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Convergence of Fourier series

Final Degree Dissertation
Degree in Mathematics

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Introduction

Motivation

The main goal of this dissertation is to study the problem of convergence of Fourier series. This problem consists on, given a function f, deciding if its Fourier series converges to f. What conditions are necessary or sufficient for that to happen? What do we actually mean with convergence? We are going to give an answer to these question in this dissertation.

The problem of convergence of Fourier series was first posed in the beginning of the XIX century, and it has evolved with the development if mathematical analysis. Several great mathematicians have worked on Fourier series, and this subject has influenced some of them. It would be bold to say that Fourier series have shaped the development of history of mathematics, but there is no doubt that it has influenced it in some way.

Abstract

This Final Degree Dissertation is organized as follows.

In Chapter 1, we define the basic concepts about Fourier series. We also give some basic properties of the Fourier coefficients.

In Chapter 2, we introduce the most classical theorems about convergence of the Fourier series, all of them proved in the XIX century.

In Chapter 3, we study summability of Fourier series. These are alternative techniques to obtain the "sum" of the Fourier series.

In Chapter 4, we prove convergence of Fourier series on the spaces L^p, for $1 < p < \infty$.

Finally, in Chapter 5, we explain the divergence of Fourier series. We give a continuous function whose Fourier series is divergent at a point, and we prove the existence of an L^1 function whose Fourier series is divergent on the L^1 norm.

Historical comment about Fourier series

The problem of Fourier series began with partial differential equations, in the middle of the XVIII century. More precisely, with the vibrating string problem. D'Alembert, Euler and Daniel Bernouilli, amongst other mathematicians, worked on the solution of the wave equation. Even though d'Alembert obtain his famous solution to the equation, Bernouilli wrote the solution as a trigonometric sine. Since then, the theory of Fourier series has had a huge development, and there are several applications to this theory.

In 1822, Joseph Fourier published his *Théorie analytique de la chaleur*. In this book, he studied the propagation of heat in a finite solid. He deduced the constitutive equation of the propagation of heat, and he solved the problem of the distribution of the temperature at a given time if he knew the distribution at an initial moment. In order to do that, he invented the method of separation of variables, also known as the Fourier method. As we know, this method requires writing a given function as the sum of a trigonometric series.

Fourier did not prove convergence in any form. He opened the problem of the expansion in a trigonometric series of a given function. There are a lot of concepts related to this problem, such as integral, sum of series, and even the mere concept of function. There is not doubt that this problem influenced the development of the mathematical analysis.

There were a few attempts to prove convergence of the series by Cauchy and Poisson, but the first theorem was proved by Dirichlet in 1829 (Theorem 2.9). He determined some sufficient conditions that assure the convergence of the Fourier series of a function to the function itself at a given point. This is the first of many similar theorems that give sufficient conditions for convergence. In this dissertation we mention theorems by Camille Jordan (Theorem 2.11, 1881), Lipschitz (1864, we see it as a consequence of Dini's theorem) and Dini (Theorem 2.12, 1880).

Riemann worked, like on most topics in mathematics developed in the XIX century, on the problem of Fourier series. He developed his theory of the integral and then applied it to the Fourier series. He realized that if a function f does not admit integration in the domain, then the Fourier series of f has not sense. He gave the famous Riemann-Lebesgue lemma 2.3, and the Riemann-localization principle 2.4.

Not everything result positive, though. Heine pointed out in 1870 that, at that point, there was not proof that a function had a unique Fourier series, and that there was no evidence that the Fourier series of a continuous function had to be uniformly convergent. Even so, both statements were widely accepted until there appeared a counterexample.

Paul du Bois-Reymond found a continuous function whose Fourier series is divergent at a point. Not only the series was not uniformly convergent, it was divergent at a point. After this counterexample, there appeared other ones, by Schwarz, Fejér and Lebesgue.

After the disappointing counterexample by du Bois-Reymond, there appeared a new way of summing the series. Fejér proved in 1900 that given a continuous function f, the sequence of the arithmetic averages of the partial sums of the Fourier series of f converges uniformly to f. This is not the only way, we also recover the original function with the Abel-Poisson summability.

In the beginning of the XX century, there appeared new theories in mathematical analysis. The theory of integration by Lebesgue, the theory of Hilbert and Banach spaces, for example. In this wider context, there was much to be done. Lebesgue applied his own theory to the Fourier series, and he proved his version of the Riemann-Lebesgue lemma 2.3.

With the Lebesgue theory, there appeared the L^p spaces. The first theorem in these spaces is called the Parseval identity, which actually was published by Parseval in 1806 (even before Fourier's publication, Theorem 4.2). Fatou proved in 1906 that it holds for every $f \in L^2$. After that came a negative result by Banach and Steinhaus. They proved in 1918 that there is no mean convergence, meaning no convergence in L^1. The last theorem we are going to explain we owe it to Marcel Riesz in 1923; this theorem states that there is convergence in L^p for every $1 < p < \infty$ (Theorem 4.10).

But there is further history of Fourier series. The stronger theorem yet about Fourier series was proved by Carleson in 1965. He proved that the Fourier series of an L^2 function converges pointwise to the function for almost every point. This came as a surprise because, although that theorem was conjectured around 1913 by Lusin, it was expected to be false. It was later expanded by Richard Hunt to every $1 < p < \infty$.

Chapter 1

Trigonometric Fourier Series

1.1 Basic notation

There are many options to define the Fourier series. I have chosen the torus one. We define the torus $\mathbb{T} = [-\pi, \pi)$. We can identify $\mathbb{T}$ with the unit circle in $\mathbb{C}$, using the exponential map e^{ix}. We will identify the space of functions defined on the torus $\mathbb{T}$ with the space of the periodic functions of period 2π defined on the real line $\mathbb{R}$. We will study both real valued and complex valued functions.

Another way of understanding this concept is considering in $\mathbb{R}$ the following equivalence relation:

$$x \sim y \Leftrightarrow x - y = 2k\pi \text{ for some } k \in \mathbb{Z}.$$

In this case, the torus would be the quotient space $\mathbb{T} = \mathbb{R}/\sim$. Both ways of understanding the torus are obviously equivalent.

This simplifies the notation, for example, when we talk about the left or right sided limits of the function at the points $x = \pm\pi$, or when we talk about continuity at those points. We will not need to consider the periodic extension of the original function.

Given a function $f \in L^1(\mathbb{T})$, we define the Fourier series of f:

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}, \tag{1.1}$$

where $\hat{f}(n)$ is the n-th Fourier coefficient of f, defined in the following way:

$$\hat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(x)e^{-inx}dx. \tag{1.2}$$

If there is no confusion about what function f is, we may call $c_n = \hat{f}(n)$.

Since we use integrals, we may need to specify what we mean. We are going to use Lebesgue measure and integral, due to the advantages it carries against the Riemann integral. There will appear the spaces $L^p(\mathbb{T})$, with $1 \leq p \leq \infty$. These are the usual L^p spaces, along with their usual norms.

We will at least require the function f to be integrable in order to be able to compute the coefficients. If f were not integrable, the coefficients would not be defined at all.

This way of describing the Fourier series is equivalent to the traditional real form of the Fourier series given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx), \tag{1.3}$$

where the coefficients are defined in the following way:

$$a_n = \frac{1}{\pi} \int_{\mathbb{T}} f(x) \cos(nx) dx, \quad n \geq 0, \tag{1.4}$$

$$b_n = \frac{1}{\pi} \int_{\mathbb{T}} f(x) \sin(nx) dx, \quad n \geq 1. \tag{1.5}$$

It can easily be checked that this relation holds for all $n \in \mathbb{Z}$:

$$c_0 = \frac{a_0}{2},$$

$$c_n = \frac{a_n - ib_n}{2}, \quad n \geq 1,$$

$$c_n = \frac{a_{-n} + ib_{-n}}{2}, \quad n \leq -1.$$

We prefer to use the complex form because it is more compact, we only need one formula and everything lies under a unique sum sign. This way we only have one type of coefficient and we have a linear map between the space $L^1(\mathbb{T})$ and the space $l(\mathbb{Z})$ of all complex sequences indexed by the integers.

Once we have defined the series, we will establish the notation for the partial sums. The N-th partial sum of the Fourier series of a function $f \in L^1(\mathbb{T})$ is defined in either of the two equivalent ways:

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}, \tag{1.6}$$

$$S_N(f)(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos nx + b_n \sin nx. \tag{1.7}$$

Our main goal is to study what happens with $S_N(f)$ when, given an "arbitrary" function f on $\mathbb{T}$, we let N tend to infinity.

In this chapter we introduce the first formal definitions of the Fourier series of a function f defined on $\mathbb{T}$.

1.2 Trigonometric series

Let f be an integrable function on $\mathbb{T}$. We define the trigonometric Fourier series of f in the following way:

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}, \tag{1.8}$$

where the coefficients $\hat{f}(n)$ are the Fourier coefficients of f, which are defined as follows:

$$\hat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)e^{-int}\,\mathrm{d}t. \tag{1.9}$$

It is clear from the expression (1.9) that the Fourier coefficients are linear maps from $L^1(\mathbb{T})$ to $\mathbb{C}$:

$$(\alpha f + \beta g)\hat{\ }(n) = \alpha\hat{f}(n) + \beta\hat{g}(n).$$

We will try to justify this definitions. Suppose that we have a function expanded in a trigonometric series of the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

and suppose that we know that the series converges uniformly. In this case, we can check that the coefficients of the series are precisely the Fourier coefficients of f:

$$\begin{aligned}
\hat{f}(k) &= \frac{1}{2\pi}\int_{\mathbb{T}} f(x)e^{-ikx}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{\mathbb{T}}\left(\sum_{n=-\infty}^{\infty} c_n e^{inx}\right)e^{-ikx}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} c_n \int_{\mathbb{T}} e^{(n-k)ix}\,\mathrm{d}x \\
&= c_k.
\end{aligned}$$

This justifies the choice of the Fourier coefficients. Notice that in order to do the computations above we do not need the uniform convergence of the series, we just need to be able to change the order of the sum and the integral.

Theorem 1.1. *If a function f defined on $\mathbb{T}$ can be expanded in a trigonometric series which converges uniformly to f, then this series is the Fourier series of f.*

Proof. The proof follows from the computations above. □

1.3 Properties of the coefficients

Let f be an integrable function on $\mathbb{T}$:

(i) The sequence $\{\hat{f}(n)\ :\ n \in \mathbb{Z}\}$ is bounded by

$$\left|\hat{f}(n)\right| \leq \frac{1}{\pi} \int_{\mathbb{T}} |f(x)|\, \mathrm{d}x.$$

(ii) *Linearity.* The coefficients are linear maps on $L^1(\mathbb{T})$:

$$(\alpha f + \beta g)\hat{}(n) = \alpha \hat{f}(n) + \beta \hat{g}(n).$$

(iii) *Derivability.* If f is continuously derivable in $\mathbb{T}$, then we can write the Fourier coefficients of the derivative in terms of the coefficients of f:

$$(f')\hat{}(n) = in\hat{f}(n),\ \forall n \in \mathbb{Z}.$$

This can easily be proved integrating by parts. It is not really required that f' be continuous, these relations also hold in more general situations.

The coefficients a_n and b_n of the real form hold these relationships:

$$\begin{aligned} a_0(f') &= 0, \\ a_n(f') &= nb_n(f), \quad n \geq 1, \\ b_n(f') &= -na_n(f), \quad n \geq 1. \end{aligned}$$

(iv) For the real form of the Fourier series:

If f is *even* (i.e. $f(-x) = f(x)$ in $\mathbb{T}$), then $b_n = 0$ for all $n \geq 1$ and we can write

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, \mathrm{d}x.$$

When f is *odd* (i.e. $f(-x) = -f(x)$ in $\mathbb{T}$), then $a_n = 0$ for all $n \geq 0$ and we can write

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\, \mathrm{d}x.$$

1.4 Bessel inequality

Another basic property of the Fourier series is the so-called Bessel inequality. This inequality is a much more general inequality and it holds for any orthogonal system in any Hilbert space. In this case the orthogonal system will be the basic trigonometric system $\{e^{ix} : k \in Z\}$ and the Hilbert space will be $L^2(\mathbb{T})$.

Let f be a function in $L^2(\mathbb{T})$ (which implies $f \in L^1(\mathbb{T})$ because $\mathbb{T}$ has finite measure). We will show that the trigonometric polynomial of degree N wich is the best approximation of f in the norm of $L^2(\mathbb{T})$ is the N-th partial sum of its Fourier series. We emphasize that this is only true in that norm, there are other trigonometric polynomials of the same degree that approximate f better in other norms. We will come back to this issue later in this dissertation.

Let p_N be a trigonometric polynomial of degree N. That is, p_N has the form:

$$p_N(x) = \sum_{k=-N}^{N} c_k e^{ikx}.$$

It follows from the orthogonality of the trigonometric system that the L^2 norm of p_N is the following:

$$||p_N||_2^2 = 2\pi \sum_{k=-N}^{N} |c_k|^2.$$

Theorem 1.2. *If $f \in L^2(\mathbb{T})$, then the trigonometric polynomial which best approximates f in the L^2 norm is the N-th partial sum of its Fourier series.*

Proof. Let p_N be a trigonometric polynomial, and let us see which is the distance from p_N to f:

$$\begin{aligned}
||f - p_N||_2^2 = \langle f - p_N, f - p_N \rangle &= ||f||_2^2 + ||p_N||_2^2 - \langle f, p_N \rangle - \langle p_N, f \rangle \\
&= ||f||_2^2 + 2\pi \sum_{k=-N}^{N} |c_k|^2 - 2\pi \sum_{k=-N}^{N} \left(\hat{f}(k)\overline{c_k} + \overline{\hat{f}(k)} c_k \right) \\
&= ||f||_2^2 + 2\pi \sum_{k=-N}^{N} \left(|\hat{f}(k)|^2 + |c_k|^2 - |\hat{f}(k)|^2 - \hat{f}(k)\overline{c_k} - \overline{\hat{f}(k)} c_k \right) \\
&= ||f||_2^2 - 2\pi \sum_{k=-N}^{N} |\hat{f}(k)|^2 + 2\pi \sum_{k=-N}^{N} (\hat{f}(k) - c_k)\overline{(\hat{f}(k) - c_k)} \\
&= ||f||_2^2 - 2\pi \sum_{k=-N}^{N} |\hat{f}(k)|^2 + 2\pi \sum_{k=-N}^{N} |\hat{f}(k) - c_k|^2.
\end{aligned}$$

It is clear that the minimum is obtained when

$$\hat{f}(k) = c_k \quad \forall k \in \mathbb{Z}. \quad \square$$

One particular consequence of this theorem is that the L^2 norm of the partial sums is bounded by the L^2 norm of f:

$$||S_N(f)||_2^2 = \sum_{k=-N}^{N} |\hat{f}(k)|^2 \leq \frac{1}{2\pi} ||f||_2^2.$$

Taking the limit as $N \to \infty$ we prove the following theorem.

Theorem 1.3 (Bessel inequality)**.** *If $f \in L^2(\mathbb{T})$, then*

$$\sum_{k=-\infty}^{\infty} |\hat{f}(n)|^2 \leq \frac{1}{2\pi} \int_{\mathbb{T}} |f|^2 = ||f||_2^2. \tag{1.10}$$

In particular, we observe that if $f \in L^2(\mathbb{T})$ then the sequence of Fourier coefficients $\{\hat{f}(k)\}$ tends to zero when $|k|$ tends to infinity. We will prove in the following chapter that this also happens in L^1.

With these simple results, we see that the more natural context of the Fourier series is the Hilbert space $L^2(\mathbb{T})$. But this space was “born” in the XX century: it is quite modern. The Fourier series are much older than Hilbert spaces. So the first important theorems about Fourier series do not use this special structure of function spaces. They focus on classical properties of functions, such as continuity, derivability and other local or global properties.

In the next chapter we will discuss these classical results, and we will move towards younger theories in the later chapters.

Chapter 2

Convergence of Fourier Series

In this chapter we are first going to study the pointwise convergence of the Fourier series of a function, and then we will try to give a criterion for the uniform convergence of the series. Finally, we explain the Gibbs phenomenon, which appears around jump discontinuities.

2.1 The Dirichlet kernel

We will introduce the Dirichlet kernel, which will simplify the computations of the partial sums and will be of much help proving the different convergence theorems.

Definition 2.1 (Dirichlet kernel)**.** *For $N \geq 0$, the Dirichlet kernel D_N is the following function:*

$$D_N(t) = \frac{1}{2} \sum_{k=-N}^{N} e^{ikt}. \tag{2.1}$$

It is very easy to verify that the following equality holds

$$D_N(t) = \frac{\sin(N + 1/2)t}{2 \sin t/2}. \tag{2.2}$$

Indeed, we just need to use the formula $2\cos\theta = e^{i\theta} + e^{-i\theta}$, multiply the expression (2.1) by $\sin t/2$ and use the trigonometric formula

$$2 \cos nt \sin t/2 = [\sin(k + 1/2)t - \sin(k - 1/2)t] .$$

We can write the partial sum $S_N(f)$ in terms of the Dirichlet kernel:

$$
\begin{aligned}
S_N(f)(x) &= \sum_{k=-N}^{N} \hat{f}(k) e^{ikx} \\
&= \sum_{k=-N}^{N} \left[\frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-ikt} \, \mathrm{d}t \right] e^{ikx} \\
&= \frac{1}{\pi} \int_{\mathbb{T}} f(t) \left[\frac{1}{2} \sum_{k=-N}^{N} e^{-ikt} e^{ikx} \right] \mathrm{d}t \\
&= \frac{1}{\pi} \int_{\mathbb{T}} f(t) \left[\frac{1}{2} + \sum_{k=1}^{N} \frac{e^{ik(x-t)} + e^{-ik(x-t)}}{2} \right] \mathrm{d}t \\
&= \frac{1}{\pi} \int_{\mathbb{T}} f(t) \left[\frac{1}{2} + \sum_{k=1}^{N} \cos k(x-t) \right] \mathrm{d}t \\
&= \frac{1}{\pi} \int_{\mathbb{T}} f(t) D_N(x-t) \, \mathrm{d}t.
\end{aligned}
$$

The Dirichlet kernel has the following basic properties, that will be of crucial importance.

Proposition 2.2. *The Dirichlet kernel has the following properties:*

(i) D_N *is a* 2π*-periodic function. We will understand that it is defined in* $\mathbb{T}$.

(ii) D_N *is even:* $D_N(t) = D_N(-t)$.

(iii) $\int_{\mathbb{T}} D_N = \pi$.

Proof. It follows directly from either of the expressions (2.1) or (2.2). □

The most important thing about the Dirichlet kernel is that it allows us to express the partial sums as a convolution:

$$S_N(f) = \frac{1}{\pi}(D_N * f). \tag{2.3}$$

These three properties allow us to express the partial sums in different ways. For example, since f is defined in $\mathbb{T}$, we understand that it is 2π-periodic. So we can write

$$S_N(f)(x) = \frac{1}{\pi} \int_{\mathbb{T}} f(x-t) D_N(t) \, \mathrm{d}t. \tag{2.4}$$

Making a change of variable in the interval $(-\pi, 0)$, we can obtain this other formula:

$$S_N(f)(x) = \frac{1}{\pi} \int_0^{\pi} \left[f(x+t) + f(x-t) \right] D_N(t) \, \mathrm{d}t. \tag{2.5}$$

2.2 The Riemann-Lebesgue lemma and some consequences

The well-known Riemann-Lebesgue lemma owes its name to both Riemann and Lebesgue. Both of them proved this result in the context of their respective theories of integral. We will show it in the context of the Lebesgue integral, since most of our work is within that theory.

Lemma 2.3 (Riemann-Lebesgue)**.** *Let f be an integrable function in $\mathbb{T}$ and $\lambda \in \mathbb{R}$ (not necessarily an integer). Then*

$$\lim_{\lambda\to\infty} \int_{\mathbb{T}} f(t) \sin \lambda t \, \mathrm{d}t = \lim_{\lambda\to\infty} \int_{\mathbb{T}} f(t) \cos \lambda t \, \mathrm{d}t = 0.$$

Proof. We will prove the limit for the sine. The limit for the cosine is analogous. We first start with the characteristic function of an interval $(a, b) \subset \mathbb{T}$. In this case

$$\left| \int_{\mathbb{T}} \chi_{(a,b)}(t) \sin(\lambda t) \, \mathrm{d}t \right| = \left| \int_a^b \sin \lambda t \, \mathrm{d}t \right| = \left| \frac{\cos \lambda b - \cos \lambda a}{\lambda} \right|,$$

which clearly tends to zero as λ tends to infinity. We deduce that the result is also true for step functions, which are finite linear combination of characteristic functions of intervals. We know, from measure theory, that this would also imply that the result is true for simple functions.

If f is an arbitrary integrable function, then given an $\epsilon > 0$, there exist a simple function g_ϵ such that

$$\int_{\mathbb{T}} |f - g_\epsilon| < \epsilon/2.$$

We can write

$$\left| \int_{\mathbb{T}} f(t) \sin \lambda t \, \mathrm{d}t \right| \leq \int_{\mathbb{T}} |f(t) - g_\epsilon(t)| \, \mathrm{d}t + \left| \int_{\mathbb{T}} g_\epsilon(t) \sin \lambda t \, \mathrm{d}t \right|.$$

It turns out that if λ is big enough, the second term is smaller than $\epsilon/2$. This completes the proof. □

The first consequence of the Riemann-Lebesgue lemma is that the sequences of the Fourier coefficients of an integrable function tend to zero. We saw, using Bessel's inequality (1.10) that this was true for functions in $L^2(\mathbb{T})$.

Using this lemma, we can prove a very important property of the Fourier series: the localization property. It means that the behaviour of the Fourier series of f at a point $x_0 \in \mathbb{T}$ depends only on the behaviour of f in a neighbourhood of x_0. This can be surprising, since the Fourier coefficients are defined integrating f in the whole torus $\mathbb{T}$.

Theorem 2.4 (Riemann's localization principle)**.** *Let x_0 be a point in $\mathbb{T}$.*

(i) *If $f \in L^1(\mathbb{T})$ is a function such that $f(x) = 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$ and some $\delta > 0$, then*

$$\lim_{N\to\infty} S_N(f)(x_0) = 0.$$

(ii) *If $f, g \in L^1(\mathbb{T})$ and $f(x) = g(x)$ for all $x \in (x_0 - \delta, x_0 + \delta)$ and some $\delta > 0$, then either $\lim_{N\to\infty} S_N(f)(x_0)$ and $\lim_{N\to\infty} S_N(g)(x_0)$ both exist and are equal or neither of them exists.*

Proof. (i) From the hypotheses and from the formula (2.4) we have

$$S_N(f)(x_0) = \frac{1}{\pi}\int_{\delta\le|t|<\pi} \frac{f(x_0-t)}{2\sin t/2}\sin(N+1/2)t\,\mathrm{d}t.$$

The function $\sin t/2$ is continuous and does not vanish in the integration domain. Since $f(x_0 - t)$ is integrable, it follows that the function

$$g(t) = \begin{cases} \frac{f(x_0-t)}{2\sin t/2}, & \delta \le |t| < \pi, \\ 0, & |t| \le \delta, \end{cases}$$

is integrable in $\mathbb{T}$. Thus, the Riemann-Lebesgue lemma implies that $\lim_{N\to\infty} S_N(f)(x_0) = 0$.

(ii) It is sufficient to notice that $(f - g)(x) = 0$ in $(x_0 - \delta, x_0 + \delta)$, and to apply (i) to the function $f - g$. □

Another simple but effective consequence of the Riemann-Lebesgue lemma is a condition for the convergence of the Fourier series at a point.

Theorem 2.5. *Let f be an integrable function in $\mathbb{T}$, and suppose that f is derivable at $x_0 \in \mathbb{T}$. Then the Fourier series of f converges to $f(x_0)$ at x_0:*

$$\lim_{N\to\infty} S_N(f)(x_0) = f(x_0).$$

Proof. Using the properties of the Dirichlet kernel from Proposition 2.2, we shall write

$$\begin{aligned} S_N(f)(x_0) - f(x_0) &= \frac{1}{\pi}\int_{\mathbb{T}} [f(x_0+t) - f(x_0)]\, D_N(t)\,\mathrm{d}t \\ &= \frac{1}{\pi}\int_{\mathbb{T}} \frac{f(x_0+t) - f(x_0)}{t}\frac{t}{2\sin t/2}\sin(N+1/2)t\,\mathrm{d}t. \end{aligned}$$

We know that the first factor is integrable in $(-\pi, -\delta) \cup (\delta, \pi)$, and it is bounded in $(-\delta, \delta)$, since f is derivable in x_0. So the first factor is integrable in $\mathbb{T}$. The second factor is continuous in $\mathbb{T}$. Thus, we can use lemma 2.3 to conclude that the limit of $S_N(f)(x_0) - f(x_0)$ is 0. □

Remark 2.6. *If f is not derivable at x_0, but its both left-sided and right-sided derivatives exist, then the conclusion of Theorem 2.5 still holds. This can easily be checked using expression* (2.5)*.*

2.3 The Dirichlet theorem

The first convergence theorem was given by Dirichlet in 1829. It requires the function f to be bounded and continuous in $\mathbb{T}$, except for a finite number of points, and to have a finite number of maxima and minima in $\mathbb{T}$. Later on, Dirichlet realized that the hypothesis of f being bounded could be replaced by the hypothesis of $|f|$ having a finite integral in $\mathbb{T}$.

Let f be a piecewise monotone function in $\mathbb{T}$ (meaning that we can decompose $\mathbb{T}$ in a finite number of smaller intervals in which f is monotone). Then f has finite left- and right-limits at every point of $\mathbb{T}$.

We will prove the Dirichlet theorem in a simplified way discovered by Bonnet. Dirichlet's original proof is much longer, but not complicated.

Lemma 2.7 (Bonnet 1850)**.** *Let g be a non-decreasing and non-negative function on $[a,b]$, and let h be continuous and with a finite number of sign changes in $[a,b]$. Then there exists some $c \in (a,b)$ such that*

$$\int_a^b g(t)h(t)\,\mathrm{d}t = g(b)\int_c^b h(t)\,\mathrm{d}t.$$

Proof. First of all, let us decompose the interval (a,b) in the smaller intervals $(a_0,a_1),(a_1,a_2),...,(a_{k-1},a_k)$ where h has constant sign. There exist $\mu_j \in [g(a_{j-1}+),g(a_j-)]$ such that

$$\int_{a_{j-1}}^{a_j} g(t)h(t)\,\mathrm{d}t = \mu_j \int_{a_{j-1}}^{a_j} h(t)\,\mathrm{d}t.$$

Define $H(x) = \int_x^b h(t)\,\mathrm{d}t$, so that

$$\begin{aligned}\int_a^b g(t)h(t)\,\mathrm{d}t &= \sum_{j=1}^{k} \mu_j\,[H(a_{j-1}) - H(a_j)] \\ &= \mu_1 H(a) + \sum_{j=1}^{k-1} (\mu_{j+1} - \mu_j)H(a_j) + (g(b) - \mu_k)H(b).\end{aligned}$$

Notice that $H(b) = 0$, so the expression is the same. All the coefficients that are multiplying H in the expression above are non-negative because g was non-negative and non-decreasing. So the sum from the third term can be expressed as the sum of all those coefficients times a number between the maximum and the minimum of H. Since H is continuous, we know that H

reaches that value in some $c \in (a,b)$:

$$\begin{aligned}\int_a^b g(t)h(t)\,\mathrm{d}t &= H(c)\left(\mu_1 + \sum_{j=1}^{k-1}(\mu_{j+1}-\mu_j) + g(b) - \mu_k\right)\\ &= H(c)g(b)\\ &= g(b)\int_c^b h(t)\,\mathrm{d}t.\end{aligned}$$ □

Lemma 2.8. *There exists $M > 0$ such that*

$$\left|\int_\eta^\delta D_N(t)\,\mathrm{d}t\right| \le M,$$

for every $0 \le \eta \le \delta \le \pi$ and for every $N \ge 0$.

Proof. We can add and subtract $t^{-1}\sin(N+1/2)t$ inside the integral and we obtain

$$\left|\int_\eta^\delta \frac{\sin(N+1/2)t}{2\sin t/2}\,\mathrm{d}t\right| \le \int_\eta^\delta \left|\frac{1}{2\sin t/2} - \frac{1}{t}\right|\,\mathrm{d}t + \left|\int_\eta^\delta \frac{\sin(N+1/2)t}{t}\,\mathrm{d}t\right|.$$

The first integral on the second term does not depend on N, and it is bounded because it is the integral of a continuous function. The second integral does depend on N. If we make the variable change $(N+1/2)t = x$, we obtain

$$\left|\int_A^B \frac{\sin x}{x}\,\mathrm{d}x\right|, \tag{2.6}$$

where $A = \eta/(N+1/2)$ and $B = \delta/(N+1/2)$. So we have to prove that (2.6) is uniformly bounded for any $0 \le A \le B < \infty$. If $B \le 1$, then $\sin t \le t$ and the integral is bounded by 1. If $A \ge 1$, then we can integrate by parts and obtain

$$\left|\int_A^B \frac{\sin t}{t}\,\mathrm{d}t\right| = \left|\frac{\cos A}{A} - \frac{\cos B}{B} - \int_A^B \frac{\cos t}{t^2}\,\mathrm{d}t\right| \le \frac{1}{A} + \frac{1}{B} + \frac{1}{A} \le 3.$$

Finally, if $A < 1 < B$, then we can bound the integrals in $(A,1)$ and $(1,B)$ separately as before. □

Theorem 2.9 (Dirichlet, 1829)**.** *Let f be a piecewise monotone and bounded function on $\mathbb{T}$. Then, for every $x \in \mathbb{T}$, $S_N(f)(x)$ converges to $(f(x+) + f(x-))/2$. In particular, if f is continuous at x, then $S_N(f)(x)$ converges to $f(x)$.*

Proof. We want to show that

$$\lim_{N\to\infty}\int_0^\pi [f(x+t) + f(x-t) - f(x+) - f(x-)]\,D_N(t)\,\mathrm{d}t = 0. \tag{2.7}$$

The function between brackets is a piecewise monotone function such that its right-side limit at 0 is 0. So we just need to prove that the following equality holds for every piecewise monotone function g on $(0, \pi)$ such that $g(0+) = 0$:

$$\lim_{N\to\infty} \int_0^\pi g(t)D_N(t)\,\mathrm{d}t = 0.$$

We may assume that g is increasing on the first interval at the right of 0, otherwise change the sign of g. Take $\epsilon > 0$, and choose $\delta > 0$ such that g is increasing in $(0, \delta)$ and $g(\delta) < \epsilon/2M$ (this M is the constant in Lemma 2.8). If we apply both Lemmas 2.7 and 2.8, we obtain

$$\left|\int_0^\delta g(t)D_N(t)\,\mathrm{d}t\right| = \left|g(\delta)\int_\eta^\delta D_N(t)\,\mathrm{d}t\right| \leq \frac{\epsilon}{2}.$$

If we follow the proof of the Localization principle 2.4, we can make the integral in (δ, π) smaller than $\epsilon/2$, choosing N big enough. This completes the proof. □

This theorem was the first theorem which proved the pointwise convergence of Fourier series. The hypotheses are quite strong. This theorem also applies if f is not a bounded function, but it has to be integrable.

Camille Jordan, while studying Fourier series in his work, came up with a new condition that would assure the pointwise convergence of the Fourier series. He called that property *bounded variation*, and it has many applications in analysis, not necessarily related to the problem of convergence of Fourier series.

Definition 2.10. *Let f be a function on $[a, b]$. We say that f has* bounded variation *if there is some constant $C > 0$ such that*

$$\sum_{i=1}^{k} |f(t_i) - f(t_{i-1})| \leq C \tag{2.8}$$

for every partition $a = t_0 < t_1 < ... < t_k = b$ of the interval. In this case we call total variation of f *the smallest of those constants C, and we denote it as $V(f)$.*

Real-valued functions of bounded variation have a very surprising property: every real-valued function of bounded variation is the difference of two non-decreasing and bounded functions. Indeed, put $Vf(x)$ the total variation of f in $[a, x]$. It is cleat that $V(f)$ is a non-decreasing function, and if $x_1 < x_2$, then

$$Vf(x_2) \geq Vf(x_1) + f(x_2) - f(x_1)$$

so $Vf(x) - f(x)$ is also a non decreasing function. And writing $f = Vf - (Vf - f)$, we prove that statement. For complex-valued functions, we take into account the real and complex parts separately. In this way, Theorem 2.9 applies also to functions of bounded variation:

Theorem 2.11 (Dirichlet-Jordan theorem). *Let f be a function of bounded variation on $\mathbb{T}$. Then $S_N(x)$ converges to $(f(x+)+f(x-))/2$ for every $x \in \mathbb{T}$.*

2.4 The Dini theorem

There are other theorems that give the pointwise convergence of the Fourier series. One of them was first proved by Dini in 1880.

This theorem is more general than a previous theorem proved by Lipschitz in 1864. This theorem has the hypothesis that f satisfies the local Lipschitz condition at a point x_0. The condition described in the theorem is known by the name *Dini condition* and, as the concept of bounded variation, it was born in the context of Fourier series.

Theorem 2.12 (Dini, 1880). *Let f be an integrable function on $\mathbb{T}$, $x_0 \in \mathbb{T}$ and $l \in \mathbb{C}$ a complex number such that the function*

$$\Phi(t) = f(x_0+t) + f(x_0-t) - 2l$$

satisfies the condition

$$\int_{(0,\delta)} \frac{|\Phi(t)|}{t}\,\mathrm{d}t < \infty$$

for some $\delta > 0$. Then, the Fourier series of f converges to l in x_0:

$$\lim_{N\to\infty} S_N(f)(x_0) = l.$$

Proof. Using formula (2.5) and the properties of the Dirichlet kernel, we can write

$$S_N(f)(x_0) - l = \frac{1}{\pi}\int_0^\pi \Phi(t)D_N(t)\,\mathrm{d}t = \frac{1}{\pi}\int_0^\pi \frac{\Phi(t)}{2\sin t/2}\sin(N+1/2)t\,\mathrm{d}t.$$

The function

$$\frac{\Phi(t)}{2\sin t/2} = \frac{\Phi(t)}{t}\frac{t}{2\sin t/2}$$

is integrable in $(0,\delta)$ because the term on the left is integrable by hypothesis and the term on the right is bounded. So it is clear that this function is integrable in $(0,\pi)$. So we can use Lemma 2.3 and we conclude that $\lim_{N\to\infty} S_N(f)(x_0) = l$. □

Remarks 2.13. *We notice the following statements.*

- *If f is continuous at the point x_0, then the only option for l is $f(x_0)$, but the mere continuity does not ensure the Dini condition.*

- *If f satisfies the local Lipschitz condition at $x_0 \in \mathbb{T}$ (and if, of course, f is integrable in $\mathbb{T}$), then we can apply Theorem 2.12 to obtain the pointwise convergence of the Fourier series at x_0.*

- *Suppose that f satisfies the uniform Hölder condition of order α on $\mathbb{T}$. That is, suppose that there exists a positive constant L and $\alpha \in (0,1]$ such that for every pair of points $x, y \in \mathbb{T}$, $|f(x) - f(y)| \le L|x-y|^\alpha$. Then, the Fourier series converges pointwise to f at every point of $\mathbb{T}$.*

We have seen two main theorems of pointwise convergence of the Fourier series. They have very different hypotheses. But neither one is stronger than the other. There exist functions that satisfy the conditions for one theorem but not for the other.

The function

$$f(x) = -\frac{1}{\log|x/2\pi|}$$

satisfies the Dirichlet conditions, but it fails the Dini condition at the origin.

Similarly, the function

$$g(x) = |x|^\alpha \sin\frac{1}{|x|}$$

satisfies the Hölder condition of order α in $\mathbb{T}$, so it satisfies the Dini condition. But g is not of bounded variation, so we cannot apply the Dirichlet-Jordan theorem.

2.5 Uniform convergence

The most simple criterion for the uniform convergence of a series of functions is Weierstrass theorem, which states that given a series of functions, if we can majorate each term by a constant in a way that the series of the constant is convergent, then the original series is uniformly convergent. This can easily be applied to some Fourier series

Proposition 2.14. *Let f be a piecewise C^1 function on $\mathbb{T}$, meaning that f is continuous in $\mathbb{T}$ and derivable except for a finite number of points and that the derivative is piecewise continuous and bounded. Then the Fourier series of f converges uniformly to f*

Proof. The relation between $\hat{f}$ and $(f')\hat{}$ is given in section 1.3:

$$(f')\hat{}(n) = in\hat{f}(n), \ \forall n \in \mathbb{Z}.$$

Now, using the simple but effective inequality $2\xi\eta \le \xi^2 + \eta^2$ we obtain

$$|\hat{f}(n)| \le \frac{1}{2}\left(\frac{1}{n^2} + n^2|\hat{f}(n)|^2\right) \le \frac{1}{2}\left(\frac{1}{n^2} + |(f')\hat{}(n)|^2\right).$$

Using Bessel inequality, we conclude that the series

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|$$

is convergent. So the Fourier series of f is uniformly convergent to f. □

Remark 2.15. *If the Fourier series of f is uniformly convergent to f, then f must be continuous, because the partial sums are all trigonometric polynomials which are always continuous. So if f is not continuous, there is no hope for the uniform convergence.*

The localization principle says that the behaviour of the series at a point depends only on the behaviour of the function on a neighbourhood of that point. We will show that we have a similar result for uniform convergence. We will begin giving a new version of the Riemann-Lebesgue lemma.

Lemma 2.16. *Let $f \in L^1(\mathbb{T})$ be a bounded function and g a bounded and piecewise monotone function. Then,*

$$\lim_{\lambda\to\infty} \int_{\mathbb{T}} f(x+t)g(t) \sin \lambda t \, \mathrm{d}t = 0$$

uniformly on x.

Proof. We can suppose without loss of generality that f and g are positive. Let M_f and M_g be the bounds of f and g respectively. Given $\epsilon > 0$, there exists a step function $h = \sum m_j \chi_{I_j}$, where I_j is an interval, that satisfies $0 \le h \le f$ and

$$\|f-h\|_1 = \int_{\mathbb{T}} f - h \le \frac{\epsilon}{2M_g}.$$

Thus

$$\int_{\mathbb{T}} f(x+t)g(t)\sin\lambda t \, \mathrm{d}t$$
$$= \int_{\mathbb{T}} (f(x+t) - h(x+t))g(t)\sin\lambda t \, \mathrm{d}t + \int_{\mathbb{T}} h(x+t)g(t)\sin\lambda t \, \mathrm{d}t.$$

On the one hand, the first integral is clearly bounded by $\epsilon/2$. On the other hand, for the second integral we have

$$\sum_{j=1}^{J} m_j \int_{I_j - x} g(t)\sin\lambda t \, \mathrm{d}t,$$

where we can apply the second mean value theorem for integrals (remember that g was piecewise monotone). We obtain the bound for the second integral

$$\frac{C M_f M_g J}{\lambda}.$$

We can make that bound smaller than $\epsilon/2$ choosing λ big enough. This was all independent from x, so we conclude the proof. □

Theorem 2.17. *Let f be a function that vanishes on $[a, b] \subset \mathbb{T}$. Then the Fourier series of f tends to zero uniformly on $[a + \delta, b - \delta]$, for $\delta > 0$.*

Proof. If $x \in [a + \delta, b - \delta]$, then

$$S_N(f)(x) = \frac{1}{\pi} \int_{|t| \geq \delta} f(x - t) \frac{\sin(N + 1/2)t}{2 \sin t/2} \, \mathrm{d}t.$$

Put $g(t) = \chi_{|t| \geq \delta} / \sin t/2$. Then we can apply Lemma 2.16 and obtain the desired result. □

Theorem 2.18. *Let f be a continuous function on $\mathbb{T}$, and suppose that f has a piecewise continuous and bounded derivative on $[a, b]$. Then the Fourier series of f tends to f uniformly on $[a + \delta, b - \delta]$, for $\delta > 0$.*

Proof. We just need to define a function g that agrees with f on $[a, b]$ and that satisfies the conditions for Proposition 2.14 outside of the interval. Then apply Theorem 2.17 to $f - g$ and we finish the proof. □

2.6 The Gibbs phenomenon

The Gibbs phenomenon appears when dealing with real valued functions that are piecewise continuous and have a (finite) jump discontinuity. For a complex valued function, we would analyse the real and the imaginary part of the function separately. For example, let us examine the first few partial sums of the sign function:

$$f(x) = \operatorname{sign} x = \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x < 0. \end{cases}$$

The Fourier series of f is, in its real form

$$\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k + 1)x}{2k + 1}.$$

We can apply the Dirichlet theorem 2.9, and we see that the series converges to f pointwise. There cannot be uniform convergence in $\mathbb{T}$ because the sign function is not continuous. But for any $\delta > 0$ there is uniform convergence in the closed interval $[\delta, \pi - \delta]$, by Theorem 2.18.

In Figure 2.1, there is a representation of S_N. We see that in the central part of each interval we are relatively close to f, and at the discontinuities we have an overshoot in the oscillations of the partial sum. In Figures 2.2

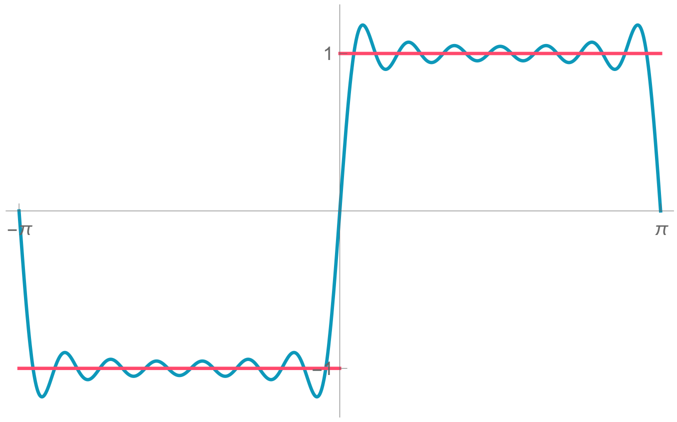

Figure 2.1: Graphic of S_6.

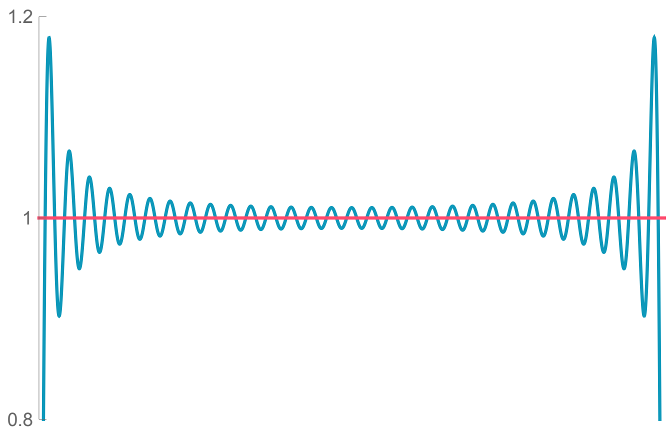

Figure 2.2: Detail of the graphic of S_{30}.

and 2.3 we see that this overshoot does not die out. Indeed, it approaches the value

$$\frac{2}{\pi}\int_0^\pi \frac{\sin x}{x}\,\mathrm{d}x \approx 1.178979744472167.$$

Let us see that this is true. First we locate the maximum of S_{2N-1}. It will be at the points at which the derivative vanishes.

$$\frac{d}{dx}S_{2N-1}(f)(x) = \frac{4}{\pi}\sum_{k=0}^{N-1}\cos{(2k+1)x} = \frac{4}{\pi}\frac{\sin 2Nx}{\sin x}.$$

This derivative vanishes at the points $x = k\pi/2N, k = 1, 2, ..., 2N-1$. The second derivative has the same sign as $\cos 2Nx$ at those points, so the fist one is a local minimum, the second one a local maximum and so on.

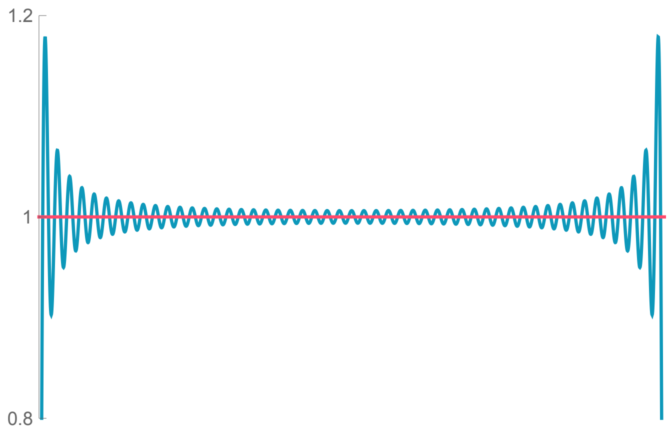

Figure 2.3: Detail of the graphic of S_{50}.

We can use the fundamental theorem of calculus to obtain

$$S_{2N-1}(f)(x) = \frac{4}{\pi}\int_0^x \frac{\sin 2Nt}{\sin t}\,\mathrm{d}t,$$

since we have that $S_{2N-1}(f)(0) = 0$. Thus, we can easily check that the global maximum is obtained in $x = \pi/2N$. We are going to evaluate the partial sum at this point and we are going to compute the limit as $N \to \infty$. We want to know the value of

$$\lim_{N\to\infty} S_{2N-1}(f)\left(\frac{\pi}{2N}\right) = \lim_{N\to\infty} \frac{4}{\pi}\sum_{k=0}^{N-1} \frac{\sin(2k+1)\pi/2N}{2k+1}.$$

In order to compute this limit, we consider the function $g(x) = \sin x/x$. We know that it is integrable in the interval $(0, \pi)$, because it is bounded. We can write the Riemann sum in the partition $x_k = k\pi/N$, for $k = 0, ..., N$. We evaluate the Riemann sum in the middle point of each interval. We obtain

$$S_{\{x_k\}}(g) = \frac{\pi}{N}\sum_{k=1}^{N-1} \frac{\sin(2k+1)\pi/2N}{(2k+1)\pi/2N} = \frac{\pi}{2}S_{2N-1}(f)\left(\frac{\pi}{2N}\right).$$

We conclude that the value of the overshoot at the discontinuity is

$$\lim_{N\to\infty} S_{2N-1}(f)\left(\frac{\pi}{2N}\right) = \frac{2}{\pi}\int_0^\pi \frac{\sin t}{t}\,\mathrm{d}t = \lambda.$$

This phenomenon is known as the Gibbs phenomenon. It also happens to other functions with jump discontinuities. Let g be a function on $\mathbb{T}$ that has a jump discontinuity at x_0. Let $2l = g(x_0+) - g(x0-)$. Then the function $g - 2lf$ is continuous at x_0, where f is the sign function above. If g satisfies one of the convergence hypotheses, then the Fourier series of $g - 2lf$ converges uniformly in a neighbourhood of x_0 and, in consequence, the behaviour of the partial sums of g is the same as the behaviour of the partial sums of $2lf$ in that neighbourhood. Thus, the value of the limit of the overshoot of the partial sums of g will be $l\lambda$.

The Gibbs phenomenon was discovered by Wilbraham in 1848, but it was forgotten until 1898, when Michelson and Stratton designed a tool that could make graphics of the partial sums of Fourier series. Gibbs successfully analysed the phenomenon in 1899, and that is why it is called the Gibbs phenomenon.

The Gibbs phenomenon is one of the causes of *ringing artefacts* in signal processing. In this field, an artefact is an error in the perception of a visual information introduced by the involved techniques. In particular, in digital image processing, ringing artefacts appear near sharp transitions in a signal. Some image compressing algorithms use Fourier analysis, so the Gibbs phenomenon appears every time there is a sharp transition in the image.

Chapter 3

Summability of Fourier series

3.1 Cesàro summability

When we are dealing with number series, we think of the series as a formula to compute the total "sum" of the terms of the series, whenever it exists. But we actually obtain that sum as the limit of the partial sums. If the sequence of partial sums fails to have a limit, we may want to find another way of giving a meaning to the sum of the series. The first way we study is called the Cesàro summability.

Given a numerical series

$$\sum_{k=0}^{\infty} a_k,$$

we define the averages of the partial sums as

$$\sigma_N = \frac{1}{N+1}\sum_{n=0}^{N} s_n = \frac{1}{N+1}\sum_{n=0}^{N}\sum_{k=0}^{n} a_k = \frac{1}{N+1}\sum_{k=0}^{N}\left(1-\frac{k}{N+1}\right) a_k. \tag{3.1}$$

Definition 3.1. *We will say that the series* $\sum_{k=0}^{\infty} a_k$ *is* Cesàro summable *to* a *if*

$$\lim_{N\to\infty} \sigma_N = a.$$

In that case, we write

$$\sum_{k=0}^{\infty} a_k = a \qquad (C).$$

Thankfully, one can check that if a series converges to a in the traditional sense, it is Cesàro summable to a. This way, we have a more general meaning of the convergence of a series.

Example 3.1. The series $\sum_{n=0}^{\infty}(-1)^n$ is divergent, but

$$\sum_{n=0}^{\infty}(-1)^n = \frac{1}{2} \quad (C).$$

We are interested in applying the Cesàro summability to Fourier series. The partial sums of a Fourier series $\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{inx}$ often do not converge as we would like, but the corresponding Cesàro partial sums may have a better behaviour. These Cesàro partial sums are

$$\sigma_N(f)(x) = \frac{s_0(f)(x) + \ldots s_N(f)(x)}{N+1} = \sum_{k=-N}^{N}\left(1 - \frac{|k|}{N+1}\right)\hat{f}(n)e^{ikx}. \quad (3.2)$$

Using formula (1.7) we can express $\sigma_N(x)$ in the trigonometric form

$$\sigma_N(f)(x) = \frac{a_0(f)}{2} + \sum_{k=1}^{N}\left(1 - \frac{k}{N}\right)(a_k(f)\cos kx + b_k(f)\sin kx). \quad (3.3)$$

We can express the partial Cesàro sum as a convolution, in a similar way as we did with the partial sum and the Dirichlet kernel in (2.3):

$$\begin{aligned}
\sigma_N(f)(x) &= \frac{1}{N+1}\sum_{k=0}^{N} S_k(f)(x) \\
&= \frac{1}{N+1}\sum_{k=0}^{N}\frac{1}{\pi}\int_{\mathbb{T}} f(t)D_k(t-x)\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_{\mathbb{T}} f(t)\left(\frac{1}{N+1}\sum_{k=0}^{N} D_k(t-x)\right)\mathrm{d}t.
\end{aligned}$$

Using the fact that $2\sin t/2\sin(k+1/2)t = \cos jt - \cos(j+1)t$, we obtain

$$\sum_{k=0}^{N} D_k(t) = \sum_{k=0}^{N}\frac{\sin(j+1/2)t}{2\sin t/2} = \frac{1-\cos(N+1)t}{4(\sin t/2)^2} = \frac{1}{2}\left(\frac{\sin(N+1)t/2}{\sin t/2}\right)^2.$$

Definition 3.2 (Fejér kernel)**.** *For $N \geq 0$, the function*

$$F_N(t) = \frac{1}{N+1}\sum_{k=0}^{N-1} D_N(t) = \frac{1}{2(N+1)}\left(\frac{\sin(N+1/2)t/2}{\sin t/2}\right)^2 \quad (3.4)$$

is called the Fejér kernel*.*

In this way, we can write the $N-$th Cesàro partial sum of the Fourier series in the following way:

$$\sigma_N(f)(x) = \frac{1}{\pi}\int_{\mathbb{T}} f(t)F_N(t-x)\,\mathrm{d}t = \frac{1}{\pi}(f * F_N)(x). \tag{3.5}$$

If we use (2.5), we obtain an analogous formula for $\sigma_N(f)$:

$$\sigma_N(f)(x) = \frac{1}{\pi}\int_0^{\pi} (f(x+t)f(x-t))\, F_N(t)\,\mathrm{d}t. \tag{3.6}$$

Proposition 3.3. *The Fejér kernel has the following properties:*

(i) *F_N is a continuous, bounded, even and non-negative function.*

(ii) *$\int_{\mathbb{T}} F_N(t)\,\mathrm{d}t = \pi$ for every N.*

(iii) *For every $\delta > 0$, $F_N(t)$ tends uniformly to zero as N tends to infinity in $\mathbb{T}\setminus(-\delta,\delta)$.*

Proof. The first statement is clear from (3.4).

The second statement is also clear from the fact that F_N is the arithmetic mean of Dirichlet kernels, whose integral is also π.

The third statement follows from the bound

$$F_N(t) \le \frac{1}{2(N+1)\sin^2\delta/2}. \qquad \square$$

With the help of the Féjer kernel, we are able to obtain summability of the Fourier series (in the Cesàro meaning, of course) for every continuous function.

Theorem 3.4 (Fejér)**.** *Let $f \in L^1(\mathbb{T})$ be an integrable function that has side-limits at the point x_0. Then*

$$\lim_{N\to\infty} \sigma_N(f)(x_0) = \frac{1}{2}\left[f(x_0+) + f(x_0-)\right].$$

In particular, if f is continuous at x_0, then

$$\lim_{N\to\infty} \sigma_N(f)(x_0) = f(x_0).$$

Proof. Let us write, using (3.6) and the second property from Proposition 3.3,

$$\begin{aligned}\sigma_N(f)(x) &- \frac{1}{2}\left[f(x+) + f(x-)\right] \\ &= \frac{1}{\pi}\int_0^{\pi} F_N(t)\left[f(x-t) - f(x-) + f(x+t) - f(x+)\right]\mathrm{d}t.\end{aligned} \tag{3.7}$$

We know the behaviour of the Fejér kernel outside a neighbourhood of zero, so we split the integral into four parts. First, we take $f(x-t) - f(x-)$:

$$\left| \int_0^\pi F_N(t) \left[f(x-t) - f(x-) \right] \mathrm{d}t \right| \leq \int_0^\delta + \int_\delta^\pi \left| F_N(t) \left[f(x-t) - f(x-) \right] \right| \mathrm{d}t$$
$$\leq \pi \sup_{0 \leq t \leq \delta} |f(x-t) - f(x-)| + \left(\sup_{\delta \leq t \leq \pi} F_N(t) \right) \left(\int_{\mathbb{T}} |f| + |f(x-)| \right)$$

Given $\epsilon > 0$, we can choose δ small enough to make the first term less than $\epsilon/4$, because we know that the left-side limit is finite. Once we have this δ fixed, we can choose N large enough to make the second term smaller than $\epsilon/4$, using the third property from Proposition 3.3, and that $f \in L^1(\mathbb{T})$. If we make a similar argument for $f(x+t) - f(x+)$, we bound (3.7) by ϵ, thus finishing the proof. □

With this proof, we see that the Fejér means have very good behaviour with continuous functions, while the traditional Fourier series behaves poorly. If we tried to copy this proof with the Dirichlet kernel, it would fail because the Dirichlet kernel is not a positive function, and the behaviour of the absolute value is not very good. In particular, the integral of the absolute value of the Dirichlet kernel is not uniformly bounded in N, as we shall see in Chapter 5.

But this theorem allows us to prove Dirichlet's theorem. A theorem by Hardy (Theorem A.4) says that if a series is Cesàro summable and its general term a_n has the property that $|na_n|$ is uniformly bounded, then the series is convergent in the traditional way. If we have a function under the hypotheses of Dirichlet's theorem (a piecewise monotone and continuous function), then its Fourier coefficients have that property (Exercise 1). So Fejér theorem is stronger than the Dirichlets theorem.

3.2 Abel-Poisson summability

Another way of making a non-convergent numerical series summable is based on a result by Abel. If a series $\sum a_n$ is convergent, then $\sum a_n r^n$ is also convergent for $0 < r < 1$, and defines a continuous function $S(r)$. Abel proved that in this case $\lim_{r \to 1-} S(r)$ coincides with the sum of the original series. But $S(r)$ and its limit can exist even if the original series is not convergent. This way of summing series is called *Abel summability*. Poisson used this fact in order to prove convergence of the Fourier series, but he did not succeed. We will study this summability for the Fourier series, and it is called in this context the Abel-Poisson summability.

We want to study the series of the form

$$\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx}r^{|k|}. \tag{3.8}$$

We want to express it in an integral form. If $f \in L^1(\mathbb{T})$, then the coefficients $\hat{f}(n)$ are bounded by the Riemann-Lebesgue lemma. So the series (3.8) is continuous for $0 < r < 1$ and uniformly convergent in $0 \leq r \leq 1-\epsilon$ for any $\epsilon > 0$. With these observations, the computations bellow are justified.

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx}r^{|k|} &= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi}\int_{\mathbb{T}} f(t)e^{-ikt}\,\mathrm{d}t\; e^{ikx}r^{|k|} \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} f(t) \sum_{k=-\infty}^{\infty} e^{-ikt}e^{ikx}r^{|k|}\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_{\mathbb{T}} f(t)\frac{1}{2} \sum_{k=-\infty}^{\infty} e^{ik(x-t)}r^{|k|}\,\mathrm{d}t.
\end{aligned}$$

Definition 3.5 (Poisson kernel)**.** *We will call* Poisson kernel *to the function*

$$P_r(t) = \frac{1}{2}\sum_{k=-\infty}^{\infty} e^{ikt}r^{|k|}. \tag{3.9}$$

This expression is very difficult to work with. We will introduce a more compact form.

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} e^{ikt}r^{|k|} &= \sum_{k=-\infty}^{0} e^{ikt}r^{-k} + \sum_{k=0}^{\infty} e^{ikt}r^{k} - 1 \\
&= \sum_{k=0}^{\infty} e^{-ikt}r^{k} + \sum_{k=0}^{\infty} e^{ikt}r^{k} - 1 \\
&= \frac{1}{1-re^{-it}} + \frac{1}{1-re^{it}} - 1 \\
&= \frac{1-r^2}{1-2\cos t + r^2}
\end{aligned}$$

If we multiply by $1/2$, we obtain the formula

$$P_r(t) = \frac{1-r^2}{2(1-2r\cos t + r^2)}. \tag{3.10}$$

With this notation we can write:

$$\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx}r^{|k|} = \frac{1}{\pi}(P_r * f)(x). \tag{3.11}$$

Proposition 3.6. *The Poisson kernel has the following properties:*

- P_r *is a continuous, bounded, even and non-negative function.*
- $\int_{\mathbb{T}} P_r(t)\,\mathrm{d}t = \pi$ *for every* $0 \le r < 1$.
- *For every* $\delta > 0$, $P_r(t)$ *tends uniformly to zero as* $r \to 1-$ *in* $\mathbb{T}\backslash(-\delta, \delta)$.

Proof. The first statement is clear from (3.10) except maybe for the non negativity. Write $1 - 2r\cos t + r^2 = (1-r)^2 + 2r(1-\cos t)$ and now it is clear that P_r is non-negative.
The second statement can be seen integrating (3.9) term by term, which can be done because we have uniform convergence.
For the third statement, we can bound the denominator by $(1-r)^2 + 2r(1-\cos\delta)$ if $\delta \le t \le \pi$. And now taking the limit as $r \to 1$, the result is clear. □

The Poisson kernel has exactly the same properties as the Fejér kernel. If we analyse the proof of Theorem 3.4, we only use the properties from Proposition 3.3, so the same conclusion will hold for the Poisson kernel.

Theorem 3.7. *Let* $f \in L^1(\mathbb{T})$ *be an integrable function that has side-limits at the point* x_0. *Then*

$$\lim_{r\to 1-} (P_r * f)(x_0) = \frac{1}{2}\left[f(x_0+) + f(x_0-)\right].$$

In particular, if f *is continuous at* x_0, *then*

$$\lim_{r\to 1-} (P_r * f)(x_0) = f(x_0).$$

Proof. Let us write the convolution, using a small change of variables and the fact that P_r is even:

$$(P_r * f)(x) = \int_{\mathbb{T}} P_r(t) f(x-t)\,\mathrm{d}t = \int_0^\pi P_r(t)\left[f(x+t) + f(x-t)\right]\,\mathrm{d}t.$$

Then,

$$\begin{aligned}(P_r * f)(x) - \frac{1}{2}\left[f(x+) + f(x-)\right] &= \\ = \frac{1}{\pi}\int_0^\pi P_r(t)\left[f(x-t) - f(x-) + f(x+t) - f(x+)\right]\,\mathrm{d}t.\end{aligned} \tag{3.12}$$

We know the behaviour of the Poisson kernel outside a neighbourhood of zero, so we split the integral into four parts. First, we take $f(x-t) - f(x-)$:

$$\left|\int_0^\pi P_r(t)\,[f(x-t)-f(x-)]\,\mathrm{d}t\right| \le \int_0^\delta + \int_\delta^\pi |P_r(t)\,[f(x-t)-f(x-)]|\,\mathrm{d}t$$
$$\le \pi \sup_{0\le t\le\delta} |f(x-t)-f(x-)| + \left(\sup_{\delta\le t\le\pi} P_r(t)\right)\int_{\mathbb{T}} |f| + \pi|f(x-)|.$$

Given $\epsilon > 0$, we can choose δ small enough to make the first term less than $\epsilon/4$, because we know that the left-side limit is finite. Once we have this δ fixed, we can choose r close enough to 1 to make the second term smaller than $\epsilon/4$, using the third property from Proposition 3.6, and that $f \in L^1(\mathbb{T})$. If we make a similar argument for $f(x+t) - f(x+)$, we bound (3.12) by ϵ. □

There is a theorem by Frobenius (Exercise 5) that says that if a series is Cesàro summable then it is Abel summable. So this would mean that Abel summability is stronger than Cesàro summability, and as a consequence, Abel-Poisson summabiliy is stronger than Fejér summability, if we consider Fourier series. But in practice, it is much easier to approximate the limit of the Fejér means than Poisson means, because the Fejér means have a discrete set of indices. Also, the Fejér means has only one limit, while the Poisson means need two limits (first, the series; then $r \to 1$), and it is not clear if we can change the order of those limits with all the freedom we want.

3.3 Approximate identities

We can do the same thing we did with Cesàro and Abel-Poisson summability in a more general context using approximate identities. If we take a closer look to the proofs of Theorems 3.4 and 3.7, we see that we have only use a few of the properties of the respective kernels. We will make a generalization of these properties using approximation identities.

We need to introduce the concept of directed sets. Informally, a directed set I is a set of indices that have a limit. More precisely, it is a set I together with a collection of subsets $\{A_i\}$ such that for every (i, j), there exists k with $A_k \subset A_i \cap A_j$.

If we have a complex valued function f defined on a directed set I, we say that f has limit L if for every $\epsilon > 0$, there exists a subset A_{j_ϵ} such that $|f(x) - L| < \epsilon$ for every $x \in A_{j_\epsilon}$.

Example 3.2. The following sets are the most common directed sets

(i) $\mathbb{N}$ is a directed set with the subsets $A_k = \{k, k+1, ...\}$. The limit is the usual one: $n \to \infty$.

(ii) The set $[0, 1)$ is also a directed set with the subsets $A_k = (1 - 1/k, 1)$. In this case the limit is $r \to 1-$.

Definition 3.8. *An approximate identity in the circle $\mathbb{T}$ is a function $k(r,\theta)$ defined for $\theta \in \mathbb{T}$ and r in some directed index set I, with the following three properties:*

$$\lim_r \frac{1}{2\pi}\int_{\mathbb{T}} k(r,\theta)\,\mathrm{d}\theta = 1, \tag{3.13}$$

$$\int_{\mathbb{T}} |k(r,\theta)|\,\mathrm{d}\theta \leq C, \tag{3.14}$$

$$\lim_r \int_{|\theta|>\delta} |k(r,\theta)|\,\mathrm{d}\theta = 0, \quad \forall \delta > 0, \tag{3.15}$$

where C is a constant independent of r.

For example, both the Poisson kernel $P_r(\theta)$ and the Féjer kernel $F_N(\theta)$ are approximate identities.

Proposition 3.9. *Suppose that $k(r,\theta)$ is an approximate identity.*

- *If $\Phi \in L^\infty(\mathbb{T})$ with $\lim_{\theta\to 0}\Phi(\theta) = L$, then*

$$\lim_r \frac{1}{2\pi}\int_{\mathbb{T}} k(r,\theta)\Phi(\theta)\,\mathrm{d}\theta = L \tag{3.16}$$

- *If, in addition, we have that for every $\delta > 0$ $\sup_{|\theta|\geq\delta}|k(r,\theta)| \to 0$, then equality* (3.16) *holds for all $\Phi \in L^1(\mathbb{T})$ with $\lim_{\theta\to 0}\Phi(\theta) = L$.*

Proof. Take $\delta > 0$. Then, using (3.13) we have

$$\frac{1}{2\pi}\int_{\mathbb{T}} k(r,\theta)\Phi(\theta)\,\mathrm{d}\theta - L = \frac{1}{2\pi}\left(\int_{|\theta|>\delta} + \int_{|\theta|\leq\delta}\right) k(r,\theta)(\Phi(\theta)-L)\,\mathrm{d}\theta + o(1).$$

By (3.15), the first integral tends to zero, for any $\delta > 0$. Given $\epsilon > 0$, the second integral can be made less than $\epsilon/2$ taking δ small enough ($\Phi \in L^\infty$ and use property 3.14), proving the first statement.

In order to prove the second statement, notice that the first integral can be bounded by $\sup_{|\theta|>\delta}|k(r,\theta)|(L+||\Phi||_1)$, which tends to zero by the extra hypothesis. We can bound the second integral by $\epsilon\int_{\mathbb{T}}|k(r,\theta)|\,\mathrm{d}\theta$ taking δ small enough. This completes the proof. □

We want to apply approximate identities to norm convergence. We introduce the following notation: $f_\phi(\theta) = f(\theta-\phi)$ is the translate of f.

Definition 3.10. *A subspace $B \subset L^1(\mathbb{T})$ is called a* homogeneous Banach subspace *if it has the following properties:*

- $\|f\|_1 \leq \|f\|_B$.

- *The map $f \to f_\theta$ is continuous in the B-norm; that is, for every $f \in B$,*

$$\lim_{\theta \to 0} \|f - f_\theta\|_B = 0.$$

- *The map $f \to f_\theta$ preserves the B-norm.*

For example, the space $C(\mathbb{T})$ with the supremum norm and the spaces $L^p(\mathbb{T})$ with $1 \leq p < \infty$ are all homogeneous Banach subspaces, but $L^\infty(\mathbb{T})$ is not an homogeneous subspace.

Theorem 3.11. *If B is a homogeneous Banach subspace of $L^1(\mathbb{T})$ and $k(r, \theta)$ is an approximate identity, then for every $f \in B$,*

$$\lim_{r} \left\| \frac{1}{2\pi} \int_{\mathbb{T}} k(r, \phi) f_\phi \mathrm{d}\phi - f \right\|_B = 0. \tag{3.17}$$

Proof. Since B is a homogeneous Banach subspace, this norm is smaller than

$$\frac{1}{2\pi} \int_{\mathbb{T}} |k(r, \phi)| \, \|f_\phi - f\|_B \, \mathrm{d}\phi,$$

which tends to zero by Proposition 3.9. □

With this theorem, we conclude that the Féjer means converge uniformly when f is a continuous function, and they also converge in the norm of $L^p(\mathbb{T})$, when $1 \leq p < \infty$.

Corollary 3.12. *The space of trigonometric polynomials is dense in the spaces $C(\mathbb{T})$ and $L^p(\mathbb{T})$ for $1 \leq p < \infty$.*

Remark 3.13. *There is no convergence in L^∞ because if there were convergence, it would be uniform convergence. This would imply that the initial function, an arbitrary function in $L^\infty(\mathbb{T})$, must be continuous, which is absolutely false.*

From this theorem we can also conclude the uniqueness of the Fourier coefficients of an integrable function:

Corollary 3.14. *If $f, g \in L^1(\mathbb{T})$, such that $\hat{f}(n) = \hat{g}(n)$ for every $n \in \mathbb{Z}$, then $f = g$ almost everywhere.*

Proof. Once again, we can suppose without loss of generality that $g \equiv 0$. So then $\hat{f}(n) = 0$, for every $n \in \mathbb{Z}$. Then, using Fejér means, we can approximate f in $L^1(\mathbb{T})$ with a sequence of trigonometric polynomials that are all zero. So $f = 0$ almost everywhere. □

Chapter 4

Fourier series in $L^p(\mathbb{T})$

We want to study the convergence of Fourier series in the $L^p(\mathbb{T})$ spaces. In these spaces we are not interested in pointwise convergence, since it does not make sense to talk about the value of a function at a point. The elements of $L^p(\mathbb{T})$ are equivalence classes of functions that agree outside a set of zero measure. Even so, one can study pointwise convergence in a different way, the so-called almost everywhere convergence.

We are interested in norm convergence. We will see that there is norm convergence in $L^p(\mathbb{T})$ when $1 < p < \infty$ but there is no convergence, in general, in both the spaces $L^1(\mathbb{T})$ and $L^\infty(\mathbb{T})$.

4.1 Parseval theorem: Fourier series in $L^2(\mathbb{T})$

We first pay attention to the space $L^2(\mathbb{T})$, which is special because it is a Hilbert space. The main goal of this section is to prove that the basic trigonometric system $\left\{e^{ikx} : k \in \mathbb{Z}\right\}$ is a complete orthogonal system in $L^2(\mathbb{T})$, meaning that every function $f \in L^2(\mathbb{T})$ can be written in a unique way as a series of e^{ikx}, the Fourier series of f.

We first show a general property of the Fourier coefficients. We have proved this property in the previous chapter, but this is a more direct proof and it does not use the theory of summability.

Proposition 4.1. *Let $f, g \in L^1(\mathbb{T})$ be two functions such that $\hat{f}(k) = \hat{g}(k)$ for every integer k. Then $f = g$ almost everywhere.*

Proof. Since the map $f \longrightarrow \hat{f}$ is a linear map, we may suppose that $g = 0$. Then $\hat{f}(k) = 0$ for every integer k.

Let us assume for a second that f is continuous, and write $f = u + iv$

where u and v are real functions. This way, for any $k \in \mathbb{Z}$,

$$\begin{aligned} 0 = 2\pi \hat{f}(k) &= \int_{\mathbb{T}} (u(x) + iv(v)) e^{-ikx} \, \mathrm{d}x \\ &= \int_{\mathbb{T}} (u(x)\cos kx + v(x)\sin kx) \, \mathrm{d}x + i \int_{\mathbb{T}} (v(x)\cos kx - u(x)\sin kx) \, \mathrm{d}x. \end{aligned}$$

We can do $k = m$ and $k = -m$ and add both equations, and we will obtain

$$\int_{\mathbb{T}} u(x)\cos mx \, \mathrm{d}x = \int_{\mathbb{T}} u(x)\sin mx \, \mathrm{d}x = 0,$$
$$\int_{\mathbb{T}} v(x)\cos mx \, \mathrm{d}x = \int_{\mathbb{T}} v(x)\sin mx \, \mathrm{d}x = 0,$$

for every $m = 0, 1, 2...$ So we reduce the problem to the case of a real-valued continuous function f such that

$$\int_{\mathbb{T}} f(x)\cos mx \, \mathrm{d}x = \int_{\mathbb{T}} f(x)\sin mx \, \mathrm{d}x = 0, \quad m = 0, 1, 2, ... \tag{4.1}$$

If f is not identically zero, there exists a point x_0 such that f is not zero in a neighbourhood of x_0, due to the continuity of f. We may assume, without loss of generality, that $x_0 = 0$ and that $f(x) \geq 1/2$ in a closed interval $I = [-\delta, \delta]$ (taking, for example, the function $f(x - x_0)/f(x_0)$).

Take the functions $t(x) = 1 + \cos x - \cos\delta$ and $T_n(x) = (t(x))^n$. Clearly $t(x) \geq 1$ on I and $|t(x)| < 1$ on $\mathbb{T} \setminus I$. So $T_n(x) \geq 1$ on I while $T_n(x) \to 0$ on $\mathbb{T} \setminus I$ if $n \to \infty$. On the one hand, since T_n is as trigonometric polynomial of degree n, from (4.1) we have

$$\int_{\mathbb{T}} f(x) T_n(x) \, \mathrm{d}x = 0.$$

On the other hand, if we use the dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_{\mathbb{T}-I} f(x) T_n(x) \, \mathrm{d}x = 0.$$

If we subtract both integrals, we obtain $\int_I f(x)T_n(x) \, \mathrm{d}x = 0$ and this contradicts the fact that $f(x)T_n(x) \geq 1/2$ on I. So $f \equiv 0$ in $\mathbb{T}$. This proves the theorem for any complex-valued $f \in C(\mathbb{T})$.

Let us suppose now that f is an arbitrary function of $L^1(\mathbb{T})$. Take

$$F(x) = \int_{-\pi}^{x} f(t) \, \mathrm{d}t,$$

which is a continuous function. Let us compute its Fourier coefficients. If $k = \pm 1, \pm 2, \pm 3, ...$, then

$$\begin{aligned}2\pi\hat{F}(k) &= \int_{\mathbb{T}} F(x)e^{-ikx}\,\mathrm{d}x = \int_{\mathbb{T}} e^{-ikx}\left(\int_{-\pi}^{x} f(t)\,\mathrm{d}t\right)\mathrm{d}x \\ &= \int_{\mathbb{T}} f(t)\left(\int_{t}^{\pi} e^{-ikx}\,\mathrm{d}x\right)\mathrm{d}t \\ &= \int_{\mathbb{T}} f(t)\frac{e^{-ik\pi} - e^{-ikt}}{-ik}\,\mathrm{d}t \\ &= 2\pi\left(\frac{e^{-ik\pi}}{-ik}\hat{f}(0) + \frac{1}{ik}\hat{f}(k)\right) = 0,\end{aligned}$$

where we have used Fubini's theorem and the fact that $\hat{f}(k) = 0$ for every k. So $\hat{F}(k) = 0$, $\forall k \in \mathbb{Z}\setminus\{0\}$.

Set now $A_0 = \frac{1}{2\pi}\int_{\mathbb{T}} F(x)\,\mathrm{d}x$, and define $G(x) = F(x) - A_0$. Clearly $\hat{G}(k) = 0$ for every $k \in \mathbb{Z}$, and G is a continuous function. So, using the first part, we conclude that $G = F - A_0 \equiv 0$. Using Lebesgue's differentiation theorem, we know that $f(x) = (d/dx)F(x) = 0$ almost everywhere, so $f \equiv 0$. This completes the proof. □

Now we can prove the famous Parseval theorem.

Theorem 4.2 (Parseval)**.** *Let $f \in L^2(\mathbb{T})$. Then the sequence of partial sums of the Fourier series is convergent to f in the L^2 norm and*

$$\frac{1}{2\pi}\int_{\mathbb{T}} |f(\theta)|^2\,\mathrm{d}\theta = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2.$$

Proof. First of all, let us check that the sequence of partial sum converges. It is a Cauchy sequence. Indeed,

$$||S_N(f) - S_M(f)||_2^2 = \int_{\mathbb{T}} |S_N(f) - S_M(f)|^2\,\mathrm{d}\theta = 2\pi \sum_{|n|=M+1}^{N} |\hat{f}(n)|^2,$$

and we know that this tends to zero as N and M tend to infinity because of Bessel inequality (1.10). Thus the partial sum sequence is a Cauchy sequence, so it is convergent to some $F \in L^2(\mathbb{T})$. Now, we will see that $F = f$ almost everywhere. We will compute the Fourier coefficients of F:

$$\begin{aligned}2\pi\hat{F}(n) &= \int_{\mathbb{T}} F(\theta)e^{-in\theta}\,\mathrm{d}\theta \\ &= \int_{\mathbb{T}} \left(F(\theta) - S_N(f)(\theta)\right)e^{-in\theta}\,\mathrm{d}\theta + \int_{\mathbb{T}} S_N(f)(\theta)e^{-in\theta}\,\mathrm{d}\theta.\end{aligned}$$

If $N > |n|$, then the second integral equals $2\pi\hat{f}(n)$. Using the Cauchy-Schwarz inequality:

$$2\pi|\hat{F}(n) - \hat{f}(n)| \le \int_{\mathbb{T}} |F(\theta) - S_N(f)(\theta)|\left|e^{-in\theta}\right|\,\mathrm{d}\theta \quad \le ||F - S_N(f)||_2^2$$

So taking $N \to \infty$, we obtain $\hat{F}(n) = \hat{f}(n)$ for all $n \in \mathbb{Z}$. Using Proposition 4.1, we obtain that $f = F$ in $L^2(\mathbb{T})$, so we have proved that the partial sums converge to f. □

Theorem 4.3 (Riesz-Fischer)**.** *Let $\{c_n\} \in l_2(\mathbb{Z})$. Then there exists a unique function $f \in L^2(\mathbb{T})$ such that $\hat{f}(n) = c_n$ for every $n \in \mathbb{Z}$.*

Proof. We know, from Proposition 4.1 that if such an f exists, then it is unique. Consider the sequence $\{T_n(\theta)\}$ in $L^2(\mathbb{T})$, where

$$T_n(\theta) = \sum_{m=-n}^{n} c_m e^{im\theta}.$$

This way, $\{T_n(\theta)\}$ is a Cauchy sequence in $L^2(\mathbb{T})$, if $M < N$:

$$||T_N - T_M||_2^2 = \sum_{M \leq |m| \leq N} |c_m|^2,$$

which tends to zero as N and M tend to infinity, because the sequence was in $l_2(\mathbb{Z})$. Call f to the limit of T_N in $L^2(\mathbb{T})$, and compute its Fourier coefficients:

$$\begin{aligned} 2\pi \hat{f}(n) &= \int_{\mathbb{T}} f(\theta) e^{-in\theta} \, \mathrm{d}\theta \\ &= \int_{\mathbb{T}} \left(f(\theta) - T_N(\theta)\right) e^{-in\theta} \, \mathrm{d}\theta + \int_{\mathbb{T}} T_N(\theta) \, \mathrm{d}\theta. \end{aligned}$$

Once again, if $N > |n|$, then the second integral equals to c_n. Thus

$$2\pi |\hat{f}(n) - c_n| = \int_{\mathbb{T}} |f(\theta) - T_N(\theta)| \, e^{-in\theta} \, \mathrm{d}\theta,$$

which tends to zero if $N \to \infty$. □

With both Theorems 4.2 and 4.3 we have constructed a linear bijective isometry between the spaces $L^2(\mathbb{T})$ and $l_2(\mathbb{Z})$:

$$\begin{aligned} L^2(\mathbb{T}) &\longrightarrow l_2(\mathbb{Z}) \\ f &\longmapsto \hat{f}. \end{aligned}$$

We now give an application of these theorems, a condition for a function to be absolutely continuous. Since we are in the space $L^2(\mathbb{T})$, such a function is equal to an actually absolutely continuous function.

Proposition 4.4. *Let $f \in L^2(\mathbb{T})$ such that the Fourier coefficients satisfy the condition*

$$\sum_{n \in \mathbb{Z}} n^2 |\hat{f}(n)|^2 < \infty.$$

Then there exists a function F absolutely continuous on $\mathbb{T}$ such that $f = F$ almost everywhere. Moreover, $F' \in L^2(\mathbb{T})$ and $(F')\hat{}(n) = in\hat{f}(n)$.

Proof. We know that the series

$$g(\theta) = \sum_{n\in\mathbb{Z}} in\hat{f}e^{in\theta}$$

defines a function in $L^2(\mathbb{T})$, using Theorem 4.3. Moreover, it is clear that

$$\hat{g}(0) = \int \mathbb{T}g(\theta)\,\mathrm{d}\theta = 0.$$

Define the function F in the following way:

$$F(\theta) = \int_{-\pi}^{\theta} g(\phi)\ \mathrm{d}\phi.$$

Clearly, F is an absolutely continuous function, and $F' = g$ a.e. Then $\hat{g}(n) = in\hat{F}(n)$ for all $n \in \mathbb{Z}$, thus $\hat{F}(n) = \hat{f}(n)$ for every non-zero integer n. That means that the function $F - f$ is constant almost everywhere, which completes the proof. □

4.2 An interpolation theorem

In order to study the convergence of Fourier series in the spaces $L^p(\mathbb{T})$, where $p \neq 2$, we use interpolation theory. The main result we are going to use we owe it to M. Riesz and Thorin.

We will first recall some basic properties of the L^p spaces. Let (M, μ) be a measure space:

- If $p_0 < p < p_1$ and $f \in L^{p_0}(M) \cap L^{p_1}(M)$, then $f \in L^p(M)$.
- If M has finite measure and $f \in L^{p_0}(M)$, then $f \in L^p(M)$ for all $p > p_0$.
- If $0 < p_0 < p < p_1 < \infty$ and $f \in L^p(M)$, then there exist $f_0 \in L^{p_0}(M)$ and $f_1 \in L^{p_1}(M)$ such that $f = f_0 + f_1$.

Suppose that we have two measure spaces, (M, μ) and (N, ν) and two pairs of indices, (p_0, q_0) and (p_1, q_1), $1 \leq p_0, q_0, p_1, q_1 \leq \infty$. Suppose that we have two linear and bounded operators

$$A_0 : L^{p_0}(M) \longrightarrow L^{q_0}(N),$$
$$A_1 : L^{p_1}(M) \longrightarrow L^{q_1}(N),$$

with norms $||A_i||_{p_i,q_i} = k_i$ for $i = 0, 1$. Furthermore, suppose that they coincide: $A_0 = A_1$ in $L^{p_0}(M) \cap L^{p_1}(M)$.

In this situation, given a $t \in (0, 1)$, we define the indices (p_t, q_t) using the convex combinations of the conjugate indices:

$$\frac{1}{p_t} = \frac{t}{p_1} + \frac{1-t}{p_0}; \quad \frac{1}{q_t} = \frac{t}{q_1} + \frac{1-t}{q_0}. \tag{4.2}$$

Theorem 4.5 (M. Riesz, Thorin)**.** *In the situation above, there exists a linear operator $A_t : L^{p_t}(M) \to L^{q_t}(N)$ that coincides with A_0 and A_1 on $L^{p_0}(M) \cap L^{p_1}(M)$ and whose norm satisfies*

$$||A_t||_{p_t,q_t} \leq k_0^{1-t} k_1^t.$$

Proof. Let $f \in L^{p_t}(M)$, where p_t is given by the convex combination 4.2 for some t. Then we can write

$$f = f\chi_{|f|\leq 1} + f\chi_{|f|>1},$$

so we define

$$Af = A\left(f\chi_{|f|\leq 1}\right) + A\left(f\chi_{|f|>1}\right).$$

In the space $L^q(N)$, we can compute the norm with the formula

$$\|h\|_q = \sup \int_N hg\, d\nu,$$

where the supremum is taken over the simple functions $g \in L^{q'}$ such that $\|g\|_{q'} \leq 1$. This way, the norm of the operator A is

$$\|A\|_{p,q} = \sup_{\|f\|_p=1,\ \|g\|_{q'}=1} \int_N (Af)g\, d\nu.$$

We extend the interpolated exponents to the complex plane by defining

$$\frac{1}{p(z)} = \frac{z}{p_1} + \frac{1-z}{p_0}, \quad \frac{1}{q'(z)} = \frac{z}{q_1'} + \frac{1-z}{q_0'}, \quad 0 \leq \mathrm{Re}(z) \leq 1.$$

If f and g are both simple functions, we write

$$f = \sum_{j=1}^N a_j e^{i\alpha_j}\chi_{A_j}, \quad f = \sum_{j=1}^N b_j e^{i\beta_j}\chi_{B_j},$$

where $a_j,\ b_j \geq 0$, $\alpha_j, \beta_j \in (0, 2\pi]$ and $A_j,\ B_j$ are pairwise disjoint measurable sets of finite measure in M and N respectively. With the help of the complex indices, we can extend the functions f and g to the strip in the complex plane. In order to do that we define $p = p_t$ and $q' = q_t'$ and we set

$$\phi(\cdot, z) = \sum_{j=1}^N a_j^{\frac{p}{p(z)}} e^{i\alpha_j}\chi_{A_j} = \sum_{j=1}^N a_j^{\frac{p}{p(z)}} \Phi_j,$$

$$\psi(\cdot, z) = \sum_{j=1}^N b_j^{\frac{q'}{q'(z)}} e^{i\alpha_j}\chi_{B_j} = \sum_{j=1}^N b_j^{\frac{q'}{q'(z)}} \Psi_j,$$

where we have set

$$\Phi_j = e^{i\alpha_j}\chi_{A_j}, \quad \Psi_j = e^{i\beta_j}\chi_{B_j}.$$

It is clear that, for each fixed z in the strip $0 \le \mathrm{Re}(z) \le 1$, we have $\phi(\cdot, z) \in L^p(M)$, $\psi(\cdot, z) \in L^{q'}(N)$ and $A\phi(\cdot, z) \in L^q(N)$. Therefore, the function

$$F(z) = \int_N A\phi(\cdot, z)\psi(\cdot, z)\, d\nu = \sum_{j,k=1}^{N} a_j^{\frac{p}{p(z)}} b_k^{\frac{q'}{q'(z)}} \int_N (A\Phi_j)\Psi_k\, d\nu$$

is a finite linear combination of exponential functions and, in particular, an analytic function in the open strip $0 < \mathrm{Re}(z) < 1$ and it is bounded and continuous on the closed strip $0 \le \mathrm{Re}(z) \le 1$. We need to compute the norms on the boundary. For example, if $z = iy$, we have

$$\frac{1}{p(iy)} = \frac{1}{p_0} + \left(\frac{1}{p_1} - \frac{1}{p_0}\right) iy,$$

and since the A_j are pairwise disjoint,

$$|\phi(\cdot, iy)| = \left| \sum_{j=1}^{N} a_j^{\frac{p}{p(iy)}} \Phi_j \right| = \sum_{j=1}^{N} |a_j|^{\frac{p}{p_0}} \chi_{A_j} = |f|^{\frac{p}{p_0}}.$$

Thus the norm will be

$$\|\phi(\cdot, iy)\|_{p_0} = \left\| |f|^{p/p_0} \right\|_{p_0} = \|f\|_{p_0}^{p/p_0} = 1.$$

In a similar way, we have

$$\begin{aligned} \|\phi(\cdot, 1+iy)\|_{p_1} &= \left\| |f|^{p/p_1} \right\|_{p_1} = \|f\|_{p_1}^{p/p_1} = 1, \\ \|\psi(\cdot, iy)\|_{q_0'} &= \left\| |g|^{q'/q_0'} \right\|_{q_0'} = \|g\|_{q_0'}^{q'/q_0'} = 1, \\ \|\psi(\cdot, 1+iy)\|_{q_1'} &= \left\| |g|^{q'/q_1'} \right\|_{q_1'} = \|g\|_{q_1'}^{q'/q_1'} = 1. \end{aligned}$$

So if we use Hölder's inequality, we can obtain from the definition of $F(z)$ the following bounds

$$\begin{aligned} |F(iy)| &\le \|A\phi(\cdot, iy)\|_{q_0} \|\psi(iy)\|_{q_0'} \le k_0 \\ |F(1+iy)| &\le \|A\phi(\cdot, 1+iy)\|_{q_1} \|\psi(1+iy)\|_{q_1'} \le k_1 \end{aligned}$$

But when $z = t \in (0,1)$, we have $\phi(x,t) = f(x)$ and $\psi(y,t) = g(y)$, so that $F(t) = \int_N (Af)g\, d\nu$. We are in the situation of the three lines theorem A.7, so we conclude that $|F(t)| \le k_0^{1-t} k_1^t$. Summarizing, we have proved that for any simple function $f \in L^{p_t}(M)$ and for any simple function $g \in L^{q_t'}(N)$ such that $\|f\|_{p_t} = \|g\|_{q_t'} = 1$, we have

$$\int_N (Af)g \, d\nu \leq k_0^{1-t} k_1^t.$$

Due to the density of the simple functions, the operator A is bounded from $L^{p_t}(M)$ to $L^{q_t}(N)$, and its norm satisfies $\|A\|_{p_t,q_t} \leq k_0^{1-t} k_1^t$. □

This theorem has many applications. We show one of them, in the context of Fourier series. Consider the measure spaces $M = \mathbb{T}$ with the Lebesgue measure and $N = \mathbb{Z}$ with the counting measure. Let $A(f)(n) = 1/2\pi \int_{\mathbb{T}} f(\theta)e^{-in\theta} \, \mathrm{d}\theta$. If we consider $(p_0, q_0) = (1, \infty)$, then A is a bounded linear operator from $L^1(\mathbb{T})$ to $L^\infty(\mathbb{Z})$, whose norm is $||A||_{1,\infty} = k_0 = 1/2\pi$. Indeed, if $f \in L^1(\mathbb{T})$, then

$$||Af||_{l_\infty(\mathbb{Z})} = \sup_{n\in\mathbb{Z}} |Af(n)| = \sup_{n\in\mathbb{Z}} |\hat{f}(n)| \leq \frac{1}{2\pi} ||f||_{L^1(\mathbb{T})}.$$

We have that $||A||_{1,\infty} \leq 1/2\pi$. We can easily check that the bound is attained with the function $f = 1$.

Now we choose the pair of indices $(p_1, q_1) = (2, 2)$. We know, from the previous section, that in this case, $||A||_{2,2} = k_1 = 1/2\pi$. Let us compute now the convex combination indices:

$$\frac{1}{p_t} = \frac{t}{1} + \frac{1-t}{2} = \frac{1+t}{2},$$
$$\frac{1}{q_t} = \frac{t}{\infty} + \frac{1-t}{2} = \frac{1-t}{2}.$$

It turns out in this case that $\frac{1}{p_t} + \frac{1}{q_t} = 1$, so they are conjugate exponents. We call them $p_t = p$ and $q_t = p'$. Since $\mathbb{T}$ has finite measure, $L^p(\mathbb{T}) \subset L^1(\mathbb{T})$ and the operator A_t must agree with A. Using Theorem 4.5, we conclude that for any $1 \leq p \leq 2$, $A : L^p(\mathbb{T}) \to L^{p'}(\mathbb{Z})$ s a bounded operator, where $1/p + 1/p' = 1$. Equivalently, if $1 \leq p \leq 2$, for all $f \in L^p(\mathbb{T})$ we have

$$||Af||_{p'} = \left(\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^{p'} \right)^{1/p'} \leq \frac{1}{2\pi} \left(\int_{\mathbb{T}} |f(\theta)|^p \, \mathrm{d}\theta \right)^{1/p}.$$

This inequality is called Hausdorff-Young inequality, and it is not true for $p > 2$.

4.3 The conjugate function and convergence in $L^p(\mathbb{T})$

Our next ingredient in proving the convergence of Fourier series in these spaces is the conjugate function. The name "conjugate function" comes from complex analysis, and it is related to the Poisson problem in the unit circle. In this context, the conjugate function operator is defined in the set $\mathcal{P}$ of trigonometric polynomials: $H : \mathcal{P} \longrightarrow \mathcal{P}$. H is given by the formula

$$H(\sum_{n\in\mathbb{Z}} c_n e^{in\theta}) = -i\sum_{n\geq 1} c_n e^{in\theta} + i\sum_{n\leq -1} c_n e^{in\theta}. \tag{4.3}$$

This operator, sometimes called the *discrete Hilbert transform*, can be used to express the projection operator P, which is defined on $\mathcal{P}$ in the following way:

$$f = \sum_{n\in\mathbb{Z}} c_n e^{in\theta} \longmapsto Pf = \sum_{n\geq 1} c_n e^{in\theta}.$$

The expression of the projection operator using the conjugate function is

$$Pf = \frac{1}{2}(f + iHf) - \hat{f}(0).$$

It can easily be checked that the conjugate function operator is skew-adjoint, meaning that if f and g are trigonometric polynomials, then

$$\int_{\mathbb{T}} Hf \cdot \overline{g} = -\int_{\mathbb{T}} f \cdot \overline{Hg}. \tag{4.4}$$

We will express the N-th Fourier partial sum as an operator using the conjugate function. If f is a trigonometric polynomial, then

$$\begin{aligned}
e^{iN\theta}H(e^{-iN\theta}f) &= e^{iN\theta}H\left(\sum_{n\in\mathbb{Z}} c_n e^{i(n-N)\theta}\right) \\
&= e^{iN\theta}H\left(\sum_{n\in\mathbb{Z}} c_{n+N} e^{in\theta}\right) \\
&= e^{iN\theta}\left(-i\sum_{n\geq 1} c_{n+N} e^{in\theta} + i\sum_{n\leq -1} c_{n+N} e^{in\theta}\right) \\
&= -i\sum_{n\geq 1} c_{n+N} e^{i(n+N)\theta} + i\sum_{n\leq -1} c_{n+N} e^{i(n+N)\theta} \\
&= -i\sum_{n>N} c_n e^{in\theta} + i\sum_{n<N} c_n e^{in\theta}.
\end{aligned}$$

In a similar way, we can check that

$$e^{-iN\theta}H(e^{iN\theta}f) = -i\sum_{n>-N} c_n e^{in\theta} + i\sum_{n<-N} c_n e^{in\theta}.$$

If we subtract both expressions, we obtain

$$e^{iN\theta}H(e^{-iN\theta}f) - e^{-iN\theta}H(e^{iN\theta}f) = 2i\sum_{n=-N}^{N} c_n e^{in\theta} - ic_N e^{iN\theta} + i_{-N}e^{-iN\theta}.$$

This allows us to express the N-th partial sum of the Fourier series of f using the conjugate function:

$$S_N(f)(\theta) = \frac{1}{2i}\left[e^{iN\theta}H(e^{-iN\theta}f) - e^{-iN\theta}H(e^{iN\theta}f)\right] + \frac{1}{2}\hat{f}(N)e^{iN\theta} + \frac{1}{2}\hat{F}(-N)e^{-iN\theta}. \tag{4.5}$$

Lemma 4.6. *The operator H is bounded on $L^2(\mathbb{T})$.*

Proof. From Parseval theorem 4.2, we have that for any trigonometric polynomial f,

$$||Hf||_2^2 = \frac{1}{2\pi}\sum_{n\neq 0}|\hat{f}(n)|^2 \leq \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}|\hat{f}(n)|^2 = \|f\|_2^2. \quad \square$$

With this lemma, we can extend the definition of H to the whole space $L^2(\mathbb{T})$. We can do this because the subspace $\mathcal{P}$ of trigonometric polynomials is dense in $L^2(\mathbb{T})$ by Theorem 4.2. So given an arbitrary $f \in L^2(\mathbb{T})$, we approximate f with its Fourier partial sums $S_N(f)$ and define

$$H(f) = \lim_{N\to\infty} H(S_N(f)).$$

So the operator H is well defined on $L^2(\mathbb{T})$, due to its continuity. Since $\mathbb{T}$ has finite measure, we know that $L^p(\mathbb{T}) \subset L^2(\mathbb{T})$ if $p > 2$, so we have H defined on the whole space $L^p(\mathbb{T})$ if $p > 2$. Let us see that it is also bounded.

Lemma 4.7. *If $k = 2, 3, ...$ there exists some constant C_{2k} such that if f is a trigonometric polynomial, then $\|Hf\|_{2k} \leq C_{2k}\|f\|_{2k}$.*

Proof. We begin the proof supposing that f is real valued, and that $\hat{f}(0) = 0$. Then its coefficients satisfy that, $\forall n \in \mathbb{Z}$,

$$\hat{f}(-n) = \overline{\hat{f}(n)}.$$

Thus, Hf is also real. We can write the projection operator:

$$Pf = \frac{1}{2}(f + iHf).$$

We now expand $(Pf)^k$ using the binomial theorem:

$$(Pf)^k = \frac{1}{2^k}(f + iHf)^k = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} f^j (iHf)^{k-j}.$$

We remark that there is no constant term (remember that f had no constant term, because $\hat{f}(0) = 0$). So we have

$$0 = \int_{\mathbb{T}} (2Pf)^{2k} = \sum_{j=0}^{2k} \binom{2k}{j} \int_{\mathbb{T}} f^j (iHf)^{2k-j}$$

When j is odd, $f^j(iHf)^{2k-j}$ is purely imaginary. So if we take the real part, and make the change $j = 2r$ we are left with

$$\begin{aligned} 0 &= \sum_{r=0}^{k} \binom{2k}{2r} (-1)^{k-r} \int_{\mathbb{T}} f^{2r} (Hf)^{2(k-r)} \\ &= (-1)^k \int_{\mathbb{T}} (Hf)^{2k} + \sum_{r=1}^{k} \binom{2k}{2r} (-1)^{k-r} \int_{\mathbb{T}} f^{2r} (Hf)^{2(k-r)}. \end{aligned}$$

Note that since $f, Hf \in L^{2k}$ then $f^{2r} \in L^{k/r}$ and $(Hf)^{2(k-r)} \in L^{k/(k-r)}$. So if we isolate the first term and apply both the triangle and Hölder's inequalities:

$$\begin{aligned} \int_{\mathbb{T}} (Hf)^{2k} &\le \sum_{r=1}^{k} \binom{2k}{2r} \int_{\mathbb{T}} f^{2r} (Hf)^{2(k-r)} \\ &\le \sum_{r=1}^{k} \left[\binom{2k}{2r} \left(\int_{\mathbb{T}} f^{2k} \right)^{\frac{r}{k}} \left(\int_{\mathbb{T}} (Hf)^{2k} \right)^{\frac{k-r}{k}} \right]. \end{aligned}$$

Take

$$X := \frac{\|Hf\|_{2k}}{\|f\|_{2k}} = \left(\frac{\int_{\mathbb{T}} (Hf)^{2k}}{\int_{\mathbb{T}} f^{2k}} \right)^{\frac{1}{2k}}.$$

We have to find C_{2k} independent from f so that $X \le C_{2k}$. We may suppose that $X > 1$. If we divide the expression above by $\|f\|_{2k}^{2k}$, we have the polynomial inequality

$$X^{2k} \le \sum_{r=1}^{k} \binom{2k}{2r} X^{2k-2r}. \tag{4.6}$$

Note that every term in the right hand side of (4.6) is bounded by X^{2k-2}. So

$$X^{2k} \le X^{2k-2} \sum_{r=1}^{k} \binom{2k}{2r} = X^{2k-2}(2^{2k} - 1).$$

Then $X^2 \le 2^{2k} - 1$. So we can choose $C_{2k} = \sqrt{2^{2k} - 1}$, finishing the proof for real valued trigonometric polynomials. □

Since the space of trigonometric polynomials is dense in the space $L^p(\mathbb{T})$ (Theorem 3.12), we obtain the following result, arguing as we did in $L^2(\mathbb{T})$.

Corollary 4.8. *The operator H is a bounded linear operator from $L^{2k}(\mathbb{T})$ to $L^{2k}(\mathbb{T})$ for any $k = 1, 2, 3, \ldots$*

With all the ingredients ready, we are prepared to prove the boundedness of the operator H.

Proposition 4.9. *H is a bounded operator from $L^p(\mathbb{T})$ to $L^p(\mathbb{T})$ whenever $1 < p < \infty$.*

Proof. If $2 < p < \infty$, then there exists $k \geq 1$ such that $2 < p < 2k$. So, using the M. Riesz-Thorin interpolation theorem 4.5 we conclude that H is bounded in $L^p(\mathbb{T})$. If $1 < p < 2$, then we will use the duality of the norms and we will obtain

$$\|Hf\|_p = \sup_{0 \neq g \in L^{p'}(\mathbb{T})} \frac{\int_{\mathbb{T}} |gHf|}{\|g\|_{p'}},$$

where we can take the supremum over the set $\mathcal{P}$ of trigonometric polynomials (since it is dense in $L^{p'}(\mathbb{T})$). Using (4.4), we manipulate the expression and obtain

$$\begin{aligned} \|Hf\|_p &= \sup_{0 \neq g \in L^{p'}(\mathbb{T})} \frac{|\int_{\mathbb{T}} gHf|}{\|g\|_{p'}} = \sup_{0 \neq g \in L^{p'}(\mathbb{T})} \frac{|\int_{\mathbb{T}} fHg|}{\|g\|_{p'}} \\ &\leq \sup_{0 \neq g \in L^{p'}(\mathbb{T})} \frac{1}{\|g\|_{p'}} \|f\|_p \|Hg\|_{p'} \\ &\leq \sup_{0 \neq g \in L^{p'}(\mathbb{T})} \|f\|_p C_{p'}, \end{aligned}$$

where we have used Hölder's inequality and the bound of H in $L^{p'}(\mathbb{T})$ ($p' > 2$). Thus, we conclude the boundedness of H in $L^p(\mathbb{T})$ if $1 < p < 2$. □

This proposition can be used to deduce the main convergence result on the $L^p(\mathbb{T})$ convergence of the Fourier series.

Theorem 4.10 (M. Riesz). *Suppose that $1 < p < \infty$ and $f \in L^p(\mathbb{T})$. Then the Fourier series of f converges in the norm of $L^p(\mathbb{T})$, that is,*

$$\lim_{N \to \infty} \|S_N(f) - f\|_p = 0.$$

Proof. It is clear that we have convergence on the dense set $\mathcal{P}$ of trigonometric polynomials. Furthermore, the partial sum operators are uniformly bounded in $L^p(\mathbb{T})$:

$$
\begin{aligned}
&||S_N(f)||_p \\
&\quad \leq \frac{1}{2}\left\|e^{iN\theta}H(e^{-iN\theta}f)\right\|_p + \frac{1}{2}\left\|e^{-iN\theta}H(e^{iN\theta}f)\right\|_p + \frac{1}{2}|\hat{f}(N)| + \frac{1}{2}|\hat{f}(-N)| \\
&\quad \leq C_p \|f\|_p + \frac{1}{2\pi}\|f\|_p = (\frac{1}{2\pi} + C_p)\|f\|_p .
\end{aligned}
$$

There exists a sequence of trigonometric polynomials $\{g_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \|f - g_n\|_p = 0$. Furthermore, we may suppose that the degree of g_n is less or equal than n. Let $\epsilon > 0$, then there is n_ϵ such that $\|f - g_n\|_p < \epsilon$ if $n > n_\epsilon$. In that case,

$$
\begin{aligned}
\|S_n(f) - f\|_p &= \|f - g_n + g_n - S_n(f)\|_p \\
&\leq \|f - g_n\|_p + \|g_n - S_n(f)\|_p \\
&= \|f - g_n\|_p + \|S_n(g_n - f)\|_p \\
&\leq \epsilon + (\frac{1}{2\pi} + C_p)\epsilon \\
&= (1 + \frac{1}{2\pi} + C_p)\epsilon.
\end{aligned}
$$

Thus, the Fourier series converges to f in the norm of $L^p(\mathbb{T})$. □

Chapter 5

Divergence of Fourier series

Throughout this dissertation we have studied the convergence of Fourier series in different senses and spaces. But we always had some restrictions:

- In order to have pointwise convergence we need, for example, bounded variation or Hölder condition (Theorems 2.11 and 2.12).
- We have norm convergence in $L^p(\mathbb{T})$ when $1 < p < \infty$, but not in $L^1(\mathbb{T})$ or $L^\infty(\mathbb{T})$.

In this chapter we will try to explain why we need some restrictions, and that there are cases in which the Fourier series does not behave the way we would like.

5.1 Pointwise divergence

Paul du Bois-Reymond built an example of a continuous function whose Fourier series diverges at a point. Once we have such a function, it is easy to construct other continuous functions with divergence in a finite number of points. Furthermore, given a set A of measure zero, there exists a continuous function whose Fourier series diverges in A. This is not a simple task.

Moreover, Kolmogorov gave an example of a function in $L^1(\mathbb{T})$ whose Fourier series is divergent almost everywhere. This example was later improved to divergence everywhere.

We will give an example, first proposed by Schwarz and then simplified by Lebesgue, of a continuous function with divergent Fourier series at $x = 0$.

Let $\{c_n\}$ be a sequence that tends to zero and $\{\nu_n\}$ an increasing sequence of odd integers. We define

$$a_n = \nu_0 \nu_1 \dots \nu_n,$$

and let $I_n = [2\pi/a_n, 2\pi/a_{n-1}]$, $n = 1, 2, 3, \dots$ We define the function f in the following way:

- If $t \in I_n$ then
$$f(t) = c_n \sin\left(\frac{a_n t}{2}\right)\frac{\sin t/2}{t}.$$
- If $t = 0$ then $f(0) = 0$.
- If $2\pi/a_0 \le t \le \pi$, then $f(t) = 0$.
- And if $t > 0$ then $f(-t) = f(t)$.

The function f is clearly even, and we can check that f is continuous in $\mathbb{T}$. Indeed, at the boundary of every I_n, f has value zero, so it is continuous in $\mathbb{T} \setminus \{0\}$. And it is continuous at $t = 0$ because the sequence $\{c_n\}$ tends to zero.

So, using formula (2.5) from Chapter 2, we can write

$$\pi S_N(f)(0) = \int_0^\pi f(t)\frac{\sin(N+1/2)t}{\sin t/2}\,\mathrm{d}t = \sum_{n=1}^{\infty} c_n \int_{I_n} \sin(\frac{a_n t}{2})\frac{\sin(N+1/2)t}{t}\,\mathrm{d}t.$$

In particular, if $N_k = (a_k - 1)/2$, (remember that a_k was an odd integer), then

$$\pi S_{N_k}(f)(0) = \int_0^{2\pi/a_k} f(t)\frac{\sin a_k t}{\sin t/2}\,\mathrm{d}t + \sum_{j=1}^{k} c_j \int_{I_j} \frac{\sin a_k t/2 \sin a_j t/2}{t}\,\mathrm{d}t. \quad (5.1)$$

On the one hand, if we use the inequality $|\sin at/\sin t| \le 2\pi a$ we obtain that the first term

$$\left|\int_0^{2\pi/a_k} f(t)\frac{\sin a_k t}{\sin t/2}\,\mathrm{d}t\right| \le \frac{\pi a_k}{2}\int_0^{2\pi/a_k} |f(t)|, dt$$

and this tends to zero by the Fundamental theorem of calculus. On the other hand, whenever $j < k$,

$$\left|c_j \int_{I_j} \frac{\sin a_k t/2 \sin a_j t/2}{t}\,\mathrm{d}t\right| \le c_j \int_{I_j} \frac{1}{t}\,\mathrm{d}t = c_j \log \nu_j;$$

and when $j = k$,

$$c_k \int_{I_k} \frac{\sin^2 a_k t/2}{t}\,\mathrm{d}t = \frac{1}{2}c_k \log \nu_k - \frac{1}{2}c_k \int_{I_k} \frac{\cos a_k t}{t}\,\mathrm{d}t,$$

and the last integral is bounded. So we conclude from (5.1) that

$$\pi S_{N_k}(f)(0) \ge \frac{1}{2}c_k \log \nu_k - \left(\sum_{j=1}^{k-1} c_j \log \nu_j + r_k\right), \quad (5.2)$$

where $\{r_k\}$ is a bounded sequence.

We can make this quantity arbitrarily large by choosing wisely the sequences $\{c_n\}$ and $\{\nu_n\}$. For example, take $c_k = 2^{-k}$ and $\nu_k = 5^{4^k}$. In this case

$$c_k \log \nu_k = 2^{-k} \log 5^{2^{2k}} = 2^k \log 5$$

and it is clear from (5.2) that $S_{N_k}(f)$ tends to infinity, so $S_N(f)(0)$ must be a divergent sequence.

5.2 Divergence in $L^\infty(\mathbb{T})$ and $L^1(\mathbb{T})$

The convergence in the spaces $L^p(\mathbb{T})$ when $1 < p < \infty$ comes from the boundedness of the conjugate function operator. We will show that this operator is not bounded in $L^1(\mathbb{T})$.

But let us first explain why we do not have convergence in $L^\infty(\mathbb{T})$. The partial sums are always continuous functions. So if we had convergence in the norm $\| \ \|_\infty$, it would be uniform convergence, so the original function would be continuous. But not every function on $L^\infty(\mathbb{T})$ is continuous, neither equal to a continuous function almost everywhere. So convergence in $L^\infty(\mathbb{T})$ is impossible.

In order to study the case of $L^1(\mathbb{T})$, we need a classical theorem in functional analysis, the uniform boundedness principle or the Banach-Steinhaus theorem:

Theorem 5.1 (Uniform Boundedness principle)**.** *Let B be a Banach space, Y a normed vector space and $\mathcal{L}$ a collection of bounded linear operators from B to Y, with the additional property that for each $f \in B$,*

$$\{\|Lf\|_Y : L \in \mathcal{L}\} < \infty. \tag{5.3}$$

Then the collection $\mathcal{L}$ is uniformly bounded, that is,

$$\sup\{\|L\|_{B,Y} : L \in \mathcal{L}\} < \infty.$$

We need a lemma before we prove the theorem:

Lemma 5.2. *Suppose that* (5.3) *holds and that*

$$\sup\{\|L\|_{B,Y} : L \in \mathcal{L}\} = \infty \tag{5.4}$$

also holds. Then for each $n \geq 1$, there exist $L_n \in \mathcal{L}$ and $f_n \in B$ such that

$$\|f_n\| = 4^{-n} \tag{5.5}$$

$$\|L_n f_n\| > \frac{2}{3} \|L_n\| \, \|f_n\| \tag{5.6}$$

$$\|L_n f_n\| > 2(M_{n-1} + n) \tag{5.7}$$

where $M_0 = 1$ and for $k \geq 1$, $M_k = \sup\{\|L(f_1 + ... + f_k)\| : L \in \mathcal{L}\}$.

Proof. From (5.4), there exists $L_1 \in \mathcal{L}$ with norm $\|L_1\| > 24$. So, from the definition of the norm of an operator, there must exist $\tilde{f}_1 \in B$ with $||\tilde{f}_1|| = 1$ and $||L_1\tilde{f}_1|| > 2/3\,\|L_1\|$. If we set $f_1 = \tilde{f}_1/4$, we have satisfied (5.5), (5.6) and (5.7) when $n = 1$.

We shall define the following f_n and L_n by induction. If we have already defined $f_1, ..., f_{n-1}$ and $L_1, ..., L_{n-1}$, choose $L_n \in \mathcal{L}$ such that

$$\|L_n\| > 3 \cdot 4^n(M_{n-1} + n),$$

which is possible by (5.4). With this L_n chosen, there exists $\tilde{f}_n \in B$ with $||\tilde{f}_n|| = 1$ and $||L_n\tilde{f}_n|| > 2/3\,\|L_n\|$. Take $f_n = \tilde{f}_n/4$ and we have (5.5). Now

$$\|L_n f_n\| > \frac{2}{3}4^{-n}\,\|L_n\| > \frac{2}{3}4^{-n} \cdot 3 \cdot 4^n(M_{n-1} + n) = 2(M_{n-1} + n).$$

So we have proved (5.6) and (5.7) for the value n. And by induction, we finish the proof. □

Now we can complete the proof of the theorem.

Proof of theorem 5.1. We will make the proof by contradiction. If

$$\sup\{\|L\|_{B,Y} : L \in \mathcal{L}\} = \infty, \tag{5.8}$$

then we can apply the lemma. We define $f = \sum_{n=1}^{\infty} f_n$, which is well defined by (5.5). Note that

$$\begin{aligned}\left\| L_N\left(\sum_{k=n+1}^{\infty} f_k\right)\right\| &\le \|L_n\| \sum_{k=n+1}^{\infty} \|f_k\| \\ &= \|L_n\| \sum_{k=n+1}^{\infty} 4^{-k} \\ &= \|L_n\| \frac{4^{-n}}{3} = \frac{1}{3}\|L_n\|\,\|f_n\|\,.\end{aligned}$$

So we can bound $\|L_n f\|$ using the triangle inequality:

$$\begin{aligned}\|L_n f\| &= \left\| L_n\left(\sum_{k=1}^{n-1} f_k + f_n + \sum_{k=n+1}^{\infty} f_k\right)\right\| \\ &\ge \|L_n f_n\| - \left\| L_n\left(\sum_{k=1}^{n-1} f_k\right)\right\| - \left\| L_n\left(\sum_{k=n+1}^{\infty} f_k\right)\right\| \\ &\ge \|L_n f_n\| - M_{n-1} - \frac{1}{3}\|L_n\|\,\|f_n\| \\ &\ge \frac{1}{2}\|L_n f_n\| - M_{n-1} \\ &\ge n,\end{aligned}$$

which proves that $\sup_n \|L_n f\| = \infty$. This contradicts (5.8), the proof is complete. $\square$

We want to use the uniform boundedness principle in order to prove the existence of an $L^1(\mathbb{T})$ divergent Fourier series. We need to compute the norm of the partial sum operator S_N, which maps $L^1(\mathbb{T})$ to $L^1(\mathbb{T})$. First, we bound $\|S_N\|$. If $f \in L^1(\mathbb{T})$, we have

$$\|S_N(f)\| = \frac{1}{\pi}\|D_N * f\| \leq \frac{1}{\pi}\|D_N\|\,\|f\|,$$

so the norm of the operator S_N is bounded by $\|D_N\|/\pi$. Second, if we take the Fejér kernel $f = F_n$ with $n \geq N$, we can apply the properties of the Fejér kernel Proposition 3.3 and write

$$\|S_N(f)\| = \frac{1}{\pi}\|D_N * F_n\| = \frac{1}{\pi}\|\sigma_n(D_N)\| \to \frac{1}{\pi}\|D_N\|, \quad n \to \infty,$$

because for any fixed N, the Fejér means of D_N converge to D_N when $n \to \infty$ in $L^1(\mathbb{T})$. So we conclude that

$$\|S_N\|_{1,1} = \frac{1}{\pi}\|D_N\|_1. \tag{5.9}$$

So we just have to compute the L^1 norm of the Dirichlet kernel. Well, we will not actually compute it, we will just show that $\|D_N\|_1 \to \infty$, as N tends to infinity.

Proposition 5.3. *Let $L_n = \|D_n\|_1/2\pi$, the Lebesgue constant. Then as $n \to \infty$, $L_n = 4\log n/\pi^2 + O(1)$, meaning*

$$\lim_{n\to\infty} \frac{\pi^2 L_n}{4\log n} = 1.$$

Proof.

$$\begin{aligned} L_n &= \frac{1}{2\pi}\int_{\mathbb{T}} |D_n(t)|\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin(n+1/2)t}{\sin(t/2)}\right|\mathrm{d}t \\ &= \frac{1}{\pi}\int_0^{\pi}\frac{|\sin(n+1/2)t|}{\sin(t/2)}\,\mathrm{d}t \\ &= \frac{2}{\pi}\int_0^{\pi}\frac{|\sin(n+1/2)t|}{t}\,\mathrm{d}t + O(1) \\ &= \frac{2}{\pi}\int_0^{(n+1/2)\pi}\frac{|\sin v|}{v}dv + O(1). \end{aligned}$$

This computation actually suffices to prove that $L_n \to \infty$ as $n \to \infty$, which is what we want. But we will finish the proof anyway. We are reduced to studying the integral of $|\sin v|/v$ in the interval $0 \leq v \leq (n+1/2)\pi$. We can decompose this interval in smaller ones of the form $(k\pi, (k+1)\pi)$ for $0 \leq k \leq n$ plus the last half interval. The integral over that last smaller interval tends to zero, so we can forget about it. In the remaining terms, we compute the integral (apart from sign) integrating by parts:

$$-\int_{k\pi}^{(k+1)\pi} \frac{\sin v}{v} dv = \frac{(-1)^{k+1}}{\pi k} + \frac{(-1)^{k+1}}{\pi(k+1)} + \int_{k\pi}^{(k+1)\pi} \frac{\cos v}{v^2} dv$$
$$= \frac{2(-1)^{k+1}}{\pi k} + O(\frac{1}{k^2}).$$

So we have

$$L_n = \frac{2}{\pi} \sum_{k=1}^{n} \left(\frac{2}{\pi k} + O(k^{-2}) \right) = \frac{4 \log n}{\pi^2} + O(1). \quad \square$$

With this proposition we see that the set of operators $\{S_N : N \geq 1\}$ is not uniformly bounded. So by the uniform boundedness principle, there must exist $f \in L^1(\mathbb{T})$ such that $\|S_N f\|$ is not bounded when $N \to \infty$. So the Fourier series of f will not converge in $L^1(\mathbb{T})$. But we have not given explicitly such a function f, not even in a constructive way.

We have shown that there is not convergence in the whole space $L^1(\mathbb{T})$. But there might be some subspaces of $L^1(\mathbb{T})$ in which there is convergence. The characterization of these subspaces is still an open question and there are research teams working on this subject.

Appendix A

Further theory

In this appendix, we give a proof of some theorems we have used or mentioned through the dissertation. These theorems don't come exclusively from the study of Fourier series, but we have used them.

A.1 A result about integration

Theorem A.1 (Second mean value theorem for definite integrals)**.** *If* $G : [a,b] \to \mathbb{R}$ *is a monotonic function and* $\phi[a,b] :\to \mathbb{R}$ *is an integrable function, then there exists* $x \in (a,b)$ *such that*

$$\int_a^b G(t)\phi(t)\,\mathrm{d}t = G(a+)\int_a^x \phi(t)\,\mathrm{d}t + G(b-)\int_x^b \phi(t)\,\mathrm{d}t$$

A.2 About numerical series

Let $\{a_k\}_{k\geq 1}$ be a sequence of complex numbers. We denote

$$s_n := \sum_{j=1}^n a_j, \quad \sigma_n := \frac{1}{n}\sum_{j=1}^n s_j = \frac{1}{n}\sum_{j=1}^n (n-j)a_j.$$

Theorem A.2 (Abel summation by parts)**.** *Let* $\{a_k\}_{k\geq 1}$ *and* $\{b_k\}_{k\geq 1}$ *be two sequences, and* $A_n = \sum_{k=1}^n a_k$*, with* $A_0 = 0$*. Then*

$$\sum_{k=m}^n a_k b_k = \sum_m^n A_k(b_k - b_{k-1}) + A_n b_n - A_{m-1} b_m.$$

Proof. Since $a_k = A_k - A_{k-1}$, we have

$$\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n} (A_k - A_{k-1}) b_k = \sum_{k=m}^{n} A_k b_k - \sum_{k=m}^{n} A_{k-1} b_k$$
$$= \sum_{k=m}^{n-1} A_k b_k + A_n b_n - \left(\sum_{k=m-1}^{n-1} A_k b_{k+1} \right)$$
$$= \sum_{m}^{n} A_k (b_k - b_{k-1}) + A_n b_n - A_{m-1} b_m.$$ □

Theorem A.3 (Dirichlet's test for uniform convergence)**.** *Let $\{f_k\}$ and $\{g_k\}$ be two function sequences in D such that*

(i) *there exists some $M > 0$ such that $|\sum_{k=1}^{n} f_k(x)| \leq n$, $\forall n \geq 1$, $x \in D$;*

(ii) *$g_{k+1}(x) \leq g_k(x)$, $\forall k$, $\forall x \in D$;*

(iii) *$\{g_k\} \to 0$ uniformly on D.*

Then $\sum_{k=1}^{\infty} f_k g_k$ is uniformly convergent in D.

Proof. Let's use the following notation: $F_n(x) = \sum_{k=1}^{n} f_k(x)$. Then $|F_n(x)| \leq M$. We will show that Cauchy's uniform condition holds: given $\epsilon > 0$, we must find $n_\epsilon \geq 1$ such that if $n, m > n_\epsilon$, then for every $x \in D$

$$\left| \sum_{k=1}^{n} f_k(x) g_k(x) - \sum_{k=1}^{m} f_k(x) g_k(x) \right| < \epsilon. \tag{A.1}$$

Suppose without loss of generality that $m \leq n$. Then

$$\left| \sum_{k=1}^{n} f_k(x) g_k(x) - \sum_{k=1}^{m} f_k(x) g_k(x) \right| = \left| \sum_{k=m}^{n} f_k(x) g_k(x) \right|.$$

Since $\{g_k\}$ is uniformly convergent to zero, there exists n_ϵ such that $|g_k(x)| < \epsilon$ if $k > n_\epsilon$. If $n, m > n_\epsilon$, we use summation by parts:

$$|\sum_{k=m}^{n} f_k(x) g_k(x)|$$
$$= \left| \sum_{m}^{n} F_k(x)(g_k(x) - g_{k-1}(x)) + F_n(x) g_n(x) - F_{m-1}(x) g_m(x) \right|$$
$$\leq \left| \sum_{m}^{n} F_k(x)(g_k(x) - g_{k-1}(x)) \right| + |F_n(x) g_n(x) - F_{m-1}(x) g_m(x)|$$

It is clear that the second term can be bounded by $2M\epsilon$. We can also bound the first term:

$$\begin{aligned}\left|\sum_m^n F_k(x)(g_k(x)-g_{k-1}(x))\right| &\leq \left|\sum_m^n M(g_k(x)-g_{k-1}(x))\right| \\ &= M\left|\sum_m^n g_k(x)-g_{k-1}(x)\right| \\ &= M\left|\sum_{k=m}^{n-1} g_k(x)-\sum_{k=m+1}^{n} g_k(x)\right| \\ &= M|g_m(x)-g_n(x)| < 2M\epsilon.\end{aligned}$$

So the uniform Cauchy condition holds. That means $\sum_{k=1}^{\infty} f_k(x)g_k(x)$ is uniformly convergent. □

Theorem A.4 (Hardy)**.** *Suppose that* $\{a_k\}_{k\geq 0}$ *is a a sequence of complex numbers such that* $k|a_k| \leq C$ *for all* $k \geq 0$ *and some* $C > 0$*. Then the convergence of the Cesàro means implies the convergence of the original partial sums: If* $\lim_n \sigma_n = a$*, then* $\lim_n s_n = a$*.*

Proof. Note that if $1 \leq h, n$, then

$$\begin{aligned}(n+h)\sigma_{n+h} - n\sigma_n &= \sum_{j=1}^{n+h}(n+h+1-j)a_j - \sum_{j=1}^{n}(n+1-j)a_j \\ &= hs_n + \sum_{j=n+1}^{n+h}(n+h+1-j)a_j.\end{aligned}$$

We now subtract ha from both sides and divide by h, and after a few manipulations, we obtain:

$$s_n - a = \frac{n+h}{h}(\sigma_{n+h}-a) - \frac{n}{h}(\sigma_n - a) - \sum_{j=n+1}^{n+h}\frac{n+h+1-j}{h}a_j.$$

In the last term we have that $|a_j| \leq C/j, \leq c/n$. We note that the coefficient for C is at $1/n$. Since there are h terms, we obtain, using the triangle inequality:

$$|s_n - a| \leq \frac{n+h}{h}|\sigma_{n+h}-a| + \frac{n}{h}|\sigma_n - a| + \frac{Ch}{n}.$$

Now we choose an $\epsilon > 0$, and we take $h = [\epsilon n]$. There is an n_ϵ such that $|\sigma_m - a| < \epsilon$ if $m > n_\epsilon$. if $n > n_\epsilon$, we now from the definition of h that both the ratios $(n+h)/h$ and n/h are bounded. Also, $|\sigma_n - a| < \epsilon$ and $|\sigma_{n+h} - a| < \epsilon$. So we have

$$|s_n - a| \leq M_1\epsilon/3 + M_2\epsilon/3 + \frac{h}{n}C.$$

But by the definition of h, $h/n \leq \epsilon$, so we have

$$|s_n - a| \leq M_1\epsilon + M_2\epsilon + C\epsilon.$$

Since ϵ was arbitrary, this completes the proof. □

In the following theorem we stablish the relation between Cesàro summability and Abel summability. The proof is given in Exercise 5

Theorem A.5 (Frobenius)**.** *If the series* $\sum_{n=0}^{\infty} a_n$ *is Cesàro summable to* a*, then it is Abel summable to* a*.*

A.3 Some results in Complex Analysis

Theorem A.6 (Maximum modulus principle)**.** *Let* f *be an holomorphic function in an open connected subset* S *of the complex plane. Then if* f *has a local maximum at some interior point of* S*, then* f *is constant. If* S *is bounded and* f *is continuous at the boundary of* S*, then* $|f(z)|$ *attains its maximum on the boundary.*

Theorem A.7 (Hadamard's three lines theorem)**.** *Let* $f(z)$ *be a bounded function of* $z = x + iy$ *defined on the strip*

$$\{x + iy : \ a \leq x \leq b\},$$

holomorphic in the interior of the strip and continuous on the hole strip. If

$$M(x) = \sup_y |f(x+iy)|,$$

then $\log M(x)$ *is a convex function on* $[a,b]$*. That means, if* $x = (1-t)a + tb$*, then*

$$M(x) \leq M(a)^{1-t} M(b)^t.$$

Proof. We may suppose that $a = 0$ and $b = 1$, applying an affine transformation if needed. Consider the function $F(z)$ defined by

$$F(z) = f(z)M(0)^{z-1}M(1)^{-z}.$$

This way, it can easily be checked that $|F(z)| \leq 1$ on the boundary of the strip. We must show that the inequality also holds in the interior of the

strip.
The function

$$F_n(z) = F(z)e^{z^2/n}e^{-1/n}$$

tends to zero as $|z|$ tends to infinity, and satisfies $|F_n(z)| \leq 1$ on the boundary of the strip. Since F_n is bounded, we can apply the maximum modulus principle A.6 and we see that $|F_n(z)| \leq 1$ on the whole strip. But $F_n(z) \to F(z)$ as n tends to infinity, so we obtain that $|F(z)| \leq 1$ on the whole strip. So

$$|f(z)| \left|M(0)^{z-1}M(1)^{z}\right| \leq 1.$$

We can write $z = x + iy = +tb + iy$, and we obtain

$$|f(x+iy)|M(0)^{t-1}M(1)^{-t} \leq 1,$$

and since this happens for all values of y, we have completed the proof. □

Appendix B

Exercises

In this appendix, there are some of the exercises I have made while studying this topic. Some of them are used in some point in the documents.

B.1 Chapter 1. Trigonometric Fourier series

Exercise 1. Let f be a bounded and piecewise monotone function in $\mathbb{T}$. Then there exists $C > 0$ such that for every $k \in \mathbb{Z}$

$$|\hat{f}(k)| \leq \frac{C}{k}.$$

Solution. Let M be a bound for f: $|f(x)| \leq M$, for all $x \in \mathbb{T}$. Since f is piecewise monotone, we can divide $\mathbb{T}$ in smaller intervals:

$$-\pi = c_0 < c_1 < ... < c_s = \pi,$$

where f is monotone in $(c_i, c_i + 1)$ for $i = 0, ..., s-1$.

$$\begin{aligned}
|\hat{f}(k)| &= \left| \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-ikt} \, \mathrm{d}t \right| \\
&= \left| \frac{1}{2\pi} \int_{\mathbb{T}} f(t) \cos kt \, \mathrm{d}t - \frac{i}{2\pi} \int_{\mathbb{T}} f(t) e^{-ikt} \, \mathrm{d}t \right| \\
&\leq \left| \frac{1}{2\pi} \int_{\mathbb{T}} f(t) \cos kt \, \mathrm{d}t \right| + \left| \frac{1}{2\pi} \int_{\mathbb{T}} f(t) \sin kt \, \mathrm{d}t \right| \\
&= A + B
\end{aligned}$$

Let us begin with the cosine:

$$\begin{aligned}
A = \left| \frac{1}{2\pi} \int_{\mathbb{T}} f(t) \cos kt \, \mathrm{d}t \right| &= \frac{1}{\pi} \left| \sum_{i=0}^{s-1} \int_{c_i}^{c_{i+1}} f(t) \cos kt \, \mathrm{d}t \right| \\
\leq \frac{1}{\pi} \sum_{i=0}^{s-1} \left| \int_{c_i}^{c_{i+1}} f(t) \cos kt \, \mathrm{d}t \right|.
\end{aligned}$$

From the second mean value theorem for the integral, there exists $\xi_i \in (c_i, c_{i+1})$ such that

$$\int_{c_i}^{c_{i+1}} f(t)\cos kt\,\mathrm{d}t = f(c_{i+1}-)\int_{\xi_i}^{c_{i+1}} \cos kt\,\mathrm{d}t + f(c_i+)\int_{c_i}^{\xi_i} \cos kt\,\mathrm{d}t.$$

This way, we have

$$\begin{aligned} A &\le \frac{1}{\pi}\sum_{i=0}^{s-1}\left| f(c_{i+1}-)\int_{\xi_i}^{c_{i+1}} \cos kt\,\mathrm{d}t + f(c_i+)\int_{c_i}^{\xi_i} \cos kt\,\mathrm{d}t\right| \\ &= \frac{1}{\pi}\sum_{i=0}^{s-1}\left| f(c_{i+1}-)\frac{1}{k}(\sin kc_{i+1} - \sin k\xi_i) + f(c_i+)\frac{1}{k}(\sin \xi_i k - \sin kc_i)\right| \\ &\le \frac{2}{\pi k}\sum_{i=0}^{s-1}|f(c_{i+1}) - f(c_i)| \le \frac{2}{\pi k}\sum_{i=0}^{s-1} 2M = \frac{1}{k}\frac{4Ms}{\pi}. \end{aligned}$$

With an analogous computation, we obtain for the sine integral

$$B \le \frac{1}{k}\frac{4Ms}{\pi}.$$

Put $C = 8Ms/\pi$, which depends only on the function f. We have shown that for every $k \in \mathbb{Z}$,

$$\hat{f}(k) \le \frac{C}{k}. \quad \square$$

Exercise 2. Show that the function

$$f(x) = |x|^\alpha \sin\frac{1}{|x|}$$

is not of bounded variation in any neighbourhood of 0, if $0 < \alpha < 1$.

Solution. Let I be a neighbourhood of zero. We may suppose that $I = (-\frac{1}{n\pi}, \frac{1}{n\pi})$ for some $n \in \mathbb{N}$. Take

$$\xi_k = \frac{1}{(k+1/2)\pi}.$$

Then, we can bound the total variation of f from below:

$$\begin{aligned} V(f) &\ge \sum_{k=n}^{\infty} |f(\xi_k)| \\ &= \sum_{k=n}^{\infty}\left(\frac{1}{(k+1/2)\pi}\right)^\alpha \\ &\ge \frac{1}{\pi}\sum_{k=n}^{\infty}\frac{1}{k^\alpha} = \infty, \end{aligned}$$

since $\alpha < 1$. So f is not of bounded variation in any neighbourhood of zero. $\square$

B.2 Chapter 2: Convergence

Exercise 3. We denote

$$A(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : \quad \sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty\},$$

the space of functions whose Fourier series is absolutely convergent. It is a normed vector space with norm

$$\|f\|_A = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|.$$

Prove the following statements:

(i) if f is continuous, piecewise differentiable and $f' \in L^2(\mathbb{T})$, then $f \in A$;

(ii) if $1/2 < \alpha \leq 1$ and f satisfies the Hölder condition with exponent α, then $f \in A$;

(iii) if $f, g \in A$, then $fg \in A$ and

$$\|fg\|_A \leq \|f\|_A \|g\|_A .$$

Solution. (i) We know that $(f')\hat{}(n) = in\hat{f}(n)$. From Parseval theorem 4.2, we have

$$\|f'\|_2^2 = \sum_{n\in\mathbb{Z}} |(f')\hat{}|^2 = \sum_{n\in Z} |in\hat{f}(n)|^2 < \infty,$$

since we know that $f' \in L^2(\mathbb{T})$. Using the Cauchy inequality, we obtain

$$\sum_{n\neq 0} |\hat{f}(n)|^2 \leq \left(\sum_{n\neq 0} \frac{1}{n^2}\right)^{1/2} \left(\sum_{n\neq 0} |n\hat{f}(n)|^2\right)^{1/2} < \infty.$$

We conclude that $f \in A$.

(ii) Given $\delta > 0$, we define $g(x) = f(x+\delta) - f(x)$. We now compute the Fourier coefficients of g:

$$\begin{aligned}\hat{g}(n) &= \frac{1}{2\pi}\int_{\mathbb{T}} f(x+\delta)e^{-inx}\,\mathrm{d}x - \hat{f}(n)\\ &= \frac{1}{2\pi}\int_{\mathbb{T}} f(z)e^{-in(z-\delta)}\,\mathrm{d}x - \hat{f}(n) = (e^{in\delta} - 1)\hat{f}(n).\end{aligned}$$

We now use again the Parseval theorem 4.2, to obtain

$$\sum_{n\in\mathbb{Z}} |(e^{in\delta} - 1)\hat{f}(n)|^2 = \int_{\mathbb{T}} |f(x+\delta) - f(x)|^2\,\mathrm{d}x.$$

Using now the Hölder condition of f, we have

$$\sum_{n\in\mathbb{Z}} |(e^{in\delta}-1)\hat{f}(n)|^2 = \sum_{n\in\mathbb{Z}} (2\sin\frac{\delta n}{2})^2 |\hat{f}(n)|^2 \leq 2\pi C^2 \delta^{2\alpha},$$

where C is the Hölder constant, and we have used the fact that $|e^{iz} - 1| = |2\sin z/2|$. We now take $\delta = \pi 2^{-r-2}$, where r is a non-negative integer. With r and δ chosen in this way, if $2^{r-1} < |n| \leq 2^r$, we have $1/2 = \sin\pi/4 \leq \sin n\delta/2 \leq \sin\pi/2$. Thus

$$\sum_{2^{r-1}<|n|\leq 2^r} |\hat{f}(n)|^2 = 2^{-2r\alpha} + O(1),$$

as $r \to \infty$. We use once again the Cauchy inequality

$$\begin{aligned}\sum_{2^{r-1}<|n|\leq 2^r} |\hat{f}(n)| &\leq \left(\sum_{2^{r-1}<|n|\leq 2^r} 1\right)^{1/2} \left(\sum_{2^{r-1}<|n|\leq 2^r} |\hat{f}(n)|^2\right)^{1/2} \\ &= 2^{r/2}\left(\sum_{2^{r-1}<|n|\leq 2^r} |\hat{f}(n)|^2\right)^{1/2} = 2^{r(\frac{1}{2}-\alpha)} + O(1).\end{aligned}$$

If we take the sum for all values of r, we obtain a geometric series which is convergent if $1/2 < \alpha \leq 1$:

$$\sum_{r=1}^{\infty} (2^{\frac{1}{2}-\alpha})^r.$$

Thus, the series

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|$$

is convergent, meaning that $f \in A$.

(iii) We begin by computing the Fourier coefficients of fg. Notice that since $f, g \in A$, all the computations bellow are justified.

$$\begin{aligned} f(x)g(x) &= \left(\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}\right)\left(\sum_{k=-\infty}^{\infty} \hat{g}(k)e^{ikx}\right) \\ &= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \hat{f}(n)\hat{g}(k)e^{i(n+k)x} \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \hat{f}(n)\hat{g}(m-n)e^{imx} \\ &= \sum_{m=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(m-n)\right)e^{imx}. \end{aligned}$$

Thus,

$$\widehat{fg}(m) = \sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(m-n).$$

It remains to show that $fg \in A$. This is a simple task:

$$\begin{aligned}\sum_{m=-\infty}^{\infty} \left| \sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(m-n) \right| &\leq \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |\hat{f}(n)\hat{g}(m-n)| \\ &= \sum_{n=-\infty}^{\infty} |\hat{f}(n)| \sum_{m=-\infty}^{\infty} |\hat{g}(m-n)| = \|f\|_A \|g\|_A < \infty. \qquad \square\end{aligned}$$

Exercise 4. Using the Dirichlet kernel, prove that

$$\int_0^{\infty} \frac{\sin x}{x}\, \mathrm{d}x = \frac{\pi}{2}.$$

Solution. We know that

$$\int_0^{\pi} D_N(t)\, \mathrm{d}t = \int_0^{\pi} \frac{\sin(N+1/2)t)}{2\sin t/2}\, \mathrm{d}t = \frac{\pi}{2}.$$

We split it into two integrals

$$\begin{aligned}\frac{\pi}{2} &= \int_0^{\pi} \left(\frac{\sin(N+1/2)t)}{2\sin t/2} + \frac{\sin(N+1/2)t}{t} - \frac{\sin(N+1/2)t}{t} \right) \mathrm{d}t \\ &= \int_0^{\pi} \frac{t - 2\sin t/2}{2t \sin t/2} \sin(N+1/2)t\, \mathrm{d}t + \int_0^{\pi} \frac{\sin(N+1/2)t}{t}\, \mathrm{d}t.\end{aligned}$$

If the first term in the first integral is bounded, the first integral will tend to zero as N tends to infinity following the Riemann-Lebesgue lemma 2.3. Indeed, we just need to show that it is bounded at $t = 0$. We use the L'Hôpital rule

$$\begin{aligned}\lim_{t\to 0} \frac{t - 2\sin t/2}{2t\sin t/2} &= \lim_{t\to 0} \frac{1 - \cos t/2}{2\sin t/2 + t\cos t/2} \\ &= \lim_{t\to 0} \frac{-1/2 \sin t/2}{\cos t/2 + \cos t/2 - t/2 \sin t/2} = 0,\end{aligned}$$

thus it is bounded. Hence, the first integral tends to zero. This means that the second integral must tend to $\pi/2$. Making a change of variables we obtain

$$\lim_{N\to\infty} \int_0^{\pi} \frac{\sin(N+1/2)t}{t}\, \mathrm{d}t = \lim_{N\to\infty} \int_0^{\pi(N+\frac{1}{2})} \frac{\sin x}{x}\, \mathrm{d}x = \frac{\pi}{2}.$$

We need to see that

$$\lim_{R\to+\infty} \int_0^{R} \frac{\sin x}{x}\, \mathrm{d}x = \frac{\pi}{2}$$

Let $\epsilon > 0$ and choose N_ϵ such that, for any $N > N_\epsilon$,

$$|\int_0^\pi \frac{\sin(N+1/2)t}{t}\,\mathrm{d}t - \frac{\pi}{2}| < \frac{\epsilon}{2}.$$

Let $M = \pi(N+1/2)$. For $R > M$ choose N_R an integer such that $|\pi(N_R + 1/2) - R| \le \pi/2$. Then

$$\begin{aligned}\left|\int_0^R \frac{\sin x}{x}\,\mathrm{d}x - \frac{\pi}{2}\right| &\le \left|\int_0^{\pi(N_R+1/2)} \frac{\sin x}{x}\,\mathrm{d}x - \frac{\pi}{2}\right| + \left|\int_{\pi(N_R+1/2)}^R \frac{\sin x}{x}\,\mathrm{d}x\right| \\ &\le \frac{\epsilon}{2} + \log\frac{R}{\pi(N_R+1/2)},\end{aligned}$$

where we can make the second term arbitrarily small, choosing R large enough. Thus,

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}. \quad \square$$

B.3 Chapter 3. Summability

Exercise 5. Show that if $\sum_{k=0}^\infty a_k = a\ (C)$, then $\sum_{k=0}^\infty a_k = a\ (A)$.

Solution. We need to show that $\lim_{x\to 1}\sum_{n=0}^\infty a_n x^n = a$. We start setting

$$s_n = \sum_{k=0}^n a_k, \qquad \sigma_n = \frac{1}{n}\sum_{k=0}^n s_k.$$

Let us write $\sum_{n=0}^\infty a_n x^n$ in terms of the Césaro means. First we make the first half of the computation:

$$\begin{aligned}\sum_{n=0}^\infty s_n x^n &= s_0 + \sum_{n=1}^\infty s_n x^n = a_0 + \sum_{n=1}^\infty s_n x^n \\ &= a_0 + \sum_{n=1}^\infty \left(\sum_{k=0}^n s_k - \sum_{k=0}^{n-1} s_k\right) x^n = \sigma_1 + \sum_{k=1}^\infty ((n+1)\sigma_{n+1} - n\sigma_n)x^n \\ &= \sum_{n=0}^\infty (n+1)\sigma_{n+1}x^n - \sum_{n=1}^\infty n\sigma_n x^n \\ &= (1-x)\sum_{n=0}^\infty (n+1)\sigma_{n+1}x^n.\end{aligned}$$

The second part of the computation is as follows:

$$\begin{aligned}\sum_{n=0}^\infty a_n x^n &= \sum_{n=0}^\infty (s_n - s_{n-1})x^n = \sum_{n=0}^\infty s_n x^n - \sum_{n=1}^\infty s_{n-1}x^n \\ &= (1-x)\sum_{n=0}^\infty s_n x^n = (1-x)^2\sum_{n=0}^\infty (n+1)\sigma_{n+1}x^n.\end{aligned}$$

We now that

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n,$$

for $|x| < 1$. This way, we have

$$\sum_{n=0}^{\infty} a_n x^n - a = (1-x)^2 \sum_{n=0}^{\infty}(n+1)(\sigma_n - a)x^n.$$

We will now show that this goes to zero as $x \to 1$ assuming that $\sigma_n \to a$ as n tends to infinity. Let $\epsilon > 0$, and $N \geq 1$ such that $|\sigma_n - a| < \epsilon$ if $n > N$. We split the preceding sum in two parts, according to N.

$$\sum_{n=0}^{\infty} a_n x^n - a = (1-x)^2 \left[\sum_{n=0}^{N-1} + \sum_{n=N}^{\infty}(n+1)(\sigma_n - a)x^n\right].$$

The first part is a finite sum, so we can use the triangle inequality.

$$|S_1| \leq (1-x)^2 \sum_{n=0}(n+1)|\sigma_{n+1} - a|,$$

uniformly in x. For the second sum, we use the Césaro convergence:

$$|S_2| \leq (1-x)^2 \sum_{n=N}^{\infty}(n+1)\epsilon x^n \leq \epsilon(1-x)^2 \sum_{n=0}^{\infty}(n+1)x^n = \epsilon.$$

So we have that

$$\left|\sum_{n=0}^{\infty} a_n x^n - a\right| \leq (1-x)^2 \sum_{n=0}(n+1)|\sigma_{n+1} - a| + \epsilon,$$

which tends to ϵ when $x \to 1$. Since ϵ was arbitrarily small, we conclude that $\sum_{k=0}^{\infty} a_k = a$ (A). □

Exercise 6. Let $0 < \alpha \leq 1$, and suppose that f satisfies the Hölder condition with exponent α

$$|f(x+t) - f(x)| < K_\alpha |t|^\alpha,$$

for some suitable K_α. Prove that

$$|\sigma_N(f)(x) - f(x)| \leq C_1 \frac{\log N}{N}, \quad \alpha = 1,$$
$$|\sigma_N(f)(x) - f(x)| \leq C_\alpha N^{-\alpha}, \quad \alpha < 1,$$

for some constants C_α.

Solution. We start making some simple computations. We take $N = m-1$ in order to simplify the notation:

$$\begin{aligned}\sigma_{m-1}(f)(x) - f(x) &= \frac{1}{2\pi m}\int_{-\pi}^{\pi} \frac{\sin^2 \frac{mt}{2}}{\sin^2 \frac{t}{2}}(f(x+t)-f(x))\,\mathrm{d}t \\ &= \frac{1}{\pi m}\int_{-\pi/2}^{\pi/2} \frac{\sin^2 mt}{\sin^2 t}(f(x+2t)-f(x))\,\mathrm{d}t \\ &= \frac{1}{\pi m}\int_{0}^{\pi/2} \frac{\sin^2 mt}{\sin^2 t}(f(x+2t)-2f(x)+f(x-2t))\,\mathrm{d}t.\end{aligned}$$

We define the function

$$\phi(t) = \int_0^t |f(u+2u) - 2f(u) + f(x-2u)| du,$$

which, since f satisfies the Hölder condition with exponent α, has the property

$$|\phi(t)| \leq \frac{4}{\alpha+1} t^{\alpha+1} = Kt^{\alpha+1}.$$

We start bounding

$$\begin{aligned}\pi|\sigma_{m-1}(f)(x) - f(x)| &\leq \frac{1}{m}\int_0^{\pi/2} \frac{\sin^2 mt}{\sin^2 t}\phi'(t)\,\mathrm{d}t \\ &= \frac{1}{m}\left(\int_0^{1/m} + \int_{1/m}^{\pi/2}\right) \frac{\sin^2 mt}{\sin^2 t}\phi'(t)\,\mathrm{d}t.\end{aligned}$$

In the first integral we use the bound $\frac{\sin^2 mt}{\sin^2 t} \leq m$ and obtain

$$\frac{1}{m}\int_0^{1/m} \frac{\sin^2 mt}{\sin^2 t}\phi'(t)\,\mathrm{d}t \leq \int_0^{1/m} (2t)^\alpha\,\mathrm{d}t = O(m^{-\alpha-1}).$$

For the second integral,we use the bound $\frac{\sin^2 mt}{\sin^2 t} \leq 1/t^2$ and integration by parts, obtaining

$$\begin{aligned}\frac{1}{m}\int_{1/m}^{\pi/2} \frac{\sin^2 mt}{\sin^2 t}\phi'(t)\,\mathrm{d}t &\leq \int_{1/m}^{\pi/2} \frac{1}{mt^2}\phi'(t)\,\mathrm{d}t \\ &= \frac{4}{m\pi^2}\phi(\frac{\pi}{2}) - m\phi(\frac{1}{m}) + \int_{1/m}^{\pi/2} \frac{3\phi(t)}{mt^3}\,\mathrm{d}t.\end{aligned}$$

The first two terms in this last inequality are $O(1/m)$, and the last integral depends on α. If $\alpha = 1$, then it is $O(\log m/m)$, and if $\alpha < 1$ then it is $O(m^{-\alpha})$. This completes the exercise. □

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