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RATIONING RULES AND STABLE COALITION STRUCTURES

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Rationing rules and stable coalition structures*

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Abstract

We consider a coalition formation model in which agents have the possibility of forming part of several coalitions but are limited to participate in only one of them. Coalitions of agents produce outputs to be distributed among their members according to their aspirations and to a rationing rule prevailing in society. The outcome of such a process is a hedonic game. Using monotonicity and consistency we characterize the continuous rationing rules that induce core-stable hedonic games.

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1 Introduction

Agents form coalitions in different environments such as private clubs, research groups, country alliances, etc. The aim of gathering together is to produce an "output" to be distributed among themselves. Usually agents have the possibility of forming part of several coalitions but are limited to participate in only one of them. In these situations, each agent estimates the payoff that could be received in the hypothetical coalitions in which she may participate, ranking them from higher to lower. In doing so she also ranks coalitions. Hence, each agent will end up with a preference relation over coalitions and the question to be answered is which coalitions will finally form.

The following example illustrates the type of problems that we are dealing with. Consider a call from a governmental institution for funding for research projects and consider that researchers form groups to submit a joint project. The granting of funding depends on the quality of the project submitted and therefore on the composition of the group. Although some researchers may have the possibility of being part of several groups, suppose that participation in only one group is a prerequisite of the call. Typically, each researcher has an "aspiration" according to her contribution to the project[1] and would like to participate in the project which guarantees the highest payoff. Indeed the existence of a predominant rule in society for dividing the funding of the groups determines the estimation of payoffs of each agent and the ranking for same will yield the final submission of the research projects in such call.

This type of coalition formation process involves agents' aspirations, coalitional outputs and a division rule arbitrating in society. The outcome of such a process is a set of coalitions which may or may not be stable. If the resulting structure of coalitions is not stable then there will be at least one coalition blocking. Hence, we believe that, given the agents' aspirations and the feasible coalitional outputs, identifying the division rules that induce stable coalition structures is an essential task.

To characterize the rules that induce stable coalition structures, we define a coalition formation problem linking the literature on rationing problems and the literature on hedonic games.

The literature on rationing problems was initiated by O'Neill (1982). In a

[1] Although researchers tend to overestimate competence and have a high perception of their own contribution, objective measures, such as CVs may limit aspirations.

rationing problem the output is insufficient to meet all agents' aspirations and a rule offers a proposal for division such that every agent receives a non-negative payoff which does not exceed her aspiration.[2]

The literature on hedonic games initiated by Drèze and Greenberg (1980) is based on the idea that each agent's preference relation over coalitions depends on the identities of their members. In these games each agent ranks coalitions. For solving hedonic games the notion of core-stability naturally applies. Informally, a coalition structure (or partition of the set of agents) is blocked by a coalition if its members strictly prefer that coalition to the ones in which they are currently participating in. A coalition structure is stable if there is no blocking coalition. There may be hedonic games which lack of stable coalition structures. Banerjee *et al.* (2001) and Bogomolnaia and Jackson (2002) introduce sufficient conditions to guarantee stability in hedonic games. In a more recent paper, Iehlé (2007) provides a necessary and sufficient condition for the existence of stable coalition structures.

The ingredients of these literatures used to model a coalition formation problem with aspirations are the following: a set of agents with their aspirations which are exogenous and equal across coalitions and a set of feasible coalitions producing outputs. The preferences that agents have over coalitions are dictated by a single rationing rule which distributes each coalitional output among its members. Once payoffs in the feasible coalitions are estimated agents rank coalitions giving rise to hedonic games.

The core question that we address in this paper is what rationing rules induce hedonic games with stable coalition structures.

Our analysis is focussed on continuous rules that satisfy the following two properties: consistency and resource monotonicity. The idea behind consistency is explained as follows: Consider a rationing problem and a distribution of an output given by a rule. Assume that some agents take their payoffs and leave while the situation of the remaining agents is reassessed. This property requires that the remaining agents should receive the same payoffs as they received initially. Resource monotonicity requires that when the output increases, each agent should receive at least as much as she was getting initially.

We find that rules that satisfy consistency and resource monotonicity are the only ones that guarantee stability. To be specific, rationing rules that in-

[2]For an extensive review on this topic see Moulin (2002) and Thomson (2015).

duce stable hedonic games satisfy the common ranking property (Farrell and Scotchmer (1988)). Thus, non-consistent rationing rules such as the Shapley value (Shapley (1953)) or the minimal overlapping rule (Thomson (2003)) do not induce stability. Parametric rules including the constrained equal awards, the constrained equal losses, the Talmud rule (Aumann and Maschler (1985)) the reverse Talmud rule (Chun *et al.* (2001) and the dictatorial rule with priority stand out, among others, as rules that induce stability. But these are not the only rules that generate stability; there are continuous non-parametric rules that also do so.

The paper is inspired by Pycia (2012) who deals with a unified framework of coalition formation problems.[3] This author introduces the property of pairwise alignment, which requires any two agents that share coalitions to order them in the same manner (indifferences are allowed). He proves that, under some mild domain and preference restrictions, this property guarantees stability in hedonic games. Next, he enriches the model by considering that coalitions of agents produce outputs to be distributed among them according to their utilities. In this setting he characterizes the bargaining rules that induce pairwise aligned preference profiles. What distinguishes Pycia's formulation from ours is the following: We do not restrict the domain or the set of coalitions under consideration and we weaken the notion of pairwise alignment. Unlike Pycia we do not postulate a utility function for each agent, but rather an aspiration. Consequently we characterize rationing rules attending to desirable properties that such rules should satisfy.

Our work can also be linked to Barberá *et al.* (2015). These authors consider societies in which each individual is endowed with a productivity level. Coalitions produce the sum of their members' productivity levels. If a coalition is formed, its members decide by majority vote between a meritocratic and egalitarian division of the output. Hence, one coalition may choose meritocracy while another chooses egalitarianism. Accordingly the size, stability and composition of coalitions is analyzed. In our work coalitional outputs do not coincide with the sum of aspirations and the family of preferences that agents may have over coalitions is dictated by a single rule prevailing in society. Both formulations analyze the core-stability of induced hedonic games.

[3] We do not consider many-to-one matching problems.

The paper is organized as follows: Section 2 contains the preliminaries on rationing problems and on hedonic games which give rise to our coalition formation model with aspirations. Section 3 contains the results. Section 4 concludes.

2 Coalition Formation Model with Aspirations

In this section we present the preliminaries of two models extensively analyzed in the literature: rationing problems and hedonic games. By combining the basic notions of these literatures we define a coalition formation model with aspirations and stick to the core-stability notion to solve it.

2.1 Rationing problems

There is an infinite set of potential agents, indexed by the natural number $\mathbb{N}$. Each given rationing problem involves a finite number of agents. Let $\mathcal{N}$ denote the class of non-empty finite subsets of $\mathbb{N}$. Given $N \in \mathcal{N}$ and $i \in N$, let d_i be agent i's aspiration and $d \equiv (d_i)_{i \in N}$ the aspirations vector and let E be the estate to be divided among the agents in N. A *rationing problem* is a pair $(d, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$, such that $\sum_{i \in N} d_i \geq E$. Let $\mathcal{B}^N$ denote the class of all problems with the set of agents N. An *allocation* for $(d, E) \in \mathcal{B}^N$ is a vector $x \in \mathbb{R}^N$ such that it satisfies the non-negativity and aspiration boundedness conditions, *i.e.* $0 \leqq x \leqq d$ and the efficiency condition $\sum_{i \in N} x_i = E$.[4] A *rationing rule* is a mapping defined on $\bigcup_{N \in \mathcal{N}} \mathcal{B}^N$ that associates an aspiration x with each $N \in \mathcal{N}$ and each $(d, E) \in \mathcal{B}^N$. In this paper, the generic notation for a rule is F.

Notation: For any coalition $S \subseteq C$, we define $x_S \equiv (x_i)_{i \in S}$, $F_S(d, E) = (F_i(d, E))_{i \in S}$ and $d_S = (d_i)_{i \in S}$.

In this paper we restrict ourselves to continuous rules. A rule F is continuous if the solution changes only slightly whenever the individual aspiration and the output change slightly.[5].

Next, we introduce the axioms that we use in the characterization of the rationing rules of the paper.

[4] The notation $x \leqq y$ means that for each $i \in N$, $x_i \leq y_i$.

[5] A rule F is continuous if for each sequence of problems (d^k, E^k) of elements de $\mathcal{B}^N$ and each $(d, E) \in \mathcal{B}^N$, if (d^k, E^k) converges to (d, E) then the solution $F(d^k, E^k)$ converges to $F(d, E)$.

- **Resource monotonicity**: When the output increases, each agent is required to receive at least as much as she got initially. Formally,

  For each $N \in \mathcal{N}$, each pair $(d,E),(d,E') \in \mathcal{B}^N$, if $E \leq E'$, then $F(d,E) \leqq F(d,E')$.

- **Consistency**: Consider a problem and an allocation given by rule F. Assume that some agents depart with their payoffs while the situation of the remaining agents is reassessed. It seems desirable that rule F should assign the same payoffs as they received initially. Formally,

  For all $M, N \in \mathcal{N}$ such that $M \subset N$, and all $(d,E) \in \mathcal{B}^N$, if $x = F(d,E)$, then $x_M = F(d_M, \sum_{i \in M} x_i)$.

- **Bilateral consistency**: The previous axiom restricted to two agents reads as follows: Whenever two agents with given aspirations share a given amount, they always share it in the same way irrespective of the other agents present.

  For all $\{i,j\}, N \in \mathcal{N}$ such that $\{i,j\} \subset N$, and all $(d,E) \in \mathcal{B}^N$, if $x = F(d,E)$, then $x_{\{i,j\}} = F(d_{\{i,j\}}, x_i + x_j)$.

2.2 Hedonic games

Given $N \in \mathcal{N}$ let $\mathcal{K} \subseteq 2^N \backslash \emptyset$ be the set of feasible coalitions containing the singletons. Each agent $i \in N$ has a preference relation over the set of coalitions that she belongs to, denoted by $\succsim_i$ so that if $i \in C \cap C'$ and $C \succsim_i C'$, agent i prefers coalition C at least as much as coalition C'. The profile of preferences of agents N over coalitions is denoted by $\succsim_N = (\succsim_i)_{i \in N}$. This is equivalent to a hedonic game denoted by $(\mathcal{K}, \succsim_N)$.[6] Note that indifferences between coalitions are allowed. Let $\mathcal{D}(P) = \bigcup_{N \in \mathcal{N}} \mathcal{B}^N$ be the class of all preference profiles. Since N is finite, $\mathcal{D}(P)$ is also finite. A *coalition structure of* N (or partition) is a set of coalitions $\{C_1, \ldots, C_k\}$ so that the intersection is empty and the union is the entire set of agents. Formally,

[6] For the sake of simplicity we write $(\mathcal{K}, \succsim_N)$ instead of $(N, \mathcal{K}, \succsim_N)$.

Definition 1 *A coalition structure (or partition) of a finite set of agents* $N = \{1, \ldots, n\}$ *is a set* $\{C_1, C_2, \ldots, C_k\}$, *(*$k \leq n$ *is a positive integer) such that*

1. *for any* $j \in \{1, \ldots, k\}$, $C_j \neq \emptyset$,
2. $\cup_{j=1}^{k} C_j = N$, *and*
3. *for any* $j, l \in \{1, \ldots, k\}$ *with* $j \neq l$, $C_j \cap C_l = \emptyset$.

A coalition structure is *blocked* by a coalition if all its members are strictly better off in the new coalition than in the coalition that they are currently in. A coalition structure which admits no blocking coalition is said to be *stable*.

In hedonic games two approaches for overcoming the lack of stability can be distinguished: One, as mentioned in the introduction, is to define sufficient conditions that guarantee stability in the entire class of hedonic games. The other consists of restricting the domain of preference profiles and the set of coalitions so that a certain property is satisfied as Pycia (2012) does. This author introduces the property of pairwise alignment which by itself does not guarantee stability in hedonic games but does so when mild restrictions are introduced.

In the present study we modify this property and define the notion of weakly pairwise aligned preference profiles (WPA) formally as follows:

Definition 2 *A preference profile is weakly pairwise aligned if* $\forall\ i, j \in C \cap C'$,

$$\neg[C \succ_i C' \Longleftrightarrow C' \succ_j C].$$

That is, it cannot happen that one agent ranks coalitions C and C' in one way while the other ranks them the opposite way. [7]

Hereafter the class of preference profiles over coalitions that satisfy weakly pairwise alignment is denoted by $\mathcal{D}(WPA)$.

The lack of stability is generated by the existence of rings in preference profiles which in turn generates cycles among coalition structures.[8] However,

[7] In Pycia's work a hedonic game $(\mathcal{K}, \succsim_N)$ is pairwise aligned if for all $i, j \in C \cap C'$, $C \succsim_i C' \Longleftrightarrow C \succsim_j C'$. This definition implies that if agent i is indifferent between two coalitions so is agent j. However, our definition allows, for instance, agent i to be indifferent between the two coalitions while agent j is indifferent and any other combination.

[8] Note that we use two different notions: rings and cycles. The notion of a ring applies to circularity in preference profiles while that of cycles applies to circularity among coalition structures. A ring induces a cycle among coalition structures but not every cycle in coalition structures is induced by a ring.

the existence of rings in preference profiles does not preclude the existence of stability. Indeed it depends on the "position" of the rings in the preference profile under consideration. This can be formally defined as follows.

Definition 3 *A ring in a preference profile over coalitions is an ordered set of coalitions $\mathcal{C} = (C_1, C_2, ..., C_k)$, $k > 2$, such that*

$$C_i \succ_S C_{i+1} \text{ where } S = C_i \cap C_{i+1} \neq \emptyset \text{ (subscript modulo } k).$$

This condition requires that all agents in the intersection of any two consecutive coalitions strictly prefer C_i to C_{i+1}.[9] We believe this is a natural definition of a ring in preference profiles. The reason is that the transition from one coalition to the next is produced by all agents in the intersection of these coalitions. It may happen that a ring contains coalitions whose intersections are singletons but, in general, it seems anomalous for only one agent to be capable of implementing such a transition unless her fellows at the intersection want to do so. This possibility occurs if a preference profile satisfies the weakly pairwise alignment property as the following remark establishes.

Remark 1 *If a preference profile over coalitions belongs to $\mathcal{D}(WPA)$ then only one agent at the intersection is needed to change from one coalition to the next.*

To conclude this section we present a class of hedonic games introduced by Farrell and Scotchmer (1988) that do not have rings in preference profiles.

Definition 4 *A preference profile $(\mathcal{K}, \succsim_N)$ satisfies the common ranking property if and only if there is an ordering $\succsim$ over the coalitions in $\mathcal{K}$ such that for each $i \in N$ if $S \succsim_i T$ then $S \succsim T$ for all $S, T \in \mathcal{K}$.*

Farrell and Scotchmer (1988) define the above property on the set of $2^N \backslash \emptyset$ coalitions, but there is no harm in defining it on a subset $\mathcal{K} \subseteq 2^n \backslash \emptyset$.

The following hedonic games illustrate some links between weakly pairwise alignment, rings and stability.

Example 1 *(i) A stable hedonic game with rings that satisfies pairwise alignment.*

$$\{134\} \succ_1 \{123\} \succ_1 \{15\} \succ_1 \{1\}$$

[9]There is another definition of rings in which only one agent is required to change from one coalition to the other (see for instance, Inal (2015)).

$$\{123\} \succ_2 \{2\}$$
$$\{134\} \succ_3 \{123\} \succ_3 \{3\}$$
$$\{45\} \succ_4 \{134\} \succ_4 \{4\}$$
$$\{15\} \succ_5 \{45\} \succ_5 \{5\}$$

This preference profile has two rings $(\{123\}, \{134\}, \{45\}, \{15\})$ *and* $(\{134\}, \{45\}, \{15\}\})$ *and it is weakly pairwise aligned. Note that only one agent is needed to move from one coalition to the next and the acquiescence of the agents at each intersection is no longer needed as Remark 1 points out. For instance, in the first ring to move from* $\{123\}$ *to* $\{134\}$*, once agent* 1 *prefers* $\{134\}$ *to* $\{123\}$*, agent* 3 *orders them in the same way due to pairwise alignment. Coalition structure* $\{\{123\}, \{45\}\}$ *is stable.*

(ii) A non-stable hedonic game with a ring which satisfies pairwise alignment.

$$\{12\} \succ_1 \{13\} \succ_1 \{123\} \succ_1 \{1\}$$
$$\{23\} \succ_2 \{12\} \succ_2 \{123\} \succ_2 \{2\}$$
$$\{13\} \succ_3 \{23\} \succ_3 \{123\} \succ_3 \{3\}$$

This preference profile has no stable coalition structure.

2.3 The model

In this subsection we present our coalition formation model with aspirations taking into account the preliminaries on rationing problems and hedonic games.

Let N be a finite set of agents and let $\mathcal{K} \subseteq 2^N \backslash \emptyset$. A *coalition formation problem with aspirations* is defined by a 3-tuple $(d_N, \mathcal{K}, E(C)_{C\in\ \mathcal{K}})$ where $d_N = (d_i)_{i\in N} \in \mathbb{R}^N_{++}$ is the vector of aspirations, $\mathcal{K}$ is the set of feasible coalitions where $\{i\}_{i\in N} \in \mathcal{K}$ and $E(C) \in \mathbb{R}_+$ is the output obtained by coalition $C \in \mathcal{K}$ satisfying $E(C) \leq \sum_{i\in C} d_i$.

To solve this model $(d_N, \mathcal{K}, E(C)_{C\in\mathcal{K}})$ we define a rule that distributes each coalitional output among its members. That is, we have a collection of rationing rules, one for each coalition: $F = (F_C)_{C\in\mathcal{K}}$. Thus, for each $(d_N, \mathcal{K}, E(C)_{C\in\mathcal{K}})$ and for each $C \in \mathcal{K}$, rule F assigns an allocation $x_C = (x_i)_{i\in C}$ where x_i is the payoff given by F_C to agent i in coalition C. Since rule F is a collection of the *same* rationing rules applied to coalitions in $\mathcal{K}$ we can abusing the notation, write that rule F satisfies a certain property, say consistency, whenever each rationing rule F_C does so. Moreover, rule F is *weakly pairwise aligned* if the preference profiles over coalitions that it generates belong to $\mathcal{D}(WPA)$.

2.4 An illustrative example

We finish this section with a numerical example that illustrates a coalition formation model with aspirations solved by three rationing rules which induces hedonic games.

Example 2 *Let* $N = \{1, 2, 3, 4\}$ *be four agents with aspirations of 100, 500, 500 and 600. Suppose that they may form the following collection of coalitions* $\{\{13\}, \{23\}, \{123\}, \{124\}, \{i\}_{i\in N}\}$ *whose coalitional outputs are displayed in the following table:*

Coalitions	$\{13\}$	$\{23\}$	$\{123\}$	$\{124\}$	$\{i\}_{i\in N}$
Outputs	*200*	*340*	*250*	*550*	*0*

Consider that the Shapley value is the rationing rule used to distribute each coalitional output among agents. To compute it, line up the agents in all possible orders. Beginning at the front of the line, pay off each agent in full until her aspiration is met. The Shapley value is the average payoff to each agent over all possible orders. This rule applied to each coalitional output induces the following hedonic game which does not have a stable coalition structure.

$$\{13\} \succ_1 \{124\} \succ_1 \{123\} \succ_1 \{1\}$$
$$\{124\} \succ_2 \{23\} \succ_2 \{123\} \succ_2 \{2\}$$
$$\{23\} \succ_3 \{13\} \succ_3 \{123\} \succ_3 \{3\}$$
$$\{124\} \succ_4 \{4\}$$

Next, consider the constrained equal awards rule which divides each coalitional output as equally as possible under the constraint that no agent receives more than her aspiration.

$$\{13\} \sim_1 \{124\} \succ_1 \{123\} \succ_1 \{1\}$$
$$\{124\} \succ_2 \{23\} \succ_2 \{123\} \succ_2 \{2\}$$
$$\{23\} \succ_3 \{13\} \succ_3 \{123\} \succ_3 \{3\}$$
$$\{124\} \succ_4 \{4\}$$

In this hedonic game with indifferences the constrained equal awards rule generates stable coalition structure $\{\{124\}\{3\}\}$.

Finally, consider the constrained equal losses rule, which divides the total loss (the difference between the sum of aspirations and the output) of each coalitional output as equally as possible under the constraint that no agent receives a negative amount.

$$\begin{aligned}
\{13\} \sim_1 \{124\} \sim_1 \{123\} \succ_1 \{1\} \\
\{124\} \succ_2 \{23\} \succ_2 \{123\} \succ_2 \{2\} \\
\{13\} \succ_3 \{23\} \succ_3 \{123\} \succ_3 \{3\} \\
\{124\} \succ_4 \{4\}
\end{aligned}$$

In this case, the constrained equal losses rule also induces stable hedonic games with indifferences with two stable coalition structures: $\{\{13\}\{2\}\{4\}\}$ $\{\{124\}\{3\}\}$.

3 Rationing rules inducing stability

In this section we characterize the rationing rules that induce stable hedonic games. The characterization consists of showing, in Proposition 1, that *only* continuous rules that satisfy consistency (to be precise bilateral consistency suffices) and resource monotonicity induce preference profiles that are weakly pairwise aligned.[10] Using this result, Proposition 2 shows that these rules do not induce rings. Thus, we find that every preference profile induced by any continuous rule satisfying consistency and resource monotonicity does not generate rings in preference profiles. To be more specific, in this setting induced hedonic games satisfy the common ranking property.

Proposition 1 *A continuous rule F is weakly pairwise aligned if and only if it satisfies consistency and resource monotonicity.*

Proof. First, we prove that if a rule F does not satisfy consistency then there is a hedonic game which does not belong to $\mathcal{D}(WPA)$. Then we prove that if a rule F does not satisfy resource monotonicity, there is a hedonic game which does not belong to $\mathcal{D}(WPA)$.

[10] Example 1 (ii) shows a preference profile that satisfies weakly pairwise alignment with rings that it does not guarantee stability.

(i) Let $(d_C, E(C)) \in \mathcal{B}^C$, with $|C| > 2$ be a rationing problem and let $F(d_C, E(C)) = x_C$ be the allocation given by F. Assume that F is not bilateral consistent. Then there exists a problem $(d_{\{i,j\}}, x_i + x_j) \in \mathcal{B}^{\{i,j\}}$ where $\{i,j\} \subset C$ such that $F(d_{\{i,j\}}, x_i + x_j) \neq (x_i, x_j)$. Define the following coalition formation problem with aspirations $(d_C, \mathcal{K}, E(C)_{C\in\mathcal{K}})$ where $\mathcal{K} = \{C, \{i,j\}\,\{k\}_{i\in C}\}$ with endowments $E(C)$, $x_i + x_j$ and 0 for singletons. In this case either agent i ranks C over $\{i,j\}$ and agent j ranks $\{i,j\}$ over C or its converse. Hence the hedonic game induced by $(d_C, \mathcal{K}, E(C)_{C\in\mathcal{K}})$ does not belong to $\mathcal{D}(WPA)$.

(ii) Let d_i and d_j be the aspirations of agents i, respectively and assume that rule F does not satisfy resource monotonicity. Then there exist two different problems $(d_{\{i,j\}}, E^1(\{i,j\}))$, $(d_{\{i,j\}}, E^2(\{i,j\})) \in \mathcal{B}^{\{i,j\}}$ with and $E^2(\{i,j\}) > E^1(\{i,j\})$ such that $F(d_{\{i,j\}}, E^1(\{i,j\})) = (x_i, x_j)$ and $F(d_{\{i,j\}}, E^2(\{i,j\}))$ $= (y_i, y_j)$ where $x_i < y_i$ and $x_j > y_j$. Let $C \subseteq N$, and consider problem $(d_C, E(C)) \in \mathcal{B}^C$, where $\{i,j\} \subset C$. Define $\alpha(E(C)) = F_i(d_C, E(C)) + F_j(d_C, E(C))$. Since F is a continuous function, $\alpha(E(C))$ is continuous on $E(C)$ so that $\alpha(0) = 0$ and $\alpha(d_1 + ... + d_c) = d_i + d_j$. Hence, there exits $0 \leq E'(C) \leq d_1 + \ldots + d_c$ such that $\alpha(E'(C)) = x_i + x_j$. Let $F_i(d_C, E'(C)) = z_i$ and $F_j(d_C, E'(C)) = z_j$. Thus, $z_i + z_j = x_i + x_j$. There are two cases:

Case 1 $(z_i, z_j) \neq (x_i, x_j)$. *Let $(d_C, \mathcal{K}, E(C)_{C\in\mathcal{K}})$ be a coalition formation problem with aspirations where $\mathcal{K} = \{C, \{i,j\}, \{k\}_{i\in C}\}$ with endowment $E'(C)$, $E^1(\{i,j\})$ and 0 for singletons. The hedonic game induced by $(d_C, \mathcal{K}, E(C)_{C\in\mathcal{K}})$ does not belong to $\mathcal{D}(WPA)$.*

Case 2 $(z_i, z_j) = (x_1, x_2)$. *Then there is a problem $(z_i, z_j) \neq (y_i, y_j)$. Let $(d_C, \mathcal{K}, E(C)_{C\in\mathcal{K}})$ be a coalition formation problem with aspirations where $\mathcal{K}$ =$\{C, \{i,j\}, \{k\}_{i\in C}\}$ with endowments $E'(C)$, $E^2(\{i,j\})$ and 0 for singletons. The hedonic game induced by $(d_C, \mathcal{K}, E(C)_{C\in\mathcal{K}})$ does not belong to $\mathcal{D}(WPA)$.*

Second, we prove that each rule F that satisfies consistency and resource monotonicity induces hedonic games in $\mathcal{D}(WPA)$.

Let C, C' be two coalitions such that agents $i, j \in C \cap C'$. Let $(d_C, E(C)) \in \mathcal{B}^C$ and $(d'_{C'}, E'(C')) \in \mathcal{B}^{C'}$ be such that $d_{\{i,j\}} = d'_{\{i,j\}}$. We need to show that either $F_{\{i,j\}}(d_C, E(C)) \geqq F_{\{i,j\}}(d'_{C'}, E'(C'))$ or $F_{\{i,j\}}(d_C, E(C)) \leqq F_{\{i,j\}}(d'_{C'}, E'(C'))$, *i.e.* agents i, j must rank coalitions C and C' in the same manner.

Let $x_C = F(d_C, E(C))$ and $y_{C'} = F(d'_{C'}, E'(C'))$ the allocations given by rule F for problems $(d_C, E(C))$ and $(d'_{C'}, E'(C'))$ respectively. Now, consider the following two auxiliary rationing problems $(d_{\{i,j\}}, (x_i + x_j))$ and $(d'_{\{i,j\}}, (y_i + y_j)) = (d_{\{i,j\}}, (y_i + y_j))$. If coalition rule F satisfies bilateral consistency then it holds that

$$x_{\{i,j\}} = F_{\{i,j\}}((d_{\{i,j\}}, (x_i + x_j))$$

and

$$y_{\{i,j\}} = F_{\{i,j\}}(d_{\{i,j\}}, (y_i + y_j)).$$

Applying resource monotonicity to problems $(d_{\{i,j\}}, (x_i + x_j))$ and $(d_{\{i,j\}}, (y_i + y_j))$, we have two cases:

(i) $x_i + x_j \geq y_i + y_j$.

$$x_{\{i,j\}} = F(d_{\{i,j\}}, (x_i + x_j)) \geqq F(d_{\{i,j\}}, (y_i + y_j)) = y_{\{i,j\}}$$

Since $x_{\{i,j\}} = F_{\{i,j\}}(d_C, E(C))$ and $y_{\{i,j\}} = F_{\{i,j\}}(d'_{C'}, E'(C'))$, then

$$F_{\{i,j\}}(d_C, E(C)) \geqq F_{\{i,j\}}(d'_{C'}, E'(C')),$$

and agents i, j prefer coalition C to coalition C', *i.e.* $C \succsim_i C' \Rightarrow C \succsim_j C'$.

(ii) $x_i + x_j \leq y_i + y_j$.

$$x_{\{i,j\}} = F(d_{\{i,j\}}, (x_i + x_j)) \leqq F(d_{\{i,j\}}, (y_i + y_j)) = y_{\{i,j\}}$$

Since $x_{\{i,j\}} = F_{\{i,j\}}(d_C, E(C))$ and $y_{\{i,j\}} = F_{\{i,j\}}(d'_{C'}, E'(C'))$, then

$$F_{\{i,j\}}(d_C, E(C)) \leqq F_{\{i,j\}}(d'_{C'}, E'(C')),$$

and agents i, j prefer coalition C' to coalition C, *i.e.* $C' \succsim_i C \Rightarrow C' \succsim_j C$.

Thus, define the coalition formation problem with aspirations where $\mathcal{K} = \{C, C', \{i, j\}, \{k\}_{i \in C \cup C'}\}$. The hedonic game induced by the above problem does not belong to $D(WPA)$. ■

Next lemma says that it is always possible to add agents to coalition C giving rise to coalition C' so that agents in C receive the same payoff that they had initially.

Lemma 1 *Let F be a continuous rule that satisfies consistency. Assume that $F(d_C, E(C)) = x_C$. Then for each $C', C \subset C'$, there is a problem $(d_{C'}, E'(C'))$ such that $F_i(d_{C'}, E'(C')) = x_i$ for all $i \in C$.*

Proof. Let $(d_C, E(C))$ be a problem such that $x_C = F(d_C, E(C))$. Let $C' \supset C$. For all problems $(d_{C'}, E(C'))$, $0 \leq E(C') \leq \sum_{i \in C'} d_i$, we define a function $\alpha(E(C')) = \sum_{i \in C} F_i(d_{C'}, E(C'))$. Note that α is a continuous function of $E(C')$ in the interval $[0, \sum_{i \in C'} d_i]$ and $\alpha(0) = 0$, $\alpha(\sum_{i \in C'} d_i) = \sum_{i \in C} d_i$. By continuity there exists $E'(C')$ such that $0 \leq E'(C') \leq \sum_{i \in C'} d_i$ such that $\alpha(E'(C')) = \sum_{i \in C} x_i$. Let $F(d_{C'}, E'(C')) = y_{C'}$. We know that $\sum_{i \in C} y_i = \sum_{i \in C} x_i$. By consistency, $F(d_C, \sum_{i \in C} x_i) = x_C$. Hence, $F_i(d_{C'}, E(C')) = x_i$ for all $i \in C$. ■

Proposition 2 *A continuous rule F that satisfies consistency and resource monotonicity does not generate rings in preference profiles.*

Proof. Let $(d_N, \mathcal{K}, E(C)_{C \in \mathcal{K}})$ be a coalition formation problem with aspirations induced by a continuous rule F that satisfies consistency and resource monotonicity which contains a ring $\mathcal{C} \subset \mathcal{K}$ of size k. Denote by C any coalition in $\mathcal{C}$. For all $C \in \mathcal{C}$, let $F(d_C, E(C)) = x_C$.

Define a new set of coalitions $\mathcal{K}'$ in which each coalition S in $\mathcal{C}$ is replaced by $C' = C \cup \{a\}$ where $a \notin N$ while the remaining coalitions are the same as in $\mathcal{K}$. By applying Lemma 1, we have that for all $C' \in \mathcal{K}'$, $F(d_{C'}, E(C')) = y_{C'}$ such that $x_C = y_{C'}$ for all $i \in C$, *i.e.*, all the agents in S receive the same payoff as initially. Therefore, the structure of the new coalition formation problem with aspirations $(d_{N'}, \mathcal{K}', E(C')_{C' \in \mathcal{K}'})$ where $N' = N \cup \{a\}$ is the following: For each agent in N, each coalition $C \in \mathcal{C}$ is replaced by the modified coalition C'. The order of the remaining coalitions does not change.

Agent a can order the coalitions she belongs to in two possible ways:

- a is indifferent between all the coalitions that she participates in.
  In this case, consider $C^* = \bigcup_{C' \in \mathcal{K}'} C'$. Applying Lemma 1, we obtain that $F(d_{C^*}, E(C^*)) = z_{C^*}$ such that $z_{C^*} = y_{C'}$ for all $i \in C'$. However, there exists at least one agent $j \in C^* \setminus C'$ such that her preferences are not transitive. Consequently, the hedonic game induced by $(d_{N'}, \mathcal{K}', E(C')_{C' \in \mathcal{K}'})$ does not belong to $\mathcal{D}(WPA)$.

- Otherwise, given that a is not indifferent between all coalitions she cannot order her preferences in a transitive way. Hence, the hedonic game induced by $(d_{N'}, \mathcal{K}', E(C')_{C' \in \mathcal{K}'})$ does not belong to $\mathcal{D}(WPA)$.

■

Next, according to Definition 4, we can state the following result:

Proposition 3 *A weakly pairwise aligned preference profile without rings induces a hedonic game that satisfies the common ranking property.*

Proof. The common ranking property may be interpreted as a social order such that every agent's order is consistent with it. The strategy of the proof is to construct a social order which ranks all coalitions in $\mathcal{K}$ respecting any preference profile that is weakly pairwise aligned without rings.

Let N be a set of agents of the hedonic game which follows order $1, 2, ..., n$.

Step 1 Set the coalitions containing agent 1 according to her own ranking.

Step 2 Consider the coalitions containing agent 2.

(i) Coalitions containing agent 1 are already ordered. We have two cases:

- Agent 2 orders these coalitions in the same way as agent 1 and we are done.
- Agent 2 orders these coalitions differently from agent 1. (For instance, agent 1 may have a strict order between two coalitions while agent 2 is indifferent.) In this case, the construction of the social order is modified so that there will be a weak preference relation between these two coalitions.

(ii) Coalitions which do not contain agent 1 are inserted respecting agent $2'$s order.

Therefore the orders of agents 1 and 2 are consistent with the social order under construction.

Step 3 Consider the coalitions containing agent 3.

(i) Coalitions containing agents 1 and/or 2 are already ordered and by weakly pairwise alignment the existing order under construction only needs to be modified if agent 3 has ordered some coalitions differently. As in the previous step the construction of the social order is modified to be consistent with agent $3's$ order.

(ii) Coalitions which do not contain agent 1 or 2 are inserted respecting agent $3's$ order.

Next, assume that the social order is $C \succ C' \succ C''$ and that $C \succ_1 C'$ and $C \succ_2 C''$. Now assume that $3 \in C', C''$ and that $C''_3 \succ C'$. In this case the sequence of coalitions $\langle C, C', C'' \rangle$ form a ring which contradicts the hypothesis.

Proceeding in this manner, given that the number of agents is finite, a social order consistent with the preferences of all agents is constructed. ■

Thus, the property of common ranking leads to stable coalition structures. Furthermore, if a rule assigns equal payoffs to coalitional outputs then it induces hedonic games with indifferences in which case the hedonic game may have several stable coalition structures.

Finally, considering propositions 1,2 and 3, the main result of our paper is stated:

Theorem 1 *All continuous rationing rules that satisfy consistency and resource monotonicity induce stable coalition structures.*

3.1 Parametric rules induce stability

There are a plethora of continuous rules that satisfy consistency and resource monotonicity, the most important of which are the class of parametric rules. In 1987, Young characterized continuous parametric rules using symmetry[11] and bilateral consistency. Recently, there has been some interest in asymmetric parametric rules (see for instance Kaminsky (2006) and Stovall (2014)). This last author characterizes a family of rules- asymmetric parametric rules- using continuity, bilateral consistency, resource monotonicity and two new axioms that generalize symmetric parametric rules.

Each member of the family of parametric rules is defined as follows:

Let f be the collection of functions $\{f_i\}_N$, where each $f_i : \mathbb{R}_{++} \times [a, b] \longrightarrow \mathbb{R}_+$ where f_i is continuous in λ, and weakly increasing in λ, $\lambda \in [a, b]$ and for each $i \in N$ and $d_i \in \mathbb{R}_{++}$ we have $f_i(d_i, a) = 0$ and $f_i(d_i, b) = d_i$.

Hence, for any f we can define a rule F for problem (d, E) as follows. For each $i \in N$,

$$F_i(d, E) \equiv f_i(d_i, \lambda) \text{ where } \lambda \text{ is chosen so that} \sum_{i \in N} f_i(d_i, \lambda) = E.$$

[11] Two agents with equal aspirations should receive equal payoffs.

In this case f is a parametric representation of F.

Note that in parametric rules the payoff given to each agent is determined by her aspiration d_i and a parameter λ. The constrained equal awards, the constrained equal losses, the Talmud and the reverse Talmud rules are symmetric parametric rules while the dictatorial rule with strict priority is an asymmetric parametric rule.

Next, we give a specific proof in which (asymmetric) parametric rules induce hedonic games that satisfies the common ranking property.

Proposition 4 *A continuous parametric rule always generates weakly pairwise aligned preference profiles.*

Proof. Let $i, j \in C \cap C'$ where $C, C' \in \mathcal{K}$. If a preference profile does not satisfy weakly pairwise alignment then $C \succ_i C'$ while $C' \succ_j C$.

By definition of parametric rule F, for each $C \in \mathcal{K}$ there exist a function f_i and a value λ such that $x_i = f_i(d_i, \lambda)$.
Let $x_{\{i,j\}} = f_{\{i,j\}}(d_{\{i,j\}}, \lambda)$, $i, j \in C$ and $y_{\{i,j\}} = f_{\{i,j\}}(d_{\{i,j\}}, \lambda')$, $i, j \in C'$. If $C \succ_i C'$ then $x_i > y_i$ and hence $\lambda > \lambda'$ because f_i is weakly monotone increasing in λ. On the other hand, if $C' \succ_j C$ then $y_j > x_j \Longrightarrow \lambda' > \lambda$ which is a contradiction. ■

Proposition 5 *A continuous parametric rule does not induce rings in preference profiles.*

Proof. Let $(d_N, \mathcal{K}, E(C)_{C \in \mathcal{K}})$ be a coalition formation problem with aspirations. Consider a continuous parametric rule F that solves the above problem then a hedonic game is induced.
Assume that the hedonic game contains a ring: $\mathcal{C} = (C_1, \ldots, C_k)$, $j = 1, ..., k$ and let $\{S_1, \ldots, S_k\}$ be the sets of agents such that $S_i \in C_i \cap C_{i+1}$ (*subscript modulo k*). By definition of parametric rule F, for each coalition C_j in the ring there exist a function f_i and a value λ such that $x_i = f_i(d_i, \lambda)$ for all agents in coalition C_j. For the sake of convenience we denote by $x_i(C_j)$ the payoff of agent i in coalition C_j and by $\lambda(C_j)$ the value of parameter λ associated to coalition C_j. Suppose the ring is formed as follows:

$$C_1 \succ_{S_1} C_2 \succ_{S_2} \ldots C_k \succ_{S_k} C_1.$$

By Remark 1 only one of the agents in the intersection between any two consecutive coalitons is considered.
Therefore,

$$x_i(C_1) > x_i(C_2) \Leftrightarrow f_i(d_i, \lambda(C_1) > f_i(d_i, \lambda(C_2))$$

for all $i \in S_1 \in C_1 \cap C_2$.
As f is weakly monotone increasing in λ, and the ring is defined only for strict preferences then

$$\lambda(C_1) > \lambda(C_2).$$

In the same way,

$$x_i(C_2) > x_i(C_3) \Leftrightarrow f_i(d_i, \lambda(C_2)) > f_i(d_i, \lambda(C_3)) \; for \; all \; i \in S_2 \Leftrightarrow \lambda(C_2) > \lambda(C_3),$$

$$\vdots$$

$$x_i(C_k) > x_i(C_1) \Leftrightarrow f_i(d_i, \lambda(C_k)) > f_i(d_i, \lambda(C_1)) \; for \; all \; i \in S_k \Leftrightarrow \lambda(C_k) > \lambda(C_1).$$

Thus, we obtain

$$\lambda(C_1) > \lambda(C_2) > \lambda(C_3) > \ldots > \lambda(C_k) > \lambda(C_1),$$

which is a contradiction. ■

Thus, we have proven that continuous parametric rules are weakly pairwise aligned and that they do not generate rings in preference profiles. Hence, as in Theorem 1, we can state that continuous (asymmetric) parametric rules induce hedonic games that satisfy the common ranking property, and the existence of at least one stable coalition structure is guaranteed. However, these rules are not the only ones which verify our results. There are continuous non-parametric rules that induce stability as Example 2 in Stovall's paper shows.

Example 3 *Let F be a rule that solves problem (d, E), so that, for $i \neq 1$,*

$$F_i(d, E) = f_i(d_i, \lambda) = \lambda d_i$$

and for $i = 1$,

$$F_1(d, E) = E - \sum_{i \in N \setminus \{1\}} F_i(d, E),$$

where λ is chosen so that $E \in \sum_{i \in N} f_i(d_i, \lambda)$.

This author shows that F has not a parametric representation and however, it satisfies continuity, consistency and resource monotonicity.

4 Concluding remarks

First, we must mention an extension of our approach to more general settings. In our modeling we assume that all coalitional outputs are insufficient to meet agents' aspirations. This assumption seems to limit the application of coalition formation with aspirations to bankruptcy situations. But we argue that our approach could be easily extend to problems in which coalitions of agents get enough profits to meet aspirations. Consider several agents who have to decide whether to invest a certain amount of money in one project out of a set of coalitional projects. To take that decision agents would estimate the profits of each project. Suppose that the agents agree to reimburse their initial investments k-times, $k \geq 1$ and on dividing the rest according to a rationing rule. After subtracting the sum of the reimbursements from the coalitional profits considering that aspirations are the initial investments we come back to a coalition formation problem with aspirations.

Finally, since we are left with a plethora of rules that induce core-stable hedonic games it seems interesting to study whether some rationing rules satisfy a stronger notion of stability. As further research we could analyze which rationing rules, if any, satisfy the notion of strong Nash stability (Karakaya, (2011))

References

[1] Aumann, R.J. and M. Maschler (1985). Game theoretical analysis of a bankruptcy problem from the Talmud. Journal of Economic Theory 36, 195-213.

[2] Banerjee, S., H. Konishi and T. Sonmez (2001). Core in a simple coalition formation game, Social Choice and Welfare,18, 135-153.

[3] Barberà S., C. Beviá and C. Ponsatí (2015). Meritocracy, egaliatarianism and the stability of majoritarian organizations. Games and Economic Behavior 91, 237-257.

[4] Bogomolnaia, A. and M.O. Jackson (2002). The stability of hedonic coalition structures, Games and Economics Behavior , 38, 201-230.

[5] Chun, Y., J. Schummer and W. Thomson (2001). Constrained egalitarianism A new solution to bankruptcy problems. Seoul Journal of Economics 14, 269-297.

[6] Dreze, J.H. and J. Greenberg, (1989). Hedonic Coalitions: Optimality and Stability, Econometrica, 48, 987-1003.

[7] Farrell, J., and S. Scotchmer, (1988). Partnerships, Quarterly Journal of Economics, 103, 279-297.

[8] Iehlé, V. (2007). The core-partition of a hedonic games, Mathematical Social Sciences, 54(2), 176-185.

[9] Inal, H. (2015). Core of coalition formation games and fixed-point methods, Social Choice and Welfare, 45(4), 745-763.

[10] Kaminsky, M. (2006). Parametric rationing methods, Games and Economic Behavior, 54, 115-133.

[11] Karakaya, M. (2011) Hedonic coalition formation games: A new stability solution, Mathematical Social Sciences, 61, 157-165.

[12] Moulin, H. (2003). Fair Division and Collective Welfare, The MIT Press

[13] O'Neill, B. (1982), A Problem of Rights Arbitration from the Talmud, Mathematical Social Sciences 2, 345-371.

[14] Pycia, M. (2012). Stability and Preference Alignment in Matching and Coalition Formation, Econometrica, 80, 1, 323-362.

[15] Shapley, LS. (1953). A Value for n-person Games, Annals of Mathematics Study No. 28, Princeton University Press, Princeton, N.J., pp. 307â€“317.

[16] Stovall, J.E (2014). Asymmetric parametric division rules, Games and Economic Behavior 84, 87-110.

[17] Thomson W. (2008). Two families of rules for the adjudication of conflicting claims, Social Choice and Welfare, 31, 667-692.

[18] Young H.P. (1987). On dividing an amount according to individual claim or liabilities. Mathematics of Operation Reseach 12, 398-414.