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# Mathematical Background of the Chaotic Dynamics in the Lorenz System

Final Degree Dissertation
Degree in Mathematics

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# Contents

# Introduction

This dissertation is devoted to the mathematical theory involved in chaotic dynamics, with the ultimate target of understanding such dynamics in the Lorenz system. This way, the foundations concerning dynamical systems are to be treated as well as a rigorous definition of chaos. Again, the main motivation of this work is the Lorenz system, thus, we shall give a brief discussion on the origin of it.

The Lorenz system was initially a crude model of fluid convection, a physical phenomenon strongly related to weather dynamics. However, its characteristic dynamical behaviour made it deserve a purely mathematical attention. Edward Lorenz, mathematician and meteorologist at Massachusetts Institute of Technology (MIT), came up with this system of differential equations:

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = \rho x - xz - y \\ \dot{z} = xy - \beta z \end{cases} \tag{1}$$

where $\rho$,$\sigma$ and $\beta$ are three positive constants.

Let us briefly discuss the derivation of (1). We are given a fluid embedded in a strip of width $h$ in the $x$-$z$ plane. The layer bellow is heated whereas the top of the strip is cooled down, creating a constant temperature difference $\Delta T$ which generates motion in the fluid. Saltzman in 1962 obtained the equations behind this physical problem. What Lorenz did, is to develop the work done by Saltzman and use approximation theory to obtain the so called Lorenz system. A deeper analysis on the original derivation of this model can be found in [10].

The constants $\sigma$ and $\rho$ are called the Prandtl number and Rayleigh number respectively. The Prandtl number is just the ratio of the viscosity and the thermal diffusion coefficient, while $\rho$ is a more intricate number which basically depends on the temperature difference $\Delta T$. The third parameter $\beta$ has no particular name and it is just a mathematical arrangement.

Although the Lorenz system is strongly related to fluids, it arises in many other physical problems [11]. In the following section we shall discuss how to obtain Lorenz-like equations out from a leaking watermill.

## 0.1 One Derivation of the Model

Professor W. Malkus at MIT came up with the Lorenz equations when modelling the motion of a leaky watermill [19].

Let $R$ be the radius of the watermill and $m(t,\theta)$ the whole mass of the watermill consisting uniquely of the water it has in it at time $t$ and angle $\theta$.

The wheel is expected to rotate, so $\omega(t)$ will be its angular velocity, and there will be frictional damping of the system proportional to $\omega(t)$.

The flux of water entering the watermill will be constant. We will also assume that the points on the wheel gain weight proportionally to its height. Due to the leaks, the watermill loses weight proportionally to $m(t,\theta)$. Finally, the following notation $\overline{f(t,\theta)}$ will mean integration over $[0,2\pi]$ in the $\theta$ variable of the function $f(t,\theta)$. Thus, $\overline{m(t,\theta)}$ is the total mass of the watermill.

As we are dealing with a rotational system we must use Newton's second law applied to the angular momentum:

$$\frac{\partial(R^2\overline{m(t,\theta)}\omega(t))}{\partial t} = -gR\overline{m(t,\theta)\cos\theta} - kR^2\overline{m(t,\theta)}\omega(t), \tag{2}$$

where $g$ and $k$ are the modulus of the gravitational force (taken constant) and the damping coefficient respectively. Now, according to the conditions given, the differential equations for $m(t,\theta)$ are:

$$\frac{\mathrm{d}(m(t,\theta))}{\mathrm{d}t} = \frac{\partial(m(t,\theta))}{\partial t} + \frac{\partial(m(t,\theta))}{\partial\theta}\cdot\frac{\partial\theta(t)}{\partial t} = A + 2B\sin\theta - Cm(t,\theta). \tag{3}$$

In the last equation, $A$,$B$ and $C$ are all positive constants which represent a proportion. $A$ represents the water income, $B$ expresses the fact that the watermill is spinning around and, finally, $C$ illustrates the proportion of water that leaks out.

If we compute the total mass in equation (2) at time $t$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{m(t,\theta)} = \int_0^{2\pi} A + 2B\sin\theta - Cm(t,\theta)\mathrm{d}\theta = 2\pi A - C\overline{m(t,\theta)}.$$

Therefore, $\overline{m(t,\theta)}$ moves towards $2\pi A/C$, where there will be an equilibrium. Let us suppose that $\overline{m(t,\theta)}$ has reached the value $2\pi A/C$, so that $\overline{m(t,\theta)}$ is constant. Then, equation (2) becomes:

$$\frac{d(\omega(t))}{dt} = -\left(\frac{gC}{2\pi RA}\right)\overline{m(t,\theta)\cos\theta} - k\omega(t). \tag{4}$$

Now, we can write the differential equations for $\overline{m(t,\theta)\cos\theta}$ and $\overline{m(t,\theta)\sin\theta}$ using equation (2):

$$\frac{d(\overline{m(t,\theta)\sin\theta})}{dt} = \int_0^{2\pi} \frac{d(m(t,\theta)\sin\theta)}{dt}\mathrm{d}\theta =$$
$$\int_0^{2\pi} \left(A\sin\theta + 2B\sin^2\theta - Cm(t,\theta)\sin\theta + \omega(t)m(t,\theta)\cos\theta\right)\mathrm{d}\theta =$$
$$\omega(t)\overline{m(t,\theta)\cos\theta} - C\overline{m(t,\theta)\sin\theta} + 2\pi B.$$

In the same way,

$$\frac{\mathrm{d}\left(\overline{m(t,\theta)\cos\theta}\right)}{\mathrm{d}t} =$$
$$\int_0^{2\pi} \left(\left(\frac{\partial(m(t,\theta))}{\partial t} + \omega(t)\frac{\partial(m(t,\theta))}{\partial\theta}\right)\cos\theta - \omega(t)m(t,\theta)\sin\theta\right)\mathrm{d}\theta =$$
$$\int_0^{2\pi} A\cos\theta + 2B\sin\theta\cos\theta - Cm(t,\theta)\cos\theta - \omega(t)m(t,\theta)\sin\theta\mathrm{d}\theta =$$
$$-C\overline{m(t,\theta)\cos\theta} - \omega(t)\overline{m(t,\theta)\sin\theta}.$$

Now we have the following system:

$$\begin{cases} \dot{\omega} = -K\omega - C_2\overline{m\cos\theta} \\ \dot{\overline{m\sin\theta}} = \omega\overline{m\cos\theta} - C_2\overline{m\sin\theta} + K \\ \dot{\overline{m\cos\theta}} = -C\overline{m\cos\theta} - \omega\overline{m\sin\theta} \end{cases} \tag{5}$$

where $K = 2\pi B$ and $C_2 = \frac{gC}{2\pi RA}$. Let us now show that (5) is a Lorenz-like system. First we consider consider the following change of variables:

$$\omega = \overline{x}, \qquad ,\overline{m\cos\theta} = \overline{y}, \qquad ,\overline{m\sin\theta} = \overline{z} + C_3,$$

where $C_3$ is a constant to be determined. Then, the system can be rewritten as:

$$\begin{cases} \dot{\overline{x}} = -K\overline{x} + C_2\overline{y} \\ \dot{\overline{y}} = -Cy + (C_3 + z)\,\overline{x} \\ \dot{\overline{z}} = K - C\left(\overline{z} + C_3\right) - \overline{xy} \end{cases}$$

and taking $C_3 = \frac{K}{C}$ it yields,

$$\begin{cases} \dot{\overline{x}} = -K\overline{x} + C_2\overline{y} \\ \dot{\overline{y}} = -Cy + \left(\frac{K}{C} + z\right)\overline{x} \\ \dot{\overline{z}} = -C\overline{z} - \overline{xy} \end{cases}$$

One last change of variables:

$$\overline{x} = x, \qquad , \overline{y} = y, \qquad \overline{z} = -z$$

gives,

$$\begin{cases} \dot{x} = -Kx + C_2 y \\ \dot{y} = -Cy + \left(\frac{K}{C} - z\right) x \\ \dot{z} = Cz - xy \end{cases}$$

which is a Lorenz-like system.

## 0.2 Dynamical Systems

In order to be able to study the Lorenz system, dynamical systems must be introduced somehow. Dynamical systems can be regarded as a very specific kind of functions which transform the domain space. Clearly, dynamical systems have a lot to do with motion and time dependent systems. Of course, time can be considered either continuous or discrete. This lead naturally to discrete dynamical systems and continuous dynamical systems.

Continuous dynamical systems are strongly related to differential equations. Indeed, the definition of a vector field induces a differential equation which models the motion of the points in the domain space. Thus, given a vector field $\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and an initial condition $\mathbf{x}_0$ we can set a straightforward differential equation:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) \\ \mathbf{x}(0) = \mathbf{x}_0. \end{cases} \tag{6}$$

This way, the study of the action of the vector field $\mathbf{f}$ is reduced to the study of Equation (6). Is the solution defined for all $t$? Does $\mathbf{f}$ induce a transformation on the domain space? How do subsets behave under this transformation? These are some questions we shall answer further in the text. For instance, let us take the vector field $\mathbf{f} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by:

$$\mathbf{f}(x, y) = (y, x(1-x)^2 + y). \tag{7}$$

This vector field induces a differential equation whose solution is plotted in Figure 1 along with the vector field.

On the other hand, discrete dynamical systems take time to be discrete. This fact can be modelled using recurrence relations and therefore finite difference schemes:

$$x_{n+1} = f(x_n) \tag{8}$$

where we know some initial condition $x_0$.

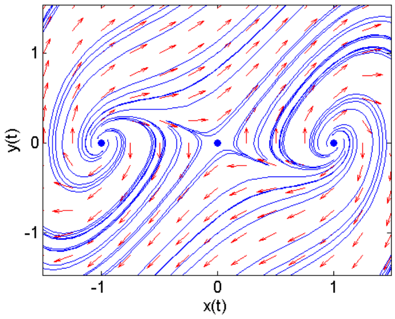



Figure 1: The red arrows represent the vector field (7) while the blue lines some solutions to the initial value problem associated to it.

It might seem that the discrete case is a simplification of the continuous version, and in some sense it is, yet the techniques involved require subtle mathematics such as metric space topology, set theory as well as symbolic dynamics. What is more, as we shall see further in this dissertation, discrete maps constitute an essential tool to study continuous dynamical systems.

The project is fundamentally divided into two sets of chapters:

- Chapters 1 and 2 are dedicated to discrete dynamical systems and to continuous dynamical systems respectively. In Chapter 1 we shall give the main results on discrete maps, symbolic dynamics and we will give a rigorous definition of chaos. On the other hand, in Chapter 2, flows, invariant sets and the Poincaré map are studied.

- Chapters 3 and 4 focus on the Lorenz system itself. Chapter 3 constitutes a discussion of the most noticeable results concerning the Lorenz system: Construction of the attractor, the Lorenz map, geometric models, etc. Chapter 4 is a rather qualitative overview of the problem of determining the dynamics of the Lorenz system with an eventual comment on the computer assisted proof of the existence of the Lorenz attractor.

# Chapter 1

# Discrete Dynamical Systems

Discrete maps are a specific kind of functions which often lead to very interesting dynamics and that give a useful tool to study continuous dynamical systems. Discrete dynamical systems are nothing else than continuous functions which are maps as well. In this chapter we shall make no difference between maps and discrete dynamical systems.

Discrete maps can be regarded as a "simplification" of continuous ones, however, in the deeper analysis of continuous dynamical systems many discrete maps arise, for example, the Poincaré map.

This way, in order to study the dynamics of the Lorenz equations, it is necessary first to go through some of the theory of discrete dynamical systems. In this chapter we give the basic results and definitions we shall need later.

## 1.1 Maps: stability and periodicity

**Definition 1.1.1.** A function whose domain space and range are the same is called a *map*.

The strict definition of map does not require any notion of distance, however, for the sake of coherence from now on we will deal will metric spaces. If $X$ is a metric space, unless it is specified, $d(\cdot,\cdot)$ will denote the respective distance and $B_\epsilon(x_0) = \{x \in X : d(x_0, x) < \epsilon\}$.

Given a map $f : X \longrightarrow X$, it makes sense to write the following difference equation:

$$x_{n+1} = f(x_n),\ \ x_n \in X.$$

This really means that given an initial value $x_0 \in X$, we can consider the way $x_0$ changes after evaluating $f$ successively. We will denote as $f^n$ the first $n$ evaluations of the function $f$, this is,

$$f^n = \overbrace{f \circ \ldots \circ f}^{n}.$$

We will assume that $f^0$ is the identity map. This idea leads to the following definition.

**Definition 1.1.2.** Given a map $f : X \longrightarrow X$ and $x_0 \in X$, the set

$$\Gamma(x_0) = \{f^n(x_0) : n \geq 0\}$$

is called the *orbit* of $x_0$ for the map $f$.

The behaviour as well as the study of discrete maps depend strictly on the nature of $f$. Indeed, if we ask a map $f$ to be linear we are actually dealing with a recurrence relation. If $f$ were not to be linear, the dynamics become more intricate.

**Definition 1.1.3.** Let $f : X \longrightarrow X$ be a map. The point $p \in X$ is said to be *periodic of period* $k \in \mathbb{N}$ if

$$p = f^k(p) \text{ and } p \neq f^r(p), \;\; 0 \leq r < k.$$

If $k = 1$, $p$ is called a *fixed point.*

This definition arises naturally, what is more, periodicity may also refer to sets of points:

**Definition 1.1.4.** Given a map $f : X \longrightarrow X$ and a $k$-periodic point $p \in X$ the set $\{p, f(p), f^2(p), \ldots, f^{k-1}(p)\}$ is called a *periodic orbit of length* $k$.

In systems of differential equations, equilibrium points (those in which the derivatives vanish) had stability properties. Fixed points of discrete maps have analogous properties:

**Definition 1.1.5.** Lef $f : X \longrightarrow X$ be a map. A fixed point $p \in X$ is said to be *stable* if there exists $r > 0$ such that,

$$\lim_{n\to\infty} f^n(x) = p, \;\; \forall x \in B_r(p).$$

When this does not happen, the fixed point is said to be *unstable.*

*A priori*, the function $f : X \longrightarrow X$ is not asked to satisfy differentiability conditions. Yet, if $X = \mathbb{R}$, provided with the euclidean topology, we can ask $f$ to be sufficiently smooth. Thus, we have the following theorem which shows how to identify stable and unstable points according to their derivative.

**Theorem 1.1.1.** *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth map and let $p \in \mathbb{R}$ be a fixed point for $f$. If $|f'(p)| < 1$, $p$ is stable, whereas if $|f'(p)| > 1$, $p$ is unstable.*

*Proof.* By definition we have,

$$|f'(p)| = \lim_{x \to p} \frac{|f(x) - f(p)|}{|x - p|} < 1$$

then, there exists $\epsilon > 0$ such that given $x$ in $B_p(\epsilon)$, there exists $\lambda < 1$ such that

$$\frac{|f(x) - f(p)|}{|x - p|} < \lambda \Longrightarrow |f(x) - p| < \lambda|x - p| < \lambda\epsilon < \epsilon. \tag{1.1}$$

This means that $f(x)$ is closer to $p$ than $x$. In particular (1.1) implies that $|f(x) - p| < \epsilon$, then we obtain in the same way that,

$$\frac{|f(f(x)) - f(f(p))|}{|f(x) - f(p)|} < \lambda.$$

This last inequality is due to the fact that $f(x)$ lies in $B_p(\epsilon)$. Thus,

$$|f^2(x) - p| < \lambda|f(x) - p| < \lambda^2|x - p|.$$

This tells us that $f^2(x)$ is even closer to $p$ than $f(x)$. We can now apply induction to $f^k(x)$:

$$\frac{|f^{k+1}(x) - f^{k+1}(p)|}{|f^k(x) - f^k(p)|} < \lambda \Longrightarrow |f^{k+1} - p| < \lambda^{k+1}|x - p|, \quad \forall k \in \mathbb{N}.$$

This means that,

$$\lim_{k \to \infty} f^k(x) = p$$

if $x \in B_\epsilon(p)$ and therefore, $p$ is stable. The case for instability analogous [11]. $\square$

We note that the case in which $|f'(p)| = 1$ is critical, in the sense that further information is needed to determine the stability or instability. For example, Figure 1.1 shows stable behaviour of the origin of the function $f(x) = x - x^2$, whose derivative at the fixed point zero is one; it computes the evolution of three different initial conditions. This figure is known as *coweb plot*. The idea of this plot is to join with straight lines the evolution of an initial condition: first $x_0$ is joined with $f(x_0)$ vertically, then $f(x_0)$ is joined with the diagonal horizontally and so forth.

For $f_\mu(x) = x - x^2$ (known as the logistic map of parameter 1), although the derivative does not tell us anything about the stability of the origin, the actual stability of the origin can be proved. Indeed, for every $x_0 \in [0, 1]$ the sequence $\{x_0, f(x_0), f^2(x_0), \ldots\}$ is decreasing and bounded, therefore, there exists a limit. Let us call $\alpha$ to that limit. Then, $\alpha$ should verify $\alpha = \alpha - \alpha^2$, which implies that $\alpha$ is zero.

The following example shows that not all fixed points at which the derivative is 1 are stable.

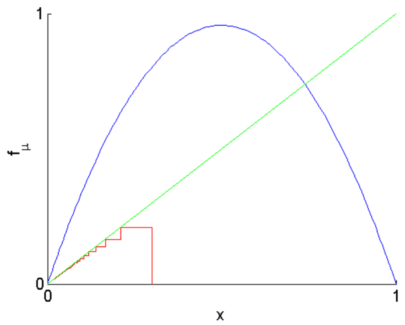



Figure 1.1: The Logistic Map for $\mu = 1$. The green line is the identity function and the red plot is the *coweb plot*.

**Example 1.1.1.** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ the map defined by $f(x) = e^x - 1$. Clearly, $f(0) = 0$, then $x = 0$ is a fixed point for the map $f$. We wonder about its stability computing the derivative:

$$f'(x) = e^x \Rightarrow f'(0) = 1.$$

Hence, we can say nothing about the stability of the fixed point $x = 0$. However, we can do further analysis to show the eventual instability.

We note that $f$ is an increasingly monotone function, thus, if we take $x_0 > 0$, it follows that $f(x_0) > x_0$. Consequently:

$$f^2(x_0) > f(x_0) \Rightarrow f^n(x_0) > f^{n-1}(x_0) \dots$$

when $n \in \mathbb{N}$. As $x_0$ is arbitrarily close to $x = 0$, every neighbourhood of $x = 0$ possesses an diverging orbit.

Notice that if $x_0$ is strictly negative,

$$\lim_{n \to \infty} f^n(x_0) = 0.$$

Anyway, as any neighbourhood of $x = 0$ has diverging orbits; the fixed point $x = 0$ is unstable.

In the same way we considered periodic points, we can also consider periodic orbits, but these are just a consequence of the definition of periodic points, indeed, let $f$ map $X$ to itself and let $p$ be a period $m$ point, then,

$$f^m(p) = p \Rightarrow f^{m+1} = f(p) \Rightarrow \dots \Rightarrow f^{2m}(p) = f^m(p) = p$$

This means that an $m$-periodic point determines a periodic orbit whose stability can be found out through its derivative.

**Proposition 1.1.2.** *Let $\{p_0, p_1 = f(p_0), \ldots, p_{m-1} = f^{m-1}(p_0)\}$ be a periodic orbit for the sufficiently smooth map $f : \mathbb{R} \longrightarrow \mathbb{R}$. Then,*

$$(f^m)'(p_0) = f'(p_0) \ldots f'(p_{m-1}).$$

*Proof.* It suffices to apply the chain rule. □

**Remark 1.1.1.** Periodic points of period $n$ arise when solving the equation $f^n(x) = x$. This is to say that period $n$ points of the map $f$ are the fixed points for the map $f^n$.

## 1.2 Sensitive Dependence on Initial Conditions

A key concept in dynamical systems is the notion of dependence on initial conditions. If we recall the main article where Lorenz showed his results [10], the most remarkable fact was that very close initial data lead to extremely different output. Moreover, a necessary condition for a map to be chaotic is to have sensitivity to initial conditions.

The following definition gives a first intuition of chaotic dynamics:

**Definition 1.2.1.** Let $X$ be a metric space, and $f : X \longrightarrow X$ a map. A point $p_0 \in X$ has *sensitive dependence on initial conditions* if there exists $R > 0$ such that given $B_\epsilon(p_0)$ there exists a point $x$ in it such that $|f^k(x) - f^k(p_0)| \geq R$, for some $k \geq 1$.

**Remark 1.2.1.** We will say that a map displays sensitivity to initial conditions if it has points which have sensitive dependence on initial conditions.

To illustrate the sensitivity on initial conditions we will consider the map $\mathrm{Mod}_3 : [0, 1] \longrightarrow [0, 1]$ defined by $\mathrm{Mod}_3(x) = 3x - \lfloor 3x \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function. If we identify the boundary points of $[0, 1]$ we can regard $\mathrm{Mod}_3$ as a continuous function which maps the unit circumference to itself.

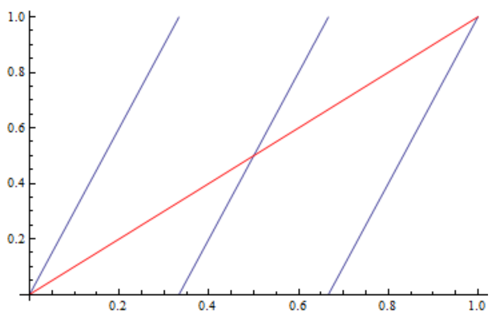



Figure 1.2: Plot of $\mathrm{Mod}_3$ and the identity map.

This map shows clear sensitive dependence on initial conditions. Indeed, we can observe this in the following table:

| $\mathrm{Mod}_3(x) = 3x - \lfloor 3x \rfloor$ | |
|---|---|
| $\mathrm{Mod}_3^n(x_0)$ | $\mathrm{Mod}_3^n(x_0 + \epsilon)$ |
| $x_0 = 0.25$ | $x_0 = 0.2501$ |
| 0.75 | 0.7503 |
| 0.25 | 0.2509 |
| 0.75 | 0.7527 |
| 0.25 | 0.2528 |
| 0.75 | 0.7743 |
| 0.25 | 0.3229 |
| 0.75 | 0.9687 |
| 0.25 | 0.9061 |

**Theorem 1.2.1.** *Given $f : X \longrightarrow X$, an unstable point has always sensitive dependence on initial conditions.*

*Proof.* Let $p_0$ be an unstable point under $f$. By definition of unstable point, for every neighbourhood $B_\epsilon(p_0)$ of $p_0$, there exists a integer $m$ such that $f^m(x)$ is not in $B_\epsilon(p_0)$ for every $x$ in $B_\epsilon(p_0)$. Then:

$$d(p_0, x) = r_0 \Rightarrow d(f^m(p_0), x) = \lambda r_0, \ \lambda > 1.$$

For instance we can choose $\lambda r_0$. $\square$

The map $f(x) = 4x(1-x)$ also has sensitive dependence on initial conditions. The following table shows it.

| $f_4(x) = 4x(1-x)$ | |
|---|---|
| $f_4^n(x_0)$ | $f_4^n(x_0 + \epsilon)$ |
| $x_0 = 0.5$ | $x_0 = 0.501$ |
| 1 | 0.9999 |
| 0 | $1.6 \cdot 10^{-5}$ |
| 0 | $6.3999 \cdot 10^{-5}$ |
| ... | ... |
| 0 | $f_4^{18}(x_0) = 0.9683$ |

## 1.3 The Logistic Map

The logistic map in $[0,1]$ is a paradigmatic map defined by a simple quadratic function $f_\mu(x) = \mu x(1-x)$, $\mu > 0$. This function models, for instance, population growth for discrete time [11]. The logistic map presents very interesting phenomena and is, traditionally, the first approach to discrete dynamical systems.

**Definition 1.3.1.** Let $f_\mu : [0,1] \longrightarrow [0,1]$ the function given by $f_\mu(x) = \mu x(1-x)$. Then, $f_\mu$ is called the *logistic map of parameter $\mu$*.

We will focus on the parameters $\mu \in (0, 4]$ because otherwise $f_\mu$ would not map $[0, 1]$ to itself. According to the previous section, the fixed points of the map will be the result of solving the following equations:

$$f_\mu(x) = x \Rightarrow \mu x(1 - x) = x \Rightarrow x \in \{0, 1 - 1/\mu\}\,.$$

Note that if $\mu$ is less than 1 the only fixed point is $x = 0$.

We may wonder about the stability of these fixed points:

$$f'_\mu(x) = \mu - \mu x \Rightarrow f'_\mu(0) = \mu,\ f'_\mu\,(1 - 1/\mu) = 1,\ \forall \mu \in (1, 4].$$

As we saw in Section 1.1, when the derivative in a fixed point is 1, nothing can be said about the stability of the point. Further study is needed.

### Bifurcation on $\mu$

Clearly, the dynamics of $f_\mu$ will depend strictly on the value of $\mu$. Therefore, it seems fair to study the stability of $f_\mu$ according to the parameter $\mu$.

If $\mu \in (0, 1]$ the origin is a stable point. When we move on to $\mu > 1$, now, the origin becomes unstable, however, a new fixed point appears, namely, $p = 1 - 1/\mu$. The point 0 is unstable since $f'_\mu(0) > 1$. However,

$$f'_\mu(1 - 1/\mu) = \mu - 2\mu(1 - 1/\mu) = 2 - \mu.$$

This shows that $p$ is stable whenever $1 < \mu < 3$. We can actually add the value $\mu = 3$ to the stability range of $p$. We can consider the following quotient,

$$\frac{f_\mu(x) - f_\mu(p)}{x - p} = \frac{f_\mu(x) - p}{x - p} = 1 - \mu x. \tag{1.2}$$

The expression on the right remains in $(-1, 1)$ when $1 < \mu \leq 2$ and $x \in (0, 1)$. Certainly, we shall exclude the trivial cases in which $x = 0$ or $x = 1$. Moreover, at $2 < \mu \leq 3$ the right hand side of Equation (1.2) remains in $(-1, 1)$ whenever $x \in (0, 2/\mu)$. If $x \in [2/\mu, 1]$, after some iterations we find that eventually $x$ lies in $[1/\mu, p]$. For this interval we take the second iteration to build an equation analogous to Equation (1.2):

$$\frac{f^2_\mu(x) - p}{x - p} = (1 - \mu x)(1 - \mu f_\mu(x)). \tag{1.3}$$

Now, the right hand side term of the equation above is in $(-1, 1)$ when $x \in (1/\mu, p)$. This latter information and the continuity of $f_\mu$ implie that

$$\lim_{n\to\infty} f^{2n}_\mu(x) = p \Rightarrow \lim_{n\to\infty} f^{2n+1}_\mu(x) = f(p) = p,$$

from where we obtain that $f_\mu(x)$ tends to $p$ when $x \in (0, 1)$. I recall again that the cases when $x = 0$ and $x = 1$ are not mentioned since these points eventually move to 0.

At $\mu > 3$, the fixed point $p$ is no longer stable. However, we can have a look at the period two points, this is, we have to solve the equation $f_\mu^2(x) = x$. This way,

$$x = \frac{1 + \mu \pm \sqrt{(\mu+1)(\mu-3)}}{2\mu}$$

from where we obtain two period two points, $q_+$ and $q_-$. An easy computation shows that $f_\mu(q_+) = q_-$. Thus, the set $\{q_+, q_-\}$ is a periodic orbit. Let us wonder about its stability. Notice that $f_\mu^2$ constitutes a map itself, therefore, applying Proposition 1.1.2 the derivative will determine the stability:

$$\left(f_\mu^2\right)'(x) = (f_\mu)'(q_+)\,(f_\mu)'(q_-) = -\mu + 2\mu + 4$$

which if $\mu \in (3, 1 + \sqrt{6})$ its absolute value is less than 1. This process can be repeated to obtain this values for higher iterations to the logistic map.

The noticeable issue here is that the initially stable fixed point $p$ ends up doubling its period. This fact is known as *period doubling*, and this process continues for greater values of $\mu$, it suggested in [11] or [20].

The period doubling phenomenon is illustrated in Figure 1.3. Whenever a period doubling is exhibited, the branches of the figure split into two.

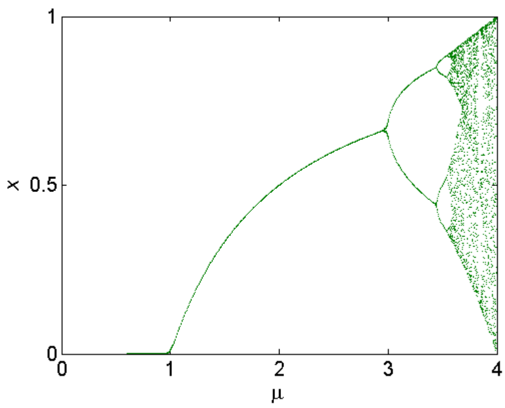


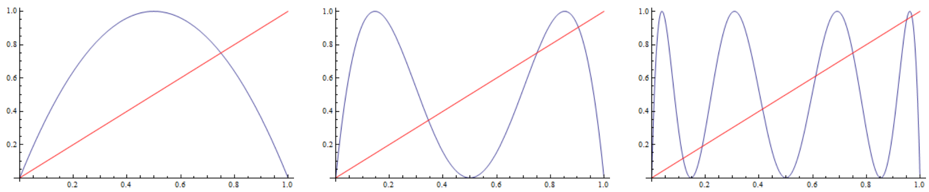

Figure 1.3: Bifucation diagram for the Logistic map $f_\mu$.

**The Logistic Map $f_4(x) = 4x(1-x)$**

This is, of course, a particular case for the general logistic map, yet this concrete example deserves special attention. We now know that both 0 and 3/4 are fixed points.

We can compute the period two points using the explanations above:

$$f_4^2(x) = x \Rightarrow 16x - 16x^2 - 64x^2 + 128x^3 - 4x^3 = x \Rightarrow$$



(a) $n = 1$ (b) $n = 2$ (c) $n = 3$

Figure 1.4: Plots of $f_4^n$ along with the identity function.

$$x \in \left\{0, 3/4, \frac{1}{8}(5-\sqrt{5}), \frac{1}{8}(5+\sqrt{5})\right\}.$$

However, we know that both 0 and 3/4 are period one orbits, therefore, $\left\{\frac{1}{8}(5-\sqrt{5}), \frac{1}{8}(5+\sqrt{5})\right\}$ is a period two orbit, this is equivalent to see that $f_4\left(\frac{1}{8}(5-\sqrt{5})\right) = \frac{1}{8}(5+\sqrt{5})$. The latter can be easily checked.

The plots above show, apparently, that for each positive integer $k$, $f_4^k$ has at most $2^k$ points of period $k$ (some of them will have smaller periods, for instance, $x = 0$).

**Lemma 1.3.1.** *Given a positive integer $k$, $f_4^k$ has $2^k$ fixed points.*

*Proof.* The result follows from the fact that $x = 1/2$ is not a fixed point, $f$ is surjective, $f([0, 1/2]) = [0, 1]$ and $f([1/2, 1]) = [0, 1]$. □

From this lemma we can now prove the following theorem:

**Theorem 1.3.2.** *$f_4$ has period $k$ orbits for each positive integer $k$.*

*Proof.* Let $p$ be a fixed point of $f_4^k$ which is not fixed in $f_4^{k-1}$. The existence of this point follows from the previous proposition. Then,

$$\left\{p, f_4(p), \ldots, f_4^{k-1}\right\}$$

is a period $k$ orbit. □

The interesting fact here is that we are in conditions of using Sharkovskii's theorem which is stated at the appendix.

## 1.4 On the Definition of Chaos

In this section we want to set when a map is considered chaotic. Sensitivity to initial conditions is something a chaotic system must show. However, this is not enough to define rigorously a chaotic system. Indeed, the function $f(x) = 5x$ has sensitivity to initial conditions but we certainly know where the orbits of this map go without doubt. In order to define chaos, several concepts must introduced.

**Definition 1.4.1.** Let $f : X \longrightarrow X$ be a map on a metric space $X$. Then, $f$ is said to be *topologically transitive* if for any given open sets $U, V \subset X$ there is a positive integer $k$ such that $f^k(U) \cap V \neq \emptyset$.

Topological transitiveness is strongly related to chaos. Indeed, topological transitiveness has a lot to do with dynamical indecomposability [23], present in the Lorenz attractor. Roughly speaking, this means that the Lorenz attractor cannot be splitted into smaller invariant pieces.

**Definition 1.4.2.** A *dense orbit* $\Gamma$ for a map $f : X \longrightarrow X$ is an orbit which satisfies that for each open set $U$ in $X$, $U \cap \Gamma \neq \emptyset$.

**Proposition 1.4.1.** *A map $f : X \longrightarrow X$ which has a dense orbit is also topologically transitive.*

*Proof.* Let $\Gamma = \{f^k(x_0) \in X : k \in \mathbb{N}\}$ be a dense orbit for the map $f$. Let $U, V$ be two open sets in $X$. As $\Gamma$ is a dense orbit, for every open subset $S$ of $X$ there exists a positive integer $k_0$ such that $f^{k_0}(x_0) \in S$. In particular, there exist two positive integers $n_U$, $n_V$ such that $f^{n_U}(x_0) \in U$ and $f^{n_V}(x_0) \in V$ respectively. Without loss of generality, we assume that $n_U \leq n_V$. Then, the result holds, as $f^k(y_0) \in V$ where $y_0 = f^{n_U}(x_0)$ and $k = n_V - n_U$. □

Sometimes different maps display similar dynamical behaviour, therefore we need to be able to identify such phenomenon. This is done through *topological equivalences*:

**Definition 1.4.3.** Two discrete dynamical systems $f_1 : X_1 \longrightarrow X_1$ and $f_2 : X_2 \longrightarrow X_2$ are said to be *topologically equivalent* if there exists a homeomorphism $h : X_1 \longrightarrow X_2$ such that for every $x \in X_1$,

$$h(f_1(x)) = f_2(h(x)).$$

Topological equivalence is a powerful tool to simplify maps, but this simplification needs to preserve the qualitative features of the original map. Certainly, the following properties are satisfied:

**Proposition 1.4.2.** *Using the same notation as in the last definition and assuming that $f_1$ and $f_2$ are topologically equivalent, the following holds:*

(i) *$x$ is a periodic point of period $n$ for $f_1$ if and only if $h(x)$ is a periodic point of period $n$ for $f_2$.*

(ii) *If $x_1$ is a stable point for $f_1$, then $h(x_1)$ is stable for $f_2$ and* vice versa.

(iii) *Topological transitiveness is preserved under topological equivalences.*

*Proof.* (i). Let $p$ be a periodic point of period $n$ of the map $f_1$. We consider $f_2^n(x)$. Due to the commutativity of the topological equivalences:

$$f_2^n(h(x)) = \overbrace{f_2 \circ \ldots \circ f_2}^{n}(h(x)) = f_2 \circ h(f_1^{n-1}(x)) = h(f_1^n(x)).$$

Therefore, $h(x)$ is a periodic point of period $n$ in $f_2$.

(ii). Let $x_1$ be now a stable point for $f_1$. This means that:

$$\lim_{n\to\infty} f_1^n(x) = x_1,$$

where $x \in B_\epsilon(x_1)$. The natural candidate for limit point is $h(x_1)$. As before,

$$\lim_{n\to\infty} f_2(h(x)) = \lim_{n\to\infty} (f_1(x)).$$

This equality holds thanks to continuity and the commutativity of topological equivalences.

(iii). Let us suppose $f_1$ is a transitive map and let $U_2$ and $V_2$ be two open sets of $X_2$. Let us suppose that

$$f_2^n(U_2) \cap V_2 = \emptyset$$

for every natural number $n$. We will eventually arrive to a contradiction.

Since $h$ is a homeomorphism, $U_1 := h^{-1}(f_2^n(U_2))$ and $V_1 := h^{-1}(V_2)$ are open sets of $X_1$. Furthermore, their intersection is empty since if $p \in U_1 \cap V_1$, implies that $h(p) \in f^n(U_2) \cap V_2$ which is not possible.

As $f_1$ is topologically transitive, there exists a positive integer $n_0$ such that, $f_1^{n_1}(U_1) \cap V_1 \neq \emptyset$. Let $p \in f_1^{n_1}(U_1) \cap V_1$. Then,

$$\begin{aligned} p \in f_1^{n_1}(U_1) \cap V_1 = f_1^{n_1}(h^{-1}(f_2(U_2))) \cap h^{-1}(V_2) \\ = f_1^{n_1}(f_1^n(h^{-1}(U_2))) \cap h^{-1}(V_2) = f_1^{n+n_1}(h^{-1}(U_2)) \cap h^{-1}(V_2) \end{aligned}$$

hence,

$$h(p) \in h(f_1^{n_1}(U_1)) \cap V_2 = f_2^{n+n_1}(U_2) \cap V_2$$

which contradicts the assumed fact that for every positive integer $n$ $f_2^n(U_2) \cap V_2 = \emptyset$.

□

There are many definitions of chaos [20],[16]. It is clear that sensitive dependence on initial conditions is crucial in chaos, in fact some authors take it as part of the definition of chaos, however, there are some others which do not link chaos to the metric structure of the space.

Sensitive dependence on initial conditions is not preserved under topological equivalences. Consider, for instance, $f_1(x) = 5x$, defined on $(0, \infty)$. This map displays sensitivity to initial conditions. Let us define $h(x) = \frac{1}{x}$, on $(0, \infty)$. Then $h$ is a homeomorphism. Now, following the notation above, the function $f_2(x) = \frac{1}{5}x$ satisfies that $h \circ f_1 = f_2 \circ h$. However, $f_2$ does not have sensitivity to initial conditions. For this reason we do not give a metric definition of chaos.

**Definition 1.4.4.** A discrete dynamical system is said to be *chaotic* if it is transitive and if the periodic points are dense.

**Corollary 1.4.3.** *Chaos is preserved under topological equivalences.*

Note that the definition above does not make use of metric intuitions, yet the following result makes a link to sensitivity:

**Theorem 1.4.4.** *Suppose $f : X \longrightarrow X$ is a chaotic dynamical system. Then it has sensitive dependence on initial conditions.*

*Proof.* The proof is thoroughly done in [20]. The idea is, firstly, to observe that there exists $\delta > 0$ such that for any point $x \in X$ there exists a periodic point whose orbit is of distance at least $4\delta$ from $x$. Moreover, we can pick two periodic points with disjoint orbits and because of the triangular inequality, at least one orbit must be at least $4\delta$ away from $x$.

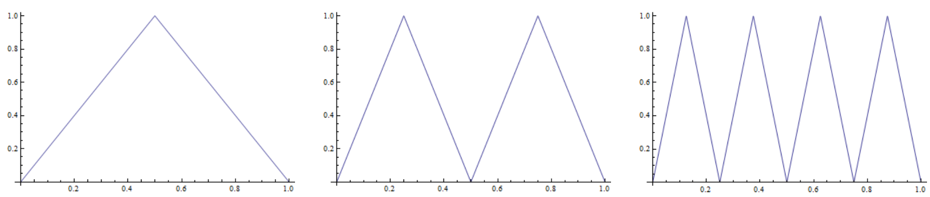
Let us fix $x \in X$ and a periodic point $q$ such that $d(x, q) \geq 4\delta$. If $\epsilon < \delta$, since periodic points are dense, there is a periodic point $x_0 \in B_\epsilon(x)$ of period $n$.

Now, we must use transitivity to find a point which is close to $x$ and simultaneously gets closer to $q$ after several iterations. Thanks to the periodicity of $x_0$, it is possible to build some $k_0$ such that $f^{k_0}(p) = p \in B_\epsilon(x)$ and that $f^{k_0}(y)$ is still close to the orbit of $q$. However, $f^{k_0}(x)$ cannot be close to both $f^{k_0}(y)$ and $f^{k_0}(p) = p$.

□

This results can be applied to paradigmatic maps, such as the tent map and the logistic map, where the tent map is

$$T(x) = \begin{cases} 2x, & 0 \leq x \leq 1/2 \\ 2 - 2x, & 1/2 \leq x \leq 1 \end{cases} \tag{1.4}$$



(a) $n = 1$ (b) $n = 2$ (c) $n = 3$

Figure 1.5: Plots of $T^n$.

**Lemma 1.4.5.** *The tent map* (1.4) *and the logistic map* $f_4(x) = 4x(1-x)$ *are topologically equivalent.*

*Proof.* We define the function $h(x) = \sin\left(\frac{\pi x}{2}\right)^2$, $x \in [0,1]$. Let us see it is a homeomorphism. Continuity is evident. In addition, $h(1) = 1$ and $h(0) = 0$, since $h$ is continuous, surjectivity holds. If $x_1, x_2 \in [0,1]$:

$$\sin\left(\frac{\pi x_1}{2}\right)^2 = \sin\left(\frac{\pi x_2}{2}\right)^2 \Leftrightarrow \sin\left(\frac{\pi x_1}{2}\right) = \sin\left(\frac{\pi x_2}{2}\right) \Rightarrow \frac{\pi x_1}{2} = \frac{\pi x_2}{2}.$$

Therefore, $h$ is injective.

The inverse function is clearly, $h^{-1}(x) = \frac{2}{\pi}\arcsin\sqrt{x}$. We now compute:

$$h(T(x)) = \sin(\pi x)^2$$

and

$$\begin{aligned} f_4(h(x)) &= f_4\left(\sin\left(\frac{\pi x}{2}\right)^2\right) = 4\sin\left(\frac{\pi x}{2}\right)^2\left(1 - \sin\left(\frac{\pi x}{2}\right)^2\right) \\ &= 4\sin\left(\frac{\pi x}{2}\right)^2\cos\left(\frac{\pi x}{2}\right)^2 = \sin\left(\pi x\right)^2. \end{aligned}$$

thus, $f_4$ and $T$ are topologically equivalent. □

**Theorem 1.4.6.** *The logistic map* $f_4(x) = 4x(1-x)$, $x \in [0,1]$ *is chaotic.*

*Proof.* As $f_4$ is topologically equivalent to the tent map $T$, it will be enough to show that $T$ exhibits chaotic behaviour, by Corollary 1.4.3.

We can obtain an explicit expression for $T^n$:

$$T^n(x) = \begin{cases} 2^n - 2i, & \frac{2i}{2^n} \leq x \leq \frac{2i+1}{2^n} \\ 2(i+1) - 2^n x, & \frac{2i+1}{2^n} \leq x \leq \frac{2(i+1)}{2^n} \end{cases}$$

where $i = 0, \ldots, 2^{n-1} - 1$. Plugging the extreme points of the intervals we observe that $\left[\frac{2i}{2^n}, \frac{2i+1}{2^n}\right]$ covers $[0,1]$ under $T^n(x)$. This means that for each interval of this kind there exists a solution to the equation

$$T^n(x) = x$$

which means that $T^n$ has, at least, a fixed point for each interval. Therefore, periodic points are dense.

Let $B_\epsilon(x)$ be an open neighbourhood of $x \in [0,1]$. If we make $n$ as large as we wish, eventually there will be an integer $i$ such that $\left[\frac{2i}{2^n}, \frac{2i+1}{2^n}\right]$ is contained in $B_\epsilon(x)$.

Thus, $T^n(B_\epsilon(x)) = [0,1]$. As a consequence, given two open sets in $[0,1]$ it follows that for an adequate positive integer $n$, $T^n(U) \cap V \neq \emptyset$, for any open sets $U$, $V$. Then, $f_4$ is chaotic. □

## 1.5 Cantor Sets and Symbolic Dynamics

Both discrete and continuous dynamics give rise to complex orbits and attracting sets. Some of them are topologically elementary, but many others show intricate structure.

**Definition 1.5.1.** A compact set which is totally disconnected and for which every point is an accumulation point is called a *Cantor set*.

Symbolic dynamics come from a very intuitive idea. For instance, in the logistic map for $\mu = 4$, we can divide $[0, 1]$ into two intervals, namely, $[0, 1/2]$ and $(1/2, 1]$, named after 1 and 0 respectively. Now, given $x_0 \in [0, 1]$ the trajectory of $x_0$ under the map can be understood as a binary sequence $\{x_{0,n}\}_n$. This is, if $x_0$ lies in $[0, 1/2]$ then $x_{0,0} = 1$, if $f(x_0)$ lies in $(1/2, 1]$ then $x_{0,1} = 0$, and so forth. For example if $x_0 = 0.25$ its associated sequence under $f_4 = 4x(1-x)$ would be $0111\ldots$.

This point of view leads to symbolic dynamics, which is a widely used tool in the study of dynamical systems [19], [20].

**Definition 1.5.2.** Let $n \in \mathbb{N} \setminus \{1\}$, then the *space of $n$ symbols* $\Sigma_n$ is defined as the set of all the sequences $\{x_k\}_k = x_0x_1x_2\ldots$ such that $x_k \in \{0, 1, 2, \ldots, n-1\}$.

**Remark 1.5.1.** $\Sigma_n$ is not a countable set. It suffices to see that all the binary and infinite sequences are contained in $\Sigma_2$.

Given $\Sigma_n$ we can provide to it a distance:

**Lemma 1.5.1.** *The function $d(x, y) = \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{n^k}$ is a distance in $\Sigma_n$.*

*Proof.* Let $x$, $y$, $z$ be sequences in $\Sigma_n$. Certainly, $d(x, y)$ is a positive real number and never diverges. Let us suppose $d(x, y) = 0$. Since every term of the sum which defines $d(\cdot, \cdot)$ is non negative, it holds that $x_k = y_k$ for every $k \in \mathbb{N}$, this is, $x = y$. Conversely, if $x = y$, trivially $d(x, y) = 0$.

Finally,

$$
\begin{aligned}
d(x, y) &= \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{n^k} = \sum_{k=0}^{\infty} \frac{|x_k - z_k + z_k - y_k|}{n^k} \\
&\leq \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{n^k} + \sum_{k=0}^{\infty} \frac{|z_k - y_k|}{n^k} = d(x, z) + d(y, z)
\end{aligned}
$$

which is what we wanted to prove. □

With this result we obtain that $\Sigma_n$ is a metric space.

**Proposition 1.5.2.** *If $x, y \in \Sigma_n$, $d(x, y) \leq n^{-k_0}$ if $x_i = y_i$ for all $i \leq k_0$ and $d(x, y) \geq n^{-k_0}$ if $x_{k'} \neq y_{k'}$ for some $k' \leq k_0$.*

*Proof.* First we set that $x_k = y_k$, for all $k \leq k_0$. Then,

$$d(x, y) = \sum_{k>k_0} \frac{|x_k - y_k|}{n^k} \leq \frac{1}{n^{k_0+1}} \sum_{k=0}^{\infty} \frac{n-1}{n^k} = \frac{1}{n^{k_0}}.$$

If $x_{k'} \neq y_{k'}$ for some $k' \leq k_0$, we have,

$$d(x, y) = \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{n^{k'}} \geq \frac{1}{n^k} \geq \frac{1}{n^{k_0}}$$

□

**Proposition 1.5.3.** $\Sigma_n$ *is a Cantor set.*

*Proof.* The proof can be found in [20]. To see that $\Sigma_n$ is compact [20] shows that every sequence contains a converging subsequence.

To see that every point in $\Sigma_n$ is an accumulation point, let $x \in \Sigma_n$. We define $x_j^k = x_j$ for $0 \leq j \leq k$ and $x_{k+1}^k \neq x_{k+1}$. Then, $\{x^k\}$ converges to $x$.

Disconnectedness is based on the fact that the function $\chi_{j_0} : \Sigma_n \longrightarrow \{0, \ldots n-1\}$ given by $\chi(x) = x_{j_0}$ is continuous. □

**Definition 1.5.3.** The function $\sigma : \Sigma_n \longrightarrow \Sigma_n$ given by $\sigma(x_0 x_1 x_2 \ldots) = x_1 x_2 \ldots$ is called the *Shift map*.

Now, the main target is to show that the dynamics within $\Sigma_n$ display chaos under the Shift map.

**Lemma 1.5.4.** *The Shift map is uniformly continuous in the topology* $(\Sigma_n, d)$.

*Proof.* Let $x = x_0 x_1 \ldots$, $y = y_0 y_1 \ldots$ be sequences in $\Sigma_n$. We must show that the distance between the two images is uniformly bounded. Indeed,

$$d(\sigma(x), \sigma(y)) = \sum_{k=0}^{\infty} \frac{|x_{k+1} - y_{k+1}|}{n^k} \leq n \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{n^k} = nd(x, y).$$

□

**Lemma 1.5.5.** *If* $x \in \Sigma_n$ *is a periodic point of period* $m$ *of the Shift map* $\sigma$ *if and only if it is a periodic sequence of period* $m$.

*Proof.* If $x = x_0 x_1 x_2 \ldots$ is a periodic point of period $m$ of $\sigma$, then $\sigma^m(x) = x_{0+m} x_{1+m} x_{2+m} \ldots = x_0 x_1 x_2 \ldots$ □

**Lemma 1.5.6.** *The Shift map has a countable number of periodic points which are dense.*

*Proof.* The periodic points of period $m$ are given by

$$\sigma^m(x) = x,$$

this is an equation which has $n^m$ solutions. Let $x \in \Sigma_n$. To prove the density we have to build a sequence $\{x^k\}_k$ in $\Sigma_n$ such that $\{x^k\} \longrightarrow x$, as $k \to \infty$. Indeed, taking $x_i^k = x_i$, $0 \leq i \leq k$, it is clear that $\{x^k\} \to x$ as $k \to \infty$. □

**Lemma 1.5.7.** *The Shift map has a dense orbit.*

*Proof.* Let $y \in \Sigma_n$. We must show that there is a sequence such that after several evaluations of the Shift map it approaches $y$ as much as we wish.

We define $x$ as follows. The first $n$ terms of $x$ will be $0, 1, 2, \ldots, n-1$. The next $n^2$ terms will be the $n^2$ combinations of the symbols, and so forth. This way, for all $m \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $\sigma^k(x)_i = y_i$ for all $i \in \{1, \ldots, m\}$. Fixing $m \in \mathbb{N}$:

$$\begin{aligned} d(y, \sigma^k(x)) &= \sum_{i=0}^{\infty} \frac{|y_i - \sigma^k(x)_i|}{n^i} = \sum_{i=m+1}^{\infty} \frac{|y_i - \sigma^k(x)_i|}{n^i} \leq \sum_{i=m+1}^{\infty} \frac{n-1}{n^i} \\ &= (n-1) \sum_{i=m+1}^{\infty} \frac{1}{n^i} \leq n^{-m} \end{aligned}$$

which is what we wanted to see. □

Taking into account the previous results, we have the following theorem:

**Theorem 1.5.8.** *The Shift map is chaotic.*

Sensitive dependence on initial conditions of the Shift map follows directly from the previous theorem along with Theorem 1.4.4. Still, this result can be obtained explicitly:

**Proposition 1.5.9.** *The Shift map displays sensitivity to initial conditions.*

*Proof.* Let $\epsilon > 0$ and $x \in \Sigma_n$. We consider $B_\epsilon(x)$. Now, let $y \in B_\epsilon(x) \setminus \{x\}$, then there exists $i_0 \in \mathbb{N}$ such that $x_{i_0} \neq y_{i_0}$ and $x_i = y_i$, $0 \leq i \leq i_0$. If we take $\delta = |y_{i_0} - x_{i_0}|$, then,

$$d(x, y) < \epsilon \text{ and } d(\sigma^{i_0}(x), \sigma^{i_0}(y)) \geq \delta.$$

□

## 1.6 The Smale Horseshoe

Stephen Smale defined his horseshoe map (Smale horseshoe) in 1963 and showed that it possessed an invariant set which exhibited chaotic dynamics [17]. The Smale horseshoe is very much of an artificial map which rarely would someone expect to find in physical models. However, experience tells us that many maps display horseshoe map-like behaviour and therefore similar dynamics can be expected from them [24].

On $D = [0,1] \times [0,1]$ we define the following regions:

$$H_0 = \{(x, y) \in \mathbb{R}^2 : x \in [0,1], \quad y \in [0, 1/\mu]\}$$

and,

$$H_1 = \{(x, y) \in \mathbb{R}^2 : x \in [0,1], \quad y \in [1 - 1/\mu, 1]\}.$$

Where $0 < \lambda < 1$ and $\mu > 1$. This idea can be translated into mathematical language. We define the function $f : D \longrightarrow \mathbb{R}^2$ which is defined by

$$f(H_0) = \{(x, y) \in \mathbb{R}^2 : x \in [0, \lambda], \quad y \in [0,1]\}$$

and,

$$f(H_1) = \{(x, y) \in \mathbb{R}^2 : y \in [0,1], \quad x \in [1 - \lambda, 1]\}.$$

It can be expressed in matrix form. If $(x, y)$ is a point in $H_0$,

$$f\,(x, y) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{1.5}$$

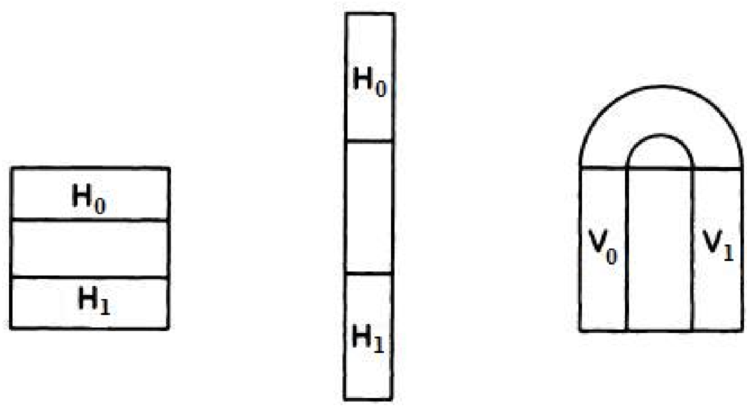



Figure 1.6: Scheme of the Horseshoe.

Analogously, if $(x, y)$ is a point in $H_1$,

$$f\,((x, y)) = \begin{pmatrix} -\lambda & 0 \\ 0 & -\mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ \mu \end{pmatrix}. \tag{1.6}$$

We also want to build an inverse image of this function. Of course, the inverse will take elements on the vertical columns $V_0$ and $V_1$ analogously as with the horizontal case. $V_0$ and $V_1$ are defined as follows:

$$V_0 = \{(x, y) \in \mathbb{R}^2 : y \in [0, 1], \quad x \in [0, \lambda]\}$$
$$V_1 = \{(x, y) \in \mathbb{R}^2 : y \in [0, 1], \quad x \in [1 - \lambda, 1]\}.$$

Figure 1.6 sketches the situation; it shows that the horseshoe map is nothing else than stretching and folding the unit square.

**Lemma 1.6.1.** *With the conditions and notation above:*

- *Let $V$ be a vertical rectangle, then $f(V) \cap D$ consists of two rectangles, one in $V_0$ and the other in $V_1$, with their widths each being equal to a factor of $\lambda$ times the width of $V$.*
- *If $H$ is a horizontal rectangle, then $f^{-1}(H) \cap D$ consists of two horizantal rectangles, one in $H_0$ and one in $H_1$, with their widths being $1/\mu$ times the width of $H$.*

### 1.6.1 Construction of the Invariant Set

If we wonder about all possible iterations of the map $f$ and $f^{-1}$, we can build an invariant set $\Lambda$ which will eventually enclose itself under any iteration of the maps $f$ or $f^{-1}$. This is defined as,

$$\Lambda = \bigcap_{n=-\infty}^{\infty} f^n(D). \tag{1.7}$$

We will now make use of $\Sigma_2 = \{0, 1\}$ the space of two symbols. Let us start with the first iteration for $f$, namely, $D \cap f(D)$:

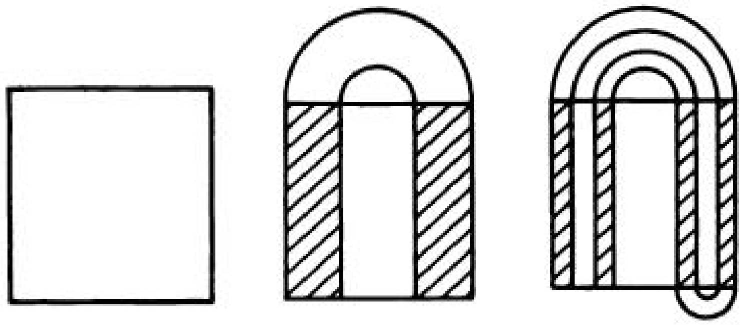
$$D \cap f(D) = \bigcup_{s_{-1} \in \Sigma_2} V_{s_{-1}} = \{(x, y) \in D : (x, y) \in V_{s_{-1}}, s_{-1} \in \Sigma_2\}. \tag{1.8}$$

The second iteration of $f$ is evaluating $f$ on $f(D) \cap D$, this is,

$$f(D \cap f(D)) \cap D = D \cap f(D) \cap f^2(D) = \bigcup_{s_{-2} \in \Sigma_2} D \cap f(V_{s_{-2}}). \tag{1.9}$$

Now, due to Lemma 1.6.1, $V_{s_{-2}}$ moves on to two rectangles contained in $V_0$ and $V_1$. Thus,

$$\bigcup_{s_{-2} \in \Sigma_2} D \cap f(V_{s_{-2}}) = \bigcup_{s_{-i} \in \Sigma_2, i=1,2} V_{s_{-1}} \cap f(V_{s_{-2}}) = D \cap f(D) \cap f^2(D). \tag{1.10}$$



(a) $D$ (b) $D \cap f(D)$ (c) $D \cap f(D) \cap f^2(D)$

Figure 1.7: Iterations of $f$, [13].

We are really looking forward to using symbolic dynamics, this way we will use the following notation:

$$D \cap f(D) \cap f^2(D) := \bigcup_{s_{-i} \in \Sigma_2, i=1,2} V_{s_{-1}s_{-2}}$$
$$= \{(x,y) \in D : (x,y) \in V_{s_{-1}}, \quad f^{-1}((x,y)) \in V_{s_{-2}}, \quad s_{-i} \in \Sigma_2, \quad i = 1,2\}$$

Using the same reasoning and proceeding inductively,

$$D \cap f(D) \cap \ldots \cap f^k(D) = \bigcup f\left(V_{s_{-2}\ldots s_{-k}}\right) \cap V_{s_{-1}} := \bigcup V_{s_{-1}\ldots s_{-k}}. \quad (1.11)$$

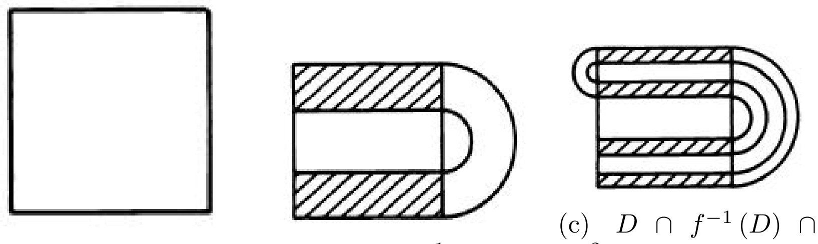
These are $2^k$ vertical rectangles each with width $\lambda^k$; thanks to Lemma 1.6.1. Moreover, each of this rectangles can univocally be determined by a sequence of symbols in $\Sigma_2$. We note also that each of these sets including $V_0$ and $V_1$ are compact, thus, the ultimate intersection is non-empty and zero volume, since $\lambda < 1$. Now, letting $k$ tend to infinity,

$$\bigcap_{n=0}^{\infty} f^n(D) = \bigcup_{s_{-i}, i=1,\ldots} f(V_{s_{-2}\ldots s_{-k}\ldots}) \cap V_{s_{-1}} := \bigcup V_{s_{-1}\ldots s_{-k}\ldots}. \quad (1.12)$$

Now, we need to construct $D \cap f^{-1}(D) \cap f^{-2}(D)\ldots$ The reasoning is completely analogous to the previous case, for $f^{-1}$ acts on the horizontal rectangles the same way $f$ acts on vertical rectangles. This way,

$$D \cap f^{-1}(D) \cap f^{-2}(D) \ldots f^{-k}(D) = \bigcup_{s_i \in \Sigma_2, i=0,\ldots,k} H_{s_0\ldots s_k}. \quad (1.13)$$

We note again that at each step $k$, $2^k$ new rectangles are created, therefore, as $k$ tends to infinity, a sequence in $\Sigma_2$ will univocally determine a horizontal line in $D$. As before,



(a) $D$ (b) $D \cap f^{-1}(D)$ (c) $D \cap f^{-1}(D) \cap f^{-2}(D)$

Figure 1.8: Interations of $f^{-1}$, [13].

$$\bigcap_{-\infty}^{n=0} f^n(D) = \bigcup_{s_i \in \Sigma_2, i=0,\ldots} f(H_{s_1\ldots s_k\ldots}) \cap H_{s_0} := H_{s_0 s_1 \ldots s_k \ldots}. \tag{1.14}$$

Thus, $\Lambda$ will consist of an infinite set of points which can be labelled uniquely by a bi-infinite sequence in $\Sigma_2$. Hence, given a point $(x, y) \in \Lambda$ , we define the map $\mathcal{S} : D \longrightarrow \Sigma_2$ defined by:

$$(x, y) \mapsto \mathcal{S}((x, y)) = \ldots s_{-k} \ldots s_{-1} s_0 s_1 \ldots s_k \ldots. \tag{1.15}$$

Given $p \in \Lambda$, its dynamics can be easily computed, since the point can be regarded as a bi-infinite binary sequence which has the information of both forward and preceding points, namely, $f^k(p)$ and $f^{-k}(p)$, $k \in \mathbb{N}$.

In order to obtain the sequence associated to $f^k(p)$, it seems logical to apply the Shift map $\sigma$, as defined in 1.5.3, $k$ times to the sequence associated to $p$. But to assure this latter conjecture, as proved in the previous section, we need to show that $\mathcal{S}$ is, indeed, a homeomorphism.

**Proposition 1.6.2.** *The map $\mathcal{S} : \Lambda \longrightarrow \Sigma_2$ is a homeomorphism.*

*Proof.* The proof can be found in [24]. □

The previous result implies in particular that $\Lambda$ has the cardinality of the continuum. However, we could have also proved that each point in $\Lambda$ is an accumulation point in order to use the following result due to Hausdorff:

**Theorem 1.6.3.** *In a complete space, every non-empty set in which each point is an accumulation point has at least the cardinality of the continuum.*

*Proof.* The proof can be found in [8]. As a remark, in [8], the fact that every point is an accumulation point is called *perfectness*. □

The following result is what we ultimately wanted to show:

**Theorem 1.6.4.** *The Smale horseshoe map $f$ displays chaotic dynamics on $D$.*

*Proof.* It is enough to observe that $\mathcal{S} \circ f = \sigma \circ \mathcal{S}$. □

In Chapter 3 we will see that there are models of the Lorenz system which display horseshoe-like maps and therefore they have attracting sets which are Cantor sets and display chaotic dynamics.

# Chapter 2

# Continuous Dynamical Systems

So far we have introduced some key aspects of chaotic dynamics through discrete dynamical systems. We wish to move forward to continuous dynamical systems.

## 2.1 Flows

We will now work with the following general equation:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) \tag{2.1}$$

We will denote by $I(\mathbf{x}_0)$ the maximal interval of uniqueness and existence to the initial value problem associated to (2.1) with initial condition $\mathbf{x}(0) = \mathbf{x}_0$:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{2.2}$$

**Definition 2.1.1.** Given (2.1), let us assume that $\mathbf{f} \in C^1(U)$, where $U$ is open in $\mathbb{R}^n$. For every $\mathbf{x}_0 \in U$, $\phi(\mathbf{x}_0, t)$ denotes the solution to (2.2) and it is called the *flow associated to* (2.1).

When the initial condition is fixed and we make $t$ vary on $I(\mathbf{x}_0)$, $\phi(t, \mathbf{x}_0)$ can be thought of the motion of a particle, if we take $S \subset U$, (2.2) models the motion of a fluid volume.

If $\mathbf{f}$ is linear, (2.1) can be equivalently written as:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}, \quad A \in \mathbb{R}^{n \times n} \tag{2.3}$$

and therefore, $\phi_t(\mathbf{x}_0) = e^{At}\mathbf{x}_0$. In fact, the following properties are satisfied by (2.3):

(i) $\phi_0(\mathbf{x}_0) = \mathbf{x}_0$

(ii) $\phi_s(\phi_t(\mathbf{x}_0)) = \phi_{s+t}(\mathbf{x}_0), \quad s, t \in I(\mathbf{x}_0) = \mathbb{R}$

(iii) $\phi_0(\mathbf{x}_0) = \mathbf{x}_0$

These group properties can be easily proved using the definition of the exponential of a matrix.

We now wish to show that these properties are satisfied for (2.1) in general. As a consequence, we need to extend the fundamental theorem of dependence on initial conditions to a global result. This last theorem is written on the appendix.

We define $\Omega = \{(\mathbf{x}_0, t) \in U \times \mathbb{R} : t \in I(\mathbf{x}_0)\}$. The target is to extend the cited results up to $\Omega$.

**Theorem 2.1.1.** *Let $\mathbf{f} \in C^1(U)$, where $U$ is an open subset of $\mathbb{R}^n$. Then $\phi \in C^1(\Omega)$ and $\Omega$ is open.*

*Proof.* Let $(\mathbf{x}_0, t_0) \in \Omega$, then $\phi(\mathbf{x}_0, t)$ is defined for every $t \in [0, t_0]$. If we recall the results of extension of solutions of differential equations, as $[0, t_0]$ is compact in $I(\mathbf{x}_0)$, there exists $t_1 \in I(\mathbf{x}_0)$ such that $t_1$ does not lay in $[0, t_0]$. Therefore, the solution can be extended to the interval $[0, t_1]$.

Again, recalling the results on dependence on initial conditions, as $\phi(\mathbf{x}_0, t)$ is now defined on a closed interval, there exists $\delta > 0$ such that for all $\mathbf{x}_1 \in B_\delta(\mathbf{x}_0)$ the associated Cauchy problem is well posed. Then, the solution is defined in $B_\delta(\mathbf{x}_0) \times (0, t + \epsilon)$, for some $\epsilon > 0$. Thus, for every point in $\Omega$ we have found an open set contained in $\Omega$ containing the point.

Now we know that $\phi$ is well defined for $\Omega$ it is clear that $\phi \in C^1(\Omega)$. □

**Corollary 2.1.2.** *Let $\mathbf{f} \in C^r(U)$, $U$ open set in $\mathbb{R}^n$. Then, $\phi \in C^r(\Omega)$. What is more, if $f$ is analytic $\phi$ is analytic.*

**Theorem 2.1.3.** *Let $\mathbf{f} \in C^1(U)$, where $U$ is an open set in $\mathbb{R}^n$. Given $(\mathbf{x}_0, t) \in \Omega$ and $s \in I(\phi_t(\mathbf{x}_0))$, then $s + t \in I(\mathbf{x}_0)$ and*

$$\phi_{s+t}(\mathbf{x}_0) = \phi_s(\phi_t(\mathbf{x}_0))$$

*Proof.* We take $0 < s \in I(\phi_t(\mathbf{x}_0))$, $t \in I(\mathbf{x}_0) = (a, b)$. We now define

$$\mathbf{x} : (a, s + t] \longrightarrow U$$

$$\mathbf{x}(r) = \begin{cases} \phi(\mathbf{x}_0, r) & \text{if } a < r \leq t \\ \phi(\phi_t(\mathbf{x}_0), r - t) & \text{if } t \leq r \leq s + t \end{cases}$$

Clearly, $\mathbf{x}(r)$ is solution to the Cauchy problem. Indeed, if $a < r \leq t$,

$$\mathbf{x}(r) = \phi(\mathbf{x}_0, r) = \phi_r(\mathbf{x}_0)$$

which is solution to the problem. On the other hand, $t \leq r \leq s+t$ implies,

$$\mathbf{x}(r) = \phi(\phi_t(\mathbf{x}_0), r-t) = \phi_{r-t}(\phi_t(\mathbf{x}_0)).$$

We have shown that $\mathbf{x}(r)$ is solution to the problem, hence, by uniqueness,

$$\mathbf{x}(s+t) = \phi_{s+t}(\mathbf{x}_0) = \phi(\phi_t(\mathbf{x}_0), s).$$

The proof for $s = 0$ and $s < 0$ is analogous. □

**Theorem 2.1.4.** *Let $\mathbf{f} \in C^1(U)$, where $U$ is an open set in $\mathbb{R}^n$. Let $(\mathbf{x}_0, t)$ be a point in $\Omega$, then there exists $\delta > 0$ such that $B_\delta(\mathbf{x}_0) \times \{t\} \subset \Omega$. Then, the set $\phi_t(U)$ is open and*

$$\phi_{-t}(\phi_t(\mathbf{x})) = \mathbf{x}, \quad \forall \mathbf{x} \in B_\delta(\mathbf{x}_0)$$

*and,*

$$\phi_t(\phi_{-t}(\mathbf{x})) = \mathbf{x}, \quad \forall \mathbf{x} \in B_\delta(\mathbf{x}_0)$$

*Proof.* The proof can be found in [13]. □

Now we have enough tools in order to define a dynamical system rigorously and eventually link the discrete theory with the continuous one. Given an open set $U \in \mathbb{R}^n$, we have de following definition:

**Definition 2.1.2.** We consider the following function:

$$\phi : U \times \mathbb{R} \longrightarrow U$$

$$(\mathbf{x}_0, t) \mapsto \phi(\mathbf{x}_0, t)$$

Then, if $\phi$ satisfies

(i) $\phi \in C^1(U \times \mathbb{R})$.

(ii) $\phi(\mathbf{x}_0, 0) = \mathbf{x}_0$, $\forall \mathbf{x}_0 \in U$.

(iii) $\phi(\phi(\mathbf{x}_0, s), t) = \phi(\mathbf{x}_0, t+s)$, $\forall \mathbf{x}_0 \in U$ and $s, t \in \mathbb{R}$.

then we call $\phi$ a *dynamical system.*

**Remark 2.1.1.** Using the notation above, it is clear that if we initially fix $t \in \mathbb{R}$, $\phi(\mathbf{x}_0, t)$ becomes a function which maps elements of $U \subset \mathbb{R}^n$ to itself.

Given a dynamical system we can easily build up a $C^1$ vector field $\mathbf{f}$ whose associated initial value problem (2.2) has a solution for all $t \in \mathbb{R}$. Indeed,

$$\mathbf{f}(\mathbf{x}) := \frac{\mathrm{d}}{\mathrm{d}t}\phi(\mathbf{x}, t = 0).$$

The inverse statement is conditionally true. Suppose we have the initial value problem (2.2). This equation determines a flow which is solution to the problem. Still this might not be defined for all $t$, therefore, it might not be a dynamical system.

The notion of topological equivalence presented in the discrete dynamical systems can be extended to continuous dynamical systems. In this project this notion will be subtle but fundamental to show that the Lorenz Attractor is well defined.

**Definition 2.1.3.** Let $\mathbf{f}_1, \mathbf{f}_2 \in C^1(U)$, where $U$ is an open set in $\mathbb{R}^n$. This vector fields lead naturally to the equations

$$\dot{\mathbf{x}}(t) = \mathbf{f}_1(\mathbf{x}) \tag{2.4}$$

and,

$$\dot{\mathbf{x}}(t) = \mathbf{f}_2(\mathbf{x}) \tag{2.5}$$

Equations (2.4) and (2.5) are said to be *topologically equivalent* if there exists an homeomorphism which maps trajectories of (2.4) to trajectories of (2.5) and it preserves the orientation in time.

**Theorem 2.1.5.** *If* $\mathbf{f} \in C^1(\mathbb{R}^n)$*, the following initial value problem*

$$\begin{cases} \dot{\mathbf{x}} = \frac{\mathbf{f}(\mathbf{x})}{1+\|\mathbf{f}(\mathbf{x})\|} \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{2.6}$$

*has a unique solution defined for all $t$.*

This theorem gives a way of defining a continuous dynamical system from a given regular vector field. This theorem gives sufficient conditions to see when the previous vector field defines a dynamical system. However, this latter vector field is topologically equivalent to $\mathbf{f}$:

**Proposition 2.1.6.** *The system* (2.6) *is topologically equivalent to* (2.1).

*Proof.* We will take $h$ to be the identity homeomorphism. We now take a change of then time variable in Equation (2.1):

$$\tau = \int_0^t (1 + \|\mathbf{f}(\mathbf{x}(s))\|)\, \mathrm{d}s.$$

Thus,

$$\frac{\partial \mathbf{x}}{\partial \tau} = \frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial t}{\partial \tau} = \frac{\partial \mathbf{x}}{\partial t} \Big/ \frac{\partial \tau}{\partial t} = \frac{\mathbf{f}(\mathbf{x})}{1 + \|\mathbf{f}(\mathbf{x})\|}.$$

□

**Theorem 2.1.7.** *If* $\mathbf{f} \in C^1(D)$ *where* $D$ *is a compact subset of* $\mathbb{R}^n$ *and it satisfies a global Lipschitz condition*

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \leq M \|\mathbf{x} - \mathbf{y}\|, \quad M > 0$$

*for every* $\mathbf{x}$, $\mathbf{y} \in D$*. Then, for* $\mathbf{x}_0 \in \mathbb{R}^n$*, the initial value problem* (2.2) *has a unique solution defined for* $t \in \mathbb{R}$ *.*

D. R. J. Chillingworth showed that the existence and uniqueness results can be stated for compact manifolds [4]:

**Theorem 2.1.8.** *The equation* (2.1) *for* $\mathbf{x} \in D$*, where* $D$ *is a compact manifold, has solutions defined for every* $t \in \mathbb{R}$*.*

*Proof.* Let $\phi(\mathbf{x}_0, t)$ be the global solution for equation (2.1). Let us suppose that $\phi$ is defined for the maximal interval $(a, b)$. We shall see first that $b = \infty$. As $b$ is finite we can find a sequence $\{t_n\}$ such that $t_n \to b$, as $n$ tends to infinity.

If $\phi(\mathbf{x}_0, t_n) = \phi(\mathbf{x}_0, t_m)$ it follows that there is a periodic orbit and therefore the solution is defined for all $t$. Hence, $\phi(\mathbf{x}_0, t_n) \neq \phi(\mathbf{x}_0, t_m)$ for every $n \neq m$. As $D$ is compact, there exists an accumulation point $\mathbf{y}$ of the sequence $\{\phi(\mathbf{x}_0, t_n)\}$. Now, the local existence and uniqueness theorem shows that unique solutions exist based at any $\mathbf{x}_1$ in some neighbourhood $B_\epsilon(y)$.

Now we can choose some $t_{n_0}$ such that $b - t_{n_0} < \epsilon$ and $\phi(\mathbf{x}_0, t_{n_0})$ lies in $B_\epsilon(y)$. Thus, by local uniqueness and the extension lemma, $\phi(\mathbf{x}_0, t)$ is extendible to a solution curve defined in $(a, t_{n_0} + \epsilon)$, and since $t_{n_0} + \epsilon > b$ we have contradicted the hypothesis. Therefore, $b = \infty$.

Similar arguments are used to show that $a = -\infty$ [4]. □

## 2.2 Limit Sets and Attractors

In order to describe the global behaviour of the Lorenz model, we need to introduce the concept of *attractor*. As well as in the discrete theory, different kinds of orbits can arise in continuous dynamical systems.

**Definition 2.2.1.** Given $p \in U$, where $U$ is an open set in $\mathbb{R}^n$, $p$ is an *ω-limit point of* $\phi(\mathbf{x}_0, t)$ (the flow associated to (2.1)) if there is a sequence such that $\{t_n\} \longrightarrow \infty$ and that

$$\lim_{n \to \infty} \phi(\mathbf{x}_0, t_n) = p$$

The set of all $\omega$-limit points is called the *ω-limit set of* $\phi(\mathbf{x}_0, t)$. It is denoted by $\omega(\phi(\mathbf{x}_0, t))$.

**Theorem 2.2.1.** *If $\phi(\mathbf{x}_0, \cdot)$ is a trajectory for* (2.1)*, the $\omega$-limit set of $\phi(\mathbf{x}_0, \cdot)$ is a closed subset of $U$. Moreover, if $\phi(\mathbf{x}_0, t)$ is contained in a closed and bounded subset of $\mathbb{R}^n$ for all $t \in \mathbb{R}$, then $\omega(\phi(\mathbf{x}_0, t))$ is non empty, connected and compact.*

*Proof.* To prove closeness, let $\{p_n\}$ be a sequence of points in $\omega(\phi(\mathbf{x}_0, t))$ such that converges to some $p$ in $\mathbb{R}^n$. Let us see if $p$ lays in $\omega(\phi(\mathbf{x}_0, t))$. As, for every $n$, $p_n \in \omega(\phi(\mathbf{x}_0, t))$, there exists an increasing sequence $\{t_{n,k}\}_k$ such that:

$$\lim_{n\to\infty} \phi(\mathbf{x}_0, t_{n,k}) = p_n$$

but this is equivalent to,

$$\|\phi(\mathbf{x}_0, t_{n,k}) - p_n\| < \frac{1}{n}$$

where $k > K(n)$ for some $K(n) \in \mathbb{N}$. Let us define $\{t_n\} = \{t_{n,K(n)}\}$. To start with, the limit of this last sequence is $\pm\infty$ and

$$\|\phi(t_n, \mathbf{x}_0) - p\| \leq \|\phi(t_n, \mathbf{x}_0) - p_n\| + \|p - p_n\| \leq \frac{1}{n} + \|p - p_n\| \longrightarrow 0$$

as $n$ tends to infinity. Let us suppose that $\omega(\phi(\mathbf{x}_0, t)) \subset K$, where $K$ is compact. Then, $\omega(\phi(\mathbf{x}_0, t))$ is compact, as we have shown that $\omega(\phi(\mathbf{x}_0, t))$ is closed. Furthermore, $\omega(\phi(\mathbf{x}_0, t))$ is non-empty since if we take any sequence in $U$, there exists a subsequence in $\omega(\phi(\mathbf{x}_0, t))$ which converges in $\omega(\phi(\mathbf{x}_0, t))$, by hypothesis.

To prove connectedness it suffices to assume that $\omega(\phi(\mathbf{x}_0, t))$ is not connected and deriving a contradiction [13].

□

**Theorem 2.2.2.** *If $p \in \omega(\phi(\mathbf{x}_0, t))$ for the flow $\phi$ associated to* (2.1)*, then all other points of $\phi(p, t)$ are also $\omega$-limit points of $\omega(\phi(\mathbf{x}_0, t))$.*

*Proof.* Let $p \in \omega(\phi(\mathbf{x}_0, t))$. Let $p'$ be a point on the trajectory $\phi(p, t)$, this is $p' = \phi(p, t')$, for some $t' \in \mathbb{R}$. Since $p$ is an $\omega$-limit point of $\phi(\mathbf{x}_0, t)$, there exists a sequence $\{t_n\}$ such that

$$\lim_{n\to\infty} \phi(\mathbf{x}_0, t_n) = p.$$

Thus, using the group properties of dynamical systems,

$$\phi(\mathbf{x}_0, t_n + t') = \phi(\phi(\mathbf{x}_0, t_n), t').$$

Moreover, since $\mathbf{f}$ in (2.1) is $C^1(U)$ and $\mathbf{x}_0 \in U$ we can use the continuous dependence on initial conditions to get:

$$\lim_{n\to\infty} \phi(\phi(\mathbf{x}_0, t_n), t') = \phi(p, t') = p'.$$

Which means that $p'$ is in $\omega(\phi(\mathbf{x}_0, t))$. □

Subsets in $\mathbb{R}^n$ can remain behave differently under the action of a flow. Let us define the concept of invariance:

**Definition 2.2.2.** A set $M \subset \mathbb{R}^n$ is *invariant* if for all $\mathbf{x} \in M$, there exists $t'$ such that $\phi(\mathbf{x}, t) \subset M$ for all $t > t'$.

Invariant sets and trapping regions often exhibit an attractive behaviour, actually, these sets may *attract* every trajectory in the phase space.

**Definition 2.2.3.** A subset $M$ of $\mathbb{R}^n$ is said to be a *globally attracting* if for every $\mathbf{x} \in \mathbb{R}^n$, there exists $t' > 0$ such that $\phi(\mathbf{x}, t) \in M$ for all $t > t'$.

**Corollary 2.2.3.** *Given a trajectory for* (2.1) *namely,* $\phi(\mathbf{x}_0, t)$*,* $\omega(\phi(\mathbf{x}_0, t))$ *is an invariant subset for the flow defined by* (2.1)*.*

*Proof.* Let $p \in \omega(\phi(\mathbf{x}_0, t))$, the there exists a sequence $\{t_n\}$ such that:

$$\lim_{n \to \infty} \phi(\mathbf{x}_0, t_n) = p.$$

We are now wondering if $\phi(p, t) \in \omega(\phi(\mathbf{x}_0, t))$, for every $t \in \mathbb{R}$. But this holds thanks to the previous theorem. □

**Definition 2.2.4.** Given the equation (2.1), an invariant set $A \subset U$, where $U$ is an open set in $\mathbb{R}^n$, is said to be an *attracting set of* (2.1) if there is an open set $V$ containing $A$ such that for all $\mathbf{x} \in V$, $\phi(\mathbf{x}, t) \in V$ and $d(\phi(\mathbf{x}, t), A) \longrightarrow 0$ as $t$ tends to infinity.

The next definition is key in dynamical systems and it is what eventually we would like to find in the Lorenz equations:

**Definition 2.2.5.** An *attractor* is an attracting set which contains a dense orbit.

**Definition 2.2.6.** A *cycle* or *periodic orbit of* (2.1) is any closed curve of (2.1) which does not contain an equilibrium point.

This definition extends the idea of stable and unstable points in the local theory. Indeed:

**Definition 2.2.7.** A periodic orbit $\Gamma$ is said to be *stable* if for every $\epsilon > 0$ there exists an open set $N$ containing $\Gamma$ such that for all $\mathbf{x} \in N$, there exist $t' > 0$ such that $d(\phi(\mathbf{x}, t), \Gamma) < \epsilon$ for all $t > t'$. If this doesn't hold $\Gamma$ is *unstable.*

**Definition 2.2.8.** A periodic orbit is said to be *asymptotically stable* if for every, $\mathbf{x} \in N$,

$$\lim_{t \to \infty} d(\phi(\mathbf{x}, t), \Gamma) = 0.$$

An $\omega$-limit set which is a periodic orbit is called an $\omega$*-limit cycle.*

**Proposition 2.2.4.** *Any $\omega$-limit cycle is an attractor.*

*Proof.* Let $\Gamma$ be an $\omega$-limit cycle. Certainly, $\Gamma$ is an attracting set, since there exists an open set $N$ containing $\Gamma$ such that for every $\mathbf{x} \in N$,

$$\lim_{t\to\infty} d(\phi(\mathbf{x},t),\Gamma) = 0.$$

What is more, $\Gamma$ is dense in itself, therefore, $\Gamma$ has a dense orbit. □

### Extension to Arbitrary Sets

We now want to extend the behaviour of flows in arbitrary sets. The $\omega$-limit set can be extended naturally to arbitrary sets. In fact, these definitions bellow are essential in order to define coherently the Lorenz attractor.

**Lemma 2.2.5.** *Arbitrary intersections and unions of invariant sets are invariant. The closure of an invariant set is invariant.*

*Proof.* Let $\{U_i\}_{i\in I}$ be a family of invariant sets. We consider the intersection:

$$A = \bigcap_{i\in I} U_i.$$

Let $\mathbf{x}_0 \in A$, and let is take its forward orbit $\phi(\mathbf{x}_0,t)$. Clearly, as $\mathbf{x}_0 \in U_i$ for all $i$, $\phi(\mathbf{x}_0,t) \in U_i$ for all $i$ and $t$. The union is still more evident.

Let $U$ be an invariant set, and $\mathbf{x}$ a point on $\overline{U}$. Let us consider a sequence in $U$, $\{\mathbf{x}_n\}$ which converges to $\mathbf{x}$. Now, as we approach $\mathbf{x}$, there exists $N \in \mathbb{N}$, such that every $t \in I(\mathbf{x})$ is in $I(\mathbf{x}_n)$, whenever $n \geq N$. Finally, thanks to continuity, $\phi(\mathbf{x},t) = \lim_{n\to\infty} \phi(\mathbf{x}_n,t) \in \overline{U}$. □

Let us recall that although we are now dealing with arbitrary sets, $\omega(\mathbf{x})$ is a closed and an invariant set. Closeness follows from Theorem 2.2.1 and invariance follows from Corollary 2.2.3.

**Definition 2.2.9.** Let $\phi$ be the flow of equation (2.1) defined on $U \subset \mathbb{R}^n$. If $X \subset U$, the *$\omega$-limit set of* $X$, $\omega(X)$, is the set of points $\mathbf{x}_0 \in U$ such that there exists a sequence $\{t_n\}_n$ and a sequence $\{\mathbf{x}_n\}_n$ such that $\lim_{n\to\infty} \phi(\mathbf{x}_n,t_n) = \mathbf{x}_0$.

**Definition 2.2.10.** We define the *orbit of a set* $X$ as:

$$\bigcup_{t\geq 0} \phi(X,t) = \bigcup_{\mathbf{x}\in X} \bigcup_{t\geq 0} \phi(\mathbf{x},t).$$

**Proposition 2.2.6.** *Given a flow $\phi(\mathbf{x},t)$, defined on $U \subset \mathbb{R}^n$ and $X \subset U$, the set $\omega(X)$ is a closed invariant set. Moreover, it is given by:*

$$\omega(X) = \bigcap_{t\geq 0} \phi(\overline{\bigcup_{t\geq 0} \phi(X,t)}, t).$$

*Proof.* Invariance follows from Lemma 2.2.5. The rest of the proof can be found in [20]. □

An analogous result for Theorem 2.2.1 can also be stated.

**Proposition 2.2.7.** *If $X$ is non empty, and $\overline{\bigcup_{t\geq 0}\phi(X,t)}$ is compact, then $\omega(X)$ is non empty and compact. If $\overline{\bigcup_{t\geq 0}\phi(X,t)}$ is connected, then $\omega(X)$ is also connected.*

**Definition 2.2.11.** A *trapping region* $D$ is an invariant set such that $\phi(\overline{D},t) \subset D$.

Proposition 2.2.7 is used in the construction of the Lorenz attractor, in fact, we will find an invariant compact set $D$ which is actually a trapping region. The following proposition will be useful.

**Proposition 2.2.8.** *Given a flow $\phi$ let $D \subset \mathbb{R}^n$ be a set which satisfies $\phi(\overline{D},t) \subset D$ for every $t \geq 0$. Then,*

$$\omega(D) = \bigcap_{t\geq 0} \phi(D,t)$$

*is a non empty, invariant, compact and connected attracting set.*

*Proof.* We first observe that, given $\epsilon > 0$, $\phi(\overline{D},t+\epsilon) \subset \phi(D,t) \subset \phi(D,t)$, then,

$$\bigcap_{t\geq 0} \phi(D,t) = \bigcap_{t\geq 0} \phi(\overline{D},t) = \bigcap_{t\geq 0} \phi(\overline{\bigcup_{t\geq 0} \phi(t,D)},t) = \omega(D).$$

It suffices now to apply Proposition 2.2.7 to show compactness, and Lemma 2.2.5 to see invariance.

Let us show that $\omega(X)$ is an attracting set. Assume it is not attracting, then there exists $\mathbf{x} \in D$ such that

$$\lim_{n\to\infty} d(\phi(\mathbf{x},t_n),\omega(D)) > R > 0.$$

As $\phi(\mathbf{x},t_n)$ is in $\overline{D}$, there exists a subsequence such that

$$\lim_{k\to\infty} \phi(\mathbf{x},t_{n,k}) = y \in \omega(\mathbf{x}).$$

However, $\omega(\mathbf{x}) \subset \omega(D)$, which contradicts the first assumption. □

## 2.3 The Poincaré Map

The concept of Poincaré Map was firstly introduced in 1881 by Henri Poincaré in his "Memoires sur les courbes définies par une equation differentelle" [14]. Poincaré maps provide a very useful tool in the study of dynamical systems.

To start with, Poincaré maps reduce the dimension of the problem, for instance, given a three dimensional dynamical system, the Poincaré map is a two dimensional discrete dynamical system. What is more, Poincaré maps can (not always) describe the global dynamics of some dynamical systems, however, in order to do this, further background on numerics is required [6],[24].

In this dissertation we shall distinguish two kinds of Poincaré maps. The first one, and the original one, which is constructed strictly to analyse the stability of periodic orbits, whereas the second one is a rather *ad hoc* construction to study global dynamics of a continuous dynamical system. The difference between these two kinds of Poincaré maps is substantial, however, they are based of the same intuitive idea.

The first kind of Poincaré map is simpler, and we shall go over the main definitions and existence theorem. The second kind is much more intricate, thus, we will just provide an example. However, we are to return to the second kind of Poincaré maps in the following chapter as this is used in the study of the dynamics of the Lorenz equations.

We will make an heuristic definition first. We wish to study the behaviour of initial conditions when time increases. Given $\mathbf{x}_0$ an initial condition for (2.1), this point lays on a plane $\Sigma$. We will ask $\phi(\mathbf{x}_0, t)$ to induce a periodic orbit $\Gamma$ of period $T$, this is, $\phi(\mathbf{x}_0, t+T) = \phi(\mathbf{x}_0, t)$ for some $T > 0$. If $\mathbf{x}$ is an initial condition for (2.1) sufficiently close to $\mathbf{x}_0$, $\phi(\mathbf{x}, t)$ will intersect $\Sigma$ again, we will call this intersection point $P(\mathbf{x})$. The function which maps $\mathbf{x}$ to $P(\mathbf{x})$ is called the Poincaré Map or first return map.

**Theorem 2.3.1.** *Let $U$ be an open subset of $\mathbb{R}^n$ and let $\mathbf{f} \in C^1(U)$. Suppose that $\phi(\mathbf{x}_0, t)$ is a periodic solution of (2.1) of period $T$ and that the set*

$$\Gamma = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \phi(\mathbf{x}_0, t), 0 \leq t \leq T\}$$

*is contained in $U$. We consider the normal plane to $\Gamma$ at $\mathbf{x}_0$:*

$$\Sigma = \{\mathbf{x} \in \mathbb{R}^n : (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{f}(\mathbf{x}_0) = 0\}.$$

*Then, there exists $\delta > 0$ and a unique function $\tau(\mathbf{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined and continuously differentiable $\in B_\delta(\mathbf{x}_0)$ such that $\tau(\mathbf{x}_0) = T$ and*

$$\phi(\mathbf{x}, \tau(\mathbf{x})) \in \Sigma, \quad \forall \mathbf{x} \in B_\delta(\mathbf{x}_0).$$

*Proof.* Let $\mathbf{x}_0 \in \Gamma \subset U$. The idea is to use the implicit function theorem. We define:

$$F : U \times \mathbb{R} \longrightarrow \mathbb{R}$$

given by $F(\mathbf{x}, t) = (\phi(\mathbf{x}, t) - \mathbf{x}_0) \cdot \mathbf{f}(\mathbf{x}_0)$. Now, as it has been proved in Theorem 2.1.1, $\phi(\mathbf{x}, t) \in C^1(U)$ (actually, $\phi(\mathbf{x}, t) \in C^1(U \times \mathbb{R})$) therefore, as

$f \in C^1(U)$, it yields that $F(\mathbf{x}, t) \in C^1(U \times \mathbb{R})$. Let $T$ be the period of the orbit, then:

$$\frac{\partial F(\mathbf{x}_0, T)}{\partial t} = \frac{\partial \phi(\mathbf{x}_0, T)}{\partial t} \cdot \mathbf{f}(\mathbf{x}_0) = \mathbf{f}(\mathbf{x}_0) \cdot f(\mathbf{x}_0) \neq 0.$$

This last inequality is due to the fact that $\mathbf{x}_0$ is not an equilibrium point. If it had been so, there would have been no periodic orbit.

Using now the implicit function theorem, there exists $\delta > 0$ such that there is a differentiable function $\tau(\mathbf{x})$ defined on $B_\delta(\mathbf{x}_0)$ that satisfies:

$$F(\mathbf{x}, \tau(\mathbf{x})) = (\phi(\mathbf{x}, \tau(\mathbf{x})) - \mathbf{x}_0) \cdot \mathbf{f}(\mathbf{x}_0) = 0$$

which implies that $\phi(\mathbf{x}, \tau(\mathbf{x})) \in \Sigma$. □

This theorem proves that the following definition is correctly done.

**Definition 2.3.1.** Let $\mathbf{x}_0 \in \mathbb{R}^n$ be an initial condition such that $\phi(\mathbf{x}_0, t)$ induces the periodic orbit $\Gamma$. Let $\Sigma$ be a transversal section to $\Gamma$ at $\mathbf{x}_0$. If $B_\epsilon(\mathbf{x}_0)$ is a neighbourhood of $\mathbf{x}_0$ such that for every $\mathbf{x} \in B_\epsilon(\mathbf{x}_0)$ the $\tau(\mathbf{x})$ function in Theorem 2.3.1 is well defined, we define the Poincaré Map as:

$$\begin{aligned} P : \Sigma \cap B_\epsilon(\mathbf{x}_0) &\longrightarrow \Sigma \\ \mathbf{x} &\mapsto \phi(\mathbf{x}, \tau(\mathbf{x})) \end{aligned}$$

The Poincaré Map need not be defined on the transversal plane to the periodic orbit at a certain point $\mathbf{x}_0$. It is actually enough to take some manifold which does not contain the vector field evaluated at $\mathbf{x}_0$. Figure 2.1 shows schematically how a starting point $\mathbf{x}_0$ flows around until it strikes $\Sigma$ again.

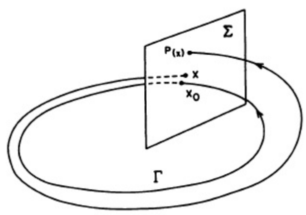



Figure 2.1: Sketch of the Poincaré map [13].

**Remark 2.3.1.** The $\tau$ function defined in Theorem 2.3.1 is commonly known as time-of-flight function. Indeed, this function essentially measures how much time it takes for a particle to return to $\Sigma$.

**Corollary 2.3.2.** *Let $\Sigma$ be a transversal section to a periodic orbit, and $U_\Sigma$ an open set in $\Sigma$ where the Poincaré map $P$ is well defined. Then, $P \in C^1(U_\Sigma)$ and it is a diffeomorphism.*

*Proof.* Differentiability follows from the fact that both $\phi$ and $\tau$ are differentiable. The existence of the inverse function follows from considering the change $t \mapsto -t$ in Equation (2.1). $\square$

Technically, the definition of the Poincaré depends wholly on the explicit expression of the solution of the differential equations unless powerful numerical methods are used.

**Example 2.3.1.** Consider the following differential equation:

$$\begin{cases} \dot{x} = x - y - x(x^2 + y^2) \\ \dot{y} = x + y - y(x^2 + y^2) \end{cases} \tag{2.7}$$

for $(x, y) \in \mathbb{R}^2$. Let us change variables into polar coordinates. This way, (2.7) becomes:

$$\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = 1. \end{cases} \tag{2.8}$$

This differential equation can be integrated directly. If $(r_0, \theta_0)$ is an initial condition , the solution is:

$$\theta(t) = t + \theta_0 \tag{2.9}$$

$$r(t) = \left(1 + \left(\frac{1}{r_0} - 1\right) e^{-2t}\right)^{1/2}. \tag{2.10}$$

Thus, the flow is:

$$\phi((r_0, \theta_0), t) = \left(\left(1 + \left(\frac{1}{r_0} - 1\right) e^{-2t}\right)^{1/2}, t + \theta_0\right). \tag{2.11}$$

As $\dot{\theta}(t) = 1$, a periodic orbit will appear when $r - r^3 = 0$, which would imply that $r = 0$ or $r = \pm 1$. In particular, $\phi((1, 0), t)$ is a periodic trajectory which gives rise to a periodic orbit. Now that we have found a periodic orbit, we have to choose a transversal section $\Sigma$. Let us define:

$$\Sigma = \{(x.y) \in \mathbb{R}^2 \times [0, 2\pi) : r \in (0, +\infty), \quad \theta = 0\}. \tag{2.12}$$

We shall verify that $\Sigma$ is indeed a cross section. To show this, we need to check that the vector field which defines the differential equation is not parallel to $\Sigma$. This is equivalent to see that the normal vector to $\Sigma$, $\mathbf{n}$, multiplied by the vectorfield is non-zero. We can choose without loss of generality $\mathbf{n} = (0, 1)$:

$$(0, 1) \cdot (r - r^3, 1) \neq 0,$$

hence $\Sigma$ is transversal. As $\dot{\theta}(t) = 1$, the $\tau$ function from definition 2.3.1 is $\tau(t) = 2\pi$. Thus, if we call the Poincaré $P$:

$$P(r_0, \theta_0) = \phi_{\tau(t)}(r_0, \theta_0) = \left( \left( 1 + \left( \frac{1}{r_0^2} - 1 \right) e^{-4\pi} \right)^{1/2}, 2\pi \right). \tag{2.13}$$

Since the variable which depends only in $\theta_0$ is constant, we can reduce the Poincaré map into a one dimensional map:

$$P(r_0) = \left( 1 + \left( \frac{1}{r_0^2} - 1 \right) e^{-4\pi} \right)^{1/2}. \tag{2.14}$$

We are in conditions of applying the results on the derivatives of discrete maps. Clearly, $r_0 = 1$ is a fixed point of the Poincaré map, thus, we can wonder about its stability:

$$P'(r_0) = \frac{e^{-4\pi}}{1 + e^{-4\pi} \left( \frac{1}{r_0^2} - 1 \right)^{3/2} r_0^3}. \tag{2.15}$$

hence,

$$P'(1) = e^{-4\pi} < 1. \tag{2.16}$$

The point 1 is stable for $P$.

The next example illustrates the second kind of Poincaré map. The difference is subtle with respect to the previous example. In the example above we first identified a periodic orbit and then we constructed the Poincaré map. On the other hand, the following example will not identify any periodic orbit (there are none) and will take the cross section $\Sigma$ conveniently.

**Example 2.3.2.** Let us consider the following damped oscillator:

$$\ddot{x} + \dot{x} + x = 0, \tag{2.17}$$

with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$. This equation can be formulated as a first order differential equation:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.18}$$

This way, we want to know the evolution of the starting point $(1, 0)$. To do so, we will make use of the second kind of Poincaré map. Let us take $\Sigma$ the cross section defined as:

$$\Sigma = \{(x, y) \in \mathbb{R}^2 : x > 0, \;\; y = 0\}. \tag{2.19}$$

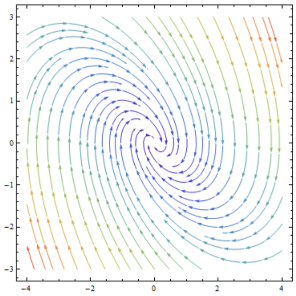


Figure 2.2: The phase portrait of equation (2.18).

The solution to (2.17) is

$$x(t) = e^{-t/2}\left(\cos\left(\frac{\sqrt{3}t}{2}\right) + \frac{\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}t}{2}\right)\right). \tag{2.20}$$

To determine an explicit formula for the trajectory $\phi((1,0),t)$ of (2.18) it is enough to compute the derivative of $x(t)$:

$$y(t) = \dot{x}(t) = -\frac{2}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}t}{2}\right). \tag{2.21}$$

It is clear that the starting point $(1,0)$ lies in $\Sigma$ and that it will take a time of $4\pi/\sqrt{3}$ for the trajectory to strike $\Sigma$ again. Thus, the Poincaré map is given by:

$$P((1,0)) = \phi((1,0), 4\pi/\sqrt{3}). \tag{2.22}$$

Several iterations of the Poincaré map suggest that the $(1,0)$ ends up at the origin:

$$\begin{aligned} P^0((1,0)) &= (1,0) \\ P((1,0)) &= (0.026579\ldots,0) \\ P^2((1,0)) &= (0.000706\ldots,0) \end{aligned}$$

Figure 2.2 shows that the origin of the phase portrait is a stable focus.

# Chapter 3

# The Lorenz Equations

In this chapter we will work almost exclusively with the Lorenz system (1), therefore, from now on **f** will denote the Lorenz vector field. As mentioned in the introduction, we are interested in the classical parameters, this way, unless it is claimed explicitly, $\sigma = 10$, $\beta = 8/3$ and $\rho = 28$.

## 3.1 Bifurcation on $\rho$

For this section, $\rho$ is let to vary on the positive real numbers. This discussion on $\rho > 0$ is not petty, since $\rho$ is a coefficient which depends on the dimensionless Rayleigh number. The latter number depends on the temperature difference which originates the convecting process. Usually, convection is modelled in laboratories and the Rayleigh number can be changed artificially.

First of all we will compute the equilibrium points. These arise when we set the gradient of the flow equal to the zero vector:

$$\sigma(y - x) = 0 \tag{3.1}$$

$$\rho x - xz - y = 0 \tag{3.2}$$

$$xy - \beta z = 0. \tag{3.3}$$

Clearly, the origin is an equilibrium point. Also, from (3.1) we know that $x = y$, therefore using (3.3) and (3.4) :

$$z = \rho - 1 \Rightarrow x = y = \pm\sqrt{\beta(\rho - 1)}.$$

If $\rho \leq 1$, the only equilibrium point is the origin. Thus, for $\rho > 1$, the equilibrium points are:

$$(0, 0, 0), \left(\pm\sqrt{\beta(\rho - 1)}, \pm\sqrt{\beta(\rho - 1)}, \rho - 1\right)$$

We first study the behaviour of the origin. Linearising at the origin, we get to this matrix:

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -\beta \end{pmatrix}_{(0,0,0)} = \begin{pmatrix} -10 & 10 & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix} \tag{3.4}$$

whose eigenvalues are all negative. Hence, the origin is a hyperbolic equilibrium point and it is a locally asymptotically stable point. In addition, it is a globally stable point:

**Proposition 3.1.1.** *If $0 < \rho < 1$, the origin is globally stable, i.e., every trajectory ends up at the origin.*

*Proof.* Let $\phi$ be the flow defined by (1).

We shall consider the function $V(x, y, z) = \rho x^2 + \sigma y^2 + \sigma z^2$. $V$ is a Lyapunov function. Indeed, $V \in C^1(\mathbb{R}^3)$ and $V(0, 0, 0) = 0$. Furthermore, $V(x, y, z) > 0$ if $(x, y, z) \neq (0, 0, 0)$. As we wanted to see.

We now take its Lie derivative:

$$\begin{aligned}
\dot{V}(x, y, z) &= \frac{\partial}{\partial t} V(\phi(t, (x, y, z))) \\
&= \nabla V(x, y, z) \cdot \mathbf{f}(x, y, z) \\
&= (2\rho, 2\sigma y, 2\sigma z) \cdot (\sigma(y - x), \rho x - y - xz, xy - \beta z) \\
&= \ldots = 2\sigma \left(2\rho xy - \rho x^2 - y^2 - \beta z^2\right) \\
&= -2\sigma \left((\rho x - y)^2 + \beta z^2\right) < 0.
\end{aligned}$$

Therefore, $\dot{V}(x, y, z) < 0$, for all $(x, y, z) \neq (0, 0, 0)$. Thus, every trajectory $\phi(t, (x, y, z))$ ends up at the origin. □

Figure 3.1a shows several trajectories converging to the origin.

When $\rho = 1$, the linearised system leads to the matrix:

$$\begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix}$$

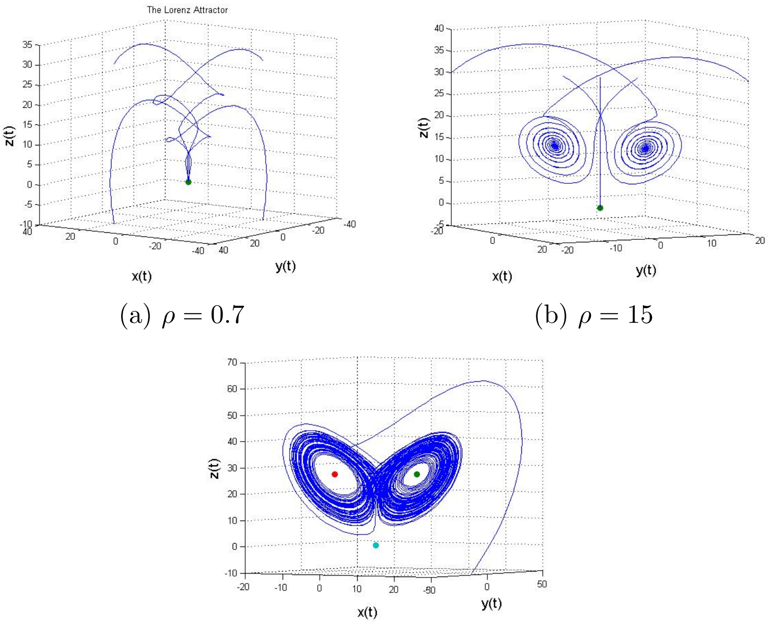
whose eigenvalues are negative but for one which is zero. This phenomenon is known as pitchfork bifurcation [6].

At $\rho > 1$, the origin is a hyperbolic equilibrium point with one-dimensional unstable subspace and a two-dimensional stable manifold. This result comes from the linearisation matrix (3.4).

Further bifurcation analysis [6] tells us that when $\rho > 1$ but $\rho < \frac{\sigma(\sigma+\beta+3)}{\sigma-\beta-1} \approx 24.7368$ the three points are asymptotically stable. See Figure 3.1b.

The value $\rho_h := 24.7368\ldots$ corresponds to a Hopf bifurcation value, i.e., when $\rho$ reaches this value the two non-trivial equilibrium points stop being asymptotically stable and turn out to be unstable. At this value, the linearisation matrix possesses two pure imaginary eigenvalues and a negative eigenvalue.

At $\rho > \rho_h$, every non-trivial equilibrium point has a two dimensional unstable manifold and a stable one dimensional manifold, which is the case of the classical parameters. Bifurcation is not the main subject of study here, the interested reader is referred to [13] for further analysis.




(a) $\rho = 0.7$ (b) $\rho = 15$

(c) $\rho = 28$

Figure 3.1: Phase space of the Lorenz equations for different values of $\rho$.

## 3.2 Construction of the Lorenz Attractor

As mentioned earlier, we will find a trapping region in which the Lorenz Attractor lays entirely. A region of this kind was found by Lorenz in [10] and a deeper discussion on this topic can be found in [19]. Indeed, Figure 3.1c suggests that every trajectory moves forward into a bounded region.

**Lemma 3.2.1.** *If $M$ is a compact subset of $\mathbb{R}^n$, let us consider a real function $L \in C^1(M)$. Suppose that there exists $R > 0$ such that the Lie derivative of $L$ with respect to $\mathbf{f}$ satisfies:*

$$\nabla(L)(x) \cdot \mathbf{f}(x) < 0, \qquad L(x) = R.$$

*Then, every connected component of $V_R = \{x \in M : L(x) \leq R\}$ is a trapping region.*

**Theorem 3.2.2.** *There exists a bounded globally trapping region for the Lorenz System.*

*Proof.* We wish to show that there is a bounded region $E$ such that every trajectory enters $E$ and never leaves it. We consider the following Lyapunov function:

$$V(x,y,z) = \rho x^2 + \sigma y^2 + \sigma(z-2\rho)^2 \tag{3.5}$$

which satisfies that

$$\begin{aligned}
\dot{V}(x,y,z) &= \frac{\mathrm{d}}{\mathrm{d}t}V(\phi(t,(x,y,z))) \\
&= \nabla V(x,y,z)\cdot \mathbf{f}(x,y,z) \\
&= (2\rho x, 2\sigma y, 2\sigma(z-2\rho))\cdot(\sigma(y-x), \rho x - y - xz, xy - \beta z) \\
&= -2\sigma\left(\rho x^2 + y^2 + \beta z^2 - 2\rho\beta z\right)
\end{aligned}$$

Clearly, the set of points which satisfies $\dot{V} \geq 0$ is bounded and its boundary is an ellipsoid. Let $D$ be the bounded region in which $\dot{V}$ is non-negative. Since it is a closed set and bounded, let $M$ be the maximum of $V$ in $D$. We shall now consider the bounded region $E$ in which $V \leq M + \epsilon$, $\epsilon > 0$.

If a point $(x,y,z)$ lies in $\mathbb{R}^3 \setminus E$, then $(x,y,z)$ lies in $\mathbb{R}^3 \setminus D$. Indeed, if $(x,y,z) \in \mathbb{R}^3 \setminus E$ then $V(x,y,z) \geq M+\epsilon$, thus, $V(x,y,z) > M$ and $(x,y,z) \in \mathbb{R}^3 \setminus D$ because $M$ is the maximum. As a consequence, $\dot{V}(x,y,z) > 0$, for all $(x,y,z) \in \mathbb{R}^3 \setminus E$.

So we have found a region where the Lyapunov function (3.5) has a negative derivative, then Lemma 3.2.1 gives the result.

$\square$

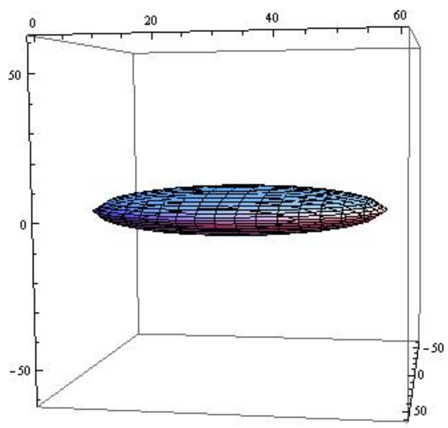



Figure 3.2: The compact subset $D$

We have now seen that there is a compact region $D$ in $\mathbb{R}^3$ where eventually every trajectory enters and never leaves. If we recall Theorem 2.1.8 as $\mathbf{f} \in C^1(D)$ we can assume that the flow determined by equation (3) is defined for all $t \in \mathbb{R}$. Therefore, a forward invariant set $\Lambda$ can be built:

$$\Lambda = \bigcap_{0 \leq t} \phi_t(D).$$

$\Lambda$ is called The Lorenz Attractor.

**Proposition 3.2.3.** *$\Lambda$ is the $\omega$-limit set of $D$ and it is, non-empty, invariant, compact and connected attracting set.*

*Proof.* The proof follows directly from Proposition 2.2.8. □

Numerical simulations in [6] show that the attractor seems to be a two dimensional surface. In fact, it can be shown that $\Lambda$ has zero volume.

**Lemma 3.2.4.** *Given Equation* (2.1) *and its associated flow $\phi$, we define the volume of a bounded set $U$ at time $t$ as:*

$$V(t) = \int_{\phi(U,t)} 1 \mathrm{d}x.$$

*With this definition,*

$$\dot{V}(t) = \int_{\phi(U,t)} \operatorname{div}(f(x)) \mathrm{d}x.$$

*Proof.* The proof can be found in [20]. □

Intuitively, if $V(t)$ is the volume enclosed by a closed surface $S(t)$ at time $t$, $V(t + \Delta t)$ is approximated by $V(t) + \int_{St} (\mathbf{f} \cdot \mathbf{n} \mathrm{d}A)$. Therefore,

$$\dot{V}(t) = \int_{S(t)} (\mathbf{f} \cdot \mathbf{n}) \, \mathrm{d}A = \int_V \operatorname{div}(\mathbf{f}). \tag{3.6}$$

**Proposition 3.2.5.** *$\Lambda$ has zero volume.*

*Proof.* Taking into account the previous Lemma 3.2.4,

$$\operatorname{div}(f) = -\sigma - 1 - \beta = -(\sigma + \beta + 1) < 0.$$

Thus,

$$\dot{V}(t) = -V(t)(\sigma + \beta + 1) \Rightarrow V(t) = V(0)e^{-(\sigma+\beta+1)}.$$

This way, as $t$ tends to infinity, any volume is contracted to a zero volume set in $\mathbb{R}^3$. □

**Remark 3.2.1.** The previous result does not mean that each piece of volume contracts in every direction, i.e., the contracted volume need not be an eventual point.

The latter result implies further information on the stability of the system. The Lorenz system cannot support unstable equilibrium points nor unstable closed orbits.

**Proposition 3.2.6.** *Volume contraction is not compatible with unstable equilibrium points.*

*Proof.* Let $\mathbf{G}$ denote a volume contracting vector field. Without loss of generality, let $S$ be a closed and conveniently small sphere which contains an unstable equilibrium point. This the dynamics within $S$ are given by the linearised vector field $\mathbf{G}_l = (\lambda_1 x, \lambda_2 y, \lambda_3 z)$, where $\lambda_1$, $\lambda_2$ and $\lambda_3$ are the strictly positive eigenvalues of the system. Recalling Lemma 3.2.4,

$$V(t) = V(0)e^{(\lambda_1+\lambda_2+\lambda_3)t}, \tag{3.7}$$

which contradicts the fact that the system is volume contracting. □

**Remark 3.2.2.** A consequence is that closed orbits cannot be unstable in the sense that every point in every neighbourhood of the orbit flies away. Yet it may be unstable and still attracting for some points. This behaviour would be analogous to that of saddle points.

## 3.3 The Lorenz Map

In his original paper, Lorenz plotted the $y(t)$ and $z(t)$ components of the solution on the plane. He observed that as a unit particle entered one of the spirals, it started to increase its height until it suddenly moved on to the other spiral.

He suspected, throughout numerical observations, that what made a particle change spiral is the local maximum on $z(t)$. Figure 3.4 shows the $z(t)$ function where we can observe clearly the local maxima to which Lorenz made reference. He also conjectured that there should be a rule, a map maybe, which relates the $n$-th local maximum with the next one. This rule is the so called Lorenz map $L$ [19].

After 6000 iterations he came up with Figure 3.5 which is the plot of the values of the local maxima in $z(t)$.

Simple observation lead Lorenz to identify his map with the tent map. Indeed, they seem to be topologically equivalent, therefore, their dynamics are expected to be similar. This fact, of course, suggests that the Lorenz map might display chaotic dynamics.

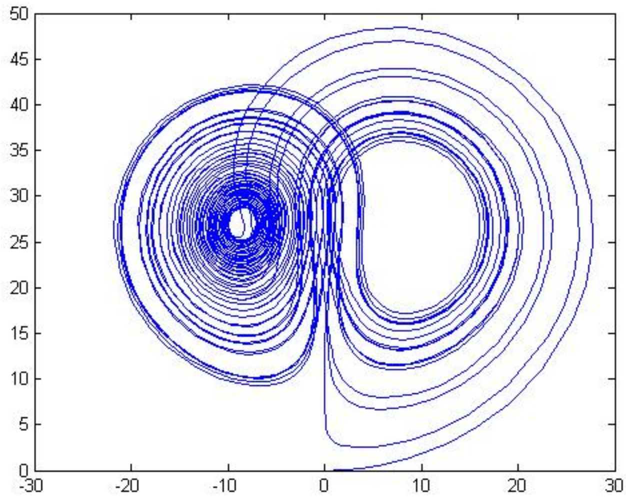



Figure 3.3: Plot of $y(t)$ vs. $z(t)$, the initial condition is $(0, 1, 0)$.

Moreover, looking at Figure 3.5 again, the Lorenz map looks as if $|L'(t)| > 1$ for all $t$. If this were to be true, the Lorenz attractor cannot be a stable limit cycle, which *a priori* could be. In fact, there is a more general result.

**Remark 3.3.1.** If $\Gamma$ is a closed orbit for (1), and under the assumption that $|L'(t)| > 1$, $\Gamma$ cannot be stable.

Heuristically, if we make a small perturbation on an initial condition in $\Gamma$ this will provoke a small perturbation on the periodic orbit $\{z_0, z_1, \ldots, z_m\}$ in the Lorenz map induced by $\Gamma$. Now, using the fact that $|L'(t)| > 1$, the accumulation of perturbations in $z_k$ is amplified. Therefore, the orbit cannot be stable.

## 3.4 A Numerical Method for the Poincaré Map

The idea of the Poincaré map in the Lorenz system is to measure somehow the deviation of perturbed initial conditions. In this section we shall give a numerical method in order to compute an approximation of the deviation in terms of the perturbation of the initial condition. This method is an elementary one but brings together the main ideas of how to obtain information about the sensitive dependence of the Lorenz equations.

We recall that the main idea of the Poincaré map is to build a convenient cross section $\Sigma$ in order to study how points in $\Sigma$ return to it after being moved by the flow.

There is no general formulation for the Poincaré maps, rather each case deserves special attention. In the case of the Lorenz equations, experiments

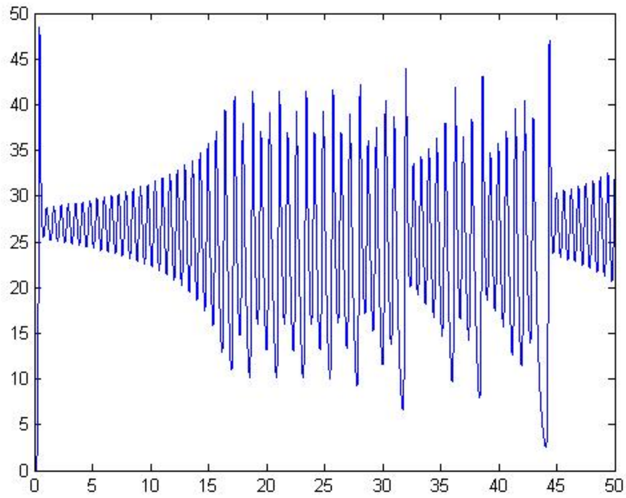



Figure 3.4: Plot of $z(t)$.

*in silico* done by [19], [6] and [21] suggest taking $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z = \rho - 1 = 27\}$ as the returning surface. Firstly, notice that the Poincaré map can be written as $P(\mathbf{x}_0) = \phi_{\tau(t)}(\mathbf{x}_0)$ where $\tau(t)$ is the $\tau$ function given in Definition 2.1, the so called time-of-flight function, and $\mathbf{x}_0 = (x_0, y_0, 27) \in \Sigma$ is the starting point. Thus,

$$\frac{\mathrm{d}\phi_{\tau(t)}(\mathbf{x}_0)}{\mathrm{d}\tau} = \mathbf{f}(\phi(\mathbf{x}_0, \tau(t))). \tag{3.8}$$

If we iterate once the Poincaré map $P$, $P(\mathbf{x}_0) = \mathbf{x}_1 = (x_1, y_1, 27)$. This means that $\mathbf{x}_0$ will return to $\Sigma$ after a time $\tau_1$. The coordinates $x_1$ and $y_1$ can be calculated with numerical integration. Notice that the target of defining the Poincaré map here is not to compute successive iterations of $\mathbf{x}_0$, rather, the successive iteration of a *perturbation* of $\mathbf{x}_0$.

Let us call $\delta\mathbf{x}_0 = (\delta x_0, \delta y_0, 0)$ a small perturbation of $\mathbf{x}_0$. This perturbation will lead to a small change in the time of flight, namely, $\delta\tau$. As a consequence it will generate a small change in the returning point $\delta\mathbf{x}_1$. We are particularly interested in $\delta\mathbf{x}_1 = (\delta x_1, \delta y_1, 0)$. Therefore, we expand using Taylor series around $\mathbf{x}_0$ and $\tau_1$ (which are known data):

$$\phi_{\tau_1+\delta\tau}(\mathbf{x}_0 + \delta\mathbf{x}_0) = \phi_{\tau_1}(\mathbf{x}_0) + \delta\tau\frac{\mathrm{d}}{\mathrm{d}t}\phi_{\tau_1}(\mathbf{x}_0) + (D\phi_{\tau_1})\,\delta\mathbf{x}_0. \tag{3.9}$$

The term $D\phi_{\tau_1}$ contains nine partial derivatives which are computed by solving the following ordinary differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(D\phi_{\tau_1}\right) = \left(D\mathbf{f}\right)_{\phi_{\tau_1}}\left(D\phi_{\tau_1}\right). \tag{3.10}$$

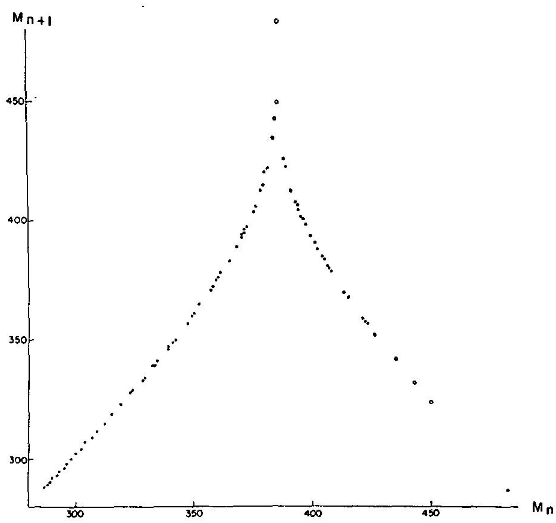



Figure 3.5: The original plot of the Lorenz map [10].

This way, if we set $\phi = (\phi_1, \phi_2, \phi_3)$, (3.10) becomes:

$$\begin{pmatrix} \phi'_{1_x} & \phi'_{1_y} & \phi'_{1_z} \\ \phi'_{2_x} & \phi'_{2_y} & \phi'_{2_z} \\ \phi'_{3_x} & \phi'_{3_y} & \phi'_{3_z} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - \phi & -1 & \phi_1 \\ \phi_2 & \phi_1 & \beta \end{pmatrix}_{\phi_{\tau_1}} \cdot \begin{pmatrix} \phi_{1_x} & \phi_{1_y} & \phi_{1_z} \\ \phi_{2_x} & \phi_{2_y} & \phi_{2_z} \\ \phi_{3_x} & \phi_{3_y} & \phi_{3_z} \end{pmatrix}. \quad (3.11)$$

We can actually rearrange (3.9) to get rid of the non interesting terms, since we are just interested in the deviation. Let us set $\mathbf{f} = (f_1, f_2, f_3)$:

$$\begin{pmatrix} \delta x_1 \\ \delta y_1 \\ 0 \end{pmatrix} = \delta\tau \begin{pmatrix} f_1(\mathbf{x}_1) \\ f_2(\mathbf{x}_1) \\ f_3(\mathbf{x}_1) \end{pmatrix} + D\phi_{\tau_1} \begin{pmatrix} \delta x_0 \\ \delta y_0 \\ 0 \end{pmatrix}. \quad (3.12)$$

This is a system of equations in three unknowns, thus, $\delta x_1$, $\delta y_1$ and $\delta\tau$ can be calculated in terms of $\delta x_0$ and $\delta y_0$. In other words, we have been able to write the eventual deviation of the first return in terms of the initial perturbation.

In fact, the time of flight is rather uninteresting, so we can obtain a reduced system. From (3.12) we get that

$$0 = \delta\tau f_3(\mathbf{x}_1) + \delta x_0 \phi_{3_x} + \delta y_0 \phi_{3_y} \Rightarrow \delta\tau = \lambda_1 \delta x_0 + \lambda_2 \delta y_0, \quad (3.13)$$

for some real numbers $\lambda_1$ and $\lambda_2$. Hence,

$$\begin{pmatrix} \delta x_1 \\ \delta y_1 \end{pmatrix} = \begin{pmatrix} f_1(\mathbf{x}_1)(\lambda_1 \delta x_0 + \lambda_2 \delta y_0) \\ f_2(\mathbf{x}_1)(\lambda_1 \delta x_0 + \lambda_2 \delta y_0) \end{pmatrix} + D\phi_{\tau_1} \begin{pmatrix} \delta x_0 \\ \delta y_0 \end{pmatrix}. \quad (3.14)$$

This method strictly depends on the election of $\mathbf{x}_0$, therefore it has a local use only.

## 3.5 Geometric Models of the Lorenz System

The *Geometric Lorenz Model* was a concept coined by J. Guckenheimer and R. F. Williams in [7] and [5]. This concept is essentially a particular way of studying the dynamics of the Lorenz equations. The key point here is the analysis of the Poincaré maps defined in different cross sections and the discrete maps they generate. In this section we shall study how to reduce the two dimensional Poincaré map into a one dimensional map as well as the conjectures made by Guckenheimer and Williams in [7].

As mentioned in Section 3.4, the returning plane $\Sigma$ will be at $z = 27$. $\Sigma$ is non bounded, however, [6] takes an adequate rectangle in $\Sigma$ such that each of the saddle points lie in opposite sides of the rectangle. For the sake of simplicity in notation, we will call this rectangle $\Sigma$. Figure 3.6 sketches the situation, where $q_+$ and $q_-$ are the saddle points of the system.

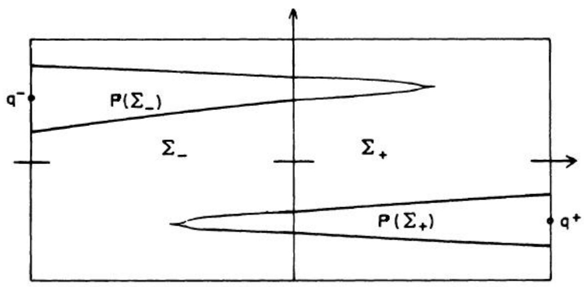



Figure 3.6: Sketch of the returning plane [6].

We note that $\dot{z} < 0$ for every point in $\Sigma$, thus, particles in the interior of $\Sigma$ move downwards. Moreover, we must now take into account the stable manifold of the origin. We recall it is a two dimensional manifold and, thus, it induces a dividing line $L$ in the middle of $\Sigma$. This means that every point which goes through $L$ will eventually converge to the origin. In Figure 3.6, $L$ would be the vertical line in the middle. Thus, $\Sigma$ consists of two connected components, $\Sigma_+$ and $\Sigma_-$.

This way, we have reduced the Lorenz equations to a two dimensional discrete map, namely, $P : \Sigma \longrightarrow \Sigma$. Numerical integration results provided by [6] and [19], show the image of $\Sigma_+$ and $\Sigma_-$ through $P$, which is the triangular regions depicted in Figure 3.6.

One of the main problems now is to reduce even more the dimension of the map we are dealing with. In order to do that, there must be an equivalence relation in $\Sigma$ which allows us to identify lines with points. We will eventually call this identification the *foliation of* $\Sigma$.

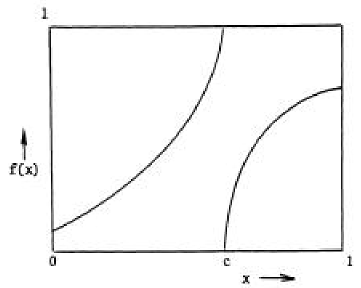



Figure 3.7: A function $f$ satisfying assumptions H1-H6.

The following discussion is just a brief summary of the mathematics Guckenheimer and Williams used to study the dynamics of the Geometric Lorenz Attractor.

**Definition 3.5.1.** The *foliation of* $\Sigma$ is a family of curves $\mathcal{F}$ in $\Sigma$ such that:

(i) $L \in \mathcal{F}$.

(ii) If $\alpha \in \mathcal{F} \setminus \{L\}$ then there exists $\beta \in \mathcal{F}$ such that $P(\alpha) \subset \beta \in \mathcal{F}$.

Thus, the eventual equivalence relation is to identify every point in the same curve. This way, we will be able to construct a one dimensional model, $f$, of the Poincaré map defined of $\Sigma$. Let us give the mathematical assumptions first:

H1 There exists a change of coordinates $(x, y) \in \Sigma$ such that the curves in $\mathcal{F}$ are given by the vertical lines $x =$constant, and $x = c$ is the element $L$.

H2 $f : [0, 1] \longrightarrow [0, 1]$ is a continuous map except for the point $x = c$.

H3 $f$ is monotonic and strictly increasing on $[0, c)$ and on $(c, 1]$.

H4 $\lim_{x \to c^-} f(x) = 1$, $\lim_{x \to c^+} f(x) = 0$ and $f(c) = 1$.

H5 $\lim_{x \to c^\pm} f'(x) = \infty$ and $f'(x) > \sqrt{2}$ whenever $x \neq c$.

H6 For every interval $I \subset [0, 1]$ there exists $k \in \mathbb{N}$ such that $f^k(I) = [0, 1]$.

Williams in [25] showed that the fact that $f'(x) > \sqrt{2}$ implies the sixth assumption, but we shall make both requirements. As stated before, from now on, we will work under the assumptions H1-H6. The idea now is to use symbolic dynamics. Let $\varphi : [0, 1] \longrightarrow \Sigma_2$ be the function given by:

$$\varphi(x) = s_0 s_1 \dots s_k \dots \tag{3.15}$$

where $s_i = 0$ if $f^i(x) < c$ and $s_i = 1$ if $f^i(x) > c$. Of course the sequence may be finite, certainly, if $f^i(x) = c$, the sequence stops.

**Lemma 3.5.1.** *Given $x \in [0,1]$, $\varphi$ is the sequence of an at most one point in $[0,1]$.*

*Proof.* If $x_1 \neq x_2$, then, by assumption H6, there is some $k \in \mathbb{N}$ such that $f^k(x_1) < c$ while $f^k(x_2) > c$, or vice versa. □

The order in the real number induces order on the sequences, however, the absent symbols must be taken into account. We shall set that the absent symbol is always between 0 and 1, for instance:

$$000\ldots < 00 < 001\ldots. \tag{3.16}$$

**Lemma 3.5.2.** *If $x \in [0,1]$, $\varphi(0) \leq \varphi(x) \leq \varphi(1)$.*

We do not know still which sequences of $\Sigma_2$ correspond to some $\varphi(x)$. Notice that:

$$\varphi(f(x)) = \sigma(\varphi(x)), \tag{3.17}$$

where $\sigma$ is the Shift map. It is clear that if $x \in [0,1]$, then $f^i(x) \in [0,1]$. Therefore, for an element in $\Sigma_2$ to be a sequence for some $x$ we must have the following condition:

$$\varphi(0) \leq \sigma^i(\varphi(x)) \leq \varphi(1), \tag{3.18}$$

for all $i \in \mathbb{N} \cup \{0\}$. This latter fact means that $\varphi(0)$ and $\varphi(1)$ determine which sequences can appear when iterating some $x \in [0,1]$ under $f$.

**Definition 3.5.2.** If $\varphi(x)$ is the sequence associated to $x$, we define $\varphi'(x)$ to be the sequence which is originated after swapping the symbols in $\varphi(x)$.

Guckenheimer and Williams in [7] proved that two functions satisfying the initial assumptions are topologically equivalent if their associated sequences are related:

**Theorem 3.5.3.** *Let $f_1$ and $f_2$ be two maps satisfying the conditions given above. Then $f_1$ and $f_2$ are topologically equivalent if and only if $\varphi_{f_1}(0) = \varphi_{f_2}(0)$ and $\varphi_{f_1}(1) = \varphi_{f_2}(1)$, or if $\varphi'_{f_1}(0) = \varphi_{f_2}(1)$ and $\varphi'_{f_1}(1) = \varphi_{f_2}(0)$.*

*Proof.* The proof can be found in [7]. □

This theorem above is the most significant in [7], since this provides the elementary tool to find simpler models of the Geometric Attractor which still display the same dynamics.

The mathematical assumptions made earlier are very artificial. However, [7], [1] and [25] proved that if the Poincaré map can be reduced to an interval map satisfying assumptions H1-H6, the Lorenz attractor $\Lambda$ exhibits chaotic dynamics.

**Remark 3.5.1.** We have seen that the original Poincaré map can be reduced to a one dimensional map after assuming some facts. This depends strictly on the election of the cross section $\Sigma$, [12] provides a counterexample in which the Poincaré map cannot be reduced.

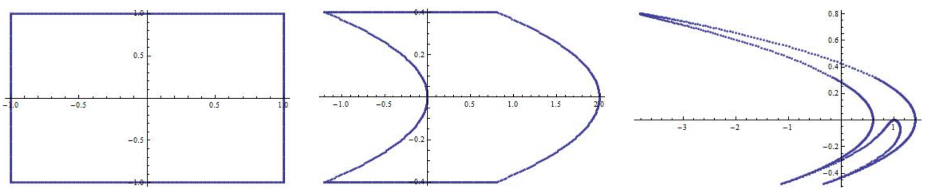
## 3.6 The Hénon Map

The Hénon Map is a two dimensional discrete map which was studied by the french astronomer Michel Hénon [9]. This map was used as a model of the Poincaré map introduced in the previous section. The model is given by the following equations:

$$\begin{cases} x_{n+1} = 1 + y_n - Ax_n^2 \\ y_{n+1} = Bx_n \end{cases} \tag{3.19}$$

where $A > 0$ and $|B| < 1$.

The interesting fact about the Hénon map is that it produces horseshoe-like images. Figure 3.8 shows it clearly. Thus, if we suppose that the Hénon map is a fair approximation of the Poincaré map of the Lorenz equations, it is reasonable to expect that the eventual attractor will be a Cantor set and it will display chaotic dynamics.



(a) $n = 0$ (b) $n = 1$ (c) $n = 2$

Figure 3.8: Iterations of a square under the Hénon map

# Chapter 4

# Historical Background and Comments on Tucker's Proof

## 4.1 Historical Remarks

As mentioned on the introduction Edward Lorenz published "A Deterministic Nonperiodic Flow" [10] in 1963 in the Journal of the Atmospheric Sciences. *Deterministic* is very much of a tricky word; what it means is that there is no stochastic process involved and if we have a solution to the initial value problem associated to (1), then, we would know exactly the long-term behaviour of the system. Lorenz was actually looking for periodicity of the system, what he did not know is that (1) displayed *a priori* chaotic behaviour.

This article does not say anything about chaos explicitly, but Lorenz observed the sensitive dependence on initial condition equation (1): *"When our results concerning the instability of nonperiodic flow are applied to the atmosphere, which is ostensibly nonperiodic, they indicate that prediction of the sufficiently distant future is impossible by any method, unless the present conditions are known exactly."*

Weather forecasting is paradigmatic when talking about chaos, and in some sense, we should acknowledge that Lorenz provided experimental (numerical) evidence. However, Henri Poincaré was also concerned earlier with this problem [15]: *"Why have meteorologists such difficulty in predicting the weather with any certainty? [...]a tenth of a degree more or less at any given point, and the cyclone will burst here and not there, and extend its ravages over districts that it would otherwise have spared. If they had been aware of this tenth of a degree, they could have known it beforehand, but the observations were neither sufficiently precise, and that is the reason why it all seems due to the intervention of chance. "*

Certainly, Lorenz found evidence of chaotic dynamics but a rigorous proof of such result was missing. The next remarkable step is the creation of the so called *Geometric Models* of the Lorenz equations. As commented previously, the idea of the geometric models is to use the Poincaré map in order to obtain global information. J. Guckenheimer and R. F. Williams provided a proof of the existence of the chaotic attractor under some assumptions [7], but still, a general proof was not found. In 1982 Colin Sparrow, professor at the University of Warwick published a comprehensive survey on the results regarding the existence of the Lorenz attractor [19]. He followed the geometric point of view and provided an exhaustive study of the problem.

Many years later, in 1998, Stephen Smale published his famous list of mathematical problems for the 21st century [18], whose fourteenth component was the proof of the existence of the Lorenz attractor. The proposed question was:

*Is the dynamics of the ordinary differential equations of Lorenz (1963) that of the geometric Lorenz attractor of Williams, Guckenheimer and Yorke*?*

This question remained unanswered until 2002, when Warwick Tucker, student at Uppsala University gave a computer assisted proof [21]. He proved the following theorem:

**Theorem 4.1.1.** *For the classical parameter values, the Lorenz equations support a robust strange attractor $\mathcal{A}$. Furthermore, the flow supports a unique SRB measure $\mu_\varphi$ with* $\operatorname{supp}(\mu_\varphi) = \mathcal{A}$.

## 4.2 Comments on Tucker's Proof

We shall focus on the first part of the theorem as it is the one with stronger relation with the dissertation. The essential concepts in Theorem 4.1.1 are the *strangeness* and *robustness*. Robustness is related to the trapping region $D$ computed in the previous chapter, indeed, an attractor is said to be *robust* if it admits a neighbourhood which is trapping or forward invariant. On the other hand, strangeness is just the fact that the Lorenz equations exhibit sensitivity to initial conditions [23].

As Guckenheimer and Sparrow, Tucker goes over the geometric model studied in Chapter 3. However, the proof of Theorem 4.1.1, provided by Tucker, is not a traditional proof, it is a *computer assisted proof*. This means that Tucker builds an algorithm which, if successfully executed, proves the existence of the strange attractor.

The approach to this result is done in two sections: the global section and the local section. The reason why to distinguish two sections is that,

*James Yorke did not work on the geometric Lorenz attractor, yet he introduced Lorenz's work to the mathematical community [21].

although we wish to know global features of the Lorenz equations, trajectories near the origin behave in such a different way they deserve special attention. The global approach involves "rigorous numerics" [21] and the local approach makes use of the *normal form theory*.

As commonly suggested, Tucker [21] works initially on the cross section $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z = \rho - 1 = 27\}$. Again, from now on $\Sigma$ will be referring to a rectangular surface within $\Sigma$. The main results Tucker proves in his paper are the following:

(i) There exists a region $N \subset \Sigma$ that is forward invariant under the Poincaré map $P$.

(ii) There exists a cone $\mathcal{C}(z)$ inside the tangent space of $\Sigma$ at each point $z$ of $N \setminus \Gamma$ such that $DP(z)\mathcal{C}(z) \subset \mathcal{C}(P(z))$ for every $z \in N \setminus \Gamma$.

(iii) Vectors inside this invariant conefield are uniformly expanded by the derivative $DP$ of the Poincaré map. This is, there exists $c > 0$ and $\delta > 1$ such that:

$$\|DP^n(z)v\| \geq c\delta^n \|v\|, \text{ for all } v \text{ in the conefield.}$$

In the following we will describe the proof of $(i)$. The proof of $(ii)$ and $(iii)$ are nontrivial consequences of $(i)$. We must underline that $(i)$ implies that there is an attracting set for the Poincaré map while $(ii)$ and $(iii)$ are concerned with the topological structure of the attractor.

### 4.2.1 Existence of a Forward Invariant Region

The first thing to notice is that the trajectory of a point in $\Sigma$ cuts it along two arcs as shown in Figure 4.1. If we pick one of these arcs we can cover it with a finite set of squares $R_i$ (see Figure 4.1 again). Tucker does it with squares of width $\delta = 0.03$. For simplicity, the sides of the squares are parallel to the sides of $\Sigma$. The idea is to prove that the union of $R_i$ are invariant under $P$. In order to prove so, Tucker creates an algorithm which needs computer assistance and strong numerical background. We shall not give the details of the algorithm, rather a comprehensive overview of it. The algorithm goes like this:

(i) Take $c_i$ the central point of $R_i$.

(ii) Compute $c_i'$ which is the point in the trajectory of $c_i$ which intersects $\Sigma'$. Where $\Sigma'$ is the cross section $\{(x, y, z) \in \mathbb{R}^3 : z = 27 - h\}$, $h = 10^{-3}$. See Figure 4.2.

(iii) As $R_i$ is taken "sufficiently small" the image on $\Sigma'$ of the points of $R_i$ can be estimated using Taylor expansions. This produces an error of $\epsilon_1 = \epsilon_1(\delta, h)$. Numerical integration also produces an error $\epsilon_2$.

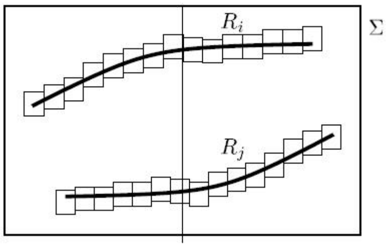



Figure 4.1: Intersection of the Lorenz flow through some point in $\Sigma$, [23].

(iv) The image on $\Sigma'$ of $R_i$ will be contained in a square of side $\epsilon_1 + \epsilon_2$ which is a neighbourhood of $c_i'$. See Figure 4.3.

(v) For the sake of accuracy, Tucker subdivides $R_i'$ into smaller squares of size at most $\delta$ and deals with them individually.

(vi) Now we create inductively extra cross sections $\Sigma'', \Sigma''' \dots$

The cross sections taken during the algorithm might be vertical, this is due to numerical reasons which are found in [23] and [21]. However, the idea is exactly the same.

Thus, if for each rectangle the algorithm finishes at finite time and returns to $\Sigma$ inside $N$, then $N$ will be a forward invariant region for the flow.

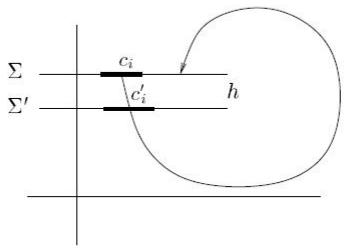



Figure 4.2: A two dimensional sketch of each step of the algorithm [23].

**Rigorous Numerics: Interval Arithmetic**

In order to compute the Poincaré map of the Lorenz equations, we recall that Tucker demanded "rigorous numerics". We shall now explain the key idea, which is the so called *interval arithmetics*.

First of all let is consider the initial value problem (2.2), where $\mathbf{f} \in C^1$. We denote $\phi(\mathbf{x}, t)$ the associated flow. It is not natural to have accurate information on the exact initial conditions of a physical system, rather, one can take a whole neighbourhood to be the initial condition. Without loss of generality we can take a $n$-cube $[\mathbf{x}_0]$ to be such neighbourhood.

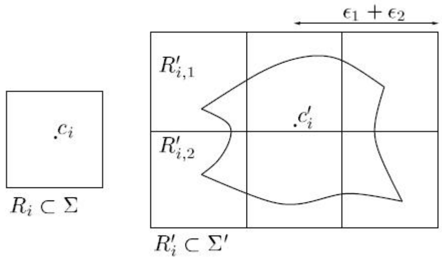



Figure 4.3: Scheme of the covering squares after one iteration [23].

Thus, we need to replace $\mathbf{f}$ by a function $\mathbf{F}$ whose variables are intervals:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{F}([\mathbf{x}]) \\ \mathbf{x}(0) \in [\mathbf{x}_0] \end{cases} . \tag{4.1}$$

This is how, Tucker introduces interval arithmetic.

Let $\mathbb{IR}$ denote the set of all closed intervals on $\mathbb{R}$. Elements in $\mathbb{IR}$ can be denoted as $[a] = [a_0, a_1]$. With this definitions, natural operations follow. If $[a], [b] \in \mathbb{IR}$:

$$[a] + [b] = [a_0 + b_0, a_1 + b_1]$$

$$[a] - [b] = [a_0 - b_0, a_1 - b_1]$$

$$[a] \times [b] = [\min\{a_0b_0, a_0b_1, a_1b_0, a_1b_1\}, \max\{a_0b_0, a_0b_1, a_1b_0, a_1b_1\}, a_1 - b_1]$$

$$[a] \div [b] = [a] \times [1/b_1, 1/b_0] \text{ if 0 is not in } [b].$$

**Remark 4.2.1.** Note that the distributive law does not hold:

$$[-1, 1] \times ([-1, 0] + [3, 4]) = [-1, 1] \times [2, 4] = [-4, 4]$$

whereas,

$$[-1, 1] \times [1, 0] + [-1, 1] \times [3, 4] = [-1, 1] + [-4, 4] = [-5, 5]$$

However, there is a weaker notion of distributive law, namely, *subdistribuity*:

$$[a] \times ([b] + [c]) \subset [a] \times [b] + [a] \times [c]. \tag{4.2}$$

What is more, $\mathbb{IR}$ is a a metric space:

**Proposition 4.2.1.** *The function* $d : \mathbb{IR} \times \mathbb{IR} \longrightarrow \mathbb{R}$ *given by*

$$d([a], [b]) = \max\{|a_0 - b_0|, |a_1 - b_1|\} \tag{4.3}$$

*is a distance.*

*Proof.* Let $[a], [b], [c]$ be elements in $\mathbb{IR}$. Clearly, $d$ is a non negative function. Moreover,

$$d([a],[b]) = 0 \Leftrightarrow a_0 = b_0 \text{ and } a_1 = b_1 \Leftrightarrow [a] = [b].$$

Finally, the triangular inequality is also satisfied:

$$\begin{aligned} d([a],[b]) =& \max\{|a_0 - b_0|, |a_1 - b_1|\} \\ \leq& \max\{|a_0 - c_0 + c_0 - b_0|, |a_1 - c_1 + c_1 - b_1|\} \\ =& \max\{|a_0 - c_0| + |c_0 - b_0|, |a_1 - c_1| + |c_1 - b_1|\} \\ =& d([a],[c]) + d([b],[c]). \end{aligned}$$

□

The following property is known as the *inclusion monotonic property*:

**Proposition 4.2.2.** *If $[a], [b], [a'], [b'] \in \mathbb{IR}$, $[a] \subset [a']$ and $[b] \subset [b']$, then,*

$$[a] \circledast [b] \subset [a'] \circledast [b']. \tag{4.4}$$

*Where $\circledast$ denotes any of the previously defined operations. For the division $0$ must not be in $[b]$ nor in $[b']$.*

The idea of interval arithmetic is to define interval valued functions. The concept of *range* is to be defined first:

**Definition 4.2.1.** Consider a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Given an $n$-cube $[a] = [a_1] \times [a_2] \times \ldots [a_n]$, we define the *range* of $f$ over $[a]$ by

$$\mathcal{R}(f;[a]) = \{f(x) : x \in [a]\}. \tag{4.5}$$

**Definition 4.2.2.** A function $\mathbf{F} : \mathbb{IR}^n \longrightarrow \mathbb{IR}^n$ is an interval extension of $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ if, for all $n$-cubes $[x] \in \mathbb{IR}$, we have $\mathcal{R}(f;[x]) \subset \mathbf{F}([x])$.

**Example 4.2.1.** Monotonic functions can be easily extended, since they preserve the ordering. For instance, let us see that the function $\mathbf{F} : \mathbb{IR} \longrightarrow \mathbb{IR}$ given by:

$$\mathbf{F}([x]) = e^{[x]} = [e^{x_0}, e^{x_1}]$$

is an interval extension of the real exponential function $f$. Let $[x] \in \mathbb{IR}$ then,

$$\mathcal{R}(f;[x]) = \{f(y) : y \in [x]\} = [e^{x_0}, e^{x_1}] = \mathbf{F}([x]).$$

Thus, $\mathbf{F}$ is an interval extension of the real exponential function.

Tucker goes even further and adapts these definitions to the floating-point arithmetic in order to construct his algorithm. This way if $\mathbb{IF}$ denotes the set of intervals with extremes in floating-point arithmetic, the extension of function we would be looking for would be of the form $\mathbf{F} : \mathbb{IF} \longrightarrow \mathbb{IF}$.

**Trajectories Close to the Origin**

When a trajectory enters a cube of size 0.2 around the origin, other techniques are required. The key concept here is the normal form of a differential equation. The target of the normal form of a differential equation is to write (1) in the form $\dot{\mathbf{x}} = A\mathbf{x} + F(\mathbf{x})$ where $A$ is a constant matrix which is associated to the linearised system. The normal form of a differential equation is not necessarily the same to the original one, however, for this purpose, Tucker refers to the normal form of the Lorenz system as the proper Lorenz system. The normal form of the Lorenz system is as follows:

$$\begin{cases} \dot{x} = 11.8x - 0.29(x+y)z \\ \dot{y} = -22.8y + 0.29(x+y)z \\ \dot{z} = -2.67z + (x+y)(2.2x - 1.3y) \end{cases} \tag{4.6}$$

The reason why to treat the trajectories close to the origin in a different way is that the algorithm shown above does not work correctly. Thus, when a trajectory flows into the cube of size 0.2 the algorithm stops and continues with the same idea but working with the normal form of the differential equations.

Using the normal form of the Lorenz system when we are close to the origin allows us to make our computations *analitically*. This is because equation (4.6) is virtually linear and it preserves the local properties. Certainly, if we linearise equation (4.6) we obtain the linearised Lorenz system. Moreover, the eigenvalues associated to the linearisation at the origin are precisely the coefficients of the linear terms. L.Perko [13] develops the normal form theory thoroughly.

# Afterword

## On Computed Assisted Proofs

As mentioned earlier, the proof of the existence of the Lorenz Attractor is not a classical mathematical proof at all. The proof given by Tucker is a *computer assisted proof* or *computer aided proof*; this means that Tucker formulated mathematical theorems in such a way that the assumptions can be verified with a computer. In general, simple systems of differential equations may lead to very complicated dynamics which demand computer aided proofs.

Computer assisted proofs bring the relevance of computing within mathematics to the spotlight. Is a computer aided proof a proof itself? Of course, computer assisted proofs are perfectly rigorous, however, these proofs cannot be entirely surveyed by a human mathematician. Although rigorous, is a computer assisted proof *less* rigorous?

Of course, the existence of the Lorenz Attractor is not the only computer assisted proof. The Four Colour Theorem (FCT) is another example amongst many others. The publication of the FCT's proof brought philosophical consequences. Thomas Tymoczko in [22] suggested that the acceptance of computer assisted proofs implied a change on the definition of mathematical proof. He argued that a classical proof should have the three following characteristics:

(i) Proofs are *convincing*†

(ii) Proofs are *surveyable*

(iii) Proofs are *formalizable*

A proof must be convincing since mathematics is a human activity, therefore, this concept is *anthropological*. Surveyablity is linked to the capacity of an external agent to check the proof; it is an *epistemological* concept. Finally, formalizability regards the *logic* of mathematics. As Tymoczko says, "a

†Ludwig Wittgenstein in [26] actually argued that convincingness is the only characteristic of a proof, since science and mathematics are only comprehended under human coherence.

proof is formalizable if there is a formal language and theory in which the informal proof can be embedded into a rigorous formal proof".

Tymoczko analyses the FCT's proof and finally concludes that the surveyability of the proof is not guaranteed if a computer is used. We must note that Tymoczko did not want to show that the proof of FCT was not really a proof. Rather, he argued that if we were to accept such proof as a *classical* proof, we should then face a paradigm shift[‡] and change our notion of proof.

[‡]The *paradigm shift*, is a concept coined by Thomas Kuhn which is referred to any substantial change in the framework of science.

# Appendix A

# Notes and Exercises on Chapter 1

**Exercise 1.** Let $\{p_0, p_1, \ldots\}$ be an orbit for $f_4(x) = 4x(1-x)$. Then,

$$p_n = \frac{1}{2} - \frac{1}{2}\cos\left(2^n \arccos(1-2p_0)\right). \tag{A.1}$$

First of all we note that

$$p_i = f_4^i(p_0). \tag{A.2}$$

To prove (A.1) we will proceed by induction. To start with, let us take $n = 1$. On the one hand,

$$f_4(p_0) = 4p_0(1-p_0)$$

on the other hand,

$$\begin{aligned} p_1 =& \frac{1}{2} - \frac{1}{2}\cos\left(2\arccos(1-2p_0)\right) \\ =& \frac{1}{2} - \frac{1}{2}\left(\cos^2(\arccos(1-2p_0)) - \sin^2(\arccos(1-2p_0))\right). \end{aligned}$$

Taking into account that $\sin(\arccos x) = \sqrt{1-x^2}$, the expression above becomes:

$$\begin{aligned} \ldots =& \frac{1}{2} - \frac{1}{2}\left((1-2p_0)(1-2p_0) - (1-(1-2p_0)^2)\right) \\ =& \frac{1}{2} - \frac{1}{2}\left(1 - 4p_0 + 4p_0 + 4p_0^2 - 1 + 1 - 4p_0 + 4p_0^2\right) = 4p_0(1-p_0) \\ =& f_4(p_0). \end{aligned}$$

Now we assume the result to be true up to $n$, and we compute $p_{n+1}$:

$$\begin{aligned}
p_{n+1} =& 4(1-p_n)p_n \\
=& 2(1-\cos(2^n \arccos(1-2p_0)))\left(1-\frac{1}{2}(1-\cos(2^n \arccos(1-2p_0)))\right) \\
=& (1-\cos(2^n \arccos(1-2p_0)))\left(\frac{1}{2}+\frac{1}{2}\cos(2^n \arccos(1-2p_0))\right) \\
=& 1-\cos^2(2^n \arccos(1-2p_0)).
\end{aligned}$$

But $\cos^2 x = (1+\cos 2x)/2$, hence,

$$\begin{aligned}
1-\cos^2(2^n \arccos(1-2p_0)) =& 1-\frac{\cos^2(2^{n+1} \arccos(1-2p_0))+1}{2} \\
=& p_{n+1}.
\end{aligned}$$

**Exercise 2.** Let $\mathrm{Mod}_2 : [0,1] \longrightarrow [0,1]$ be the map given by $\mathrm{Mod}_2(x) = 2x - \lfloor 2x \rfloor$. Show that it is chaotic.

We have to prove that $\mathrm{Mod}_2$ possesses dense periodic points and that it is topologically transitive. Let us show the topological transitiveness first. It will be enough to show it has a dense orbit.

Let $x \in [0,1]$ an arbitrary point. We consider the point $z \in [0,1]$ with the following binary expansion:

$$z = \sum_{i=1}^{\infty} \frac{a_i}{2^i} \Rightarrow (z)_2 = 0.\overbrace{01}\overbrace{00011011}\overbrace{000001010100011101110111}\ldots \tag{A.3}$$

In addition, $x$ can also be expanded binarily:

$$x = \sum_{i=1}^{\infty} \frac{b_i}{2^i} \Rightarrow 0.b_1 b_2 b_3 \ldots \tag{A.4}$$

Let $\epsilon > 0$, then, there exists $n_0 \in \mathbb{N}$ such that $1/n_0 < \epsilon$. What is more, $b_1 \ldots b_{n_0}$ must appear somewhere in the binary expansion of $z$. Therefore, there exists $k \in \mathbb{N}$ such that

$$\left(\mathrm{Mod}_2^k = 0.b_1 b_2 \ldots b_{n_0} c_{k+n_0+1}\right). \tag{A.5}$$

We shall now look at the difference:

$$|\mathrm{Mod}_2^k(z) - x| = \left| \sum_{i=n_0+1}^{\infty} \frac{b_i}{2^i} - \sum_{i=n_0+1}^{\infty} \frac{a_{k+i}}{2^i} \right| = \sum_{i=n_0+1}^{\infty} \frac{1}{2^i} = \frac{1}{n_0} < \epsilon.$$

Thus, the orbit of $z$ is dense.

We now turn to prove that the periodic points are dense. Let $x \in [0,1]$, then we can write its binary expansion:

$$x = \sum_{i=1}^{\infty} \frac{c_i}{2^i} \Rightarrow (x)_2 = 0.c_1c_2\ldots \tag{A.6}$$

Furthermore,

$$\mathrm{Mod}_2^k(x) = \mathrm{Mod}_2^k\left(\sum_{i}^{\infty} \frac{c_i}{2^i}\right) = \sum_{i=1}^{\infty} \frac{a_{k+i}}{2^i}. \tag{A.7}$$

If $x$ is rational, the binary expansion has repeating finite-length-sequences, therefore, every rational point is periodic and consequently, the periodic points are dense.

### A.0.2 On Sharkovskii's Theorem

The number of periodic orbit of period $k$ a continuous map has might determine its dynamics. Indeed, Sharkovskii came up with a result which gave a condition in order to determine the existence of $k$-periodic orbits in the map:

**Theorem A.0.3.** *Consider the following ordering of the natural numbers (see Exercise 3) :*

$$(2n+1)2^0 \succ (2n+1)2^1 \succ \ldots \succ 2^n \succ \ldots 2 \succ 1. \tag{A.8}$$

*Let $f : I \longrightarrow I$ be a continuous function defined on the interval $I$ and suppose $m \succ n$. If $f$ has a point of at least period $n$, then $f$ also has periodic points of at least period $m$.*

The most interesting result is the one that follows. It is the so called "3 implies chaos" theorem, which is merely a consequence of Sharkovskii theorem:

**Corollary A.0.4.** *If a continuous map has orbits of period 3, then it has periodic orbits of any order and it will display chaotic dynamics.*

**Exercise 3.** The fact that (A.8) is indeed an ordering follows from the fact that all the natural numbers are considered once in the ordering and every number in the ordering is a natural number. Certainly, all the odd numbers are taken into account. What is more, every even number $N$ can be written as $2^k q$, and this latter expression is considered in (A.8). Conversely, every number in (A.8) is natural, therefore, there is a one-to-one relation between the numbers considered in (A.8) and $\mathbb{N}$.

The ordering $\succ$ is not a well-ordering. For example, $S = \{3, 5, 7, 9, 11, \ldots\}$ does not have a least element. If it had, it would be an odd number $K$, but $K \succ K + 2$.

# Appendix B

# Notes and Examples on Chapter 2

In Section 2.1 the main target was to define a dynamical system and it has been necessary to extend the following theorem concerning the dependence on initial conditions:

**Theorem B.0.5.** *Let $U$ be an open subset of $\mathbb{R}^n$ containing $\mathbf{x}_0$ and assume that $\mathbf{f} \in C^1(U)$. Then, there exists an $a > 0$ and a $\delta > 0$ such that for all $\mathbf{y} \in B_\delta(\mathbf{x}_0)$ the initial value problem*

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) \\ \mathbf{x}(0) = y \end{cases} \tag{B.1}$$

*has a unique solution $\mathbf{u}(\mathbf{y}, t)$ with $\mathbf{u} \in C^1(G)$ where $G = B_\delta(\mathbf{x}_0) \times [-a, a]$. Furthermore, for each $\mathbf{y} \in B_\delta(\mathbf{x}_0)$, $\mathbf{u}(\mathbf{y}, t)$ is a twice continuously differentiable function of $t$, when $t$ varies in $(-a, a)$.*

## B.1 Another Example of Poincaré Map

### B.1.1 Example

Consider $\Sigma = \{(x, y) \in \mathbb{R}^2 : x \in [0, 1], \quad y = 0\}$, and the following differential equation:

$$\begin{cases} \dot{x} = -y - x\sqrt{x^2 + y^2} \\ \dot{y} = -x - y\sqrt{x^2 + y^2} \end{cases} \tag{B.2}$$

If we change variables into polar coordinates, system (B.2) becomes:

$$\begin{cases} \dot{r} = -r^2 \\ \dot{\theta} = 1 \end{cases} . \tag{B.3}$$

We will ask system (B.3) to verify the initial condition $(r_0, \theta_0) = (1, 0)$. Equation (B.3) can be solved directly:

$$\dot{\theta}(t) = 1 \Rightarrow \theta(t) = t + \theta_0 = t$$
$$\dot{r}(t) = -r^2(t) \Rightarrow \frac{1}{r(t)} = t + r_0 \Rightarrow r(t) = \frac{1}{t+1}.$$

From the solution, we obtain that the trajectories flow around the origin with a period of $2\pi$. Thus, the flow is defined as:

$$\phi((1,0),t) = (r(t,\theta(t))) = \left(\frac{1}{t+1}, t\right). \tag{B.4}$$

However, as we know that the returns occur when $\theta = k\pi$, $k \in \mathbb{N}$, the Poincaré map can be simplified into:

$$r_n = \frac{1}{1+2n\pi}. \tag{B.5}$$

Hence, we can express the $(n+1)$-th term in terms of the previous one:

$$\begin{aligned} r_{n+1} &= \frac{1}{1+2(n+1)\pi} = \frac{1}{1+2\pi n+2\pi} \\ &= \frac{r_n}{r_n(1+2n\pi+2\pi)} = \frac{r_n}{1+2\pi r_n}. \end{aligned}$$

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