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Universidad del País Vasco / Euskal Herriko Unibertsitatea

ZTF-FCT
Zientzia eta Teknologia Fakultatea
Facultad de Ciencia y Tecnología

Gradu Amaierako Lana / Trabajo Fin de Grado
Fisikako Gradua / Grado en Física

Quantum Computation With Superconductors

Egilea/Autora:
Amaia Irastorza Gabilondo
Zuzendaria/Director:
Enrique Rico Ortega
Zuzendarikidea/ Co-Director:
Enrique Solano Villanueva

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Introduction

Back in the 1980s some physicists started questioning if quantum mechanics could be applied to computer science and information theory. For the first time scientists did not just observe quantum phenomena, they started designing quantum systems, creating them looking for possible applications. Quantum computation is based in quantum bits or qubits, two-level quantum systems that are analogous of the classical bits in quantum computation. Quantum computers are thought to bring several advantages comparing to classical computers, even though it is not yet known the advances quantum computers will bring. For example, some quantum algorithms are thought to be exponentially faster than the classical ones. Qubits that build up quantum computers must be strongly coupled among themselves, but decoupled from environment, except for writing, control and readout. Most proposals to build qubits are microscopic, such as atoms or ions and electron or nuclei spin. However, the superconducting qubits studied in this work are macroscopic.

Superconducting qubits are an interesting candidate for the construction of the quantum computer. But not just that, their macroscopic size has led scientists to study the limits of quantum mechanics. Coherence and quantum noise are essential when studying superconducting qubits, and open new areas of research. Superconducting qubits have many advantages such as lithographic scalability, compatibility with microwave control and operability at nanosecond time scales. Lithography makes superconducting qubits easy to build and couple, being the same system used to build microchips. Microwave control allows us to change the state of a qubit by exciting it with a photon with the right frequency. We can distinguish three types of superconducting qubits depending on their quantized state: charge, flux and phase. They are built using superconducting circuits, and one of their advantages is the easy coupling using a capacitor or an inductor. Also, as their degrees of freedom are macroscopic currents and voltages they are easy to read out.

The work can be divided in to parts: first, an introduction is made to quantum computation and the theory necessary to understand superconducting qubits is studied. Then, in the second part, we are ready to study superconducting qubits and use them to do a simple logical operation. In chapter 1, the idea of qubit will be introduced. After knowing what a qubit is, the basics of quantum computation and the general purpose of quantum information will be described in chapter 2. In chapter 3, several qubits that are candidates for building a quantum computer will be stated. Among all the qubits, our chosen ones to develop further are superconducting qubits. In chapter 4 quantum phenomena necessary to understand the limitations of superconducting qubits will be studied. In chapter 5 we will set the basis of superconducting circuits so that in chapter 6 we are ready to understand superconducting qubits, how they work, their measurement, and their decoherence mechanisms. Once we know how to manage superconducting qubits, we are ready to couple them in chapter 7 to perform logical operations in chapter 8.

1 General description of a qubit

The qubit (quantum bit) is the fundamental element of quantum information and quantum computation, analogous to the classical bit. The same way the classical bit can be in two states, 0 and 1, the qubit can also be in two possible states $|0\rangle$ and $|1\rangle$ which are the computational basis states. The difference is that the qubit can be in a superposition of its basis states, which is a fundamental property of quantum computing. Its state can be written as a linear combination of the basis states

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \tag{1}$$

When we measure the state of the qubit there is a $|\alpha|^2$ probability of measuring $|0\rangle$ and $|\beta|^2$ probability of measuring $|1\rangle$, being $|\alpha|^2 + |\beta|^2 = 1$. We can also write α and β like

$$\alpha = \cos\left(\frac{\theta}{2}\right) \tag{2}$$

and

$$\beta = e^{i\varphi}\sin\frac{\theta}{2} \tag{3}$$

where the angles θ and φ define the Bloch sphere. Classical bits can only be in the north or south poles of the sphere, while a qubit can be in any point of the sphere.

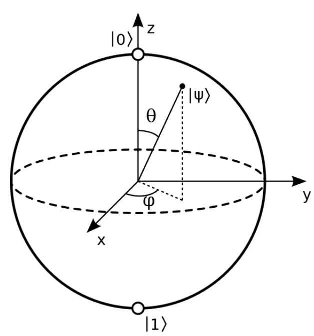

Figure 1: Graphical representation of a qubit using the Bloch sphere. Besides the classically possible states $|0\rangle$ and $|0\rangle$ superposed $|\Psi\rangle$ states are also possible.

Measurements change the state of the qubit, this being a fundamental postulate of quantum mechanics. Thus, when we make a measurement in a qubit and it gives $|1\rangle$ it will change its state to $|1\rangle$.

Even if a qubit can be in a superposition of its basis states, a single measurement can only give us one bit of information, either $|0\rangle$ or $|1\rangle$[1].

2 Quantum computation and quantum information

A quantum computation is a physical process based in the laws of quantum mechanics where an input state is transformed in an output state of a system. The input and output states are given by the quantum states of the system. Quantum computation is performed by qubits, which are the system that gives us the input and output and where operations are performed. Qubits can encode any number of inputs and outputs using the binary notation and quantum entanglement between qubits. That is, the base of quantum computation are quantum superposition and quantum entanglement[2]. The mentioned operations are performed by quantum gates, which are quantum processes that transform qubit states. Quantum computation is thought to give many advantages comparing to classical computing, as the laws of quantum mechanics give the ability to do more than just with the laws of classical mechanics. Quantum computers are thought to be able to solve problems that classical computers cannot solve, as it is thought to be exponentially faster and takes less resources. This is based in the idea of quantum parallelism. In a classical computer we perform an operation in a bit 0 and another in a bit 1. In a quantum computer just one operation would be needed, because a qubit is in a superposition of both states $|0\rangle$ and $|1\rangle$. This way we perform the same operation using just one qubit instead of the two classical qubits. More formally, in a classical computer one must run it n times to compute a function $f : N \rightarrow N$ for the numbers $1, 2, .., n$. With a quantum computer it is possible to do the same by

$$|1\rangle |0\rangle \rightarrow |1\rangle |f(1)\rangle \tag{4}$$

$$|2\rangle |0\rangle \rightarrow |2\rangle |f(2)\rangle \tag{5}$$

$$...$$

$$|n\rangle |0\rangle \rightarrow |n\rangle |f(n)\rangle \tag{6}$$

But, we can also prepare the following input state, being a quantum superposition of the previous states

$$|\psi\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} |k\rangle |0\rangle \tag{7}$$

Running this state just once in a quantum computer we obtain the state

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} |k\rangle |f(k)\rangle \tag{8}$$

Thus, we get all the values of f in a quantum superposition by running the quantum computer just once. The problem is that it is not possible to read out all of the states in a measurement, it is only possible to read out certain information.[1][3]

The purpose of quantum computers is the management of quantum information, the information encoded in qubits. They are thought to be able to solve algorithms that are impossible for classical computers. They should solve problems that need a huge amount of operations, that classical computers would need a impossible amount of time and resources to solve. Another advantage is that they would be able to simulate quantum phenomena that is impossible to simulate using classical computers.

The study of the manipulation of this quantum information is referred to as quantum information science (QIS). The first definition of QIS was given in a workshop with the same name by the National Science Foundation of the United States in 1991[4]:

Quantum information science (QIS) is a new field of science and technology, combining and drawing out disciplines of physical science, mathematics, computer science, and engineering. Its aim is to understand how certain fundamental laws of physics discovered earlier in this century [20th] can be harnessed to dramatically improve the acquisition, transmission, and processing of information.

Quantum information science has had many successes, not necessarily related to the construction of quantum computers. It has shown that quantum mechanics is more than what it was thought when it was first studied. It has proved to be a theory of transmission of information in addition to just a theory of matter and energy as it was first thought. Also, it has brought discussions of what we understand as the quantum world. Finally, it has defined a set of experimentally used metrics, such as fidelity or decoherence time, making possible discussion and comparison of quantum system that was previously impossible.

3 Types of qubits

Any two-level quantum system can represent a qubit, as well as any system that can be approximated to a two-level system. There are many different ways to implement qubits, each with its advantages and disadvantages. Only time will tell which one will win the race to building a quantum computer. In the following section some of the most promising possible candidates will be introduced.

3.1 Neutral atoms

Atoms can be used as qubits. For that, an atom trap is built using lasers, trapping just one atoms in each lattice site. Needless to say that as atoms are very small, it is possible to fit thousands of them in a machine. As the atoms are trapped, they do not interact with each other unless we make them to. It is possible to set the atoms to the lowest state by cooling them using a laser beam. The atoms absorb the light moving against them, using resonance and the Doppler effect. It is known that atoms always vibrate, unless their temperature is 0K. As the atom vibrates, a laser beam is sent towards the atom with a frequency that only is in resonance with the atom when they are moving in opposite directions. As the photons have momentum as well as the atom, they collide slowing the atom. It is also possible to excite the atoms using circularly polarized beams. These beams do not only excite the target atom, but many surrounding atoms are shifted too, but the target atom is excited twice as much at least. This allows to use microwaves to change the target atom. The qubit states are given by the electronic hyperfine sublevels. One of the pros of atoms is that they have very long coherence times, and that they have no scalability problems as we can make the lattice as big as we want. But, the qubits lose coherence during gate operations.

3.2 Trapped ions

Same as the neutral atoms, ions can be used as qubits, however, the mechanism is somewhat different. The ion is not trapped by laser beams, but between four electrodes arranged as the corners of a square. The electrodes are 3cm long and have 1mm diameter. The ions are confined between the electrodes, trapped by the fast oscillating potential that alternates the confinement between the axis x and y. The qubit is confined along one of those axes and expelled along the other. Thus, the ion has zero average potential energy. This explains the confinement in the xy plane, but it can still scape along z. This is prevented by putting fixed positive voltage at either side of the trap. The motion of the ion is controlled by laser beams. The ions are cooled down the same way as the neutral atoms, by using a laser beam and taking advantage of the resonance and the Doppler effect. As ions are tiny magnets, their qubit states are given by the spin down $|\downarrow\rangle$ and spin up $|\uparrow\rangle$. As an ion is a quantum system, it can be in the superposition of all its states. The superposition of this states is caused by the laser light fields. If the initial state is known, choosing a given duration and frequency it is possible to calculate the final state accurately.

3.3 Photon qubits

Photons can be used to build qubits. They have some advantages, like being free from decoherence and being able to travel long distances without interacting with their surroundings. They are also easy to manage: they can be guided using optical fibers, delayed using phase shifters and combined using beamsplitters. On the other hand, they give problems with scalability, as they do not interact with each other. Photons can be used in different ways to build qubits, as they have many properties that can be exploited. Here the photon polarization qubit and the dual-rail qubit will be briefly introduced.

In photon polarization qubits the quantum information is stored in the polarization state of the photon, $|0\rangle$ for vertical polarization and $|1\rangle$ for horizontal polarization. The problem of this type of qubits is that the information is very difficult to store because of birefringence and because some materials only absorb one polarization.

The dual-rail qubit is a qubit with two spatial modes. This spatial modes correspond to cavities that can be empty $|0\rangle$ or have one photon $|1\rangle$. When we have two spatial modes, the two possible states are $|01\rangle$ and $|10\rangle$, both with total energy $\hbar\omega$.

As photons do not interact with each other, optical cavity quantum electrodinamics(QED) can be used to manage photons. The electric-field in the cavity is very high, and thus, there is a strong dipole coupling between the atom and the field. Photons interact with the atom, and they can do it many times before escaping the cavity. The objective of cavity QED is to transfer the state of the photons to and from the atoms. To do this, interaction between photons and atoms must be controlled. Qubit representation used in cavity QED can be both polarization or dual-rail.

3.4 Quantum dot qubits

Quantum dots are semiconducting nanostructures with diameters about 2-10nm (10-50 atoms) wide. Quantum dots confine the movement of electrons or electron holes in the conduction band in all three spatial dimensions. The quantum dot is an artificial atom with discreet energy spectrum and thus can be operated as a qubit. The size of the dot is highly controllable during the fabrication process. The smaller the size of the dot the bigger the energy difference between the ground and first excited states. Applying small voltages to the leads that build the dots, they can be controlled and precise measurements of the electron spins can be made.

4 Quantum coherence and quantum entanglement

4.1 Entanglement

Quantum entanglement occurs when a system of more than one particle cannot be described by the individual states of each particle, but with the quantum state of the system as a whole. As a result, a measurement of one of the particles affects the rest of the ensemble. Entangled states cannot be written as product states. Lets take for example the state $|A\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. this state is a linear combination of product states $|0\rangle \otimes |1\rangle = |01\rangle$ and $|1\rangle \otimes |0\rangle = |10\rangle$. Still, the state $|A\rangle$ as a whole cannot be written as a product state of $|0\rangle$ and $|1\rangle$. If we look again at the state $|A\rangle$, we can see that if we measure the first state and get, for example, 0 (1), we know that the second state is 1 (0). That is, the measurement of the first particle conditions the state of the second.

4.2 Coherence

According to quantum mechanics, all objects have wave-like properties. If an object's wave is split (by diffraction, for instance), those waves might coherently interfere with each other to form a single state that is a superposition of all the states.

A coherent state is a quantum state which maintains its relative complex phase during a period of time. Loss of coherence or decoherence explains why a quantum system begins to obey classical probability rules after interacting with its environment. Coherence is lost when our system couples to its immediate surroundings, acquiring phases from the environment and no longer maintaining its complex relative phase.

Coherence can be described by a system coupled to a heat bath. Imagine our qubit is in a pure state, with density matrix $\rho = |g\rangle \langle g|$ in time $t = 0$. At $t > 0$, we let the system evolve coupled to a reservoir at temperature $1/k_B T = \beta$. The state is not pure any more, as it mixes with the first excited state. The stationary state of the system is given by

$$\rho = \frac{1}{1 + e^{-\beta\omega}} |g\rangle \langle g| + \frac{e^{-\beta\omega}}{1 + e^{-\beta\omega}} |e\rangle \langle e| \tag{9}$$

If the pure state we have at the beginning corresponds to the first excited state $|e\rangle$, it will also decay populating the ground state and creating a mixed state, an thus losing coherence.

When we talk about quantum computing, quantum states experience decoherence when their "quantumness" is lost, that is, when it becomes a classical superposition of states as we introduced above (9). It is important that a qubit stays in a coherent superposition of states not to lose its quantum properties that are necessary for quantum computation. To avoid coherence loss, the external environment would not be able to interact with qubits, but that means that neither would we. The ideal qubit would be unperturbed by its environment but still have a port open through which its state could be controlled and measured.

When we talk about the causes of decoherence, the decohering elements can be extrinsic and intrinsic. The extrinsic decoherence is caused the already mentioned coupling to the environment, and the solution will be isolating our system as much as possible (as it must be still readable). Intrinsic decoherence is caused by noise coming from the superconducting circuit. Given the importance of this noise it will be better explained later.

4.2.1 Decoherence rates

Long enough coherence times are essential for quantum computing, the qubit must remain long enough in a coherent state so that measurement and implementation of qubit gates is possible. Qubits are characterized by two times, T_1 and T_2. T_1 is the time required by a qubit to relax from the first excited state to the ground state[5], that is, the decay time, which can be referred to as dissipation[6] or relaxation. T_2 is the average time in which the energy-level splitting remains unchanged or the time it takes for the phase difference between two eigenstates to become random[5], the dephasing time[6]. It is necessary to specify the type of experiment used to measure T_2, as the dephasing time is not a unique phase coherence time[7]. Both relaxation and dephasing arise from the weak coupling to quantum noise in the environment. The relaxation rate comes from the fluctuations at the frequency of the energy-level splitting. It can be written as the sum of the transition rates from one eigenstate to the other[7]

$$\frac{1}{T_1} = \Gamma_{\uparrow} + \Gamma_{\downarrow} \tag{10}$$

The dephasing rate has two contributions

$$\frac{1}{T_2} = \frac{1}{2T_1} + \frac{1}{\tau_{\Phi}} \tag{11}$$

The first contribution comes from the relaxation process, it represents the destruction of the coherent superposition when the qubit transitions from one eigenstate to the other[7], as we have introduced in (9). The difference is that, as we will explain later, the temperature in which the qubits will perform will be of the order of millikelvin, meaning that the temperature will not be a cause of decoherence. Still, (9) is the basis that illustrates our further understanding of quantum decoherence. Decoherence will come from the noise that will be introduced in the next chapter. The second contribution is given by the 'pure dephasing' τ_{Φ}, and comes from low-frequency fluctuations with exchange of infinitesimal energy[5].

We can also define the coherence quality factor $Q = \pi T_2 \omega_0$, the number of one-qubit operations achievable before the system loses coherence[8].

Experimentally, the longest coherence times achieved for the simplest qubits before 2008 can be seen in Table.1 [5]. The coherence quality factor, on the other hand, is about 10^5[8].

Qubit	$T_1(\mu s)$	$T_2(\mu s)$
Charge	2.0	2.0
Flux	4.6	1.2
Phase	0.5	0.3

Table 1: [5]Longest reported values of T_1 and T_2 until 2008

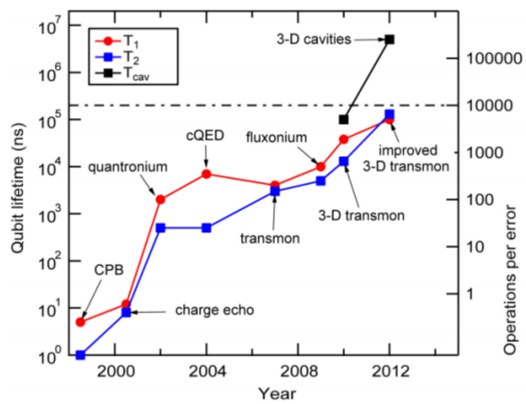

Figure 2: [9]This graph shows the improvement made in decoherence rates in the last years.

4.3 Noise

We have already mentioned that the loss of coherence does not come from temperature, as it is negligible. Instead, the phenomenology above described occurs due to microscopic low-frequency noise, also called 1/f. Introducing it briefly, 1/f noise comes from low-frequency fluctuations. The origin of 1/f noise in superconducting qubits is not yet clear and it is a subject of study, it is one of the physical questions that has arisen from the study of superconducting quantum computation.

Even if the origin of 1/f noise in superconducting qubits is not completely clear, educated guesses have been made based on phenomenological models[8]. There are thought to be three sources of 1/f noise in superconducting circuits[5]. The first is critical current fluctuations, caused by the trapping and untrapping of electrons in defects of the tunnel barrier. A trapped charge locally modifies the height of the tunnel barrier changing the critical current of the junction[10]. This noise affects all superconducting qubits. The slow fluctuations modify the energy level splitting and thus each measurement gives a slightly different measurement. The resultant phase errors lead to decoherence.
The second source are charge fluctuations. This noise is produced by electrons jumping

from one trap to other in the surface of the superconductor. This induces charges in nearby superconductors. This decoherence mechanism affects charge qubits particularly, but in the degeneracy point. With the increase of the E_C/E_J value the charge qubit becomes less susceptible to this noise.
The third source is magnetic-flux fluctuations. It is thought it arises from spin diffusion[1] on the superconductor surface generated by the exchange mediated by the conduction electrons[11]. This decoherence mechanism affects flux qubits, except in the degeneracy point.

Low-frequency noise only affects decoherence or dephasing, it does not affect relaxation because relaxation requires energy exchange with the environment, and 1/f noise is an intrinsic feature of the circuit. Noise can be reduced improving the quality of the materials used in junctions and capacitors and thus reducing the number of defects that are the cause of noise.

Dephasing times for the CPB, for example, are of the order of nanoseconds[12]. To enhance dephasing times two strategies can be followed. First, materials and properties of the junction can be improved in order to eliminate intrinsic 1/f noise. Secondly, qubits can be operated at their optimum working points and thus eliminate linear noise sensitivity. A combination of both may be necessary in the future to build scalable quantum computers[5]

[1] continuous exchange of energy between individual nuclear spins

5 Superconducting circuits

Superconducting circuits are the basis of superconducting qubits, as they are built using superconducting elements. It is advisable to first check the basic theory of superconducting circuits before scaling up to superconducting qubits. Our purpose for now is to study simple superconducting circuits that behave quantum mechanically. For a circuit to behave quantum mechanically there must be no dissipation, that is, the circuit must have zero resistance at the (qubit) operating temperature. This is essential to preserve quantum coherence. Non-linearity is also essential, because we want to approximate our system to a two level qubit, meaning that the energy level cannot be uniformly spaced. The only element which is both non-dissipative and non-linear is a Josephson junction. This makes Josephson junctions the essential element of superconducting qubits.

5.1 Quantum LC oscillator

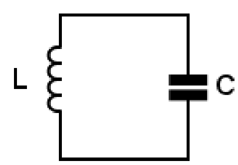

Figure 3: The LC circuit is the simplest possible quantum circuit, consisting of an inductor connected to a capacitor using superconducting wire.

The first quantum circuit we will introduce is the LC oscillator, as it is the simplest example of a quantum integrated circuit. Studying the simplest case will help us understand the circuits of the superconducting qubits and their Hamiltonians. LC circuits consist of an inductor L connected to a capacitor C. All of the wires connecting the elements must be superconducting for the circuit to be quantum, this way the energy levels in the superconducting gap will be discrete. The LC circuit obeys the equations of motion of the linear harmonic oscillator. The flux in the inductor Φ is analogous to the position coordinate and the charge Q on the capacitor is analogous to the conjugate momentum. The $\hat{\Phi}$ and $\hat{Q}$ variables are conjugate quantum operators, which do not commute $[\hat{\Phi}, \hat{Q}] = i\hbar$ [13]. Even if the circuit has a huge amount of electrons, the degrees of freedom have been reduced to one, the Cooper-pair fluid moving back and forth in the circuit[7]. The Hamiltonian of the circuit is

$$\hat{H} = \frac{\hat{\Phi}^2}{2L} + \frac{\hat{Q}^2}{2C} \tag{12}$$

The inductance L of the system can be thought as the 'mass' and the inverse of de capacitance 1/C as the 'spring constant'. Knowing the LC oscillator is analogous to the harmonic oscillator we can write the Hamiltonian in terms of the raising and lowering operators $\omega = 1/\sqrt{LC}$ being the resonance frequency of the circuit

$$\hat{H} = \frac{1}{2}\hbar\omega \left\{ \hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger \right\} \tag{13}$$

Using the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ we can rewrite de Hamiltonian as

$$\hat{H} = \hbar\omega \left\{ \hat{a}^\dagger \hat{a} + \frac{1}{2} \right\} \tag{14}$$

where the raising and lowering operators are, consecutively

$$\hat{a}^\dagger = -i\frac{1}{\sqrt{2C\hbar\omega}}\hat{Q} + \frac{1}{\sqrt{2L\hbar\omega}}\hat{\Phi} \tag{15}$$

$$\hat{a} = i\frac{1}{\sqrt{2C\hbar\omega}}\hat{Q} + \frac{1}{\sqrt{2L\hbar\omega}}\hat{\Phi} \tag{16}$$

Solving the Hamiltonian of the LC circuit (14) the difference between two consecutive energy levels is always $\Delta E = \hbar\omega$, same as in the the linear harmonic oscillator. This means that the energy levels of the LC circuit are regularly spaced and it is impossible to approximate it to a two level system using the lowest two levels. That is why we can not use LC circuits as qubits, even if they are able to preserve coherence due to superconductivity, as they do not satisfy both conditions we have established before. This means that we must introduce non-linear elements such as Josephson junctions in the circuit. Nevertheless, the Hamiltonian of the LC circuit gives us the basis to construct the Hamiltonians of the superconducting qubits that will be later introduced.

5.2 Superconductivity

To understand the Josephson effect, it is advisable to first acquire some notion about the origin of superconductivity. Macroscopically, superconductivity manifests itself as a lack of resistance, so that the current flows with no obstacles. Microscopically, it is the condensation of paired electrons(Cooper pairs) into a Bose-Einstein condensate, a boson-like state. This is caused by an attractive potential. An electron moving through a conductor attracts positive charges in the lattice. For this reason, the lattice suffers a deformation, attracting another electron with opposite spin due to the newly created high positive charge density region. Thus, the electrons become correlated forming a Cooper pair. This pairing causes a excitation gap. The pairs energy must be higher than the gap before separating into quasiparticles again. Above the gap quasiparticle states form a continuum. The single particle spacing will be only noticeable when the electrode is very small, of the order of nanometres.

Superconductivity reduces a huge Hilbert space to a single quantum state $|N\rangle$, the number of pairs moving. This state can be described by a many-body wave function where all pairs have the same phase and energy

$$\Psi(\vec{r}, t) = \Psi_0(\vec{r}, t) e^{i\theta(\vec{r}, t)} \tag{17}$$

5.3 Josephson junctions

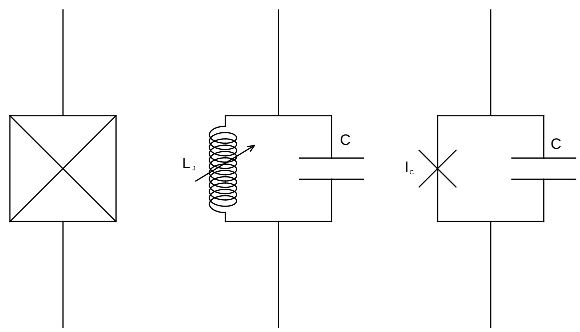

Figure 4: Circuit representation of a Josephson junction. There are many ways to represent a Josephson junction using circuits. The one of the left is the one adopted generally as it is a simplification of the others.

Josephson junctions are the essential element of superconducting qubits, as they introduce the needed non-linearity to the system. They consist of two superconductor separated by an insulating barrier. The dimension of the barrier is of the order of nanometres[5]. The origin of the non-linearity of the Josephson element is associated with the discreteness of charge that tunnels across the thin insulating barrier. Hence the junction is characterized by only one degree of freedom $N(t)$, the number of Cooper pairs having tunnelled across the barrier. The charge that has flown across the element is $Q_J(t) = -2eN(t)$. In the junction, the charge $Q(t)$ in the capacitance does not need to equal $Q_J(t)$ when the junction is connected to an electrical circuit. The Josephson element can also be described by the flux Φ_J

$$\Phi_J = \int_{-\infty}^{t} v(t')\,\mathrm{d}t' \tag{18}$$

where v is the voltage across the element. The current through the inductor is proportional to the branch flux

$$I(t) = \frac{1}{L}\Phi(t) \tag{19}$$

For the Josephson element this relationship can be written as[13]

$$I(t) = I_0 sin\left[\frac{2e}{\hbar}\Phi_J(t)\right] = I_0 sin\left[2\pi\frac{\Phi_J(t)}{\Phi_0}\right] \tag{20}$$

where I_0 is the critical current through the element and $\Phi_0 = \frac{h}{2e}$ is the superconducting flux quantum with value $\Phi_0 = 2.07\text{x}10^-15\text{Wb}$. The discreteness of Cooper pairs tunnelling through the barrier causes a periodic flux dependence of the current, with a period given by the superconducting flux quantum Φ_0. $\delta = 2\pi\frac{\Phi_J(t)}{\Phi_0}$ is the gauge-invariant phase-difference or simply the phase. This variable is just the electromagnetic flux in dimensionless units. The real phase difference between the two superconducting electrodes is given by

$$\varphi = \delta mod 2\pi \tag{21}$$

It denotes the phase difference between the wave functions ψ_R and ψ_L, which describe the Cooper pairs residing in the right and left superconductors of the junction.
The Josephson element can also be described by two other parameters. The first is the Josephson effective inductance $L_{J0} = \frac{\Phi_0}{2\pi I_0}$. Thus the Josephson phase dependent inductance is

$$L_J(\delta) = \left(\frac{\partial I}{\partial \Phi}\right)^{-1} = \frac{L_{J0}}{\cos\delta} \tag{22}$$

The second parameter is the Josephson energy

$$E_J = \frac{\Phi_0}{2\pi}I_0 \tag{23}$$

the required energy to store a Φ_0 flux quantity in the Josephson inductor. If we calculate the energy stored in the junction we get

$$E(t) = -E_J\cos\left[2\pi\frac{\Phi(t)}{\Phi_0}\right] \tag{24}$$

The variable that gives the number of Cooper-pairs across the junction N should be treated as a discrete operator, as we have said that the charge tunnelling across the barrier is discrete

$$\hat{N} = \sum_N N\left|N\right\rangle\left\langle N\right| \tag{25}$$

The two superconducting electrodes form a capacitor, but we will ignore the Coulomb energy that builds up as Cooper pairs tunnel from one side to the other. The tunnelling of electrons through the barrier couples the $|N\rangle$ states. The coupling energy is given by the Hamiltonian

$$\hat{H}_J = -\frac{E_J}{2} \sum_N |N\rangle \langle N+1| + |N+1\rangle \langle N| \tag{26}$$

The Hamiltonian can be written in terms of the phase difference across the junction. We introduce new basis states

$$|\varphi\rangle = \sum_N e^{iN\varphi} |N\rangle \tag{27}$$

Where $\varphi \to \varphi + 2\pi$ leaves the $|\varphi\rangle$ unaffected. $\hat{\varphi}$ and $\hat{N}$ operators are conjugate variables, with uncertainty relation $\Delta\hat{N}\Delta\hat{\varphi} \geq 1$ and $\left[\hat{\varphi}, \hat{N}\right] = i$. The gauge invariant phase difference is analogous to the position operator, and $\hat{N} = -i\frac{\partial}{\partial\varphi}$ is analogous to the momentum operator. Conversely, the $|N\rangle$ state is given by the Fourier transformation of the phase state

$$|N\rangle = \frac{1}{2\pi} \int_0^{2\pi} d\varphi e^{iN\varphi} |\varphi\rangle \tag{28}$$

We introduce the operator

$$e^{i\hat{\varphi}} = \frac{1}{2\pi} \int_0^{2\pi} d\varphi e^{i\varphi} |\varphi\rangle \langle\varphi| \tag{29}$$

Which acts in $|N\rangle$ as

$$e^{i\hat{\varphi}} |N\rangle = |N-1\rangle \tag{30}$$

We can obtain the expression for the coupling Hamiltonian (26) in this new basis as[14]

$$\hat{H}_J = -\frac{E_J}{2}(e^{i\hat{\varphi}} + e^{-i\hat{\varphi}}) = -E_J \cos\hat{\varphi} \tag{31}$$

We see that equation (31) is the quantum equivalent of equation (24). More correctly, if we solve the Hamiltonian in the right $\hat{\varphi}$ basis, the eigenvalues we get are given by equation (24).

6 Superconducting qubits

Among all types of qubits, superconducting qubits will be the chosen ones to be studied here. Superconducting qubits are promising candidates for implementing quantum computers. As we will now see, they have demonstrated large coherence times of the order of nanoseconds and promising scalability. One of the most important features that differentiates them from the other types of qubits, however, is their macroscopic dimension and the easy manufacture. Their size is of the order of micrometres and can be built using electron beam lithography, the same process used in microelectronics. In this section the basic construction and operation of superconducting qubits will me introduced. We will study different types of superconducting qubits, depending on their quantized states: charge, flux or phase. We will study some options of how each qubit can be measured and also how they are affected by external noise and their ability to maintain coherence.

6.1 Charge qubit

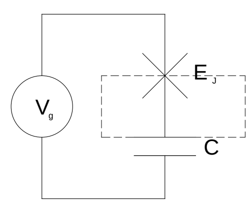

Figure 5: Circuit representation of a charge qubit. The region which lies inside the dashed lines is the Cooper-pair box.

The charge qubit, also known as Cooper pair box, consists of a small superconducting island coupled by a Josephson junction to a superconducting reservoir (the superconducting wire acts as the Cooper pair reservoir). The superconducting island is located between one of the plates of the capacitor and the insulating barrier of the Josephson junction. In this type of qubit the charge is the controlled variable, controlled by the voltage source, inducing a charge difference between both sides of the Josephson junction. The Hamiltonian of the circuit is

$$\hat{H} = E_C(\hat{N} - N_g)^2 - E_J \cos\hat{\varphi}. \tag{32}$$

The first term is the electrostatic energy (Coulomb energy) operator

$$\hat{U} = E_C(\hat{N} - N_g)^2 = E_C \sum_N (N - N_g)^2 \left|N\right\rangle \left\langle N\right| \tag{33}$$

where $N_g = C_g \frac{V_g}{2e}$ is the dimensionless gate charge or offset charge, the polarization charge induced by the voltage on the gate capacitor. $E_C = \frac{(2e)^2}{2(C_J+C_g)}$ is the electrostatic Coulomb energy, the energy used to store a Cooper pair in the capacitor. The second term belongs to the Josephson coupling energy (31) introduced before. The charge qubit is in the charge regime, $E_C >> E_J$, which means that the variable $\hat{N}$ is well defined and the phase $\hat{\varphi}$ fluctuates. $\hat{N}$ is a discreet variable representing the quantity of Cooper pairs on the island which can only take integer values. The offset charge N_g is, on the other side, a continuous variable.

The charge states are given by the excess of Cooper pairs that have tunnelled to the island. For a qubit, all the states over $|2\rangle$ can be ignored when the offset charge is about the same charge as an electron, $N_g = 1/2$, because of a great energy difference between the first and second states. Thus, we can make a two level approximation.

The qubit basis states are $|0\rangle$ and $|1\rangle$, the first corresponding to the lack of Cooper pairs in the superconducting island and the second corresponding to the presence of a single Cooper pair. When $N_g = 1/2$, the energy is the same for both charge states $|0\rangle$ and $|1\rangle$ leading to degeneracy. For that reason an energy splitting occurs in that region and the energy eigenstates become linear combinations of the charge states. At the degeneracy point the energy eigenstates are $|g\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$ and $|e\rangle = 1/\sqrt{2}(|0\rangle - |1\rangle)$[2]. The energy splitting in this point is Δ, but we can write the general energy splitting as

$$\Delta E = \sqrt{\Delta^2 + \varepsilon^2} \tag{34}$$

where $\varepsilon = E_C(N_g - 1/2)$.

Making the two level approximation, we can write the qubit reduced Hamiltonian as[15]

$$\hat{H}_{qubit} = E_C \begin{pmatrix} N_g^2 & 0 \\ 0 & (1-N_g)^2 \end{pmatrix} - \frac{E_J}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{35}$$

If we change the zero of energy of the system to $E_0 = E_C(1/2 - N_g)^2$ we can rewrite the reduced Hamiltonian as

$$\hat{H}_{qubit} = E_C \hat{\sigma}_Z - \frac{E_J}{2} \hat{\sigma}_X \tag{36}$$

The eigenvalues and eigenstates of the Hamiltonian are solutions to the Mathieu equation. The level of anharmonicity of the system depends on the relation E_J/E_C. When it gets higher, the qubit spectrum becomes more and more uniformly spaced and the system starts behaving like a linear harmonic oscillator.

[2] g stands for ground state and e stands for first excited state

Charge dispersion determines the variation of energy levels. It differs depending on the environment offset charge and the gate voltage, and determines the sensitivity of the CPB to charge noise. The dispersion can be defined as[16]

$$\epsilon_m = E_m(N_g = 1/2) - E_m(N_g = 0) \tag{37}$$

Where E_m are the energy eigenvalues. The smaller the charge dispersion, the less the qubit frequency will change in response to gate charge fluctuations. The charge dispersion decreases rapidly with E_J/E_C. When the CPB works in the flux regime with $E_J >> E_C$, the decrease of charge dispersion and the uniformity of the qubit spectrum influence the operation of the system as a qubit.

To read-out the state of the CPB qubit a single-electron transistor (SET) is used. The SET is a sensitive electrometer. It consists of a superconducting island connected by two Josephson junctions. When the voltages across the junctions are near the degeneracy point $N_g = 1/2$, charges cross the junctions inducing a current flow through the SET. If we couple the CPB island to the SET island using a capacitance, a contribution to the SET voltage is made and thus the SET current depends on the CPB state. When the Josephson junctions are small the phase coherence is destroyed by environmental fluctuations, that is, the current flow is dissipative. For this reason the read-out is made by measuring the resistance of the SET.

As it has been said before, the CPB is sensitive to low-frequency noise, which reduces its coherence times. To reduce this problem, other qubits based on the CPB have been designed, such as the quantronium and the transmon.

6.1.1 Quantronium

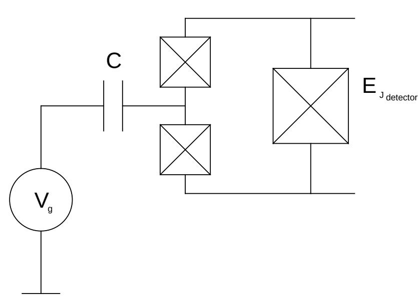

Figure 6: Circuit representation of a quantronium qubit.

The quantronium qubit consists of two Josephson junctions and a capacitance C_g separated by a superconducting island. The two Josephson junctions are connected to a bigger one with a higher critical current, forming a closed superconducting circuit. The big junction is used to measure the state of the qubit. A external flux Φ_e is applied through the loop. The quantronium was the first qubit to use the charge degeneracy point to mitigate low-frequency noise and thus extend dephasing times. Since it is a closed circuit with applied external flux, this is a new noise channel that should be operated at the flux degeneracy point.

In the double degeneracy point[3] the qubit is insensitive to low-frequency (1/f) charge and flux noise. Linear noise is not able to change the qubit transition frequency, because the eigenenergies have no slope in the degeneracy point. This point is called the optimum working point. Success of the optimum working point has been shown experimentally [5]. The insensitivity to both charge and flux means that the two qubit states can not be distinguished at the degeneracy point.

To measure the qubit state a current pulse is applied moving the qubit away from the flux degeneracy point. This pulse produces a clockwise or anticlockwise current in the loop, depending on the qubit state. The direction of the current is detected by the third junction. The new current adds or subtracts from the induced current pulse. This affects the read-out junctions bias-current making it higher or lower. Thus, the junction switches out of the zero-voltage state. Measuring the voltage and seeing if it positive or negative we can determine the state of the qubit.

With quantronium qubits much longer relaxation and decoherence times are obtained comparing to a Cooper-pair box. Saclay group extended the dephasing time T_2* about three orders of magnitude ($\sim 500ns$)[17].

6.1.2 Transmon

The transmon consists of two superconducting islands coupled through two Josephson junctions. It is basically a CPB, only that it is operated in the flux regime $E_J >> E_C$, normally $E_J/E_C \sim 50$. This increases dephasing times in various orders, as much as three orders of magnitude (from nanoseconds to microseconds) [18].The Cooper pair box is sensitive to 1/f charge noise even in the degeneracy point, when its affected by external charge fluctuators. Operating the transmon at the flux regime solves this problem.

The Hamiltonian of the transmon is the same as in the CPB (32), considering that both junctions are identical, with the difference that we have an additional capacitance, meaning that $E_C = (2e)^2/(C_J + C_B + C_g)$.

[3] That is $N_g = 1/2$ and $\Phi_e = m\Phi_0/2$, where m is an integer.

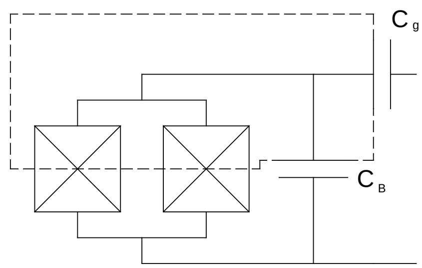

Figure 7: Circuit representation of a transmon qubit. The region which lies inside the dashed lines is the Cooper-pair box.

As E_J/E_C increases, the eingenenergies flatten, meaning that every point can be considered an optimum operating point. Charge dispersion decreases exponentially with $\sqrt{E_J/E_C}$, the dispersion being $\epsilon_m \propto e^{-\sqrt{8E_J/E_C}}$, as the anharmonicity decreases slower. Thus, the qubit transition frequency is very stable respect to charge noise. As the qubit is immune to charge noise, coherent control is possible and dephasing times $T_2^* \sim 2\mu s$ approaching $T_2^* = 2T_1$ are possible. In the large E_J/E_C limit, the charge fluctuations are given by[18]

$$\Delta\hat{N} = \sqrt{<\hat{N}^2>_m - <\hat{N}>_m^2} \simeq (m+1/2)^{1/2}\left(\frac{E_J}{8E_C}\right)^{1/4} \tag{38}$$

6.2 Flux qubit

The flux qubit is constituted by a superconducting loop of inductance L interrupted by one (rf-SQUID) or three Josephson junctions. When a weak magnetic field is applied through the superconducting loop, a current is induced. This happens even if there is a Josephson junction interrupting the superconducting loop.

Like it might be guessed from its name, flux qubits work in the flux regime, that is, $E_J > E_C$, more accurately, $E_J/E_C \sim 50$, where the phase degree of freedom becomes dominant, which means that in this case the phase is the discrete variable. Even if it works in the same regime as the transmon qubit, the flux qubit is completely different while both the superconducting circuit and the discrete variable are different. The macroscopic degree of freedom is the flux Φ induced in the loop, that is, the integral of the voltage across the inductance L (18). The flux, like the charge in the charge qubit, is quantized. The quanta is called the fluxon and quantization comes from the relation between the phase and the total flux $(\Phi_0/2\pi)\varphi + \Phi_e + \Phi = m\Phi_0$, where m is an integer. The continuous variable is the charge Q on the capacitance C_J, which in this case is the continuous variable,

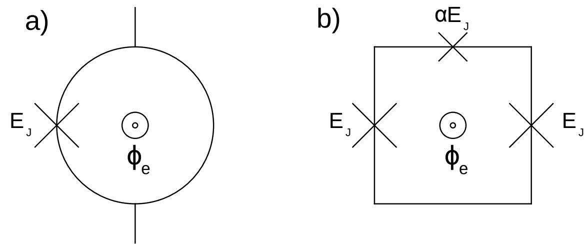

Figure 8: a) Circuit representation of a rf-SQUID, which is a superconducting loop interrupted by a Josephson junction. The loop is controlled by an externally applied flux Φ_e. **b)** Circuit representation of a three-junction flux qubit. Two of the junctions are identical, while the third is smaller. Like the rf-SQUID, the flux qubit is controlled by an externally applied flux Φ_e.

the conjugate variable of the flux. The qubit is controlled by the externally applied flux Φ_e.

First, the simplest flux qubit will be described, the case in which the superconducting loop is interrupted by just one Josephson junction. This circuit is called radio frequency superconducting quantum interference device (rf-SQUID).

The superconducting loop is represented by an asymmetric two well potential as shown in Fig.9. The well is only symmetrical in the degeneracy point $\Phi_e = m\Phi_0/2$. The two well potential is given by the sum of the magnetic energy and the Josephson coupling energy

$$U = \frac{(\Phi - \Phi_e)^2}{2L} - E_J \cos\left(2\pi\frac{\Phi}{\Phi_0}\right) \tag{39}$$

The Hamiltonian is

$$H = \frac{Q^2}{2C_J} + \frac{(\Phi - \Phi_e)^2}{2L} - E_J \cos\left(2\pi\frac{\Phi}{\Phi_0}\right) \tag{40}$$

The qubit can be in the superposition of two basis states, in the case of the flux qubit being an anti-clockwise super-current or a clockwise super-current in the loop, which are classical states. This states correspond to the center-of-mass motion of all the Cooper pairs in the system. Analogously, the states can be represented as magnetic flux pointing up $|\uparrow\rangle$ and magnetic flux pointing down $|\downarrow\rangle$. The weak coupling of the Josephson junction (or the coupling of the three of them) allows transitions between the states. This is a manifestation of quantum mechanical behaviour of a macroscopic system.

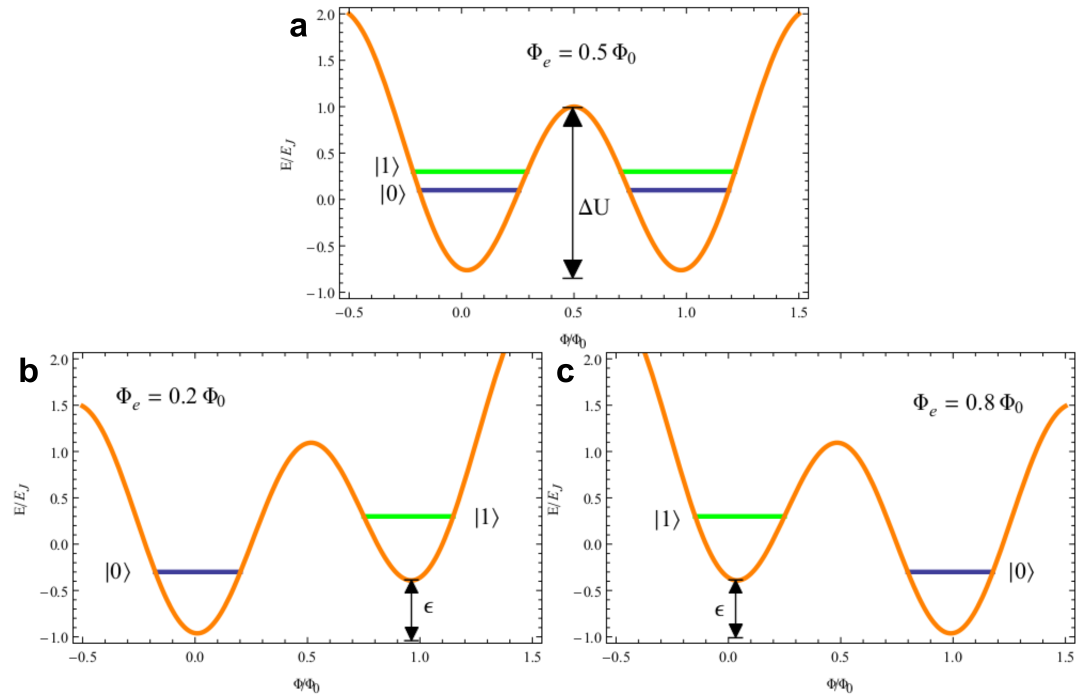

Figure 9: Schematic potential energy of the flux-qubit for different values of Φ_e. The first two energy eigenvalues are given in the qubit approximation, as the second excited energy is much higher due to anharmonicity.**a)** Symmetrical two well potential for the degeneracy point $\Phi_e = \Phi_0/2$.

The superposition of this states could be seen in an anticrossing, as shown in Fig.10. The diagonal dashed lines represent the classical energies, which would become degenerate in absence of coherence. But coherent tunnelling lifts the degeneracy and in the degeneracy point $\Phi = \Phi_0/2$ we have a symmetric state $\frac{1}{\sqrt{2}}(|\downarrow\rangle + |\uparrow\rangle)$ and antisymmetric state $\frac{1}{\sqrt{2}}(|\downarrow\rangle - |\uparrow\rangle)$. The symmetric state is the ground state $|0\rangle$ and the antisymmetric state is the first excited state $|1\rangle$. The splitting between energy levels in the degeneracy point $\Phi_e = \frac{\Phi_0}{2}$ is Δ, also called the tunnel splitting, which is the splitting between the ground and first excited states. The energy levels vary with the externally applied flux, the difference being $\Delta E = \sqrt{\Delta^2 + \varepsilon^2}$, where ε is the energy bias $\varepsilon = 2I_c(\Phi_e - \Phi_0/2)$, which can also be understood as the depth difference between the wells as shown in Fig.9, with the critical current being $I_c = 2\pi \frac{E_J}{\Phi_0}$. The energy bias is controlled by the externally applied flux Φ_e.

The rf-SQUID, having only one Josephson junction interrupting the loop, requires a relatively large loop inductance, it is thus more susceptible to external flux variations (magnetic-field noise). That is why a three junction flux qubit is a more useful structure, as it is less susceptible to magnetic noise fluctuations. For a loop interrupted by three

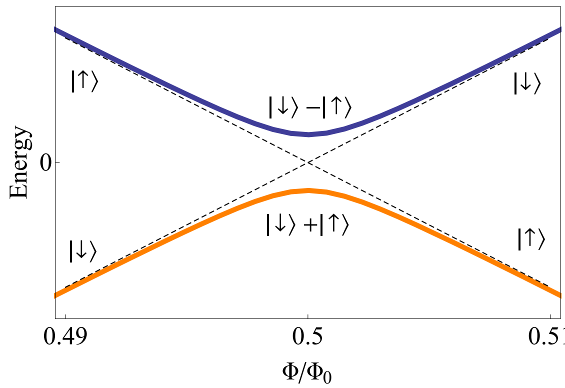

Figure 10: Coherence of the states lifts the energy degeneracy of the classical states forming an anti-crossing in the degeneracy point $\Phi_e = \frac{\Phi_0}{2}$, producing a ground state(orange) and a first excited state(blue)

junctions, two of them are identical, with the same coupling energy and capacitance. The other one has smaller capacitance and coupling energy αE_J. The two larger junctions play the role of the loop inductance in the rf-SQUID. The equations that describe this circuit are the same as in the simplest case, with the difference being that each junction has its own phase and charge variables.

The state of the flux qubit can be measured using a dc-SQUID. The dc-SQUID consists of two Josephson junctions connected in parallel in a superconducting loop. The critical current of the dc-SQUID changes periodically with the external flux Φ_e, with period Φ_0. The dc-SQUID is placed near the flux qubit, not directly connected. When it is biased with a constant magnetic flux about $\Phi_0/2$, it is possible to measure the changes in flux caused by a nearby qubit by measuring the critical current. Experimentally, a current pulse is applied to the dc-SQUID. The exact value of the switching current depends on the state of the qubit, and the pulse is chosen so that the dc-SQUID switches for one state but not for the other. In response to the pulse, it can remain in the zero-voltage state or switch to the non-zero-voltage state. The dc-SQUID remains in this state while the pulse is applied, being possible to determine if it has switched. Repeating the experiment many times an estimate of the probability of each state can be obtained.

The flux qubit can be also measured using a resonant readout technique. The circuit consists of a flux qubit geometrically coupled to a dc-SQUID. An external flux is created using a DC current through the qubit and the dc-SQUID. By carefully choosing the relative area between the dc-SQUID and the qubit, the qubit can be biased at the degeneracy point for a value of dc-SQUID switching current that is approximately midway between the maximum and minimum values. The resonance frequency depends on the dc-SQUID critical current $I_c(\Phi_e)$ which depends on the dc-SQUID inductance L_J.

6.2.1 Fluxonium

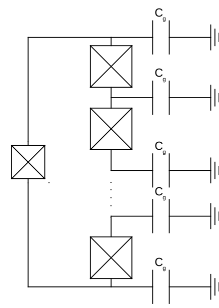

Figure 11: Circuit representation of the fluxonium qubit. The loop is formed by a small Josephson junction shunted by the an array of junction of bigger area. The islands formed between the larger junctions are connected to ground by capacitances C_g.

The fluxonium qubit is based on superinductance. Superinductance has two essential properties: it must superconduct direct current (DC) and it must present the impedance of a frequency independent inductance L to an alternating current (AC).The impedance must have a stray capacitance C_s small enough so that $\sqrt{L/C_s} > R_Q$ where $R_Q = h/(2e)^2 \simeq 6.5k\Omega$ is the superconducting impedance quantum.

The fluxonium qubit consists of a small Josephson junction shunted by an array of larger area junctions. A simple way to understand it is to think of it as a superconducting loop interrupted by $N+1$ Josephson junctions, one of them being weaker than the rest, called the black-sheep junction. As in the flux qubit, the quantum states are given by the fluxons, the flux quanta trapped in the loop. Fluxons can escape the loop across one of the junctions by a phase-slip. The junction array stores the inductive energy associated with the phase-slip, the energy needed to charge the loop with a fluxon. The black-sheep junction allows the fluxon number to change coherently by a unity[19].

The Hamiltonian is given by

$$\hat{H} = E_C \hat{N}^2 + \frac{1}{2} E_L \hat{\varphi}^2 - E_J \cos\left(\hat{\varphi} - 2\pi \frac{\Phi_e}{\Phi_0}\right) \tag{41}$$

Where the three energies are $E_L = \frac{(\Phi_0/2\pi)^2}{L_A}$, $E_J = \frac{(\Phi_0/2\pi)^2}{L_J}$ and $E_C = e^2/(2C_J)$.

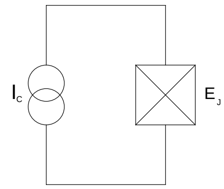

Figure 12: **a)**Circuit representation of the phase qubit, which consists of a current-biased Josephson junction.

6.3 Phase qubit

The phase qubit consists of a current-biased Josephson junction. The dominant degree of freedom is the phase as the reader might expect, and the ratio E_J/E_C is several orders of magnitude bigger than in the flux qubit, being of the order $E_J/E_C \sim 10^6$.

The Hamiltonian of the circuit is given by

$$\hat{H} = \frac{\hat{Q}^2}{2C} - I\frac{\Phi_0}{2\pi}\delta - E_J \cos\delta \tag{42}$$

Using the Taylor series approximation, we can write the potential in a cubic form. This can be done when I approaches I_0 and thus the phase is $\delta \approx 2/\pi$

$$U = \frac{\Phi_0}{2\pi}(I_0 - I)(\delta - 2/\pi) - \frac{E_J}{6}(\delta - 2/\pi)^3 \tag{43}$$

From the potential we can get the equation for the barrier height, which depends on the bias-current, which must be $I \lesssim I_0$ because of the earlier approximation and because I_0 is the maximum possible current.

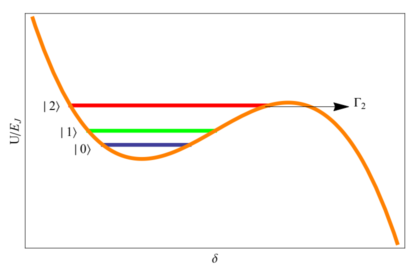

Figure 13: Potential of the phase qubit making the cubic approximation. The first three states are drawn in the image. The second excited state is very close to the height of the barrier and can easily jump across it.

$$\Delta U = \frac{2\sqrt{2}\Phi_0 I_0}{3\pi}\left(1 - \frac{I}{I_0}\right)^{3/2} \tag{44}$$

The approximated potential has one well because of its cubic nature. Instead, if we plot the original potential, we get a tilted washboard potential with periodic wells. In the well live three phase states, the second excited state's energy being far enough to have a two-level qubit approximation, but still not too widely separated. Due to the size of the spacing, the state of the qubit can jump to the third level. Even if this might seem a problem, it can be turned into an advantage in order to determine the state of the qubit.

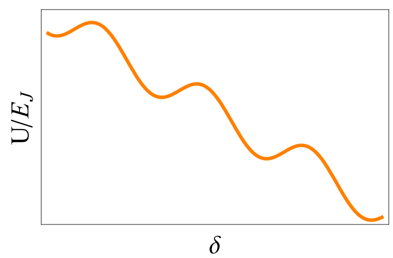

Figure 14: Tilted potential of the phase qubit before the cubic approximation.

If we make a two-level approximation and reduce the Hamiltonian ignoring the rest of the states, we get the reduced Hamiltonian[13]

$$\hat{H}_{qubit} = \frac{\hbar\omega_{01}}{2}\hat{\sigma}_Z + \sqrt{\frac{\hbar}{2\omega_{01}C}}\left(\hat{\sigma}_X + \sqrt{\frac{\hbar\omega_{01}}{3\Delta U}}\hat{\sigma}_Z\right) \tag{45}$$

To measure the quantum state of the qubit, a pulse with frequency $(E_2 - E_1)/2$ must be applied. If the qubit is in the state $|1\rangle$, it will be excited to the state $|2\rangle$. When in the state $|2\rangle$ it is probable to tunnel across the barrier. A voltage across the junction is induced when the particle tunnels, therefore, measuring this voltage it is possible to deduce the state of the qubit, being non-zero when the qubit is initially in the state $|1\rangle$. On the other hand, if the qubit is in the state $|0\rangle$, nothing will happen, meaning that the measured voltage will be null.

There is a different but similar way to measure the state of a phase qubit. The barrier height in the potential can be controlled by the bias-current. It is possible to tilt the potential allowing direct tunnelling from state $|1\rangle$. Again, it will be possible to measure a non-zero voltage when the qubit was in the state $|1\rangle$ and nothing will happen if it was in the state $|0\rangle$.

7 Coupling of superconducting qubits

After achieving coherence times long enough to perform operations and readout on qubits, the next step is implementing a two-qubit coupling, to then go adding more qubits to build a quantum computer. Scalability in one of the main characteristics qubits should have in order to build a quantum computer, as coupling of qubits is necessary for their architecture. However, scalability to many qubits is still one of the goals before achieving quantum computing.

The main purpose of coupling qubits is getting entangled states, because as we have stated above, entanglement is one of the bases of quantum computation.When two qubits are coupled, there are four entangled states, where the first and the second might be degenerate. This states can be described using a 4x4 density matrix

$$\rho = \sum_{i=0}^{4} w_i \left| i \right\rangle \left\langle i \right| \tag{46}$$

Where the energy states are linear combinations of the coupled qubit states

$$\{\left|0,0\right\rangle, \left|1,0\right\rangle, \left|0,1\right\rangle, \left|1,1\right\rangle\}$$

and w_i is the weight of each state. The $\left|01\right\rangle$ and $\left|10\right\rangle$ are degenerate states having the same energy.

One of the characteristics the coupling must have, is that it must be possible to turn the coupling on and off when desired. When turned on, the coupling allows coherent transfer of quantum states between the qubits.

Qubits can be coupled just by placing near each other, as they interact due to their charge or flux. But this only allows coupling a qubit to its nearest neighbour. Coupling to not neighbouring qubits is a condition for scalability making this system obsolete. The simplest method to overcome this, is connecting all the qubits using an inductor L. This way two not neighbouring qubits can be effectively coupled. If the frequency of the resulting LC circuit ω_{LC} is large enough, $\hbar\omega_{LC} \gg E_J, k_B T$, the fast oscillations couple the qubits

$$H_{int} = \sum_{i<j} \frac{E_J^i E_J^j}{E_L} \sigma_y^i \sigma_y^j \tag{47}$$

The interaction Hamiltonian of any two coupled qubits, performing a two-qubit operation gives[20]

$$H_{int} = -\frac{E_J^1}{2}\sigma_x^1 - \frac{E_J^2}{2}\sigma_x^2 + \frac{E_J^1 E_J^2}{E_L}\sigma_y^1 \sigma_y^2 \tag{48}$$

Another possibility is coupling the qubits capacitively. The most suitable method should be chosen depending on each qubit.

For charge qubits, a scalable way of coupling them is connecting N Cooper-pair boxes using an inductor L. Islands are coupled in parallel amongst them, and in series with a dc-SQUID in each connection to the inductor. The two dc-SQUIDs connecting the CPB to the circuit are identical, and all Josephson junctions in them have the same Josephson energy and capacitance. To couple to particular qubits i and j the dc-SQUIDs connecting this qubits must be switched on. The switching on and off condition is satisfied by the dc-SQUID, which are easily switchable. Thus, the current through the inductor L has contributions from all the coupled qubits. When all the Cooper-pair boxes are at their degeneracy point, with their dc-SQUIDs also in the degeneracy point $\Phi_e = \Phi_0/2$, except one qubit i, the inductor is only connected to the ith island[21].

Two flux qubits can also be coupled by a superconducting loop surrounding them. A change in the state of one qubit induces a current in the loop that induces itself a flux in the other qubit. The loop is called a flux transformer. Flux transformers can be interrupted by Josephson junctions, which enable to turn on or off the interaction between the qubits. One device for such purpose is the dc-SQUID. The inductance between the qubits has two components: that of the direct coupling between the qubits and the one of the coupling through the dc-SQUID. The self-inductance of the dc-SQUID can be negative for certain values of applied bias current and flux, having the opposite sign of the inductance of the direct coupling between the qubits. This gives us the ability to switch the coupling off when the inductances cancel each other, and on when they do not, tuning the coupling to be stronger or weaker.

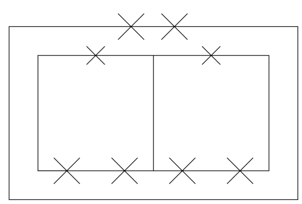

Figure 15: [5]Two flux qubits coupled by a dc-SQUID surrounding them.

A similar way to couple flux qubits inductively is to directly coupled them by sharing one of the legs of the loop, as well as surrounded by a dc-SQUID as shown in Fig.15. The energy eigenstates are similar to the classical states far from the flux degeneracy point. Near the degeneracy point, the states are superpositions of the classical states due to tunnel coupling[22].

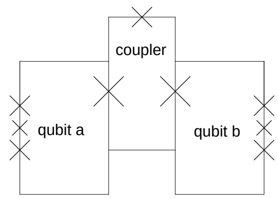

Figure 16: Two flux qubits coupled by another flux qubit used as a coupler.

A variant of the method above is inductively coupling the flux qubits by inserting an additional coupler loop between them as seen in Fig.16. The coupled qubits are three Josephson junction loops, as well as the coupler. The coupler is connected to each loop by one of its bigger junctions. The fluxes through the loops are controlled by the bias currents. The coupling is switch on and off via the coupler's flux bias, which tunes the coupling strength[23].

Phase qubits are most commonly capacitively coupled. They can be directly connected using a capacitance. The qubit bias currents are used to control the interaction between them by tuning the energy level spacings of the junctions in and out of resonance. When brought into resonance, the interaction produces an oscillation between the $|01\rangle$ and $|10\rangle$ states lifting degeneracy. The strength of the coupling is set by the capacitive relation $\frac{C_c}{C_c+C_J}$, so the coupling strength is proportional to the coupling capacitance. Capacitive coupling mixes the uncoupled states lifting the energy degeneracy[2][24].

8 Logical operations with superconducting qubits

In the process of building a quantum computer, the purpose of coupling qubits to get entangled states and the achievement of long[4] coherent states is the manipulation of quantum information via quantum gates. This quantum computer is built from a quantum circuit composed of wires and quantum gates[25]. The wires carry around the quantum information that is then manipulated by quantum gates. Quantum gates are operations that transform qubit states. To understand how quantum gates can be achieved using the previously studied superconducting qubits, first is necessary a theoretical introduction of quantum gates.

8.1 Quantum gates

There are two types of quantum gates: single quantum gates, operated in a single qubit, and multiple qubit quantum gates, operated in more than one qubit. A important characteristic of quantum gates is that they must be invertible, so that there is no loss information, that is, we can be able to know the input once we have the output. The quantum gates are represented by matrices. The matrices U describing the gates must be unitary, that is, the must satisfy

$$U^{\dagger}U = I \tag{49}$$

where $U^{\dagger}$ is the adjoint of U and I is a unitary matrix. The constraint of being unitary comes from the need to conserve the probabilities. This unitarity constraint of quantum gates is the only constraint, any unitary matrix can be a quantum gate.

If we use the Bloch sphere, the unitary matrices correspond to rotations and reflections of the sphere,

To illustrate the idea of a quantum gate, we will set the example of a NOT gate. Classically, the NOT gate interchanges the 0 and 1 states, $0 \to 1$ and $1 \to 0$. The quantum analogous of the classical NOT gate, the quantum NOT gate, operates linearly, as it must transform the linear combination of the states $|0\rangle$ and $|1\rangle$. Thus, if we have the state

$$\alpha\,|0\rangle + \beta\,|1\rangle \tag{50}$$

The NOT gate interchanges the states $|0\rangle$ and $|1\rangle$ obtaining

$$\alpha\,|1\rangle + \beta\,|0\rangle \tag{51}$$

As we have states above, quantum gates can be represented by matrices. We can write the NOT gate using the matrix X

[4] Of the order of microseconds

$$X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{52}$$

Notice that the matrix above is Pauli matrix $\hat{\sigma}_X$. We can also write the input state (50) in vector notation

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{53}$$

And thus we can write the quantum gate operation as a multiplication of the gate and the input state

$$X \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix} \tag{54}$$

Apart from the NOT gate, there are many other important single-qubit gates. Like the NOT gate, the other Pauli matrices represent quantum gates too

$$X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ; Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} ; Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{55}$$

Another important gate is the Hadamard gate, given by the matrix

$$H \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{56}$$

This is one of the most useful single-qubit gates. The Hadamard gate turns $|0\rangle$ into $(|0\rangle + |1\rangle)/\sqrt{2}$ and the $|1\rangle$ into $|0\rangle$ into $(|0\rangle - |1\rangle)/\sqrt{2}$. In the Bloch sphere picture, the Hadamard gate is a rotation of 90 around the $\hat{y}$ axis followed by a rotation of 180 around the $\hat{x}$ axis[1]. Another single-qubit gate, and the last one we will introduce is the phase shift gate Φ, with matrix representation[26]

$$\Phi \equiv \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix} \tag{57}$$

Besides the single-qubit gates, we have multiple-qubit gates. To perform multi-qubit gates, entanglement amongst the qubits is essential. There are many multiple-qubit gates, but here only the controlled-NOT or CNOT gate will be introduced. The CNOT gate is the prototypical multi-qubit quantum logic gate. It has two input qubits: the control qubit and the target qubit. If the control qubit is set to 0, then nothing happens to the target, but if the control qubit is set to 1, the the target qubit's state is switched

$$|00\rangle \rightarrow |00\rangle ; |01\rangle \rightarrow |01\rangle ; |10\rangle \rightarrow |11\rangle ; |11\rangle \rightarrow |10\rangle \tag{58}$$

The matrix representation of the gate is[1]

$$U_{CN} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{59}$$

Any multiple qubit logic gate can be composed from CNOT, Hadamard and phase shift gates. This makes the set $\{H, \Phi, CNOT\}$ set a universal gate set, as any gate can be performed from the composition of this gates alone[1].

8.2 CNOT gate using charge qubits

After studying how to couple superconducting qubits and introducing the basic theory about quantum gates we are ready to perform a gate operation using superconducting qubits. As the CNOT gate is a two-qubit gate we need to couple two qubits in order to perform the gate operation. We will study the case where two charge qubits are coupled by a capacitor. The (1) qubit will be the control qubit while the (2) qubit will be the target qubit. The system has two pulse gates, to address each qubit individually. The Hamiltonian of the system is given by

$$\begin{aligned} H = \sum_{N1,N2=0,1} E_{N1N2} \left|N1, n2\right\rangle \left\langle N1, N2\right| - \frac{E_{J2}}{2} \sum_{N2=0,1} (\left|0\right\rangle \left\langle 1\right| + \left|1\right\rangle \left\langle 0\right|) \otimes \left|N2\right\rangle \left\langle N2\right| \\ - \frac{E_{J1}}{2} \sum_{N1=0,1} \left|N1\right\rangle \left\langle N1\right| \otimes (\left|0\right\rangle \left\langle 1\right| + \left|1\right\rangle \left\langle 0\right|) \end{aligned} \tag{60}$$

where E_{Ji} is the Josephson energy of each qubit and $E_{N1N2} = E_{c1}(N_{g1} - N_1)^2 + E_{c2}(N_{g2} - N_2)^2 + E_m(N_{g1} - N_1)(N_{g2} - N_2)$ is the total electrostatic energy of the system. E_{ci} are the charging energies of the islands and E_m is the coupling energy. When N_{g1} and N_{g1} are far away from their degeneracy point 1/2 we have four independent entangled states. If we fix one of the offset charges, $N_{g1} = C$ for example, our system is divided into a pair of independent two-level systems $\left|00\right\rangle, \left|01\right\rangle$ and $\left|10\right\rangle, \left|11\right\rangle$. The charging energies of each of the two-level systems degenerate at different values of N_{g2}, N_{g2L} for $\left|00\right\rangle, \left|01\right\rangle$ and N_{g2U} for $\left|10\right\rangle, \left|11\right\rangle$. Applying a pulse in the control qubit shifts the system maintaining N_{g2} constant. If our input state is $\left|00\right\rangle$ and a pulse is applied, the system is brought to the degeneracy point N_{g2L}, evolving with frequency $\Omega = \frac{E_{J2}}{\hbar}$. Thus, the state of the system evolves as

$$\cos\left(\frac{\Omega \Delta t}{2}\right) \left|00\right\rangle + \sin\left(\frac{\Omega \Delta t}{2}\right) \left|01\right\rangle \tag{61}$$

so, if we send a π pulse, $\Omega \Delta t = \pi$, we get the output state $\left|01\right\rangle$. If we, on the other hand, have the input state $\left|10\right\rangle$ and apply the same pulse, the system does not reach the degeneracy point N_{g2U}. IF E_m remains large enough, the state remains almost unchanged. Using the exact same method, we can perform the transition from $|01\rangle$ to $|00\rangle$ and suppress the transition out of $|11\rangle$. So, we have described a CNOT gate where the state of the target qubit is flipped only when the state of the control qubit is $|0\rangle$[27]. We can express this operation using a matrix very similar to (59)

$$U_{CN} \equiv \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{62}$$

Conclusions

We have seen the viability of superconducting qubits as a strong option for implementing quantum computation. Superconducting qubits fulfil the requirements needed to build a quantum computer. They are able to maintain long enough coherence times to perform logical operations on them, they are easily scalable and capable of performing logical operations.

Many different superconducting qubits have been studied. They have been classified according to their discreet variable. If we compare the Hamiltonian for each of this qubits, it can be seen that they are the same. The main difference lies in the values for E_J and E_C, which give us the regime in which the qubit is operated. Also, the variables used are not the same, sometimes we use the flux and others the phase, but we have learned that they give the same information.

Even if it has been demonstrated that quantum computation using superconducting qubits is possible, it will be a long time until a functional quantum computer is built using superconducting qubits. But not just that, there are many other qubits, as we have seen in chapter 3, that may win the race to build a quantum computer. There is still lots of research to be made and only time will determine which will be the optimal option.

This work is limited in the sense of deepness. The presented objects and methods are the simplest in order to stablish the basis of quantum computation with superconducting qubits. As the subject is object of current research, there are lots of recent papers that have not been included, meaning that the presented information is not the most recent or advanced. The reason for omitting some circuits and methods is the complication of them, as the aim of this work is not to acquire expertise, but to build an understanding that can lead to possible further research.

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