$$-u'(w - \alpha m) + u'(w - \alpha m - (1-\alpha)L) = 0$$

As $u'(\cdot)$ is strictly decreasing, this equation implies $w - \alpha m = w - L + \alpha(L - m) \Rightarrow \alpha = 1$. Hence, when $m = pL$ a risk averse decision maker will buy full insurance. Note that if $m = pL$ it follows from (2) that for any α the expected final wealth of the decision maker is $w - pL$ (it is the same with any level of insurance as without insurance). Hence, when $m = pL$ insurance reduces the variance of the decision maker's wealth without changing his expected final wealth. As that reduction in dispersion is greater the greater is α ($w - \alpha pL$ decreases when α increases and $w - \alpha pL - (1-\alpha)L$ increases with α) the decision maker decides $\alpha = 1$.

When $m > pL$ insurance reduces the variance of the decision maker's wealth, but it also reduces his expected final wealth as from (2) it follows that:

$$w - pL - \alpha(m - pL) < w - pL$$

Moreover, $m > pL \Rightarrow m(1-p) = m - pm > pL - pm = p(L-m)$. Hence, if there is an interior solution when $m > pL$ it will be, from (3):

$$\Rightarrow u'(w - \alpha m - (1-\alpha)L) = \frac{m(1-p)}{p(L-m)}u'(w - \alpha m) > u'(w - \alpha m)$$

²³Note that the second order condition is satisfied as $u''(z) < 0$.

$$\Rightarrow w - \alpha m - (1 - \alpha)L < w - \alpha m \Rightarrow \alpha < 1.$$

When insurance is not actuarially fair, the decision maker will not buy full insurance, even if his degree of risk aversion is very high. Any risk averse decision maker prefers to retain some risk and increase his expected final wealth by saving in policy premium payment. Moreover, when $m > pL$ it will be obtained $\alpha = 0$ as a corner solution when the first derivative of the maximization problem of the decision maker is negative for all α such that $0 \leq \alpha \leq 1$.

Remark: Consider an alternative setting where the decision maker may buy insurance and the insurance premium is e per unit of loss covered. The decision maker has to decide the number β of units of loss to insure. If $\beta = L$ there will be full insurance and if $\beta < L$ there will be partial insurance. This situation is analogous to the one just considered with $\alpha = \frac{\beta}{L}$ and $m = eL$.

Exercise F: Full and partial insurance

An individual owns a house with value equal to 300000. There is a probability equal to 0.05 that the house will burn down completely in a fire. The individual can insure his house against a loss from this fire. There is only one risk neutral insurance company and the premium that it charges for full insurance is 17000. The individual has other wealth equal to 100000 in non-risky assets.

i) What is the expected profit to the insurance company from this full insurance policy? Is this insurance policy actuarially fair?

ii) If the individual were risk neutral, would he buy this full insurance policy?

iii) If the individual had utility function $u(z) = \sqrt{z}$, would he buy this full insurance policy? Which is the maximum premium that he would be willing to pay for full insurance? If, instead, the individual had utility function

$u(z) = \ln z$, which would be the maximum premium that he would be willing to pay for full insurance?

iv) If the individual had utility function $u(z) = \sqrt{z}$, which would be the maximum premium that he would be willing to pay for a policy of full insurance with a deductible equal to 50000?

v) Consider that the individual could buy partial insurance α , with $0 \leq \alpha \leq 1$, such that if he paid a premium equal to 17000α he would receive a compensation from the insurance company, in the event of fire, equal to 300000α . What level of partial insurance would select an individual with utility function $u(z) = \sqrt{z}$? How does α depend on the level of the other wealth of the individual? If, instead, the individual had utility function $u(z) = \ln z$, would he select a greater level of partial insurance?

vi) Consider that the individual has utility function $u(z) = \sqrt{z}$ and that he can buy partial insurance. What level of the insurance premium m for full insurance of the risk faced by that individual would be selected by the insurance company?

Solution

i) Insurance is actuarially fair when the insurance premium is equal to the expected compensation from the insurance company. As $17000 > 0.05(300000) = 15000$, this insurance policy is not actuarially fair. The expected profit of the insurance company is $17000 - 0.05(300000) = 2000$.

ii) As $0.95(400000) + 0.05(100000) = 385000 > 400000 - 17000 = 383000$, a risk neutral individual would not buy that insurance policy (the expected final wealth is smaller under the insurance policy).²⁴

²⁴A risk neutral individual never buys insurance when $m > pL$ as his expected final wealth decreases if he buys insurance.

iii) As $0.95\sqrt{400000} + 0.05\sqrt{100000} = 616.64$ and $\sqrt{383000} = 618.87$, the individual would buy that full insurance policy. For this risk averse individual the reduction in exposure to risk makes up for the loss in expected final wealth under the insurance policy. The maximum premium m he would be willing to pay for full insurance is:

$$\sqrt{400000 - m} = 0.95\sqrt{400000} + 0.05\sqrt{100000} \Leftrightarrow m = 19750.$$

As the expected loss is $0.05(300000) = 15000$, the individual is willing to pay for full insurance more than the expected loss because he is risk averse (he is willing to reduce his expected final wealth to insure against the loss he faces).

As $0.95 \ln(400000) + 0.05 \ln(100000) = 12.83$ and $\ln(383000) = 12.856$, the individual would also buy that full insurance policy when $u(z) = \ln z$. The maximum premium m that the individual would pay for full insurance in this case would solve:

$$\ln(400000 - m) = 0.95 \ln(400000) + 0.05 \ln(100000) \Leftrightarrow m = 26787.$$

The individual is willing to pay more for full insurance when $u(z) = \ln z$ than when $u(z) = \sqrt{z}$. An individual with $u(z) = \ln z$ is more risk averse than an individual with $u(z) = \sqrt{z}$, as $r_A(z, \ln z) = \frac{1}{z} > \frac{1}{2z} = r_A(z, \sqrt{z})$, and, therefore, the former individual is willing to pay more for full insurance.

Remark to iii): Note that the certainty equivalent of the risk of fire is the negative of the maximum premium that the individual is willing to pay for full insurance. The certainty equivalent is negative as the risk of fire is an undesirable risky alternative for the individual.

iv) The maximum premium m that the individual would pay for full insurance would solve:

$$\begin{aligned} 0.95\sqrt{400000 - m} + 0.05\sqrt{400000 - m - 50000} \\ = 0.95\sqrt{400000} + 0.05\sqrt{100000} \Leftrightarrow m = 17167 \end{aligned}$$

Note that m decreases with the level of the deductible. For instance, the decision maker would pay at most a premium of 15529 if the deductible were 80000.

v) This is a particular case of the analysis presented at the beginning of this Section where $w = 400000$, $L = 300000$, $p = 0.05$ and $m = 17000$. Hence, when $u(z) = \sqrt{z}$ the individual will solve:

$$\max_{\alpha} (0.95\sqrt{400000 - 17000\alpha} + 0.05\sqrt{100000 + 300000\alpha - 17000\alpha})$$

subject to $0 \leq \alpha \leq 1$. The first order condition for an interior solution (first derivative of the objective function with respect to α equal to 0) is in this case:

$$-2.5 \frac{323\sqrt{(1000 + 2830\alpha)} - 283\sqrt{(4000 - 170\alpha)}}{\sqrt{(4000 - 170\alpha)}\sqrt{(1000 + 2830\alpha)}} = 0.$$

Therefore, the individual will decide $\alpha^* = 0.699$.

As $u(z) = \sqrt{z}$ implies decreasing absolute risk aversion (see section 3.4) the individual will buy a smaller percentage of insurance coverage as his level of other wealth increases. For instance if the individual has other wealth equal to 140000 he will decide $\alpha^* = 0,668$.

When $u(z) = \ln z$ the individual will solve:

$$\max_{\alpha} (0.95 \ln(400000 - 17000\alpha) + 0.05 \ln(100000 + 300000\alpha - 17000\alpha))$$

subject to $0 \leq \alpha \leq 1$. The first order condition for an interior solution is:

$$\frac{-4045 + 4811\alpha}{(-400 + 17\alpha)(100 + 283\alpha)} = 0 \Rightarrow \alpha^* = 0.841.$$

The individual buys a greater level of insurance coverage when $u(z) = \ln z$ than when $u(z) = \sqrt{z}$. An individual with $u(z) = \ln z$ is more risk averse than an individual with $u(z) = \sqrt{z}$, and, therefore, buys a greater amount of insurance (prefers to face a less risky situation).

vi) For any m decided by the insurance company the individual would solve:

$$\max_{\alpha} (0.95\sqrt{400000 - \alpha m} + 0.05\sqrt{100000 + 300000\alpha - \alpha m})$$

subject to $0 \leq \alpha \leq 1$. The first order condition for an interior solution (first derivative of the objective function with respect to α equal to 0) is:

$$\frac{7500 - 0.025m}{\sqrt{100000 + 300000\alpha - \alpha m}} - \frac{0.475m}{\sqrt{400000 - \alpha m}} = 0.$$

Therefore, the individual will decide $\alpha^*(m) = \frac{60000m+8.925m^2-9(10^9)}{-22500m-26.925m^2+9(10^{-5})m^3}$. This is the best response function of the individual to any value m decided by the insurance company. Taking into account this response from the individual, the risk neutral insurance company will select m that solves:

$$\max_m \alpha^*(m)(m - 0.05(300000))$$

From the first order condition we obtain that the solution to this problem is $m^* = 20145$. Hence the individual will decide $\alpha^*(m^*) = 0.392$.

The expected utility of the individual in this solution is:

$$0.95\sqrt{400000 - (0.392)(20145)} + 0.05\sqrt{100000 + 300000(0.392) - (0.392)(20145)} = 617.77.$$

This expected utility is greater than the expected utility he would obtain without insurance, as $0.95\sqrt{400000} + 0.05\sqrt{100000} = 616.64$. Nevertheless, if the insurance company increased m in an attempt to approach the expected utility of the individual to 616.64 then the individual would change α according to his best response function $\alpha^*(m)$ and the insurance company would not maximize its profits. \square

4.4 Other applications

4.4.1 Taxes and income underreporting

Exercise G: Taxes and income underreporting

The income of a risk averse individual is taxed at a rate t . He has earned some extra income in an amount equal to y and he is considering not to report that extra income to avoid the corresponding tax payment. If he is caught underreporting his income he will have to pay βt for every unit of income he failed to report (the taxes owed plus a fine), with $\beta > 1$. If he underreports his income, the probability of being caught is q . Answer the following questions, in the context of the expected utility theory:

i) To avoid underreporting of earned income the government is considering two policies: an increase of a 10% in q and an increase of 10% in β . Which policy has more possibilities of reducing underreporting by the individual we have considered?

ii) If the utility function of the individual is $u(z) = \ln(z)$, $t = 0.40$, $y = 10000$ and the individual does not obtain any other income and his initial wealth is 0, obtain the amount of extra income that the individual will fail to report as a function of β and q .

iii) Consider that the individual has initial wealth equal to w . Otherwise, the situation is the same as in ii). How does the extra income that the individual will fail to report depends on w ? Obtain that extra income.

Solution

i) Consider that the individual, in case of income underreporting, does not report his extra income at all. The cases where the individual may underreport a fraction of his extra income will be considered in ii) and iii) below. Under the initial levels of the policies the expected net additional income of the individual, if he decides not to report his extra income, is: $(1 - q)y + q(y - \beta ty) = y - q\beta ty$. Under any of the two new levels of the policies the expected net additional income of the individual, if he decided not to report his extra income, would be: $y - 1.1q\beta ty$. However, when there is a 10% increase in q the net additional income of the individual if he is caught underreporting will be $y - \beta ty$, and when there is a 10% increase in β the net additional income of the individual if he is caught underreporting will be $y - 1.1\beta ty$. We know that a risk averse individual who faces two risky alternatives with the same expected gains, two possible outcomes and the same value of the best outcome selects the alternative with less dispersion of outcomes (i.e., with the higher level of the worst outcome). Hence, the individual in this case prefers a 10% increase in q . As a consequence, the government must select a 10% increase in β as this policy will have more possibilities of reducing underreporting.²⁵ If the individual were risk neutral, the effect on income disclosure of the two policy changes would be the same.

²⁵If the costs of implementation of the two policies were different, the government might

ii) Let s be the amount of extra income that the individual will fail to report. If he is not caught underreporting, his final wealth will be $(y - s)(1 - t) + s = y(1 - t) + ts$. If he is caught underreporting, his final wealth will be $(y - s)(1 - t) + s - \beta ts = y(1 - t) + t(1 - \beta)s$.

In this case the individual will solve:²⁶

$$\max_s (1 - q) \ln(6000 + 0.4s) + q \ln(6000 + 0.4(1 - \beta)s)$$

subject to $s \leq 10000$. From the first order condition the interior solution is $s^* = \frac{15000(1-\beta q)}{\beta-1}$. This interior solution requires $0 \leq \frac{15000(1-\beta q)}{\beta-1} \leq 10000$. Note that $s^* > 0 \Leftrightarrow \beta q < 1$. Moreover, it is $\frac{\partial s^*}{\partial q} < 0$ and $\frac{\partial s^*}{\partial \beta} = 15000 \frac{q-1}{(-1+\beta)^2} < 0$. It follows that if $\frac{15000(1-\beta q)}{\beta-1} \geq 10000$ then the individual will not report any income to the tax administration and that the individual will report all his income when $\beta q \geq 1$.

Let us compare, as in i), the case in which q is increased in 10% and the case in which β is increased in 10%. The interior solutions in those cases will be $s^*(1.1q, \beta) = \frac{15000(1-1.1\beta q)}{\beta-1}$ and $s^*(q, 1.1\beta) = \frac{15000(1-1.1\beta q)}{1.1\beta-1}$. As $s^*(1.1q, \beta) > s^*(q, 1.1\beta)$ the government prefers a 10% increase in β , as in i).

iii) When the individual has initial wealth equal to w , he will solve:

$$\max_s (1 - q) \ln(w + 6000 + 0.4s) + q \ln(w + 6000 + 0.4(1 - \beta)s)$$

From the first order condition the solution is $s^* = \frac{(2.5w+15000)(1-\beta q)}{\beta-1}$. Note that $s^* > 0 \Leftrightarrow \beta q < 1$, as in ii). When $\beta q < 1$ we have $\frac{\partial s^*}{\partial w} > 0$ and the individual will fail to report more extra income when his wealth increases. As this individual has a utility function with decreasing absolute risk aversion,

also take into account the difference in implementation costs when selecting the policy to reduce income underreporting.

²⁶Note that this situation could be considered a particular case of Exercise A discussed in Section 4.1.2 with $w = y(1 - 0.4)$, $x_1 = 0.4$, $x_2 = 0.4(1 - \beta)$, $r = 0$, and the risky alternative (underreporting the income y) pays $1 + x_1$ with probability $1 - q$ per unit invested and $1 + x_2$ with probability q per unit invested (the individual faces a risky gamble that provides a net gain of 0.4 per unit invested with probability $1 - q$ and a net loss of $0.4(\beta - 1)$ per unit invested with probability q). Note that $x_1 > r > x_2$, as required, and that $(1 - q)x_1 + qx_2 > r \Leftrightarrow \beta q < 1$.

he will invest more in the risky gamble (he will underreport more income and will take the risk of paying a greater fine if he is caught underreporting his income) as his wealth increases.

The result on the comparison between an increase of a 10% in q and an increase of 10% in β goes in the same direction as in part ii). \square

4.4.2 Bank solvency and deposit insurance

Exercise H: Bank solvency and deposit insurance

A decision maker with utility function $u(z) = \sqrt{z}$ has a one-year deposit in a bank. The decision maker has the right to receive 20000 from the bank within a year for this deposit. The depositor thinks that a year from now the bank will be solvent with a probability p . The deposit could be withdrawn from the bank at a cost of 300 (it is the cost for early withdrawal). The depositor is insured for γ per cent of the deposit. Considering that there is no discounting of the future, answer the following questions:

- i) Study the decision to withdraw the deposit as a function of γ if $p = 0.95$.
- ii) Study the decision to withdraw the deposit as a function of p if $\gamma = 0.8$.

Solution

- i) The decision maker will withdraw the deposit if:

$$\begin{aligned} \sqrt{20000 - 300} &> 0.95\sqrt{20000} + 0.05\sqrt{\gamma 20000} \\ \Leftrightarrow 140.36 &> 134.35 + 7.0711\sqrt{\gamma} \Leftrightarrow \gamma < 0.7224, \end{aligned}$$

and he will maintain the deposit if $\gamma \geq 0.7224$.

- ii) The decision maker will withdraw the deposit if:

$$\sqrt{20000 - 300} > p\sqrt{20000} + (1 - p)\sqrt{(0.8)20000}$$

$$\Leftrightarrow 140.36 > 126.49 + 14.93p \Leftrightarrow p < 0.929,$$

and he will maintain the deposit if $p \geq 0.929$. \square

4.4.3 The value of information

Exercise I: The value of information

A risk averse decision maker with utility function $u(z) = \sqrt{z}$ and wealth equal to 500 has to decide whether to invest or not to invest in a project. There is uncertainty about the future economic situation. It can be better than today (state E_1), the same as today (state E_2) or worst than today (state E_3). The net gain (or loss) with the investment is 300 if E_1 , 100 if E_2 and -200 if E_3 . The decision maker believes that the probabilities of the states are: $\Pr(E_1) = 0.1$, $\Pr(E_2) = 0.5$ and $\Pr(E_3) = 0.4$. Considering that there is no discounting of the future, answer the following questions:

i) Will this decision maker invest in the project?

ii) What is the maximum amount that this decision maker would be willing to pay for a complete information service that eliminates all uncertainty about the future economic situation (an information that informs about the state that will occur)?

iii) What is the maximum amount that this decision maker would be willing to pay for an incomplete information service that only points out if E_2 will occur or if E_2 will not occur (in this latter case E_1 or E_3 may occur)?

Solution

i) The decision maker will not invest in the project as the expected utility when the investment is made is smaller than the expected utility when the investment is not made:

$$0.1\sqrt{500 + 300} + 0.5\sqrt{500 + 100} + 0.4\sqrt{500 - 200} = 22.004 < \sqrt{500} = 22.361$$

Hence, his expected utility without information will be 22.361.

ii) With complete information the decision maker will know the future economic state. If the message of the complete information service indicates that E_i , with $i = 1, 2, 3$, will occur the future economic state will be E_i . The decision maker does not know which will be the message that the information will provide but, according to his initial beliefs on the probabilities of the states, he thinks that 0.1 is the probability that the message will say that state E_1 will occur, that 0.5 is the probability that the message will say that state E_2 will occur and that 0.4 is the probability that the message will say that state E_3 will occur.

The decision maker will invest in the project if the message of the complete information service indicates that the state will be E_1 or if it indicates that the state will be E_2 . Instead, if the message of the complete information service indicates that the state will be E_3 the decision maker will not invest in the project. Hence, the expected utility with that complete information service is:

$$0.1\sqrt{500 + 300} + 0.5\sqrt{500 + 100} + 0.4\sqrt{500} = 24.02$$

As we know from i) that without information the decision maker will not invest in the project, the maximum amount that this decision maker would be willing to pay for a complete information service will be B such that:

$$0.1\sqrt{500 + 300 - B} + 0.5\sqrt{500 + 100 - B} + 0.4\sqrt{500 - B} = \sqrt{500}$$

that is, $B = 76.537$ (if the decision maker pays this price for the complete information service then his expected utility with that service is the same as his expected utility without information; if the price paid for the complete information service is smaller than 76.537 then his expected utility with that service would be greater than his expected utility without information).

Remark to ii): The certainty equivalent of the, a priori, risky situation corresponding to a complete information service is c such that:

$$\sqrt{500 + c} = 0.1\sqrt{500 + 300} + 0.5\sqrt{500 + 100} + 0.4\sqrt{500} \Rightarrow c = 76.967$$

We have that $c = 76.967 > 76.537$ as in ii) the decision maker does not have access to the (a priori risky) complete information service unless it pays for it (see the definition of certainty equivalent in Section 3.2 and parts ii) to iv) of Exercise C in Section 4.1.2). If in i) the decision maker would have decided to invest in the project then, for the same reason, we would obtain that the maximum amount that the decision maker would have been willing to pay for a complete information service would be less than the difference between the certainty equivalents of the risky situations with complete information and without information.

iii) The decision maker will think, according to his initial beliefs on the probabilities of the states, that 0.5 is the probability that the message will say that state E_2 will occur and that 0.5 (0.1+0.4) is the probability that the message will say that state E_2 will not occur. If the message of the incomplete information service indicates that the state will be E_2 the decision maker will invest in the project.

If the message of the incomplete information service indicates that the state will not be E_2 then E_1 or E_3 may occur. As the initial beliefs on the probabilities of the states indicated that E_3 was four times more likely than E_1 , the decision maker will maintain that proportion in the revised probabilities of those two states when he receives the message indicating that E_2 will not occur (this message says nothing to induce the decision maker to change the relative probabilities between E_1 and E_3). As a consequence, the revised probabilities of the the decision maker on the states will be: $\Pr(E_1) = \frac{0.1}{0.1+0.4} = 0.2$, $\Pr(E_2) = 0$ and $\Pr(E_3) = \frac{0.4}{0.1+0.4} = 0.8$. With these revised probabilities the decision maker will not invest in the project as the expected utility when the investment is made is smaller than the expected utility when the investment is not made:

$$0.2\sqrt{500 + 300} + 0\sqrt{500 + 100} + 0.8\sqrt{500 - 200} = 19.513 < \sqrt{500} = 22.361$$

Hence, the expected utility of the decision maker with this incomplete information service is:

$$0.5\sqrt{500 + 100} + (0.1 + 0.4)\sqrt{500} = 23.428$$

The maximum amount that this decision maker would be willing to pay for that incomplete information service will be B such that:

$$0.5\sqrt{500 + 100 - B} + (0.1 + 0.4)\sqrt{500 - B} = \sqrt{500}$$

that is, $B = 48.75$. As expected, the decision maker is willing to pay less for this incomplete information service than for a complete information service.

□

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