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Sobolev Spaces and Partial Differential Equations

Final Degree Dissertation
Degree in Mathematics

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Preface

This Final Degree Dissertation is intended as an introduction to Sobolev spaces, with the objective of applying abstract results of Functional Analysis and Sobolev Spaces results to the study of *Partial Differential Equations* (PDEs).

The Dissertation is divided in two main sections. The first section introduces Sobolev spaces, and it will cover the main results that will be used in the second part of the Dissertation. This second part is divided as well in several different subsections, each one devoted to a certain type of Partial Differential Equations. The objective of this section is to study existence, uniqueness, regularity and other results of weak solutions, using different techniques.

The text is based on several books about Partial Differential Equations and Functional Analysis, although it has been highly influenced by Haim Brezis' *Functional Analysis, Sobolev Spaces and Partial Differential Equations* and Lawrence C. Evans' *Partial Differential Equations*. However, the Dissertation includes some original content or exercises. Whenever this happens, a black circle (•) will precede such content, which may take the form of a proposition, the proof of a proposition, an exercise, etc.

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Notation

In general, and if it is not specified otherwise, Ω will denote an open set $\Omega \subset \mathbb{R}^N$.

Given a functional $f \in E^\star$, with E a Banach space, for each $x \in E$ we denote $f(x)$ as $\langle f, x \rangle$. Moreover, if H is a Hilbert space, its scalar product will be denoted as $(\cdot, \cdot)$.

For each $1 \leq p \leq \infty$, the number p' will be given by the unique $1 \leq p' \leq \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Given a Banach space E, the closed unit ball of E will be denoted by B_E. That is, $B_E = \{x \in E \big| \|x\| = 1\}$.

Given an open set $\Omega \subset \mathbb{R}^N$ and an open set $\omega \subset \mathbb{R}^N$, we say that ω is strongly included in Ω if $\overline{\omega} \subset \Omega$ and $\overline{\omega}$ is compact. In this case, we write $\omega \subset\subset \Omega$.

Chapter 1

Sobolev Spaces

In this chapter the theory of Sobolev Spaces will be developed, which will be the core of the study of elliptic, parabolic and hyperbolic PDEs in the second chapter, and even some nonlinear PDEs.

In a certain manner, Sobolev spaces are analogous to the Hölder spaces $C^{k,\alpha}(\overline{\Omega})$ with $\Omega \subset \mathbb{R}^N$ an open set, where the usual differentiability is replaced by weak differentiability.

1.1 Motivation and definition of Sobolev Spaces

Let $C_c^\infty(\Omega)$ be the set of infinitely differentiable functions with compact support $\varphi : \Omega \to \mathbb{R}$. A function $\varphi \in C_c^\infty(\Omega)$ will often be called a *test function*. For each $u \in C^1(\Omega)$, and taking into account that every $\varphi \in C_c^\infty(\Omega)$ has compact support, Green's identity yields

$$\int_\Omega u\varphi_{x_i} = -\int_\Omega u_{x_i}\varphi \qquad \forall \varphi \in C_c^\infty(\Omega), \quad \forall i = 1, \ldots, n$$

This motivates the definition of Sobolev spaces as follows:

Definition 1.1.1. For each $1 \leq p \leq \infty$, the Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \middle| \overset{\exists g_1,\ldots,g_N \in L^p(\Omega) \text{ such that}}{\int_\Omega u\varphi_{x_i} = -\int_\Omega g_i\varphi \quad \forall \varphi \in C_c^\infty(\Omega)} \right\}$$

We denote by $u_{x_i} = g_i$ the *weak* derivative of $u \in W^{1,p}(\Omega)$, which are unique as we will see now. Also, the *gradient* of u is defined by $\nabla u = (u_{x_1}, \ldots, u_{x_N})$. If $p = 2$, we write $H^1(\Omega) = W^{1,2}(\Omega)$.

In order to prove that the weak derivative is unique, we shall first state the following well known lemma, about L^p spaces:

Lemma 1.1.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in L^1_{loc}(\Omega)$ be such that*

$$\int_\Omega u\varphi = 0 \quad \forall \varphi \in C_c^\infty(\Omega)$$

Then, $u = 0$ a.e. on Ω.

Now, we can proof the uniqueness of weak derivative, as follows:

Proposition 1.1.2. *Let $u \in L^p(\Omega)$ be a function that has a weak derivative u_{x_i}. Then, this weak derivative is unique except in a set of zero measure, that is, if $g, h \in L^p(\Omega)$ are two functions such that $\int_\Omega u\varphi_{x_i} = -\int_\Omega g\varphi = -\int_\Omega h\varphi$ for every $\varphi \in C_c^\infty(\Omega)$, then $g = h$ a.e.*

• *Proof.* [1] Suppose that there exist $g, h \in L^p$ such that $\int_\Omega u\varphi_{x_i} = -\int_\Omega g\varphi = -\int_\Omega h\varphi$, $\forall \varphi \in C_c^\infty(\Omega)$. Let $\psi = g - h$. We shall see that $\psi = 0$ a.e. We have that

$$\int_\Omega \psi\varphi = 0 \quad \forall \varphi \in C_c^\infty(\Omega)$$

Since $\psi \in L^p(\Omega)$, in particular $\psi \in L^p_{\text{loc}}(\Omega) \subset L^1_{\text{loc}}(\Omega)$. Therefore, using the previous lemma, $\psi = 0$ a.e., and therefore the weak derivative is well defined (up to a set of zero measure).

□

There is an alternative way to define the Sobolev Spaces. Given a function $f \in L^p(\mathbb{R}^N)$, and e_i the i-th vector of the canonical basis of $\mathbb{R}^N$, we say that the *i-th partial derivative of f* exists in the L^p sense and equals f_{x_i}, if $\epsilon^{-1}(\tau_{\epsilon e_i} f - f) \to -f_{x_i}$ in $L^p(\mathbb{R}^N)$, when $\epsilon \to 0$. The function $\tau_{\epsilon e_i}$ is defined by $(\tau_{\epsilon e_i} f)(x) = f(x + \epsilon e_i)$.

With these definitions, we can alternatively define the Sobolev space as

$$W^{1,p}(\mathbb{R}^N) = \{f \in L^p(\mathbb{R}^N) \big| f_{x_i} \text{ exists in the } L^p \text{ sense for every } i = 1, \dots, n\}$$

Both definitions of the partial derivative f_{x_i} can be proved to be equal. In what follows, the first definition will be used, instead of the alternative definition.

For each $u \in W^{1,p}(\Omega)$, we define the norm of u by

$$\|u\|_{W^{1,p}} = \|u\|_p + \sum_{i=1}^N \|u_{x_i}\|_p \tag{1.1}$$

Moreover, $H^1(\Omega)$ is a Hilbert space equipped with the following scalar product:

[1]As mentioned in the Preface, whenever a black circle precedes some content, this content is original.

$$(u, v)_{H^1} = (u, v)_{L^2} + \sum_{i=1}^{N} (u_{x_i}, v_{x_i})_{L^2} \tag{1.2}$$

Since $H^1(\Omega)$ is a Hilbert space, (1.2) induces a norm in $H^1(\Omega)$,

$$\|u\|'_{H^1} = \left(\|u\|_2^2 + \sum_{i=1}^{N} \|u_{x_i}\|_2^2 \right)^{1/2}$$

This norm is equivalent to (1.1) for $p = 2$.

1.2 First properties of Sobolev Spaces

Proposition 1.2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then, the following statements hold:*

(i) *For each $1 \leq p \leq \infty$, $W^{1,p}(\Omega)$ is a Banach space.*

(ii) *For each $1 < p < \infty$, $W^{1,p}(\Omega)$ is reflexive.*

(iii) *For each $1 \leq p < \infty$, $W^{1,p}(\Omega)$ is separable.*

Proof. (i) Let $\{u_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $W^{1,p}(\Omega)$, with $1 \leq p \leq \infty$. Then, from (1.1) it follows that $\{u_n\}_{n\in\mathbb{N}}$ and $\{(u_n)_{x_i}\}_{n\in\mathbb{N}}$, with $1 \leq i \leq N$, are Cauchy sequences in L^p. Thus, since L^p is a Banach space, it follows that $u_n \to u$ and $(u_n)_{x_i} \to g_i$ in L^p, with $u, g_i \in L^p$. Therefore, since

$$\int_\Omega u_n \varphi_{x_i} = -\int_\Omega (u_n)_{x_i} \varphi \quad \forall \varphi \in C_c^\infty(\Omega)$$

Letting $n \to \infty$,

$$\int_\Omega u \varphi_{x_i} = -\int_\Omega g_i \varphi \quad \forall \varphi \in C_c^\infty(\Omega)$$

Therefore, we obtain that $u \in W^{1,p}$, $u_{x_i} = g_i$ and thus $\|u_n - u\|_{W^{1,p}} = \|u_n - u\|_p + \sum_{i=1}^N \|u_n - g_i\|_p \to 0$, as desired.

(ii) Consider the space $E = L^p(\Omega) \times L^p(\Omega)^N$, which is reflexive since it is the product of reflexive spaces. Set the operator $T : W^{1,p}(\Omega) \to E$ defined by $Tu = (u, \nabla u)$. Then, T is an isometry, and since $W^{1,p}(\Omega)$ is a Banach space, $M = T(W^{1,p}(\Omega))$ is a closed subspace of E. Now, since E is reflexive, B_E is compact in the weak topology $\sigma(E, E^\star)$, and M is closed in the topology $\sigma(E, E^\star)$. Therefore, B_M is compact in $\sigma(E, E^\star)$, and therefore $T(W^{1,p})$ is reflexive. As a consequence, $W^{1,p}$ is also reflexive.

(iii) Under the notation of (ii), and taking into account that E is separable, it follows that $T(W^{1,p}(\Omega))$ is separable and therefore $W^{1,p}(\Omega)$ is also separable.

□

Under some conditions, one can *think of* a function $u \in W^{1,p}(\Omega)$ as a function in $u \in C^1(\Omega)$. Indeed, if $u \in W^{1,p}(\Omega)$ for a certain $1 \leq p \leq \infty$, and $u_{x_i} \in C(\Omega)$ for each $1 \leq i \leq N$, where the partial derivative is a weak partial derivative, then it can be proven that there exists $v \in C^1(\Omega)$ such that $u = v$ a.e.

Moreover, we can stablish the following density result, although we will later prove a stronger result under some more assumptions. But first, we need to introduce the following lemma.

Lemma 1.2.2. *Set $\rho \in L^1(\mathbb{R}^N)$, $v \in W^{1,p}(\mathbb{R}^N)$ with $1 \leq p \leq \infty$. Then, $\rho \star v \in W^{1,p}(\mathbb{R}^N)$ and for each $i = 1, \ldots, N$, we must have that $(\rho \star v)_{x_i} = \rho \star (v)_{x_i}$.*

Proposition 1.2.3 (Friedrichs). *Let $u \in W^{1,p}(\Omega)$, with $1 \leq p \leq \infty$. Then, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^N)$ such that*

$$u_n|_\Omega \to u \quad \text{in } L^p(\Omega)$$

and

$$\nabla u_n|_\omega \to \nabla u|_\omega \quad \text{in } L^p(\omega)^N \text{ for all } \omega \subset\subset \Omega$$

If $\Omega = \mathbb{R}^N$, then there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^N)$ so that

$$u_n \to u \quad \text{in } L^p(\mathbb{R}^N)$$

and

$$u_n \to \nabla u \quad \text{in } L^p(\mathbb{R}^N)^N$$

Proof. Set $\overline{u} = u\chi_\Omega$, and take $v_n = \rho_n \star \overline{u}$ with ρ_n a sequence of mollifiers. Then, $v_n \in C^\infty(\mathbb{R}^N)$ and, moreover, $v_n \to \overline{u}$ in $L^p(\mathbb{R}^N)$. We must see that for each $\omega \subset\subset \Omega$, $\nabla v_n|_\omega \to \nabla u|_\omega$ in $L^p(\omega)^N$.

Let $\omega \subset\subset \Omega$, and take a function $\alpha \in C_c^1(\Omega)$ such that $0 \leq \alpha \leq 1$ and $\alpha|_\omega = 1$. It is easy to check that such function exists. Then, for n large enough,

$$\operatorname{supp}(\rho_n \star (\overline{\alpha u}) - \rho_n \star \overline{u}) = \operatorname{supp}(\rho_n \star (1 - \overline{\alpha})\overline{u}) \subset \overline{\operatorname{supp} \rho_n + \operatorname{supp}((1 - \overline{\alpha})\overline{u})}$$
$$\subset \overline{B(0, 1/n) + \operatorname{supp}(1 - \overline{\alpha})} \subset (\omega)^c$$

And, therefore,

$$\rho_n \star (\overline{\alpha u}) = \rho_n \star \overline{u} \quad \text{on } \omega \tag{1.3}$$

Using the previous lemma, we have

$$(\rho_n \star \overline{\alpha u})_{x_i} = \rho_n \star (\overline{\alpha u})_{x_i} = \rho_n \star (\overline{\alpha u_{x_i} + \alpha_{x_i} u})$$

The last equality follows from the fact that for each $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \overline{\alpha u} \varphi_{x_i} = \int_\Omega \alpha u \varphi_{x_i} = \int_\Omega u[(\alpha\varphi)_{x_i} - \alpha_{x_i}\varphi] = -\int_\Omega (u_{x_i}\alpha\varphi + u\alpha_{x_i}\varphi) =$$

$$-\int_{\mathbb{R}^N} (\overline{\alpha u_{x_i} + \alpha_{x_i} u})\varphi$$

As a consequence, it follows that

$$(\rho_n \star \overline{\alpha u})_{x_i} \to \overline{\alpha u_{x_i} + \alpha_{x_i} u} \quad \text{in } L^p(\mathbb{R}^N)$$

After a restriction to ω, we have that

$$(\rho_n \star \overline{\alpha u})_{x_i} \to u_{x_i} \quad \text{in } L^p(\omega)$$

Taking into account (1.3),

$$(\rho_n \star \overline{u})_{x_i} \to u_{x_i} \quad \text{in } L^p(\omega)$$

We shall define a certain sequence of cut-off functions ζ_n now. Fix a function $\zeta \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \zeta \le 1$, and

$$\zeta(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| \ge 2 \end{cases}$$

We define the sequence of cut-offs $\zeta_n(x) = \zeta(x/n)$. Now, using the dominated convergence theorem the sequence $u_n = \zeta_n v_n$ satisfies that $u_n \to u$ in $L^p(\Omega)$, and $\nabla u_n \to \nabla u$ in $(L^p(\omega))^N$. If $\Omega = \mathbb{R}^N$, the sequence defined by $u_n = \zeta_n(\rho_n \star u)$ satisfies the desired properties. □

The following proposition offers a characterization of the elements of $W^{1,p}$.

Proposition 1.2.4. *Let $u \in L^p(\Omega)$, with $1 < p \le \infty$. The following properties are equivalent:*

(i) $u \in W^{1,p}(\Omega)$.

(ii) *There exists a constant $C > 0$ such that*

$$\left| \int_\Omega u\varphi_{x_i} \right| \le C\|\varphi\|_{L^{p'}(\Omega)} \quad \forall \varphi \in C_c^\infty(\Omega), \quad \forall i = 1, 2, \ldots, N$$

(iii) *there exists a constant $C > 0$ so that for every $\omega \subset\subset \Omega$ and $h \in \mathbb{R}^N$ such that $|h| < dist(\omega, \partial\Omega)$ we have*

$$\|\tau_h u - u\|_{L^p(\omega)} \leq C|h|$$

Where τ_h is defined by $\tau_h u(x) = u(x+h)$.

If $\Omega = \mathbb{R}^N$, we have

$$\|\tau_h u - u\|_{L^p(\mathbb{R}^N)} \leq |h| \|\nabla u\|_{L^p(\mathbb{R}^N)}$$

Proof. (i) $\Rightarrow$ (ii). Since $u \in W^{1,p}(\Omega)$, for each $\varphi \in C_c^\infty(\Omega)$ and $i = 1, \dots, N$,

$$\left| \int_\Omega u\varphi_{x_i} \right| = \left| \int_\Omega u_{x_i}\varphi \right| \leq \|u_{x_i}\|_p \|\varphi\|_{p'}$$

(ii) $\Rightarrow$ (i). Given $i \in \{1, \dots, N\}$, Consider the linear functional

$$\varphi \in C_c^\infty(\Omega) \mapsto \int_\Omega u\varphi_{x_i}$$

This linear functional is defined on a dense subspace of $L^{p'}$, since $p' < \infty$ as $1 < p$. Moreover, this functional is continuous for the norm in $L^{p'}$ because of (ii). Thus, we may apply Hahn–Banach theorem in order to extend this functional to a bounded linear functional F that is defined in all of $L^{p'}$. Applying the Riesz representation theorem, there must exist a function $g \in L^p$ such that

$$\langle F, \varphi \rangle = \int_\Omega g\varphi \quad \forall \varphi \in L^{p'}$$

In particular,

$$\int_\Omega u\varphi_{x_i} = \int_\Omega g\varphi \quad \forall \varphi \in C_c^\infty$$

And therefore $u \in W^{1,p}$.

(i) $\Longrightarrow$ (iii). Suppose that $u \in C_c^\infty(\mathbb{R}^N)$. A density argument will be used to prove the general case. Set $h \in \mathbb{R}^N$, and set $v(t) = u(x+th)$, for each $t \in \mathbb{R}$. Clearly $v'(t) = h\nabla u(x+th)$, and hence

$$u(x+h) - u(x) = v(1) - v(0) = \int_0^1 v'(t)dt = \int_0^1 h \cdot \nabla u(x+th)dt$$

Therefore, for each $1 \leq p < \infty$,

$$|\tau_h u(x) - u(x)|^p \leq |h|^p \int_0^1 |\nabla u(x+th)|^p dt$$

Integrating on ω, we reach to

$$\int_\omega |\tau_h u(x) - u(x)|^p dx \leq |h|^p \int_0^1 dt \int_\omega |\nabla u(x+th)|^p dx = |h|^p \int_0^1 dt \int_{\omega+th} |\nabla u(y)|^p dy$$

Now, take $|h| < \text{dist}(\omega, \partial\Omega)$. Then, clearly there exists an open set $\omega' \subset\subset \Omega$ such that $\omega + th \subset \omega'$ for all $t \in [0,1]$. Therefore,

$$\|\tau_h u - u\|^p_{L^p(\omega)} \leq |h|^p \int_{\omega'} |\nabla u|^p$$

Which proves the case where $u \in C_c^\infty(\mathbb{R}^N)$, $1 \leq p < \infty$. Using Theorem (1.2.3), the general case follows.

(iii) $\Longrightarrow$ (ii). Take $\varphi \in C_c^\infty(\Omega)$. We may pick an open set ω that is contained *in between* $\operatorname{supp}\varphi$ and Ω, that is, $\operatorname{supp}\varphi \subset \omega \subset\subset \Omega$. Now, we proceed as follows: pick $h \in \mathbb{R}^N$ with $|h| < \text{dist}(\omega, \partial\Omega)$. We are now under the hypotheses of (iii), so that

$$\left| \int_\Omega (\tau_h u - u)\varphi \right| \leq C|h| \|\varphi\|_{L^{p'}(\Omega)}$$

Now, we have that

$$\int_\Omega (u(x+h) - u(x))\varphi(x) dx = \int_\Omega u(y)(\varphi(y-h) - \varphi(y)) dy$$

We conclude that

$$\int_\Omega u(y) \frac{(\varphi(y-h) - \varphi(y))}{|h|} dy \leq C\|\varphi\|_{L^{p'}(\Omega)}$$

(ii) follows from letting $h = te_i$ and taking $t \to 0$.

□

Weak derivatives enjoy some properties that are analogous to the case of C^1 functions, such as the differentiation of a product and composition, and the change of variables formula. The following results, whose proofs are omitted, state these properties (See H. Brezis [1], Chapter 9, for the corresponding proofs).

Proposition 1.2.5. *Let $1 \leq p \leq \infty$. Then, $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is closed under multiplication, that is, for every $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, its product $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Moreover,*

$$(uv)_{x_i} = u_{x_i} v + u v_{x_i}, \quad i = 1, 2, \ldots, N$$

Proposition 1.2.6. *Let $G \in C^1(\mathbb{R})$ be a differentiable continuous function such that $G(0) = 0$ and $\|G'\|_\infty \leq M$, with $M \geq 0$. Then, for each $u \in W^{1,p}(\Omega)$ $(1 \leq p \leq \infty)$ the composition of G and u belongs to $W^{1,p}$. That is, $G \circ u \in W^{1,p}(\Omega)$, and moreover*

$$(G \circ u)_{x_i} = (G' \circ u) u_{x_i}, \quad i = 1, 2, \dots, N$$

Proposition 1.2.7. *Let $\Omega, \Omega' \subset \mathbb{R}^N$ be two open sets, and $H : \Omega' \to \Omega$ a bijective map of class C^1, such that $H^{-1} \in C^1(\Omega)$, $JacH \in L^\infty(\Omega')$ and $JacH^{-1} \in L^\infty(\Omega)$, where Jac denotes the Jacobian matrix. Then, $u \circ H \in W^{1,p}(\Omega')$ and*

$$(u(H(y)))_{y_j} = \sum_i u_{x_i}(H(y))(H_i(y))_{y_j}, \quad j = 1, 2, \dots, N$$

1.3 $W^{m,p}(\Omega)$ spaces

After defining the $W^{1,p}(\Omega)$ spaces, we can define the general $W^{m,p}$ spaces recursively. Let $m \geq 2$ be an integer, and $1 \leq p \leq \infty$. Then, we define

$$W^{m,p}(\Omega) = \left\{ u \in W^{m-1,p}(\Omega) \;\middle|\; u_{x_i} \in W^{m-1,p}(\Omega) \quad \forall i = 1, 2, \dots, N \right\}$$

An equivalent way of defining these Sobolev spaces is defining $W^{m,p}$ as

$$W^{m,p}(\Omega) = \left\{ u \in L^p(\Omega) \;\middle|\; \begin{array}{l} \forall \alpha \text{ with } |\alpha| \leq m, \exists g_\alpha \in L^p(\Omega) \text{ such that} \\ \int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega g_\alpha \varphi \quad \forall \varphi \in C_c^\infty(\Omega) \end{array} \right\}$$

The multi-index notation have been used. That is, $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ and $|\alpha| = \sum_i \alpha_i$. Moreover,

$$D^\alpha \varphi = \frac{\partial^{|\alpha|} \varphi}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}}$$

We denote $D^\alpha u = g_\alpha$. Then, it can be proved (although its proof will be omitted) that $W^{m,p}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{m,p}} = \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_p \tag{1.4}$$

Just like in the $W^{1,2}$ case, $W^{m,2}(\Omega)$ is a Hilbert space that is denoted by $H^m(\Omega)$, and its scalar product is

$$(u, v)_{H^m} = \sum_{0 \leq |\alpha| \leq m} (D^\alpha u, D^\alpha v)_{L^2}$$

Again, the norm arising from this scalar product is equivalent to (1.4).

1.4 Extension Operators and Sobolev Inequalities

One may wonder if given a function $u \in W^{1,p}(\Omega)$ with $\Omega \subsetneq \mathbb{R}^N$, there exists an *extension* $\tilde{u} \in W^{1,p}(\mathbb{R}^N)$. That is, a function $\tilde{u} \in W^{1,p}(\mathbb{R}^N)$ such that $\tilde{u}_{|\Omega} = u$ a.e. This is not true in general, unless we require more hypotheses to Ω. If the domain is *smooth* enough, a concept that will be defined now, the result is actually true.

Notation. Let $x \in \mathbb{R}^N$. We write x as $x = (x', x_N)$, with $x' \in \mathbb{R}^{N-1}$. Moreover, we denote $|x'| = \|(x_1, \ldots, x_{N-1})\|_2$, where $\|\cdot\|$ is the euclidean norm of R^{N-1}. Finally, we define the following sets:

(i) $\mathbb{R}^N_+ = \{(x', x_N) \in \mathbb{R}^N \big| x_N > 0\}$

(ii) $Q = \{(x', x_N) \in \mathbb{R}^N \big| |x'| < 1 \text{ and } |x_N| < 1\}$

(iii) $Q_+ = Q \cap \mathbb{R}^N_+$

(iv) $Q_0 = \{(x', 0) \in \mathbb{R}^N \big| |x'| < 1\}$

Definition 1.4.1. An open set $\Omega \subset \mathbb{R}^N$ is said to be of class C^1 if for every $x \in \partial\Omega = \Gamma$ there exists a neighborhood U of x in $\mathbb{R}^N$ and a bijective map $H : Q \to U$ such that

$$H \in C^1(\overline{Q}), \quad H^{-1} \in C^1(\overline{U}), \quad H(Q_+) = U \cap Q, \quad \text{and} \quad H(Q_0) = U \cap \Gamma$$

Under these conditions, H is said to be a local chart.

The following theorem assures the existence of an *extension operator* that extends any function $u \in W^{1,p}(\Omega)$ to a function $\tilde{u} \in W^{1,p}(\mathbb{R}^N)$, as long as Ω is of class C^1:

Theorem 1.4.1. *Let $\Omega \subset \mathbb{R}^N$ be a domain of class C^1 with $\Gamma = \partial\Omega$ bounded or $\Omega = \mathbb{R}^N_+$. Then, there exists a linear operator*

$$P : W^{1,p}(\Omega) \longrightarrow W^{1,p}(\mathbb{R}^N)$$

with $1 \leq p \leq \infty$, that fulfills the following properties for each $u \in W^{1,p}(\Omega)$:

(i) $(Pu)_{|\Omega} = u$,

(ii) $\|Pu\|_{L^p(\mathbb{R}^N)} \leq C\|u\|_{L^p(\Omega)}$,

(iii) $\|Pu\|_{W^{1,p}(\mathbb{R}^N)} \leq C\|u\|_{W^{1,p}(\Omega)}$,

with $C \geq 0$ a constant that depends only on Ω. P is the extension operator we mentioned before.

Using this result, we may prove a density result regarding $W^{1,p}$ spaces.

Corollary 1.4.2. *Suppose Ω is of class C^1. Then, the restrictions to Ω of functions in $C_c^\infty(\mathbb{R}^N)$ form a dense subspace of $W^{1,p}(\Omega)$.*

Proof. Let $u \in W^{1,p}(\Omega)$. First, we will assume that Γ is bounded. Using the previous theorem, there exists an extension operator P. Let

$$u_n = \zeta_n(\rho_n \star Pu)$$

With ζ_n the cut-off functions previously mentioned, and ρ_n a sequence of mollifiers. Then, $u_n \in C_c^\infty(\mathbb{R}^N)$ and $u_{n|\Omega} \to u$ in $W^{1,p}(\Omega)$.

If, on the other hand, Γ is not bounded, we consider the sequence $\zeta_n u$. Then, $\zeta_n u \to u$ in $W^{1,p}(\Omega)$, so we may pick $n_0 \geq 1$ so that $\|\zeta_{n_0} u - u\|_{W^{1,p}} < \epsilon$. Using the case where Γ is bounded, we can construct an extension $v \in W^{1,p}(\mathbb{R}^N)$ of $\zeta_{n_0} u$. Finally, using (1.2.3) we pick $w \in C_c^\infty(\mathbb{R}^N)$ such that $\|w - v\|_{W^{1,p}(\mathbb{R}^N)} < \epsilon$. Then,

$$\begin{aligned} \|w_{|\Omega} - u\|_{W^{1,p}(\Omega)} &\leq \|w_{|\Omega} - \zeta_{n_0} u\|_{W^{1,p}(\Omega)} + \|\zeta_{n_0} u - u\|_{W^{1,p}(\Omega)} \\ &\leq \|w - v\|_{W^{1,p}(\mathbb{R}^N)} + \epsilon < 2\epsilon \end{aligned}$$

□

The following corollary generalizes a classical result of C^1 functions: if u is of class C^1 and its partial derivatives vanish in an open connected set U, then u is constant on U.

Corollary 1.4.3. *Let $\Omega \subset \mathbb{R}^N$ be a domain such that $\Omega = \mathbb{R}^N$, or Ω is of class C^1 with Γ bounded. If $u \in W^{1,p}(\Omega)$ satisfies that*

$$u_{x_i} = 0 \quad \text{on } U \subset \Omega, \quad \forall\ 1 \leq i \leq N$$

with $U \subset \Omega$ an open connected set, then $u_{|U}$ is constant.

• *Proof.* Assume that $\Omega = \mathbb{R}^N$ first. Let $\{\rho_n\}$ be a sequence of mollifiers such that $\rho_n \star u \to u$ in $W^{1,p}(\mathbb{R}^N)$. Using Lemma (1.2.2),

$$(\rho_n \star u)_{x_i} = \rho_n \star (u_{x_i}) = 0 \quad \text{on } U, \quad \forall\ 1 \leq i \leq N, n \geq 1$$

Since $\rho_n \star u \in C^\infty(\mathbb{R}^N)$, $\rho_n \star u$ has to be constant on U, and taking into account that $(\rho_n \star u)_{|U} \to u_{|U}$, u is constant on U.

If $\Omega \subset \mathbb{R}^N$ is a domain of class C^1 with Γ bounded, using the Extension Operator Theorem (1.4.1), we extend u to a function in $W^{1,p}(\mathbb{R}^N)$ using the extension operator $P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$. Then, $(Pu)_{x_i|U} = u_{x_i|U} = 0$. Thus, using the case where $\Omega = \mathbb{R}^N$ we just proved, necessarily $Pu_{|U}$ has to be constant, and as a consequence $u_{|U}$ is constant. □

In numerous occasions, it is useful to embed Sobolev spaces into other spaces. Namely, it is important to know if we can embed a Sobolev space in some L^q space, or even in the space of continuous functions. Moreover, it is useful to determine when these embeddings are continuous or compact.

The dimension of the space will play a key role here, and the space where the Sobolev space is embedded will in general depend on the dimension of the space. Whether Ω is a proper subset of $\mathbb{R}^N$ or not will also be important.

1.4.1 When $\Omega = \mathbb{R}^N$

We will begin stating the following lemma, that will be used to prove a theorem by Sobolev, Gagliardo and Nirenberg:

Lemma 1.4.4. *Let $N \geq 2$, and set $f_1, \ldots, f_N \in L^{N-1}(\mathbb{R}^{N-1})$. Given $x \in \mathbb{R}^N$, we denote $\tilde{x}_i$ as*

$$\tilde{x}_i = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in \mathbb{R}^{N-1}$$

Then,

$$f(x) = f_1(\tilde{x}_1) \ldots f_N(\tilde{x}_N) \in L^1(\mathbb{R}^N)$$

And we have the estimate

$$\|f\|_{L^1(\mathbb{R}^N)} \leq \prod_{i=1}^{N} \|f_i\|_{L^{N-1}(\mathbb{R}^{N-1})}$$

Now, the following theorem gives us a first result about when a Sobolev space is included in a L^p space.

Theorem 1.4.5 (Sobolev, Gagliardo, Nirenberg)**.** *For each $1 \leq p < N$, let $p^\star$ be the unique number that is defined by $\frac{1}{p^\star} = \frac{1}{p} - \frac{1}{N}$. Then, we have the inclusion*

$$W^{1,p}(\mathbb{R}^N) \subset L^{p^\star}(\mathbb{R}^N)$$

Moreover, there exists a constant C that only depends on p and N, such that

$$\|u\|_{p^\star} \leq C\|\nabla u\|_p$$

Proof. Assume that $p = 1$ and $u \in C_c^1(\mathbb{R}^N)$ first. Then,

$$|u(x_1, x_2, \ldots, x_N)| = \left| \int_{-\infty}^{x_1} u_{x_1}(t, x_2, \ldots, x_N) dt \right| \leq \int_{-\infty}^{\infty} |u_{x_1}(t, x_2, \ldots, x_N)| dt \tag{1.5}$$

Let $f_i(\tilde{x_i}) = \int_{-\infty}^{\infty} |u_{x_i}(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_N)|dt$. Then, proceeding as in (1.5), we have that

$$|u(x_1, \ldots, x_N)| \leq f_i(\tilde{x_i}), \quad i = 1, \ldots, N$$

It follows, using the previous lemma, that

$$\int_{\mathbb{R}^N} |u(x)|^{N/(N-1)} dx \leq \prod_{i=1}^{N} \|f_i\|_{L^1(\mathbb{R}^{N-1})}^{1/(N-1)} = \prod_{i=1}^{N} \|u_{x_i}\|_{L^1(\mathbb{R}^N)}^{1/(N-1)}$$

That is,

$$\|u\|_{L^{N/(N-1)}}(\mathbb{R}^N) \leq \prod_{i=1}^{N} \|u_{x_i}\|_{L^1(\mathbb{R}^N)}^{1/N} \tag{1.6}$$

Which is precisely what we wanted to prove, since if $p = 1$, $p^\star = N/(N-1)$. For the general case $1 < p < N$ (although with $u \in C_c^1(\mathbb{R}^N)$), we proceed as follows. Let $m \geq 1$. Applying (1.6) to $|u|^{m-1}u$, we have that

$$\|u\|_{mN/(N-1)}^m \leq m \prod_{i=1}^{N} \||u|^{m-1} u_{x_i}\|_1^{1/N} \leq m \|u\|_{p'(m-1)}^{m-1} \prod_{i=1}^{N} \|u_{x_i}\|_p^{1/N} \tag{1.7}$$

Since m is arbitrary, we may pick m so that $mN/(N-1) = p'(m-1)$, obtaining $m = (N-1)p^\star/N$. Notice that $m \geq 1$ since $1 < p < N$. Then,

$$\|u\|_{p^\star} \leq m \prod_{i=1}^{N} \|u_{x_i}\|_p^{1/N}$$

And thus $\|u\|_{p^\star} \leq C\|\nabla u\|_p \quad \forall u \in C_c^1(\mathbb{R}^N)$. When $u \in W^{1,p}(\mathbb{R}^N)$, we use density to conclude the result. Take $\{u_n\} \subset C_c^1(\mathbb{R}^N)$ a sequence converging to u in $W^{1,p}(\mathbb{R}^N)$. We can assume that $u_n \to u$ a.e. Otherwise, just take a subsequence that meets this requirement.

Then, we obtain

$$\|u_n\|_{p^\star} \leq C\|\nabla u_n\|_p$$

The conclusion is immediate: Using Fatou's lemma, $u \in L^{p^\star}$ and $\|u\|_{p^\star} \leq C\|\nabla u\|_p$. □

Remark 1.4.1. In the theorem, we can take $C = C(p, N) = (N-1)p/(N-p)$. However, this is not the optimal constant. It is possible to calculate the optimal one, although the procedure is not simple at all. See G. Talenti [9] for a statement and proof of the corresponding result.

Corollary 1.4.6. *If* $1 \leq p < N$, *then*

$$W^{1,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N) \quad \forall q \in [p, p^\star]$$

with a continuous injection.

Proof. Let $q \in [p, p^\star]$. Write

$$\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p^\star}, \qquad \text{for some } \alpha \in [0,1]$$

It can be checked that $\|u\|_q \leq \|u\|_p^\alpha \|u\|_{p^\star}^{1-\alpha} \leq \|u\|_p + \|u\|_{p^\star}$. Young's inequality has been used in here. Therefore, using the theorem that was just proved,

$$\|u\|_q \leq C\|u\|_{W^{1,p}} \quad \forall u \in W^{1,p}(\mathbb{R}^N)$$

□

A good question that arises from the Sobolev, Gagliardo and Nirenberg is what happens when $p = N$. The following corollary answer this question:

Corollary 1.4.7. *The following embedding holds:*

$$W^{1,N}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N) \quad \forall q \in [N, +\infty)$$

Proof. As usual, we assume that $u \in C_c^1(\mathbb{R}^N)$ first. We can apply (1.7) with $p = N$, obtaining

$$\|u\|_{mN/(N-1)}^m \leq m\|u\|_{(m-1)N/(N-1)}^{m-1}\|\nabla u\|_N \quad \forall m \geq 1$$

Using Young's inequality,

$$\|u\|_{mN/(N-1)} \leq C(\|u\|_{(m-1)N/(N-1)} + \|\nabla u\|_N) \quad \forall m \geq 1$$

In the previous equation we can pick $m = N$. Then,

$$\|u\|_{N^2/(N-1)} \leq C\|u\|_{W^{1,N}}$$

Using Gagliardo–Nirenberg interpolaiton inequality (see L. Nirenberg [5]), we conclude that

$$\|u\|_q \leq C\|u\|_{W^{1,p}}$$

For every q such that $N \leq q \leq N^2/(N-1)$. We repeat the argument with $m = N+1, N+2, \ldots$ and we finally get

$$\|u\|_q \leq C\|u\|_{W^{1,N}} \quad \forall u \in C_c^1(\mathbb{R}^N)$$

For every $q \geq N$. Repeating the usual density argument, the corollary is proved. □

Lastly, we have the following embedding result by Morrey, for the case where $p > N$.

Proposition 1.4.8 (Morrey)**.** *If $p > N$, then*

$$W^{1,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$$

The injection is continuous, and moreover for every $u \in W^{1,p}(\mathbb{R}^N)$, and if we define α as $\alpha = 1 - N/p$, we have

$$|u(x) - u(y)| \leq C|x-y|^\alpha \|\nabla u\|_p \quad a.e.\ x, y \in \mathbb{R}^N \tag{1.8}$$

where C is constant and depends only on p and N.

For a proof of this theorem, see H. Brezis [1].

Let us emphasize an implication of the previous theorem. Let $\Lambda \subset \mathbb{R}^N$ be a set of zero measure such that the inequality (1.8) is satisfied in $\mathbb{R}^N \setminus \Lambda$. Then, we can extend the function $u_{|\mathbb{R}^N \setminus A}$ to a continuous function in $\mathbb{R}^N$, and given the fact that $\mathbb{R}^N \setminus \Lambda$ is dense in $\mathbb{R}^N$, this extension is unique. That is, we can replace u by a continuous representative.

Using repeatedly the theorems and corollaries that were stated previously, we obtaing the following corollary:

Corollary 1.4.9. *Let $m \in \mathbb{N}$ and $1 \leq p < \infty$. Then, we have the following continuous injections:*

$$\begin{aligned}
&W^{m,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N), \quad \textit{with } \frac{1}{q} = \frac{1}{p} - \frac{m}{N},\ \textit{if } \frac{1}{p} - \frac{m}{N} > 0 \\
&W^{m,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N), \quad \forall p \leq q < \infty,\ \textit{if } \frac{1}{p} - \frac{m}{N} = 0 \\
&W^{m,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N), \quad \textit{if } \frac{1}{p} - \frac{m}{N} < 0
\end{aligned} \tag{1.9}$$

Moreover, we have that $W^{m,p}(\mathbb{R}^N) \subset C^k(\mathbb{R}^N)$, with $k = [m - (N/p)]$ ($[\cdot]$ denotes the integer part).

1.4.2 When $\Omega \subset \mathbb{R}^N$

In what follows, Ω will be considered to be a domain of class C^1, with $\Gamma = \partial\Omega$ bounded, or $\Omega = \mathbb{R}^N_+$. In this section the Theorem (1.4.1) will play a crucial role. The general idea that will appear in the proofs of this section is to extend the functions of $W^{1,p}(\Omega)$ to a function of $W^{1,p}(\mathbb{R}^N)$ using Theorem (1.4.1), in order to use the results from the previous section.

Proposition 1.4.10. *For each $1 \leq p \leq \infty$, we have the following continuous injections:*

$$\begin{aligned} &W^{1,p}(\Omega) \subset L^{p^\star}(\Omega), \quad \textit{where } \frac{1}{p^\star} = \frac{1}{p} - \frac{1}{N}, \textit{ if } p < N \\ &W^{1,p}(\Omega) \subset L^{q}(\Omega), \quad \forall p \le q < \infty, \textit{ if } p = N \\ &W^{1,p}(\Omega) \subset L^{\infty}(\Omega), \quad \textit{if } p > N \end{aligned} \tag{1.10}$$

Proof. Using Theorem (1.4.1), we take the extension operator $P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$. Then, we apply the different results from the previous section where we studied the case of $\mathbb{R}^N$, in order to conclude the desired results after a restriction to Ω. □

Corollary 1.4.11. *If $p > N$, for every $u \in W^{1,p}(\Omega)$ we have*

$$|u(x) - u(y)| \le C\|u\|_{W^{1,p}}|x - y|^\alpha \ \textit{a.e. } x, y \in \Omega \tag{1.11}$$

with $\alpha = 1 - N/p$, and C is a constant depending on Ω, p and N. Thus, $W^{1,p} \subset C(\overline{\Omega})$.

Proof. In a similar way as in the proof of the previous theorem, take the extension operator P and apply Morrey's Theorem (1.8). □

Corollary 1.4.12. *The conclusions of Corollary* (1.4.9) *are still true if $\mathbb{R}^N$ is replaced by Ω.*

The following result shows different cases in which the injection is compact, instead of just continuous. It will be very useful in the second part of the Dissertation, where PDEs will be studied.

Theorem 1.4.13 (Rellich–Kondrachov)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^1. Then, the following injections are compact:*

$$\begin{aligned} &W^{1,p}(\Omega) \subset L^{q}(\Omega) \quad \forall q \in [1, p^\star), \quad \textit{where } \frac{1}{p^\star} = \frac{1}{p} - \frac{1}{N}, \textit{ if } p < N \\ &W^{1,p}(\Omega) \subset L^{q}(\Omega) \quad \forall q \in [p, +\infty), \textit{ if } p = N \\ &W^{1,p}(\Omega) \subset C(\overline{\Omega}) \quad \textit{ if } p > N \end{aligned} \tag{1.12}$$

Proof. Let $p > N$. Let $\mathcal{H}$ be the unit ball in $W^{1,p}(\Omega)$. Using Proposition (1.4.10) the injection $W^{1,p}(\Omega) \subset L^\infty(\Omega)$ is continuous. Thus, there exists a constant $M > 0$ such that $\|u\|_\infty \le M\|u\|_{W^{1,p}}$. Therefore, the set $\mathcal{H}$ is bounded. Now, we will prove that $\mathcal{H}$ is equicontinuous. Given $\epsilon > 0$, set $\delta = (\epsilon/C)^{1/\alpha}$, where C is the constant mentioned by (1.11). Then, for each $x, y \in \Omega$ such that $|x - y| < \delta$,

$$|u(x) - u(y)| \le C\|u\|_{W^{1,p}}|x - y|^\alpha \le \epsilon, \quad \forall u \in \mathcal{H}$$

So, indeed, $\mathcal{H}$ is an equicontinuous family. Using Ascoli–Arzelà's theorem, $\mathcal{H}$ has compact closure and therefore the the injection is compact.

The case $p = N$ reduces to the case $p < N$, so we will study the case where $p < N$ now.

Again, we denote by $\mathcal{H}$ the unit ball in $W^{1,p}(\Omega)$. Using the Theorem (1.4.1), we consider the extension operator P. Set $\mathcal{F} = P(\mathcal{H})$, so that $\mathcal{H} = \mathcal{F}_{|\Omega}$. We will use Kolmogorov–Riesz compactness theorem to prove that $\mathcal{H}$ has compact closure in $L^p(\Omega)$, for $q \in [1, p^\star)$. We can assume that $q \geq p$, since Ω is bounded. Clearly, $\mathcal{F}$ is bounded in $W^{1,p}(\mathbb{R}^N)$ (the prove is similar to the case $p > N$), and therefore it is also bounded in $L^q(\mathbb{R}^N)$, by Corollary (1.4.6). In order to use Kolmogorov–Riesz's theorem, we have to check that

$$\lim_{|h|\to 0} \|\tau_h f - f\|_{L^q(\mathbb{R}^N)} = 0 \text{ uniformly in } f \in \mathcal{F} \tag{1.13}$$

Using Proposition (1.2.4),

$$\|\tau_h f - f\|_{L^p(\mathbb{R}^N)} \leq |h| \|\nabla f\|_{L^p(\mathbb{R}^N)} \quad \forall f \in \mathcal{F}$$

We write

$$\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p^\star} \quad \text{for some} \quad \alpha \in (0, 1]$$

Since $p \leq q < p^\star$. Using Gagliardo–Nirenberg interpolation inequality we conclude that

$$\begin{aligned} \|\tau_h f - f\|_{L^q(\mathbb{R}^N)} &\leq \|\tau_h f - f\|^{\alpha}_{L^p(\mathbb{R}^N)} \|\tau_h f - f\|^{1-\alpha}_{L^{p^\star}(\mathbb{R}^N)} \\ &\leq |h|^\alpha \|\nabla f\|^\alpha_{L^p(\mathbb{R}^N)} (2\|f\|_{L^{p^\star}(\mathbb{R}^N)})^{1-\alpha} \leq C|h|^\alpha \end{aligned} \tag{1.14}$$

Where C is a constant that does not depend on $\mathcal{F}$, because as we have proved $\mathcal{F}$ is bounded in $W^{1,p}$. Then, (1.13) holds, and thus using Kolmogorov–Riesz's compactness theorem the injection is compact. □

• **Corollary 1.4.14.** *Let* $\{u_n\}$ *be a bounded sequence in* $W^{1,p}(\Omega)$, $1 \leq p < \infty$, *such that* $u_n \rightharpoonup u$ *weakly on* $W^{1,p}(\Omega)$, *with* Ω *bounded and of class* C^1. *Then, there exists a subsequence that converges strongly to* u *in* $L^p(\Omega)$, *and in particular there exists a subsequence that converges a.e. to* u.

• *Proof.* Using Theorem (1.13), the injection $W^{1,p}(\Omega) \subset L^p(\Omega)$ is compact. That is, the injection operator $i : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact. Thus, and taking into account that $\{u_n\}$ is bounded, $i(u_n) = u_n$ has a convergent subsequence in $L^p(\Omega)$. Therefore, due to the uniqueness of the limit, we must have that $u_n \to u$ strongly in $L^p(\Omega)$. Moreover, since $u_n \to u$ in $L^p(\Omega)$, we can extract again a subsequence that converges a.e. to u. □

1.5 $W_0^{m,p}(\Omega)$ space and its dual

Definition 1.5.1. We denote by $W_0^{1,p}(\Omega)$, with $1 \leq p < \infty$, the closure of $C_c^1(\Omega)$ in $W^{1,p}(\Omega)$. We write $H_0^1(\Omega) := W_0^{1,2}(\Omega)$. We can equip the space $W_0^{1,p}$ with the norm of $W^{1,p}$. Then, the space is a separable Banach space, and it is reflexive for $1 < p < \infty$. If $p = 2$, the space H_0^1 is a Hilbert space, equipped with the scalar product of H^1. An immediate observation is that given the fact that $C_c^1(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$, we obtain $W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$.

Similarly, we define de space $W_0^{m,p}(\Omega)$ as the closure of $C_c^m(\Omega)$ in $W_0^{m,p}(\Omega)$.

Intuitively, a function of $W_0^{1,p}(\Omega)$ *vanishes* on $\Gamma = \partial\Omega$. This is not accurate at all, since a function $u \in W^{1,p}(\Omega)$ is well defined up to a set of zero measure (it is defined a.e.), so it does not make sense to say that u vanishes on Γ. However, the next result will formalize this idea. Similarly, we can think of a function $u \in W_0^{m,p}(\Omega)$ as a function $u \in W^{m,p}$ such tat $D^\alpha u = 0$ on Γ, for every multi-index α with $|\alpha| \leq m - 1$.

Theorem 1.5.1. *Let Ω be a domain of class C^1, and let $1 \leq p < \infty$. Set $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. Then, $u = 0$ on Γ if and only if $u \in W_0^{1,p}(\Omega)$.*

The proof will be omitted[2]. This kind of results belong to the Theory of traces. Roughly speaking, the trace of u on Γ, denoted by $u_{|\Gamma}$, is a linear operator, defined in an appropiate way, from $W^{1,p}(\Omega)$ into $L^p(\Gamma)$.

A corollary of this result is Poincaré's inequality, that estimates the norm of a function $u \in W_0^{1,p}(\Omega)$ in terms of its gradient:

Corollary 1.5.2 (Poincaré's inequality)**.** *Let $1 \leq p < \infty$, and Ω a bounded domain. Then, there exists a constant C such that*

$$\|u\|_p \leq C\|\nabla u\|_p \quad \forall u \in W_0^{1,p}(\Omega)$$

The constant C depends only on Ω and p. Therefore, $\|\nabla u\|_{L^p(\Omega)}$ is a norm on $W_0^{1,p}(\Omega)$ equivalent to (1.1). For the Hilbert space $H_0^1(\Omega)$, the scalar product $\sum_i \int_\Omega u_{x_i} v_{x_i}$ induces the norm $\|\nabla u\|_2$, and it is equivalent to $\|u\|_{H^1}$.

The dual of $W_0^{1,p}(\Omega)$ will be denoted by $W^{-1,p'}(\Omega)$. If $p = 2$, we write $H^{-1}(\Omega) := W^{-1,2}(\Omega)$. If Ω is bounded, we have the following continuous and dense injections:

$$W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,p'}(\Omega) \quad \text{if } 2N/(N+2) \leq p < \infty$$

If Ω is not bounded the injections are still continuous and dense, but only if $2N/(N+2) \leq p \leq 2$. Therefore, in particular

[2]See H. Brezis [1] for a proof of the theorem.

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$$

With continuous and dense injections, for every domain $\Omega \subset \mathbb{R}^N$. The following proposition gives a better insight of the elements of $W^{-1,p'}$.

Proposition 1.5.3. *Given $f \in W^{-1,p'}(\Omega)$, there exist $f_0, f_1, \ldots, f_N \in L^{p'}(\Omega)$ such that*

$$\langle f, v\rangle = \int_\Omega f_0 v + \sum_{i=1}^N \int_\Omega f_i v_{x_i} \quad \forall v \in W_0^{1,p}(\Omega)$$

and $\|f\| = \max_{0\le i\le N}\|f_i\|_{p'}$. Moreover, we may pick $f_0 = 0$ if Ω is bounded.

Proof. Set $E = L^p(\Omega)^{N+1}$. E is a Banach space equipped with the norm

$$\|h\| = \sum_{i=0}^N \|h_i\|_p, \quad h = (h_0, h_1, \ldots, h_N)$$

We define the following map:

$$\begin{aligned} T : W_0^{1,p}(\Omega) &\longrightarrow E \\ u &\mapsto (u, u_{x_1}, u_{x_2}, \ldots, u_{x_N}) \end{aligned} \tag{1.15}$$

Taking into account (1.1), T is a isometry. Set $G = T(W_0^{1,p}(\Omega))$, and set $S = T^{-1} : G \to W_0^{1,p}(\Omega)$. The map $h \in G \mapsto \langle f, Sh\rangle$ is a bounded linear functional on G. We may use Hahn–Banach theorem now, and extend it to a bounded linear functional Φ that is defined on all of E, and $\|\Phi\|_{E^\star} = \|F\|$.

Using Riesz representation theorem, there exist functions $f_0, f_1, \ldots, f_N \in L^{p'}$ such that

$$\langle \Phi, h\rangle = \sum_{i=0}^N \int_\Omega f_i h_i \quad \forall h \in E$$

Moreover, $\|\Phi\|_{E^\star} = \max_{0\le i\le N}\|f_i\|_{p'}$. Therefore, if $u \in W_0^{1,p}$,

$$\langle \Phi, Tu\rangle = \langle f, u\rangle = \int_\Omega f_0 u + \sum_{i=1}^N \int_\Omega f_i u_{x_i} \tag{1.16}$$

If Ω is bounded we may equip the space $W_0^{1,p}(\Omega)$ with the following norm:

$$\|u\|_{W^{1,p}} = \sum_{i=1}^N \|u_{x_i}\|_p$$

and repeating the same argument with $E = L^p(\Omega)^N$, we conclude that we can take $f_0 = 0$ in (1.16). $\square$

Chapter 2

Partial Differential Equations

Sobolev Spaces are a powerful tool that can be used to study various properties of Partial Differential Equations. This chapter will be focused on the solvability of mainly second-order linear partial differential equations, with both Dirichlet and Neumann boundary conditions. A distinction will be made between different types of PDEs, and each of them will be studied separately.

The linear second-order partial differential operator will be used frequently, which will be denoted by L. This operator may have the divergence form

$$L \equiv -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a^{ij}(x)\frac{\partial}{\partial x_i}\right) + \sum_{i=1}^{N} b^i(x)\frac{\partial}{\partial x_i} + c(x) \tag{2.1}$$

or in the nondivergence form

$$L \equiv -\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x)\frac{\partial}{\partial x_i} + c(x) \tag{2.2}$$

Notice that if the coefficients a^{ij} are of class C^1, we may write an operator in divergence form (2.1) in the nondivergence form (2.2), and reciprocally. We will assume that the coefficients a^{ij} are symmetric, in the sense that $a^{ij} = a^{ji}$ for $1 \leq i, j \leq N$.

2.1 Second-order elliptic equations

This chapter will study the equation $Lu = f$ in a certain open and bounded domain $\Omega \subset \mathbb{R}^N$, with some boundary conditions on $\partial\Omega$, where $f : \Omega \to \mathbb{R}$ is a function in a certain space of functions.

Throughout this section, we will assume that the operator L satisfies the uniform ellipticity condition, that is, there exists a certain constant $\theta > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \quad \text{a.e. } x \in \Omega \tag{2.3}$$

An example of a partial differential operator L that satisfies the ellipticity condition is the Laplacian changed of sign, that is, $L = -\Delta$. In this case, $b^i \equiv 0$ for $i = 1, \dots, N$, $c \equiv 0$ and $a^{ij} = 0$ if $i \neq j$, and $a^{ii} = 1$. Then, the condition (2.3) is satisfied trivially.

Remark 2.1.1. The uniform ellipticity condition can be seen as imposing that the matrix $(a^{ij}(x))$ is positive definite a.e. $x \in \Omega$, and its smallest eigenvalue is greater or equal to θ.

2.1.1 Weak solutions of the Dirichlet problem

Our objective now is to define a weak solution of the problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

In order to motivate the definition weak solutions of (2.4), we will first assume that the coefficients a^{ij}, b and c as well as f and u are smooth enough. Then, multiplying the PDE by a function $\varphi \in C_c^\infty(\Omega)$ (such functions are usually called *test functions*), integrating over Ω and integrating by parts, we obtain

$$\int_\Omega \left(\sum_{i,j=1}^{N} a^{ij} u_{x_i} \varphi_{x_j} + \sum_{i=1}^{N} b^i u_{x_i} \varphi + cu\varphi \right) = \int_\Omega f\varphi$$

Notice that this equality still makes sense when $u, \varphi \in H_0^1(\Omega)$. Moreover, the homogeneous Dirichlet boundary condition is implicit in the choice of the space $H_0^1(\Omega)$. Thus, we can introduce the following definitions:

Definition 2.1.1. The bilinear form B is defined as follows:

$$B : H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow \mathbb{R}$$

$$B[u, \varphi] := \int_\Omega \left(\sum_{i,j=1}^{N} a^{ij} u_{x_i} \varphi_{x_j} + \sum_{i=1}^{N} b^i u_{x_i} \varphi + cu\varphi \right)$$

Definition 2.1.2. A function $u \in H_0^1(\Omega)$ will be called a *weak solution* of (2.4) if $B[u, \varphi] = (f, \varphi)$ for every test function $\varphi \in H_0^1(\Omega)$. Here, $(\cdot, \cdot)$ denotes the usual inner product in $L^2(\Omega)$.

If we have the more general problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{2.5}$$

In many cases we may suppose that $g \equiv 0$. Indeed, assume that $u \in H^1(\Omega)$ is a weak solution of the inhomogeneous problem and $\partial\Omega$ is C^1. Suppose that there exists a function $\tilde{g} \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $\tilde{g} = g$ on $\partial\Omega$. Then, $v = u - g \in H_0^1(\Omega)$ is a weak solution of

$$\begin{cases} Lv = \tilde{f} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{2.6}$$

With $\tilde{f} = f - Lg$. Thus, we will only consider the case where $g \equiv 0$.

Existence and uniqueness of weak solutions using Lax–Milgram Theorem

In order to obtain the first results about weak solutions, we will first introduce some estimates of the bilinear form $B[\cdot,\cdot]$, known as *energy estimates.* They will play a key role when it comes to the proof of existence of weak solutions.

Proposition 2.1.1 (Energy estimates)**.** *Let $B[\cdot,\cdot]$ be the bilinear form defined in Definition* (2.1.1)*. Then,*

$$|B[u,\varphi]| \leq \alpha \|u\|_{H_0^1(\Omega)} \|\varphi\|_{H_0^1(\Omega)}$$

for every $u, \varphi \in H_0^1(\Omega)$, where $\alpha > 0$ is a certain constant. Moreover, there exist constants $\beta > 0$ and $\gamma \geq 0$ such that for every $u, \varphi \in H_0^1(\Omega)$,

$$\beta \|u\|^2_{H_0^1(\Omega)} \leq B[u,u] + \gamma \|u\|^2_{L^2(\Omega)}$$

Proof. We will first prove the former inequality of the proposition. From the definition of $B[\cdot,\cdot]$ and taking into account Hölder's inequality, it follows that for each $u, \varphi \in H_0^1(\Omega)$,

$$\begin{aligned} |B[u,\varphi]| &\leq \sum_{i,j=1}^{N} \|a^{ij}\|_\infty \int_\Omega |\nabla u||\nabla\varphi| + \sum_{i=1}^{N} \|b^i\|_\infty \int_\Omega |\nabla u||\varphi| + \|c\|_\infty \int_\Omega |u||\varphi| \\ &\leq \max\left\{ \sum_{i,j=1}^{N} \|a^{ij}\|_\infty, \sum_{i=1}^{N} \|b^i\|_\infty, \|c\|_\infty \right\} \left(\int_\Omega (|\nabla u||\nabla\varphi| + |\nabla u||\varphi| + |u||\varphi|) \right) \\ &\leq \alpha \|u\|_{H_0^1(\Omega)} \|\varphi\|_{H_0^1(\Omega)} \end{aligned} \tag{2.7}$$

for some suitable $\alpha > 0$, and therefore the first inequality of the proposition is proved.

We shall prove the second ineqality now. The uniform ellipticity hypothesis (2.3) will be used in this part. Let $\theta > 0$ be the constant mentioned in the definition of uniform ellipticity (2.3). Then, given $u \in H_0^1(\Omega)$ we apply (2.3) with $\xi = \nabla u$:

$$\theta|\nabla u|^2 \leq \sum_{i,j=1}^{N} a^{ij} u_{x_i} u_{x_j}$$

Integrating over Ω we obtain that

$$\theta \int_\Omega |\nabla u|^2 \leq \int_\Omega \sum_{i,j}^{N} a^{ij} u_{x_i} u_{x_j} = B[u,u] - \int_\Omega \left(\sum_{i=1}^{N} b^i u_{x_i} u + cu^2 \right)$$

$$\leq B[u,u] + \sum_{i=1}^{N} \|b^i\|_\infty \int_\Omega |\nabla u||u| + \|c\|_\infty \int_\Omega u^2 \tag{2.8}$$

Recall that Cauchy's inequality for a given $\epsilon > 0$ states that for each $a, b > 0$,

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}$$

Applying this inequality with $a = |\nabla u|$ and $b = |u|$ and integrating over Ω, we get that

$$\int_\Omega |\nabla u||u| \leq \int_\Omega \left(\epsilon |\nabla u|^2 + \frac{1}{4\epsilon} u^2 \right)$$

Using this last inequality in (2.8) with ϵ such that

$$0 < \epsilon < \frac{\theta}{2 \sum_{i=1}^{n} \|b^i\|_\infty}$$

We conclude that for an appropiate constant C,

$$\frac{\theta}{2} \int_\Omega |\nabla u|^2 \leq B[u,u] + C \int_\Omega u^2$$

From Poincaré's inequality (1.5.2), we know that $\|u\|_{L^2(\Omega)} \leq C' \|\nabla u\|_{L^2(\Omega)}$ for a constant $C' > 0$. Thus, we finally obtain that for some suitable constants $\beta > 0, \gamma \geq 0$,

$$\beta \|u\|^2_{H_0^1(\Omega)} \leq B[u,u] + \gamma \|u\|^2_{L^2(\Omega)}$$

□

Before proving the first result on existence and uniqueness of weak solutions, we shall introduce a powerful tool from Hilbert spaces first, known as the Lax–Milgram Theorem.

Theorem 2.1.2 (Lax–Milgram Theorem)**.** *Let H be a real Hilbert space. Let $B : H \times H \longrightarrow \mathbb{R}$ be a bilinear form such that*

$$|B[u,v]| \leq \alpha \|u\| \|v\|, \quad \forall u, v \in H$$

$$\beta \|u\|^2 \leq B[u,u], \quad \forall u \in H$$

for some constants $\alpha, \beta > 0$. Then, for every $\varphi \in H^\star$, there exists a unique element $u \in H$ such that

$$B[u,v] = \langle \varphi, v \rangle, \quad \forall v \in H$$

Moreover, if B is symetric, we may characterize u as the unique $u \in H$ such that

$$\frac{1}{2} B[u,u] - \langle \varphi, u \rangle = \min_{v \in H} \left\{ \frac{1}{2} B[v,v] - \langle \varphi, v \rangle \right\}$$

For a proof of the theorem, see H. Brezis [1]. Now, we may state and prove our first result on existence and uniqueness of weak solutions:

Proposition 2.1.3. *There exists a number $\gamma \geq 0$ such that for each $\mu \geq \gamma$ and $f \in L^2(\Omega)$, the boundary-value problem*

$$\begin{cases} Lu + \mu u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.9}$$

has a unique weak solution $u \in H_0^1(\Omega)$.

Proof. Let γ be the constant from Theorem (2.1.1). Then, for every $\mu \geq \gamma$ we define

$$B_\mu[u,\varphi] = B[u,\varphi] + \mu(u,\varphi) \quad \forall u, \varphi \in H_0^1(\Omega)$$

Which corresponds to the second-order partial differential operator $L_\mu u := Lu + \mu u$. Here, $(\cdot,\cdot)$ denotes de inner product in $L^2(\Omega)$. Then, B_μ satisfies the hypotheses of the Lax–Milgram Theorem, due to the energy estimates. We define now the bounded linear functional φ_f given by $\langle \varphi_f, v \rangle := (f,v)$. Applying Lax–Milgram Theorem, there exists a unique function $u \in H_0^1(\Omega)$ such that $B_\mu[u,v] = \langle \varphi_f, v \rangle$, for all $v \in H_0^1(\Omega)$. Thus, $B_\mu[u,v] = (f,v)$ for every $v \in H_0^1(\Omega)$ and hence u is the unique weak solution we were looking for. □

• **Remark 2.1.1.** Notice that if $b^i \equiv 0$ for each $i = 1, \ldots, N$, the bilinear form $B[\cdot, \cdot]$ is symmetric. Thus, we may make use of the characterization of the weak solution u that provides the Lax–Milgram Theorem. That is, the unique solution of (2.9) will be characterized as the unique $u \in H_0^1(\Omega)$ such that

$$\frac{1}{2}B[u,u] - (f,u) = \min_{v \in H_0^1(\Omega)} \left\{ \frac{1}{2}B[v,v] - (f,v) \right\}$$

That is, u is the unique function of $H_0^1(\Omega)$ at which the previous minimum is achieved. Even more, from this characterization and the energy estimates, we deduce that there exist constants $\beta > 0$ and $\gamma \geq 0$ such that

$$\beta \|u\|_{H_0^1(\Omega)^2} \leq (f,u) + \gamma \|u\|_{L^2(\Omega)}^2$$

This estimate is easily deduced inserting $v = 0$ in the previous characterization, obtaining therefore that $B[u,u] \leq 2(f,u)$, and using this inequality in the energy estimates.

Existence of solutions using fixed point theorems

There is another approach of proving the existence, but not uniqueness, of weak solutions of the homogeneous Dirichlet problem. This approach is based on a fixed point theorem, called the Schauder's fixed point theorem:

Theorem 2.1.4 (Schauder's fixed point theorem)**.** *Let X be a Banach space, $f : X \to X$ a continuous compact map and assume that*

$$F = \{x \in X \big| x = \lambda f(x) \text{ for some } \lambda \in [0,1]\}$$

is bounded. Then f has a fixed point.

See V. Pata [10] for a proof of this theorem. We will use this theorem to prove existence of the homogeneous Dirichlet problem. In fact, our objective will be the proof of existence of solutions of the following non-linear problem:

$$\begin{cases} Lu + B(u) = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.10}$$

Under some hypothesis for the mapping B and Ω. First, we shall prove the following lemma:

Lemma 2.1.5. *Let X, V be Banach spaces with compact and dense embeddings $V \hookrightarrow X \hookrightarrow V^\star$. Assume we are given a bounded, surjective linear operator $A : V \longrightarrow V^\star$ and a, possibly nonlinear, continuous map $B : X \longrightarrow V^\star$ that carries bounded sets into bounded sets, such that*

$$\langle Au, u \rangle \geq \epsilon \|u\|_V^2 \quad \forall \epsilon \in V \tag{2.11}$$

and

$$\langle B(u), u\rangle \geq -c(1+\|u\|_V^{\alpha}) \quad \forall u \in X \tag{2.12}$$

for some $\epsilon > 0, c \geq 0$ *and* $\alpha \in [0,2)$. *Then, the equation*

$$Au + B(u) = g \tag{2.13}$$

with $g \in V^{\star}$, *admits a solution* $u \in V$.

Proof. From (2.11), A is injective and thus bijective. By the open mapping theorem, $A^{-1} \in \mathcal{L}(V^{\star}, V)$. Therefore, for each $v \in V$, we may define

$$w = A^{-1}(f - B(v)) \in V$$

which is a solution of $Aw = g - B(v)$. Let $f : X \to X$ be defined as $f(v) = A^{-1}(g - B(v))$. Notice that f is continuous and compact. Suppose there exists $u_\lambda \in X$ such that $u_\lambda = \lambda f(u_\lambda)$, with $\lambda \in [0,1]$. Then

$$Au_\lambda + \lambda B(u_\lambda) = \lambda g$$

Therefore, $\langle Au_\lambda, u_\lambda\rangle + \lambda\langle B(u_\lambda), u_\lambda\rangle = \lambda\langle g, u_\lambda\rangle$, so using (2.11) and (2.12),

$$\lambda\|g\|_{V^\star}\|u\|_V \geq \langle g, u_\lambda\rangle \geq \epsilon\|u_\lambda\|_V^2 - \lambda c(1+\|u_\lambda\|_V^{\alpha})$$

Thus,

$$\epsilon\|u_\lambda\|_V^2 \leq \lambda c(1+\|u_\lambda\|_V^{\alpha}) + \lambda\|g\|_{V^\star}\|u_\lambda\|_V$$

Recall Young's inequality

$$ab \leq \nu a^p + K(\nu, p) b^q$$

Where $a, b \geq 0, \nu > 0$, $K(\nu, p) = (\nu p)^{-q/p} q^{-1}$, $1 < p, q < \infty$ and $1/p + 1/q = 1$. Therefore, we obtain the following estimate

$$\|u_\lambda\|_V^2 \leq \frac{2}{\epsilon}\left(c + K(\epsilon/4, 2/\alpha)c^{2/(2-\alpha)} + \frac{1}{\epsilon}\|g\|_{V^\star}^2\right)$$

Using Schauder's fixed point theorem, f has a fixed point u, which is clearly a solution of (2.13).

□

Notice that this proof also provides an estimate of the solution u. We shall now prove the following existence result:

• **Proposition 2.1.6.** *Let $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ be a bounded domain with smooth boundary $\partial\Omega$. If $N = 2$, let $q \geq 2$, and if $N > 2$, let $q \in [1, 6)$. Let $B : L^q(\Omega) \to H^{-1}(\Omega)$ be a continuous map that carries bounded sets into bounded sets, such that*

$$\langle B(u), u\rangle_{H^{-1}(\Omega)} \geq -c(1 + \|u\|^{\alpha}_{H^1_0(\Omega)}) \quad \forall u \in L^q(\Omega)$$

For some $c \geq 0$, $\alpha \in [0, 2)$. Then, the nonlinear elliptic problem

$$\begin{cases} Lu + B(u) = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.14}$$

with $g \in H^{-1}(\Omega)$ admits a weak solution $u \in H^1_0(\Omega)$.

• *Proof.* Using Rellich-Kondrachov's Theorem (1.4.13), and given the hypothesis on q, the embedding $H^1_0(\Omega) \hookrightarrow L^q(\Omega)$ is compact. $A = L$ satisfies the hypotheses of the preceeding Lemma. Using the previous Lemma we conclude that (2.14) admits a weak solution.

□

• **Corollary 2.1.7.** *Let $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, be a bounded domain with smooth boundary. Given $1 \leq n \leq 5$, or $n \geq 1$ if $N = 2$, the nonlinear elliptic problem*

$$\begin{cases} Lu + u^n = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $g \in H^{-1}(\Omega)$, admits a weak solution.

Proof. It follows from the preceeding Proposition, using $q = n + 1$. □

• **Corollary 2.1.8.** *Let $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ be a bounded domain with smooth boundary. Then, there exists a weak solution of*

$$\begin{cases} Lu + u = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

For each $g \in H^{-1}(\Omega)$.

Regularity

Now, we will study if given a function $u \in H^1(\Omega)$ such that $Lu = f$, we can show that u is in fact more regular than just $u \in H^1(\Omega)$. We will have to require some extra conditions in order to assure this, both on the coefficients a^{ij}, b^i and c, as well as on the function f.

The following theorem shows a first result on regularity of weak solutions.

Theorem 2.1.9. *Let $u \in H^1(\Omega)$ be a weak solution of $Lu = f$ in a certain bounded and open set $\Omega \subset \mathbb{R}^N$. Assume that the coefficients a^{ij}, b^i and c satisfy*

$$a^{ij} \in C^1(\Omega), \quad b^i, c \in L^\infty(\Omega)$$

for each $1 \leq i, j \leq N$ and $f \in L^2(\Omega)$. Then, we in fact have that $u \in H^2_{loc}(\Omega)$. Moreover,

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)})$$

for every open subset $V \subset\subset \Omega$, where C is a constant depending only on V, Ω, a^{ij}, b^i and c.

The proof of the theorem is quite long and it has many technical steps (see L. C. Evans [2] for a proof). It uses the *difference quotients* $D^h_k u$, where $h \in \mathbb{R} \setminus \{0\}$ and $1 \leq k \leq N$, which are defined as

$$D^h_k u(x) = \frac{u(x + he_k) - u(x)}{h}$$

The general idea of the proof consists on rewriting $B[u, \varphi] = (f, \varphi)$, with u a weak solution of $Lu = f$, as

$$\sum_{i,j=1}^{N} \int_\Omega a^{ij} u_{x_i} \varphi_{x_j} = \int_\Omega \tilde{f} \varphi$$

With $\tilde{f} = f - \sum_{i=1}^N b^i u_{x_i} - cu$. Then, some estimates are found for both sides of the previous equality, in order to use these estimates to prove that $\nabla u \in H^1_{\text{loc}}(\Omega; \mathbb{R}^N)$. The fact that $u \in H^2_{\text{loc}}(\Omega)$ follows immediately.

This theorem can be used in order to prove higher regularity of weak solutions, making more assumptions on the coefficients of L and f. Our objective will be to use the previous theorem iteratively, in order to prove the mentioned regularity result:

Theorem 2.1.10. *Let m be a nonnegative integer. Suppose that $a^{ij}, b^i, c \in C^{m+1}(\Omega)$ for $1 \leq i, j \leq N$, and assume that $f \in H^m(\Omega)$. If $u \in H^1(\Omega)$ is a weak solution of $Lu = f$, then in fact $u \in H^{m+2}_{loc}(\Omega)$, and moreover for each $V \subset\subset \Omega$ there exists a constant C, which depends only on m, Ω, V and the coefficients of L, such that*

$$\|u\|_{H^{m+2}(V)} \leq C(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)})$$

Proof. If $m = 0$, we are in the case of the previous theorem, so the result is proved. We will use induction to prove the general statement. Assume that the result is valid for a nonnegative integer m and all open sets Ω. Now, suppose that $a^{ij}, b^i, c \in C^{m+2}(\Omega)$ and $f \in H^{m+1}(\Omega)$. Let $u \in H^1(\Omega)$

be a weak solution of $Lu = f$. By induction, it follows immediately that $u \in H^{m+2}_{\text{loc}}(\Omega)$, and moreover we have the estimate

$$\|u\|_{H^{m+2}(W)} \leq C(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)}) \tag{2.15}$$

for any open set $W \subset\subset \Omega$ and a constant C depending on m, Ω, W and the coefficients of L. Given $V \subset\subset \Omega$, fix an open set W such that $V \subset\subset W \subset\subset \Omega$. Take now a multi-index α with $|\alpha| = m+1$, and a test function $\tilde{\varphi} \in C_c^\infty(W)$. Set $\varphi := (-1)^{|\alpha|} D^\alpha \tilde{\varphi}$ and plug it into the equality $B[u, \varphi] = (f, \varphi)_{L^2(\Omega)}$. Then, after integrating by parts, we obtain that

$$B[\tilde{u}, \tilde{\varphi}] = (\tilde{f}, \tilde{\varphi})$$

with $\tilde{u} = D^\alpha u \in H^1(W)$, and

$$\begin{aligned} \tilde{f} = D^\alpha f - \sum_{\beta \leq \alpha, \beta \neq \alpha} \binom{\alpha}{\beta} \Bigg[&- \sum_{i,j=1}^N (D^{\alpha-\beta} a^{ij} D^\beta u_{x_i})_{x_j} \\ &+ \sum_{i=1}^N D^{\alpha-\beta} b^i D^\beta u_{x_i} + D^{\alpha-\beta} c D^\beta u \Bigg] \end{aligned} \tag{2.16}$$

Since $B[u, \tilde{\varphi}] = (\tilde{f}, \tilde{\varphi})$ holds for every $\tilde{\varphi} \in C_c^\infty(W)$, then $\tilde{u}$ is a weak solution of $L\tilde{u} = \tilde{f}$.

Since $a^{ij}, b^i, c \in C^{m+2}(\Omega)$, $f \in H^{m+1}(\Omega)$, $u \in H^{m+2}_{\text{loc}}(\Omega)$, the estimate (2.15) and the previous definition of $\tilde{f}$, we obtain that

$$\|\tilde{f}\|_{L^2(W)} \leq C(\|f\|_{H^{m+1}(\Omega)} + \|u\|_{L^2(\Omega)})$$

and thus $\tilde{f} \in L^2(W)$. Using the previous regularity theorem, and since as we mentioned $\tilde{u}$ is a weak solution of $L\tilde{u} = \tilde{f}$, we see that $\tilde{u} \in H^2(V)$, and moreover we have the estimate

$$\|\tilde{u}\|_{H^2(V)} \leq C(\|\tilde{f}\|_{L^2(W)} + \|\tilde{u}\|_{L^2(W)}) \leq C(\|f\|_{H^{m+1}(\Omega)} + \|u\|_{L^2(\Omega)})$$

Since the election of the multiindex α was arbitrary (as long as $|\alpha| = m+1$), we must have that $u \in H^{m+3}(V)$ and

$$\|u\|_{H^{m+3}} \leq C(\|f\|_{H^{m+1}(\Omega)} + \|u\|_{L^2(\Omega)})$$

□

From the previous theorem we may obtain an immediate corollary that assures that if both the coefficients of L and f are smooth, then the solution is also smooth:

Corollary 2.1.11. *Suppose that $a^{ij}, b^i, c \in C^\infty(\Omega)$ for $1 \le i, j \le N$. Moreover, assume that $f \in C^\infty(\Omega)$. Then, if $u \in H^1(\Omega)$ is a weak solution of $Lu = f$, then $u \in C^\infty(\Omega)$.*

Proof. Using Theorem (2.1.10), we deduce that $u \in H^m_{\text{loc}}(\Omega)$ $\forall m \in \mathbb{N}$. Thus, using Corollary (1.4.12), we conclude that $u \in C^k(\Omega)$ for each $k \in \mathbb{N}$ and thus $u \in C^\infty(\Omega)$. □

Remark 2.1.2. Notice that from the previous corollary we may deduce that the regularity of u in the boundary does not play any role when it comes to the regularity of u in Ω. That is, even if u has some singularities on $\partial\Omega$, these singularities do not propagate into the interior.

Remark 2.1.3. Another conclusion of the previous corollary is that if $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ in Ω with $a^{ij}, b^i, c, f \in C^\infty(\Omega)$, then u is actually a strong solution of the PDE. Indeed, we deduce from the previous corollary that $u \in C^\infty(\Omega)$. Moreover, since u is a weak solution of $Lu = f$, we then have that

$$B[u, \varphi] = (f, \varphi) \quad \forall \varphi \in C_c^\infty(\Omega)$$

Since u is smooth, we may integrate by parts in order to obtain that $B[u, \varphi] = (Lu, \varphi)$ for each $\varphi \in C_c^\infty(\Omega)$. Thus, $(f, \varphi) = (Lu, \varphi)$ and therefore

$$(Lu - f, \varphi) = 0 \quad \forall \varphi \in C_c^\infty(\Omega)$$

Using Lemma (1.1.1) we conclude that $Lu = f$ a.e. However, since $u, f \in C_c^\infty(\Omega)$, we deduce that $Lu = f$ and thus u is a strong solution of the PDE.

The previous regularity results provide assertions about regularity *in the interior* of Ω. We did not make any assumptions on Ω, rather than the mere supposition that it is bounded and open. We did not ask any boundary conditions to the PDE $Lu = f$ neither. With some hypotheses on the coefficients of L and f, we concluded that u is more regular than just $H^1(\Omega)$, but just in subsets $V \subset\subset \Omega$. That is, we reached to conclusions such as $u \in H^2_{\text{loc}}(\Omega)$, $u \in H^m_{\text{loc}}(\Omega)$, etc. Now, we will obtain regularity up to the boundary. However, some more hypotheses will be needed on Ω, as we will wee.

Theorem 2.1.12. *Suppose that $a^{ij} \in C^1(\overline{\Omega}), b^i, c \in L^\infty(\Omega)$ for $1 \le i, j \le N$. Moreover, assume that $f \in L^2(\Omega)$. Lastly, suppose that $\partial\Omega$ is C^2. Then, if $u \in H^1_0(\Omega)$ is a weak solution of the following boundary-value problem*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

we actually have that $u \in H^2(\Omega)$, and

$$\|u\|_{H^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}$$

with C a constant that only depends on Ω and a^{ij}, b^i and c.

Using this result inductively, we reach to the following higher boundary regularity result.

Theorem 2.1.13. *Let $m \in \mathbb{N}$, and suppose that $a^{ij}, b^i, c \in C^{m+1}(\overline{\Omega})$ for $1 \leq i, j \leq N$. Moreover, suppose that $f \in H^m(\Omega)$, and that $\partial\Omega$ is C^{m+2}. Then, if $u \in H_0^1(\Omega)$ is a weak solution of*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

then $u \in H^{m+2}(\Omega)$. Morever, the following inequality is satisfied:

$$\|u\|_{H^{m+2}(\Omega)} \leq C(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)})$$

Just like in the case of interior regularity, if the coefficients of L and f are smooth, then any weak solution must also be smooth:

Theorem 2.1.14. *Suppose that $a^{ij}, b^i, c \in C^\infty(\overline{\Omega})$ for $1 \leq i, j \leq N$. Moreover, suppose that $f \in C^\infty(\overline{\Omega})$, and that $\partial\Omega$ is C^∞. Then, if $u \in H_0^1(\Omega)$ is a weak solution of*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

then $u \in C^\infty(\overline{\Omega})$.

2.1.2 Weak solutions of the homogeneous Neumann problem

Now the boundary conditions will be replaced by Neumann boundary conditions. That is, the problem we will study now is

$$\begin{cases} Lu = f & \text{in } \Omega \\ a\nabla u \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases} \tag{2.17}$$

where $a\nabla u \cdot \mathbf{n} = \sum_{i,j=1}^N a_{ij} u_{x_j} n_i$, and $\mathbf{n}$ denotes the outward normal vector of Ω, at each point. In a similar way as in the Dirichlet boundary conditions case, one may see after multiplying (2.17) by a function φ and integrating over Ω, that every smooth solution of (2.17) satisfies

$$\int_\Omega \left(\sum_{i,j=1}^N a^{ij} u_{x_i} \varphi_{x_j} + \sum_{i=1}^N b^i u_{x_i} \varphi + cu\varphi \right) = \int_\Omega fv \quad \forall \varphi \in H^1(\Omega)$$

thus, weak solutions of (2.17) will be defined as follows:

Definition 2.1.3. A weak solution of (2.17) is a function $u \in H^1(\Omega)$ such that $B[u, \varphi] = (f, \varphi)$, for every function $\varphi \in H^1(\Omega)$, where $B[\cdot, \cdot]$ is the bilinear form defined in (2.1.1).

Given the similarity between the homogeneous Neumann problem and the homogeneous Dirichlet problem, many of the results from the Neumann problem are analogous to ones we obtained in the previous section, where the Dirichlet problem was studied. For instance, we have the following existence and uniqueness result.

Theorem 2.1.15. *For each $f \in L^2(\Omega)$, there exists a unique weak solution $u \in H^1(\Omega)$ of the boundary-value problem* (2.17)*.*

Proof. It follows from Lax–Milgram's theorem, with $H = H^1(\Omega)$. □

We also have the following regularity result, similar to the one obtained for the Dirichilet problem:

Theorem 2.1.16. *Let m be a nonnegative integer, and let Ω be a bounded open set with $\partial\Omega$ of class C^{m+2}. Assume that $a^{ij}, b^i, c \in C^{m+1}(\overline{\Omega})$ and $f \in H^m(\Omega)$. Then, if $u \in H^1(\Omega)$ is a weak solution of* (2.17)*, then $u \in H^{m+2}(\Omega)$. In particular, if $\partial\Omega$ is C^∞, $a^{ij}, b^i, c \in C^\infty(\overline{\Omega})$ and $f \in C^\infty(\overline{\Omega})$, then $u \in C^\infty(\overline{\Omega})$.*

2.2 Second-order parabolic equations

In this section parabolic equations will be studied. Parabolic equations are PDEs of the form $u_t + Lu = f$ in a certain domain, where L is the operator defined in the previous section. However, the coefficients of L will now depend on both time space, that is, $a^{ij} = a^{ij}(x,t), b^i = b^i(x,t)$ and $c = c(x,t)$. We will consider the following problem,

$$\begin{cases} u_t + Lu = f & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega \times [0,T] \\ u = g & \text{on } \Omega \times \{0\} \end{cases} \tag{2.18}$$

where $\Omega_T = \Omega \times (0,T]$ with $T > 0$, and $f : \Omega_T \longrightarrow \mathbb{R}$ and $g : \Omega \longrightarrow \mathbb{R}$ are given functions.

Definition 2.2.1. The partial differential operator $\frac{\partial}{\partial t} + L$ is said to be uniformly parabolic if given a fixed time $0 \leq t \leq T$, the operator L is uniformly elliptic. That is, $\frac{\partial}{\partial t} + L$ will be called uniformly parabolic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|^2 \quad \forall (x,t) \in \Omega_T, \xi \in \mathbb{R}^N \tag{2.19}$$

A classical example of a parabolic equation is the heat equation $u_t - \Delta u = f$. Here, $b^i \equiv c \equiv 0$, and $a^{ij} = -\delta_{ij}$, where δ_{ij} denotes the Kronecker Delta. As we will see later, many peculiarities of the heat equations will also apply to general parabolic equations, and that is why parabolic equations are considered a natural generalization of the heat equation.

We will use two different methods to study solutions of the parabolic equation. The first method, called *Galerkin approximations*, uses finite-dimensional spaces in order to build a sequence that converges to the desired solution. The second method will only be used for the heat equation, and it will be based on the Hille–Yosida theorem.

2.2.1 Galerking approximations

Instead of considering the function $u(x,t)$, we will consider $\mathbf{u} : [0,T] \longrightarrow H_0^1(\Omega)$. That is, given $t \in [0,T]$, $\mathbf{u}(t)$ will be a *function* defined as $x \mapsto u(x,t)$. Similarly, we will suppose that $\mathbf{f} : [0,T] \longrightarrow L^2(\Omega)$ too, so $\mathbf{f}(t)$ will be the mapping $x \mapsto f(x,t)$.

Under this notation, we will introduce the following bilinear form, which is similar to the one defined for elliptic PDEs. Throughout this section we will assume that $a^{ij}, b^i, c \in L^\infty(\Omega_T)$, for $1 \leq i,j \leq N$, the coefficients a^{ij} are symmetric, $f \in L^2(\Omega_T)$ and $g \in L^2(\Omega)$.

Definition 2.2.2. The bilinear form $B[\cdot,\cdot]$ associated with the operator $\frac{\partial}{\partial t} + L$ is defined by

$$B[\mathbf{u},\varphi;t] = \int_\Omega \left(\sum_{i,j=1}^{N} a^{ij}(\cdot,t)(\mathbf{u}(t))_{x_i}\varphi_{x_j} + \sum_{i=1}^{N} b^i(\cdot,t)(\mathbf{u}(t))_{x_i}\varphi + c(\cdot,t)\mathbf{u}(t)\varphi \right)$$

with $\mathbf{u} \in L^2(0,T;H_0^1(\Omega))$, $\varphi \in H_0^1(\Omega)$ and a.e. $0 \leq t \leq T$.

We will try to define the notion of *weak solution* of the problem (2.18) now. We can proceed as we did with the elliptic problem, assuming that u is smooth first and multiplying the equation (2.18) by a function $\varphi \in C_c^\infty(\Omega)$ in order to integrate over Ω then. This motivates the following definition of weak solutions:

Definition 2.2.3. A function $\mathbf{u} \in L^2(0,T;H_0^1(\Omega))$ with $\mathbf{u}' \in L^2(0,T;H^{-1}(\Omega))$ is a weak solution of (2.18) if the following conditions are fulfilled:

(i) $\langle \mathbf{u}', \varphi \rangle + B[\mathbf{u},\varphi;t] = (\mathbf{f},\varphi)$ for each $\varphi \in H_0^1(\Omega)$ and a.e. $t \in [0,T]$.

(ii) $\mathbf{u}(0) = g$

It can be proved (see L. C. Evans [2]) that (i) in the previous definition implies that in fact $\mathbf{u} \in C([0,T];L^2(\Omega))$, so that (ii) actually makes sense.

Let $\{w_k\}_{k=1}^{\infty}$ a family of functions such that $\{w_k\}$ is an orthonormal basis of $H_0^1(\Omega)$ and $L^2(\Omega)$. We will try to find a sequence of functions $\{\mathbf{u}_m\}_{m=1}^{\infty}$, with $\mathbf{u}_m : [0,T] \longrightarrow H_0^1(\Omega)$, such that each $\mathbf{u}_m(t)$ is a linear combination of $\{w_1, \ldots, w_m\}$. That is, we are looking for functions $\mathbf{u}_m$ such that

$$\mathbf{u}_m(t) = \sum_{k=1}^{m} d_m^k(t) w_k \tag{2.20}$$

However, we are not looking for *any* linear combination of the type (2.20), we want to impose certain conditions on the coefficients d_m^k. These conditions are gathered in the following theorem:

Theorem 2.2.1. *Let $m \in \mathbb{N}$. Then, there exists a unique function $\mathbf{u}_m$ of the type* (2.20) *such that*

$$d_m^k(0) = (g, w_k) \quad k = 1, 2, \ldots, m \tag{2.21}$$

and

$$(\mathbf{u}_m', w_k) + B[\mathbf{u}_m, w_k; t] = (\mathbf{f}, w_k) \quad \forall\ 0 \leq t \leq T, k = 1, 2, \ldots, m \tag{2.22}$$

Proof. Since $\{w_k\}_{k=1}^{\infty}$ is an orthonormal basis of $L^2(\Omega)$, we deduce that $(\mathbf{u}_m'(t), w_k) = {d_m^k}'(t)$. Moreover, given the linearity of $B[\cdot, \cdot; t]$ we get that

$$B[\mathbf{u}_m, w_k; t] = \sum_{j=1}^{m} B[w_j, w_k; t] d_m^j(t)$$

Thus, (2.22) and (2.21) become in the following linear system of ODE:

$$\begin{cases} {d_m^k}' + \sum_{j=1}^m B[w_j, w_k; t] d_m^j(t) = (\mathbf{f}(t), w_k), & k = 1, 2, \ldots, m \\ d_m^k(0) = (g, w_k), \quad k = 1, 2, \ldots, m \end{cases}$$

From ODE theory we conclude that there exists a unique solution of the previous system of linear ODEs, and this solution is absolutely continuous. Thus, there exists a unique function $\mathbf{u}_m$ that satisfies conditions (2.21) and (2.22).

□

As mentioned earlier, our objective is to pass to the limit $m \to \infty$ and see if $\mathbf{u}_m$ (or a subsequence) converges to a solution of the parabolic PDE (2.18). But first, we will introduce some estimates.

Lemma 2.2.2 (Gronwall's inequality)**.** *Let η be a nonnegative absolutely continuous function on $[0,T]$, such that*

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t)$$

for a.e. $t \in [0,T]$*, with* ϕ *and* ψ *nonnegative function in* $L^1([0,T])$*. Then,*

$$\eta(t) \leq e^{\int_0^t \phi(s)ds} \left[\eta(0) + \int_0^t \psi(s)ds\right]$$

Theorem 2.2.3. *We have the following estimate*

$$\begin{aligned} \max_{0\leq t\leq T} \|\mathbf{u}_m(t)\|_{L^2(\Omega)} + \|\mathbf{u}_m\|_{L^2(0,T;H_0^1(\Omega))} + \|\mathbf{u}'_m\|_{L^2(0,T;H^{-1}(\Omega))} \\ \leq C(\|\mathbf{f}\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{L^2(\Omega)}) \end{aligned} \tag{2.23}$$

for $m = 1, 2, \ldots$*, and with* C *a constant depending on* Ω*,* T *and the coefficients of* L*.*

Proof. Multiplying (2.22) by d_m^k and summing from $k = 1$ to $k = m$, we obtain that

$$(\mathbf{u}'_m, \mathbf{u}_m) + B[\mathbf{u}_m, \mathbf{u}_m; t] = (\mathbf{f}, \mathbf{u}_m)$$

Using the energy estimates (2.1.1), we see that there exists a constant $\beta > 0$ and a constant $\gamma \geq 0$, such that

$$\beta\|\mathbf{u}_m\|^2_{H_0^1(\Omega)} \leq B[\mathbf{u}_m, \mathbf{u}_m; t] + \gamma\|\mathbf{u}_m\|^2_{L^2(\Omega)}, \quad \forall 0 \leq t \leq T, m = 1, 2, \ldots$$

Also, from the fact that $(\mathbf{f} + \mathbf{u}_m, \mathbf{f} + \mathbf{u}_m) \geq 0$ we deduce that $|(\mathbf{f}, \mathbf{u}_m)| \leq \frac{1}{2}\|\mathbf{f}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\mathbf{u}_m\|^2_{L^2(\Omega)}$. Also, we have that $(\mathbf{u}'_m, \mathbf{u}_m) = \frac{d}{dt}(\frac{1}{2}\|\mathbf{u}_m\|^2_{L^2(\Omega)})$, for a.e. $0 \leq t \leq T$. Thus, joining the previous inequalities,

$$\frac{d}{dt}\left(\|\mathbf{u}_m\|^2_{L^2(\Omega)}\right) + 2\beta\|\mathbf{u}_m\|^2_{H_0^1(\Omega)} \leq C_1\|\mathbf{u}_m\|^2_{L^2(\Omega)} + C_2\|\mathbf{f}\|^2_{L^2(\Omega)} \tag{2.24}$$

for a.e. $0 \leq t \leq T$ and some constants C_1, C_2.

When defining $\eta(t) = \|\mathbf{u}_m(t)\|^2_{L^2(\Omega)}$ and $\xi(t) = \|\mathbf{f}\|^2_{L^2(\Omega)}$, this inequality becomes

$$\eta'(t) \leq C_1\eta(t) + C_2\xi(t)$$

Using Gronwall's inequality of the previous lemma, we obtain

$$\eta(t) \leq e^{C_1 t}\left(\eta(0) + C_2\int_0^t \xi(s)ds\right) \quad (0 \leq t \leq T) \tag{2.25}$$

Now, since $d_m^k(0) = (g, w_k)$, we obtain that $\eta(0) = \|\mathbf{u}_m(0)\|^2_{L^2(\Omega)} \leq \|g\|^2_{L^2(\Omega)}$. As a consequence, (2.25) becomes

$$\max_{0\leq t\leq T}\|\mathbf{u}_m(t)\|^2_{L^2(\Omega)} \leq C(\|g\|^2_{L^2(\Omega)} + \|\mathbf{f}\|^2_{L^2(0,T;L^2(\Omega))})$$

Using this inequality and (2.24), we obtain

$$\|\mathbf{u}_m\|^2_{L^2(0,T;H^1_0(\Omega))} = \int_0^T \|\mathbf{u}_m\|^2_{H^1_0(\Omega)} \leq C(\|g\|^2_{L^2(\Omega)} + \|\mathbf{f}\|^2_{L^2(0,T;L^2(\Omega))})$$

Take now $v \in H^1_0(\Omega)$ with $\|v\|_{H^1_0(\Omega)} \leq 1$. If we write $v = v_1 + v_2$ with $v_1 \in \text{span}\{w_k\}_{k=1}^m$ and $(v_2, w_k) = 0$ for $k = 1, \ldots, m$, and taking into account that the functions $\{w_k\}_{k=1}^\infty$ are orthogonal in $H^1_0(\Omega)$, we have that $\|v_1\|_{H^1_0(\Omega)} \leq \|v\|_{H^1_0(\Omega)} \leq 1$.

Using (2.22) we conclude that for a.e. $t \in [0, T]$,

$$(\mathbf{u}'_m, v_1) + B[\mathbf{u}_m, v_1; t] = (\mathbf{f}, v_1)$$

Thus,

$$\langle \mathbf{u}'_m, v\rangle = (\mathbf{u}'_m, v) = (\mathbf{u}'_m, v_1) = (\mathbf{f}, v_1) - B[\mathbf{u}_m, v_1; t]$$

So, since $\|v^1\|_{H^1_0(\Omega)} \leq 1$,

$$|\langle \mathbf{u}'_m, v\rangle| \leq C(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{u}_m\|_{H^1_0(\Omega)})$$

Thus,

$$\|\mathbf{u}'_m\|_{H^{-1}(\Omega)} \leq C(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{u}_m\|_{H^1_0(\Omega)})$$

So we conclude that

$$\int_0^T \|\mathbf{u}'_m\|^2_{H^{-1}(\Omega)} \leq C\int_0^T (\|\mathbf{f}\|^2_{L^2(\Omega)} + \|\mathbf{u}_m\|^2_{H^1_0(\Omega)}) \leq C(\|g\|^2_{L^2(\Omega)} + \|\mathbf{f}\|_{L^2(0,T;L^2(\Omega))})$$

□

Now, we have the following existence theorem:

Theorem 2.2.4. *The problem*

$$\begin{cases} u_t + Lu = f & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega \times [0, T] \\ u = g & \text{on } \Omega \times \{0\} \end{cases}$$

with the hypotheses on f, g and the coefficients of L mentioned at the beginning of this section, has a weak solution.

See L. C. Evans [2] for a proof of the theorem. The general idea of the proof is to see that $\{\mathbf{u}_m\}_{m=1}^{\infty}$ is bounded in $L^2(0,T;H_0^1(\Omega))$ using energy estimates. Thus, we may extract a subsequence that converges weakly in $L^2(0,T;H_0^1(\Omega))$ to a certain function $\mathbf{u} \in L^2(0,T;H_0^1(\Omega))$. Then, the idea is to check that this function solves the problem (2.18).

We also have the following result, that assures uniqueness of solutions:

Theorem 2.2.5. *The weak solution of the parabolic problem* (2.18)*, whose existence is guaranteed by the previous theorem, is unique.*

Proof. If $\mathbf{u}_1$ and $\mathbf{u}_2$ are two solutions of (2.18), then $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ is a weak solution of (2.18) with $\mathbf{f} \equiv 0$, $g \equiv 0$. Thus, we only have to see that $\mathbf{u} \equiv 0$ is the only weak solution of (2.18) with $\mathbf{f} \equiv 0$, $g \equiv 0$.

Let $\mathbf{u}$ be a weak solution of (2.18) with $\mathbf{f} \equiv 0$, $g \equiv 0$. Then,

$$\langle \mathbf{u}', \varphi \rangle + B[\mathbf{u}, \varphi; t] = 0$$

For each $\varphi \in H_0^1(\Omega)$, and a.e. $0 \leq t \leq T$. Picking $\varphi = \mathbf{u}$ in particular, and taking into account that $\langle \mathbf{u}', \mathbf{u} \rangle = \frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}\|_{L^2(\Omega)}^2\right)$, we obtain that

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}\|_{L^2(\Omega)}^2\right) + B[\mathbf{u}, \mathbf{u}; t] = 0$$

Now, using energy estimates (2.1.1), we have that

$$B[\mathbf{u}, \mathbf{u}; t] \geq \beta \|\mathbf{u}\|_{H_0^1(\Omega)}^2 - \gamma \|\mathbf{u}\|_{L^2(\Omega)}^2 \geq -\gamma \|\mathbf{u}\|_{L^2(\Omega)}^2$$

Thus,

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}\|_{L^2(\Omega)}^2\right) \leq \gamma \|\mathbf{u}\|_{L^2(\Omega)}^2$$

Using Gronwall's inequality (2.2.2) and the fact that $\mathbf{u}(t) = 0$, we deduce that $\mathbf{u} \equiv 0$.

□

2.2.2 Hille-Yosida Theorem

Hille–Yosida's theorem offers a result of existence and uniqueness of the following evolution problem in abstract Banach spaces:

$$\begin{cases} u'(t) + Au(t) = 0 \quad \text{on } [0, +\infty) \\ u(0) = u_0 \end{cases} \tag{2.26}$$

With $u : [0, +\infty) \longrightarrow H$, where H denotes a Hilbert space, and $A : D(A) \subset H \longrightarrow H$ is a certain unbounded linear operator. We will use this theorem in order to study the heat equation. However, before stating the actual theorem, we will begin with some definitions.

Definition 2.2.4. Let H be a Hilbert space, and let $A : D(A) \subset H \longrightarrow H$ be an unbounded linear operator. A is monotone if

$$(Av, v) \geq 0 \quad \forall v \in D(A)$$

Where $(\cdot, \cdot)$ denotes de scalar product of H. If A also satisfies that $\forall f \in H$ there exists $u \in D(A)$ such that $u + Au = f$, then A is said to be maximal monotone.

With these definitions, Hille–Yosida's theorem asserts the following:

Theorem 2.2.6 (Hille–Yosida)**.** *Let H be a Hilbert space, and let $A : D(A) \subset H \longrightarrow H$ be a maximal monotone operator. Set $g \in D(A)$. Then, there exists a unique function $u \in C^1([0, +\infty); H) \cap C([0, +\infty); D(A))$ such that*

$$\begin{cases} u'(t) + Au(t) = 0 & \textit{on } [0, +\infty) \\ u(0) = g \end{cases} \tag{2.27}$$

Moreover,

$$|u(t)| \leq |g| \quad \textit{and} \quad \left|u'(t)\right| = |Au(t)| \leq |Ag| \quad \forall t \geq 0$$

Corollary 2.2.7. *Let A be a self-adjoint maximal monotone operator. Given $g \in H$, there exists a unique function*

$$u \in C([0, +\infty); H) \cap C^1((0, +\infty); H) \cap C((0, +\infty); D(A))$$

so that

$$\begin{cases} u'(t) + Au = 0 & \textit{on } (0, +\infty) \\ u(0) = g \end{cases}$$

Moreover, $u \in C^k((0, +\infty); D(A^\ell)) \quad \forall k, \ell \in \mathbb{N}$, and

$$|u(t)| \leq |g| \quad \textit{and} \quad |u'(t)| = |Au(t)| \leq \frac{1}{t}|g| \quad \forall t > 0$$

Now, Hille–Yosida's theorem will be used to prove existence and uniqueness of solutions of the heat equation.

Lemma 2.2.8. *If A is a maximal monotone symmetric operator, A is self-adjoint.*

Theorem 2.2.9. *Let $g \in L^2(\Omega)$. Then, there exists a unique weak solution $\mathbf{u}(t)$ of the heat equation*

$$\begin{cases} u_t - \Delta u = 0 \quad \text{in } \Omega \times (0,+\infty) \\ u = 0 \quad \text{on } \partial\Omega \times (0,+\infty) \\ u(x,0) = g \quad \text{on } \Omega \end{cases}$$

Moreover, $\mathbf{u}$ *satisfies*

$$\mathbf{u} \in C([0,+\infty); L^2(\Omega)) \cap C((0,+\infty); H^2(\Omega) \cap H_0^1(\Omega))$$
$$\mathbf{u} \in C^1((0,+\infty); L^2(\Omega))$$
$$\mathbf{u} \in C^\infty(\overline{\Omega} \times [\epsilon,+\infty)), \quad \forall \epsilon > 0$$
$$\mathbf{u} \in L^2(0,+\infty; H_0^1(\Omega))$$

We also have that

$$\frac{1}{2}|\mathbf{u}(T)|^2_{L^2(\Omega)} + \int_0^T |\nabla \mathbf{u}(t)|^2_{L^2(\Omega)} dt = \frac{1}{2}|g|^2_{L^2(\Omega)} \quad \forall T > 0 \tag{2.28}$$

Proof. Let $H = L^2(\Omega)$. We will define the operator $A : D(A) \subset H \longrightarrow H$. We set $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, and for each $\mathbf{u} \in D(A)$, we set $A\mathbf{u} = -\Delta\mathbf{u}$.

Given $\mathbf{u} \in D(A)$,

$$(A\mathbf{u}, \mathbf{u})_{L^2(\Omega)} = \int_\Omega (-\Delta\mathbf{u})\mathbf{u} = \int_\Omega |\nabla\mathbf{u}|^2 \geq 0$$

and therefore A is monotone. In order to check that it is maximal monotone, we have to check that given $f \in L^2$ there exists a unique solution $\mathbf{u} \in H^2 \cap H_0^1$ of the problem $f = \mathbf{u} + A\mathbf{u} = \mathbf{u} - \Delta\mathbf{u}$. However, we know from the section of elliptic PDEs that such solution exists, and thus A is maximal monotone.

In order to prove that A is self-adjoint, using the preceeding lemma we only have to see that A is symmetric. Let $\mathbf{u}, \mathbf{v} \in D(A)$. Then,

$$(A\mathbf{u}, \mathbf{v})_{L^2(\Omega)} = \int_\Omega (-\Delta\mathbf{u})\mathbf{v} = \int_\Omega \nabla\mathbf{u} \cdot \nabla\mathbf{v} = \int_\Omega \mathbf{u}(-\Delta\mathbf{v}) = (\mathbf{u}, A\mathbf{v})_{L^2(\Omega)}$$

so A is symmetric and thus self-adjoint. Using Corollary (2.2.7), there exists a solution $\mathbf{u}$ of the heat equation with $\mathbf{u} \in C^k((0,+\infty); D(A^\ell))$, for every $k, \ell \in \mathbb{N}$. Taking into account that $D(A^\ell) \subset H^{2\ell}(\Omega)$ with continuous injection, we conclude that

$$\mathbf{u} \in C^k((0,\infty); H^{2\ell}(\Omega)) \quad \forall k, \ell \in \mathbb{N}$$

Using Corollary (1.4.12) it follows that $\mathbf{u} \in C^k((0,\infty); C^k(\overline{\Omega}))$, for each $k \in \mathbb{N}$. Let $\varphi(t) = \frac{1}{2}|u(t)|^2_{L^2(\Omega)}$. It is of class C^1 in $(0,\infty)$, and for every $t > 0$,

$$\varphi'(t) = \big(\mathbf{u}(t), \mathbf{u}'(t)\big)_{L^2(\Omega)} = (\mathbf{u}(t), \Delta\mathbf{u}(t))_{L^2(\Omega)} = -\int_\Omega |\nabla \mathbf{u}(t)|^2$$

Let $0 < \epsilon < T < \infty$ now. Then,

$$\varphi(T) - \varphi(\epsilon) = \int_\epsilon^T \varphi'(t)dt = -\int_\epsilon^T |\nabla u(t)|^2_{L^2(\Omega)} dt$$

and (2.28) follows if we let $\epsilon \to 0$, taking into account that $\varphi(\epsilon) \to \frac{1}{2}|g|^2$ and thus $\mathbf{u} \in L^2(0, \infty; H_0^1(\Omega))$.

□

2.2.3 Regularity

The next step that should be taken after proving existence and uniqueness of weak solutions is checking how regular these weak solutions are. In a similar way as we did in elliptic PDEs, we will see that under some conditions the solution $\mathbf{u}$ is guaranteed to have some regularity.

Theorem 2.2.10. *Let $g \in H_0^1(\Omega)$ and $\mathbf{f} \in L^2(0,T;L^2(\Omega))$. Assume that $\mathbf{u} \in L^2(0,T,H_0^1(\Omega))$ with $\mathbf{u}' \in L^2(0,T;H^{-1}(\Omega))$ solves*

$$\begin{cases} u_t + Lu = f & \textit{in } U_T \\ u = 0 & \textit{on } \partial U \times [0,T] \\ u = g & \textit{on } U \times \{t = 0\} \end{cases}$$

Then, we have that

$$\mathbf{u} \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)), \quad \mathbf{u}' \in L^2(0,T;L^2(\Omega))$$

Moreover, there exists a constant C that depends on Ω, T and the coefficients of L, such that

$$\begin{aligned} \textit{ess sup}_{0\le t\le T}\|\mathbf{u}(t)\|_{H_0^1(\Omega)} + \|\mathbf{u}\|_{L^2(0,T;H^2(\Omega))} + \|\mathbf{u}'\|_{L^2(0,T;L^2(\Omega))} \\ \le C(\|\mathbf{f}\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{H_0^1(\Omega)}) \end{aligned} \tag{2.29}$$

Assume now that $g \in H^2(\Omega)$ and $\mathbf{f}' \in L^2(0,T;L^2(\Omega))$. Then,

$$\mathbf{u} \in L^\infty(0,T;H^2(\Omega)), \quad \mathbf{u}' \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)),$$

$$\mathbf{u}'' \in L^2(0,T;H^{-1}(\Omega))$$

with the following estimate:

$$\text{ess } sup_{0\le t\le T}(\|\mathbf{u}(t)\|_{H^2(\Omega)}+\|\mathbf{u}'(t)\|_{L^2(\Omega)})+\|\mathbf{u}'\|_{L^2(0,T;H_0^1(\Omega))}+\|\mathbf{u}''\|_{L^2(0,T;H^{-1}(\Omega)}$$

$$\le C(\|\mathbf{f}\|_{H^1(0,T;L^2(\Omega))}+\|g\|_{H^2(\Omega)})$$

Now, based on this result one can prove the following higher regularity result:

Theorem 2.2.11. *Let $g \in H^{2m+1}(\Omega)$ and $\frac{d^k\mathbf{f}}{dt^k} \in L^2(0,T;H^{2m-2k}(\Omega))$ for $k = 0,\dots,m$. Moreover, assume that we have the following compatibility conditions:*

$$\begin{cases} g_0 := g \in H_0^1(\Omega) \\ g_1 := \mathbf{f}(0) - Lg_0 \in H_0^1(\Omega) \\ g_2 := \dfrac{d\mathbf{f}}{dt}(0) - Lg_{m-1} \in H_0^1(\Omega) \\ \quad\vdots \\ g_m := \dfrac{d^{m-1}\mathbf{f}}{dt^{m-1}}(0) - Lg_{m-1} \in H_0^1(\Omega) \end{cases}$$

Then, we have that in fact

$$\frac{d^k\mathbf{u}}{dt^k} \in L^2(0,T;H^{2m+2-2k}(\Omega)) \quad k = 0,\dots,m+1$$

And we also have the following estimate:

$$\sum_{k=0}^{m+1}\left\|\frac{d^k\mathbf{u}}{dt^k}\right\|_{L^2(0,T;H^{2m+2-2k}(\Omega))} \le C\left(\sum_{k=0}^{m}\left\|\frac{d^k\mathbf{f}}{dt^k}\right\|_{L^2(0,T;H^{2m-2k}(\Omega)}+\|g\|_{H^{2m+1}(\Omega)}\right)$$

with C depending on m, U, T and the coefficients of L.

The proof uses induction on m, since the case $m = 0$ corresponds precisely to Theorem (2.2.10). See L. C. Evans [2] for a proof of both theorems. Now, from Theorem (2.2.11) one can deduce that if g and f are smooth enough, the solution is in fact of class C^∞. But before stating the result, we will need the following Lemma:

Lemma 2.2.12. *Let Ω be open and bounded, with $\partial\Omega$ smooth. Assume that m is a nonnegative integer, and that*

$$\mathbf{u} \in L^2(0,T;H^{m+2}(\Omega)), \text{ with } \mathbf{u}' \in L^2(0,T;H^m(\Omega))$$

Then,

$$\mathbf{u} \in C([0,T];H^{m+1}(\Omega))$$

Theorem 2.2.13. *Let Ω be open and bounded, and $\partial\Omega$ smooth. Let $g \in C^\infty(\overline{\Omega})$ and $f \in C^\infty(\overline{\Omega}_T)$. Suppose that the compatibility conditions of Theorem (2.2.11) hold for every $m \geq 0$. Then, the parabolic problem (2.18) has a unique solution $u \in C^\infty(\overline{\Omega}_T)$.*

• *Proof.* Given the regularity of g and f, we may apply Theorem (2.2.11) for $m = 0, 1, \ldots$. Thus, we deduce that

$$\frac{d^k\mathbf{u}}{dt^k} \in L^2(0,T;H^{2m}(\Omega)), \quad \forall m \geq 1, \forall 1 \leq k \leq m+1$$

thus, using the previous Lemma, we see that

$$\frac{d^k\mathbf{u}}{dt^k} \in C([0,T];H^{2m+1}(\Omega))$$

so, from this inclusion and Corollary (1.4.12), we have that

$$\mathbf{u} \in C^\infty([0,T];C^\infty(\overline{\Omega}))$$

that is, $u \in C^\infty(\overline{\Omega}_T)$.

□

2.3 Dirichlet problem for the Stokes system

Now, the problem that will be studied is the following:

$$\begin{cases} -\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\ \text{div } \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = 0 & \text{on } \partial\Omega \end{cases} \tag{2.30}$$

With $\Omega \subset \mathbb{R}^N$, where $N = 2$ or $N = 3$. If $N = 3$ and Ω is unbounded, we will also require that $\mathbf{u}(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$. Here, $p : \mathbb{R}^N \longrightarrow \mathbb{R}$ and $\mathbf{f} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$, and we are looking for solutions $\mathbf{u} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ that belong to a certain function space.

We will try to define a notion of weak solutions of the Dirichlet problem for the Stokes system (2.30). But first, we shall define a few concepts.

Definition 2.3.1. The set of smooth compactly supported functions in Ω that are divergence free is denoted by $C^\infty_{0,0}(\Omega;\mathbb{R}^N)$. That is,

$$C^\infty_{0,0}(\Omega;\mathbb{R}^N) := \{\varphi \in C^\infty_0(\Omega;\mathbb{R}^N) \big| \text{div}\varphi = 0 \text{ in } \Omega\}$$

On the other hand, we will denote by $L^{s,k}(\Omega;\mathbb{R}^N)$ the functions with weak derivatives of order k belonging to $L^s(\Omega;\mathbb{R}^N)$. The norm of this space is defined by

$$\|u\|_{L^{s,k}(\Omega;\mathbb{R}^N)} = \|\nabla^k u\|_{L^s(\Omega)}$$

Then, we define $J_0^{r,1}(\Omega;\mathbb{R}^N)$ as the completion of $C_{0,0}^{\infty}(\Omega;\mathbb{R}^N)$ in $L^{r,1}(\Omega;\mathbb{R}^N)$, that is,

$$J_0^{r,1}(\Omega;\mathbb{R}^N) := [C_{0,0}^{\infty}(\Omega;\mathbb{R}^N)]^{L^{r,1}(\Omega;\mathbb{R}^N)}$$

Similarly,

$$L_0^{s,k}(\Omega;\mathbb{R}^N) := [C_0^{\infty}(\Omega;\mathbb{R}^N)]^{L^{s,k}(\Omega;\mathbb{R}^N)}$$

Then, we define

$$\widehat{J}^{r,1}(\Omega;\mathbb{R}^N) := \{\varphi \in L_0^{r,1}(\Omega;\mathbb{R}^N) \big| \mathrm{div}\varphi = 0 \text{ in } \Omega\}$$

It is clear that $J_0^{r,1}(\Omega;\mathbb{R}^N) \subseteq \widehat{J}^{r,1}(\Omega;\mathbb{R}^N)$.

Under these definitions, we will try to deduce an appropriate statement of weak solutions. Formally, multiplying (2.30) by a test function $\varphi \in C_{0,0}^{\infty}(\Omega;\mathbb{R}^N)$,

$$\int_\Omega (-\Delta \mathbf{u} + \nabla p) \cdot \varphi = \int_\Omega \sum_{i,j=1}^{N} \frac{\partial \mathbf{u}_i}{\partial x_j} \frac{\partial \varphi_i}{\partial x_j} = (\mathbf{f}, \varphi)$$

Where $(\cdot,\cdot)$ is the standard scalar product of $L^2(\Omega;\mathbb{R}^N)$. Then, if we introduce the bilinear form

$$B[\mathbf{u}, \varphi] = \int_\Omega \sum_{i,j=1}^{N} \frac{\partial \mathbf{u}_i}{\partial x_j} \frac{\partial \varphi_i}{\partial x_j}$$

we may define weak solutions of (2.30) as follows:

Definition 2.3.2. Let $\mathbf{f} \in L^2(\Omega;\mathbb{R}^N)$. A function $u \in \widehat{J}^{2,1}(\Omega;\mathbb{R}^N)$ is said to be a weak solution of (2.30) if

$$B[u, \varphi] = (\mathbf{f}, \varphi) \quad \forall \varphi \in C_{0,0}^{\infty}(\Omega;\mathbb{R}^N)$$

Theorem 2.3.1 (Existence). *Let* $\mathbf{f} \in L^2(\Omega;\mathbb{R}^N)$*. Then, there exists at least a weak solution* $\mathbf{u}$ *of* (2.30)*, and moreover*

$$\|\nabla \mathbf{u}\|_{L^2(\Omega;\mathbb{R}^N)} \leq \|\mathbf{f}\|_{L^2(\Omega;\mathbb{R}^N)}$$

• *Proof.* The bilinear form $B[\mathbf{u}, \mathbf{v}]$ is a scalar product in $\widehat{J}^{2,1}(\Omega;\mathbb{R}^N)$. Indeed, clearly B is symmetric and $B[\mathbf{u}, \mathbf{u}] \geq 0$ for all $u \in \widehat{J}^{2,1}(\Omega;\mathbb{R}^N)$. Assume now that $B[\mathbf{u}, \mathbf{u}] = 0$. Then, necessarily $\nabla \mathbf{u} = 0$ in Ω, and therefore $\mathbf{u}$ belongs to the equivalence class $[0]$ of $J^{1,2}(\Omega;\mathbb{R}^N)$, as we wanted to see.

Let l be the linear functional on $J_0^{2,1}(\Omega;\mathbb{R}^N)$ defined by $\langle l, \mathbf{v}\rangle = (\mathbf{f}, \mathbf{v})$. Then,

$$|\langle l, \mathbf{v}\rangle| \le \|\mathbf{f}\|_{L^2(\Omega;\mathbb{R}^N)}\|\mathbf{v}\|_{L_0^{2,1}(\Omega;\mathbb{R}^N)}$$

And thus l is a bounded linear functional. We may apply Hahn–Banach's theorem in order to extend l to a bounded linear functional L on $\widehat{J}^{2,1}(\Omega;\mathbb{R}^N)$. Using Riesz representation theorem, there must exist $\mathbf{u} \in \widehat{J}^{2,1}(\Omega;\mathbb{R}^N)$ such that $B[\mathbf{u},\mathbf{v}] = \langle L, \mathbf{v}\rangle$ for every $\mathbf{v} \in \widehat{J}^{2,1}(\Omega;\mathbb{R}^N)$. Moreover,

$$B[\mathbf{u},\varphi] = \langle L, \varphi\rangle = (\mathbf{f}, \varphi) \quad \forall \varphi \in C^\infty_{0,0}(\Omega;\mathbb{R}^N)$$

So $\mathbf{u}$ is a weak solution of the Stokes system. Moreover,

$$\|\mathbf{u}\|_{\widehat{J}^{2,1}(\Omega;\mathbb{R}^N)} = \|\nabla \mathbf{u}\|_{L^2(\Omega;\mathbb{R}^N)} \le \|f\|_{L^2(\Omega;\mathbb{R}^N)}$$

□

We can't assure in general that the weak solution is unique. However, under some more hypotheses, we can guarantee its uniqueness:

Theorem 2.3.2 (Uniqueness)**.** *Let $f \in L^2(\Omega;\mathbb{R}^N)$, and assume that $\widehat{J}^{2,1}(\Omega;\mathbb{R}^N) = J_0^{2,1}(\Omega;\mathbb{R}^N)$. Then, the solution provided by the previous theorem is unique.*

Proof. Suppose that there exist two solutions $\mathbf{u}_1, \mathbf{u}_2 \in \widehat{J}^{2,1}(\Omega;\mathbb{R}^N)$. Then, $\mathbf{w} := \mathbf{u}_1 - \mathbf{u}_2$ satisfies

$$(\nabla \mathbf{w}, \nabla \varphi) = 0 \quad \forall \varphi \in C^\infty_{0,0}(\Omega;\mathbb{R}^N)$$

As a consequence, since by hypothesis $\widehat{J}^{2,1}(\Omega;\mathbb{R}^N) = J_0^{2,1}(\Omega;\mathbb{R}^N)$, we deduce that

$$\|\nabla \mathbf{w}\|_{L^2(\Omega;\mathbb{R}^N)} = 0$$

If $N = 3$ or $N = 2$ and $\Omega \ne \mathbb{R}^2$, taking into account the boundary conditions we conclude that $\mathbf{w} = 0$ and thus $\mathbf{u}_1 = \mathbf{u}_2$. However, if $N = 2$ and $\Omega = \mathbb{R}^2$, we can only assert that $\mathbf{w} \in [0]$, where $[0]$ is the equivalence class of 0, consisting of all functions that are constant in $\mathbb{R}^2$. Thus, if $\Omega = \mathbb{R}^2$ we can only say that $\mathbf{u}_1 - \mathbf{u}_2 \in [0]$. □

Remark 2.3.1. It can be proved (see G. Seregin [7]) that if $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{R}^N_+$, with $N = 2, 3$, then $J_0^{1,2}(\Omega;\mathbb{R}^N) = \widehat{J}^{1,2}(\Omega;\mathbb{R}^N)$, so the previous theorem may be applied to this case.

Finally, we have the following local regularity result (see G. Seregin [7]):

Proposition 2.3.3. *Let B be the unit ball in $\mathbb{R}^N$, and let $\mathbf{u} \in H^1(B)$, $p \in L^2(B;\mathbb{R}^N)$, $\mathbf{f} \in L^2(B;\mathbb{R}^N)$ and $g \in H^1(B)$ satisfy*

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{f} & \textit{in } B \\ \textit{div } \mathbf{u} = g & \textit{in } B \end{cases}$$

Then,

$$\nabla^2\mathbf{u}, \nabla q \in L^2(B(\epsilon);\mathbb{R}^N) \quad \forall \epsilon \in (0,1)$$

Moreover, we have the following estimate:

$$\|\nabla^2\mathbf{u}\|_{L^2(B(\epsilon);\mathbb{R}^N)} + \|\nabla p\|_{L^2(B(\epsilon);\mathbb{R}^N)} \leq$$
$$C\left(\|\mathbf{f}\|_{L^2(B;\mathbb{R}^N)} + \|p\|_{L^2(B;\mathbb{R}^N)} + \|g\|_{H^1(B)} + \|\mathbf{u}\|_{H^1(B)}\right)$$

with C a constant that depends only on ϵ.

2.4 Stationary Navier–Stokes system

Navier–Stokes equations describe the motion of viscous fluid substances. There are many open questions involving Navier–Stokes equations, and in fact the Clay Mathematics Institute offers a prize of 1,000,000$ to whoever proves a certain result on existence and regularity of solutions.

In this section we shall study a special form of the Navier–Stokes system, the *stationary* system:

$$\begin{cases} -\nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \\ \text{div } \mathbf{u} = 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.31}$$

With $\nu > 0$. Just like in the case of Stokes system, $\Omega \subset \mathbb{R}^N$ with $N = 2, 3$, $\mathbf{u}, \mathbf{f} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ and $p : \mathbb{R}^N \longrightarrow \mathbb{R}$. We will also assume that Ω is bounded.

Multiplying the equation by a function $\varphi \in \widehat{J}^{2,1}(\Omega;\mathbb{R}^N)$ and integrating over Ω, we may reach to the following definition of weak solutions:

Definition 2.4.1. A function $\mathbf{u} \in \widehat{J}^{2,1}(\Omega;\mathbb{R}^N)$ is said to be a weak solution of (2.31) if

$$B[\mathbf{u},\varphi] + b[\mathbf{u},\mathbf{u},\varphi] = (\mathbf{f},\varphi) \quad \forall \varphi \in \widehat{J}^{2,1}(\Omega;\mathbb{R}^N)$$

With

$$B[\mathbf{u},\varphi] = \nu\int_\Omega \sum_{i,j=1}^N \frac{\partial \mathbf{u}_i}{\partial x_j}\frac{\partial \varphi_i}{\partial x_j}$$

And

$$b[\mathbf{u},\mathbf{v},\varphi] = \int_\Omega \mathbf{u}\cdot\nabla\mathbf{v}\cdot\varphi$$

We will introduce the following lemmas now, that will be useful when it comes to proving existence of weak solutions of (2.31):

Lemma 2.4.1. *For $N = 2, 3$, the trilinear form $b(\mathbf{u}, \mathbf{v}, \varphi)$ is bounded on $H_0^1(\Omega)^3$.*

The proof follows directly from Hölder's inequality's repeated application:

Proof. If $N = 2$,

$$\left|\int_\Omega u_j (v_i)_{x_j} \varphi_i\right| \leq C\|u_j\|_{L^4(\Omega)}\|(v_i)_{x_j}\|_{L^2(\Omega)}\|\varphi_i\|_{L^4(\Omega)}$$

If $N = 3$,

$$\left|\int_\Omega u_j (v_i)_{x_j} \varphi_i\right| \leq C\|u_j\|_{L^6(\Omega)}\|(v_i)_{x_j}\|_{L^2(\Omega)}\|\varphi_i\|_{L^3(\Omega)}$$

So the lemma follows, considering that from Sobolev theory we know that

$$\|u\|_{L^q(\Omega)} \leq C'\|u\|_{H^1(\Omega)} \quad \forall 1 \leq q < \infty, N = 2$$
$$\|u\|_{L^6(\Omega)} \leq C'\|u\|_{H^1(\Omega)} \quad \forall 1 \leq q < \infty, N = 3$$

□

Lemma 2.4.2. *Let $\mathbf{u}, \mathbf{v} \in \widehat{J}^{2,1}(\Omega; \mathbb{R}^N)$ and $\varphi \in C_{0,0}^\infty(\Omega; \mathbb{R}^N)$. Then, $b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0$.*

Proof.

$$\int_\Omega u_j (v_i)_{x_j} v_i = \frac{1}{2}\int_\Omega u_j[(v_i)]_{x_j} = -\frac{1}{2}\int_\Omega (u_j)_{x_j}(v_i)^2$$

so, as a consequence,

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = -\frac{1}{2}\int_\Omega \text{div } \mathbf{u} \, |\mathbf{v}|_{\mathbb{R}^N} = 0$$

□

An immediate consequence of the previous lemma is that $b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v})$.

We shall define the following operator $\mathcal{A} : \widehat{J}^{2,1}(\Omega; \mathbb{R}^N) \longrightarrow (\widehat{J}^{2,1}(\Omega; \mathbb{R}^N))^\star$ by

$$\mathcal{A} : \widehat{J}^{2,1}(\Omega) \longrightarrow (\widehat{J}^{2,1}(\Omega))^\star$$
$$\mathcal{A}(\mathbf{u})(\varphi) = B[\mathbf{u}, \varphi] + b(\mathbf{u}, \mathbf{u}, \varphi)$$

Definition 2.4.2. An operator $\mathcal{A} : E \longrightarrow E^\star$, with E a Banach space, is said to be of Type M if $u_n \rightharpoonup u$, $Au_n \rightharpoonup f$ and $\limsup Au_n \leq f(u)$ implies that $Au = f$.

Our objective will be to prove that $\mathcal{A}$ defined above is of Type M, bounded and coercive. We have already shown in Lemma (2.4.1) that $\mathcal{A}$ is bounded.

Lemma 2.4.3. *The operator $\mathcal{A}$ is coercive.*

Proof. Let $\mathbf{u} \in \widehat{J}^{2,1}(\Omega; \mathbb{R}^N)$. Then,

$$\mathcal{A}(\mathbf{u})(\mathbf{u}) \geq B[\mathbf{u}, \mathbf{u}] + b(\mathbf{u}, \mathbf{u}, \mathbf{u}) \geq C\|\mathbf{u}\|^2$$

With $C > 0$ a constant. □

Lemma 2.4.4. *The mapping $\mathbf{u} \mapsto b(\mathbf{u}, \mathbf{u}, \cdot)$ is a weakly continuous mapping from $\widehat{J}^{2,1}(\Omega; \mathbb{R}^N)$ to $\widehat{J}^{2,1}(\Omega; \mathbb{R}^N)^\star$. In other words, if $\{\mathbf{u}_m\}$ converges to $\mathbf{u}$ weakly in $J^{2,1}(\Omega; \mathbb{R}^N)$, then $b(\mathbf{u}_m, \mathbf{u}_m, \mathbf{w}) \longrightarrow b(\mathbf{u}, \mathbf{u}, \mathbf{w})$ for every $\mathbf{w} \in \widehat{J}^{2,1}(\Omega; \mathbb{R}^N)$.*

Proof. Using Rellich–Kondrachov's Theorem (1.4.13), $\mathbf{u}_m \to \mathbf{u}$ strongly in $L^2(\Omega; \mathbb{R}^N)$. Let $\mathbf{w} \in \widehat{J}^{2,1}(\Omega; \mathbb{R}^N)$. Then,

$$\begin{aligned} &b(\mathbf{u}_m, \mathbf{u}_m, \mathbf{w}) = -b(\mathbf{u}_m, \mathbf{w}, \mathbf{u}_m) = \\ &= -\int_\Omega (u_m)_j (w_i)_{x_j} (u_m)_i \longrightarrow -b(\mathbf{u}, \mathbf{w}, \mathbf{u}) = b(\mathbf{u}, \mathbf{u}, \mathbf{w}) \end{aligned} \tag{2.32}$$

□

As a consequence, we have the following corollary:

Corollary 2.4.5. *The operator $\mathcal{A} : \widehat{J}^{2,1}(\Omega; \mathbb{R}^N) \longrightarrow \widehat{J}^{2,1}(\Omega; \mathbb{R}^N)$ is weakly continuous, and thus, of Type M.*

Finally, the existence of weak solutions of the stationary Navier–Stokes equations will directly follow from the following result:

Proposition 2.4.6. *Let E be a reflexive Banach space, and let $\mathcal{A} : E \longrightarrow E^\star$ be a bounded and coercive operator of Type M. Then, $\mathcal{A}$ is surjective.*

See D. Holland [4] for a proof of this result.

Theorem 2.4.7. *Let $\mathbf{f} \in \widehat{J}^{2,1}(\Omega; \mathbb{R}^N)^\star$. Then, the problem (2.31) has a solution $\mathbf{u} \in \widehat{J}^{2,1}(\Omega; \mathbb{R}^N)$.*

Proof. Using the previous Lemmas, the operator $\mathcal{A} : \widehat{J}^{2,1}(\Omega;\mathbb{R}^N) \longrightarrow \widehat{J}^{2,1}(\Omega;\mathbb{R}^N)$ is of Type M, bounded an coercive, and using the previous Proposition, it is also surjective. Thus, there exists $\mathbf{u} \in \widehat{J}^{2,1}(\Omega;\mathbb{R}^N)$ such that $\mathcal{A}(\mathbf{u}) = \mathbf{f}$. In other words,

$$\mathcal{A}(\mathbf{u})(\varphi) = B[\mathbf{u},\varphi] + b(\mathbf{u},\mathbf{u},\varphi) = (\mathbf{f},\varphi) \quad \forall \varphi \in \widehat{J}^{2,1}(\Omega;\mathbb{R}^N)$$

And therefore $\mathbf{u}$ is a weak solution of (2.31). □

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