

A formalism based on moments for classical and quantum cosmology

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Abstract. We compare the classical and quantum description of systems with one degree of freedom with an effective formalism based on statistical moments. Within this formalism the similarities and differences between the classical and quantum evolution of an initial probability distribution are made apparent. We revisit the application of this formalism to a simple cosmological model: the homogeneous and isotropic universe with a massless scalar field as matter content and with positive cosmological constant.

1. Introduction

The classical limit of a quantum system is a very subtle concept and much has been written about it. One usually has in mind that a semiclassical state, that is, a state that behaves almost classically, is given by a coherent state whose centroid follows a classical trajectory in phase space. In addition, this state is usually requested to be peaked, that is, to have small fluctuations both in position and momentum.

In this talk we will follow an idea pushed forward by Ballentine, Yang and Zibin [1] that the classical limit of a quantum system is not a single classical trajectory but an ensemble of classical trajectories described by a probability distribution on phase space. As it is well known, the evolution of such a distribution will be given by Liouville equation. We will see that indeed a system can behave almost classically even if the centroid of its wave function does not follow a classical orbit.

Another reason to consider classical probability distributions is that even if it is ideally possible to have exact initial conditions of the classical system under analysis (which would be given by a point on the phase space), in practice there are always errors in these initial data. These errors can then be described as a probability distribution. This might be of particular importance in quantum cosmology because, aside from multiverse theories, there is only one realization of the universe and if we were able to obtain, let us say the fluctuations of the volume of the universe, we would need to know whether these fluctuations are purely of quantum origin or they are due to experimental errors.

Therefore a question arises: how do we know if a dynamically evolving system (described by a probability distribution) is following its corresponding classical or quantum equations of motions?

In order to shed some light on this issue we will make use of a formalism based on statistical moments. Instead of using directly the probability distribution itself, which is not observable, we will decompose it into an infinite set of moments and study the evolution of these observable



quantities directly. We will do so both for quantum and classical probability distributions and will point out their similarities and differences. In particular, special Hamiltonians regarding their classical and quantum behavior will be analyzed. Finally, applications of this formalism to a particular cosmological models will be considered.

2. General formalism

In the next subsection the general formalism for quantum moments as developed in [2, 3] will be briefly summarized. The classical counterpart of this formalism was presented in [4, 5] and will be reviewed in Subsection 2.2.

2.1. Quantum moments

Let us assume a quantum system with one degree of freedom described by the basic operators (\hat{q}, \hat{p}) and Hamiltonian operator $\hat{H}(\hat{q}, \hat{p})$. The observable information of this system is extracted through expectation values. Therefore, we define the following infinite set of moments

$$G^{a,b} := \langle (\hat{p} - p)^a (\hat{q} - q)^b \rangle_{\text{Weyl}}, \quad (1)$$

with $p := \langle \hat{p} \rangle$ and $q := \langle \hat{q} \rangle$ for all nonnegative integers a and b . The subscript Weyl stands for completely symmetric ordering, which makes all moments to be real. The sum of the two indices $(a + b)$ of a given moment $G^{a,b}$ will be referred as its order.

The complete information of the state is contained in this infinite set of moments. The evolution is given by the following effective Hamiltonian that is defined as the expectation value of the Hamiltonian operator and then Taylor expanded around the position of the centroid (q, p) :

$$H_Q(q, p, G^{a,b}) := \langle \hat{H}(\hat{q}, \hat{p}) \rangle = H(q, p) + \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \frac{1}{a!b!} \frac{\partial^{a+b} H}{\partial p^a \partial q^b} G^{a,b}. \quad (2)$$

The dynamics generated by this Hamiltonian is completely equivalent to the Schrödinger flow of states. The equations of motion for expectation values (q, p) and moments are obtained by making use of the general relation $\{\langle \hat{X} \rangle, \langle \hat{Y} \rangle\} = -i\hbar \langle [\hat{X}, \hat{Y}] \rangle$ for any two operators \hat{X} and \hat{Y} :

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} + \sum_{a+b \geq 2} \frac{1}{a!b!} \frac{\partial^{a+b+1} H}{\partial p^{a+1} \partial q^b} G^{a,b}, \quad (3)$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} - \sum_{a+b \geq 2} \frac{1}{a!b!} \frac{\partial^{a+b+1} H}{\partial p^a \partial q^{b+1}} G^{a,b}, \quad (4)$$

$$\frac{dG^{a,b}}{dt} = \sum_{c+d \geq 2} \frac{1}{c!d!} \frac{\partial^{c+d} H}{\partial p^c \partial q^d} \{G^{a,b}, G^{c,d}\}. \quad (5)$$

As can be seen, generically this infinite set of equations are highly coupled. In particular there are moments in the equations of motion for the position q and for the momentum p . [Note that without this moment terms Eqs. (3-4) are the usual Hamilton equations which describe the classical orbits on phase space.] This is the quantum back-reaction which prevents the centroid (q, p) from following a classical orbit on phase space.

Due to the infinite number of equations present in the system (3-5), usually a truncation is necessary for practical purposes. In fact this formalism is good as long as the state remains peaked so that the high-order moments are not too large and the commented truncation is valid.

2.2. Classical moments

Let us assume a classical system with one degree of freedom with Poisson brackets $\{\tilde{q}, \tilde{p}\}_c = 1$ and Hamiltonian $H(\tilde{q}, \tilde{p})$. A classical ensemble is then described by a probability distribution $\rho(\tilde{q}, \tilde{p}, t)$. The evolution of this distribution is given by the Liouville equation:

$$\frac{\partial \rho}{\partial t} = -\{\rho, H(\tilde{q}, \tilde{p})\}_c. \quad (6)$$

With the probability distribution $\rho(\tilde{q}, \tilde{p}, t)$ at hand, it is possible to define the expectation value for any function f on phase space as follows:

$$\langle f(\tilde{q}, \tilde{p}) \rangle_c := \int d\tilde{q} d\tilde{p} f(\tilde{q}, \tilde{p}) \rho(\tilde{q}, \tilde{p}, t). \quad (7)$$

In this way, one can define the classical moments in complete analogy to their quantum counterparts:

$$C^{a,b} := \langle (\tilde{p} - p)^a (\tilde{q} - q)^b \rangle_c, \quad (8)$$

with $p := \langle \tilde{p} \rangle_c$ and $q := \langle \tilde{q} \rangle_c$.

As in the quantum case, the effective Hamiltonian is defined as the expectation value of the Hamiltonian and it is Taylor expanded around the position of the centroid (q, p) :

$$H_C(q, p, C^{a,b}) := \langle H(\tilde{q}, \tilde{p}) \rangle_c = H(q, p) + \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \frac{1}{a!b!} \frac{\partial^{a+b} H}{\partial p^a \partial q^b} C^{a,b}. \quad (9)$$

The evolution equations for the moments can then be obtained by making use of the relation $\{\langle f \rangle_c, \langle g \rangle_c\} = -i\hbar \langle \{f, g\}_c \rangle_c$:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} + \sum_{a+b \geq 2} \frac{1}{a!b!} \frac{\partial^{a+b+1} H}{\partial p^{a+1} \partial q^b} C^{a,b}, \quad (10)$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} - \sum_{a+b \geq 2} \frac{1}{a!b!} \frac{\partial^{a+b+1} H}{\partial p^a \partial q^{b+1}} C^{a,b}, \quad (11)$$

$$\frac{dC^{a,b}}{dt} = \sum_{c+d \geq 2} \frac{1}{c!d!} \frac{\partial^{c+d} H}{\partial p^c \partial q^d} \{C^{a,b}, C^{c,d}\}. \quad (12)$$

The only difference between the quantum (3-5) and the classical (10-12) set of equations is contained in the Poisson brackets between moments. The brackets between any two quantum moments is given by,

$$\{G^{a,b}, G^{c,d}\} = a d G^{a-1,b} G^{c,d-1} - b c G^{a,b-1} G^{c-1,d} + \sum_{n \in 2\mathbb{N}+1} \left(\frac{i\hbar}{2}\right)^{n-1} K_{abcd}^n G^{a+c-n, b+d-n}, \quad (13)$$

with constants K_{abcd}^n , whereas for classical moments,

$$\{C^{a,b}, C^{c,d}\} = a d C^{a-1,b} C^{c,d-1} - b c C^{a,b-1} C^{c-1,d} + (b c - a d) C^{a+c-1, b+d-1}. \quad (14)$$

In fact, this last equation is obtained from the previous one just by taking $\hbar = 0$.

Hence, it is possible to define two kind of quantum effects. On the one hand, there are *distributional* effects due to the fact that all moments can not be vanishing since this would violate the Heisenberg uncertainty relation. These effects are also present in the classical setting. On the other hand, there are *non-commutativity* or *purely-quantum* effects, which only appear in the quantum equations of motion as explicit \hbar^{2m} factors. The origin of such terms lies in the non-commutativity of the basic operators. They first appear at third order and thus the evolution of the quantum and classical system is completely equivalent when truncating at second order.

3. Special Hamiltonians

In this section we will analyze two classes of Hamiltonians that have very peculiar properties regarding the classical and quantum dynamics they generate.

3.1. Harmonic Hamiltonians

We define a harmonic Hamiltonian as being at most quadratic in both q and p . For such kind of Hamiltonians, the equations of motion (3-5) read

$$\begin{aligned}\frac{dq}{dt} &= \frac{\partial H}{\partial p}, \\ \frac{dp}{dt} &= -\frac{\partial H}{\partial q}, \\ \frac{dG^{a,b}}{dt} &= b\frac{\partial^2 H}{\partial p^2}G^{a+1,b-1} + (b-a)\frac{\partial^2 H}{\partial p\partial q}G^{a,b} - a\frac{\partial^2 H}{\partial q^2}G^{a-1,b+1}.\end{aligned}$$

In this case all orders decouple. In particular there is no back reaction of the moments on the expectation values and therefore q and p follow a classical orbit on phase space. In addition, everything is independent of \hbar and thus there is no purely-quantum effects. Therefore, the classical (distributional) evolution is exactly the same as the quantum one. This means that for harmonic Hamiltonians the classical and quantum evolutions are completely indistinguishable.

3.2. Linear Hamiltonians

Let us now analyze the case of a Hamiltonian that is linear in one of the basic variables, for instance on the position q ,

$$H = q \varphi(p) + \xi(p)$$

with generic functions φ and ξ . In such a case there is indeed back reaction of the moments on the expectation values and the centroid does not follow a classical orbit not in the quantum neither in the classical setting. Nevertheless, the set $(q, p, G^{a,0}, G^{a,1})$, for all a , forms a closed (infinite) subsystem of equations with no \hbar terms. Therefore, the departure of the centroid from the classical trajectory is uniquely due to distributional effects since the classical set $(q, p, C^{a,0}, C^{a,1})$ will follow exactly the same evolution as its quantum counterpart. Hence, in order to find purely-quantum effects in this system, the evolution of other moments must be checked.

4. Application to a cosmological model

In this section we will revisit the homogeneous and isotropic cosmological model with a positive cosmological constant Λ and filled with a scalar matter field ϕ studied in Ref. [6]. In that reference, the quantum evolution of this model was considered. Here, the classical distributional evolution will be studied and compared with the quantum one.

The Friedmann equation of the model under consideration is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{4\pi G}{3} \frac{p_\phi^2}{a^6} + \Lambda. \quad (15)$$

The canonical gravitational variables are defined as $q := 2a^3/3$ and $p := \dot{a}/a$, which are essentially the volume of the universe and the Hubble parameter. The theory can then be deparameterized by using the scalar field ϕ as an internal time and solving the above equation

for the conjugate momentum of the scalar field p_ϕ . Then this latter variable turns out to be the physical Hamiltonian for the system:

$$H := p_\phi = \frac{3}{2}q\sqrt{p^2 - \Lambda}. \quad (16)$$

This Hamiltonian is linear in the position variable q and thus all the properties commented in Subsection 3.2 apply. The limit of a vanishing cosmological constant ($\Lambda = 0$) leads to a harmonic Hamiltonian which is linear in both position and momentum variables. In this latter case, as commented in Subsection 3.1, the classical (distributional) and the quantum evolutions are completely equivalent.

Before further analyzing the distributional evolution of this system, let us briefly comment the behavior of the classical phase space orbits. Starting from finite values $q = q_0$ and $p = p_0$ at a time $\phi = \phi_0$, the position q increases reaching $q = \infty$, whereas p decreases up to $p = \Lambda^{1/2}$, at a finite value $\phi = \phi_{div}$.

4.1. Results: the harmonic ($\Lambda = 0$) case

As always happens when dealing with a harmonic Hamiltonian, in this case equations at different orders decouple. There is thus no quantum back-reaction on the classical orbits, and the solution at any order reads,

$$q(\phi) = q_0 \exp\left[\frac{3}{2}(\phi - \phi_0)\right], \quad (17)$$

$$p(\phi) = p_0 \exp\left[-\frac{3}{2}(\phi - \phi_0)\right], \quad (18)$$

$$G^{a,b}(\phi) = G^{a,b}(\phi_0) \exp\left[\frac{3}{2}(b-a)(\phi - \phi_0)\right]. \quad (19)$$

As can be seen in this solution, a given moment $G^{a,b}$ increases (decreases) exponentially if $b > a$ ($b < a$), whereas moments of the form $G^{a,a}$ are kept constant under evolution. In addition the combinations $\sqrt[b]{|G^{0,b}|}/q$ and $\sqrt[a]{|G^{a,0}|}/p$ are also constants of motion. Classical moments $C^{a,b}$ have exactly the same behavior as their quantum counterparts.

4.2. Results: the anharmonic ($\Lambda > 0$) case

In this ($\Lambda > 0$) case, the Hamiltonian is no longer harmonic but it is still linear in the position q . Therefore, as commented in Subsection 3.2, $q(\phi)$ and $p(\phi)$ have exactly the same evolution under the classical (distributional) or quantum dynamics. In addition to the expectation values, the following infinite set of moments have also the same classical and quantum behavior. That is, given the same initial distribution for both sectors,

$$G^{n,0}(\phi) = C^{n,0}(\phi) \quad \text{and} \quad G^{n,1}(\phi) = C^{n,1}(\phi), \quad (20)$$

for all integer n and all times ϕ .

In order to extract physical consequences for the rest of the moments, the numerical resolution for an initial unsqueezed Gaussian state has been performed. For such a state, the only non-vanishing moments are those with two even indices $G^{2n,2m}$. The numerical resolution has been done with different cut-offs, from zero up to tenth order, and the convergence of the method with the truncation order has been verified. Furthermore, several values of the cosmological constant have been considered.

The obtained results can be summarized as follows. Except moments of the form $G^{0,2n+1}$, the rest $G^{a,b}$ qualitatively have the same behavior as their classical counterparts $C^{a,b}$. Even so, odd fluctuations of the position $G^{0,2n+1}$ present also asymptotically the same evolution as their corresponding classical moments $C^{0,2n+1}$.

For a quantitative comparison, we define the ratio $r_{a,b} := G^{a,b}/C^{a,b}$. In this way, three different behaviors (independent of the value of Λ) have been observed:

- Moments with two even ($G^{2n,2m}$) and with two odd ($G^{odd,odd}$) indices have a very small ratio $r_{a,b} \approx 1 \pm 10^{-6}$. This kind of quantum moments show thus the smallest departure from their classical counterparts but this difference increases with time.
- Moments of the form $G^{2n+1,2m}$ are smaller than their classical counterparts and therefore $r_{a,b} < 1$. Interestingly this ratio is kept approximately constant under evolution.
- Finally, moments of the form $G^{2n,2m+1}$ are larger than their classical counterparts at the beginning of the evolution ($r_{a,b} > 1$), but this ratio decreases with time.

Note that the last two cases correspond to moments that are initially vanishing. Therefore, as soon as the evolution is started these moments are excited to certain value. But purely-quantum effects affect in an opposite way moments of the form $G^{2n+1,2m}$ and $G^{2n,2m+1}$. While for moments $G^{2n+1,2m}$ excitation happens to a smaller value than their classical counterparts, and thus purely-quantum effects reduce this excitation, the initial value for $G^{2n,2m+1}$ is enhanced by purely-quantum effects.

5. Conclusions

A formalism based on a decomposition of moments of the corresponding probability distributions has been presented for classical and quantum systems with one degree of freedom. This framework is very useful to compare both evolutions since it makes transparent the parallelism and differences between both. Harmonic and linear Hamiltonians have been analyzed and their special properties regarding the classical (distributional) and quantum evolution they generate have been discussed. Finally an application of this formalism to particular cosmological model has been considered.

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