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Connectivity and spaninng bipartite subgraphs

Final Degree Dissertation
Degree in Mathematics

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Contents

Introduction

0.1 Content of the work

In this work we study an especific branch of Graph Theory: connectivity. The idea is that someone who is not familiar with Graph Theory learns enough of it in order to understand cutting edge results. Nonetheless, it is assumed that the reader is comfortable with inductive and Reductio ad Absurdum reasonings.

The notes are organized in three chapters:

In the first chapter, the reader is introduced to the basic tools of Graph Theory. In addition, we explain the notation that we are going to use throughout the notes. Since in this chapter there are several definitions not used until almost the end of the notes, the majority of the defined words have been included in the index. Anything in this chapter (except probably the line graph) is usually explained in any basic course on Graph Theory. In fact, the chapter has been written in order to make the notes self-contained.

In the second chapter, classical notions of connectivity are introduced. All the notions in this chapter are covered in any thematic course.

In the third chapter we treat cutting-edge results. Firstly, we find spanning bipartite subgraphs in any sufficiently well-connected graph. Then, we find that the same result is false for directed graphs. Finally, we present an application to a branch of the theory of connectivity: proper connectivity.

Throughout the notes we use the perspective of pure graph theory. This mainly means two things. On the one hand, we do not care what the vertices and edges are, that is, we give them generic names like v, u, w... In fact, we are interested in properties that do not change under isomorphism. On the other hand, the proofs are not necessarily constructive. We may prove that a graph might have a subgraph with an especific property, but we may not show an algorithm to get it. The reason for doing this is that it simplifies the proofs. Of course, by tracking back the lemmas used in the proofs, the reader might get a constructive algorithm, probably not as efficient as possible. Algorithm Design and Algorithm Analysis goes beyond the objective of this work.

0.2 Motivation for doing this work

I came up with the idea of doing my Bachellor's Thesis about connectivity in the summer school that was held in the Eötvös Loránd University in Budapest in June, 2014. There, I met a PhD student who told me she was reading [2]. So, I decided to read it too and try to make some original apportations. Since the results I got were marginal, I decided to present them in form of a Bachellor's Thesis. I proposed myself that the reader of these notes understands what I have proposed, and some results of [2], in the sense that the reader understands how to prove it. Later on, I asked Luis Martínez and Josu Sangróniz to be my supervisors and they agreed.

0.3 Personal work and acknowledgments

As a general rule, I write at the beginning of each chapter or section the used sources of external information. For the majority of the short proofs, I usually read them, thought about them, and waited some days to write them. In particular, I have homogenized the notation, since different sources use different notation. I also split the proofs in several lemmas whenever possible, which makes the majority of the proofs in the notes short. In addition, I explain in detail the proof of Lemma 2.21 given in [5], where it appears summarized in 6 lines because the author intended to give a short proof. This proof has ended up being the longest proof of the whole notes. Moreover, I came up with the idea of stating and proving Theorem 3.4 before I was told that it had already been proved by Thomassen in [8]. I have written both proofs so that the reader has an example of how constructive proofs are not necessarily better. As far as I am concerned section 3.2, is an original piece of work, that is, it has not been published anywhere. Finally, the application in Section 3.3 is also a (minor) contribution.

I want to acknowledge Josu Sangroniz and Luis Martínez for their many corrections and suggestions. In particular, Josu Sangroniz suggested the idea of doing the notes only about simple graphs, which simplifies everything without losing the core of the proofs. He also suggested me Remark 1.17, which I think is an important remark because it makes some ideas more natural. I also want to acknowledge the UFI of Mathematics and Applications of the University of Basque Country for financial support to attend the Eötvös Loránd University summer school.

0.4 Notation

In general, we use $v, u, w, x, y, z, v_i, u_i, \ldots$ for vertices. Moreover, V is the vertex set, and U, W, X, Y, Z some subsets. Normally, the order given is maintained, giving priority to forcing the letter for the set and its elements to be the same one. S, S_i and T are used for edge sets, and e, f for edges. Moreover, P, Q and R are used for paths. In order to use a fixed number, k is the most frequently used letter, followed by i and j which are preferably used for indexing. This is because n and m are used to indicate the number of vertices and edges of a graph. Any additional notation is explained throughout the notes.

Chapter 1

Basic concepts

In this chapter we explain some basic concepts of Graph Theory. This chapter is divided into four sections: basic definitions, connectedness, some families of graphs and the line graph. The whole chapter is mainly based on Balakrishnan's and Ranganathan's book [1, Chapters 1 and 2] adapting the definitions. In particular, they define the notions for multigraphs, and here they are adapted for simple graphs for simplification. The beauty of the proofs remains intact.

1.1 Basic definitions

Let us first define what a graph is:

Definition 1.1. A *graph* is an ordered duple $G = (V(G), E(G))$, where $V(G)$ is a nonempty finite set and $E(G)$ is a family of subsets of two elements of $V(G)$. Elements of $V(G)$ are called *vertices* (or *nodes* or *points*) of G, and elements of $E(G)$ are called *edges* (or *lines*) of G. Finally, $V(G)$ and $E(G)$ are the *vertex set* and *edge set* of G, respectively.

It should be remarked that *simple graph* is a more correct name for what we have defined as a graph. In fact, the term *graph* is a more general word which includes many generalizations and variations. But, when the context is clear, graph is often used to refer to simple graphs.

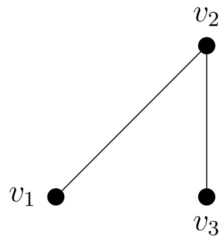


Figure 1.1: Diagrammatic representation of G.

One such generalization is the family of *multigraphs*. In a multigraph, $E(G)$ is an arbitrary set disjoint from $V(G)$ and there is an incidence function I_G which matches each edge to a pair of vertices. The generalization to multigraphs of the concepts in these notes is straight so, in order to simplify notation, we are going to present it for simple graphs.

As for notation, we usually omit the curly brackets and the comma when we are denoting an edge, that is, the edge $\{v, u\}$ is usually denoted by vu (notice that with this notion $uv = vu$).

Example 1.2. If $V(G) = \{v_1, v_2, v_3\}$, $E(G) = \{v_1v_2, v_2v_3\}$, then $G = (V, E)$ is a graph (see Figure 1.1).

Usually, graphs are represented by diagrams in the plane. In each diagram, each vertex is represented by a point, with distinct vertices being represented as distinct points. Each edge is a simple Jordan arc joining two vertices. Two edges may intersect at a point which is not a vertex. Moreover, given a diagram with the name of the vertices written on it, we can easily get the vertex set and the edge set.

It is important to know when two graphs are "the same", that is, when we have only changed the names of the vertices. The criterion is the existance of a pair of mutually inverse functions that preserve connections.*

Definition 1.3. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. A *graph isomorphism* from G to H is a bijective function $\varphi : V(G) \to V(H)$ with the property that $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(H)$.

* Notice that it is the same criterion as in group isomorphisms or homeomorphisms, asking in those situations to preserve the operation or the open sets.

Now we introduce some general terminology of edges and vertices:

Definition 1.4. If $e = vu$ is an edge, then the vertices v and u are called the *end vertices* or *ends* of e. Each edge is said to *join* its ends; in this case, we say that e is *incident* with each one of its ends. A vertex v is *neighbour* of u in G, if vu is an edge of G. The set of all neighbours of v is called its *(open) neighbourhood*, denoted as $N(v)$. The set $N[v] = N(v) \cup \{v\}$ is its *closed neighbourhood*. Vertices v and u are *adjacent* to each other when there is an edge of G with v and u as its ends. A vertex without adjacent vertices is called an *isolated point*. Two distinct edges e and f are said to be *adjacent* when they have a common end vertex.

Now we introduce some notation useful to denote the edges between especific vertex sets.

Definition 1.5. Let G be a graph and U, W proper disjoint subsets of V. Then, $[U, W]$ denotes the set of all edges of G that have one end in U and the other in W. Moreover, if U is a subset of some vertex set V, $\overline{U}$ denotes $V - U$. When we write throughout these notes $\left[U, \overline{U}\right]$, we assume that U is a proper nonempty subset of V.

Now we introduce another family of graphs, which is used to allow going from v to u, but not necessarily from u to v. Just imagine streets in a modern city: they are usually one way. This graph can also be used to analyze information of social situations. For example, the fact that Alice can ask Bob a favour does not necessarily mean that Bob can ask Alice a favour. It could be the case, for example, that Alice is Bob's boss.

Definition 1.6. A *directed graph*, also called a *digraph*, D has the same structure as a graph, but $E(D)$ is a set of duples of distinct elements of $V(D)$. Each element of $E(D)$ is called an *(oriented) edge* or an *arrow*. Each arrow (v, u) is usually denoted by vu. If $v, u \in V(D)$ and $e = vu$, v is called the *tail* of e, and u is called the *head* of e. The arrow e is said to join v with u, and v and u are called the ends of e. Its diagrammatic representation is made with arrows. With each digraph D, we can associate a graph G (or $G(D)$) on the same vertex set as follows: corresponding to each arrow of D, there is an edge of G with the same ends.† This graph G is called the *underlying graph* of the digraph D. It is unique up to isomorphism. In addition, if G is the underlying graph of a digraph D, D is called an *orientation* of G.

†Its underlying graph is usually a multigraph.

There are many notions about graphs that can be extended to digraphs in a unique logical way (like the notion of isomorphism). Unless otherwise specified throughout these notes, we assume that extensions exists for digraphs. For example, directed multigraphs exist, but we are not using them in these notes.

Now we define some parameters of a graph:

Definition 1.7. Let G be a graph and $v \in V$. The number of edges incident with v in G is called the *degree* (or *valency*) of the vertex v in G and is denoted by $d_G(v)$ or $d(v)$. The minimum (respectively, maximum) degree of the vertices of a graph G is denoted by $\delta(G)$ or δ (respectively, by $\Delta(G)$ or Δ). The number of vertices, called the *order*, is denoted by $n(G)$ or n, and the number of edges, called *size*, by $m(G)$ or m.

Finally, we define the idea of substructure in a graph:

Definition 1.8. A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. If H is a subgraph of G, then G is called an *overgraph* of H. A subgraph H of G is a *spanning subgraph* of G if $V(H) = V(G)$. If $U \subset V$, the *subgraph of G induced by U*, namely $G[U]$, is the graph whose vertex set is U and whose edge set is formed by the edges of G which join vertices in U. Similarly, if S is a set of edges the *subgraph of G induced by S*, denoted by $G[S]$, is the subgraph (U, S), where U is the set of the end vertices of the edges of S. If $U \subset V$, we denote $G[V - U]$ by $G - U$. In addition, if $S \subset E$, we denote the graph $(V, E - S)$ by $G - S$. Finally, if v is a vertex or e is an edge we denote $G - \{v\}$ by $G - v$ and $G - \{e\}$ by $G - e$.

Remark 1.9. Even if $U \subseteq V(G)$ and $S \subseteq E(G)$, (U, S) may not be a graph, so in particular it may not be a subgraph.

1.2 Walks and connectedness

In this section we explain the basic notions of walks in a graph and connectedness between vertices. In addition, we prove a useful fact concerning parity.

Definition 1.10. A *walk* in a graph G is a sequence $W = v_0 v_1 v_2 \dots v_n$ of vertices such that $v_i v_{i+1} \in E$ for $i = 0, \dots, n-1$. We say that the walk *uses* the edge $v_i v_{i+1}$. The walk is *closed* if $v_0 = v_n$ and is *open* otherwise. The *inverse walk* of W is the walk $W^{-1} = v_n v_{n-1} \dots v_0$. A walk is called a *path* if all the vertices are distinct. A *cycle* is a closed walk in which only the first and last vertices are equal. The *length* of a walk is the number of vertices minus 1.[‡] It is denoted by $l(W)$. A walk of length 0 consists just of a single vertex. An *odd/even* walk is a walk of odd/even length. If we have two walks $W_1 = v_0 \dots v_n$ and $W_2 = u_0 \dots u_m$, we can define the *union* of W_1 and W_2 if $v_n = u_0$, getting the walk $v_0 \dots v_n u_1 \dots u_m$. It is denoted by $W_1 W_2$. It can also be denoted by $v_0 \dots v_n W_2$ or $W_1 u_0 \dots u_m$, as is usually done in walks of length 1.[§] Moreover, W_1 and W_2 are *internally disjoint* if $v_i = u_j$ implies $i = 0$ or $i = n$ and $j = 0$ or $j = m$. Finally, W_1 and W_2 are *edge disjoint* if each edge is used by at most one walk.

Remark 1.11. Two internally disjoint walks are edge disjoint, but the converseis not necessarily true.

Remark 1.12. In a digraph, a cycle can consists just of two vertices For instance, $C_2 = (\{v, u\}, \{vu, uv\})$ (see Figure 1.2). In a graph, we do not have this possibility.

Lemma 1.13. *Let G be a graph. G has an odd cycle if and only if it has an odd closed walk.*

Proof. Necessity follows from their definitions. As for sufficiency, we use induction on the lenght of the walk. The case $k = 3$ is trivial. If we have $v_0 v_1 v_2 v_0$, necessarily, $v_0 \neq v_1$, $v_1 \neq v_2$ and $v_2 \neq v_0$, so the walk is an odd cycle. Let us consider the closed walk $v_0 v_1 \dots v_k$, being $k \geq 5$ odd. If it is not a cycle, it is because there are some vertices $v_i = v_j$, with $0 \leq i < j < n$. If i and j have the same parity, then the subsequence $v_0 \dots v_i v_{j+1} \dots v_k$ is an odd closed walk of smaller order. Otherwise, the subsequence $v_i v_{i+1} \dots v_{j-1} v_j$ is an odd closed walk of smaller order. In both cases we can apply the inductive hypothesis, and hence G has an odd cycle. □

Using induction in a similar way, the following proposition can be proved:

Proposition 1.14. *Let G be a graph and $v, u \in V$. There is a path in G from v to u if and only if there is a walk from v to u.*

[‡]The reason of this formula is that represents the number of edges, counting repetitions, that must be used for the walk in the graph.

[§]It is an interesting remark that the vertices of a graph with walks as arrows form a category, though it is not going to be used in these notes.

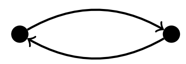


Figure 1.2: C_2

Using the concept of walk, we can define what we understand by being connected:

Definition 1.15. Let G be a graph. Two vertices v and u of G are said to be *connected* if there is a walk from v to u in G. It is clearly an equivalence relationship on $V(G)$. We can denote the equivalence class by $C[v]$ if we want to emphasize that it is the equivalence class of an especific vertex. Let $V_1, \ldots, V_k$ be the equivalence classes. Any of the subgraphs $G[V_i]$ is called a *(connected) component* of G. If G has a single component it is *connected*; otherwise, it is *disconnected*.

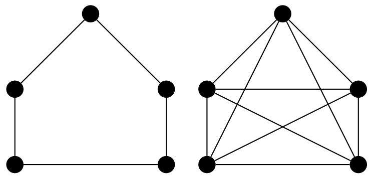
Remark 1.16. There is no edge between distinct connected components since any edge has both ends in the same component.

Remark 1.17. Proposition 1.14 is the reason why in the majority of the books the word path is used instead of walk in order to determine whether two vertices are connected. The point is that it is much more intuitive the idea of walk, and it makes it easier to prove transitivity. Moreover, conceptually, it makes more sense to talk about walks, because we want to reflect the idea that we can reach v from u, having much less importance how we can reach it. This is more clear in a more general context than simple graphs. If Alice wants to ask her boss Bob a favour, she probably does not care if she has to talk with her supervisor Charles more than once as long as she gets Bob's help.

Remark 1.18. For digraphs, we say that a vertex v is *reachable* from u if there is a walk from u to v. In the literature the word "connected" often means to be connected in the underlying graph. If we mean that each vertex is reachable from any other we should use the word "*diconnected*" or "*strongly-connected*". Again, it is clearly an equivalence relation, and we should use the word "*dicomponent*". Nonetheless, throughout the notes we use these word "connected" and "component", as we are mainly interested in the strong connectedness.

1.3 Some families of graphs

In this section we present some important families of graphs. We insist on bipartite graphs as they play an important role in Chapter 3.



(a) C_5 (b) K_5

Figure 1.3: Special graphs

Definition 1.19. A graph which consists of a cycle of length n is denoted C_n.

Remark 1.20. In the literature, P_i is used to denote a graph which consists on a path of length i and also to denote the i-th path in a sequence of paths. In these notes, we are going to use it only with the second meaning. (In the literature you can usually derive the meaning from context.)

Definition 1.21. A graph G is said to be *complete* if every pair of distinct vertices of G is adjacent in G. The complete graph of order n is denoted K_n.

In Figure 1.3 we have the diagrammatic representation of C_5 and K_5.

Definition 1.22. A graph $K_1 = (\{v\}, \varnothing)$ is said to be the *trivial graph* or the *point graph* or, when we see K_1 as a subgraph of some graph G, a *point*.

Note that these graphs are unique up to isomorphisms.

Definition 1.23. A graph is *bipartite* if its vertex set can be partitioned into two nonempty subsets U and W such that each edge of G has one end in U and the other in W. Any such pair (U, W) is called a *bipartition* of G. A graph G with a bipartition (U, W) is denoted by $G(U, W)$. A bipartite graph $G(U, W)$ is *complete* if $uw \in E$ for all $u \in U$, $w \in W$. If $G(U, W)$ is complete with $|U| = p$ and $|W| = q$, then $G(U, W)$ is denoted by $K_{p,q}$. Finally, if G is a graph and U, W are disjoint subsets of vertices, $G[U, W]$ denotes the bipartite subgraph $(U \cup W, [U, W])$.

Bipartite graphs have an important characterization:

Theorem 1.24. *A graph G is bipartite if and only if it contains no odd cycle.*

Proof. Let us assume it contains some odd cycle $v_0 v_1 \dots v_n, v_n = v_0$. Let us consider some partition (U, W). We can assume $v_0 \in U$. So, since it is a bipartition we must have $v_1 \in W$.

Similarly, v_2 should be U and in general, U contains the vertices v_i with even subindex and W those with odd subindex. But $v_n = v_0$ is in U and n is odd. So, (U, W) is not a bipartition.

We now assume that G is connected and contains no odd cycle. Let us pick some $u \in V$. Let U be the set of the vertices such that there is an even path from u to those vertices, and let W be the set of the vertices such that there is an odd path from u to those vertices. As G is connected, every vertex must be in one of the sets. If a $v \in V$ is in U and W, then the union of the odd walk and the inverse of the even walk from u to v is an odd closed walk in G. Hence, by Lemma 1.13, it contains an odd cycle, contradicting our hypothesis. So, (U, W) is a partition. If two vertices in U or in W are connected, we find similarly an odd closed walk, so (U, W) is a bipartition.

Finally, let us assume that G contains no odd cycle and has $G[V_1], \dots, G[V_k]$ as its connected components. Since each $G[V_i]$ is a connected graph, applying what we have proved in the previous paragraph, we get the bipartitions (U_i, W_i) for $1 \leq i \leq k$. Then, the duple $(\cup_{i=1}^k U_i, \cup_{i=1}^k W_i)$ is a bipartition of the whole graph. It is a partition because each vertex is in only one V_i, so it is either in U_i or in W_i and only in one of them. It is a bipartition because an edge has an end in U_i if and only if it has the other end in W_i, which follows from Remark 1.16. □

Remark 1.25. We have proved that given any two connected vertices of a bipartite graph, all walks joining them have the same parity.

Remark 1.26. For directed graphs we can only ensure that if D contains an odd cycle, then D is not bipartite. We show in the Figure 1.4 that the converse is not true. Nonetheless, we do have a characterization: a digraph D is bipartite if and only if its underlying graph G is bipartite.

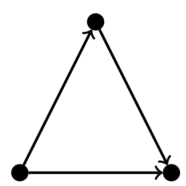
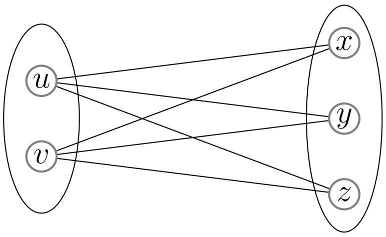
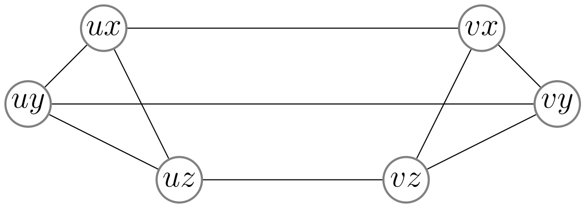


Figure 1.4: A non-bipartite digraph without a cycle



(a) $K_{2,3}$



(b) $L(K_{2,3})$

Figure 1.5: The line graph of $K_{2,3}$

1.4 The line graph

The line graph is a natural construction which creates short cuts to proving facts for edges using facts that we already know for vertices. In fact, this construction will be needed to prove Theorem 3.4.

Definition 1.27. Let G be a graph. The *line graph* $L(G)$ is the graph which has $E(G)$ as its vertex set and whose edge set is

$$\{ef : e, f \in E(G) \text{ and they are adjacent}\}.$$

In order to avoid confusion, we will name to a vertex of $L(G)$ an *edge*, and to an edge of $L(G)$ a *link*.

In Figure 1.5 the diagram of $L(K_{2,3})$ is shown.

Other examples are that $L(C_n) = C_n$, $L(K_{1,3}) = C_3$, $L(K_n) = K_{n(n-1)/2}$. In particular, the first two example show that two graphs may have isomorphic line graphs without being isomorphic.

One important property of the line graph is that it preserves connectedness:

Proposition 1.28. *Let G be a graph and $L(G)$ its line graph. Let $v, u \in V(G)$ vertices that are not isolated points. Then, the following conditions are equivalent:*

(i) *v and u are connected in G*

(ii) *Any pair of edges of the form vx,yu are connected in $L(G)$*

(iii) *A pair of edges of the form vx,yu are connected in $L(G)$.*

Proof. Let us suppose that v and u are connected in G by the walk $v_0, \dots v_n$. Any pair of edges of the form vx, yu are connected in $L(G)$. For instance, if $x \neq v_1$ and $y \neq v_{n-1}$ we have the following walk:

$$(vx)(v_0v_1)(v_1v_2)\dots(v_{n-1}v_n)(yu).$$

Moreover, since v and u are not isolated points, there are some edges vx, yu, so 2 implies 3. Finally, let us suppose that some vx and yu are connected in $L(G)$. Let us suppose that some edge wz of that walk is in $E(G[C[v]])$ (the set of edges of the connected component of v). Then, the following edge is either incident to w or to z, so it is in $E(G[C[v]])$. Since $vx \in E(G[C[v]])$, the last edge, yu, is in $E(G[C[v]])$. So, $u \in C[v]$, which means that v and u are connected. □

Finally, here come some facts about line graphs. The proofs are interesting and they are going to be used later on.

Proposition 1.29. *Let G, H be graphs and $e, f \in E(G) \cap E(H)$. Then, they are neighbours in $L(G)$ if and only if they are neighbours in $L(H)$.*

Proof. By definition, e and f are sets of two elements of $V(G) \cap V(H)$. Moreover, the criterion to be neighbours in $L(G)$ or $L(H)$ is the same: if and only if they share an element. □

Proposition 1.30. *Let G be a graph and $S \subset E$. Then $L(G[S]) = L(G)[S]$.*

Proof. First, we must realize that both graphs have the same vertex set. The one in the left is the edge set of $G[S]$, which is S by definition, and the one in the right is S by definition. As for the edge set, the one in the left, by Proposition 1.29, has the links in $L(G)$ with both ends in S. Moreover, the one in the right, by defintion, has as edge set the links in $L(G)$ whose both ends are in S. So, $L(G[S]) = L(G)[S]$. □

Remark 1.31. Let D be a digraph. It has to be clarified that in its line digraph $L(D)$ there is an arrow from $e = vu$ to $f = wx$ if and only if $u = w$.

Chapter 2

Classical notions of connectivity

In Chapter 1 we have seen what a connected graph is. Nonetheless, not all the graphs are equally "well connected". There is a clear difference between C_{10} and K_{10}, since C_{10} becomes disconnected when (any) two edges are removed, whereas K_{10} remains connected even without any 8 edges. In this chapter, we analyse how to measure how well connected a graph is, showing some equivalent ways of describing it. In order to do so, elementary proofs have been prioritized.

2.1 Measuring connectivity

This section is an adaptation of [1, Chapter 3].

Definition 2.1. A subset U of the vertex set V of a connected graph G is a *vertex cut* of G if $G - U$ is disconnected or K_1. A vertex v is a *cut vertex* of G if $\{v\}$ is a vertex cut of G.

The reason for considering K_1 as a special case is to have the order relation given by Theorem 2.16. This definition can be unified if we consider the trivial graph not connected, so, henceforth we consider K_1 disconnected. Normally, it would not be considered it because the majority of situations in Graph Theory (trees, number of connected components, path graphs, etc.) ask for this graph to be connected.

Definition 2.2. Let G be a nontrivial connected graph and $S \subset E$. S is an *edge cut* of G if $G - S$ is disconnected. An edge e is a *cut edge* (or a *bridge edge*) of G if $\{e\}$ is an edge cut of G.

Remark 2.3. Here there are some straight properties:

(i) If G is a graph, any subset of E of the form $[U, \overline{U}]$ is an edge cut. The converse is not true (in $C_3 = (V, E)$, E is an edge cut but does not have that form).

(ii) The removal of a cut edge splits the graph into two connected components. This fact is not necessarily true for a cut vertex. One such example is the star graph $K_{k,1}$, which has some vertex v connected to other k vertices. When v is removed, k connected components are obtained.

(iii) If U is a proper nonempty subset of V, then any walk from some vertex $u \in U$ to some other vertex $v \in \overline{U}$ uses at least an edge of $[U, \overline{U}]$.

Next we see how a minimal edge cut looks like. We say minimal in the sense that it does not contain a proper edge cut.

Proposition 2.4. *Let G be a connected graph and $S \subset E$ a minimal edge-cut and $H = G - S$. Then, there is a proper non-empty subset $U \subset V$ such that H consists of the components $H[U]$ and $H[\overline{U}]$ and such that $[U, \overline{U}] = S$.*

Proof. If H had 3 or more connected components, for any $e \in S$, $S - \{e\}$ would be an edge cut since adding an edge can increase the number of connected components at most by 1, so S would not be minimal. As H is disconnected, there is some proper subset $U \subset V$ such that H consists of the components $H[U]$ and $H[\overline{U}]$. If one edge e of S had both ends in U or in $\overline{U}$, then S would not be a minimal edge cut as $S - \{e\}$ would be an edge cut too. So, $S \subset [U, \overline{U}]$. The other inclusion is clear because U and $\overline{U}$ are disconnected in H. Indeed, if some edge of $[U, \overline{U}]$ did not belong to S, then $H[U]$ and $H[U]$ would be connected. □

Remark 2.5. The converse of the previous proposition is not true. Not every edge cut of the form $[U, \overline{U}]$ is minimal and it can split G in more than two components. For instance, we can consider $K_{3,1} = (\{v_1, v_2, v_3, u\}, \{v_1u, v_2u, v_3u\})$, and $U = \{u\}$. Then $[U, \overline{U}] = \{v_1u, v_2u, v_3u\}$ is not a minimal edge cut because removing one of those edges suffices to disconnect the graph. Moreover, $G - [U, \overline{U}]$ has four connected components (the four points).

The notions of cut vertex and cut edge have the next characterizations:

Proposition 2.6. *A vertex v of a connected graph G with at least three vertices is a cut vertex of G if and only if there exist vertices $u, w \in G$ distinct from v such that any walk from u to w has v as an internal vertex.*

Proof. If v is a cut vertex, its removal produces at least two connected components. Pick u and w from different connected components. If there is in G some walk from u to w that does not use v, they are in the same component; so any walk from u to w has v as an internal vertex.

As for sufficiency, if v is not a cut vertex, the graph G is still connected when it is deleted. So, for any $u, w \in G$ there is a walk from u to w in which v is not an internal vertex (the path in $G - v$). □

We have a similar proposition for edges, whose proof is analogous.

Proposition 2.7. *An edge $e = vu$ is a cut edge of a connected graph G if and only if there exist vertices x and y such that e belongs to every walk from x to y.*

Next, there is a second characterization of a cut edge that is useful to determine when e is not a cut edge.

Proposition 2.8. *An edge $e = vu$ of a connected graph G is a cut edge if and only if e belongs to no cycle of G.*

Proof. If e is not a cut edge of a connected graph G, then v and u are connected in $G - e$ by some path. The union of that path and e creates a cycle.

If e is in cycle, there is some path from v to u which does not use e. So, for any vertices $x, y \in G$ there is some walk from x to y which does not use e (substitute e by the path if needed), and hence they are connected even if e is removed. So e is not a cut edge. □

Once these equivalences are clear, we define the connectivity of a graph:

Definition 2.9. For a nontrivial, non-complete connected graph G the minimum k for which there exists a k-vertex cut is called the *vertex connectivity* or simply the *connectivity* of G, denoted by $\kappa(G)$ or κ. The connectivity of a trivial or disconnected graph is taken to be 0. Any vertex cut of cardinal $\kappa(G)$ is called a *minumum vertex cut.*

Example 2.10. All the cycles $C_n : v_0 \dots v_{n-1} v_0$ have connectivity 2, $n \geq 3$. If $n = 3$, we have $\kappa(C_3) = \kappa(K_3) = 2$. If $n \geq 4$, by removing v_0 and v_2 we get a disconnected graph, because v_1 is not connected to any vertex. Moreover, if we remove a vertex, the graph is still connected. So $\kappa(C_n) = 2$.

Definition 2.11. The *edge-connectivity* of a non-trivial graph G is the smallest k for which there exists a k-edge cut. The edge-connectivity of a trivial or disconnected graph is taken to be 0. The edge-connectivity of G is denoted by $\lambda(G)$ or λ. Any edge cut of cardinal $\lambda(G)$ is called a *minumum edge cut*.

Remark 2.12. If H is a spanning subgraph of G, then $\kappa(H) \leq \kappa(G)$ and $\lambda(H) \leq \lambda(G)$.

Definition 2.13. A graph G is *k-connected* if $\kappa(G) \geq k$. Also, G is *k-edge-connected* if $\lambda(G) \geq k$.

Let us remark an easy but important characterization of 2-edge-connectivity, which follows from Proposition 2.8:

Theorem 2.14. *A connected graph G is 2-edge-connected if and only if all its edges are in cycles.*

Remark 2.15. The characterizations of Proposition 2.8 and Theorem 2.14 cannot be generalized to digraphs. One counterexample is the directed cycle. The "reason" is that in simple graphs the reverse path always exists.

Next, we present a theorem which relates edge and vertex connectivity by an inequality. This theorem can be seen as a direct corollary of Menger's Theorems, which are presented in the next section. Nonetheless, it is worth to present a short independent elementary proof.

Theorem 2.16. *For a connected graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

Proof. The second inequality is quite clear, because if we remove all the edges incident in a vertex, G is disconnected. This is also true for a vertex of smallest degree.

As for the first, let $\{v_1u_1, \dots, v_\lambda u_\lambda\}$ be a λ-edge cut. By Proposition 2.4, it is of the form $[W, \overline{W}]$. Let us call $U = \{u_1, \dots, u_\lambda\}$. We can assume by renaming vertices that $U \subset W$. If $G - U$ is not connected, then U is a vertex cut of at most λ vertices, so $\kappa \leq \lambda$ by definition. Otherwise, $W = \{u_1, \dots, u_\lambda\}$ by Remark 2.3 (3). So, u_1 has λ neighbours (by the second inequality it cannot have less). This is because we can find a surjective function from $\{1, \dots, \lambda\}$ to $N(u_1)$. For $i = 1, \dots, n$, if $u_i = u_1$, we define $f(i) = v_i$. Otherwise, $f(i) = u_i$. Hence, $\kappa(G) \leq d_G(x_1) \leq \lambda$. □

A fact about this proof that should be pointed out is that not necessarily all u_i or all v_i have to be distinct vertices. This is because u_i is one of the vertices that the i-th edge of the edge cut is adjcent to. So, it could be $u_2 = u_1$ or any similar equality. One such example is $K_{k,1}$, in which all the v_i are connected to a single vertex $u = u_i$

Finally, it is clear that when a vertex is removed from a graph, its connectivity decreases at most by 1. Similarly, when an edge is removed, its edge-connectivity decreases at most by 1. In the following proposition we analyze what happens to vertex connectivity when an edge is removed:

Proposition 2.17. *Let G be a graph with connectivity $\kappa \geq 1$. Then, for any edge $e = uv$, the subgraph $H = G - e$ has connectivity κ or $\kappa - 1$.*

Proof. Let W be a minimum vertex cut of H. Then, if $H - W$ has at least three connected components, $G - W$ has at least two connected components, so W is a vertex cut of G. Moreover, if u or v is in W, $H - W = G - W$, so W is a vertex cut of G too. Otherwise, if $H - W$ has two connected components and $u, v \notin W$, either $W \cup \{u\}$ or $W \cup \{v\}$ is a vertex cut of G. In either of the three ways, $\kappa(G) \leq \kappa(H) + 1$. Moreover, as H is a spanning subgraph of G, $\kappa(H) \leq \kappa(G)$. As they are natural numbers, the only two possibilities are $\kappa(H) = \kappa(G)$ and $\kappa(H) = \kappa(G) - 1$. □

2.2 Menger's Theorems

In this section we present an important characterization of being κ-connected and another of being λ-edge-connected. These characterizations have to do with the largest number of internally disjoint and edge disjoint paths from one vertex to another. Having

κ internally disjoint path joining each pair of vertices seems at first a stronger condition than having connectivity κ. Assuming there are κ internally-disjoint paths, even if you remove $\kappa - 1$ vertices, there is still a path connecting each pair of remaining vertices. Similarly, if we have λ edge-disjoint paths connecting any two vertices, edge-connectivity must be at least λ. The objective of this section is to prove these are equivalent facts, which is one of the main statements of Menger's Theorems.

Menger's Theorems are known since 1927 [6]. From then on, many proofs have been given. Some of them, like the one provided in [1], prove more general facts about Network Theory. Nonetheless, here we adapt the elementary proof written by Goring [5] for the lemma and we conclude the theorems as suggested by Diestel in [3]. We explain their respective proofs in detail, although some easy explanations have been omitted.

We now present some definitions that are specifically needed for Lemma 2.21:

Definition 2.18. Let G be a graph and $U, W \subset V$. Then U and W are *connected* if there is some path starting in a vertex in U and ending in a vertex in W. We say that removing a set of vertices X *disconnects* U and W if $U - X$ and $W - X$ are not connected in $G - X$. In that case, X is a *UW-separator (in G)*. Nonetheless, if $v, u \in V$, a set X disconnects them if $v, u \notin X$ and if X disconnects $\{v\}$ and $\{u\}$.

Remark 2.19. Note that a different definition is given to"disconnect" two sets or two vertices. In addition, note that if U and W have a vertex in common, they are connected by a path of length 0. Moreover, it follows from the definition that the empty set is disconnected with any subset; hence, U and W are trivially UW-separators.

Definition 2.20. A family of paths is *disjoint* if they have no vertex in common (including the start vertices and the end vertices).

Lemma 2.21. *Let G be a graph, $U, W \subset V$ and k the smallest number of vertices that must be deleted in order to disconnect U and W. Then, there are k paths from U to W that are disjoint.*

Proof. We prove this fact by using induction on the number of edges of G. More specifically, we prove that if G has n edges, the statement of the lemma is true. The base case is when G is edgeless. Then, for any $U, W \subset V$, the smallest k is the number of common vertices. We get the k paths by taking those vertices as 0-length paths.

Now, let us prove the inductive case. Let G be any graph with n edges, $U, W \subset V$ and k the smallest number of vertices that must be deleted in order to disconnect U and W. Let us select some edge e. We consider the graph $H = G - e$, in which we can apply the inductive hypothesis. If it is still necessary to delete k vertices to disconnect U and W, we have by the inductive hypothesis k disjoint paths in H from U to W, which are also paths in G.

If that is not the case (see Figure 2.1), there is a UW-separator X for H, with $|X| = k-1$. Moreover, in $G - X$ there is a path from U to W that uses e, which is a bridge edge of its connective component. So, one of the ends of e, namely y, is connected to U in $H - X$, and the other, namely z, to W in $H - X$. Furthermore, $Y = X \cup \{y\}$ and $Z = X \cup \{z\}$ are UW-separators of k vertices.

To continue with, we are going to prove that a UY-separator in H, namely A, is a UW-separator in G. First, let us see it is a UW-separator in H. If in $H - A$ there was a path from U to W, then it could not use any vertex of X because by the restriction we would get a path from U to Y. But any path from U to W must use e or a vertex of X. So, there is no path from U to W in $H - A$, and A is a UW-separator in H. Let us suppose for the sake of contradiction that there is some path from U to W in $G - A$. This means that there must be some path from U to z that does not use any vertex in Y, then go to y via e, and then to W by another path. But that also means that in $H - X$ there is a path from U to z, and, by prolonging it with the path from z to W in $G - X$, that we know it does not use X, so U and W would be connected in $G - X$, which contradicts our hypothesis. Therefore, A is a UW-separator in G.

So, it is necessary to delete at least k vertices to disconnect U and Y. As $|Y| = k$, k is the minimum number of vertices needed to be removed to disconnect U and Y. By a similar reasoning, k is the smallest number of vertices needed to delete to disconnect Z and W.

So, by applying the inductive hypothesis, there are k vertex disjoint paths from some vertices of U to the vertices of Y, and others from the vertices of Z to W. It is not possible that one of the paths P from U to Y and other of the paths Q from W to Z has any vertex $v \notin X$ because it would mean that deleting X and the edge e does not disconnect

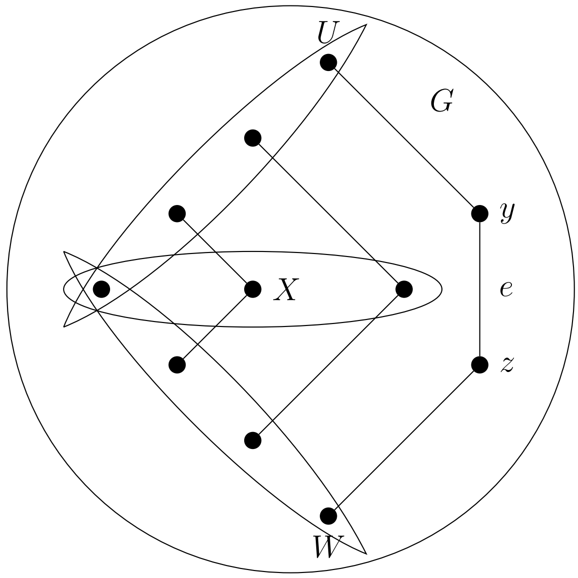


Figure 2.1: Illustration of the proof of Lemma 2.21

U and W (there would be the path that goes from U to v in P and then follow from v to W in Q). So, joining the paths, we get the k paths we are looking for. □

Using this lemma, we can prove the following theorems almost as a corollary:

Theorem 2.22 (Menger's Theorem for vertex connectivity)**.** *Let G be a graph and $v, u \in V$. We assume that v and u are not neighbours in V. Let κ be the smallest number of vertices needed to be removed to disconnect v and u. Then, there are κ internally-disjoint paths joining them.*

Proof. Let us consider the sets $N(v)$ and $N(u)$. Disconnecting v and u is equivalent to disconnecting $N(v)$ and $N(u)$. So, the minimum number of vertices needed to disconnect $N(v)$ and $N(u)$ is κ. Then, there are κ disjoint paths from $N(v)$ to $N(u)$. Joining them to v and u, we get the desired internally disjoint paths. □

There is an analogous theorem for edges:

Theorem 2.23 (Menger's Theorem for edge-connectivity)**.** *Let G be a graph and $v, u \in V$. Let λ be the smallest number of edges which must be removed to disconnect v and u. Then, there are λ edge-disjoint paths joining them.*

Proof. Let us consider the line graph $L(G)$. Due to Proposition 1.28, if $E(v)$ is the set of edges incident to v and $E(u)$ to u, then it is necessary to remove λ vertices in $L(G)$ to disconnect $E(v)$ from $E(u)$. So, we have the disjoint paths $S_1, \ldots, S_\lambda$ joining $E(v)$ and $E(u)$ viewed as sets of edges. Let us consider $G[S_i]$. Then, $v, u \in V(G[S_i])$ because the initial and final edges of S_i are incident to v and u, respectively. Moreover, these edges are connected in $L(G[S_i])$ since $L(G[S_i]) = L(G)[S_i]$ (Proposition 1.30). So, because of Proposition 1.28, there is a path P_i from v to u in every $G[S_i]$. Since $\{E(G[S_i])\} = \{S_i\}$ is a family of disjoint edge sets, $\{P_i\}_{i=1}^{\lambda}$ is family of λ edge-disjoint paths from v to u. □

Now, we are ready to prove the global version of Menger's Theorems:

Theorem 2.24 (Global version of Menger's Theorems)**.** *Let G be a connected graph:*

(i) *It is κ-connected if and only if it contains κ internally disjoint paths between any two vertices.*

(ii) *It is λ-edge-connected if and only if it contains λ edge disjoint paths between any two vertices.*

Proof. Sufficiency has been explained in both cases before. As for necessity, in the first case, if a pair $v, u \in V$ is not neighbour, by Theorem 2.22 we have the desired κ paths. If they are neighbours, we remove the edge joining them. According to Proposition 2.17, the connectivity of this new graph is greater than $\kappa - 1$, so we have that number of internally disjoint paths, and so we have together with the edge between u and v the κ paths. Necessity in the second case follows from Theorem 2.23. □

Finally, we are going to present another consequence of Lemma 2.21 that will be needed for proving Theorem 3.3.

Corollary 2.25. *Let G be a κ-connected graph, H a subgraph with at least κ vertices and $v \notin V(H)$. Then, there are κ paths that start in v, that end in distinct vertices of H, and that are internally disjoint with H.*

Proof. Since the graph is κ-connected, one needs to remove κ vertices to disconnect $N(v)$ and $V(H)$ (both sets have at least κ element, and you need to remove κ vertices in order to disconnect G). So, there will be κ disjoint paths from $N(v)$ to $V(H)$. Hence, by joining those paths with v and by deleting from each path anything after their first vertex in H, we get the desired κ-paths. □

Remark 2.26. For digraphs all these theorems have analogous statements and proofs. In fact, the proof in Göring's article [5] was done for digraphs and we have adapted it to graphs.

2.3 Ear decomposition

For 2-connected and 2-edge-connected graphs there is a constructive characterization which is frequently used. Moreover, there is also a similar characterization for connected digraphs. In the next chapter we analyze a property of digraphs so, in order to allow the reader to get used to them, this time we prove the characterization for digraphs. The other two proofs are analogous, so they will be omitted. We have used the statements based on [4], which by the way contain some additional easily derivable characterizations.

Theorem 2.27. *Let G be a graph, $|V(G)| \geq 3$. Then G is 2-connected if and only if there is a sequence of subgraphs $G_0, \dots, G_n = G$, where G_0 is a cycle, and each G_i arises from G_{i-1} by adding a path P_i for which the end vertices belong to G_{i-1} while the inner vertices do not.*

Remark 2.28. In the decomposition of Theorem 2.27 we ask the P_i to be open ears, that is, to have distinct initial and final vertices. This is a consequence of the definition of path.

Remark 2.29. Each P_i can consist of just one edge, so it is possible that $V(G_i) = V(G_{i+1})$ or $V(D_i) = V(D_{i+1})$. It is also true for the next theorems.

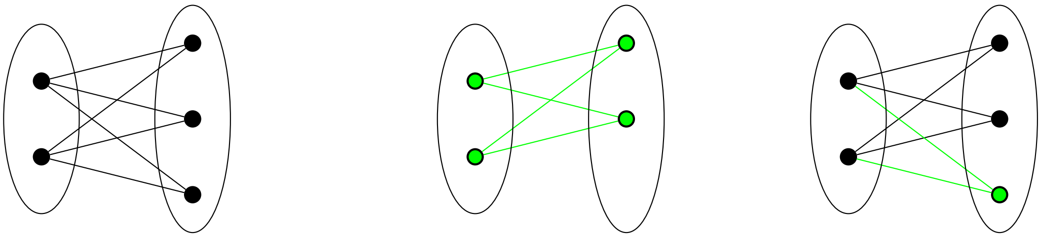
Theorem 2.30. *Let G be a graph. Then G is 2-edge-connected if and only if there is a sequence of subgraphs $G_0, \dots, G_n = G$, where G_0 is a cycle, and each G_i arises from G_{i-1} by adding a path or a cycle P_i for which the end vertices belong to G_{i-1} while the inner vertices do not.*

Theorem 2.31. *Let D be a digraph, $|D| \geq 2$. Then D is connected if and only if there is a sequence of subgraphs $D_0, \dots, D_n = D$, where D_0 is a directed cycle, and each D_i arises from D_{i-1} by adding a directed path or a directed cycle P_i for which the end vertices belong to D_{i-1} while the inner vertices do not.*

Remark 2.32. When P_i is a cycle, the vertex of G_{i-1}/D_{i-1} is considered as both the initial and final vertex, whereas the others are considered as internal vertices.

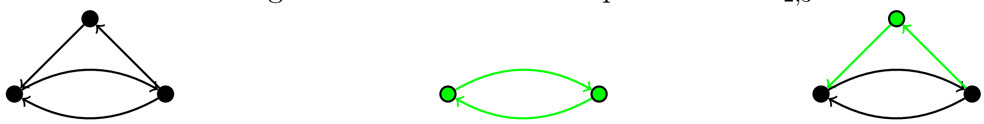
In both cases each P_i is called an *ear*, and the sequence $G_0, \ldots, G_n$ $/D_0, \ldots, D_n$ is called an *ear decomposition of* G/D. When the P_i is a path, we say it is an *open ear*.

As a toy example of Theorem 2.27 we present an ear decomposition of $K_{2,3}$ in Figure 2.2. Moreover, in Figure 2.3 and 2.4 we present a toy example of a digraph, which shows that ear decompositions need not be unique.



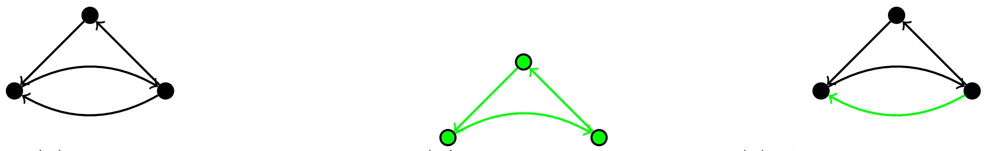
(a) $K_{2,3}$ (b) The initial ear (c) An additional open ear.

Figure 2.2: The ear decomposition of $K_{2,3}$



(a) The digraph (b) The initial ear (c) An additional open ear.

Figure 2.3: The ear decomposition of a digraph



(a) The digraph (b) The initial ear (c) An additional open ear.

Figure 2.4: The ear decomposition of a digraph

Proof of Theorem 2.31. Let us first prove sufficiency. We prove by induction that any D_i is connected. The base case, D_0, is connected because it is a cycle. Let us suppose that D_{i-1} is connected, and let us called v and u the initial and final vertex of P_i. Firstly, there is a walk from any vertex of D_{i-1} to any other of D_{i-1} because we have not removed any arrow. Secondly, there is a walk from any vertex of D_{i-1} to any vertex of P_i by going first to v and then using the arrows of P_i until we have reached the target vertex. Thirdly, we can go from any vertex of P_i to any vertex of D_{i-1} by going first to u and then using the walk inside D_{i-1}. Finally, we can go from any vertex of P_i to any vertex of P_i by going first to u, and then using the walk from u to the target vertex (the second case). So, each D_i is connected, and, in particular, D_n is connected.

As for necessity, let us pick a vertex v. As it is connected, there is an arrow vu going out of v. In addition, there is a path P back from u to v. The union of the edge and the path makes the cycle D_0. As for the others, we use induction. If D_{i-1} does not have all the vertices, let us pick an arrow xy leaving D_{i-1}. Moreover, there is path Q_i from y back to D_{i-1}, and we take $P_i = xyQ_i$. If D_{i-1} has all the vertices, P_i can be any arrow that is not in D_{i-1}. As V and E are finite, this process must finish. □

Remark 2.33. In the ear decomposition (of the three cases) it can be added the restriction that the order of the G_i/D_i increases strictly until some G_j/D_j, which is a spanning subgraph.

Chapter 3

Searching spanning bipartite subgraphs that preserve connectivity and some applications

In this chapter we analyse some recent results about bipartite subgraphs of graphs and how they reflect the connectivity of the graph. In particular, we are interested in finding a spanning subgraph that has at least half the connectivity of the graph. The case of edge-connectivity was solved by Thomassen in 2008 [8, Proposition 1], whereas for connectivity, to ht ebest of my knowledge, it is only known for 3-connected graphs [7, Theorem 3.3]. Later on, we see with a counterexample that in a connected digraph we cannot expect to find a bipartite spanning connected subgraph, let alone that has half the connectivity. Finally, we present an application to proper connection made in [2] and even suggest how the result could be improved.

3.1 Bipartite spanning subgraphs of simple graphs that preserve connectivity or edge-connectivity

Throughout this section, we are interested in seeing that if G is a $(2k-1)$-connected graph, then there is a k-connected spanning bipartite subgraph. Similarly, if G is a $(2k-1)$-edge-connected graph, then there is a k-edge-connected spanning bipartite subgraph. For connectivity, we are only able to prove the cases $k = 1, 2$. As for edge-connectivity, a proof for the general case is given, as well as additional proofs for the cases $k = 1$ and $k = 2$.

The case $k = 1$ is a classical result:

Proposition 3.1. *Any connected graph has a spanning bipartite connected subgraph.*

Proof. Let G be a connected graph and H a minimal spanning connected subgraph, minimal in the sense that if we remove any edge, it is disconnected. H cannot have a cycle, since removing an edge of a cycle does not disconnect it. So, by Theorem1.24, H is bipartite. □

Remark 3.2. The previous proposition serve as a proof in the case of 1-edge-connected and 1-connected, which is equivalent of being connected.

The next step is to prove it for 3-connected graphs. We use the sketch of the proof in [7, Theorem 3.3], which was later on explained in more detail by Čada et al. [9, Lemma 3.1.]. The proof here is very similar to theirs.

Theorem 3.3 ([7] and [9])**.** *Let G be a 3-connected graph. Then, G has a 2-connected spanning bipartite subgraph.*

Proof. First, we find an ear decomposition of that bipartite subgraph. First, let us pick any two $v, u \in V$. By Menger's Theorem, there are three internally disjoint paths from v to u. Moreover, two of them have the same parity. So joining one with the inverse of the other path, we get an even cycle, namely H_0.

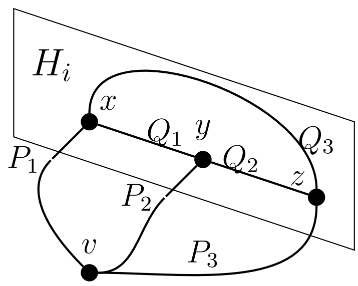


Figure 3.1: Graph illustrating the proof of Theorem 3.3

Now, let us suppose that H_i is a bipartite 2-connected subgraph which is not a spanning subgraph (from now on see Figure 3.1). Let $v \notin H_i$. By Corollary 2.25, there are three internally disjoint paths P_1, P_2, P_3 that start in v and end in 3 different vertices of H_i, namely x, y, z. Let us pick 3 paths inside H_i: Q_1, that goes from x to y; Q_2, that goes from y to z, Q_3, that goes from z to x. Now, since H_i is bipartite, the closed walk $Q_1Q_2Q_3$ is even, so

$$l(P_1Q_1P_2^{-1}) + l(P_2Q_2P_3^{-1}) + l(P_3Q_3P_1^{-1}) = 2l(P_1) + 2l(P_2) + 2l(P_3) + l(Q_1Q_2Q_3) \equiv 0 \mod 2.$$

This means that one of the three cycles is even, let us say $P_1Q_1P_2^{-1}$. So, we add to H_i the open ear $R_i = P_2^{-1}P_1$ in order to define H_{i+1}.

Let us remark that H_{i+1} is still bipartite. Any odd cycle must use an edge of R_i, so necessarily all edges R_i. But any path in H_i between x and y has the same parity of Q_1 (Remark 1.25), which is the parity of R_i. So any odd cycle, must be an even cycle. So, H_{i+1} does not have an odd cycle, and therefore, is bipartite. So, we can use this process to expand H_0 to a spanning bipartite subgraph H_k via the ear decomposition $H_0, \dots, H_k$, adding each time $R_1, \dots, R_k$ (since V is finite, it must terminate). So, H_k is a spanning bipartite 2-connected subgraph of G. □

This proof has an analogous for edge-connectivity. Although dealing with edges makes it nastier, we can see that the main ideas remain.

Theorem 3.4. *If G is a 3-edge-connected graph, then G has a spanning 2-edge-connected bipartite subgraph.*

Proof. First, we prove that G has an even cycle H_0. Certainly G has a cycle. If it is odd, pick some vertex v in the cycle. There is a third path going from v to some vertex $u \neq v$ in the cycle which is internally disjoint to the cycle. The reason is that G is connected after deleting the two edges of the cycle adjacent to v. So, we have 3 internally-disjoint paths from v to u. Two of these paths have lengths of the same parity and they give an even cycle.

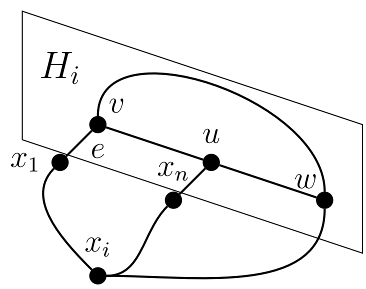


Figure 3.2: Graph illustrating the proof of Theorem 3.4

Next (from now on see Figure 3.1), we prove that if we have some not-spanning bipartite subgraph H_i, then H_i can be expanded by an ear to some overgraph H_{i+1}, also bipartite, which has more vertices. Let us pick some edge vx_1 such that $v \in H_i$ and $x_1 \notin V(H_i)$. Since vx_1 is in a cycle, there is some path (it might be closed, that is, it might be $u = v$) $vx_1, \dots, x_n u$, being $u \in V(H_i)$ and internally-disjoint from H_i. Let us denote $\{x_1, \dots, x_n\}$ by X. Since G is 3-edge-connected, there is some i such that there is a path P_2 from x_i to some $w \in V(H_i)$ internally-disjoint from $(\{v, x_1, \dots, x_n, u\}, \{vx_1, \dots, x_n u\}) \cup H_i$. This is because in $G - \{vx_1, x_n u\}$ X is connected to $V(H_i)$, so we get the desired path by considering a path that starts in X and ends in $V(H_i)$. We take the restriction since it last leaves X until it first enters $V(H_i)$. Comparing Figure 3.2 and Figure 3.1, we are in a very similar situation to the one in the proof of the previous theorem, so it can be concluded in the same way the existence of the ear and H_{i+1}. Moreover, we get the 2-edge-connected spanning bipartite subgraph starting from an even cycle and expanding it with ears. □

Remark 3.5. In the above proof u, v, w might be equal or different vertices, so the ears might be open or close.

So, we see that the difference is that the given paths are not vertex disjoint; so, in order to look for the ear, we must look in relatively close vertices. This is due to the fact that edge-connectivity is weaker. Moreover, both proofs have the disadvantage that they depend on a characterization exclusive of 2-connectedness and 2-edge-connectedness. This is because those proofs are constructive, and being constructive is not always an advantage. In fact, pointing a subgraph with an especific property as in Proposition 3.1 can be a more effective way. That is how Thomassen proves the following fact about edge-connectivity [8, Proposition 1]:

Theorem 3.6 ([8])**.** *Let G be a $2k-1$-edge-connected graph. Then G has a spanning k-edge-connected bipartite subgraph.*

Proof. Let U be maximal in the sense that $H = G[U, \overline{U}]$ has as much edges as possible. Let us suppose, for the sake of contradiction, that $\lambda(H) < k$. Then, there is a $W \subset V$ such that $[W, \overline{W}]_H$ has less than k edges (Proposition 2.4). Let us consider now

$$I = G[(U \cap W) \uplus (\overline{U} \cap \overline{W}), (U \cap \overline{W}) \uplus (\overline{U} \cap W)].$$

We prove that

$$E(I) = (E(H) - [W, \overline{W}]_H) \uplus ([W, \overline{W}]_G - [W, \overline{W}]_H),$$

being the union disjoint and being the differences removal of a subset. This is because

$$
\begin{aligned}
E(I) &= [U\cap W, \overline{U}\cap W] \uplus [U\cap W, U\cap\overline{W}] \uplus [\overline{U}\cap\overline{W}, \overline{U}\cap W] \uplus [\overline{U}\cap\overline{W}, U\cap\overline{W}]\\
E(H) &= [U\cap W, \overline{U}\cap W] \uplus [U\cap\overline{W}, \overline{U}\cap W] \uplus [U\cap W, \overline{U}\cap\overline{W}] \uplus [U\cap\overline{W}, \overline{U}\cap\overline{W}]\\
[W,\overline{W}]_H &= [U\cap W, \overline{U}\cap\overline{W}] \uplus [U\cap\overline{W}, \overline{U}\cap W]\\
[W,\overline{W}]_G &= [U\cap W, \overline{U}\cap\overline{W}] \uplus [U\cap\overline{W}, \overline{U}\cap W] \uplus [U\cap W, U\cap\overline{W}] \uplus [\overline{U}\cap\overline{W}, \overline{U}\cap W].
\end{aligned}
$$

Since $[W,\overline{W}]_G$ has at least $2k-1$ edges, that means that $[W,\overline{W}]_G - [W,\overline{W}]_H$ has at least k edges, so we add at least k edges, and remove at most $k-1$. That means that I has more edges than H, which goes against the hypothesis. Therefore H is k-edge-connected, and consequently, is the subgraph we are looking for. □

A critical reader may think that the proof is so easy that there must be a trick. The trick is that although we have proved the existence, getting the subgraph H is not easy at all. There are $2^{n-1}-1$ possible candidates. This is why this proof cannot be considered a constructive proof, but rather a proof that points out an especific subgraph which we know that must exist. In fact, these proofs tend to be much cleaner precisely because they hide how to get the graph (note that there is a constructive algorithm: get the $2^{n-1}-1$ subgraphs, count the edges and get the subgraph of a maximum number of edges).

Remark 3.7. As the example C_4 shows, Proposition 3.1 and Theorem 3.6 do not produce the same subgraph. In the first case we get P_3, and in the second case C_4. More generally, the three proofs are not deterministic proofs: since there are decisions to be made, you can get different subgraphs.

3.2 Digraphs and bipartite subgraphs

With digraphs, the situation is completely different. First, it is rather straightforward that a connected graph may not have a connected bipartite subgraph. One such example is the directed cycle $C_3 = (\{v,u,w\},\{vu,uw,wv\})$. Nonetheless, one may argue that if we increase the connectivity sufficiently then perhaps the graphs has a spanning bipartite connected subgraph. The answer is negative and we prove it next.

Definition 3.8. Let D be a digraph and let U, W be disjoint subsets of $V(D)$. We denote by $U\times W$ the set of ordered pairs with the first element in U and the second one in W, that is, to the cartesian product.

Theorem 3.9. *For all $k \in \mathbb{N}$ there is a digraph D_k with connectivity k and no connected spanning bipartite subgraph.*

Proof. Let us consider the digraph whose vertex set is the union of $U = \{u\}$, $W = \{w\}$, $X = \{x_1, \ldots, x_k\}$, $Y = \{y_1, \ldots, y_k\}$ and $Z = \{z_1, \ldots, z_k\}$ and whose edge set is the union of $U \times X$, $Z \times X$, $X \times W$, $W \times Y$, $X \times Z$, $Z \times U$, $Y \times Z$ and $Z \times Y$ (see Figure 3.3 for $k = 2$).

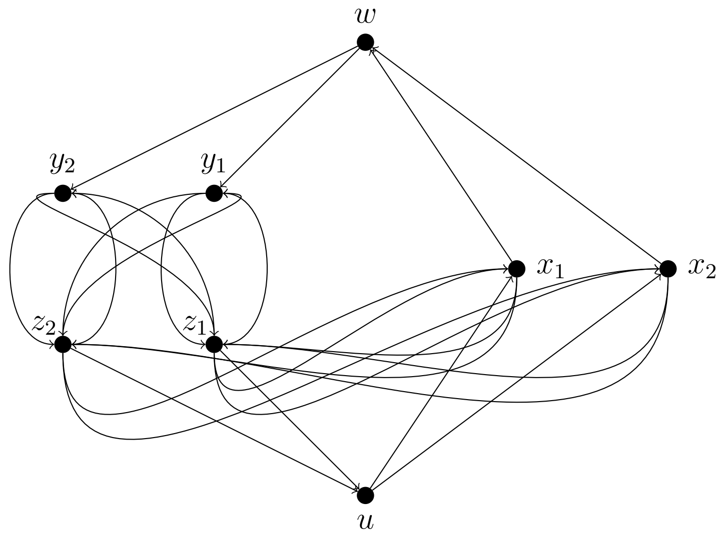


Figure 3.3: Graph illustrating Theorem 3.9 for $k = 2$

Let us first prove that any path from u to w is even, and any path from w to u is odd. Indeed, in any path from u to w we are in X or Y after an odd number of steps, and in Z after an even number, as it can be proved by induction. Similarly, in any path from w to u, we are in X or Y after an odd number of steps, and in Z after an even number. So, being a subgraph connected implies that is has an odd closed walk. Therefore, it cannot have a spanning connected bipartite subgraph.

As for connectivity, let us suppose that we remove $k - 1$ vertices. If u and w are not removed, we have some cycle $ux_{i_1}wy_{i_2}z_{i_3}u$. In addition, any y_j or x_j are connected to z_{i_3} by edges in two directions. Finally, any z_j is connected to x_{i_1} by edges in two directions. So the remaining graph is connected. Otherwise, if u or w are removed, pick a cycle $x_{i_1}z_{i_2}y_{i_3}z_{i_4}x_{i_1}, i_2 \neq i_4$. Proceeding as before, we can see that the graph is still connected, so $\kappa \geq k$. Since only k arrows have u as its tail, we have the equality $\kappa = k$. □

Remark 3.10. The theorem is true if we change the word "connected" by "edge-connected". In fact, as D_k is k-connected and as only k arrows have u as its tail D_k is k-edge-connected.

3.3 An application to proper connectivity

In this final section we see that all this theory can be applied in other areas of Graph Theory, in particular, in the field of proper connectivity. Moreover, we sharpen a conjecture proposed in [2].

Definition 3.11. Let G be a graph and c a function from E to some (finite) set C. Then (G, c) or (G, E, c) is an *edge-colored graph*. Any element of C is called a *color*. Moreover, we use the verb *color* to mean that we assign the function of c.

Definition 3.12. Let G be an edge-colored graph. A *proper path* is a path in which every two adjacent edges differ in color. Moreover, G is *proper connected* if any two vertices are connected by an internally pairwise-disjoint proper paths. If G is a connected graph, we define the *proper connection number*, denoted $\mathrm{pc}(G)$, as the smallest number of colors needed to make G k-proper connected. If $k = 1$ we usually omit it and write $\mathrm{pc}(G)$.

Borozan et al. proved the following theorem in [2, Theorem 3], which gives the proper connection number of 2-edge-connected bipartite graphs:

Theorem 3.13 ([2])**.** *Let G be a graph. If G is 2-edge-connected and bipartite, then* $\mathrm{pc}(G) = 2$. *Moreover, there exists a 2-coloring of G that makes it properly connected such that for any pair of vertices u, v there are two paths from u to v such that they begin with a different color and they end with a different color.*

Proof. Let us suppose that $G_0, \ldots, G_k = G$ is an ear decomposition with succesive ears $P_1, \ldots, P_k$. We prove by induction that we can color each G_i as we claim. The base case is an even cycle (it cannot be odd because G is bipartite), so by coloring alternatively the edges we get a coloring as desired.

Let us suppose we add the ear P_i to G_{i-1}. The color of the edges of G_i is the same as the edges of G_{i-1}. If P_i is just an edge, we can color that edge red or blue. We can find the two paths of G_i by using the paths in G_{i-1}. Otherwise, let us call $P_i = v_0 \ldots v_r$. We paint v_0v_1 by red, v_1v_2 by blue, v_2v_3 by red, and so on. We still have those two paths from vertex of G_{i-1}. Moreover from a vertex v_i we can find those two paths to a vertex $u \in V(G_i)$. First we go to v_0. There is a proper path Q_1 from v_0 to u whose first edge is blue. Moreover, there is another proper path Q_2 from v_r to u whose first color is the opposite of $c(v_n v_{n-1})$. So, the paths are $v_i \dots v_0 Q_1$ and $v_i \dots v_r Q_2$. They clearly start with an edge of a different color. They must end with alternating colors because, otherwise, we have an odd closed walk from v_i to u. Finally, we can get two paths for vertices in P_i by considering the proper cycle $v_r \dots v_0 Q$, where Q is an alternating path from v_0 to v_r in G_{i-1} whose first edge is blue. The last edge of Q must be of different color of $c(v_r v_{r-1})$ because, otherwise, we have an odd closed walk. So, G_i is colored as stated. □

They also prove in [2, Theorem 8] :

Theorem 3.14 ([2])**.** *If G is a connected non-complete graph with $n \geq 68$ vertices and $\delta(G) \geq n/4$, then* $\mathrm{pc}(G) = 2$.

Since they use theory related with Hamiltonian cycles, we do not prove it in these work. In addition, they pose the following conjecture in [2, Conjecture 3]:

Conjecture 3.15 ([2])**.** *If $\kappa(G) = 2$ and $\delta(G) \geq 3$, then* $\mathrm{pc}(G) = 2$.

Because of our previous work, we can make the following contribution:

Corollary 3.16. *Let G be a graph. If G is 3-edge-connected and non-complete, then* $\mathrm{pc}(G) = 2$. *Moreover, there exists a 2-coloring of G such that for any pair of vertices u, v there are two paths from u to v such that they begin with a different color and they end with a different color.*

Proof. This is a straightforward consequence of Theorem 3.4 or Theorem 3.6 and Theorem 3.13. □

So, the current unknown cases for Conjecture 3.15 are $\kappa = 2$, $\lambda = 2$ and $\delta \geq 3$ (and $\delta < n/4$ if $n \geq 68$).

Bibliography

[1] R. Balakrishnan and K. Ranganathan. *A Textbook on Graph Theory*. Springer-Verlag, second edition, 2012.

[2] V. Borozan, S. Fujita, A. Gerek, C. Magnant, Y. M, L. Montero, and Z. Tuza. Proper connection of graphs. *Discrete Math.*, 312(17):2550–2560, 2012. doi: 10.1016/j.disc.2011.09.003.

[3] R. Diestel. *Graph Theory*, chapter 3, pages 50–55. Springer-Verlag, electronic edition, 2000.

[4] A. Frank. *Connections in Combinatorial Optimization*, chapter 2, pages 45–58. Oxford University Press, first edition, 2011.

[5] F. Goring. Short proof of menger's theorem. *Discrete Mathematics*, Vol 219:295–296, 2000.

[6] K. Menger. Zur allgemeinen kurventhorie. *Fund. Math*, Vol 10:96–115, 1927.

[7] P. Paulraja. A characterization of hamiltonian prisms. *J. Graph Theory*, 17(2):161–171, 1993. doi: 10.1002/jgt.3190170205.

[8] Carsten Thomassen. Edge-decompositions of highly connected graphs into paths. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 78(1):17–26, 2008.

[9] R. Čada, T. Kaiser, M. Rosenfeld, and Z. Ryjáček. Hamiltonian decompositions of prisms over cubic graphs. *Discrete Math.*, 286(1):45–56, 2004. doi:10.1016/j.disc.2003.11.044.

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