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# Matching of spacetimes theory applied to rotating stars and quadratic gravity

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Thesis submitted for the degree of Doctor of Physics
2016

# Agradecimientos

La pole position de estos agradecimientos se la debo a Raül, por haber sido parte imprescindible para que esta tesis vea la luz, por todo el apoyo y ayuda brindados durante el transcurso de ésta y por hacer fácil lo difícil.

De la misma forma, agradezco a Marc Mars sus consejos científicos, que no han sido pocos, y que han sido esenciales para la obtención de varios de los resultados de esta tesis, así como a José Senovilla, por sus cursos de RG y por haberme brindado la oportunidad de hacer algún que otro cálculo geométrico en gravedad (en gravedad modificada, nótese).

Asímismo, me gustaría dedicar un hueco en estas líneas a todos aquellos con los que he tenido la suerte de colaborar y de intercambiar ideas y opiniones sin las cuales muchos de estos trabajos que componen la tesis no habrían visto la luz. Me adentré en el mundo de la cuerdología cósmica gracias a Jon Urrestilla, Kepa Sousa y Jose Juan Blanco-Pillado, quien semana tras semana nos da vidilla con el *Cosmology Journal Club*, y que con o sin Donuts se agradece. Hablando de momentos en los que uno ataca bandejas de dulces, quiero mencionar a Michael Bradley por todas sus sugerencias y comentarios recibidos en los coffe breaks de los ERE y otros congresos. Para dar forma a una de esas sugerencias sobre estrellas homogéneas conté con la inestimable ayuda de Irene Sendra, que es capaz de comunicarse con el Mathematica a través de comandos en forma de símbolos que sólo ella conoce. Y hablando de códigos, sus notas sobre la resolución numérica del TOV me facilitaron mucho la vida para empezar a jugar con el Fortran. Así que gracias por todo eso y por muchas otras cosas que no tienen cabida aquí.

Si hoy en día sé que una estrella de las de esta tesis tiene una masa de un Sol y medio y un radio que va desde mi casa hasta la uni es gracias a Nico Sanchís y Toni Font (y a toda la tropa valenciana), porque además de hacerme sentir como en casa el tiempo que pasé allí, me dieron la oportunidad de hacer un código modesto, pero del que me siento orgulloso. De igual manera, quería acordarme aquí de Cuchí por sus apariciones *estelares* en varios momentos a lo largo de la tesis. Y para terminar este repaso científico, quiero volver a los inicios y acordarme de Desiree, Mendi y Mari Jose porque consiguieron que en el colegio me lo pasara bien aprendiendo matemáticas, física y química.

También me gustaría expresar mi gratitud a Iñaki Garay por recibirme bien todas las veces que he entrado en su oficina dando la paliza, por sus consejos de tres palabras, por sus pendrives con libros bonitos, por el curso de RG para tesinandos y por las cañas a destiempo.

Durante esta tesis he tenido el placer de conocer y compartir comidas, oficina, discusiones y muchos momentos divertidos y/o memorables con Carlos, Uri, Lluc, Iker, Nastassja, Pablo, Asier, Joanes, Laura, Santi, María, Iagoba, Raúl, Olatz, Diego y en general, con todos los demás estudiantes y postdocs de por aquí. No me olvido de João, al que además de lo anterior quería agradecerle la lectura detallada, con detección de *typos* incluida, del capítulo de perturbaciones. Con Diego Regalado no he compartido la tesis, pero sí la carrera y le menciono por haberme enseñado mucha física, casi sin querer, durante aquellos maravillosos años. Y por seguirme el ritmo con las cheese burger y las bacon cheesefries.

También quiero agradecer al Departamento de Física Teórica e Historia de la Ciencia de la UPV/EHU el haberme acogido unos cuantos años que allí he pasado muy a gusto. En especial, a Montse por ayudarme con el papeleo y a Alex Feinstein y Ruth Lazcoz por incluirme en sus proyectos. Y hablando de economía, tengo que agradecer al Gobierno Vasco la beca predoctoral con la que he podido realizar esta tesis. También me ha servido para aprender que la burocracia es poca broma.

Y ahora le toca el turno a esa clase de gente que vas conociendo de aquí y de allá y que ya te va acompañando por la vida: Luisao, Pumel, Bustape, Bako, Vesti, Selfantonio, Polux, Rolando, Peebolin, Zatik, Falven, Nolteko, Tito, Gala, Artxenemy, Arrate, Lutxi, Zoni, Patri y todos los demás. Definitivamente lo de destrozar los nombres se nos fué de las manos hace tiempo. Por algún motivo difícil de comprender, no consigo ahuyentaros y ojalá siga siendo así por mucho tiempo.

Por último, si una tesis se dedica, ésta va para Víctor, Ama, Ane, Nerea y Amaia, por haberme apoyado tanto durante esta carrera de fondo. Y en general, por todos los motivos por los que uno puede dar las gracias.

# Resumen

Este resumen contiene un repaso breve de las actividades de investigación que se han llevado a cabo durante el desarrollo de la presente tesis doctoral, realizada con la ayuda predoctoral del Gobierno Vasco (BFI-2011-250) durante los años 2012-2015 en el departamento de Física Teórica e Historia de la Ciencia de la UPV/EHU, bajo la dirección de R.Vera. La primera parte del resumen está dedicada a explicar los problemas que se han abordado, cómo se ha hecho o qué métodos se han empleado y qué resultados hemos obtenido. La lista de publicaciones se incluye en la segunda sección.

## Líneas de investigación

Esta tesis se enmarca dentro del área de la Relatividad General. El tema central de estudio son los cuerpos compactos aislados en rotación, aunque como consecuencia de los métodos matemáticos empleados para su estudio, también ha dado lugar a otros trabajos estrechamente relacionados con las *condiciones de enlace*. Todos estos temas se explican en las siguientes subsecciones.

### Cuerpos compactos en rotación

Una comprensión adecuada de los cuerpos compactos en rotación en Relatividad General (RG) es fundamental para muchas situaciones astrofísicas. El tratamiento relativista original de estrellas compactas en rotación en equilibrio se debe a Hartle [57], que comenzó una serie de artículos al respecto en el año 1967. Su trabajo constituye la base de la mayoría de los enfoques analíticos para construir modelos numéricos con simetría axial [104].

El modelo de Hartle describe configuraciones de cuerpos compactos aislados rotando en equilibrio, esto es, en el régimen estacionario y dentro del marco de la teoría de perturbaciones hasta segundo orden en RG. “Aislado y compacto” quiere decir que la estrella termina en una superficie que separa su interior, que se suele modelizado como un fluido (perfecto), de un exterior de vacío, que se escoge asintóticamente plano de manera que el campo gravitatorio decae a cero a medida que uno se aleja del objeto compacto. El esquema perturbativo se elabora sobre una configuración esférica sin rotación, en otros términos, sobre “una pelota en el vacío”. Sobre ésta se toman perturbaciones estacionarias y axisimétricas a primer y segundo orden. El modelo de Hartle se construye sobre una serie de suposiciones implícitas, que resultan razonables en la mayoría de los casos: el interior del cuerpo es un fluido perfecto con una ecuación de estado barotrópica, rota uniformemente (no hay movimientos convectivos y la rotación es rígida), y la configuración completa admite simetría axial y ecuatorial.

Bajo estas suposiciones, las perturbaciones a primer orden están descritas por una única función, que origina el “arrastre” del espaciotiempo (*rotational dragging of inertial frames*). Las perturbaciones de segundo orden vienen descritas por tres funciones. Los valores de estas funciones en la superficie de la estrella, calculados desde el interior dada una ecuación de estado y condiciones de regularidad en el origen, se utilizan para obtener la deformación y la masa total en términos de la densidad central y la rotación de la estrella. El “cambio en la masa”, definido como la contribución a la masa debido a la rotación, se calcula mediante la comparación de las masas del sistema en rotación y el estático dada una densidad central fija.

Sin embargo, al margen de las suposiciones explícitas, el modelo se construye sobre otra premisa implícita; la continuidad de las funciones que describen las perturbaciones a través de la superficie de la estrella, en términos de un sistema particular de coordenadas. El modelo de Hartle se basa esencialmente en enlazar el interior y exterior igualando las condiciones de contorno en la frontera común, la superficie de la estrella. En RG esta situación se contempla bajo la teoría de enlace de espaciotiempos. Supongamos dos espaciotiempos con frontera, de forma que las fronteras de uno y otro se puedan identificar, que sean *difeomorfas*. Ahora imaginemos que queremos formar un espaciotiempo que resulte de la unión de estos dos iniciales a través de la frontera común, de manera que la geometría esté bien definida en todo el espaciotiempo, incluyendo en la frontera común, y que podamos formular las ecuaciones de Einstein (en el ámbito de las distribuciones). Esto es posible si se cumplen una serie de condiciones geométricas, conocidas como condiciones de enlace. Se sabe que una vez que dos espaciotiempos las satisfacen, existe un sistema de coordenadas (conocido como admisible en el sentido Lichnerowicz) en el que las funciones métricas y sus primeras derivadas son continuas. Sin embargo, desconocemos cómo este hecho se traslada a un esquema perturbativo. Ni siquiera la continuidad de las funciones que describen las perturbaciones está garantizada a priori. En cualquier caso, una elección explícita de coordenadas (y *gauge*) en el cual las perturbaciones satisfacen ciertas condiciones de continuidad y diferenciabilidad puede constituir una suposición implícita que, en principio, podría restar generalidad al modelo. Aún peor, podría ser incorrecta y conducir a resultados erróneos.

Por otro lado, el tratamiento del enlace de espaciotiempos en el marco de la teoría de perturbaciones se complica, puesto que donde en el problema exacto había espaciotiempos individuales ahora encontramos familias uniparamétricas de espaciotiempos. La identificación de éstos entre sí pone de manifiesto una libertad inherente a la teoría de perturbaciones, conocida como libertad de *gauge* espaciotemporal. Además de esta libertad en la identificación de espaciotiempos, surge una libertad adicional correspondiente la identificación de las fronteras. En [79], M.Mars (USAL) analizó este problema de forma general y consistente, formulando las condiciones de enlace perturbadas hasta segundo orden de manera independiente de las coordenadas empleadas y de las libertades inherentes a la teoría de perturbaciones, sin que tampoco sea necesario recurrir a la formulación de cantidades invariantes *gauge*.

Con el fin de establecer hasta qué punto la hipótesis original de “continuidad” de las funciones en el esquema de Hartle tiene consecuencias, diseñamos un programa destinado a poner rigor en el modelo, basándonos en primeros principios. En nuestro trabajo [95] empezamos revisando el modelo de Hartle dentro de la teoría presentada en [79]. Hasta donde somos conscientes, es la primera vez que se emplea [79] para realizar enlaces perturbados a segundo orden. Hemos demostrado que los valores de las funciones que describen las perturbaciones se pueden ajustar para coincidir en la superficie, tal y como se da por hecho en el modelo de Hartle, a excepción de una de las funciones en las perturbaciones a segundo orden. Esta función presenta un salto en la superficie que es proporcional al valor de la densidad de energía allí. La presión debe anularse en la superficie, como consecuencia de las condiciones de enlace, pero no la densidad de energía, en general. Esta discontinuidad contribuye al cálculo del cambio en la masa y por lo tanto, a la masa total de la configuración rotante. El resto de las cantidades, como las que originan el arrastre del espaciotiempo o determinan la deformación de la estrella, no necesitan ninguna modificación.

La expresión original del cambio en la masa dada en [57], debe ser, por tanto, modificada con este término adicional. Sin embargo, dado que este término se anula si la densidad de energía es cero en la superficie, los modelos más comunes de estrellas de neutrones, así como cualquier otro basado en ecuaciones de estado politrópicas (densidad de energía proporcional a potencias de la presión) no se ven afectados, y el cálculo de la masa de la estrella en rotación no requiere ninguna corrección. Sin embargo, los modelos de estrellas en rotación *à la Hartle*, en los cuales la densidad de energía no se anula en la superficie, tales como estrellas homogéneas (de densidad constante) o de materia extraña (estrellas de quarks), han de ser corregidos. Los diagramas que típicamente se emplean para caracterizar estos cuerpos compactos muestran la masa de la estrella frente a su densidad central o su radio. Estos se verán modificados por el efecto del término corrector, pero no es posible estimar a priori si de forma relevante o no. Para ello hemos retomado el artículo clásico de Chandrasekhar y Miller sobre estrellas homogéneas [30] y hemos calculado el cambio en la masa con la expresión correcta en [93]. También hemos estudiado el caso de estrellas de materia extraña, contemplado originalmente por Miller y Colpi en [34], y aunque obtenemos diferencias considerables, aún no hemos publicado los resultados. En ambos casos la contribución del término corrector al cambio en la masa resulta ser importante y para nada despreciable. Cabe destacar que los códigos numéricos desarrollados para la realización de este trabajo son fácilmente ampliables para situaciones más realistas, como pueden ser interiores estelares compuestos por varios fluidos.

En el momento de escribir nuestro artículo [95] no reparamos en que el término corrector a la masa contribuye al límite newtoniano del modelo. En un artículo [96], complementario a [95], calculamos dicho límite y mostramos cómo ese término aparece, aunque de manera implícita, en el trabajo original de Chandrasekhar sobre polítropos en rotación en el marco de la gravedad newtoniana [23]. Como la mayoría de los modelos de estrellas son polítropos, la aparición de este término había sido de alguna manera olvidada, incluso en la revisión del enfoque newtoniano que se presenta en el trabajo de Hartle [57]. Las condiciones de enlace generales en el marco de la gravedad newtoniana no se formulan en el trabajo original de Chandrasekhar [23], así que el modelo se construye asumiendo la continuidad del potencial y de su derivada a través de la superficie de la estrella sin deformar. Posteriormente, publicó otro artículo [29] aplicando nuevas condiciones de enlace: continuidad de funciones y derivadas primeras a través de la superficie de la estrella deformada. Sorprendentemente los resultados no se alteraban. En [96] revisamos este asunto y formulamos las condiciones de enlace newtonianas generales. Al particularizarlas para un fluido, comprendimos que los resultados de [23] y [29] coinciden, de nuevo, porque en ambos se emplea un polítropo como ecuación de estado.

Sin embargo, queda un aspecto final argumentado en el artículo original [57] que necesita ser demostrado rigurosamente: las funciones en las perturbaciones de segundo orden no contienen sectores con $l > 2$ en una expansión en polinomios de Legendre. Este es un resultado común al modelo newtoniano para polítropos de Chandrasekhar [23] aunque no fue demostrado rigurosamente hasta 35 años después, por Kovetz en [72]. De vuelta al modelo de Hartle [57], este asunto se discute empleando, de nuevo, argumentos que se basan en el carácter global y la continuidad de las funciones perturbativas, en tanto que la dependencia angular de éstas queda determinada por su comportamiento en el centro de la estrella, donde se exige regularidad, y por su comportamiento muy lejos de la estrella, donde se pide asintoticidad plana. Nosotros hemos abordado este estudio caracterizando los problemas para el interior y exterior por separado en términos de los operadores y funciones adecuadas, las cuales han de satisfacer ciertas condiciones impuestas por el enlace, además de pertenecer a espacios de funciones apropiados. Dicha tarea ha sido desarrollada en colaboración con M. Mars (USAL), en un artículo que pronto debería estar terminado. En la tesis escrita se incluye la descripción del problema y demostramos que efectivamente, la única estructura angular posible de las perturbaciones son las propuestas en [57].

Por último, cabe destacar que el enlace perturbado que se calcula en nuestro trabajo [95] se lleva a cabo primero en un marco púramente geométrico, sin usar las ecuaciones de campo. Por lo tanto, puede ser usado en situaciones más generales, como en otras teorías alternativas a la RG para las cuales el modelo de Hartle ya se ha generalizado en la literatura, y encontrar así las correspondientes correcciones a la masa.

## Condiciones de enlace en teorías cuadráticas de gravedad

Las condiciones de enlace en RG han sido ampliamente investigadas. En [15] se incluye una presentación rigurosa del formalismo de enlace de dos espaciotiempos con frontera a través de la frontera común, teniendo ésta la libertad de ser una hipersuperficie de carácter causal arbitrario, e incluso de cambiarlo de punto a punto.

No obstante, aparte de la RG existe un amplio espectro de teorías geométricas de gravedad alternativas a ésta, candidatas para la explicación satisfactoria de fenómenos como, por ejemplo, la energía oscura. Entre el amplio espectro de estas teorías, podríamos seleccionar dos tipos ampliamente estudiados en la bibliografía. Uno de ellos son las teorías $F(R)$, que sustituyen el escalar de curvatura $R$ en la Lagrangiana de Einstein-Hilbert de la RG por una función arbitraria de éste, de ahí que sean conocidas por el nombre de teorías $F(R)$. El otro tipo de teorías a las que hacemos alusión se conocen como teorías de gravedad cuadrática y son aquellas que resultan de considerar una lagrangiana que incluye invariantes de curvatura cuadráticos. En particular, tenemos términos del tipo $R^2$, $R_{ab}R^{ab}$ y $R_{abcd}R^{abcd}$, siendo estos dos últimos los tensores de Ricci y Riemann respectivamente.

Entonces, un tema de interés es conocer las condiciones de enlace en este tipo de teorías de gravedad alternativas a la RG. En primer lugar, permitiría establecer similitudes (o diferencias) entre la RG y estas teorías modificadas, con lo que podríamos alcanzar una mejor comprensión de éstas. Por otro lado, la correcta modelización de cuerpos compactos en rotación, en el marco de teorías alternativas, combinada con datos observacionales constituye una herramienta importante para encontrar restricciones sobre estas teorías, bien ajustando los valores de los parámetros libres que contengan o en casos más drásticos, descartando la validez de la teoría.

En dos artículos recientes [99] y [98] J. M. M. Senovilla (UPV/EHU) ha obtenido las condiciones de enlace para las teorías $F(R)$ generales, así como las ecuaciones que satisface la distribución superficial de materia, generalizando así las ecuaciones de Israel de la RG. Cabe destacar que salvando algún caso particular, éstas son comunes a RG y teorías $F(R)$. De este trabajo se desprenden dos conclusiones importantes que desarrollamos a continuación.

La primera de ellas es que el tensor energía momento puede presentar contribuciones de tipo doble-capa o dipolares (de tipo $\delta'$). Elaboremos un poco más este punto. Para ello, consideremos el escenario de la electrodinámica clásica, donde estas contribuciones están asociadas a cambios muy abruptos del potencial eléctrico. En particular, un campo dipolar viene generado por una configuración de cargas positiva y negativa muy concentradas, separadas por una pequeña distancia, tan pequeña que matemáticamente se formula como el límite cuando ésta tiende a cero y podemos hablar entonces de una distribución superficial. La correspondiente ecuación de Poisson, que relaciona derivadas segundas del potencial eléctrico con la densidad de carga, adecuada para describir esta configuración requiere, precisamente, de un perfil de densidad de tipo $\delta'$. Este objeto matemático es una distribución que aplicada a una función de prueba, retorna el valor de la derivada normal de la función de prueba allí donde la $\delta'$ tiene soporte (bajo integración, como es usual en teoría de distribuciones). Es importante destacar aquí que no sólo es relevante el valor de la función de prueba en el soporte de la $\delta'$, sino que también es necesario conocer su comportamiento en un entorno.

Volviendo a teorías de gravedad, este tipo de comportamiento no es el esperado, ya que sólo existen masas positivas, de forma que la gravedad es atractiva. De hecho, en RG no existen dipolos, lo que resulta razonable. En cambio, las teorías modificadas sí que admiten contribuciones dipolares localizadas en hipersuperficies de enlace. Además, la contribución de estas doble-capas resulta esencial para que el tensor energía momento se conserve.

En segundo lugar, cabe destacar que, en general, una solución enlazada en RG de manera que no contenga distribuciones de materia superficiales, no sigue siendo una solución del mismo tipo en teorías $F(R)$, sino que presentará distribuciones superficiales de materia y contribuciones del tipo doble-capa.

Motivados por estos resultados, hemos realizado un trabajo análogo para teorías cuadráticas de gravedad generales [94]. Las conclusiones son similares a las obtenidas en [99] y [98]. Encontramos que las teorías cuadráticas de gravedad también presentan contribuciones de tipo dipolo y verificamos la conservación del tensor energía momento, para lo cual las doble-capas siguen siendo indispensables. Además, generalizamos las ecuaciones de Israel, que siguen siendo idénticas a las de RG cuando se prescinden de las contribuciones de tipo doble-capa. De hecho, discutimos los posibles escenarios que se pueden dar sobre la hipersuperficie de enlace y estudiamos las condiciones para que se produzca cada uno de ellos. Esta casuística abarca casos como un buen enlace sin ningún tipo de distribución superficial de energía o dobles capas, que haya distribuciones superficiales sin dobles capas, o que haya dobles capas puras.

Asimismo, hemos dedicado una serie de secciones a derivar con detalle ciertos resultados geométricos que se obtienen en teoría de distribuciones, como la identidad de Ricci. Por otro lado, también discutimos los problemas que ocasiona una técnica habitual en la bibliografía para enlazar espaciotiempos en el marco de teorías alternativas, y que consiste en el uso de coordenadas gaussianas adaptadas a la hipersuperficie de enlace. En estos casos, la interpretación de los términos $\delta'$ no es correcta.

# Publicaciones

Artículos u otras publicaciones realizadas en el transcurso del disfrute de la beca predoctoral.

## Published Papers

- **Supermassive Cosmic String Compactifications**

  J.J. Blanco-Pillado, B. Reina, K. Sousa, J. Urrestilla. Journal of Cosmology and Astroparticle Physics, **1406** 001 (2014)

- **Revisiting Hartle's model using perturbed matching theory to second order: amending the change in mass**

  B. Reina, R. Vera. Classical and Quantum Gravity **32** 155008 (2015).

  Esta publicación ha sido incluida en IOPselect y se le ha dedicado una reseña en la web CQG+ bajo el título *On the mass of compact rotating stars.*

- **Slowly rotating homogeneous masses revisited**

  B. Reina. Monthly Notices of the Royal Astronomical Society **455** 4512-4517 (2016)

- **On the mass of rotating stars in Newtonian gravity and GR**

  B. Reina, R. Vera. Classical and Quantum Gravity **33** 017001 (2016)

- **Junction conditions in quadratic gravity: thin shells and double layers**

  B. Reina, J. M. M. Senovilla, R. Vera. Classical and Quantum Gravity, in press (2016)

## Proceedings

- **Revisiting Hartle's model for relativistic rotating stars** (ERE2012)

  B. Reina, R. Vera. Progress in Mathematical Relativiy, Gravitation and Cosmology, Springer Proceedings in Mathematics and Statistics **60** 377 (2014).

- **Hartle's model within the general theory of perturbative matchings: the change in mass** (ERE2014)

  B. Reina, R. Vera. Journal of Physics: Conference Series **600** 012013 (2015)

# Contents

# 1

# Introduction

General Relativity (GR) is the theory of gravitation proposed by Einstein 100 years ago, back in 1915. He came up with the observation that a gravitational field and an accelerating frame of reference are locally equivalent. Hence, gravity can be locally removed. This fact suggests that it is definitely not like other interactions, such as the electromagnetism for instance. Rather than an interaction that occurs in the spacetime, it stands as something imprinted into the spacetime itself. In absence of forces, test particles move along extremal curves, known as geodesics, in a 4-dimensional Lorentzian manifold. Einstein's field equations relate manifestly the geometry of the spacetime with the distribution of energy and momentum of matter.

The theory has a great predictive power and it is well supported by observations. For instance, the so called *classical tests* account for novel effects with respect to Newtonian gravity that arise from the analysis of the geodesics in the Schwarschild spacetime. From the study of null geodesics, one encounters the bending of light by a massive body. The study of timelike orbits explains the precession of the perihelium of Mercury, and the study of emission of signals between two static observers reveals the gravitational redshift effect.

Beyond these classical tests, many others have been proposed. Experimental data collected from measurements taken at the level of the Solar System is used to constrain theories of gravity. For instance, restrictions on the PPN (parametrized post Newtonian) parameters show the best agreement with GR [112]. These mentioned tests are valid to explore GR in the weak field approximation. However, the strong regime of gravity can also be explored by tests based on compact binaries and gravitational waves. Hulse and Taylor received the 1993 Nobel prize for the study of the PSR B1913+16 binary system, composed by a pulsar and a companion neutron star. Exploring the changes in the radio emissions of the pulsar, they gave account of the time dilation effect. But the big deal was the study of the advance of the periastron of the pulsar. Not only they noticed that the periastron advanced $\sim 4.2$ degrees per year, but also that each year the pulsar reached the periastron earlier than expected. This shrinking of the orbit has been perfectly explained by the energy loss by emission of gravitational radiation, which constitutes an indirect detection of gravitational waves.

The first direct detection of gravitational waves has been recently announced, on February 2016, by the LIGO collaboration [1]. The astrophysical system that emitted the signal was a binary black hole merger. The analysis of the signal [2] is fully compatible with GR, although given the limitations of the experiment they have not been able to determine whether the degrees of polarization of the wave were just the two predicted by the GR or there exists any other as predicted by alternative theories of gravity. The hope is that data based on gravitational waves will provide new constraints on theories of gravity over the next years.

GR has proven to be essential in order to describe successfully high energy astrophysical events. Most of them, such as the gravitational collapse of massive stars, the coalescence of binary systems composed of neutron stars and black holes, or the phenomena related to accretion disks require an accurate relativistic description of compact objects (see [49] for a review of hydrodynamics in GR).

## On relativistic stars

Zwicky and Baade adressed in the year 1934 [7] and [6] the possibility of stars composed mainly of neutrons, born as a result of supernova processes. The mechanisms that generate thermal pressure and hold ordinary stars against collapse due to gravity is not present in neutron stars, which support themselves due to the neutron degeneracy pressure. They described neutron stars as small and very dense objects. In order to have a picture in mind, we could imagine an spherical object with a typical mass of $1.4 M_{\odot}$ packed within a radius of 10 km. Taking into account the mass of the neutron, we can estimate for these values that these compact objects consist of $10^{57}$ neutrons packed together by gravitational interactions. An estimation of the density results in $6.65 \cdot 10^{14}$ g cm$^{-3}$, which is above the density of the atomic nucleus $2.3 \cdot 10^{14}$ g cm$^{-3}$.

At the same time, in the year 1930, Chandrasekhar claimed that there was a mass limit for white dwarf stars, at which electron degeneracy pressure was no longer sufficient to balance the gravitational force. Later on, in the year 1933, he published a series of works under the title “The equilibrium of distorted polytropes” [23], [24], [25], [26], where he developed a perturbation formalism over spherically symmetric isolated fluid balls with a polytropic equation of state (EOS) in Newtonian gravity. These were aimed at studying modifications in the shape produced by slow rotation or/and tides due to a second companion body. In the first one [23], and most relevant for this thesis, slowly and uniformly rotating equilibrium configurations are constructed. The model of a spherically symmetric fluid ball with a polytropic equation of state reduces to the study of one function, called the Emden function. Taking this setting as the background configuration, perturbations of the Emden potential driven by a slow rotation parameter are developed.

Since the model describes isolated bodies, an exterior vacuum needed to be considered. The gravitational potential generated by the fluid and the potential of the vacuum region were matched at the undeformed surface of the star (the spherical boundary) in [23]. However, Jardetzky pointed out that the matching method was inaccurate [69] and suggested that the matching should be performed in the distorted surface of the star. The Newtonian polytropic stars were retaken by Chandrasekhar and Lebovitz in 1961, to analyze the problem of oscillations and stability in the works [27], [28] and [29]. The effects of rotation and modes of pulsation, computed in terms of some specific functions encoded in a *superpotential*, become completely determined after demanding that this last is continuous and has a continuous radial derivative, this time at the distorted surface of the polytrope. They found that the results of the matching were unaltered from [23], so that they found no difference between imposing the continuity of the functions in the distorted and undeformed boundary of the star. In this way, they answered the objection raised by Jardetzky.

In these models for polytropes the angular velocity couples only to the modes $l = 0, 2$ of a Legendre polynomials expansion of the gravitational potential (see for instance eq. (4) in [72]) and it is reasonable to think that the rest of the modes, that are not excited by rotation, vanish. Nonetheless, this analysis was ommited by Chandrasekhar and Lebovicz, and was retaken by Kovetz in [72], where a consistent proof was given.

Regarding the analysis of compact objects in General Relativity, Tolman studied in [107], back in 1939, a variety of static spherically symmetric metrics that included the Schwarzschild interior solution with constant density. In the same year, Oppenheimer and Volkoff [90] wrote the equations that describe interior configurations in the standard form. Their objective was to find a kind of Chandrasekhar limit for neutron stars. They used the equation of state for a cold Fermi gas and obtained a value for the maximum mass of $\sim 0.7M_{\odot}$.

In 1967, Hartle and Sharp published a contribution [63] with the foundations to develop a relativistic model of stellar rotation. They formulated a relativistic variational principle for stationary and axially symmetric configurations of matter ruled by a barotropic equation of state, with a fixed number of baryons and angular momentum. Taking variations with respect to the fluid flow and the baryon density, they found that the configuration extremizing their functional rotates rigidly and has a constant injection energy (so that it satisfies the hydrostatic equilibrium first integral). This was the starting point of a remarkable series of, at least, 9 articles, [57], [64], [65], [58], [59], [61], [60], [62], [66], that settled a whole perturbational approach to describe slowly rotating and oscillating stars in GR.

Hartle developed in [57] his classical model to describe slowly rotating stars in equilibrium perturbatibely to second order in some rotation parameter. The model describes perfect fluid stars, with arbitrary barotropic equation of state, rotating rigidly in equilibrium and provides expressions to compute the properties of the compact body, such as the mass, angular momentum and quadrupole moment or the distortion of the surface due to rotation. It constitutes the basis of most of the analytical approaches and is widely used to construct numerical schemes [104] in axial symmetry.

In collaboration with Thorne, Hartle computed explicit numerical solutions for various equations of state [64] and with Friedman [61] he used the equation of state of a $n = 3/2$ polytrope. The model was extended to account for the third order perturbations in [59]. The work [58] is aimed at formulating a procedure to compute the mass of rotating configurations, without the need of computing the second order perturbations (developed in [57]), and it is claimed to be valid for both differential and uniform rotation. The remaining works are devoted to study the development of a formalism to treat radial, or quasiradial, oscillations and to formulate a criterion for the stability of the rotating stars.

Nowadays Hartle's model stands as one of the most used schemes for the study of slowly rotating stars. In fact, it is one of the few analytical approaches that we have for the problem, although the final equations must be solved by numerical integration. Regarding the slow rotation approximation, Hartle sets a scale [57] of angular velocities given by $\Omega^* \equiv \sqrt{GM/a^3}$, where $M$ and $a$ are the mass and radius of the static configuration. The requirement that angular velocities are much smaller than $\Omega^*$ implies that every particle in the fluid must move at non-relativistic velocities. The quantity $\Omega^*$ is closely related with the mass shedding limit, that occurs when the angular velocity of the star reaches the velocity of a particle in a circular Keplerian orbit at the equator. In the Newtonian regime, the mass shedding velocity corresponds to $(2/3)^{3/2}\Omega^*$ [104], [13]. With the numbers for neutron stars given above, the frequency corresponds to about $f^* = \Omega^*/2\pi = 2000\,\mathrm{Hz}$.

Berti *et. al* made a quantitative study in [13], and shed some light onto the question of how slow is the slow rotation approximation. For this, they compared the quadrupole moment of rotating configurations computed by using numerical approaches for rapidly rotating stars (CST-rns) to Hartle's approach, sharing the mass and angular momentum. Note that these two quantities are enough to adjust the two parameters that must be specified in Hartle's model, from where the quadrupole moment can be computed. The results depend on the equation of state used, but not *very strongly* (see Table 6 in [13]).

Again, to have some numbers in mind, we can round off the results to think of deviations that go from 10% to 20% in the value of the quadrupole moment for angular velocities of 20% of $\Omega^*$, i.e. 433 Hz with the numbers above. Thus, Hartle's model is a good approximation to study typical pulsars, since statistical studies of the angular velocities reveal neutron stars and pulsars with typical values of hundreds of Hz (see [92] and references therein). However, the approximation turns out to be inaccurate for rapidly rotating pulsars, such as PSR J1748-2446ad, the fastest pulsar known rotates at 716 Hz.

The analytical character of Hartle's model has paved the path to many works that consist of some generalization of the fluid, vacuum or both. For instance, Bradley et al. revisited Hartle's formalism in [16], but they substituted the barotropic equation of state by the Petrov D condition for the interior region. They concluded that some Petrov type D perfect fluids can be matched to an asymptotically flat vacuum, under some restrictions in the parameters of the model (see Figure 2 therein). As expected, Wahlquist is not among those interiors. However, some of the succesfully matched interiors yielded a reasonable equation of state, i.e. with subluminal speed of sound. On the other hand, they also considered the case of non-asymptotically flat vacuum, for which the matching with perfect fluid interiors was succesfully performed.

Out of the context of General Relativity, Hartle's model has been generalized to describe rotating compact objects in alternative theories of gravity, such as $F(R)$ [101], Einstein-Dilaton-Gauss-Bonnet gravity [91] or Chern-Simons theory [3].

Apart from Hartle's model, other analytical perturbation methods to describe isolated rotating compact objects have been developed. For instance, the so called CMMR approach [21] is a double perturbative scheme to describe rotating stars in equilibrium. It is based on a post Minkowskian expansion, where some parameter $\lambda$ controls the deviation from the flat spacetime, and a slow rotation (or slow deformation) approximation, controlled by the parameter $\Omega$. This latter is analogous to the slow rotation parameter in Hartle's model. In this formalism a global spacetime is built by matching a perfect fluid, with an asymptotically flat vacuum making use of harmonic (and quoharmonic) coordinates. A hypersurface of vanishing pressure is identified and matching conditions are imposed there. One of the main advantages of the method is that one can reach high orders in the rotational approximation. It has been applied to several equations of state: constant density in [21], polytropes in [84] and linear equations of state [36]. For instance, in [39] an explicit model for a linear equation of state is computed to orders $\lambda^{9/2}$ and $\Omega^3$. Thanks to the analiticity of the model, several questions such as the impossibility of i) matching the perfect fluid interior to a Kerr vacuum and ii) matching the Wahlquist solution to an asymptotically flat vacuum, have been worked out in [84] and [38] respectively. For more applications of the CMMR scheme, see [45].

Out of the perturbation arena, the problem of a rotating star has been studied under several numerical approaches, such as the BI and KEH schemes, or the BGSM scheme (see [104]). Butterworth and Ipster formulated the field equations for the metric potentials as a set of three elliptic equations plus one quadrature, supplemented with boundary conditions at infinity that ensure aymptotic flatness [20]. The KEH scheme [70], [71] formulates these equations as integral equations by means of the appropriate Green's functions and impose the asymptotic flatness conditions truncating the spacetime. The CST scheme [35] avoids this problem by making a coordinate transformation that compactifies the radial coordinate, so that higher accuracy is obtained. Stergioulas and Friedman implemented the CST scheme in the *rns* code [105], able to compute sequences of rapidly rotating stars, with uniform rotation. Bonazzola et al. [14] worked in a 3+1 decomposition that takes advantage of the stationarity of the model, in the so called Maximal Slicing Quasi Isotropic coordinates.

However, as far as we understand, these models do not describe compact bodies in the sense that there is not a hypersurface that separates the fluid and vacuum regions. The fluid extends all over the spacetime, satisfying some suitable decays at infinity, so that the spacetime is asymptotically flat.

## On Hartle's model

In this thesis we revisit the perturbational approach by Hartle in [57]. Hartle's scheme depicts the equilibrium (stationary regime) configurations of rotating isolated compact bodies in perturbation theory up to second order in GR. "Isolated and compact" means that the star finishes at a surface that separates its interior from a vacuum exterior, which is assumed to be asymptotically flat (the gravitational field decays to zero as one moves away). The perturbative scheme is based on a spherical (non-rotating) background configuration (a ball in vacuum), on top of which first and second order stationary and axisymmetric perturbations are computed. Hartle's model carries some explicit assumptions, which are expected to hold eventually in most cases; the interior of the body is a perfect fluid with a barotropic equation of state, rotates uniformly (no convective motions and rigid rotation), and the whole configuration admits axial and equatorial symmetries.

Given these assumptions, the first order perturbation is driven by a single function that accounts for the rotational dragging of inertial frames. The second order is described by three functions. The values of these functions at the surface of the star, computed from the interior given an equation of state and conditions at the centre, are used to obtain the deformation and the total mass in terms of the central density and the rotation of the star. The "change in mass" $\delta M$, defined as the contribution to the mass due to the rotation, is then computed by comparing the masses between the rotating and the static configurations given, e.g., a fixed central density.

Apart from the explicit assumptions mentioned above, the model is constructed upon another implicit premise; the continuity across the surface of the star of those functions driving the perturbation, in terms of a particular coordinate system (a class, in fact). Hartle's scheme is essentially based on joining the interior and the exterior problems by properly "matching" the boundary conditions at the common boundary, the surface of the star. In GR that accounts for the matching of the spacetimes concerning the two problems. It is known that once two spacetimes are matched there exist a coordinate system (called Lichnerowicz admissible) in which the metric functions and their first derivatives are continuous.

However, how this fact translates to a perturbative scheme remains to be settled. Even the continuity of the functions driving the perturbations is not ensured a priori. In any case, an explicit choice of coordinates (and gauge) in which the perturbations satisfy certain continuity and differentiability conditions may constitute an implicit assumption that, in principle, could subtract generality to the model. Worse, it could turn out to be a wrong choice, and lead to wrong outcomes.

In order to establish up to which extent the original "continuity" assumption had any consequence, or none at all, we have devised a programme aimed at putting the whole model on firm grounds, based on first principles. This is the main aim of this thesis. After the preliminar Chapter 2 devoted to the matching of spacetimes, in Chapter 3 we describe the general and consistent theory of perturbative matchings to second order devised in [79], independent of the coordinates and gauges used, with no need of constructing gauge-independent quantities (which may lead to problems [80]). We have also collected some results given in [80]. The next chapters, Chapter 5 to 7, are devoted to revisiting Hartle's model within this theory to carry out the perturbed matching to second order. To this aim, in Chapter 5 we introduce the set up of the perturbed configurations needed for the (stationary and axially symmetric) geometries that are going to be used for the interior and exterior regions, together with the perturbed matching hypersurface. We present, in the form of two propositions, the necessary and sufficient conditions that the first and second order perturbations of the geometries at either side and the perturbed hypersurface must satisfy in order to match. In this first step, the perturbative matching is computed on a purely geometric setting in a first step, without using any field equations. In Chapter 6 the Einstein's field equations are obtained in terms of some convenient quantities. Finally, in Chapter 7 the interior and exterior problems at first and second order are imposed using Hartle's model explicit assumptions: perfect fluid interior with barotropic equation of state rotating rigidly and absent of convective motions, asymptotically flat vacuum exterior, and global equatorial symmetry. We also assume, in this chapter, the angular structure of the perturbations argued in [57]. The particularization of the previous propositions to Hartle's setting is then analysed in detail. The result concerning the interior and exterior problems at second order is finally given in the form of a Theorem, in which the equations the functions at either side must satisfy together with their corresponding matching conditions are given in full.

We prove that the values of the functions driving the perturbations can be set to coincide on the surface, as assumed originally in [57], except for one at second order. This function presents a jump at the surface proportional to the value of the energy there. The pressure needs indeed vanishing at the surface, as a consequence of the matching, but not the energy density in general. This jump contributes to the calculation of $\delta M$, and therefore the total mass of the rotating configuration. The rest of the quantities, like the frame dragging and the deformation of the star, need no modification. The fact that some relevant second order function may had a jump across the boundary has appeared previously in [16] (see equation (46) there) and in [46], where a correct expression of $\delta M$ is given. Nonetheless, the exact relationship of their functions with the original functions in [57] and thus the discrepancy in the computation of $\delta M$ in [57], had not been realised at the moment.

The original expression of $\delta M$, therefore, has to be amended with this additional term. Nevertheless, since that term vanishes whenever the energy density is zero at the surface, the standard neutron star models, and any other consisting of polytropes in particular (energy density proportional to the pressure to some power), are not affected and the computation of the mass of the rotating star needs no correction. However, rotating star models (based on Hartle's scheme) with non vanishing energy density at the surface, such as homogeneous stars or strange quark stars, need a re-calculation of the curves representing the mass in terms of the central density. Those curves have been recalculated in Chapter 10 for homogeneous star models, completely described by Chandrasekhar and Miller in their classical paper [30], and for stars with an equation of state often used to represent quark matter. This is part of an ongoing work in collaboration with J.A. Font and N. Sanchís-Gual (UV), where we have developed a numerical code to compute Hartle's model with the amended mass for the most common EOS: polytropes, constant density, linear EOS and tabulated EOS. We expect to complete it soon including multilayer interiors. The contribution of the amending term to $\delta M$ has been found to be far from negligible for constant density stars and strange quark matter stars.

In Chapter 9 we compute the Newtonian limit of the amending term and show how that term appears indeed, although implicitly, in the original work on Newtonian rotating polytropes by Chandrasekhar [23]. Since most models are polytropes, the appearance of that term had been somehow forgotten, even in the review of the Newtonian approach in Hartle's work [57]. We also discuss the perturbed Newtonian matching conditions for the problem of a rotating star, resolving the problem raised by Jardetzky about the matching procedure in Chandrasekhar's work.

Still, a final aspect mentioned earlier needs to be rigorously proven given the present state of things: it is argued in the original paper [57] that the function at first order perturbations depends only on the radial coordinate and the functions at second order contain no $l > 2$ sectors after an expansion in Legendre polynomials. This is the aim of Chapter 8, where we provide a proof based on maximum principles. The results therein generalize those of Kovetz for Newtonian polytropes [72], not only to a relativistic context, but also for any barotropic equation of state. This work has been done in collaboration with Marc Mars.

## Modified gravity

Apart from the astrophysical phenomena, General Relativity can also be applied to describe the large structure of the Universe showing good agreements with observations. In fact, GR started a new branch in science on its own: Cosmology. Nowadays the most accepted description of the Universe is achieved by the standard or *concordance model* in Cosmology, also known by ΛCDM model. Its main ingredients are a FLRW spacetime, spatially homogeneous and isotropic. The energy momentum tensor corresponds to a perfect fluid that incorporates components of radiation and matter, being this baryonic but mostly non-baryonic cold dark matter, and dark energy. The dark energy component, introduced to explain the accelerated expansion of the universe, is encoded in a cosmological constant type energy momentum. The ΛCDM model together with its perturbations manages to explain most of the current cosmological observations, showing great compliance with the data. However, there are aspects that remain unclear in this setting.

In the realm of the "small" and to account for alternative explanations to dark matter and the acceleration of the rate of expansion, actions more general than the Einstein Hilbert Lagrangian density (which gives rise to GR) are often considered. One of the simplest modifications to GR consists of $F(R)$ theories, where the Ricci scalar $R$ in the Einstein Hilbert action is substituted by a suitable function $F(R)$. One of the most relevant models inside this class of theories was proposed by Starobinsky with a Lagrangian density $F(R) = R + \alpha R^2$, with $\alpha > 0$. This Lagrangian has good properties from the point of view of field theory (see [4] and references therein). Other type of modification of the Einstein-Hilbert Lagrangian comes from the consideration of other invariants constructed from the curvature tensors, for example by means of contractions of the Riemann and the Ricci tensors. In fact, effective actions in string theory contain infinite series of higher curvature corrections to Einstein-Hilbert action. Among them, in this thesis we focus on quadratic theories of gravity, with a Lagrangian density of the form $a_1R^2 + a_2R_{ab}R^{ab} + a_3R_{abcd}R^{abcd}$. These can be seen as an effective theory truncated to second order in the curvature and presents nice properties regarding the quantum regime [4], [47]. However, these theories include some inconvenient extra degrees of freedom apart from the graviton. A particular combination of the constants $a_1 = 1$, $a_2 = -4$, $a_3 = 2$ leads to Gauss-Bonnet gravity, whose field equations contain second derivatives of the metric at most. This theory is equivalent to GR in 4 dimensions but not in higher dimensions. Precisely, high dimensional settings are often contemplated in modified gravity scenarios applied for Cosmology. In these, either the extra dimensions are compactified, or the physical fields are confined in four dimensional hypersurfaces embedded in a higher dimensional spacetime. The study of this *branes* requires a well understanding of the junction conditions in the corresponding theory of gravity.

The theory of matching of spacetimes in General Relativity, considered a product of the theory of hypersurfaces in geometry, contemplates the situation where two, a priori, independent spacetimes are joined across a common boundary to form a single spacetime. To be able to treat the curvature of spacetime in a distributional sense (at least), one requires that the metric is $C^0$ and piecewise $C^2$. The junction conditions contemplate the case where a $\delta$-type contribution is present in the energy momentum tensor, with support in some locallized hypersurface. This allows to model thin shells, surface layers of matter or impulsive gravitational waves. On the other hand, further conditions can be imposed in order to produce a proper matching, i.e. with an energy momentum tensor that contains discontinuities at most. This is used to build models of compact bodies surrounded by vacuum. In [82], a rigorous development of the matching of spacetimes theory was presented to deal with matching hypersurfaces of general causal character. We have included a summary of the matching of spacetimes theory in Chapter 2, collecting some of the results and conclusions regarding general (character-wise) boundaries [82], [83] and when symmetries are present [108].

In the context of $F(R)$ theories the junction conditions were developed in [44] by the use of convenient Gauss coordinates adapted to the matching hypersurface. However, this approach presents disadvantages, that we discuss in Appendix B. A development of the matching conditions for $F(R)$ using properly distribution theory was not satisfactorily done until [99] and [98], in which the field equations on the shell were provided, generalizing Israel equations from GR. Two main conclusions drop from these works.

The first one is that, in general, the distributional energy momentum tensor presents a *double layer* contribution. In analogy with electrostatics, one can think of this as a dipole contribution. This is surprising, because there are no negative masses. In addition, the presence of the double layer is needed for the conservation of the energy momentum tensor. The other relevant conclusion is that, in general, a solution generated by a proper matching in GR will present surface layers and double layers when set into a $F(R)$ theory.

Motivated by these results, we have carried out the analogous work for quadratic theories of gravity [94] mentioned above. We conclude that in these theories double layers may arise in matching hypersurfaces, and we verify that they are neccesary for the conservation of the energy momentum tensor. We also find the generalized Israel conditions, which in absence of double layers hold identical to those in GR. A detailed case study is performed analyzing the proper matchings, matchings with pure double layers and matchings with surface distributions. We have included our results regarding the junction conditions in quadratic theories of gravity in Chapter 11, supplemented by Appendix A that contains a collection of the most relevant computations used there.

## Notation, conventions and terminology

In this section we give the basic notation that will be used all throughout this work. Further notation is introduced in the body of the text when needed.

In this thesis we will restrict to $C^k$ spacetimes: Hausdorff connected oriented $n+1$ dimensional $C^{k+1}$ manifolds $\mathcal{V}$ endowed with a Lorentzian $C^k$ metric $g$. We will use the signature $-,+,\dots,+$.

Following standard definitions and notations we will use:

Equality

by definition $:=$, or identity: $\equiv$

The symbol $\stackrel{\Sigma}{=}$ denotes the equality of the involved quantities after performing the pullback onto a hypersurface $\Sigma$.

Indices:

We use greek indices $\alpha, \beta, \gamma, \dots = 0, 1, 2, 3, \dots, n$ for spacetime objects.

Latin indices $a, b, c, ... = 1, 2, 3, \dots, n$ refer to objects relative to hypersurfaces

The usual symmetrization and antisymmetrization will be denoted by ( ) and [ ] respectively.

Vectors: $\vec{v}$

One-forms: $\boldsymbol{v}$

Scalar product: $(\vec{v}, \vec{w})_g := v^\alpha w^\beta g_{\alpha\beta}$. One-forms operating on vectors denoted as $\boldsymbol{v}(\vec{w})$

Exterior product (of one-forms): $\wedge$ such that $\boldsymbol{v} \wedge \boldsymbol{w} = \boldsymbol{v} \otimes \boldsymbol{w} - \boldsymbol{w} \otimes \boldsymbol{v}$, where $\otimes$ is the tensor product.

Exterior derivative: d

Lie derivative with respect a vector field $\vec{\xi}$: $\mathcal{L}_{\vec{\xi}}$

Covariant differentiation: $\nabla$

Partial differentiation with respect to $x$ or to an indexed coordinate $x^\alpha$: $\partial/\partial x$, $\partial_x$, and the subscript $_{,x}$, or $\partial/\partial x^\alpha$ and $_{,\alpha}$ respectively.

We will denote partial derivatives with respect to the first argument of a function, usually the radial coordinate, by a prime.

Riemann tensor: $(\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu) \, w_\lambda = R^\sigma{}_{\lambda\mu\nu} w_\sigma$

Ricci tensor: $R_{\alpha\beta} \equiv R^\sigma{}_{\alpha\sigma\beta}$

We will use $|_\Sigma$ to denote the restriction of a spacetime object to points on an embedded hypersurface $\Sigma$. If used on a function it can also denote the pullback of the function to $\Sigma$. This should be understood by the context.

Unless otherwise stated, in this thesis we will use $G = c = 1$. When numerical solutions are presented, the following values in the S.I. have been used

$$\begin{aligned} M_\odot &= 1.9891 \cdot 10^{30}\,\mathrm{kg}, \\ G &= 6.67384 \cdot 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}, \\ c &= 299792458\,\mathrm{m\,s^{-1}}. \end{aligned}$$

# 2

# Matching of spacetimes

A well known problem in electromagnetism consists of finding the relation that the electric and magnetic fields keep on either side of a given surface that separates two different regions, even in presence of surface charge distributions or surface currents. Maxwell's equations, in the integral form applied in some neighbourhood of the separating surface, provide the relations of the normal and tangent components of the fields. Thus, the normal component of the electric field has a discontinuity across the surface proportional to a surface density of charge, while its tangential component remains continuous. For the magnetic field the situation is the opposite, its normal component to the separating surface is continuous and the tangential presents a discontinuity that depends on the surface current density.

An analogous problem arises in General Relativity when we have two spacetimes with boundary, independent of each other a priori, and we want to match them across the common boundary in order to give rise to a matched spacetime. Being the spacetime Lorentzian, the common boundary is a hypersurface that may have different nature: it can be timelike, spacelike, null, or even general (i.e. changing from point to point). This chapter is intended to provide a brief introduction to the theory of matching of spacetimes. There are many works covering this topic in the literature, see for instance [9, 10, 15, 32, 40, 67, 73, 88, 106], but they are restricted to hypersurfaces with fixed character (spacelike, timelike or null everywhere). In the development presented in [78, 82] boundaries of general character are considered, so that their causal character may vary from point to point. An appropriate treatment of hypersurfaces of general character requires a geometric construction based on *rigging* vectors, which have the property of being transverse to the hypersurface everywhere. The equations relating the ambient curvature with the intrinsic curvature of a hypersurface, the so called Gauss-Codazzi equations, must be also generalized to cover this situation. In the first part of the Chapter, the requirements in order to have a well constructed geometry around the common boundary are devised.

A second part of the Chapter is devoted to the study of the structure of the curvature of the matched spacetimes. This must now be described in this framework by tensor distributions instead of tensors. In particular, the class of metrics that is $C^2$ everywhere except in some hypersurface, where they are $C^0$, is contemplated. Tensor distributions were identified in [53]. The minimum requirement of continuity of the metric is imposed in order to formulate the Einstein equations in the distributional sense. An underlying fundamental requirement is that we want to avoid product of Dirac-delta type distributions because these are not well defined in general, unless we resort to more general structures (see [102] and references therein) that are out of the scope of this thesis. Thus, we restrict ourselves to the standard distribution theory. A general presentation of the theory can be found in [31] and a self explanatory introduction in the Appendix of [82] (see also [94]). A remarkable result in [78, 82] is that the Bianchi identity holds in the distributional sense, leading to good properties of the energy momentum tensor distribution. Some results therein can be extended to other theories of gravity, like $F(R)$ gravity [98, 99, 100], or quadratic gravity [94], involving higher order field equations. A further chapter (Chapter 11) will be dedicated to these modified theories of gravity.

## 2.1 Preliminary junction conditions

Let $(\mathcal{V}^{\pm}, g^{\pm}, \Sigma^{\pm})$ be two $(n+1)$-dimensional, $C^2$ spacetimes with oriented $C^3$ boundaries $\Sigma^{\pm}$. Require also that the boundaries $\Sigma^{\pm}$ are identified through some diffeomorphism $\Phi : \Sigma^- \to \Sigma^+$. Then the matched spacetime $\mathcal{V}$ is the disjoint union of $\mathcal{V}^+$ and $\mathcal{V}^-$, with $\Sigma^-$ and $\Sigma^+$ identified through $\Phi$, and such that the junction conditions (to be introduced) are satisfied.

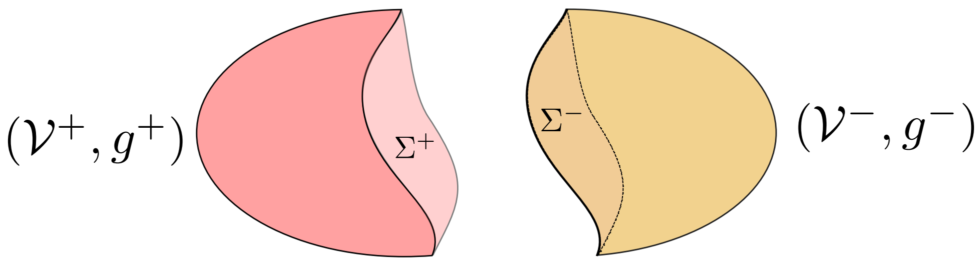


Figure 2.1: Two spacetimes $(\mathcal{V}^{\pm}, g^{\pm})$ with diffeomorphic boundaries $\Sigma^{\pm}$. These can be timelike, spacelike or null, and even change from one point to another.

Since $\Sigma^+$ and $\Sigma^-$ are diffeomorphic, they are also diffeomorphic to an abstract $n$ dimensional $C^3$ manifold, $\Sigma$. Let us introduce a local coordinate system $\{\xi^a\}$ in $\Sigma$, where latin indices run from 1 to $n$, and also in $\mathcal{V}^{\pm}$, these denoted by $\{x^{\pm\alpha}\}$, with greek indices ranging from 0 to $n$. The coordinates $\{x^+\}$ and $\{x^-\}$ do not need to be related in any way. The embeddings from $\Sigma$ to $\mathcal{V}^\pm$ are given by the $C^3$ maps

$$\begin{aligned} \Phi^\pm : \quad & \Sigma \to \mathcal{V}^\pm, \\ & \xi^a \to x^{\pm\alpha} = \Phi^{\pm\alpha}(\xi^a), \end{aligned} \tag{2.1}$$

such that $\Sigma^\pm = \Phi^\pm(\Sigma)$ and the diffeomorphism from $\Sigma^+$ to $\Sigma^-$ is just $\Phi := \Phi^- \circ {\Phi^+}^{-1}$. The pullbacks of $\Phi^\pm$ are denoted by $\Phi^{\pm *}$ and the pushforwards by $d\Phi^\pm$. Along this section, no restrictions are taken about the causal character of the hypersurfaces $\Sigma^\pm$. These can be spacelike, timelike or null, and they are even allowed to change the character from point to point.

The matching procedure involves two main stages. In the first one, the two spacetimes $(\mathcal{V}^\pm, g^\pm, \Sigma^\pm)$ are glued through their "common" boundaries and we construct a single matched spacetime $(\mathcal{V}, g)$ with a well defined metric everywhere: in the regions $\mathcal{V}^+$ and $\mathcal{V}^-$ it corresponds to $g^+$ and $g^-$ respectively, being these of class $C^2$ and in $\Sigma^\pm$ it is only $C^0$ . This is commonly known as the "gluing" and it entails the *preliminary matching conditions*. Once this has been achieved, the second task is devoted to obtain a set of well defined field equations everywhere, in absence of singular terms. This will be guaranteed when the so called *matching conditions* hold.

The boundaries $\Sigma^\pm$ have been identified pointwise by (2.1), but in order to obtain a well defined geometry it remains to be specified how the tangent spaces at points on $\Sigma^\pm$ are identified [32]. The equality of the first fundamental forms in $\Sigma^\pm$ inherited from $\mathcal{V}^\pm$ through $\Phi^\pm$ allows the identification of the tangent vectors to $\Sigma^\pm$ , i.e.

$$h^+ := {\Phi^+}^*(g^+) = {\Phi^-}^*(g^-) =: h^-. \tag{2.2}$$

These are the so called *preliminary junction conditions*.

These can be written in terms of the coordinates $\{\xi^a\}$ in $\Sigma$ as follows. The image of the natural basis $\{\partial/\partial\xi^a\}$ at the tangent spaces $T_p\Sigma$ for every $p \in \Sigma$ gives a set of $n$ independent tangent vectors in each spacetime $\{\vec{e}^{\,\pm}_a\}$ through $d\Phi^\pm$. Explicitly this is

$$\vec{e}^{\,\pm}_a := d\Phi^\pm\left(\frac{\partial}{\partial\xi^a}\right) = \frac{\partial\Phi^{\pm\alpha}(\xi)}{\partial\xi^a}\left.\frac{\partial}{\partial x^\alpha}\right|_{\Sigma^\pm} = e^{\pm\alpha}{}_a \left.\frac{\partial}{\partial x^{\pm\alpha}}\right|_{\Sigma^\pm},$$

at every point on $\Sigma\pm$. The vectors $\{\vec{e}_a\}$ provide an explicit expression for the pullback $\Phi^*$ and pushforward $d\Phi$ (in any of $\mathcal{V}^+$ or $\mathcal{V}^-$). A $s-$contravariant tensor $\Delta$ defined in $\Sigma$ with components $\Delta^{a_1 \dots a_s}$ in the basis $\{\partial/\partial\xi^a\}$ can be promoted to the spacetimes $\mathcal{V}^\pm$ by means of the pushforward $d\Phi^\pm$ so that its components in the basis $\partial/\partial x^{\pm\alpha}$ read

$$[d\Phi^\pm(\Delta)]^{\alpha_1 \dots \alpha_s} = \Delta^{a_1 \dots a_s} e^{\pm\alpha_1}{}_{a_1} \dots e^{\pm\alpha_s}{}_{a_s}. \tag{2.3}$$

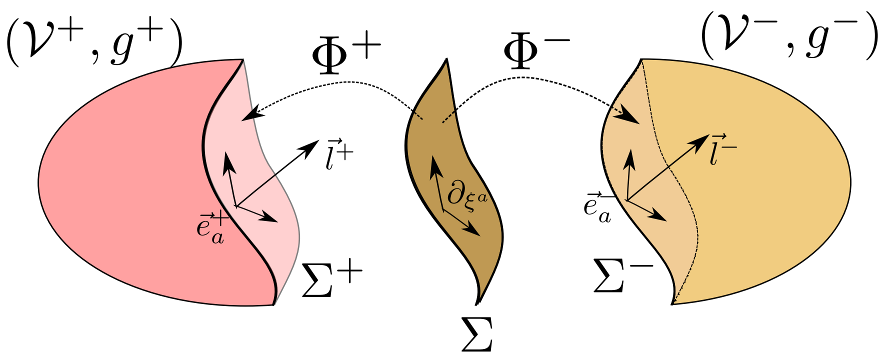


Figure 2.2: The gluing procedure. The boundaries are identified via the embeddings as $\Sigma^- = \Phi^- \circ \Phi^{+-1}(\Sigma^+)$. The tangent basis in the spacetimes are given by $\vec{e}_a{}^{\pm} = d\Phi^{\pm}(\partial_{\xi^a})$.

Any $s$-covariant tensor $\Theta^{\pm}$ in $\mathcal{V}^{\pm}$ with components $\Theta^{\pm}{}_{\alpha_1 \ldots \alpha_s}$ in the basis $\{dx^{\pm\alpha}\}$ can be projected to $\Sigma$ by means of the pullback $\Phi^{\pm *}$ and its components in the basis $d\xi^a$ read

$$[\Phi^{\pm *}(\Theta^{\pm})]_{a_1 \ldots a_s} = \Theta^{\pm}{}_{\alpha_1 \ldots \alpha_s} e^{\pm \alpha_1}_{\ a_1} \ldots e^{\pm \alpha_s}_{\ a_s}$$

Applying this last expression to the metric tensor itself, we obtain the first fundamental form and the preliminary junction conditions written in the coordinates $\{\xi^a\}$ as

$$h^+_{ab} := g^+_{\alpha\beta} e^{+\alpha}_a e^{+\beta}_b \overset{\Sigma}{=} g^-_{\alpha\beta} e_a^{-\alpha} e_b^{-\beta} =: h^-_{ab}. \tag{2.4}$$

Once the preliminary junction conditions are satisfied, the tangent vectors $\{\vec{e}^+_a\}$ and $\{\vec{e}^-_a\}$ can be identified, and we refer to $h^+ = h^-$ simply by $h$. Then $(\Sigma, h)$ is an oriented manifold with a well defined metric.

Now, in order to identify the full tangent spaces at diffeomorphic points on $\Sigma^+$ and $\Sigma^-$, we need to identify one transverse vector to $\Sigma^+$ with another transverse vector to $\Sigma^-$. We define a transverse vector by means of a normal form $\boldsymbol{N}^+$ on $\Sigma^+$, which is determined by $\boldsymbol{N}^+(\vec{e}^+_a) = 0$ for $a = 1, ..., n$ as follows. The associated "normal vector", $\vec{N}^+$, $N^{+\alpha} = g^{\alpha\beta} N_\beta$, is not transverse to $\Sigma^+$ in general. At null points on $\Sigma^+$, it satisfies $\boldsymbol{N}^+(\vec{N}^+) = 0$, so that the "normal vector" becomes tangent to $\Sigma^+$. Therefore $N^{+\alpha} = N^{+a} e^{+\alpha}_a$ and the set $\{\vec{N}^+, \vec{e}^+_a\}$ is not a basis of the tangent space at null points. Thus, we choose a $C^2$ vector field $\vec{l}^+$ which is transverse to $\Sigma^+$ everywhere, i.e. $\boldsymbol{N}^+(\vec{l}^+) \neq 0$. This vector field is called *rigging* in the literature [78, 82]. This same construction is taken, for the spacetime $(\mathcal{V}^-, g^-)$.

We intend to complete the identification of the tangent spaces on $\Sigma^{\pm}$ by identifying $\vec{l}^+$ and $\vec{l}^-$. This is achieved by imposing first the following conditions

$$g^+{}_{\mu\nu} l^{+\mu} l^{+\nu} \overset{\Sigma}{=} g^-{}_{\mu\nu} l^{-\mu} l^{-\nu}, \quad g^+{}_{\mu\nu} l^{+\mu} e^{+\nu}_a \overset{\Sigma}{=} g^-{}_{\mu\nu} l^{-\mu} e^{-\nu}_a. \tag{2.5}$$

The first one equates the norms of $l^{\pm}$ and the second one does the same for the components of the one-forms $l_a = l^{\alpha} e^{\alpha}_{a}$, as viewed from the + and − sides. Moreover, the rigging vectors must agree in their orientation to be properly identified: this accounts to showing that choosing $\vec{l}^{+}$ to point towards the interior of $\mathcal{V}^{+}$, a $\vec{l}^{-}$ satisfying (2.5) can be found to point towards the exterior of $\mathcal{V}^{-}$. It is only then that we can identify the bases of the whole $n$ dimensional tangent spaces of $\mathcal{V}^{\pm}$ at $\Sigma$, so that $\{\vec{l}, \vec{e}_a\} := \{\vec{l}^{+}, \vec{e}^{+}_{a}\} = \{\vec{l}^{-}, \vec{e}^{-}_{a}\}$. The "gluing" procedure is summarized in the following Theorem

**Theorem 1** ***(Mars, Senovilla, Vera, 2007 [83])*** *Let $(\mathcal{V}^{\pm}, g^{\pm})$ be two $n$+1-dimensional $C^2$ oriented spacetimes with boundary, with respective $C^3$ boundaries $\Sigma^{\pm}$ such that the preliminary matching conditions (2.2) hold on $\Sigma$. Assume further that there exist transverse vector fields $\vec{l}^{\pm}$ on $\Sigma^{\pm}$ satisfying the scalar product conditions (2.5) and such that $\vec{l}^{+}$ points towards $\mathcal{V}^{+}$ and $\vec{l}^{-}$ points outwards from $\mathcal{V}^{-}$.*

*Then, there exists a unique, maximal, $C^3$ differentiable structure on $\mathcal{V} = \mathcal{V}^{+} \cup \mathcal{V}^{-}$ (with their points on $\Sigma^{+}$ and $\Sigma^{-}$ identified), and a unique continuous metric $g$ which coincides with $g^{+}$ on $\mathcal{V}^{+}$ and with $g^{-}$ on $\mathcal{V}^{-}$.*

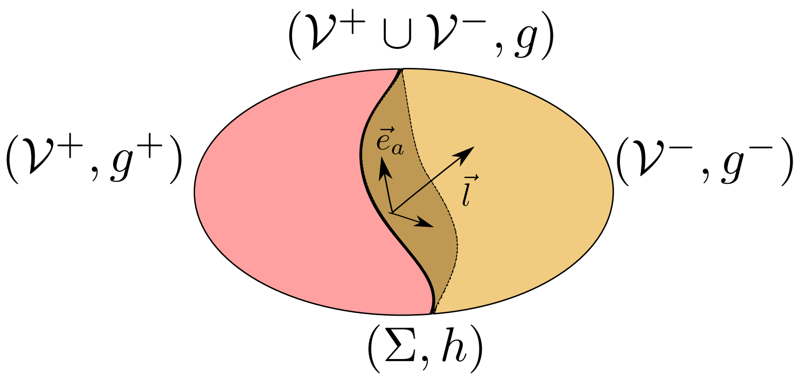


Figure 2.3: The matched spacetime $(\mathcal{V}, g)$. Now, due to the preliminary matching conditions, there is a well defined metric on $\Sigma$, given by $h$. In this picture we have identified the abstract $\Sigma$ with the equated images $\Sigma^{+}$ and $\Sigma^{-}$ as an embedded hypersurface $\Sigma$ in $\mathcal{V}$.

Given any $\vec{l}^{+}$ with the fixed orientation of the Theorem 1, the existence of a $\vec{l}^{-}$ with the desired orientation is not guaranteed. The cases for null and non null points at $\Sigma^{-}$ are considered separately in the following lemmas:

**Lemma 1** ***(Mars, Senovilla, Vera, 2007 [83])*** *Let $\mathcal{V}^{\pm}$ be two spacetimes with boundary satisfying the preliminary matching conditions (2.4). Let $\Sigma^{-}$ be non null at $p^{-} \in \Sigma^{-}$ and set $p^{+} = \Phi^{-1}(p^{-})$. Choose any transverse vector field $\vec{l}^{+}|_{p^{+}}$ pointing towards $\mathcal{V}^{+}$. Then there is at most one solution of (2.5) for $\vec{l}^{-}|_{p^{-}}$ pointing outwards from $\mathcal{V}^{-}$.*

**Lemma 2** ***(Mars, Senovilla, Vera, 2007 [83])*** *With the same notation as in Lemma 2, assume now that $\Sigma^-$ is null at $p^-$. Then the solution of the algebraic equations (2.5) at $p^-$ is unique, if it exists.*

At non-null points the unit normals are riggings, say $\vec{n}^+$, $-\vec{n}^+$, $\vec{n}^-$, $-\vec{n}^-$. In that case, Lemma 1 states that given $\vec{n}^+$, the solution to (2.5) with the appropriate orientation for $\vec{n}^-$ is unique. Conversely, at null points the system (2.5) fixes uniquely $\vec{l}^-$, including its orientation. Moreover, this result does not depend on the choice of $\vec{l}^+$, and thus, there can be spacetimes with boundaries containing null points satisfying the preliminary matching conditions (2.4) that cannot be glued by any means (see the example in Fig.1 in [83]).

Finally, let us build the dual cobasis to the tangent planes of $\mathcal{V}^\pm$ at $\Sigma$. To treat the two $\pm$ spacetimes at once, we drop the superscripts $\pm$, not to overload the notation. We choose a normalisation of the normal form with the aid of the rigging as

$$\boldsymbol{n} = \frac{\boldsymbol{N}}{N_\alpha l^\alpha},$$

which readily implies that $n_\alpha l^\alpha = 1$. The dual bases are completed with the tangent forms $\boldsymbol{\omega}^a$ defined by

$$\omega^a_\alpha l^\alpha = 0, \quad \omega^a_\alpha e^\alpha_b = \delta^a_b.$$

Note that the forms $\boldsymbol{\omega}^a$ depend on the rigging. Due to the identification of the riggings and tangent vectors, the normal and the tangent forms can also be identified, so that $\{\boldsymbol{n}, \boldsymbol{\omega}^a\} := \{\boldsymbol{n}^+, \boldsymbol{\omega}^{+a}\} = \{\boldsymbol{n}^-, \boldsymbol{\omega}^{-a}\}$. Thus, for a given $\vec{l}$, the maps $T_l : T_p\mathcal{V} \to T_p\Sigma$ and $\Lambda_l : T^*_p\Sigma \to T^*_p\mathcal{V}$ for all $p \in \Sigma$ can be defined [78, 82] as follows. $T_l$ projects a $s$-contravariant tensor $\Theta$ in $\mathcal{V}$ to a $s$-contravariant tensor in $\Sigma$ whose components in the basis $\{\xi^a\}$ are

$$[T_l(\Theta)]^{a_1 \dots a_s} = \Theta^{\alpha_1 \dots \alpha_s} \omega^{a_1}_{\alpha_1} \dots \omega^{a_s}_{\alpha_s}, \tag{2.6}$$

while $\Lambda_l$ associates a $s$-contravariant tensor $\Delta$ in $\mathcal{V}$ tangent to $\Sigma$ to a $s$-contravariant tensor in $\Sigma$ with components $\Delta_{a_1 \dots a_s}$ in the basis $\{d\xi^a\}$ as

$$[\Lambda_l(\Delta)]_{\alpha_1 \dots \alpha_s} = \Delta_{a_1 \dots a_s} \omega^{a_1}_{\alpha_1} \dots \omega^{a_s}_{\alpha_s}. \tag{2.7}$$

In particular, we can use these maps to obtain a 2-contravariant symmetric tensor on $\Sigma$ associated to the inverse of the spacetime metric $g^{-1}$. It reads $g^{ab} = \omega^a_\alpha \omega^b_\beta g^{\alpha\beta}$. Note however, that $g^{ab}$ is not the inverse of $h_{ab}$ in general, since the first fundamental form is degenerate at points of $\Sigma$ where $\vec{n}$ is null. The contraction between them is easily computed to yield $g^{ab} h_{bc} = \delta^a_c - n^a l_c$. Another relevant contraction is $\omega^a_\alpha e^\rho_a = \delta^\rho_\alpha - n_\alpha l^\rho =: h^\rho_\alpha$.

At points where $\Sigma$ turns null, if any, $h$ is degenerate and thus it is not possible to define a connection associated to it. To overcome this we use the connection in $\mathcal{V}$ to construct another in $\Sigma$. Consider two vector fields $\vec{x}$, $\vec{y}$ in $\Sigma$, promote them to the spacetime via (2.3) to obtain the respective spacetime vectors $\vec{X}$ and $\vec{Y}$, and take the (spacetime) covariant derivative of one of them along the other one, $\nabla_{\vec{Y}}\vec{X}$. In general this vector will have a component tangent to $\Sigma$ and some other component in the direction of the rigging. Take the tangential part of the derivative and project it to $\Sigma$ via (2.6) to define

$$\overline{\nabla}_{\vec{y}}\vec{x} := \left(\nabla_{\vec{Y}}\vec{X}\right)_{||} = T_l(\nabla_{\vec{Y}}\vec{X}). \tag{2.8}$$

This is called the rigged connection in [78, 82]. It has no torsion, but in general it is not a metric connection. Its corresponding Christoffel symbols are given in terms of the vectors $\vec{e}_a$ and the forms $\boldsymbol{\omega}^a$ by

$$\Gamma^a_{bc} = \omega^a_\rho e^\alpha_b \nabla_\alpha e^\rho_c = \left[T_l\left(\nabla_{\vec{e}_b}\vec{e}_c\right)\right]^a, \quad \Gamma^a_{bc} = \Gamma^a_{cb}. \tag{2.9}$$

Introducing the following objects in $\Sigma$

$$\Psi^a_b := \omega^a_\mu e^\nu_b \nabla_\nu l^\mu, \quad \varphi_a := n_\mu e^\nu_a \nabla_\nu l^\mu, \quad \kappa_{ab} := e^\alpha_a e^\beta_b \nabla_\alpha n_\beta, \tag{2.10}$$

we can cast the Gauss equation for general hypersurfaces [78, 82]

$$\omega^d_\alpha R^\alpha_{\beta\gamma\delta} e^\beta_a e^\gamma_b e^\delta_c = \overline{R}^d_{abc} - \kappa_{ac}\Psi^d_b + \kappa_{ab}\Psi^d_c, \tag{2.11}$$

where $\overline{R}^d_{abc}$ is the Riemann tensor associated to the connection (2.8), and the Codazzi equations (1,2,3 respectively) [78, 82]

$$n_\mu R^\mu_{\beta\lambda\nu} e^\beta_a e^\lambda_b e^\nu_c = \overline{\nabla}_c \kappa_{ba} - \overline{\nabla}_b \kappa_{ca} + \kappa_{ba}\varphi_c - \kappa_{ca}\varphi_b, \tag{2.12}$$

$$\omega^c_\mu R^\mu_{\beta\lambda\nu} l^\beta e^\lambda_a e^\nu_b = \overline{\nabla}_a \Psi^c_b - \overline{\nabla}_b \Psi^c_a + \varphi_b \Psi^c_a - \varphi_a \Psi^c_b, \tag{2.13}$$

$$n_\mu R^\mu_{\beta\lambda\nu} l^\beta e^\lambda_a e^\nu_b = \overline{\nabla}_a \varphi_b - \overline{\nabla}_b \varphi_a + \kappa_{cb}\Psi^c_a - \kappa_{ca}\Psi^c_b. \tag{2.14}$$

To sum up, we have so far constructed the matched spacetime $(\mathcal{V}, g)$ with a hypersurface $\Sigma \subset \mathcal{V}$ that splits it into the two open sets $\mathcal{V}^\pm$ with common boundary $\Sigma$ [1]. In each $\mathcal{V}^\pm$ the metric is $C^2$, and $C^0$ on $\Sigma$. We have also constructed a basis of the tangent space of $\mathcal{V}$ at points of $\Sigma$, given by $\{\vec{l}, \vec{e}_a\}$ and also for its dual space, given by $\{\boldsymbol{n}, \boldsymbol{\omega}^a\}$.

[1] In the beginning of the chapter we denoted the boundaries of $\mathcal{V}^+$ and $\mathcal{V}^-$ by $\Sigma^+$ and $\Sigma^-$ respectively. These are diffeomorphic to each other and thus to an abstract manifold $\Sigma$ (recall (2.1)). This abstract manifold is embedded in the matched spacetime $(\mathcal{V}, g)$ as a hypersurface and abusing of notation, we refer to this hypersurface embedded in $\mathcal{V}$ simply by $\Sigma$.

## 2.2 On the curvature tensors and field equations

Let us now assume that the construction of the matched spacetime summarized above has been carried out. Then, the Einstein equations are well defined in the distributional sense. Throughout this section, the indexes $\alpha$, $\beta$ will refer to any admissible coordinates or continuous basis for which $[g_{\alpha\beta}] = 0$. An introduction with the basic concepts of the theory of distributions can be found, for instance, in the Appendix of [82] (see also [74, 106]). For completeness of this brief introduction on the topic, let us include some definitions and results presented therein in order to keep the exposition of the junction conditions as self-contained as possible.

Let $\mathcal{D}(\mathcal{V})$ be the set of test tensor fields: $C^\infty$ tensor fields of any order with compact support in $\mathcal{V}$. Denote by $\mathcal{D}^q_p$ the subset of $p$-covariant $q$-contravariant tensor fields in $\mathcal{D}(\mathcal{V})$.

**Definition 1 (Tensor distribution)** *The $p$-covariant $q$-contravariant tensor distributions $\chi^q_p$ are the linear and continuous functionals*

$$\begin{aligned} \chi^q_p : \quad & \mathcal{D}^p_q \to \mathbb{R} \\ & Y^p_q \to \chi^q_p(Y^p_q) := \left\langle \chi^q_p, Y^p_q \right\rangle \end{aligned} \tag{2.15}$$

The set of tensor distributions constitutes a vector space (the sum of tensor distributions and the product of a tensor distribution with a real number are well defined and are tensor distributions).

A locally integrable $p$-covariant $q$-contravariant tensor field $T^q_p$ defines uniquely a tensor distribution $\underline{T}^q_p$ by

$$\begin{aligned} \underline{T}^q_p : \quad & \mathcal{D}^p_q \to \mathbb{R} \\ & Y^p_q \to \left\langle \underline{T}^q_p, Y^p_q \right\rangle := \int_{\mathcal{V}} T^{\alpha_1 \dots \alpha_q}_{\beta_1 \dots \beta_p} Y^{\beta_1 \dots \beta_p}_{\alpha_1 \dots \alpha_q} \boldsymbol{\eta}, \end{aligned} \tag{2.16}$$

being $\boldsymbol{\eta}$ the volume element of $(\mathcal{V}, g)$.

**Definition 2 (Tensor distribution components)** *The components of a $p$-covariant $q$-contravariant tensor distribution $\chi$ in a dual basis $\{\{\vec{e}_\mu\}, \{\boldsymbol{\theta}^\mu\}\}$ are scalar distributions $\chi^{\alpha_1 \dots \alpha_q}_{\beta_1 \dots \beta_p}$ defined by*

$$\left\langle \chi^{\alpha_1 \dots \alpha_q}_{\beta_1 \dots \beta_p}, Y \right\rangle := \left\langle \chi^q_p, Y \boldsymbol{\theta}^{\alpha_1} \otimes \dots \otimes \boldsymbol{\theta}^{\alpha_q} \otimes \vec{e}_{\beta_1} \otimes \dots \otimes \vec{e}_{\beta_p} \right\rangle,$$

*where $Y$ is a test function.*

Thus the following expression follows

$$\left\langle \chi_p^q, Y_q^p \right\rangle = \left\langle \chi_{\beta_1\ldots\beta_p}^{\alpha_1\ldots\alpha_q}, Y_{\alpha_1\ldots\alpha_q}^{\beta_1\ldots\beta_p} \right\rangle .$$

Now that the components of a tensor distribution are defined, we can define the contraction as

$$\left\langle [C_j^i\left(\chi_q^p\right)]_{p-1}^{q-1}, Y_{q-1}^{p-1} \right\rangle = \left\langle \chi_{\beta_1\ldots\beta_{j-1}\mu\beta_{j+1}\ldots\beta_p}^{\alpha_1\ldots\alpha_{i-1}\mu\alpha_{i+1}\ldots\alpha_q}, Y_{\alpha_1\ldots\alpha_{i-1}\alpha_{i+1}\ldots\alpha_q}^{\beta_1\ldots\beta_{j-1}\beta_{j+1}\ldots\beta_p} \right\rangle .$$

This definition of contraction is independent of the basis chosen.

**Definition 3 (Support of tensor distributions)** *The support of a tensor distribution $\chi_p^q$ is the complement in $\mathcal{V}$ of the union of all open sets where $\chi_p^q$ vanishes.*

**Definition 4 (Tensor product by tensor fields)** *The tensor product of a tensor distribution $\chi_p^q$ by a tensor field $T_s^r$, defined on a neighbourhood of the support of $\chi_p^q$, is the $(p+s)$-covariant $(q+r)$-contravariant tensor distribution acting as follows*

$$\left\langle T_s^r \chi_p^q, Y_{r+q}^{s+p} \right\rangle := \left\langle \chi_p^q, (T,Y)_q^p \right\rangle ,$$

*where, in any basis, we define*

$$(T,Y)_{\nu_1,\ldots\nu_q}^{\mu_1\ldots\mu_p} := T_{\alpha_1\ldots\alpha_s}^{\rho_1\ldots\rho_r} Y_{\rho_1\ldots\rho_r\nu_1\ldots\nu_q}^{\alpha_1\ldots\alpha_s\mu_1\ldots\mu_p} .$$

Sometimes, for this product to make sense it is enough that the tensor field $T_s^r$ is defined only on the support of $\chi_p^q$. However, this will not be the case when derivatives are involved.

Tensor distributions can be differentiated. The definition is the following

**Definition 5 (Covariant derivative of a tensor distribution)** *The covariant derivative $\nabla\chi_p^q$ of a $(p,q)$-tensor distribution $\chi_p^q$ is the $(p+1,q)$-tensor distribution defined by*

$$\left\langle \nabla\chi_p^q, Y_q^{p+1} \right\rangle := -\left\langle \chi_p^q, (DY)_q^p \right\rangle ,$$

*where $(DY)_{\nu_1\ldots\nu_q}^{\mu_1\ldots\mu_p} = \nabla_\rho Y_{\nu_1\ldots\nu_q}^{\rho\mu_1\ldots\mu_p}$.*

The components of the covariant derivative $\nabla\chi_p^q$, in any basis, are the scalar distributions $\nabla_\rho\chi_{\beta_1,\ldots\beta_p}^{\alpha_1\ldots\alpha_q}$ acting on test functions as

$$\begin{aligned}\left\langle \nabla_\rho\chi_{\beta_1,\ldots\beta_p}^{\alpha_1\ldots\alpha_q}, Y \right\rangle &= \left\langle \chi_{\beta_1,\ldots\beta_p}^{\alpha_1\ldots\alpha_q}, \partial_\rho Y + \Gamma_{\sigma\rho}^{\sigma} Y \right\rangle - \sum_{i=1}^{p} \left\langle \chi_{\beta_1,\ldots,\beta_{i-1}\sigma\beta_{i+1}\ldots\beta_p}^{\alpha_1\ldots\alpha_q}, \Gamma_{\beta_i\rho}^{\sigma} Y \right\rangle \\ &\quad + \sum_{j=1}^{q} \left\langle \chi_{\beta_1,\ldots,\beta_p}^{\alpha_1\ldots\alpha_{j-1}\sigma\alpha_{j+1}\ldots\alpha_q}, \Gamma_{\sigma\rho}^{\alpha_j} Y \right\rangle .\end{aligned}$$

This definition of the covariant derivative for tensor distributions agrees with the generalization to a distribution of the covariant derivative of a tensor field, so that $\nabla \underline{T} = \underline{\nabla T}$.

Back to the problem of the matching of spacetimes, the glued spacetime $(\mathcal{V}, g)$ contains a $C^2$ metric everywhere except on a hypersurface $\Sigma$, where it is only continuous. Thus, the curvature tensors are not defined in $(\mathcal{V}, g)$ as ordinary tensor fields, but as tensor distributions. In order to characterize them, let us first introduce two important distributions. The first one is associated to the Heaviside function $\theta$ of $\Sigma$

$$\theta = \begin{cases} 1 & \text{in } \mathcal{V}^+, \\ \frac{1}{2} & \text{in } \Sigma\,, \\ 0 & \text{in } \mathcal{V}^-. \end{cases} \tag{2.17}$$

This function is locally integrable, and therefore it defines a scalar distribution $\underline{\theta}$ as

$$\langle \underline{\theta}, Y \rangle := \int_{\mathcal{V}^+} Y \boldsymbol{\eta}.$$

Now, a function $f$ discontinuous in $\Sigma$, but differentiable everywhere else and with well defined limits on $\Sigma$, defines a scalar distribution that can be expressed in terms of the Heaviside distribution as

$$\underline{f} = f^+ \cdot \underline{\theta} + f^- \cdot (\underline{1} - \underline{\theta}), \tag{2.18}$$

where $f^\pm$ thus corresponds to the restrictions of $f$ to $\mathcal{V}^\pm$ respectively.

In order to take the derivative of (2.18) let us first define a volume element $dv$ on $\Sigma$ by

$$dv = l^\alpha dv_\alpha = l^\alpha \eta_{\alpha\beta_1\ldots\beta_n} e_1^{\beta_1} \ldots e_n^{\beta_n} d\xi^1 \wedge \cdots d\xi^n,$$

with $l^\alpha dv_\alpha > 0$, or in other words, $dv_\alpha$ points from $\mathcal{V}^-$ to $\mathcal{V}^+$. Note that $dv_\alpha = n_\alpha dv$ by construction. Thus, the covariant derivative of $\underline{\theta}$ is a one-form distribution with support on $\Sigma$ acting as

$$\left\langle \nabla \underline{\theta}, \vec{Y} \right\rangle = -\left\langle \underline{\theta}, D\vec{Y} \right\rangle = -\int_{V^+} \nabla_\alpha Y^\alpha \boldsymbol{\eta} = \int_\Sigma Y^\alpha dv_\alpha = \int_\Sigma Y^\alpha n_\alpha dv,$$

where we have used Gauss's theorem in the third equality. It arises a natural scalar distribution $\delta^\Sigma$ with support on $\Sigma$ defined by

$$\left\langle \delta^\Sigma, Y \right\rangle := \int_\Sigma Y dv \ . \tag{2.19}$$

This distribution can be multiplied by any smooth and locally integrable tensor field defined only on $\Sigma$. Observe in particular, from the above, that

$$\boldsymbol{\delta} := \nabla_\alpha \underline{\theta}. \tag{2.20}$$

defines a one covariant distribution $\boldsymbol{\delta}$ intrinsic to $\Sigma$ as

$$\left\langle \boldsymbol{\delta}, \vec{Y} \right\rangle = \int_\Sigma Y^\alpha dv_\alpha = \int_\Sigma Y^\alpha n_\alpha dv.$$

From the identity $\boldsymbol{\delta} = \delta^\Sigma \boldsymbol{n}$ it is obvious that the scalar distribution $\delta^\Sigma$ depends on the rigging through the normal form $\boldsymbol{n}$.

The derivative of (2.18) thus reads [78, 82]

$$\underline{\nabla f} = \nabla f^+ \cdot \underline{\theta} + \nabla f^- \cdot (\underline{1} - \underline{\theta}) + [f]\boldsymbol{\delta}, \tag{2.21}$$

where we have followed the usual notation to denote the discontinuity of any quantity with well defined limits on $\Sigma$,

$$\forall q \in \Sigma, \qquad [f]\,(q) \equiv \lim_{\substack{x \to q \\ \mathcal{V}^+}} f^+(x) - \lim_{\substack{x \to q \\ \mathcal{V}^-}} f^-(x) \ . \tag{2.22}$$

Note that the discontinuity function $[f]$ is defined only on $\Sigma$ and it is smooth there by definition (if $f^+$ and $f^-$ are). The generalization of the previous constructions from functions to tensors now follows. Let $T$ be any $(p,q)$-tensor field which (i) may be discontinuous across $\Sigma$, (ii) is differentiable on $\mathcal{V}^+$ and $\mathcal{V}^-$, and (iii) such that $T$ and its covariant derivative have definite limits on $\Sigma$ coming from both $\mathcal{V}^+$ and $\mathcal{V}^-$. Using the notation $T^\pm$ for the restriction of $T$ to $\mathcal{V}^\pm$ respectively, we can construct

$$T := T^+\theta + T^-\,(1-\theta) \quad \text{in } \mathcal{V}.$$

In particular, at each point of $\Sigma$

$$T^\Sigma := T|_\Sigma = \frac{1}{2}\left( \lim_{\substack{x \to \Sigma \\ \mathcal{V}^+}} T^+(x) + \lim_{\substack{x \to \Sigma \\ \mathcal{V}^-}} T^-(x) \right) . \tag{2.23}$$

Since $T$ in $\mathcal{V}$ is locally integrable, in $\mathcal{V}$, it defines a distribution given by

$$\underline{T} = T^+\underline{\theta} + T^-\,(\underline{1} - \underline{\theta}) \ . \tag{2.24}$$

Generalizing (2.21) appropriately, the covariant derivative of $\underline{T}$ can be shown to be [82]

$$\underline{\nabla T} = \nabla T^+ \underline{\theta} + \nabla T^- (\underline{1} - \underline{\theta}) + [T] \otimes \boldsymbol{\delta}^\Sigma \tag{2.25}$$

where $[T]$ is the $(p,q)$-tensor field defined only on $\Sigma$, called the "jump" or "discontinuity" of $T$ at $\Sigma$ defined as

$$\forall q \in \Sigma, \qquad [T]\,(q) \equiv \lim_{\substack{x \to q \\ \mathcal{V}^+}} T^+(x) - \lim_{\substack{x \to q \\ \mathcal{V}^-}} T^-(x) \ . \tag{2.26}$$

The index version of (2.25) reads

$$\nabla_\mu \underline{T}^{\alpha_1\ldots\alpha_q}_{\beta_1\ldots\beta_p} = \nabla_\mu T^{+\alpha_1\ldots\alpha_q}_{\beta_1\ldots\beta_p}\underline{\theta} + \nabla_\mu T^{-\alpha_1\ldots\alpha_q}_{\beta_1\ldots\beta_p}(\underline{1}-\underline{\theta}) + \left[T^{\alpha_1\ldots\alpha_q}_{\beta_1\ldots\beta_p}\right] n_\mu \delta^\Sigma. \tag{2.27}$$

The curvature tensors can be defined in a distributional sense. By Theorem 1, the metric $g$ of $\mathcal{V}$ is a $C^0$ tensor and therefore it is only differentiable in the distributional sense. Thus we write it as a function and distribution respectively

$$g = g^+\theta + g^-(1-\theta), \quad \underline{g} = g^+\cdot\underline{\theta} + (\underline{1}-\underline{\theta})\cdot g^-. \tag{2.28}$$

Recalling the standard definition of the Christoffel symbols and taking the derivative of the metric tensor distribution (2.28) using the general formula (2.21), plus $[g_{\alpha\beta}] = 0$, the Christoffel symbols are found to be (as a distribution)

$$\underline{\Gamma}^\alpha_{\beta\gamma} = \Gamma^{+\alpha}_{\ \ \beta\gamma}\cdot\underline{\theta} + \Gamma^{-\alpha}_{\ \ \beta\gamma}\cdot(\underline{1}-\underline{\theta}). \tag{2.29}$$

Now the Christoffel symbols can be defined as functions. The scalar distributions (2.29) are associated to locally integrable functions given by

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^{+\alpha}_{\ \ \beta\gamma}\theta + \Gamma^{-\alpha}_{\ \ \beta\gamma}(1-\theta)\,. \tag{2.30}$$

These functions may be discontinuous across $\Sigma$ and, as in the general case (2.23), we have

$$\Gamma^\alpha_{\beta\gamma}|_\Sigma = \frac{1}{2}\left(\Gamma^{+\alpha}_{\ \ \beta\gamma}|_\Sigma + \Gamma^{-\alpha}_{\ \ \beta\gamma}|_\Sigma\right). \tag{2.31}$$

Using now the standard formula the Riemann tensor distribution is defined [78, 82] by

$$\underline{R}^\alpha_{\beta\lambda\mu} = \partial_\lambda\underline{\Gamma}^\alpha_{\beta\mu} - \partial_\mu\underline{\Gamma}^\alpha_{\beta\lambda} + \underline{\Gamma}^\alpha_{\lambda\rho}\underline{\Gamma}^\rho_{\beta\mu} - \underline{\Gamma}^\alpha_{\mu\rho}\underline{\Gamma}^\rho_{\beta\lambda}.$$

First of all, we observe that the products of $\underline{\Gamma}$'s are well defined because $\underline{\Gamma}^\alpha_{\beta\gamma}$ are distributions associated to locally integrable functions, and actually they become (upon using $\underline{\theta}\cdot\underline{\theta} = \underline{\theta}$)

$$\underline{\Gamma}^\alpha_{\lambda\rho}\underline{\Gamma}^\rho_{\beta\mu} = \Gamma^{+\alpha}_{\ \ \lambda\rho}\Gamma^{+\rho}_{\ \ \beta\mu}\underline{\theta} + \Gamma^{-\alpha}_{\ \ \lambda\rho}\Gamma^{-\rho}_{\ \ \beta\mu}(\underline{1}-\underline{\theta})$$

On the other hand, we have from (2.29), as in (2.27):

$$\partial_\mu\underline{\Gamma}^\alpha_{\beta\lambda} = \partial_\mu\Gamma^{+\alpha}_{\ \ \beta\lambda}\underline{\theta} + \partial_\mu\Gamma^{-\alpha}_{\ \ \beta\lambda}(\underline{1}-\underline{\theta}) + \left[\Gamma^\alpha_{\beta\gamma}\right] n_\mu\delta^\Sigma$$

so that the final expression for the Riemann tensor distribution reads [78, 82]

$$\begin{aligned}\underline{R}^\alpha_{\beta\mu\nu} &= R^{+\alpha}_{\ \ \beta\mu\nu}\cdot\underline{\theta} + R^{-\alpha}_{\ \ \beta\mu\nu}\cdot(\underline{1}-\underline{\theta}) + (n_\mu[\Gamma^\alpha_{\beta\nu}] - n_\nu[\Gamma^\alpha_{\beta\mu}])\cdot\delta^\Sigma, &(2.32)\\ &= R^{+\alpha}_{\ \ \beta\mu\nu}\cdot\underline{\theta} + R^{-\alpha}_{\ \ \beta\mu\nu}\cdot(\underline{1}-\underline{\theta}) + H^\alpha_{\beta\mu\nu}\cdot\delta^\Sigma, &(2.33)\end{aligned}$$

where we have encoded the singular term in (2.32) in a tensor

$$H^{\alpha}_{\beta\mu\nu} := n_\mu[\Gamma^{\alpha}_{\beta\nu}] - n_\nu[\Gamma^{\alpha}_{\beta\mu}] \tag{2.34}$$

called, precisely, the singular part of the Riemann tensor distribution. This structure is shared by the Ricci tensor distribution

$$\underline{R}_{\beta\nu} = R^{+}{}_{\beta\nu} \cdot \underline{\theta} + R^{-}{}_{\beta\nu} \cdot (\underline{1} - \underline{\theta}) + H_{\beta\nu} \cdot \delta^{\Sigma}, \tag{2.35}$$

where its singular part is $H_{\beta\nu} := H^{\alpha}_{\beta\alpha\nu}$, and by the Ricci scalar distribution

$$\underline{R} = R^{+} \cdot \underline{\theta} + R^{-} \cdot (\underline{1} - \underline{\theta}) + H \cdot \delta^{\Sigma}, \tag{2.36}$$

with $H := H^{\alpha}{}_{\alpha}$. Finally, the Einstein tensor distribution thus gets the form

$$\underline{G}_{\beta\nu} = G^{+}{}_{\beta\nu} \cdot \underline{\theta} + G^{-}{}_{\beta\nu} \cdot (\underline{1} - \underline{\theta}) + \mathcal{G}_{\beta\nu} \cdot \delta^{\Sigma}. \tag{2.37}$$

The singular part of the curvature tensor distribution depends on the discontinuity of $\Gamma^{\alpha}_{\beta\gamma}$, by (2.34). The next step is to characterize this jump in terms of some geometric objects related to $\Sigma$. To this purpose, let us consider the discontinuity of the derivative of a function $f$, which can be decomposed into its transverse and tangent part to $\Sigma$ as

$$[\partial_\alpha f] = n_\alpha l^\beta [\partial_\beta f] + \omega^a_\alpha \partial_a [f].$$

Applying this formula to the metric tensor, taking into account $[g_{\alpha\beta}] = 0$, we obtain

$$[\partial_\alpha g_{\mu\nu}] = \zeta_{\mu\nu} n_\alpha,$$

where $\zeta_{\mu\nu}$ is a symmetric tensor defined at points on $\Sigma$, which can be proven not to depend on the rigging [78, 82]. Thus the discontinuity in the Christoffel symbols reads in terms of $\zeta_{\mu\nu}$ as

$$[\Gamma^{\alpha}_{\mu\nu}] = \frac{1}{2}(n_\mu \zeta^{\alpha}_{\nu} + n_\nu \zeta^{\alpha}_{\mu} - n^\alpha \zeta_{\mu\nu}). \tag{2.38}$$

Therefore, the explicit decomposition of the object $\zeta_{\alpha\beta}$ in the basis $\{\boldsymbol{n}, \boldsymbol{\omega}^a\}$

$$\zeta_{\alpha\beta} = \zeta^{\vec{l}} n_\alpha n_\beta + \zeta^{\vec{l}}_{\alpha} n_\beta + \zeta^{\vec{l}}_{\beta} n_\alpha + \zeta^{\vec{l}}_{\alpha\beta}, \quad l^\alpha \zeta^{\vec{l}}_{\alpha\beta} = 0, \quad l^\alpha \zeta^{\vec{l}}_{\alpha} = 0, \tag{2.39}$$

where the superscript $\vec{l}$ is used to indicate that the objects depend on $\vec{l}$, allows us to write (2.38) as

$$[\Gamma^{\alpha}_{\mu\nu}] = \frac{1}{2}\left(n^\alpha n_\mu n_\nu \zeta^{\vec{l}} + 2 n_\mu n_\nu \zeta^{\vec{l}\,\alpha} + n_\mu \zeta^{\vec{l}\,\alpha}{}_{\nu} + n_\nu \zeta^{\vec{l}\,\alpha}{}_{\mu} - n^\alpha \zeta^{\vec{l}}{}_{\mu\nu}\right). \tag{2.40}$$

This latest expression, combined with formula (2.32), shows explicitly that only the component of $\zeta_{\alpha\beta}$ (completely) tangent to $\Sigma$ enters $H^{\alpha}_{\beta\mu\nu}$, i.e.

$$H^{\alpha}_{\beta\mu\nu} = \frac{1}{2}\left(n^{\alpha}(n_{\mu}\zeta^{\vec{l}}_{\nu\beta} - n_{\nu}\zeta^{\vec{l}}_{\mu\beta}) + n_{\beta}(n_{\mu}\zeta^{\vec{l}\ \alpha}_{\ \nu} - n_{\nu}\zeta^{\vec{l}\ \alpha}_{\ \mu})\right). \tag{2.41}$$

In order to present the so called junction conditions in terms of geometric objects of $\Sigma$, we introduce the tensor $\mathcal{H}_{ab}$ in $\Sigma$ constructed as [78, 82]

$$\mathcal{H}_{ab} = e^{\alpha}_{a} e^{\beta}_{b} \nabla_{\alpha} l_{\beta}. \tag{2.42}$$

Note that this object is not necessarily symmetric. In addition, it does not agree as computed from $\mathcal{V}^{+}$ or $\mathcal{V}^{-}$, and recalling its definition and taking into account that the bases $\{\vec{l}^{\pm}, \vec{e}^{\pm}_{a}\}$ have been identified for the construction of the continuous basis, the difference of the $\mathcal{H}^{\pm}_{ab}$ is given by

$$[\mathcal{H}_{ab}] := \mathcal{H}^{+}_{ab} - \mathcal{H}^{-}_{ab} = -l_{\alpha}[\Gamma^{\alpha}_{\mu\nu}] e^{\mu}_{a} e^{\nu}_{b} = \frac{1}{2}\zeta_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b}. \tag{2.43}$$

Although $\mathcal{H}$ is a tensor defined on $\Sigma$, we still keep the brackets in order to denote its difference as computed using the (different) connections of $\mathcal{V}^{\pm}$ respectively. The promotion of $[\mathcal{H}_{ab}]$ to $\mathcal{V}$ via $\Lambda_{l}$ is $[\mathcal{H}_{\alpha\beta}] = \zeta^{\vec{l}}_{\alpha\beta}/2$. Let us stress some properties of $[\mathcal{H}_{ab}]$ that can be observed in (2.43). First, it is proportional to the tangential part $\zeta^{\vec{l}}_{\alpha\beta}$ of $\zeta_{\alpha\beta}$. Secondly, although $\mathcal{H}_{ab}$ depends on the choice of the rigging, the difference does not (see Theorem 3.4 in [78] or Theorem 8 in [82]). Finally, $[\mathcal{H}_{ab}]$ is a symmetric tensor. The combination of (2.41) and the spacetime version of (2.43) provides an explicit expression for the singular part of the Riemann tensor distribution in terms of the $[\mathcal{H}_{ab}]$, which reads

$$H^{\alpha}_{\beta\mu\nu} \;=\; n^{\alpha}([\mathcal{H}_{\beta\mu}]n_{\nu} - [\mathcal{H}_{\beta\nu}]n_{\mu}) + n_{\beta}([\mathcal{H}^{\alpha}_{\nu}]n_{\mu} - [\mathcal{H}^{\alpha}_{\mu}]n_{\nu}). \tag{2.44}$$

In view of this expression above, we have

**Theorem 2** ***(Mars, Senovilla, 1993 [82])*** *The singular part of the Riemann tensor distribution vanishes if and only if $[\mathcal{H}_{\alpha\beta}] = 0$, or equivalently, iff $[\mathcal{H}_{ab}] = 0$.*

Thus, the condition $[\mathcal{H}_{ab}] = 0$ is equivalent to impose that the curvature tensor is free from singular terms and presents, at most, finite discontinuities on $\Sigma$. These are called the *matching conditions*, and from the above, they do not depend on the choice of the rigging.

Similar expressions to (2.44) can be found for the rest of the curvature tensors, leading to

$$
\begin{aligned}
H_{\beta\nu} &= -[\mathcal{H}_{\beta\nu}]n^\alpha n_\alpha + [\mathcal{H}_{\alpha\beta}]n^\alpha n_\nu + [\mathcal{H}_{\alpha\nu}]n^\alpha n_\beta - [\mathcal{H}^\alpha_\alpha]n_\beta n_\nu, && (2.45)\\
H &= -2n^\alpha n_\alpha[\mathcal{H}^\beta_\beta] + 2n^\alpha n^\beta[\mathcal{H}_{\alpha\beta}], && (2.46)\\
\mathcal{G}_{\beta\nu} &= -n^\alpha n_\alpha[\mathcal{H}_{\beta\nu}] - [\mathcal{H}^\alpha_\alpha]n_\beta n_\nu + [\mathcal{H}_{\alpha\beta}]n^\alpha n_\nu + [\mathcal{H}_{\alpha\nu}]n^\alpha n_\beta + \\
&\quad -g_{\beta\nu}|_\Sigma\left([\mathcal{H}_{\alpha\mu}]n^\alpha n^\mu - [\mathcal{H}^\alpha_\alpha]n^\mu n_\mu\right). && (2.47)
\end{aligned}
$$

Notice that the necessary conditions for which the Ricci singular part vanishes vary depending on the causal character of $\Sigma$. On the one hand, at points where $\Sigma$ is null, $H_{\beta\nu}$ vanishes iff $n^\alpha[H_{\alpha\beta}] = [H^\alpha_\alpha] = 0$ and thus the matching conditions are a larger set of conditions. On the other hand, at points where $\Sigma$ is not null the Ricci singular part vanishes iff the matching conditions hold. The singular part of the Einstein tensor distribution vanishes iff the singular part of the Ricci tensor distribution vanishes and finally, the singular part of the Ricci scalar distribution vanishes iff $(\vec{n}\cdot\vec{n})[\mathcal{H}^\alpha_\alpha] = [\mathcal{H}_{\alpha\beta}]n^\alpha n^\beta$ (see Theorem 3.3 in [78], or the corresponding Theorem 7 in [82]).

Apart from the implications that the matching conditions have on the singular parts of the curvature tensor distributions, they also impose restrictions on the possible discontinuities that these can present. A detailed study on the continuities and discontinuities in the curvature tensors is presented in [78, 82] for the case of a four dimensional spacetime, but the results are independent of the dimensionality of the spacetime. A straightforward method consists of taking the differences of Gauss-Codazzi equations for general hypersurfaces (2.11)-(2.14) coming from $\mathcal{V}^+$ and $\mathcal{V}^-$. Recall that these are written in terms of the objects $\Psi^b_a$, $\varphi_a$ and $\kappa_{ab}$, whose differences, when the matching conditions hold, read [78, 82]

$$[\Psi^a_b] = 0, \quad [\varphi_a] = 0, \quad [\kappa_{ab}] = 0. \tag{2.48}$$

It can be proven [78, 82] that $[\Gamma^a_{bc}] = 0$ , and therefore the connection $\overline{\nabla}$ agrees as taken from $\mathcal{V}^+$ and $\mathcal{V}^-$ respectively. It follows that

$$[R^\mu_{\alpha\beta\gamma}]e^\beta_a e^\gamma_b = 0. \tag{2.49}$$

This relation shows that, once the matching conditions hold, the only allowed discontinuitites of the Riemann tensor must adopt the form

$$[R_{\alpha\beta\mu\nu}] = B_{\beta\nu}n_\alpha n_\mu + B_{\alpha\mu}n_\beta n_\nu - B_{\alpha\nu}n_\beta n_\mu - B_{\beta\mu}n_\alpha n_\nu, \tag{2.50}$$

for some symmetric 2-covariant tensor $B_{\alpha\beta}$ defined on $\Sigma$ up to transformations of the type

$$B'_{\alpha\beta} = B_{\alpha\beta} + n_\alpha X_\beta + n_\beta X_\alpha, \tag{2.51}$$

for an arbitrary one-form $X_\alpha$, so that $n+1$ components out of the $(n+1)(n+2)/2$ components of $B_{\alpha\beta}$ can be removed using this freedom. We can use this freedom to remove any component of $B_{\alpha\beta}$ non-tangential to $\Sigma$, fixing completely the one-form $X_\alpha$ by setting $X_\alpha = -l^\mu l^\beta B_{\mu\beta} n_\alpha/2 - l^\mu e^\beta_a B_{\mu\beta}\omega^a_\alpha$. Therefore $B_{\alpha\beta}$ and its pullback to $\Sigma$, i.e. $B_{ab} = B_{\alpha\beta} e^\alpha_a e^\beta_b$, encode the same information and we can use them indistinctly.

Now the discontinuity of the Ricci tensor can be expressed in terms of $B_{ab}$. An explicit decomposition in the basis $\{\boldsymbol{n}, \boldsymbol{\omega}^a\}$ of the suitable trace of (2.50) yields

$$[R_{\alpha\beta}] \;=\; g^{ab} B_{ab} n_\alpha n_\beta - 2n^a B_{ab} n_{(\alpha}\omega^b_{\beta)} + n^\mu n_\mu B_{ab}\omega^a_\alpha \omega^b_\beta. \tag{2.52}$$

Note that $g^{ab}B_{ab} = B^\alpha_\alpha$. Taking the trace of (2.52), the jump in the Ricci scalar is found to be

$$[R] = 2(g^{ab}B_{ab} n^\mu n_\mu - n^a n^b B_{ab}). \tag{2.53}$$

Combining (2.52) and (2.53) it is now straightforward to obtain the relation

$$n^\alpha [G_{\alpha\beta}] = 0. \tag{2.54}$$

The $n+1$ equations in (2.54) are the generalized Israel conditions [67] for general matching hypersurfaces (when the matching conditions hold).

Thus, we have introduced the curvature tensor distributions relevant in order to obtain the field equations as objects defined in the whole spacetime $\mathcal{V}$. The Einstein field equations $\underline{G}_{\alpha\beta} = 8\pi \underline{T}_{\alpha\beta}$ lead to a distributional energy momentum tensor of the form

$$\underline{T}_{\beta\nu} = T^+{}_{\beta\nu}\cdot\underline{\theta} + T^-{}_{\beta\nu}\cdot(\underline{1}-\underline{\theta}) + \tau_{\beta\nu}\cdot\delta^\Sigma. \tag{2.55}$$

The tensor fields $T^\pm$ correspond to the energy momentum tensor defined on each region $\mathcal{V}^\pm$, whereas $\tau_{\alpha\beta}$ is the singular part, with support on $\Sigma$. It is used to model physical situations like surface layers as crusts in star models [55] or braneworlds [83]. The generalized Israel conditions are translated to the energy momentum tensor by means of the field equations so that $n^\alpha[T_{\alpha\beta}] = 0$. Note that $\tau_{\alpha\beta}$ depends on $\boldsymbol{n}$ (only $\tau_{\alpha\beta}\delta^\Sigma$ is intrinsically defined).

As a consequence of the matching conditions, it is shown in [78, 82] that it is possible to construct a local system of coordinates in which the metric is $C^1$. To this aim, consider a $C^1$ change of local system of coordinates $x(x')$. Note that the preliminary matching conditions require that the metric is $C^0$ and thus under this change of coordinates the metric is still continuous (at least). However, we are interested in the derivative of the metric, and thus the following calculation must be understood in the distributional sense, although I will not denote it explicitly not to overload the expressions. At points of $\Sigma$, the transformation of the first derivative of metric is given by

$$\begin{aligned}\frac{\partial g_{\alpha'\beta'}}{\partial x^{\mu'}} &= \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} g_{\alpha\beta}\Big|_{\Sigma} \frac{\partial}{\partial x^{\mu'}}\left(\frac{\partial x^{\beta}}{\partial x^{\beta'}}\right) + \frac{\partial x^{\beta}}{\partial x^{\beta'}} g_{\alpha\beta}\Big|_{\Sigma} \frac{\partial}{\partial x^{\mu'}}\left(\frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\right) \\ &+ \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\frac{\partial x^{\beta}}{\partial x^{\beta'}}\Big|_{\Sigma} \frac{\partial g_{\alpha\beta}}{\partial x^{\mu}}\end{aligned}$$

Since we are interested in the jump of this derivative at $\Sigma$, we compute first the bracket of the derivative of the jacobian transformation to find

$$\left[\frac{\partial}{\partial x^{\mu'}}\left(\frac{\partial x^{\beta}}{\partial x^{\beta'}}\right)\right] = n_{\mu'}T^{\beta}_{\beta'} + \omega^{a}_{\mu'} e^{\nu'}_{a} \frac{\partial}{\partial x^{\nu'}}\left[\frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\right] = n_{\mu'}T^{\beta}_{\beta'} = n_{\mu'} n_{\beta'} T^{\beta},$$

for some arbitrary vector $T^{\beta}$, defined at points of $\Sigma$. In the last step the symmetry of the second derivatives has been used. Hence, the jump in the (primed) derivative of the (primed) metric yields

$$\begin{aligned}\left[\frac{\partial g_{\alpha'\beta'}}{\partial x^{\mu'}}\right] &= \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} g_{\alpha\beta}\Big|_{\Sigma} \left[\frac{\partial}{\partial x^{\mu'}}\left(\frac{\partial x^{\beta}}{\partial x^{\beta'}}\right)\right] + \frac{\partial x^{\beta}}{\partial x^{\beta'}} g_{\alpha\beta}\Big|_{\Sigma} \left[\frac{\partial}{\partial x^{\mu'}}\left(\frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\right)\right] \\ &+ \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\frac{\partial x^{\beta}}{\partial x^{\beta'}}\Big|_{\Sigma} \left[\frac{\partial g_{\alpha\beta}}{\partial x^{\mu}}\right] = n_{\mu'}\left(n_{\alpha'}T_{\beta'} + n_{\beta'}T_{\alpha'} + \zeta_{\alpha'\beta'}\right)\end{aligned}$$

Taking into account this last expression, the decomposition of $\zeta_{\alpha'\beta'}$ in tangent and normal components to $\Sigma$ as in (2.39) and the matching conditions given in Theorem 2, that imply $\zeta^{\vec{l}}_{\alpha\beta} = 0$ (recall (2.43)), we find that for a change of coordinates with $T_{\alpha} = -\zeta^{\vec{l}}/2n_{\alpha} - \zeta^{\vec{l}}_{\alpha}$, the metric becomes $C^1$ at $\Sigma$. This result recovers the matching conditions in the Lichnerowicz sense [73].

## Timelike matching hypersurfaces

This framework to match spacetimes does not assume any condition on the causal character of the matching hypersurface at any point. However, along this thesis we will restrict ourselves to matching hypersurfaces that are **timelike** everywhere. Therefore, any normal vector $\vec{n}$ is transverse to $\Sigma^{\pm}$ everywhere, and we can set $\vec{l}^{\pm} = \vec{n}^{\pm}$, so that $\boldsymbol{n}^{\pm}(\vec{n}^{\pm}) = 1$ and $\{\vec{n}, \vec{e}_a\}$ is clearly a basis of $T_p\mathcal{V}$ at points of $\Sigma$. The dual tangent space is built now with respect to $\vec{n}$. According to Lemma 1, given $\vec{n}^{+}$ the system (2.5) provides a unique solution with the correct orientation for $\vec{n}^{-}$. Moreover, the first fundamental form $h_{ab}$ becomes non degenerate and its inverse is just $h^{ab} = g^{ab}$. The spacetime version of the first fundamental form, unique due to (2.2), is given by the projector to $\Sigma$ (defined only on $\Sigma$)

$$h_{\mu\nu} = g_{\mu\nu}|_{\Sigma} - n_{\mu}n_{\nu}. \tag{2.56}$$

Notice that

$$n^\mu h_{\mu\nu} = 0, \qquad h_{\mu\rho} h^\rho{}_\nu = h_{\mu\nu}, \qquad h^\mu{}_\mu = n, \qquad h_{\mu\nu} e^\mu_a e^\nu_b = h_{ab}$$

and that

$$e^\mu_a = h_{ab}\, \omega^b_\nu\, g^{\nu\mu}|_\Sigma, \qquad e^\mu_c \omega^c_\nu = h^\mu_\nu \, .$$

Despite all the above, the extrinsic curvatures, or second fundamental forms, inherited by $\Sigma$ from both sides $\mathcal{V}^\pm$ will be, in principle, different, because the derivatives of the metric are not continuous in general. The tensor $\mathcal{H}_{ab}$ defined in (2.42) corresponds now to the second fundamental form $\kappa_{ab}$, defined as

$$\kappa^\pm_{ab} := e^\alpha_a e^\beta_b \nabla^\pm_\alpha n_\beta, \quad \kappa^\pm_{ab} = \kappa^\pm_{ba}, \tag{2.57}$$

or written in the spacetime version $\kappa^\pm_{\mu\nu} := \omega^a_\mu \omega^b_\nu \kappa^\pm_{ab}$

$$\kappa^\pm_{\mu\nu} = h^\rho{}_\nu h^\sigma_\mu \nabla^\pm_\rho n_\sigma, \qquad \kappa^\pm_{\mu\nu} = \kappa^\pm_{\nu\mu}$$

where only tangent derivatives are involved. Obviously $n^\mu \kappa^\pm_{\mu\nu} = 0$ by construction, thus only the $n(n+1)/2$ components tangent to $\Sigma$ are non-identically vanishing. In terms of the embeddings (2.1) these components are given by

$$\kappa^\pm_{ab} = -n^\pm_\mu \left( \frac{\partial^2 \Phi^\mu_\pm}{\partial \xi^a \partial \xi^b} + \Gamma^{\pm\mu}_{\alpha\beta} \frac{\partial \Phi^\alpha_\pm}{\partial \xi^a} \frac{\partial \Phi^\beta_\pm}{\partial \xi^b} \right), \tag{2.58}$$

which is adapted to explicit calculations.

Using (2.58) together with (2.40) we deduce

$$\kappa^+_{ab} - \kappa^-_{ab} = -n_\mu \left[\Gamma^\mu_{\rho\sigma}\right] e^\rho_a e^\sigma_b = \frac{1}{2} \zeta^{\vec{n}}_{\rho\sigma} e^\rho_a e^\sigma_b, \tag{2.59}$$

that is to say, the tangent part of $\zeta_{\mu\nu}$ is characterized by the difference of the two $\pm$-second fundamental forms. Thus, defining the *jump* on $\Sigma$ of the second fundamental form as usual

$$[\kappa_{\mu\nu}] := \kappa^+_{\mu\nu} - \kappa^-_{\mu\nu}, \qquad n^\mu \left[\kappa_{\mu\nu}\right] = 0 \tag{2.60}$$

we can rewrite the singular part of the Riemann tensor distribution (2.41) as

$$H_{\alpha\beta\mu\nu} = n_\alpha([\kappa_{\beta\mu}]\, n_\nu - [\kappa_{\beta\nu}]\, n_\mu) + n_\beta([\kappa_{\alpha\nu}]\, n_\mu - [\kappa_{\alpha\mu}]\, n_\nu). \tag{2.61}$$

Compare this expression with the analogous one given for general hypersurfaces (2.44). It is manifest that the role of $[\mathcal{H}_{\alpha\beta}]$ is now played by the jump of the second fundamental form. In fact, in order to derive the matching conditions, we can either start from (2.61) or take the expressions developed for general hypersurfaces and substitute the tensor $[\mathcal{H}_{ab}]$

there by $[\kappa_{ab}]$. It can be shown [78, 82] that in general hypersurfaces the proportionality relation $[\kappa_{ab}] = (\vec{n}\cdot\vec{n})[\mathcal{H}_{ab}]$ holds, and therefore the matching conditions for timelike hypersurfaces become $[\kappa_{ab}] = 0$.

The expressions (2.45), (2.46) and (2.47) that describe the singular parts in the curvature tensor distributions remain valid with the change $[\mathcal{H}_{ab}] \to [\kappa_{ab}]$, since $[H_{ab}]$ does not depend on the choice of $\vec{l}$. We include them for completeness. After some simplifications they read

$$
\begin{aligned}
H_{\beta\nu} &= -[\kappa_{\beta\nu}] - [\kappa^{\alpha}_{\alpha}]n_{\beta}n_{\nu}, && (2.62)\\
H &= -2[\kappa^{\beta}_{\beta}], && (2.63)\\
\mathcal{G}_{\beta\nu} &= -[\kappa_{\beta\nu}] + [\kappa^{\alpha}_{\alpha}](g_{\beta\nu}|_{\Sigma} - n_{\beta}n_{\nu}), && (2.64)\\
&\Rightarrow \; n^{\beta}\mathcal{G}_{\beta\nu} = 0. && (2.65)
\end{aligned}
$$

Note that in General Relativity the singular part $\tau_{\alpha\beta}$ of the energy momentum tensor (2.55) is now tangent to $\Sigma$ due to (2.65). In order to compute the jumps of the curvature tensors in terms of the extrisic curvature, let us remark that the connection $\overline{\nabla}$ introduced in (2.8) for general hypersurfaces is now constructed with respect to a normal vector to $\Sigma$, and it is the unique metric connection associated with the first fundamental form $h$. The two objects defined in (2.10) become $\Psi^{b}_{a} = \kappa^{b}_{a}$ and $\varphi_a = 0$. Thus, the Gauss equation (2.11) takes the standard form

$$
\omega^{d}_{\alpha}R^{\alpha}_{\beta\gamma\delta}e^{\beta}_{a}e^{\gamma}_{b}e^{\delta}_{c} = \overline{R}^{d}_{abc} - \kappa_{ac}\kappa^{d}_{b} + \kappa_{ab}\kappa^{d}_{c}, \qquad (2.66)
$$

while the Codazzi equations (2.12) and (2.13) collapse to the single equation

$$
n_{\mu}R^{\mu}_{\beta\lambda\nu}e^{\beta}_{a}e^{\lambda}_{b}e^{\nu}_{c} = \overline{\nabla}_{c}\kappa_{ba} - \overline{\nabla}_{b}\kappa_{ca}, \qquad (2.67)
$$

and Codazzi 3 equation (2.14) vanishes identically.

The aforementioned discontinuities, letting aside that of the Riemann tensor, which is still given by (2.50), follow from (2.52) and (2.53) just setting $g^{ab} = h^{ab}$ plus $n^{a} = 0$. Alternatively one can take traces of (2.50) using $n^{\alpha}B_{\alpha\beta} = 0$, and they read now

$$
\begin{aligned}
&[R_{\alpha\beta}] = h^{ab}B_{ab}n_{\alpha}n_{\beta} + B_{ab}\omega^{a}_{\alpha}\omega^{b}_{\beta} = B^{\rho}_{\rho}n_{\alpha}n_{\beta} + B_{\alpha\beta}n^{\rho}n_{\rho}, && (2.68)\\
&[R] = 2h^{ab}B_{ab} = 2B^{\alpha}_{\alpha}, && (2.69)\\
&[G_{\alpha\beta}] = (B_{ab} - h^{cd}B_{cd}h_{ab})\omega^{a}_{\alpha}\omega^{b}_{\beta} = B_{\alpha\beta} - B^{\rho}_{\rho}h_{\alpha\beta}, && (2.70)\\
&\quad\Rightarrow \quad n^{\alpha}[G_{\alpha\beta}] = 0. && (2.71)
\end{aligned}
$$

The physical interpretation of the *matching conditions* becomes more clear now. The independent discontinuities are encoded in the jump of the Einstein tensor, completely tangent to $\Sigma$. The field equations propagate (2.71) to the energy momentum tensor and we find $n^\alpha[T_{\alpha\beta}] = 0$. The decomposition of the energy momentum tensor in the basis $\{\boldsymbol{n}, \boldsymbol{\omega}^a\}$ at points of $\Sigma$

$$T_{\alpha\beta} = T^{\perp} n_\alpha n_\beta + n_\alpha T^{|}_\beta + n_\beta T^{|}_\alpha + T^{||}_{\alpha\beta}, \quad n^\alpha T^{|}_\alpha = 0, \quad n^\alpha T^{||}_{\alpha\beta} = 0$$

holds for of the + and − spacetimes. Thus, the energy density, energy fluxes and pressure components tangent to $\Sigma$, reflected in the tangent part $T^{||}_{\alpha\beta}$ of the energy momentum tensor, are allowed to be discontinuous, encoding in addition all the independent discontinuities. Conversely, the normal pressure to $\Sigma$, encoded in $T^{\perp}$ and the energy flux through $\Sigma$, in $T^{|}_\alpha$, must be continuous.

## 2.3 Matchings preserving the symmetries

In many physical situations it is required that the whole matched spacetime exhibits some symmetries, and for this, the independent spacetimes to be matched must share these symmetries, which, in addition, are asked to be inherited by the matching hypersurface $\Sigma$. In rough words, this amounts to requiring that the restrictions to the matching hypersurface $\Sigma$ of the generators of the symmetries (wanted to be preserved) in the spacetimes $(\mathcal{V}^\pm, g^\pm)$ are tangent to $\Sigma$. The precise definition of a matching preserving the symmetries, given in [108], is

**Definition 6** *Let $(\mathcal{V}, g)$ be a spacetime arising from the matching of two oriented $C^2$ spacetimes $(\mathcal{V}^\pm, g^\pm)$ admitting a $G_{n^+}$ and $G_{n^-}$ local group of symmetries, respectively, and with respective boundaries $\Sigma^\pm$ given by the embeddings (2.1). Then $(\mathcal{V}, g)$ preserves the symmetry defined by the subgroup $G_m$ with $m \leq min\{n^+, n^-\}$ when first, this group is admitted by both $(\mathcal{V}^\pm, g^\pm)$, and second, the differential maps $d\Phi^\pm$ send $m$ vector fields $\vec{\gamma}_A$ $(A = 1, ..., m)$ on $\Sigma$ to the restrictions of the generators $\vec{K}^\pm_A$ of $G_m$ to $\Sigma^\pm$.*

One of the first consequences of the symmetry preserving matchings is that if the preserved symmetries are generated by conformal Killing vector fields these satisfy

**Lemma 3** ***(Vera, 2002 [108])*** *Let $\vec{\zeta}^+$ and $\vec{\zeta}^-$ be two conformal Killing vector fields acting on $(\mathcal{V}^+, g^+)$ and $(\mathcal{V}^-, g^-)$ respectively, so that $\mathcal{L}_{\vec{\zeta}^\pm} g^\pm = \alpha^\pm g^\pm$ for some functions $\alpha^\pm$, allowed to be zero. If $(\mathcal{V}^+, g^+)$ and $(\mathcal{V}^-, g^-)$ are matched across a matching hypersurface $\Sigma \equiv \Sigma^+ = \Sigma^-$ diffeomorphic by (2.1) such that there is a vector field $\vec{\gamma}$ satisfying $d\phi^\pm(\vec{\gamma}) = \vec{\zeta}^\pm|_\Sigma$, then $\alpha^+|_\Sigma = \alpha^-|_\Sigma$.*

This kind of matchings is appropriate for the study of isolated bodies rotating in equilibrium: the models are constructed matching a spacetime containing an interior fluid with an exterior vacuum spacetime. Both are stationary and axially symmetric, admitting thus a $G_2$ group of isometries acting on $T_2$ surfaces. The matching is asked to preserve these symmetries. Axial symmetry forces the $G_2$ on $T_2$ groups to be Abelian in both interior and exterior spacetimes [81]. On the other hand, the orthogonal transitivity property is only guaranteed in vacuum, although assumed in the fluid region imposing the circularity condition.

However, the property of orthogonal transitivity of a $G_2$ group in one of the spacetimes is propagated to the matching hypersurface when the matching preserves the symmetries. Let us summarize two important results that apply to any two-dimensional $G_2$ local group of symmetries acting on non null surfaces. For this, consider the generators $\{\vec{\xi}^+, \vec{\eta}^+\}$ and $\{\vec{\xi}^-, \vec{\eta}^-\}$ in $(\mathcal{V}^+, g^+)$ and $(\mathcal{V}^-, g^-)$ respectively. Their associated one-forms are $\{\boldsymbol{\xi}^+, \boldsymbol{\eta}^+\}$ and $\{\boldsymbol{\xi}^-, \boldsymbol{\eta}^-\}$ so that the 4-forms $\boldsymbol{\xi}^\pm \wedge \boldsymbol{\eta}^\pm \wedge \mathbf{d}\boldsymbol{\eta}^\pm$ and $\boldsymbol{\xi}^\pm \wedge \boldsymbol{\eta}^\pm \wedge \mathbf{d}\boldsymbol{\xi}^\pm$, define, through the Hodge dual $\star$, two $n-3$-forms that in 4 dimensional spacetimes are just functions.

**Theorem 3** ***(Vera, 2002 [108])*** *Given a matching preserving the symmetry of a $G_2$ local conformal group, not necessarily proper, and choosing $\{\vec{\xi}^+, \vec{\eta}^+\}$ and $\{\vec{\xi}^-, \vec{\eta}^-\}$ as the sets of generators of the $G_2$ groups at $(\mathcal{V}^+, g^+)$ and $(\mathcal{V}^-, g^-)$ respectively such that $d\Phi^\pm(\vec{\gamma}_1) = \vec{\xi}^\pm\big|_{\Sigma^\pm}$ and $d\Phi^\pm(\vec{\gamma}_2) = \vec{\eta}^\pm|_{\Sigma^\pm}$, for a pair $\vec{\gamma}_1$ and $\vec{\gamma}_2$, of vectors on $\Sigma$, then*

$$\begin{aligned} \star\left(\boldsymbol{\eta}^+ \wedge \boldsymbol{\xi}^+ \wedge \mathbf{d}\boldsymbol{\xi}^+\right) &\stackrel{\Sigma}{=} \star\left(\boldsymbol{\eta}^- \wedge \boldsymbol{\xi}^- \wedge \mathbf{d}\boldsymbol{\xi}^-\right), \\ \star\left(\boldsymbol{\xi}^+ \wedge \boldsymbol{\eta}^+ \wedge \mathbf{d}\boldsymbol{\eta}^+\right) &\stackrel{\Sigma}{=} \star\left(\boldsymbol{\xi}^- \wedge \boldsymbol{\eta}^- \wedge \mathbf{d}\boldsymbol{\eta}^-\right). \end{aligned}$$

In particular, Theorem 3 implies, when one of the $G_2$ groups, say that of the spacetime $+$, acts orthogonally transitively on $\Sigma$ the following

**Corollary 3.1** ***(Vera, 2002 [108])*** *Given a matching preserving a $G_2$ local conformal group – not necessarily proper – as defined above and such that the $G_2$ acts orthogonally transitively at one side of $\Sigma$, say at $(\mathcal{V}^+, g^+)$ then*

$$\begin{aligned} \star\left(\boldsymbol{\eta}^- \wedge \boldsymbol{\xi}^- \wedge \mathbf{d}\boldsymbol{\xi}^-\right) &\stackrel{\Sigma}{=} 0, \\ \star\left(\boldsymbol{\xi}^- \wedge \boldsymbol{\eta}^- \wedge \mathbf{d}\boldsymbol{\eta}^-\right) &\stackrel{\Sigma}{=} 0. \end{aligned}$$

A similar result regarding the integrability of (conformal) Killing vectors holds, provided that the matching preserves the symmetries. Although it was not presented in [108] it can be proven using the same construction needed for Theorem 3. It reads

**Lemma 4** *Given a symmetry preserving matching, let $\vec{\zeta}^\pm$ be (conformal) Killing vector fields in $(\mathcal{V}^+, g^+)$ and $(\mathcal{V}^-, g^-)$ respectively, such that $d\Phi^\pm(\vec{\gamma}) = \vec{\zeta}^\pm|_\Sigma$, for some vector $\vec{\gamma}$ in $\Sigma$. Then*

$$\boldsymbol{\zeta}^+ \wedge \mathbf{d}\boldsymbol{\zeta}^+ \stackrel{\Sigma}{=} \boldsymbol{\zeta}^- \wedge \mathbf{d}\boldsymbol{\zeta}^-. \tag{2.72}$$

**Corollary 3.2** *And if $\vec{\zeta}^+$ is integrable, it follows that*

$$\boldsymbol{\zeta}^- \wedge \mathbf{d}\boldsymbol{\zeta}^- \stackrel{\Sigma}{=} 0. \tag{2.73}$$

**Proof:** We include here the proof for the result (2.72), which follows the proof of Theorem 3 given in [108]. The explicit decomposition of any one-form $\boldsymbol{\zeta}$ at points of $\Sigma$ in the cobasis $\{\boldsymbol{n}, \boldsymbol{\omega}^a\}$ reads

$$\boldsymbol{\zeta}|_\Sigma = \Xi \boldsymbol{n} + \zeta_a \boldsymbol{\omega}^a, \qquad \Xi := \boldsymbol{\zeta}(\vec{l}), \;\; \zeta_a := \boldsymbol{\zeta}(\vec{e}_a).$$

The exterior derivative of $\zeta$ decomposes as

$$\mathbf{d}\boldsymbol{\zeta}|_\Sigma = A_{ab}\boldsymbol{\omega}^a \wedge \boldsymbol{\omega}^b + B_a \boldsymbol{n} \wedge \boldsymbol{\omega}^a,$$

where we have defined

$$\begin{aligned} A_{ab} &:= \vec{e}_{[a}\left(\zeta_{b]}\right), \quad A_{(ab)} = 0; \\ B_a &:= 2 l^\alpha e^\beta_a \nabla_{[\alpha}\zeta_{\beta]} = l^\alpha e^\beta_a \nabla_\alpha \zeta_\beta - \vec{e}_a(\Xi) + \zeta_\alpha e^\beta_a \nabla_\beta l^\alpha. \end{aligned}$$

Thus the 3-form resulting from the wedge product at points of $\Sigma$ reads

$$\boldsymbol{\zeta} \wedge \mathbf{d}\boldsymbol{\zeta}|_\Sigma = (\zeta_a A_{bc})\, \boldsymbol{\omega}^a \wedge \boldsymbol{\omega}^b \wedge \boldsymbol{\omega}^c + (\Xi A_{ab} - \zeta_a B_b)\, \boldsymbol{n} \wedge \boldsymbol{\omega}^a \wedge \boldsymbol{\omega}^b. \tag{2.74}$$

We are interested in computing the difference of this last product as seen from the two spacetimes + and −. Note first that since the matching preserves the symmetries $[\boldsymbol{\zeta}] = 0$, so that $[\Xi] = [\zeta_a] = 0$. As a consequence, the tangential derivatives of $[\Xi]$ and $[\zeta_a]$ also agree on $\Sigma$. Recalling the definition of $A_{ab}$, this implies $[A_{ab}] = 0$. Finally, for $B_a$ we find

$$[B_a] = l^\alpha e^\beta_a [\nabla_\alpha \zeta_\beta] + \zeta_\alpha e^\beta_a [\nabla_\beta l^\alpha] = l^\alpha e^\beta_a [\nabla_\alpha \zeta_\beta] + \zeta^b [\mathcal{H}_{ab}],$$

and after a little manipulation the first summand can be written as

$$l^\alpha e^\beta_a [\nabla_\alpha \zeta_\beta] = l^\alpha e^\beta_a [\mathcal{L}_{\vec{\zeta}} g_{\alpha\beta}] + \zeta^\alpha e^\beta_a [\nabla_\beta l_\alpha].$$

Therefore, the difference of (2.74) as seen from the sides + and − is given by

$$[\boldsymbol{\zeta} \wedge \mathbf{d}\boldsymbol{\zeta}] = -\zeta_a (l^\alpha e^\beta_b [\mathcal{L}_{\vec{\zeta}} g_{\alpha\beta}] + 2\zeta^c [\mathcal{H}_{bc}]) \boldsymbol{n} \wedge \boldsymbol{\omega}^a \wedge \boldsymbol{\omega}^b,$$

which after the imposition of the matching conditions, $[\mathcal{H}_{ab}] = 0$, becomes

$$[\boldsymbol{\zeta} \wedge \mathbf{d}\boldsymbol{\zeta}] = -\zeta_a l^\alpha e^\beta_b [\mathcal{L}_{\vec{\zeta}} g_{\alpha\beta}] \boldsymbol{n} \wedge \boldsymbol{\omega}^a \wedge \boldsymbol{\omega}^b.$$

This expression vanishes by assumption due to Lemma 3. ■

# 3

# Perturbed matching of spacetimes

The previous chapter was devoted to construct a spacetime by joining two spacetimes across a common boundary. After some development, the matching conditions are formulated as a set of equations involving the first and second fundamental forms, which depend not only on the spacetimes to be matched but also on the matching hypersurface itself.

However, there are some problems in GR for which a perturbative approach is convenient. For instance, the only known solution of a fluid ball rotating in equilibrium and immersed in vacuum has been only achieved in perturbation schemes [16, 21, 57]. The matching conditions can be formulated in perturbation theory, but in the process an additional complication arises. Perturbation theory carries an inherent freedom known as the *gauge* freedom that affects, in particular, the fundamental objects of the matching theory, the *perturbed* first and second fundamental forms. A second freedom of this type arises from the identification of the boundaries themselves. To sum up, two levels of gauge freedom are inherent to the perturbations of hypersurfaces and result relevant, for instance, in the determination of their deformation. A successful theory of perturbed matchings has to be independent of these freedoms, and free of gauge choices that may restrict its applicability. This was achieved in full generality in [79], where first and second order perturbations of hypersurfaces were considered. The analysis of the gauge freedoms, although already present in [79], is retaken in a subsequent work [80] where also the consequences that symmetries of the background spacetime have in the perturbation method are studied.

There are other approaches mainly devoted to spherically symmetric background spacetimes. The classical papers [51, 52] discuss a general framework around spherical symmetry, but their approach to the matching conditions contains imprecissions (see [80]). This formalism is revisited in [85] with the aim at justifying the claims in [51, 52]. In [85] the formulation of the matching conditions is built upon a class of (spacetime) gauges that maps the perturbed matching hypersurface to the background matching hypersurface, thus introducing the concept of surface comoving gauges[1]. Although the matching conditions are eventually provided in terms of (spacetime) gauge independent quantities, some imprecissions and implicit assumptions are still present, which may give rise to problems by not properly using the framework, as discussed with detail in [80]. In particular, these approaches ignore the hypersurface gauge freedom, which may have some subtle importance, for instance when showing the existence of the perturbed matching.

In [87] the perturbed matching conditions are found for background geometries with a high degree of symmetry, and in that case the first order matching conditions are presented in terms of double gauge invariants, i.e. quantities that are both spacetime and hypersurface gauge invariants. It was shown in [80] that that set of matching conditions is not sufficient to ensure the perturbed matching, since it does not cover the $l = 0, 1$ sectors of the matching conditions (in a decomposition using spherical harmonics). Perturbations of hypersurfaces to second order haven been also treated in [11] in the context of cosmic strings and branes.

In this thesis we will follow the consistent and general theory of perturbed matchings to second order provided in [79], which formulates the set of matching conditions independetly of the gauges used at either side of the matching and provides the deformation with respect to those gauges.

This chapter is devoted to summarize the main ingredients and results of [79] (and [80]). It is divided in 5 sections. In first place (Sections 3.1 and 3.2), we present the metric perturbations as a problem for two-covariant symmetric tensors defined in a fixed spacetime (the background) and the deformation of the hypersurfaces encoded in a vector field defined in an (unperturbed) hypersurface embedded in that background spacetime. Section 3.3 is devoted to the construction of tensorial objects, also defined in the unperturbed hypersurface in the background spacetime, that represent the perturbed first and second fundamental forms induced by the metric and hypersurface perturbations. Finally, the perturbed matching conditions are presented as equations for those tensorial objects in Section 3.4. The last section (3.5) is devoted to summarize some peculiarities that arise when the background spacetime has symmetries.

[1] We will refer to the gauges used in [85] as "surface comoving gauges", although in that work this is referred to as "surface gauges". We keep the "surface gauge" term when we require also a pointwise identification. We explain these gauges just after Proposition 4.

## 3.1 Metric perturbations

Let us consider a one parameter family of $n+1$ dimensional spacetimes $(\mathcal{V}_\varepsilon, \hat{g}_\varepsilon)$ diffeomorphic to each other. From this family of spacetimes we single out one element, $(\mathcal{V}_0, g_0)$ as the background spacetime. For the sake of notation, in most places we drop the subindex 0 to denote objects related to the background, and hence we will refer to this spacetime simply by $(\mathcal{V}, g)$. By assumption there is some $\varepsilon$-dependent diffeomorphism $\psi_\varepsilon$

$$\psi_\varepsilon : \mathcal{V} \to \mathcal{V}_\varepsilon. \tag{3.1}$$

The diffeomorphism $\psi_\varepsilon$ allows us to define a one parameter family of metrics [2] $g_\varepsilon$ on $\mathcal{V}$ related to $\hat{g}_\varepsilon$. In terms of the pullback $\psi^*_\varepsilon$ these can be expressed as $g_\varepsilon := \psi^*_\varepsilon(\hat{g}_\varepsilon)$, so that $g = g_{\varepsilon=0}$. The first and second order perturbation tensors are symmetric 2-covariant tensors in the spacetime $(\mathcal{V}, g)$ defined as follows

$$K_1 := \left.\frac{dg_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0}, \quad K_2 := \left.\frac{d^2 g_\varepsilon}{d\varepsilon^2}\right|_{\varepsilon=0}. \tag{3.2}$$

Perturbation theory to second order consists in the study of these two tensors.

Tensorial objects in contravariant form can also be pushforwarded to $(\mathcal{V}, g)$ through the inverse map $\psi^{-1}_\varepsilon$, so that, for instance, a one parameter family of two contravariant symmetric tensors can be defined on $\mathcal{V}$ in terms of the inverse metrics $\hat{g}^{-1}_\varepsilon$ via the pushforward $d\psi^{-1}_\varepsilon$ so that

$$\left(g^{-1}_\varepsilon\right)^{\alpha\beta} := \left(d\psi^{-1}_\varepsilon \left(\hat{g}^{-1}_\varepsilon\right)\right)^{\alpha\beta}. \tag{3.3}$$

Note that $(g^{-1}_\varepsilon)_{\alpha\beta} = g_{\alpha\mu} g_{\beta\nu} (g^{-1}_\varepsilon)^{\mu\nu}$ is not $g_{\varepsilon\,\alpha\beta}$. Taking $\varepsilon$-derivatives at $\varepsilon = 0$ the perturbation of the inverse metrics are found to be

$$K^{-1\,\alpha\beta}_1 := \left.\frac{dg^{-1\,\alpha\beta}_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = -K^{\alpha\beta}_1, \quad K^{-1\,\alpha\beta}_2 := \left.\frac{d^2 g^{-1\,\alpha\beta}_\varepsilon}{d\varepsilon^2}\right|_{\varepsilon=0} = -K_2{}^{\alpha\beta} + 2K_1{}^{\rho\alpha} K_1{}^{\beta}{}_{\rho}.$$

Note that $K_1$, $K_2$, $K^{-1}_1$ and $K^{-1}_2$ are all tensors in the background spacetime $(\mathcal{V}, g)$ and thus their indices are raised and lowered with the metric $g$, or its inverse. In a more general situation, given a family of covariant tensors $\hat{T}_\varepsilon$ in $(\mathcal{V}_\varepsilon, \hat{g}_\varepsilon)$, a corresponding family $T_\varepsilon$ can be obtained in the background spacetime $(\mathcal{V}, g)$ via the pull back $\psi^*_\varepsilon$. Similarly, a family of contravariant tensors can be pushed forward by $d\psi^{-1}_\varepsilon$ in order to define a family of contravariant tensors in $(\mathcal{V}, g)$. Once the $T_\varepsilon$ are constructed in $(\mathcal{V}, g)$, the corresponding first and second order perturbations are defined by

$$T^{(1)} := \left.\frac{dT_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0}, \qquad T^{(2)} := \left.\frac{d^2 T_\varepsilon}{d\varepsilon^2}\right|_{\varepsilon=0}. \tag{3.4}$$

[2]These are $C^3$, nondegenerate, symmetric, two-covariant tensor fields in $\mathcal{V}$. The construction is $C^2$ with respect to $\varepsilon$.

Let us consider the family of tensors $g_\varepsilon$ and its corresponding family of associated connections, $\nabla^\varepsilon$. The relation of the covariant derivatives of $g_\varepsilon$ and $g(= g_{\varepsilon=0})$, denoting $\nabla := \nabla^{\varepsilon=0}$, is given by the standard formula

$$\nabla^\varepsilon{}_\mu T^\alpha_\beta = \nabla_\mu T^\alpha_\beta - C_\varepsilon{}^\nu_{\beta\mu} T^\alpha_\nu + C_\varepsilon{}^\alpha_{\nu\mu} T^\nu_\beta, \tag{3.5}$$

where $C_\varepsilon{}^\alpha_{\beta\gamma} = (1/2) g_\varepsilon{}^{\alpha\mu}(\nabla_\beta g_{\varepsilon\mu\gamma} + \nabla_\gamma g_{\varepsilon\mu\beta} - \nabla_\mu g_{\varepsilon\beta\gamma})$. It is convenient to define here the first and second order perturbations of $C_\varepsilon{}^\alpha_{\beta\gamma}$. These read

$$\begin{aligned} C^{(1)}{}^\alpha_{\beta\gamma} &:= \left.\frac{dC_\varepsilon{}^\alpha_{\beta\gamma}}{d\varepsilon}\right|_{\varepsilon=0} = \frac{1}{2}\left(\nabla_\beta K_1{}^\alpha_\gamma + \nabla_\gamma K_1{}^\alpha_\beta - \nabla^\alpha K_{1\beta\gamma}\right), \\ C^{(2)}{}^\alpha_{\beta\gamma} &:= \left.\frac{d^2C_\varepsilon{}^\alpha_{\beta\gamma}}{d\varepsilon^2}\right|_{\varepsilon=0} = S^\alpha_{\beta\gamma} - 2K_1{}^\alpha_\rho C^{(1)}{}^\rho_{\beta\gamma}, \end{aligned} \tag{3.6}$$

with

$$S^\alpha_{\beta\gamma} := \frac{1}{2}\left(\nabla_\beta K_2{}^\alpha_\gamma + \nabla_\gamma K_2{}^\alpha_\beta - \nabla^\alpha K_{2\beta\gamma}\right). \tag{3.7}$$

Back to relation (3.5), it allows us to express the first and second derivatives with respect to $\varepsilon$ at $\varepsilon = 0$ of the $\varepsilon$-covariant derivative of a one parameter family of tensors $T_\varepsilon{}^\alpha_\beta$ in terms of objects defined in the background spacetime. We find, explicitly

$$\left.\frac{d\nabla^\varepsilon{}_\mu T_\varepsilon{}^\alpha_\beta}{d\varepsilon}\right|_{\varepsilon=0} = \nabla_\mu T^{(1)}{}^\alpha_\beta - T^\alpha_\nu C^{(1)}{}^\nu_{\beta\mu} + T^\nu_\beta C^{(1)}{}^\alpha_{\nu\mu}, \tag{3.8}$$

$$\left.\frac{d^2\nabla^\varepsilon{}_\mu T_\varepsilon{}^\alpha_\beta}{d\varepsilon^2}\right|_{\varepsilon=0} = \nabla_\mu T^{(2)}{}^\alpha_\beta - T^\alpha_\nu C^{(2)}{}^\nu_{\beta\mu} + T^\nu_\beta C^{(2)}{}^\alpha_{\nu\mu} - 2T^{(1)}{}^\alpha_\nu C^{(1)}{}^\nu_{\beta\mu} + 2T^{(1)}{}^\nu_\beta C^{(1)}{}^\alpha_{\nu\mu}. \tag{3.9}$$

We can now obtain the field equations that the perturbation tensors $K_1$ and $K_2$ satisfy in terms of background objects. To this aim, let us apply relation (3.5) repeatedly to an arbitrary one form $\boldsymbol{\omega}$ to find

$$\begin{aligned} \nabla^\varepsilon{}_\nu \nabla^\varepsilon{}_\mu \omega_\beta &= \nabla_\nu\nabla_\mu\omega_\beta - (\nabla_\nu C_\varepsilon{}^\rho_{\beta\mu})\omega_\rho + C_\varepsilon{}^\lambda_{\beta\nu} C_\varepsilon{}^\rho_{\lambda\mu}\omega_\rho - C_\varepsilon{}^\rho_{\mu\nu}(\nabla_\rho\omega_\beta - C_\varepsilon{}^\lambda_{\beta\rho}\omega_\lambda) \\ &- 2C_\varepsilon{}^\rho_{\beta(\mu}\nabla_{\nu)}\omega_\rho. \end{aligned} \tag{3.10}$$

The antisymmetrization of this last expression in $\{\mu\nu\}$ provides the following relation between the Riemann tensor of $g_\varepsilon$ and that of $g$

$$R_{\mu\nu\beta}{}^\rho(g_\varepsilon) = R_{\mu\nu\beta}{}^\rho(g) - 2\nabla_{[\mu} C_\varepsilon{}^\rho_{\nu]\beta} + 2C_\varepsilon{}^\lambda_{\beta[\mu} C_\varepsilon{}^\rho_{\nu]\lambda}. \tag{3.11}$$

Taking the first and second derivatives with respect to $\varepsilon$ at $\varepsilon = 0$ of the previous expression we find that the perturbations of the Riemann tensor read

$$R^{(1)}{}_{\mu\nu\beta}{}^\rho := \left.\frac{dR_{\mu\nu\beta}{}^\rho(g_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = -2\nabla_{[\mu} C^{(1)}{}^\rho_{\nu]\beta}, \tag{3.12}$$

$$R^{(2)}{}_{\mu\nu\beta}{}^\rho := \left.\frac{d^2R_{\mu\nu\beta}{}^\rho(g_\varepsilon)}{d\varepsilon^2}\right|_{\varepsilon=0} = -2\nabla_{[\mu} C^{(2)}{}^\rho_{\nu]\beta} + 4C^{(1)}{}^\lambda_{\beta[\mu} C^{(1)}{}^\rho_{\nu]\lambda}. \tag{3.13}$$

Now it is direct to find the perturbations of the Ricci tensor

$$
\begin{aligned}
{R^{(1)}}_{\mu\beta} &:= \left.\frac{dR_{\mu\beta}(g_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = -2\nabla_{[\mu}{C^{(1)}}^{\rho}{}_{\rho]\beta} \\
&= \frac{1}{2}\left(-\nabla_\mu\nabla_\beta {K_1}^{\rho}_{\rho} - \nabla_\rho\nabla^\rho K_{1\beta\mu} + 2\nabla_\rho\nabla_{(\beta}{K_1}^{\rho}_{\mu)}\right), \qquad (3.14)\\
{R^{(2)}}_{\mu\beta} &:= \left.\frac{d^2R_{\mu\beta}(g_\varepsilon)}{d\varepsilon^2}\right|_{\varepsilon=0} = -2\nabla_{[\mu}{C^{(2)}}^{\rho}{}_{\rho]\beta} + 4{C^{(1)}}^{\lambda}{}_{\beta[\mu}{C^{(1)}}^{\rho}{}_{\rho]\lambda} \\
&= \frac{1}{2}\left(-\nabla_\mu\nabla_\beta {K_2}^{\rho}_{\rho} - \nabla_\rho\nabla^\rho K_{2\beta\mu} + 2\nabla_\rho\nabla_{(\beta}{K_2}^{\rho}_{\mu)}\right) + 4{C^{(1)}}^{\lambda}{}_{\beta[\mu}{C^{(1)}}^{\rho}{}_{\rho]\lambda} \\
&\quad +4\nabla_{[\mu}({C^{(1)}}^{\eta}{}_{\rho]\beta}{K_1}^{\rho}_{\eta}), \qquad (3.15)
\end{aligned}
$$

and of the Ricci scalar

$$
\begin{aligned}
R^{(1)} &:= \left.\frac{dR(g_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \left.\frac{d{g_\varepsilon}^{\mu\beta}R_{\mu\beta}(g_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = -{K_1}^{\mu\beta}R_{\mu\beta} + g^{\mu\beta}R_{1\mu\beta} \\
&= -{K_1}^{\mu\beta}R_{\mu\beta} - \Box {K_1}^{\rho}_{\rho} + \nabla_\rho\nabla^\mu {K_1}^{\rho}_{\mu}, \qquad (3.16)\\
R^{(2)} &:= \left.\frac{d^2R(g_\varepsilon)}{d\varepsilon^2}\right|_{\varepsilon=0} = \left.\frac{d^2{g_\varepsilon}^{\mu\beta}R_{\mu\beta}(g_\varepsilon)}{d\varepsilon^2}\right|_{\varepsilon=0} \\
&= -{K_2}^{\mu\beta}R_{\mu\beta} - \Box {K_2}^{\rho}_{\rho} + \nabla_\rho\nabla^\mu {K_2}^{\rho}_{\mu} \\
&\quad +2{K_1}^{\rho\beta}{K_1}^{\mu}{}_{\rho}R_{\mu\beta} - {K_1}^{\mu\beta}\left(-\nabla_\mu\nabla_\beta {K_1}^{\rho}_{\rho} - \nabla_\rho\nabla^\rho K_{1\beta\mu} + 2\nabla_\rho\nabla_{(\beta}{K_1}^{\rho}_{\mu)}\right) \\
&\quad +4g^{\mu\beta}\left({C^{(1)}}^{\lambda}{}_{\beta[\mu}{C^{(1)}}^{\rho}{}_{\rho]\lambda} + \nabla_{[\mu}({C^{(1)}}^{\rho}{}_{\rho]\beta}{K_1}^{\rho}_{\eta})\right), \qquad (3.17)
\end{aligned}
$$

The perturbations of the Riemann tensor (3.12) and (3.13) display the same structure for higher order terms, i.e. the term $C^{(2)}$ enters (3.13) exactly as the $C^{(1)}$ does in (3.12), although (3.13) includes, additionally, inhomogeneous terms coming from the first order. This structure is propagated to the perturbations of the Ricci tensor (3.14), (3.15), where the contributions of $K_2$ (the first line of (3.15)), are identical to those of $K_1$ in (3.14). The same can be observed for the Ricci scalar in (3.16) and (3.17).

With the expressions above, the perturbations of the Einstein tensor read

$$
\begin{aligned}
{G^{(1)}}_{\beta\mu} &= {R^{(1)}}_{\beta\mu} - \frac{1}{2}\left(K_{1\beta\mu}R + g_{\beta\mu}R^{(1)}\right), \qquad (3.18)\\
{G^{(2)}}_{\beta\mu} &= {R^{(2)}}_{\beta\mu} - \frac{1}{2}\left(K_{2\beta\mu}R + g_{\beta\mu}R^{(2)} + 2K_{1\beta\mu}R^{(1)}\right). \qquad (3.19)
\end{aligned}
$$

For a matter configuration described by an energy momentum tensor $T_{\varepsilon\beta\mu}$, defining as usual its first and second order perturbations ${T^{(1)}}_{\beta\mu}$ and ${T^{(2)}}_{\beta\mu}$ respectively, the field equations read ${G^{(i)}}_{\beta\mu} = 8\pi {T^{(i)}}_{\beta\mu}$, for $i = 1, 2$. For vacuum, the perturbed field equations to order $i = 1, 2$ reduce to ${R^{(i)}}_{\beta\mu} = 0$.

The perturbative approach described so far relies on the identification of $\mathcal{V}_\varepsilon$ with $\mathcal{V}$ via some diffeomorphism (3.1). However, there exists the freedom of taking any other different identification. This freedom arises naturally if we take a diffeomorphism in each manifold $\mathcal{V}_\varepsilon$ before the identification of spacetimes, for instance the original identification (3.1), is carried through. Thus, it is clear that the identification is non unique. We can deal with this situation by taking into consideration a $\varepsilon$-dependent diffeomorphism in $\mathcal{V}$ before applying $\psi_\varepsilon$. Let us denote this diffeomorphism in the background spacetime $(\mathcal{V}, g)$ by $\Omega_\varepsilon$. It generates a new identification of spacetimes driven by $\psi_\varepsilon^{(g)} := \psi_\varepsilon \circ \Omega_\varepsilon$ and it induces a new family of tensors $g_\varepsilon^{(g)} = \psi_\varepsilon^{(g)^*}(\hat{g}_\varepsilon) = \Omega_\varepsilon^*(g_\varepsilon)$ on $\mathcal{V}$. Now, the *gauged* metric perturbations are constructed analogously to (3.2) and denoted by a $^{(g)}$ superscript, so that

$$K_1{}^{(g)} := \left.\frac{dg_\varepsilon^{(g)}}{d\varepsilon}\right|_{\varepsilon=0}, \quad K_2{}^{(g)} := \left.\frac{d^2 g_\varepsilon^{(g)}}{d\varepsilon^2}\right|_{\varepsilon=0}. \tag{3.20}$$

The relation between the perturbation tensors $K_1{}^{(g)}$, $K_2{}^{(g)}$ and $K_1$, $K_2$ is addressed in the following proposition, in terms of the first and second order (spacetime) *gauge vectors* $\vec{S}_1$ and $\vec{S}_2$ defined as follows

$$\vec{S}_1 := \left.\frac{\partial\Omega_\varepsilon}{\partial\varepsilon}\right|_{\varepsilon=0}, \tag{3.21}$$

$$\vec{S}_2 := \vec{V}_2 + \nabla_{\vec{S}_1}\vec{S}_1, \quad \vec{V}_\varepsilon := \left.\frac{\partial(\Omega_{\varepsilon+h}\circ\Omega_\varepsilon^{-1})}{\partial h}\right|_{h=0}, \quad \vec{V}_2 := \left.\frac{\partial\vec{V}_\varepsilon}{\partial\varepsilon}\right|_{\varepsilon=0}. \tag{3.22}$$

**Proposition 1** ***(Bruni et al, 1997 [19]; Mars, 2005 [79])*** *Under a gauge transformation defined by the vectors $\vec{S}_1$ and $\vec{S}_2$, the first and second order perturbation tensors transform as*

$$K_1{}^{(g)}{}_{\alpha\beta} = K_{1\alpha\beta} + \mathcal{L}_{\vec{S}_1} g_{\alpha\beta}, \tag{3.23}$$

$$K_2{}^{(g)}{}_{\alpha\beta} = K_{2\alpha\beta} + \mathcal{L}_{\vec{S}_2} g_{\alpha\beta} + 2\mathcal{L}_{\vec{S}_1} K_{1\alpha\beta} - 2S_1^\mu S_1^\nu R_{\alpha\mu\beta\nu} + 2\nabla_\alpha S_1^\mu \nabla_\beta S_{1\mu}, \tag{3.24}$$

Note that if $\vec{S}_1 = 0$, the gauge transformation of the second order perturbation tensor becomes linear, in the sense that it is analogous to a first order gauge transformation.

## 3.2 Perturbation of hypersurfaces

### Deformation vectors

Consider now a family of spacetimes with boundary $(\mathcal{V}_\varepsilon, \hat{g}_\varepsilon, \hat{\Sigma}_\varepsilon)$. The setting for the description of the deformation of the boundary is constructed as follows. Assume that the $\mathcal{V}_\varepsilon$ are submanifolds of a larger manifold without boundary $\mathcal{W}_\varepsilon$, so that for each $\varepsilon$, $\hat{\Sigma}_\varepsilon$

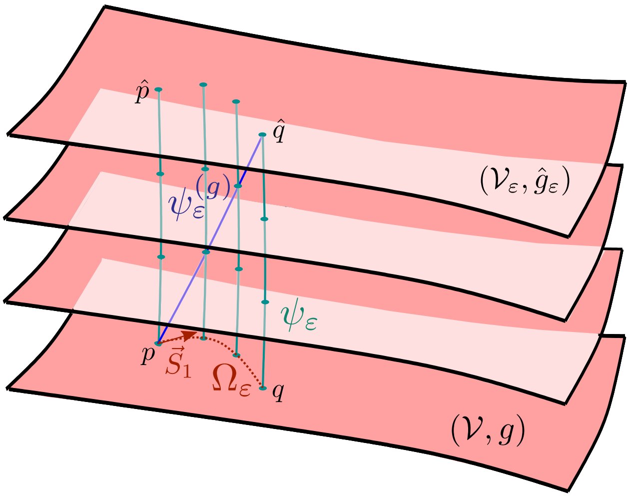


Figure 3.1: Graphic description of the spacetime gauge freedom. Let the spacetimes $(\hat{\mathcal{V}}_\varepsilon, \hat{g}_\varepsilon)$ be identified by the diffeomorphism $\psi_\varepsilon$. It is clear from the picture that $\hat{q} = \psi_\varepsilon(q)$ and $\hat{p} = \psi_\varepsilon(p)$. Consider now a diffeomorphism $\Omega_\varepsilon$ in the background, so that the point $p$ in the background spacetime is mapped to $q = \Omega_\varepsilon(p)$. This defines the new identification $\psi_\varepsilon^{(g)}$, by $\hat{q} = \psi_\varepsilon(q) = \psi_\varepsilon \circ \Omega_\varepsilon(p) \equiv \psi_\varepsilon^{(g)}(p)$. The gauge transformation is defined to first order by the vector $\vec{S}_1$.

is an embedded hypersurface in $\mathcal{W}_\varepsilon$. The whole construction will be independent on the choice of extension used to construct $\mathcal{W}_\varepsilon$. Assume now that $(\mathcal{W}_\varepsilon, \hat{g}_\varepsilon)$ are diffeomorphically related by some $\psi_\varepsilon$. We also assume that $\hat{\Sigma}_\varepsilon$ are timelike everywhere (in [79] the whole formalism is developed demanding that these are simply non-null everywhere). Each $\hat{\Sigma}_\varepsilon$ is projected to the background spacetime ($\mathcal{W} = \hat{\mathcal{W}}_{\varepsilon=0}, g$) via the map $\psi_\varepsilon$, generating there a one parameter family of hypersurfaces $\Sigma_\varepsilon$. The deformation of $\Sigma_0 \equiv \hat{\Sigma}_0$ as $\varepsilon$ varies, as a set of points, is encoded in this family of hypersurfaces embedded in $\mathcal{W}$. It is important to note here that *the deformation is referred to an specific choice of gauge*, since the whole construction depends on $\psi_\varepsilon$.

At this point an additional freedom of the process arises. In order to know how points in $\Sigma_0$ are mapped to $\Sigma_\varepsilon$, we need to identify the $\hat{\Sigma}_\varepsilon$ among themselves. This prescription is known as the *hypersurface gauge freedom* and it is driven by a map $\phi_\varepsilon : \Sigma \to \hat{\Sigma}_\varepsilon$, where $\Sigma$ is an abstract copy of one element of the family, say $\Sigma_0$. Now, using the spacetime identification $\psi_\varepsilon$ we construct the family of embeddings $\Phi_\varepsilon := \psi_\varepsilon^{-1} \circ \phi_\varepsilon$, which map the abstract hypersurface $\Sigma$ to the embedded hypersurfaces $\Sigma_\varepsilon$. Introducing local coordinates $\{\xi^a\}$ in $\Sigma$, where the index $a$ ranges from 1 to $n$, and $\{x^\alpha\}$ in $\mathcal{W}$, where $\alpha$ goes from 0 to

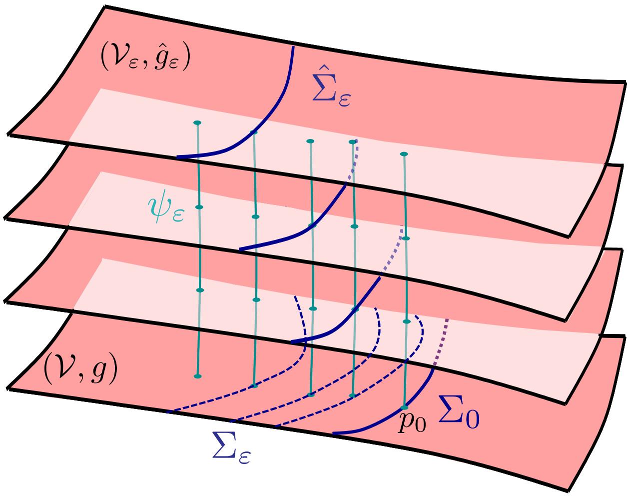


Figure 3.2: The hypersurfaces $\hat{\Sigma}_\varepsilon$ are projected down onto the background $(\mathcal{V}, g)$ by the map $\psi_\varepsilon$ to generate the family $\Sigma_\varepsilon$. This family describes how the background $\Sigma_0$ changes as $\varepsilon$ varies as a set of points on $\mathcal{V}$. But this is not enough to take $\varepsilon$-derivatives. We still need to prescribe how a given point $p_0 \in \Sigma_0$ is mapped onto $\Sigma_\varepsilon$.

$n$, we can write the embedding $\Phi_\varepsilon$ in local form

$$\begin{aligned} \Phi_\varepsilon \quad : \quad & \Sigma \to \mathcal{W} \\ & \xi^a \to x^\alpha = \Phi^\alpha(\xi^a, \varepsilon). \end{aligned} \tag{3.25}$$

We single out the embedding $\Phi_{\varepsilon=0}$ as the unperturbed embedding, so that it embeds $\Sigma$ into $(\mathcal{W}, g)$ as $\Sigma_0 = \Phi_0(\Sigma)$. The hypersurface $\Sigma$ is equipped with a nondegenerate metric $h := \Phi_0^*(g)$ and its associated covariant derivative $\overline{\nabla}$. We define the tangent basis of $\Sigma_0$ in $\mathcal{W}$ by $\vec{e}_a := d\Phi_0(\partial/\partial\xi^a)$ and denote by $\vec{n}$ a normal spacelike unit vector to $\Sigma_0$. The dual basis $\{\boldsymbol{n}, \boldsymbol{\omega}\}$ is defined as in Chapter 2. The manifold $\Sigma$ inherits a second fundamental form given by $\kappa := \Phi_0^*(\nabla \boldsymbol{n})$. Recall that the projection tensor to the hypersurface $\Sigma_0$ in terms of the tangent and dual bases is $h^\alpha_\beta := e^\alpha_a \omega^a_\beta$.

For a fixed point of the hypersurface $\Sigma$, the embedding $\Phi_\varepsilon$ generates a curve on $\mathcal{W}$ as $\varepsilon$ varies, that starts at $p_0 \equiv \Phi_0(p) \in \Sigma_0$. The tangent vector to this curve at points of $\Sigma_0$, and its acceleration, define respectively the first and second order perturbation vectors $\vec{Z}_1$ and $\vec{Z}_2$ as follows

**Proposition 2** ***(Mars, 2005 [79])*** *The first and second order perturbation vectors*

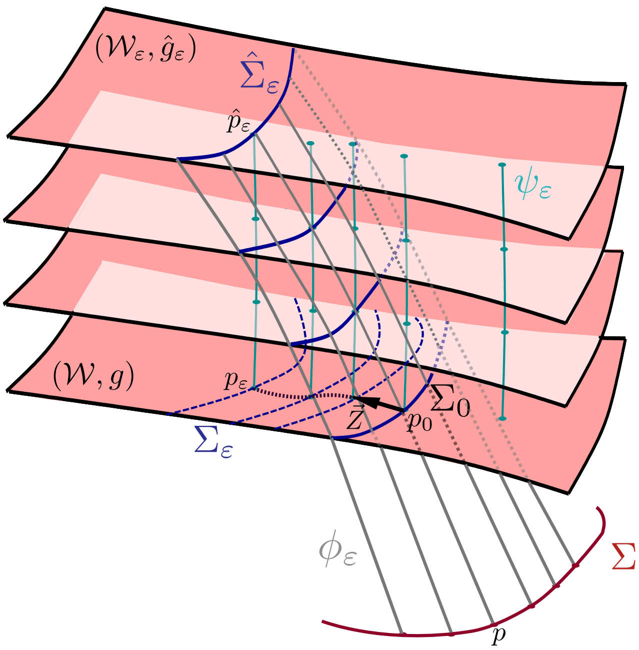


Figure 3.3: Take a point $p$ in $\Sigma$. This corresponds to $p_0$ in $\Sigma_0$. Via the diffeomorphism $\phi_\varepsilon$ (in gray) that identifies the hypersurfaces $\hat{\Sigma}_\varepsilon$ (in thick blue lines) among themselves, $p_0$ is mapped to $\hat{p}_\varepsilon$, in the spacetime $(\mathcal{W}_\varepsilon, \hat{g}_\varepsilon)$. This point $\hat{p}_\varepsilon$ is now mapped via $\psi_\varepsilon^{-1}$ (in turquoise) to the background spacetime so that $p_\varepsilon = \psi_\varepsilon^{(-1)}(\hat{p}_\varepsilon)$. The embedding $\Phi_\varepsilon$, defined as the composition $\Phi_\varepsilon := \psi_\varepsilon^{-1} \circ \phi_\varepsilon$ identifies pointwise the hypersurfaces $\Sigma_\varepsilon$ (in blue, dotted) and for a fixed $p_0 \in \Sigma_0$ it produces a curve as $\varepsilon$ varies. The tangent vector and the acceleration of this curve at $p_0$ are precisely $\vec{Z}_1$ and $\vec{Z}_2$.

*$\vec{Z}_1(\xi)$ and $\vec{Z}_2(\xi)$ of the hypersurface $\Sigma$ read*

$$Z_1^\alpha(\xi^a) = \left.\frac{\partial \Phi^\alpha(\xi^a, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}, \tag{3.26}$$

$$Z_2^\alpha(\xi^a) = \left.\frac{\partial^2 \Phi^\alpha(\xi^a, \varepsilon)}{\partial \varepsilon^2}\right|_{\varepsilon=0} + \Gamma^\alpha_{\beta\gamma}(x^{(0)}(\xi^a)) Z_1^\beta(\xi^a) Z_1^\gamma(\xi^a), \tag{3.27}$$

*where $x^{(0)}(\xi^a)$ is the local form of the (unperturbed) embedding $\Phi_0$.*

Let us decompose $\vec{Z}_1$ and $\vec{Z}_2$ into normal and tangent parts to $\Sigma_0$, i.e.

$$\vec{Z}_1 = Q_1 \vec{n} + \vec{T}_1, \quad \vec{Z}_2 = Q_2 \vec{n} + \vec{T}_2. \tag{3.28}$$

In what follows we may use $\vec{Z}$, $Q$, $\vec{T}$ to refer to both $\vec{Z}_1$, $Q_1$, $\vec{T}_1$ or $\vec{Z}_2$, $Q_2$, $\vec{T}_2$. Also, we will refer to the vectors $T^\alpha$ and function $Q$ (3.28) defined at points on $\Sigma \subset \mathcal{V}$ and to the corresponding vector $T^a$ and function $Q$ defined in $\Sigma$ by $\vec{T}$ and $Q$ indistinctively.

The deformation vectors depend on both spacetime and hypersurface gauges. Let us describe first their dependence in the latter. We have prescribed how the hypersurfaces

$\hat{\Sigma}_\varepsilon$ are identified among themselves via some diffeomorphism $\phi_\varepsilon$. The freedom inherent here relies, again, on the possibility of taking diffeomorphisms within each $\hat{\Sigma}_\varepsilon$ before performing the identification among themselves, generating thus a new identification. This is the essence of the hypersurface gauge freedom and it is best described at the level of the embedding $\Phi_\varepsilon$. Let us then consider a diffeomorphism $\chi_\varepsilon$ on $\Sigma$ previous to the application of the map $\Phi_\varepsilon$. The coordinated form is a $\varepsilon$-change of the type $\hat{\xi}_a = \hat{\xi}_a(\xi_b, \varepsilon)$,

$$\begin{aligned} \chi_\varepsilon \ &: \ \Sigma \to \Sigma, \\ &\xi^a \to \hat{\xi}^a = \xi^a(\xi^b, \varepsilon). \end{aligned} \tag{3.29}$$

As a result, a new family of embeddings $\Phi^{(h)}_\varepsilon = \Phi_\varepsilon \circ \chi_\varepsilon$ can be constructed. The gauge vectors are defined analogously as the *spacetime gauge vectors* (see (3.22)), but the role played by the diffeomorphism $\Omega_\varepsilon$ in (3.22) corresponds now to $\chi_\varepsilon$. Thus we introduce the hypersurface gauge vectors in terms of $\chi_\varepsilon$ as

$$\vec{u}_1 := \left.\frac{\partial \chi_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad \vec{u}_2 := \frac{\partial}{\partial \varepsilon}\left(\left.\frac{\partial(\chi_{\varepsilon+h} \circ \chi_\varepsilon^{-1})}{\partial h}\right|_{h=0}\right)\Bigg|_{\varepsilon=0} + \overline{\nabla}_{\vec{u}_1}\vec{u}_1. \tag{3.30}$$

Note that $\vec{u}_1$ and $\vec{u}_2$ are vectors defined on $\Sigma$, but can be promoted to spacetime vectors (that we shall still denote by $\vec{u}_1$ and $\vec{u}_2$) tangent to $\Sigma_0$.

**Proposition 3** ***(Mars, 2005 [79])*** *Under a gauge transformation on $\Sigma$ defined by gauge vectors $\vec{u}_1$ and $\vec{u}_2$, $\vec{Z}_1$ and $\vec{Z}_2$ at any point $p \in \Sigma_0$ transform as*

$$\vec{Z}^{(h)}_1 = \vec{Z}_1 + \vec{u}_1, \tag{3.31}$$

$$\vec{Z}^{(h)}_2 = \vec{Z}_2 + \vec{u}_2 + 2\nabla_{\vec{u}_1}\vec{Z}_1 - (\kappa_{ab} u^a_1 u^b_1)\vec{n}, \tag{3.32}$$

*where $\vec{n}$ is a unit normal, $(\vec{n}, \vec{n}) = 1$, and $\kappa_{ab}$ is the second fundamental form of $\Sigma_0$.*

It is clear from these rules of transformations that the hypersurface gauge can be used to set to zero the tangent parts $\vec{T}_{1/2}$ of $\vec{Z}_{1/2}$. Let us discuss now the normal component of the perturbation vectors. The transformation rule to first order (3.31) does not involve $Q_1$ at all. A hypersurface gauge change does not modify the hypersurfaces $\Sigma_\varepsilon$ as sets of points, it just varies how these are identified pointwise, and thus $Q_1$ is not sensitive to such changes. However, the effect of a hypersurface gauge transformation is more involved to second order due to the fact that $\vec{Z}_2$ measures accelerations, including those coming from the first order perturbations, leading to the not obvious effect that $Q_2$ is not gauge invariant with respect to $\vec{u}_1$. Still, (3.32) shows that $Q_2$ is clearly invariant under a change driven by $\vec{u}_2$. If we include transformations driven by some $\vec{u}_1$ and $\vec{u}_2$, a straightforward calculation using (3.31) and (3.32) leads to

$$Q^{(h)}_2 = Q_2 + 2\vec{u}_1(Q_1) + \kappa_{\alpha\beta} u^\alpha_1 (u^\beta_1 - 2T^{(h)\beta}_1).$$

This result suggests the construction of the (hypersurface) gauge invariant quantity

$$\hat{Q}_2 := Q_2 + \kappa(\vec{T}_1, \vec{T}_1) - 2\vec{T}_1(Q_1). \tag{3.33}$$

We consider now the action of the spacetime gauge on the perturbation vectors $\vec{Z}$. Recall the definition of the embedding $\Phi_\varepsilon = \psi_\varepsilon^{-1} \circ \phi_\varepsilon$ and consider a change of spacetime gauge driven by a diffeomorphism $\Omega_\varepsilon$ in the background spacetime. It follows that the embedding transforms as $\Phi_\varepsilon^{(g)} = {\psi^{(g)}}_\varepsilon^{-1} \circ \phi_\varepsilon = \Omega_\varepsilon^{-1} \circ \psi_\varepsilon^{-1} \circ \phi_\varepsilon = \Omega_\varepsilon^{-1} \circ \Phi_\varepsilon$. This is the starting point to show the following.

**Proposition 4** ***(Mars, 2005 [79])*** *Under a spacetime gauge transformation defined by $\vec{S}_1$ and $\vec{S}_2$, the first and second order perturbation vectors of $\Sigma$ transform as*

$$\vec{Z}_1^{(g)} = \vec{Z}_1 - \vec{S}_1, \tag{3.34}$$

$$\vec{Z}_2^{(g)} = \vec{Z}_2 - \vec{S}_2 - 2\nabla_{\vec{Z}_1}\vec{S}_1 + 2\nabla_{\vec{S}_1}\vec{S}_1. \tag{3.35}$$

There is a particular class of gauges known as *surface comoving gauges* where each hypersurface $\Sigma_\varepsilon$ agrees with $\Sigma_0$ as a set of points in $\mathcal{W}$, so that the matching hypersurfaces are seen *unperturbed* on those gauges. To first order this amounts to requiring $Q_1 = 0$, whereas for the second order it cannot be expressed simply as $Q_2 = 0$ since $Q_2$ carries not only information about the deformation of the hypersurface to second order but also from the acceleration of the first order deformation vector[3] . Thus, provided $Q_1 = 0$ one could ask for the combination $Q_2 - \boldsymbol{n}(\nabla_{\vec{T}_1}\vec{T}_1)$ to vanish. It is easily checked that this condition is equivalent to $\hat{Q}_2 = 0$.

A subclass of this type of gauges are the *surface gauges*, which are defined by the vanishing of the full vector $\vec{Z}$.

## 3.3 Perturbations of the first and second fundamental forms

We have endowed the hypersurface $\Sigma$, and thus the embedded $\Sigma_0$, with a metric inherited from the ambient spacetime $(\mathcal{W}, g)$. This same construction holds for $\Sigma_\varepsilon$ for small values of $\varepsilon$ and thus a family of metrics can be defined on $\Sigma$ by $h_\varepsilon = \Phi_\varepsilon^*(g_\varepsilon)$. Consider now the

[3] In [79] the second order perturbation vector is chosen to be $\vec{Z}_2$ by convenience, instead of $\vec{W} = \partial_\varepsilon(\partial_h \Psi_h^\varepsilon)_{\varepsilon=h=0}$, where $\Psi_h^\varepsilon$ is a diffeomorphism in $(\mathcal{V}, g)$ satisfying $\Phi_{\varepsilon+h} = \Psi_h^\varepsilon \circ \Phi_\varepsilon$. The difference between them is the acceleration of $\vec{Z}_1$. The point is that $\vec{W}$, contrary to $\vec{Z}_2$, cannot be expressed solely in terms of the embeddings $\Phi_\varepsilon$ and it depends on the diffeomorphism $\Psi_\varepsilon^h$, which is non-unique. Thus, $\vec{Z}_2$ results more convenient. For clarifying discussions about this issue see the remark notes in Sections 4 and 7 in [79].

family of unit normal forms $\boldsymbol{n}_\varepsilon$ to $\Sigma_\varepsilon$ with respect to $g_\varepsilon$. These define a family of second fundamental forms in $\Sigma$ by $\kappa_\varepsilon = \Phi_\varepsilon^*(\nabla^\varepsilon \boldsymbol{n}_\varepsilon)$. The first and second order perturbations of the first and second fundamental forms are

$$h^{(1)} = \left.\frac{dh_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0}, \qquad h^{(2)} = \left.\frac{d^2 h_\varepsilon}{d\varepsilon^2}\right|_{\varepsilon=0}, \tag{3.36}$$

$$\kappa^{(1)} = \left.\frac{dk_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0}, \qquad \kappa^{(2)} = \left.\frac{d^2 k_\varepsilon}{d\varepsilon^2}\right|_{\varepsilon=0}. \tag{3.37}$$

These objects admit explicit expressions in terms of background objects. In order to present them, it is convenient to decompose the first and second order perturbation tensor in normal and tangent parts to $\Sigma_0$

$$K_{1\alpha\beta} = K_1{}^{\perp} n_\alpha n_\beta + K_1{}^{|}_{\alpha} n_\beta + K_1{}^{|}_{\beta} n_\alpha + K_1{}^{||}_{\alpha\beta}, \quad K_{2\alpha\beta} = K_2{}^{\perp} n_\alpha n_\beta + K_2{}^{|}_{\alpha} n_\beta + K_2{}^{|}_{\beta} n_\alpha + K_2{}^{||}_{\alpha\beta},$$

where $K_1{}^{\perp} = n^\alpha n^\beta K_{1\alpha\beta}$, $K_1{}^{|}_{\alpha} = n^\beta h^\mu_\alpha K_{1\mu\beta}$, and $K_1{}^{||}_{\alpha\beta} = h^\mu_\alpha h^\nu_\alpha K_{1\mu\nu}$ by definition, and analogous for the second order perturbation tensor.

**Proposition 5** ***(Battye, Carter, 2001 [12]; Mars, 2005 [79])*** *Let $(\mathcal{W}, g)$ be a $C^2$ spacetime of any dimension and $\Sigma_0$ an arbitrary non-degenerate hypersurface defined by an embedding $\Phi_0 : \Sigma \to \mathcal{W}$. Let $h$ be the induced metric, $\kappa$ the extrinsic curvature and $\vec{n}$ the unit normal vector to the hypersurface $\Sigma_0$. If the metric $g$ is perturbed to first order with $K_1$ and the hypersurface is perturbed to first order with a vector field $\vec{Z}_1 = Q_1\vec{n} + \vec{T}_1$, where $\vec{T}_1$ is tangent to $\Sigma_0$, then the induced metric and extrinsic curvature are perturbed to first order as*

$$h^{(1)}{}_{ab} = \mathcal{L}_{\vec{T}_1} h_{ab} + 2Q_1\kappa_{ab} + K_{1\alpha\beta} e^\alpha_a e^\beta_b, \tag{3.38}$$

$$\kappa^{(1)}{}_{ab} = \mathcal{L}_{\vec{T}_1} \kappa_{ab} - \overline{\nabla}_a \overline{\nabla}_b Q_1 + Q_1(-n^\mu n^\nu R_{\alpha\mu\beta\nu} e^\alpha_a e^\beta_b + \kappa_{ac}\kappa^c_b) + \frac{1}{2} K_1{}^{\perp} \kappa_{ab} - n_\mu C^{(1)\mu}{}_{\alpha\beta} e^\alpha_a e^\beta_b, \tag{3.39}$$

*where $C^{(1)\mu}{}_{\alpha\beta}$ is given in (3.6).*

**Proposition 6** ***(Mars, 2005 [79])*** *With the same assumptions and notation as in Proposition 5, if the metric is perturbed to second order with $K_2$ and the hypersurface is perturbed to second order with $\vec{Z}_2 = Q_2\vec{n} + \vec{T}_2$ (with $\vec{T}_2$ orthogonal to $\vec{n}$) then the induced* *metric and extrinsic curvature are perturbed to second order as*

$$
\begin{aligned}
{h^{(2)}}_{ab} &= \mathcal{L}_{\vec{Z}_2} h_{ab} + 2Q_2 \kappa_{ab} + K_{2\alpha\beta} e^\alpha_a e^\beta_b + 2\mathcal{L}_{\vec{T}_1} {h^{(1)}}_{ab} - \mathcal{L}_{\vec{T}_1}\mathcal{L}_{\vec{T}_1} h_{ab} + \\
&+ \mathcal{L}_{2Q_1 \vec{K}_1^{|} - 2Q_1\kappa(\vec{T}_1) - \overline{\nabla}_{\vec{T}_1}\vec{T}_1} h_{ab} + 2\left( T^l_1 T^m_1 \kappa_{lm} - 2\vec{T}_1(Q_1) + 2Q_1 {K_1}^{\perp} \right)\kappa_{ab} + \\
&+ 2Q_1^2 \left( -n^\mu n^\nu R_{\alpha\mu\beta\nu} e^\alpha_a e^\beta_b + \kappa_{al}\kappa^l_b \right) + 2\overline{\nabla} Q_1 \overline{\nabla} Q_1 - 4Q_1 n_\mu {C^{(1)}}^\mu{}_{\alpha\beta} e^\alpha_a e^\beta_b, \qquad (3.40) \\
{\kappa^{(2)}}_{ab} &= \mathcal{L}_{\vec{T}_2}\kappa_{ab} - \overline{\nabla}_a \overline{\nabla}_b Q_2 - Q_2 n^\mu n^\nu R_{\alpha\mu\beta\nu} e^\alpha_a e^\beta_b + Q_2 \kappa_{al}\kappa^l_b - n_\mu S^\mu_{\alpha\beta} e^\alpha_a e^\beta_b + \\
&+ 2\mathcal{L}_{\vec{T}_1} {\kappa^{(1)}}_{ab} + \kappa_{ab}\left( \frac{1}{2} {K_2}^{\perp} - \frac{1}{4}({K_1}^{\perp})^2 - ({K_1}^{|}_l + \overline{\nabla}_l Q_1)({K_1}^{|l} + \overline{\nabla}^l Q_1) \right. \\
&+ \left. 2Q_1 n_\mu n^\rho n^\delta {C^{(1)}}^\mu{}_{\rho\delta} \right) + \left( {K_1}^{\perp} n_\mu + 2{K_1}^{|}_\mu + 2\overline{\nabla}_\mu Q_1 \right) {C^{(1)}}^\mu{}_{\alpha\beta} e^\alpha_a e^\beta_b \\
&- 2Q_1 n_\mu n^\nu \left( \nabla_\nu {C^{(1)}}^\mu{}_{\alpha\beta} \right) e^\alpha_a e^\beta_b - 2n_\mu n^\nu {C^{(1)}}^\mu{}_{\alpha\nu} e^\alpha_a \overline{\nabla}_b Q_1 - 2n_\mu n^\nu {C^{(1)}}^\mu{}_{\alpha\nu} e^\alpha_b \overline{\nabla}_a Q_1 \\
&- 2Q_1 n_\mu {C^{(1)}}^\mu{}_{\alpha\beta} e^\alpha_a e^\beta_l \kappa^l_b - 2Q_1 n_\mu {C^{(1)}}^\mu{}_{\alpha\beta} e^\alpha_b e^\beta_l \kappa^l_a \\
&+ \mathcal{L}_{grad\left(\vec{T}_1(Q_1)\right) - \frac{1}{2} grad\left(T^l_1 T^m_1 \kappa_{lm}\right) - \frac{1}{2}{K_1}^{\perp} grad(Q_1) + 2Q_1\kappa(grad Q_1)} h_{ab} \\
&+ \left( 2\vec{T}_1(Q_1) - T^l_1 T^m_1 \kappa_{lm} - Q_1 {K_1}^{\perp} \right) \left( n^\mu n^\nu R_{\alpha\mu\beta\nu} e^\alpha_a e^\beta_b - \kappa_{al}\kappa^l_b \right) - 2Q_1 \mathcal{L}_{grad(Q_1)}\kappa_{ab} \\
&+ \frac{1}{2}\left( \overline{\nabla}_a Q_1 \overline{\nabla}_b {K_1}^{\perp} + \overline{\nabla}_b Q_1 \overline{\nabla}_a {K_1}^{\perp} \right) - \mathcal{L}_{\vec{T}_1}\mathcal{L}_{\vec{T}_1}\kappa_{ab} - \mathcal{L}_{2Q_1\kappa\left(\vec{T}_1\right) + \overline{\nabla}_{\vec{T}_1}\vec{T}_1}\kappa_{ab} \\
&- Q_1^2 \left( n^\mu n^\nu n^\delta (\nabla_\delta R_{\alpha\mu\beta\nu}) e^\alpha_a e^\beta_b + 2n^\mu n^\nu R_{\delta\mu\alpha\nu} e^\delta_l e^\alpha_b \kappa^l_a + 2n^\mu n^\nu R_{\delta\mu\alpha\nu} e^\delta_l e^\alpha_a \kappa^l_b \right), \qquad (3.41)
\end{aligned}
$$

*where $S$ and $C^{(1)}$ are given in (3.6) and (3.7) and, for any tangent vector $\vec{V}$, $(\kappa(\vec{V}))^a = {\kappa^a}_b V^b$ and $(grad\, f)^a = h^{ab}\partial_b f$.*

## 3.4 Perturbed matching conditions

The matching of two spacetimes with boundary in the exact sense (as depicted in Chapter 2), say $(\mathcal{V}^+, g^+, \Sigma^+)$ and $(\mathcal{V}^-, g^-, \Sigma^-)$, requires an identification of the boundaries, $\Sigma^+$ and $\Sigma^-$. The identification of the boundaries allows the construction of an abstract manifold $\Sigma$ on which the first and second fundamental forms as coming from both sides, $h^\pm$ and $\kappa^\pm$, are pulled back so that they can be compared. If the boundaries are nowhere null (non-degenerate) the matching conditions (in full, so that the global Riemann tensor shows no Dirac-delta term) demand the existence of one such identification for which the first and second fundamental forms $h^\pm$ and $\kappa^\pm$ agree. In particular, $\Sigma$ is endowed with the metric $h(= h^+ = h^-)$.

To study the matching of spacetimes in perturbation theory one can use again the same picture. We assume two families of spacetimes with boundary [4] $(\mathcal{V}^\pm_\varepsilon, \hat{g}^\pm_\varepsilon, \hat{\Sigma}^\pm_\varepsilon)$ are

[4] We refer to [79] for a proper discussion on the subtleties involved in the definition of families of

matched across their respective boundaries $\hat{\Sigma}^{\pm}_{\varepsilon}$ for each $\varepsilon$, so that there exists a corresponding family of diffeomorphically related hypersurfaces $\hat{\Sigma}_{\varepsilon}$ on which the first and second fundamental forms from each side are equated, $\hat{h}^{+}_{\varepsilon} = \hat{h}^{-}_{\varepsilon}$, $\hat{\kappa}^{+}_{\varepsilon} = \hat{\kappa}^{-}_{\varepsilon}$. The matching hypersurface of the background configuration is $(\Sigma, h)$, where $\Sigma \equiv \hat{\Sigma}_0$ and $h = \hat{h}^{+}_{0} = \hat{h}^{-}_{0}$. The idea is to construct, from those tensors on $\hat{\Sigma}_{\varepsilon}$, corresponding families $h^{\pm}_{\varepsilon}$ and $\kappa^{\pm}_{\varepsilon}$ on $(\Sigma, h)$ containing also the information about how $\Sigma^{\pm}_{0}$ are perturbed with respect to the spacetime gauges defined at each side $\psi^{\pm}_{\varepsilon}$, and the hypersurface gauge $\phi_{\varepsilon}$, which we want to keep free.

Following the procedure described in Section 3.3 the corresponding $\varepsilon$-families of first and second fundamental forms in $\Sigma$, $h^{\pm}_{\varepsilon} = \Phi^{\pm *}_{\varepsilon}(g^{\pm}_{\varepsilon})$ and $\kappa^{\pm}_{\varepsilon} = \Phi^{\pm *}_{\varepsilon}(\nabla^{\pm\varepsilon} \boldsymbol{n}^{\pm}_{\varepsilon})$ are constructed. Consider the setting adressed in Figure 3.4, where we have represented the embeddings $\hat{\Phi}^{+}_{\varepsilon}$ and $\Phi^{+}_{\varepsilon}$ and the diffeomorphisms $\psi^{+}_{\varepsilon}$, $\phi_{\varepsilon}$ and $\phi^{+}_{\varepsilon} := \Phi^{+}_{\varepsilon} \circ \phi_{\varepsilon}$ . Given that $\hat{h}^{+}_{\varepsilon} = \hat{\Phi}^{+*}_{\varepsilon}(\hat{g}^{+}_{\varepsilon})$ and $h^{+}_{\varepsilon} = \Phi^{+*}_{\varepsilon}(g^{+}_{\varepsilon})$, the tensors $\hat{h}^{+}_{\varepsilon}$ and $h^{+}_{\varepsilon}$ are related by the diffeomorphism $\phi_{\varepsilon}$. It can be shown as follows

$$
\begin{aligned}
\phi^{*}_{\varepsilon}(\hat{h}^{+}_{\varepsilon}) &= \phi^{*}_{\varepsilon} \circ \hat{\Phi}^{+*}_{\varepsilon}\left(\hat{g}^{+}_{\varepsilon}\right) = (\hat{\Phi}^{+}_{\varepsilon} \circ \phi_{\varepsilon})^{*}\left(\hat{g}^{+}_{\varepsilon}\right) \\
&= \left(\phi^{+}_{\varepsilon} \circ (\phi_{\varepsilon})^{-1} \circ \phi_{\varepsilon}\right)^{*}\left(\hat{g}^{+}_{\varepsilon}\right) = \phi^{+*}_{\varepsilon}\left(\hat{g}^{+}_{\varepsilon}\right) = \phi^{+*}_{\varepsilon}\left((\hat{\psi}^{+}_{\varepsilon})^{-1*}(g^{+}_{\varepsilon})\right) \\
&= \left(\hat{\psi}^{+-1}_{\varepsilon} \circ \phi^{+}_{\varepsilon}\right)^{*}(g^{+}_{\varepsilon}) = \Phi^{+*}_{\varepsilon}(g^{+}_{\varepsilon}) = h^{+}_{\varepsilon}. \qquad (3.42)
\end{aligned}
$$

This also applies for the construction with the $-$ spacetimes, so that $\phi^{*}_{\varepsilon}(\hat{h}^{-}_{\varepsilon}) = h^{-}_{\varepsilon}$. Since $\hat{h}^{+}_{\varepsilon} = \hat{h}^{-}_{\varepsilon}$ we have that $h^{+}_{\varepsilon} = h^{-}_{\varepsilon}$ by construction, and analogously for the $\varepsilon-$family of second fundamental forms.

Therefore, the matching conditions for each $\varepsilon$ consist of imposing

$$
h^{+}_{\varepsilon} = h^{-}_{\varepsilon}, \qquad \kappa^{+}_{\varepsilon} = \kappa^{-}_{\varepsilon}. \qquad (3.43)
$$

The first and second $\varepsilon$ derivatives of (3.43), evaluated at $\varepsilon = 0$, provide the perturbed matching conditions. The explicit expressions are found in Section 3.3, in Propositions 5 and 6. We only have to particularise them to the $+$ and $-$ configurations respectively. Take for instance the $+$ side, then we have to substitute the (required) background quantities, $K_1$, $Q_1$, $\vec{T}_1$, and $K_2$, $Q_2$, $\vec{T}_2$ by the corresponding background quantities for the $+$ side, ${K_1}^{+}$, $Q^{+}_{1}$, $\vec{T}^{+}_{1}$, and ${K_2}^{+}$, $Q^{+}_{2}$, $\vec{T}^{+}_{2}$ in the expressions given in Propositions 5 and 6. For the $-$ side, we proceed analogously.

The perturbed matching conditions are formulated in terms of the background configuration quantities and ${K_1}^{\pm}$, $Q^{\pm}_{1}$, $\vec{T}^{\pm}_{1}$, plus ${K_2}^{\pm}$, $Q^{\pm}_{2}$, $\vec{T}^{\pm}_{2}$ in the following Theorem:

spacetimes with boundary. Also, we need only to consider non-degenerate hypersurfaces $\hat{\Sigma}_{\varepsilon}$, without loss of generality. Their orientation will extend through $\varepsilon$ by continuity.

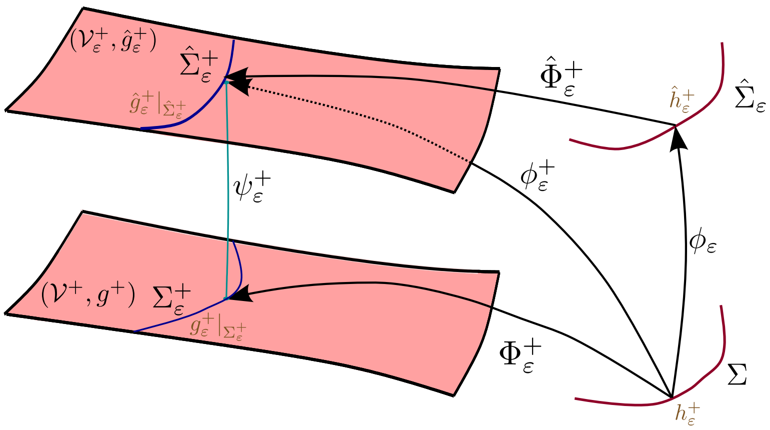


Figure 3.4: The background spacetime $(\mathcal{V}^+, g^+)$ and any other representative of the uniparametric family $(\mathcal{V}_\varepsilon^+, \hat{g}_\varepsilon^+)$. The hypersurface $\hat{\Sigma}_\varepsilon^+$ in $(\mathcal{V}_\varepsilon^+, \hat{g}_\varepsilon^+)$ is projected to $(\mathcal{V}^+, g^+)$ via $\psi_\varepsilon^+$ to obtain the hypersurface $\Sigma_\varepsilon^+$. On the right hand side, we have $\hat{\Sigma}_\varepsilon$ and $\Sigma$, the abstract copies of $\hat{\Sigma}_\varepsilon^+$ (via $\hat{\Phi}_\varepsilon^+$) and $\Sigma_\varepsilon^+$ (via $\Phi_\varepsilon^+$) respectively. The tensor $\hat{g}_\varepsilon^+$ induces $\hat{h}_\varepsilon^+$ via $\hat{\Phi}_\varepsilon^+$, and $g_\varepsilon^+$ induces $h_\varepsilon^+$ via $\Phi_\varepsilon^+$.

**Theorem 4** ***(Mars, 2005 [79])*** *Let $(\mathcal{V}, g)$ be a spacetime constructed by joining two spacetimes with boundary $(\mathcal{V}^+, g^+)$ and $(\mathcal{V}^-, g^-)$ across their corresponding boundaries $\Sigma_0^+$ and $\Sigma_0^-$. Let $\Sigma$ be an abstract copy of $\Sigma_0^+$ and $\Phi_0^\pm : \Sigma \to \mathcal{V}^\pm$ be the embeddings defining the background matching. Let also $K_1^\pm$ and $K_2^\pm$ be first and second order metric perturbations in $\mathcal{V}^\pm$. The first order perturbed matching conditions are fulfilled if and only if there exist two scalars $Q_1^\pm$ and two vectors $\vec{T}_1^\pm$ on $\Sigma$ for which*

$$h^{(1)}{}^+_{ij} = h^{(1)}{}^-_{ij}, \qquad \kappa^{(1)}{}^+_{ij} = \kappa^{(1)}{}^-_{ij}, \tag{3.44}$$

*holds, where $h^{(1)\pm}$ and $\kappa^{(1)\pm}$ are given in Proposition 5 after the substitution $Q_1 \to Q_1^\pm$, $\vec{T}_1 \to \vec{T}_1^\pm$, $g \to g^\pm$, $K_1 \to K_1{}^\pm$ and $\vec{e}_i \to \vec{e}_i^{\,\pm}$. The second order perturbed matching conditions are satisfied if and only if there exist two scalars $Q_2^\pm$ and two vector fields $\vec{T}_2^\pm$ on $\Sigma$ such that*

$$h^{(2)}{}^+_{ij} = h^{(2)}{}^-_{ij}, \qquad \kappa^{(2)}{}^+_{ij} = \kappa^{(2)}{}^-_{ij}, \tag{3.45}$$

*where these objects are obtained from Proposition 6 after similar substitutions.*

Hence, fulfilling the matching conditions requires showing the existence of $\vec{Z}_1^\pm$ and $\vec{Z}_2^\pm$, such that equations (3.44) and (3.45) are satisfied. Note that the structure of the linear matching conditions (3.44), given the explicit expressions for the perturbed first and second fundamental forms (3.38) and (3.39) is such that only the values of $Q_1^\pm$ and the differences $[\vec{T}_1]$, but not $\vec{T}_1{}^+$ and $\vec{T}_1{}^-$, can be determined, provided that the background has been already matched.

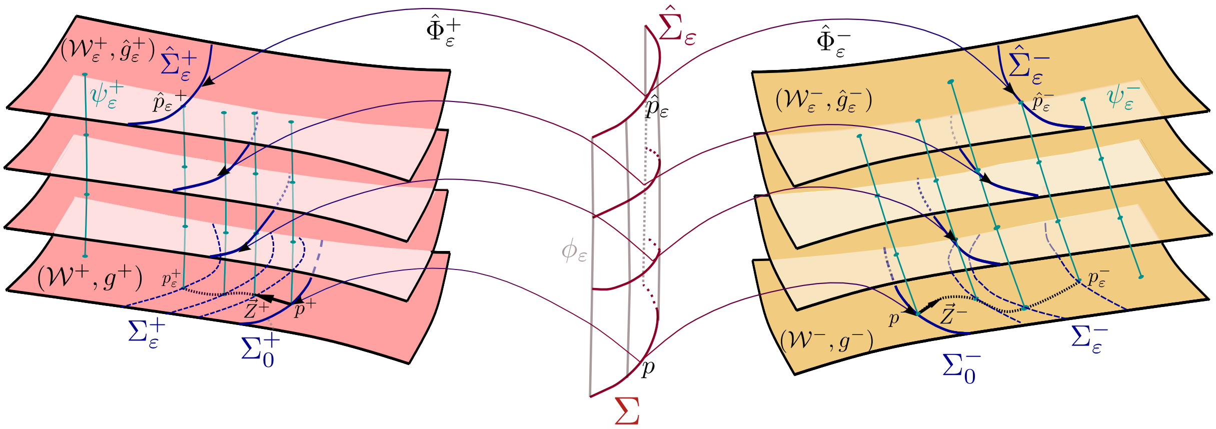


Figure 3.5: This scheme combines the settings in Fig. 3.2 and Fig. 3.4 and adapts them into a matching situation. The abstract hypersurfaces $\hat{\Sigma}_\varepsilon$ are identified among themselves via $\phi_\varepsilon$, which determines the hypersurface gauge. The maps $\hat{\Phi}^\pm_\varepsilon$ embed the $\hat{\Sigma}_\varepsilon$ into the respective spacetimes $(\mathcal{W}^\pm_\varepsilon, \hat{g}^\pm_\varepsilon)$. The perturbed matching conditions (3.43) are formulated in $\Sigma$.

It must be stressed that the objects $h^{(1)}$, $h^{(2)}$, $\kappa^{(1)}$ and $\kappa^{(2)}$ are invariant under spacetime gauge transformations by construction, but they are not hypersurface gauge invariant. However, the set of equations (3.44)-(3.45) is gauge invariant under both spacetime and hypersurface gauge transformations, provided the background is matched [87, 79].

We discussed how the vectors $\vec{Z}^\pm$ depend on both the spacetime and the hypersurface gauges. The hypersurface gauge, common to the two families of hypersurfaces $\hat{\Sigma}^\pm$, can be used to set to zero the tangent part $\vec{T}^+$ or $\vec{T}^-$, but not both. The spacetime gauge freedom at either side can be exploited to fix either or both pairs $\vec{Z}^+_{1/2}$ and $\vec{Z}^-_{1/2}$ independently a priori, but this has to be carefully analyzed if additional spacetime gauge choices are made. In fact, the vectors $\vec{Z}^+_{1/2}$ and $\vec{Z}^-_{1/2}$ can be set to zero simultaneously using the spacetime gauge freedom conveniently.

At either side, say $+$, we will call a gauge $\psi^+_\varepsilon$ "surface-comoving" if the hypersurfaces $\Sigma^+_\varepsilon$ do not vary, and thus agree with $\Sigma^+_0$, as sets of points in $\mathcal{V}^+_0$. At first order that is equivalent to $Q^+_1 = 0$, but at second order $Q^+_2$ carries more information coming from the first order. Recall the introduction of the quantity $\hat{Q}^+_2$ in Section 3.2. The gauges referred to as "surface gauges" in previous works, e.g. [18, 85], require the vanishing of the whole perturbation vector $\vec{Z}$ (more precisely $Q = 0$ and $[\vec{T}] = 0$ is enough).

## 3.5 Freedom in the matching due to symmetries

In a general situation $[\vec{T}_{1/2}]$ are completely determined by the matching conditions. However, if there are isometries in the background spacetime, this may not be the case.

The following discussion applies to the linear perturbations in any of the spacetimes $\pm$. Consider an isometry in the background spacetime which is preserved by the background matching, i.e. generated by a Killing vector field $\vec{\xi}$ that is tangent to $\Sigma_0$, so that $\boldsymbol{n}(\vec{\xi}|_{\Sigma_0}) = 0$. Then, it can be shown, using the commutation of the pullback and the Lie derivative, that $\vec{\gamma}$, where $\vec{\xi}|_{\Sigma_0} = d\Phi(\vec{\gamma})$, is an isometry in $(\Sigma, h_{ab})$, i.e.

$$\mathcal{L}_{\vec{\gamma}} h_{ab} = e_a^\alpha e_b^\beta \, \mathcal{L}_{\vec{\xi}} g_{\alpha\beta}\Big|_{\Sigma_0} = 0,$$

so that $\Sigma$ preserves that symmetry [108]. A similar calculation [80] shows that also $\mathcal{L}_{\vec{\gamma}} \kappa_{ab} = 0$. Thus, the perturbed first (3.38) and second (3.39) fundamental forms, $h^{(1)}$ and $\kappa^{(1)}$, remain invariant under a change on $\Sigma$ of the form $\vec{T}_1^\pm \to \vec{T}_1^\pm + C^\pm \vec{\gamma}$, for any constants $C^\pm$, i.e. invariant under a change of the vectors $T_1^\pm$ along the direction of any isometry of the background configuration (preserved by the matching). On the embedded $\Sigma_0$ on $\mathcal{V}$ this means that the difference transforms as $[\vec{T}_1] \to [\vec{T}_1] + \vec{\xi}^+ - \vec{\xi}^-$. Since the linearized matching conditions remain invariant under this class of transformations, $[\vec{T}_1]$ can *slide* freely along the directions of the (tangent) isometries. Therefore, an important consequence is that the matching conditions cannot determine $[\vec{T}_1]$ in these cases [80].

This can be fixed making use of the spacetime gauge freedom, that can be adjusted, at least partially (in the tangent part to $\Sigma_0$ of any of the $\vec{S}_1^\pm$), to fix $[\vec{T}_1]$ and thus relate $\vec{T}_1^+$ and $\vec{T}_1^-$ succesfully. Note, again, that the hypersurface gauge cannot help us to fix $[\vec{T}_1]$, since this is invariant under hypersurface gauge transformations.

# 4

# Hartle's model

Hartle's model [57] constitutes the basis of most of the analytical studies performed to study slowly rotating stars in General Relativity (GR). The formalism provides a method to construct numerical schemes in axial symmetry [104]. The model describes the axially symmetric equilibrium configuration of a rotating isolated compact body and its vacuum exterior in perturbation theory in GR. The interior of the body is a perfect fluid equipped with a barotropic equation of state. It does not have convective motions and it rotates rigidly. This is matched to a stationary and axisymmetric asymptotically flat vacuum exterior region across a timelike hypersurface, and the whole model is assumed to have equatorial symmetry. By matching we mean that there is no shell of matter on the surface of the star. The approach is analytic, and makes use of a perturbative method for slow rotation around a spherically symmetric static configuration driven by a single parameter $\Omega^H$[1].

The first order perturbation, driven by a single function $\omega^H$, accounts for the rotational dragging of inertial frames. It does not change the shape of the surface of the star. The second order perturbation, in contrast, does affect the original spherical shape of the body, in agreement with the fact that this must be independent of the sense of rotation. The second order perturbation of the metric is described by three functions, $h^H$, $m^H$ and $k^H$. In addition to the change in the shape, these functions provide the relation between the central density of the star and the change in mass $\delta M$ between the perturbed and the static background configuration needed to keep the central density of the star unchanged, in analogy to the Newtonian approach (see [23, 29] or Chapter 9). This is how the total mass of the rotating configuration is found in terms of the central density of the star. There is one further property of the compact body determined by the second order perturbations: the quadrupole moment of the star.

[1] In order to ease the comparison with the original paper [57] we will use a superscript $^H$ to indicate that any object $f^H$ here refers to $f$ in [57].

This Chapter is a summary of the original works [57, 65], respecting the notation and coventions therein as much as possible.

## 4.1 Hartle's model in brief

Hartle's model is a perturbative approach in which the functions in the metric and in the energy momentum tensor are expanded in powers of the constant angular velocity $\Omega^H$ (the rotation of the fluid measured by a distant observer). The background configuration is a static and spherically symmetric fluid ball immersed in an asymptotically flat vacuum. Provided a barotropic equation of state, i.e. an equation of state for which the energy is a function of the pressure alone, the only parameter that must be specified to determine completely the configuration is the value of the central density. The first and second order perturbations in the rotational model are proportional to $\Omega^H$ and its square, respectively. The structure of the equations is such that given an explicit model computed with a particular value of $\Omega^H$, models for other angular velocities can be found by scaling. A common choice in the literature to compute a rotational configuration is the velocity close to the equatorial mass shedding, so that $\Omega^{H*} \approx \sqrt{Ma^{-3}}$, where $M$ accounts for the mass and $a$ for the radius of the spherical model.

The model is based upon the following metric (to second order) [57]

$$\begin{aligned} g^H &= -e^{\nu(r)}\left(1+2h^H(r,\theta)\right)dt^2 + e^{\lambda(r)}\left(1+2\frac{e^{\lambda(r)}}{r}m^H(r,\theta)\right)dr^2 \\ &\quad + r^2(1+2k^H(r,\theta))\left[d\theta^2+\sin^2\theta(d\varphi-\omega^H(r,\theta)dt)^2\right] + \mathcal{O}(\Omega^3), \end{aligned} \tag{4.1}$$

written globally in terms of a single set of spherical-like coordinates $\{t, r, \theta, \phi\}$ that covers both the interior region (star) and the exterior vacuum, so that the domain of the radial coordinate is $r \in (0, \infty)$. It is implicitly assumed that in this set of coordinates the metric (4.1) is at least continuous. The static and spherically symmetric background is described in terms of the functions $\lambda(r)$ and $\nu(r)$. In the background, the common boundary of the interior and exterior is located at $r = a$, so that the fluid extends in the region $r \in (0, a]$ and the vacuum in $r \in [a, \infty)$. The first order perturbation is described by the function $\omega^H(r, \theta)$ and the second order perturbation by $h^H(r, \theta)$, $m^H(r, \theta)$ and $k^H(r, \theta)$. An additional second order function is used in order to measure the deformation of the sphericall ball of fluid due to the rotation. It is denoted by $\xi^H$.

We denote the Einstein's equations computed in the coordinates $\{r, \theta\}$ of (4.1) by $\mathcal{G}(g^H)_{\alpha\beta} = 8\pi\mathcal{T}_{\alpha\beta}$, where $\mathcal{G}$ is the Einstein tensor associated to the metric $g^H$ and $\mathcal{T}$ is the energy momentum tensor of a (rigidly rotating) perfect fluid, i.e.

$$\mathcal{T}_\alpha{}^\beta = (\mathcal{E} + \mathcal{P}) u_\alpha u^\beta + \mathcal{P} \delta_\alpha{}^\beta,$$

where $\mathcal{E}$ and $\mathcal{P}$ are respectively the *total* energy density and pressure of the fluid with unit fluid flow $u^\alpha$. These are expanded as

$$\mathcal{E}(r,\theta) = E(r) + E^{(2)}(r,\theta) + O(\Omega^{H4}), \qquad \mathcal{P}(r,\theta) = P(r) + P^{(2)}(r,\theta) + O(\Omega^{H4}), \quad (4.2)$$

where $E$ and $P$ are the energy density and pressure of the static star and $E^{(2)}$ and $P^{(2)}$ denote their perturbations to second order in $\Omega^H$. The lack of a first order term in the expansions (4.2) is justified by demanding that the energy density and pressure cannot depend on the sense of rotation. The vector $\vec{u}$, under the assumption of circularity and rigid rotation, explicitly reads [65]

$$\begin{aligned} \vec{u} &= \sqrt{-(g_{tt} + 2\Omega^H g_{t\varphi} + \Omega^{H2} g_{\varphi\varphi})}(\partial_t + \Omega^H \partial_\varphi), \\ &= e^{-\nu/2}(1 + \frac{r^2 e^{-\nu} \sin^2\theta}{2}(\Omega^H - \omega^H)^2 - h^H)(\partial_t + \Omega^H \partial_\varphi) + O(\Omega^{H3}). \end{aligned}$$

In a perturbation scheme the first contribution that distorts the shape of the star from its spherical nonrotating background configuration comes from the second order perturbations. The strategy followed in [57] to determine this deformation consists of resorting to coordinate systems where the surfaces of constant density are located at a constant value of the radial coordinate $r$ of (4.1). Let us consider the surfaces of constant density $E$ in the nonrotating configuration. The star ends at a radius $r = a$ where the pressure vanishes. Therefore, the condition $P(a) = 0$ selects, through the barotropic equation of state, the constant energy surface of $E(P(a))$ that separates the fluid interior from the vacuum.

In the rotating configuration the surfaces of constant energy density are displaced from their spherical shape of the background and they are determined by $r = f(R,\theta)$, where $f$ satisfies

$$\mathcal{E}(f(R,\theta),\theta) = E(R), \quad \text{for } R \in (0,a]. \qquad (4.3)$$

The expansion of $f$ in powers of $\Omega^H$, i.e. $r = f(R,\theta) = R + \xi^H(R,\theta) + O(\Omega^{H4})$, defines $\xi^H$, that accounts for the second order term in $\Omega^H$. Back to the computation of the deformation of the rotating star, the equation of state is assumed to hold in the rotating configuration. Therefore, condition (4.3) can be formulated for the pressure, and it identifies the constant energy density surface that limits the fluid ball by

$$\mathcal{P}(f(R,\theta),\theta) = P(R), \qquad (4.4)$$

which expanded to second order yields

$$P(R) + P^{(2)}(R,\theta) + \frac{dP}{dR}\xi^H(R,\theta) = P(R), \tag{4.5}$$

so that the surfaces of constant pressure, are now identified by

$$P^{(2)}(R,\theta) + \frac{dP}{dR}\xi^H(R,\theta) = 0. \tag{4.6}$$

An equivalent treatment for the energy density yields an expression for the surfaces of constant energy density, given by

$$E^{(2)}(R,\theta) + \frac{dE}{dR}\xi^H(R,\theta) = 0. \tag{4.7}$$

In particular, the surface that separates the rotating fluid from the vacuum is found by particularizing expression (4.6) to the value $R = a$ obtained from the relation $P(a) = 0$. This determines the deformation by

$$P^{(2)}(a,\theta) + \left.\frac{dP}{dR}\right|_{R=a} \xi^H(a,\theta) = 0. \tag{4.8}$$

Thus, the function $\xi^H$ measures the deviation of the surfaces of constant energy density (or constant pressure) of the rotating configuration from the spheres in the static configuration (see figure 4.1). In addition, its value at $R = a$, i.e. $\xi^H(a,\theta)$, determines the shape of the surface of the star to second order.

## 4.2 Background configuration

The background configuration for the interior region described by $\lambda(r)$, $\nu(r)$, $E(r)$ and $P(r)$ satisfies the equations of general relativistic hydrostatics addressed in [57]. A function $M$ is defined by

$$1 - \frac{2M(r)}{r} := e^{-\lambda(r)}, \tag{4.9}$$

so that the four equations that determine the static configuration are cast in the form

$$\frac{dM(r)}{dr} = 4\pi r^2 E(r), \tag{4.10}$$

$$\frac{dP(r)}{dr} = -\frac{(E(r)+P(r))(M(r)+4\pi r^3 P(r))}{r(r-2M(r))}, \tag{4.11}$$

$$\frac{d\nu(r)}{dr} = -\frac{2}{E(r)+P(r)}\frac{dP}{dr}, \tag{4.12}$$

plus a barotropic equation of state $E(P)$. The system can be solved for a given value of the energy density at the origin. The integration of equation (4.12) determines the function $\nu(r)$ up to an additive constant.

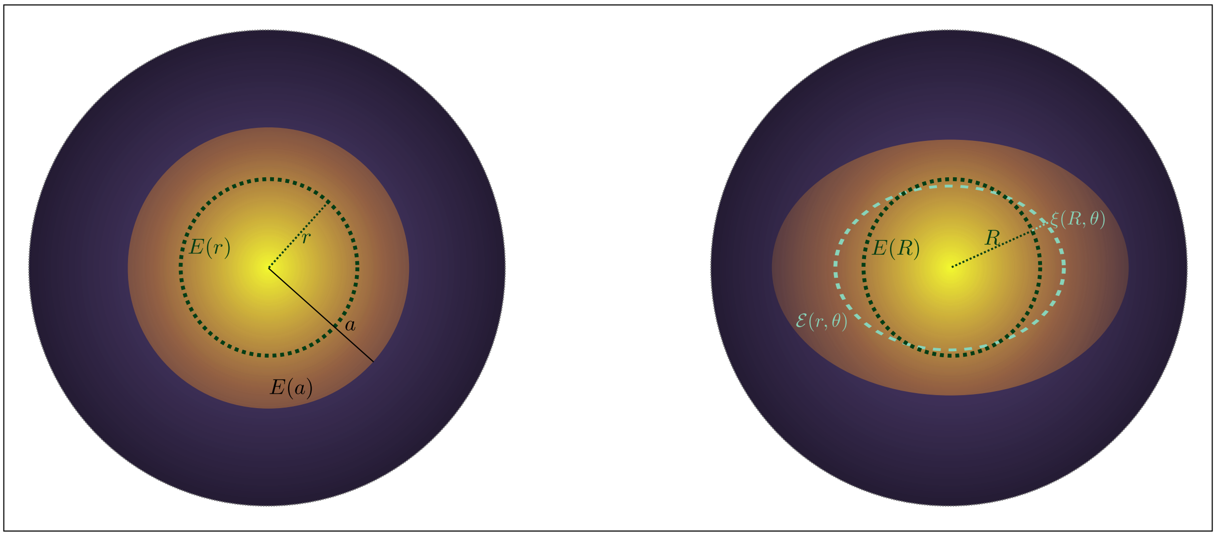


Figure 4.1: The left figure shows, in dark green, a surface of constant energy density, with value $E$, inside the star in the nonrotating configuration. It is labeled by the radial coordinate $r = R$. The right figure shows, in turquoise, a surface with the same value of the energy density, $\mathcal{E}(= E)$, in the rotating configuration. It is labeled now by the radial coordinate $r = R + \xi^H(R, \theta)$.

The exterior asymptotically flat vacuum requires $E = P = 0$, thus $M(r) = M$ is a constant and the metric functions are given by

$$e^{\nu(r)} = e^{-\lambda(r)} = 1 - \frac{2M}{r}. \tag{4.13}$$

The constant $M$ is recognized as the mass of the nonrotating star by the assumption of the continuity of the function $\lambda(r)$ at $r = a$. The continuity of $\nu(r)$ there fixes the additive constant resulting from the integration of (4.12). Finally the assumption $P(a) = 0$, which implies the continuity of $\nu'(r)$ at $r = a$, fixes the radius of the star.

It is useful to define the function

$$j(r) := e^{-(\lambda(r)+\nu(r))/2}$$

in order to cast the equations for the perturbations in a compact form. In vacuum, $j(r) = 1$.

## 4.3 First order perturbations

The only first order field equation $\{t, \varphi\}$ provides the following PDE for $\omega(r, \theta)$

$$\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4 j \frac{\partial \omega^H}{\partial r}\right) + \frac{4}{r}\frac{dj}{dr}(\omega^H - \Omega^H) + \frac{je^\lambda}{r^2}\frac{1}{\sin^3\theta}\frac{\partial}{\partial \theta}\left(\sin^3\theta \frac{\partial \omega^H}{\partial \theta}\right) = 0. \tag{4.14}$$

The equation for vacuum is recovered by setting $j = 1$ and $e^\lambda$ from (4.13). At this point $\omega^H(r, \theta)$ is implicitly assumed in [57] to be $C^1$. Regularity conditions at the origin together with asymptotic flatness are then used to argue that $\omega^H$ must be a function of $r$ alone, so that the only field equation becomes an ODE for $\omega^H(r)$. The function $\tilde{\omega}^H(r) := \Omega^H - \omega^H(r)$ is introduced in order to cast (4.14) as an homogeneous equation, that explicitly reads [57]

$$\frac{1}{r^3}\frac{d}{dr}\left(r^4 j \frac{d\tilde{\omega}^H}{dr}\right) + 4\frac{dj}{dr}\tilde{\omega}^H = 0. \tag{4.15}$$

This equation is integrated from the origin outwards, given the aforementioned regularity conditions there. This implies that there is only one parameter that must be provided to completely determine the interior solution. A convenient choice is the value of $\tilde{\omega}^H$ at the origin, denoted by $\tilde{\omega}^H_c$. However, as commented at the beggining of this chapter, it is customary in the literature to specify the critical angular velocity $\Omega^H_{e.s.}$ [64]. The choice is not relevant, since the solution can be scaled to reflect any of the models, but it is important to remark that only one (first order) parameter is needed to determine completely the configuration.

Equation (4.15) in the vacuum region holds for $j = 1$, which leads to

$$\tilde{\omega}^H(r) = \Omega^H - \frac{2J}{r^3}, \tag{4.16}$$

after imposing asymptotic flatness, for some constants $\Omega^H$ and $J$. The constant $J$ corresponds to the total angular momentum of the star [57]. Together with the constant $\Omega^H$, it characterizes the exterior solution (4.16). These constants are determined at $r = a$ assuming the continuity of $\omega^H(r)$ and its first derivative. The explicit relations are

$$J = \frac{1}{6}a^4\left(\frac{d\tilde{\omega}^H}{dr}\right)_{r=a}, \qquad \Omega^H = \tilde{\omega}^H(a) + \frac{2J}{a^3}. \tag{4.17}$$

The moment of inertia $I$ is found in terms of these two constants as $I = J/\Omega^H$.

The constant $\Omega^H$ is described in [57] to be the angular velocity of the fluid measured by an observer at rest with respecto to $(r, \theta)$ in the interior region, and $\omega^H(r, \theta)$ is the (first order) angular velocity acquired by a free falling oberver from infinity to $(r, \theta)$. Thus, the function $\tilde{\omega}^H$ is, by construction, the coordinate angular velocity of the fluid measured by the free falling observer.

## 4.4 Second order perturbations

The second order perturbation functions are argued in [57] to admit a finite expansion in Legendre polynomials containing only terms in $l = 0, 2$ due to the non-dependency of the first order function $\omega$ on any angular variable, and from the global equatorial symmetry of the model. Thus, the second order functions become

$$
\begin{aligned}
h^H(r,\theta) &= h_0^H(r) + h_2^H(r)P_2(\cos\theta), \\
m^H(r,\theta) &= m_0^H(r) + m_2^H(r)P_2(\cos\theta), \\
k^H(r,\theta) &= k_2^H(r)P_2(\cos\theta).
\end{aligned}
\tag{4.18}
$$

Note that it is also imposed that the function $k^H$ does not have spherical component, so that $k_0^H(r) = 0$. This fixes the radial coordinate to second order.

According to the discussion at the beggining of this chapter about the determination of the shape of the star and its relation to the radial coordinates used at each order, the second order functions should be expressed in terms of the radial coordinate $R$, instead of $r$. However, the second order equations are formally equivalent as computed either using $r$ or $R$. This is discussed at p.1018 in [57] and the argument given there consists in considering the series expansions in $\Omega^H$ of the field equations expressed in the different systems of coordinates. In $\{r,\theta\}$ (of (4.1)) one finds

$$
\mathcal{G}(r,\theta) = 8\pi\mathcal{T}(r,\theta) \Rightarrow G^{(2)}(r,\theta) = 8\pi T^{(2)}(r,\theta), \tag{4.19}
$$

where $G^{(2)}$ is the second order term of $\mathcal{G}$, and the same for $T^{(2)}$ and $\mathcal{T}$. Now, the same expansion is considered in the coordinates $\{R,\theta\}$ to obtain [57]

$$
\begin{aligned}
\mathcal{G}(r(R,\theta),\theta) &= G(R,\theta) + \underbrace{G^{(2)}(R,\theta) + \xi^H(R,\theta)\frac{\partial G(R,\theta)}{\partial R}}_{\Delta G(R,\theta)} = \\
= 8\pi\mathcal{T}(r(R,\theta),\theta) &= 8\pi T(R,\theta) + \underbrace{8\pi T^{(2)}(R,\theta) + 8\pi\xi^H(R,\theta)\frac{\partial T(R,\theta)}{\partial R}}_{8\pi\Delta T(R,\theta)},
\end{aligned}
$$

where the perturbed Einstein and energy momentum tensors in the coordinates $\{R,\theta\}$ have been defined as $\Delta G$ and $\Delta T$ respectively. These fulfill

$$
\Delta G(R,\theta) = 8\pi\Delta T(R,\theta). \tag{4.20}
$$

Note that in $\Delta T$ any term in $E^{(2)}$ or $P^{(2)}$ is anihilated because of equations (4.6) and (4.7). Take for instance, the timelike component of the perturbation of the energy momentum tensor

$$\begin{aligned}\Delta T_{tt} &= T^{(2)}_{tt}(R,\theta) + \xi^H(R,\theta)\frac{\partial T_{tt}(R,\theta)}{\partial R}\\ &= \left[e^\nu E^{(2)} + 2E(e^\nu h^H + 2\omega^{H2} r^2\sin^2\theta) + 2\Omega^{H2}(P-E)r^2\sin^2\theta\right] + e^\nu(-E^{(2)} + \nu' E\xi^H)\\ &= 2E(e^\nu h^H + 2\omega^{H2} r^2\sin^2\theta) + 2\Omega^{H2}(P-E)r^2\sin^2\theta + \nu' E\xi^H, \end{aligned} \tag{4.21}$$

where the two contributions in the first line correspond to the term in brackets and the term in parenthesis in the second line respectively. The field equations in the background applied to the left hand side of (4.20) imply

$$\begin{aligned}\Delta G_{tt} &= G^{(2)}_{tt}(R,\theta) + \xi^H(R,\theta)\frac{\partial G_{tt}(R,\theta)}{\partial R} = G^{(2)}_{tt}(R,\theta) + 8\pi\xi^H(R,\theta)\frac{\partial T_{tt}(R,\theta)}{\partial R}\\ &= G^{(2)}_{tt}(R,\theta) + 8\pi e^\nu \xi^H(R,\theta)(E' + E\nu') \end{aligned} \tag{4.22}$$

The terms $e^\nu\nu' E\xi^H$ in (4.21) and (4.22) cancel out at the time of imposing the field equations (4.20), but the contribution $8\pi e^\nu\xi^H(R,\theta)E'$ in (4.22) survives. By means of (4.7), this term, transported to the right hand side of the field equations, plays the role of the perturbation of the energy density. This kind of arguments lead to the statement in [57] (p.1018), as well as in [64] (p.810), that the equations (4.19) and (4.20) are formally equivalent, given the relation (4.6) for the pressure and (4.7) for the energy density. In modern terminology, this is just a consequence of a change of spacetime gauge driven by a second order vector $\vec{V}_2 \propto \xi^H\partial_r$ (see equation (6.7) and the related discussion).

Thence, the perturbed field equations given in [57] are computed explicitly by considering (4.19) as equations in the coordinates $\{R,\theta\}$ (instead of $\{r,\theta\}$) and substituting any term in $E^{(2)}$ of $P^{(2)}$ by the corresponding expression in $\xi^H$ as dictated by (4.6) and (4.7).

Thus, the equations and results in [57] for the second order will be presented here in terms of the radial coordinate $r$, whose domain of definition is $(0,a]\cup[a,\infty)$, covering the fluid and the vacuum regions respectively.

There are two further remarks about the field equations to second order. On the one hand, they do not mix the contributions $l=0$ and $l=2$, so that they can be studied separately. On the other hand, they propagate the angular structure of the metric functions to the quantities in the energy momentum tensor, so that one finds

$$\begin{aligned}E^{(2)}(r,\theta) &= E^{(2)}_0(r) + E^{(2)}_2(r)P_2(\cos\theta),\\ P^{(2)}(r,\theta) &= P^{(2)}_0(r) + P^{(2)}_2(r)P_2(\cos\theta),\\ \xi^H(r,\theta) &= \xi^H_0(r) + \xi^H_2(r)P_2(\cos\theta).\end{aligned}$$

The functions involved in the $l=0$ sector are $m^H_0$ and $h^H_0$, coming from the metric, and $\xi^H_0$, measuring the spherical deformation of the star. For convenience, this last function is substituted by the pressure perturbation factor defined as ((87) in [57])

$$p_0^{H*}(r) := \frac{P_0^{(2)}(r)}{E(r)+P(r)} = -\frac{P'(r)}{E(r)+P(r)}\xi_0^H(r). \tag{4.23}$$

The field equations can be arranged to provide a system of first order inhomogeneous ODE's for the set $\{m_0^H, p_0^{H*}\}$ and an algebraic equation for $h_0^H$. The system for $\{m_0^H, p_0^{H*}\}$ reads ((97) and (98) in [57])

$$\begin{aligned}
\frac{dm_0^H}{dr} &= 4\pi r^2\frac{dE}{dP}(E+P)p_0^{H*} + \frac{1}{12}j^2r^4\left(\frac{d\tilde{\omega}}{dr}\right)^2 - \frac{2}{3}r^3 j\frac{dj}{dr}\tilde{\omega}^2, && (4.24)\\
\frac{dp_0^{H*}}{dr} &= -\frac{4\pi(E+P)r^2}{r-2M(r)}p_0^{H*} - \frac{m_0^H r^2}{(r-2M(r))^2}\left(8\pi P+\frac{1}{r^2}\right)\\
&\quad + \frac{r^4j^2}{12(r-2M(r))}\left(\frac{d\tilde{\omega}}{dr}\right)^2 + \frac{1}{3}\frac{d}{dr}\left(\frac{r^3j^2\tilde{\omega}^2}{r-2M(r)}\right). && (4.25)
\end{aligned}$$

The boundary conditions are the vanishing of $m_0^H$ and $p_0^{H*}$ at the origin. Apart from ensuring regularity there, these conditions imply $P_0^{(2)}(r) = 0$ as $r \to 0$, so that the central pressure of the nonrotating configuration is preserved in the rotating model (see below).

The algebraic equation for $h_0^H$, given by (90) in [57], corresponds to the *hydrostatic equilibrium* first integral, which reads explicitly

$$p_0^{H*} + h_0^H - \frac{1}{3}r^2e^{-\nu}\tilde{\omega}^2 = \gamma, \tag{4.26}$$

where $\gamma$ is a (second order) constant. Note that $\gamma$ equals the value of $h_0^H$ at the origin, $\gamma = h_0^H(0)$.

The set of functions that determines the exterior configuration to second order is $\{m_0^H, h_0^H\}$. The field equations are (4.24), with $j = 1$ and $\tilde{\omega}$ given by (4.16), and the following first order equation for $h_0^H$

$$\frac{dh_0^H}{dr} = \frac{m_0^H}{(r-2M)^2} - \frac{3J^2}{r^4(r-2M)}.$$

The asymptotically flat vacuum solution thus reads ((105) and (106) in [57])

$$\begin{aligned}
m_0^H(r) &= \delta M - \frac{J^2}{r^3}, && (4.27)\\
h_0^H(r) &= -\frac{\delta M}{r-2M} + \frac{J^2}{r^3(r-2M)}, && (4.28)
\end{aligned}$$

for some arbitrary constant $\delta M$. This constant is identified as the change in mass due to the perturbations by taking the limiting behaviour of the spherical part of $g_{rr}^H$ in (4.1) as $r \to \infty$, this is,

$$\lim_{r\to\infty} g_{rr}^H\big|_{sph} = \lim_{r\to\infty} e^{\lambda(r)}\left(1+2\frac{e^{\lambda(r)}m_0^H}{r}\right) \approx 1 + 2\frac{M+\delta M}{r}. \tag{4.29}$$

Note that in order to add the quantities $M$ and $\delta M$ these must have been computed given common boundary data. This is commonly achieved by using a fixed value of the central energy density $E_c$ common to the nonrotating and rotating configurations. In this way, the total mass (to second order in $\Omega^H$) can be constructed as $M_{total}(E_c) = M(E_c) + \delta M(E_c)$.

At this point the interior solution $\{m_0^H, p_0^{H*}\}$ has been completely determined, while the exterior solution for $m_0^H$ is determined up to $\delta M$. In order to fix it, in [57] the interior and the exterior solutions are related at $r = a$ *assuming the continuity of the function* $m_0^H$. This, using (4.27), fixes the constant $\delta M$ as ((107) in [57])

$$\delta M = m_0^H(a) + \frac{J^2}{a^3}. \tag{4.30}$$

Note that the continuity of $h_0^H$ cannot be used to obtain $\delta M$, since $h_0^H$ is determined up to an additive constant ($\gamma$) in the interior. The $l = 0$ sector also determines the spherical change in the shape of the star to second order. After using (4.12), (4.9), (4.23) and $P(a) = 0$, this deformation is found to be

$$\xi_0^H(a) = -a(a - 2M)p_0^{H*}(a)/M. \tag{4.31}$$

The $l = 2$ sector involves the functions $h_2^H$, $k_2^H$ and $m_2^H$, coming from the metric perturbation, and the function $\xi_2^H$, which will account for the ellipticity of the star. As in the $l = 0$ sector, $\xi_2^H$ is substituted by $p_2^{H*}$ by a relation analogous to (4.23). Explicitly,

$$p_2^{H*}(r) := \frac{P_2^{(2)}(r)}{E(r) + P(r)} = -\frac{P'(r)}{E(r) + P(r)}\xi_2^H(r). \tag{4.32}$$

A convenient function $v^H := h_2^H + k_2^H$ is introduced to substitute $k_2^H$. For the fluid configuration, the field equations can be arranged as a system of two first order inhomogeneous ODE's for the set $\{h_2^H, v^H\}$ and two algebraic equations for $m_2^H$ and $p_2^{H*}$.

The system for $h_2^H$ and $v^H$ is given in [57] by the equations (125) and (126), whose explicit form is

$$\frac{dv^H}{dr} = -\nu' h_2^H + \left(\frac{1}{r} + \frac{\nu'}{2}\right)\left(-\frac{2}{3}r^3 j j' \tilde{\omega}^2 + \frac{1}{6}r^4 j^2 \tilde{\omega}'^2\right), \tag{4.33}$$

$$\begin{aligned} \frac{dh_2^H}{dr} = {} & \left(-\nu' + \frac{r}{\nu'(r-2M)}\left(8\pi(E+P) - \frac{4M}{r^3}\right)\right) h_2^H \\ & - \frac{4}{r(r-2M)\nu'} v^H + \frac{r^3 j^2}{6}\left(\frac{r\nu'}{2} - \frac{1}{(r-2M)\nu'}\right)\tilde{\omega}'^2 \\ & - \frac{2r^2 j j'}{3}\left(\frac{r\nu'}{2} + \frac{1}{(r-2M)\nu'}\right)\tilde{\omega}^2. \end{aligned} \tag{4.34}$$

The boundary conditions that ensure regularity at the origin are simply $h_2^H = v^H = 0$ as $r \to 0$. The field equations also provide the following algebraic equation for $m_2^H$ ((120) in

[57])

$$\frac{m_2^H(r)}{r-2M(r)} = -h_2^H(r) - \frac{2r^3 j(r) j'(r)}{3}\tilde{\omega}^2(r) + \frac{r^4 j^2(r)}{6}\tilde{\omega}'^2(r). \tag{4.35}$$

Finally, the equation defining the perturbation of the pressure, or equivalently the $l = 2$ component of the hydrostatic equilibrium first integral (90) in [57], provides the following relation, that determines $p_2^{H*}$

$$p_2^{H*}(r) + h_2^H(r) + \frac{r^2 e^{-\nu(r)}}{3}\tilde{\omega}^2(r) = 0. \tag{4.36}$$

In vacuum only the first three equations apply, and given the asymptotic behaviour as $r \to \infty$ the set of functions $\{h_2^H, v^H, m_2^H\}$ is integrated to

$$h_2^H(r) = KQ_2^2\left(\frac{r}{M} - 1\right) + J^2\left(\frac{1}{Mr^3} + \frac{1}{r^4}\right), \tag{4.37}$$

$$v^H(r) = \frac{2MK}{\sqrt{r(r-2M)}}Q_2^1\left(\frac{r}{M} - 1\right) - \frac{J^2}{r^4}, \tag{4.38}$$

$$\frac{m_2^H(r)}{r-2M} = KQ_2^2\left(\frac{r}{M} - 1\right) + \frac{J^2}{r^3}\left(\frac{1}{M} - \frac{5}{r}\right), \tag{4.39}$$

where $Q_n^m$ denote the associated Legendre polynomials of the second kind.

The interior solution is determined up to a constant, associated to the homogeneous solution of the system (4.33) and (4.34), and the exterior solution is determined up to the constant $K$, explicit in the expressions (4.37)-(4.39). These two constants are determined in [57] assuming that $h_2^H$ and $v^H$ are continuous at $r = a$.

There are two physical quantities of interest arising from the $l = 2$ sector of the second order perturbations. On the one hand, the constant $K$ in the vacuum solution is related to the quadrupole moment of the star $Q$ by (26) in [65]

$$Q = \frac{8KM^3}{5} + \frac{J^2}{M}.$$

On the other hand, the ellipticity $e$ of the configuration is related to the pressure perturbation factor $p_2^{H*}$ and it can be expressed in terms of $\xi_2^H$ using the relation (4.32) as given by (25c) in [65]

$$e = \sqrt{-3\left(v^H(a) - h_2^H(a) + \frac{\xi_2^H(a)}{a}\right)}. \tag{4.40}$$

5

# Axially symmetric and stationary matchings in perturbation theory

This is the first one of a series of three chapters (5, 6, and 7) aimed at putting Hartle's model [57] on firm grounds. In these, we construct the model using the perturbation theory [79] reviewed in Chapter 3.

In this chapter we construct an axially symmetric and stationary configuration by matching perturbatively to second order two configurations $(\mathcal{V}^{\pm}, g^{\pm}, \Sigma_0^{\pm}, \{g_\varepsilon^{\pm}\})$. The background matched configuration $(\mathcal{V}, g)$ is chosen to be static and spherically symmetric. The embedded matching hypersurface $\Sigma_0$ is timelike and preserves the symmetries of the spacetimes. On top of $(\mathcal{V}^+, g^+)$ and $(\mathcal{V}^-, g^-)$, axially symmetric and stationary metric perturbations to first and second order are developed from the families of metrics $\{g_\varepsilon^{\pm}\}$, being these stationary, axially and equatorially symmetric. The metric (4.1) used in Hartle's model is included within $\{g_\varepsilon^+\}$ and $\{g_\varepsilon^-\}$ respectively. In this chapter we do not impose field equations of any kind. In this way, the results can be used in more general situations, such as other theories of gravity for which Hartle's model has been generalised already in the literature.

Regarding our assumptions, although the deformation of the boundaries is left as an unknown, we assume that it is axially symmetric. The perturbed matched spacetime thus retains the axial symmetry. This requires that the normal components of the deformation vectors, $Q^{\pm}$ do not depend on the axial coordinate. Given the axial symmetry of the background, we can also ask $\vec{T}$ to have no axial component without loss of generality (see Section 2.3).

The structure of this chapter is the following. Firstly, we identify the neccesary ingredients in the perturbation scheme: the background spacetimes $(\mathcal{V}^{\pm}, g^{\pm})$ as two generic static and spherically symmetric spacetimes, the family of tensors $\{g_\varepsilon^{\pm}\}$ inspired by the metric $g^H$ (4.1) introduced in Hartle's model, from where the perturbation tensors to first and second order are obtained. Our $\{g_\varepsilon^\pm\}$ are chosen so that the two gauges that correspond to the two "*systems of coordinates*" used in the original model are included within the family.

Secondly, we perform the three matchings order by order. For the background matching we use the formalism introduced in Chapter 2 and for the perturbed matching we use the theory from Chapter 3. The results obtained from the perturbed matchings are summarised in the form of two propositions (Proposition 7 for the first order matching and Proposition 8 for the second order matching).

## 5.1 Family of metrics

The original "perturbed" metric in [57] is given by (4.1) assuming also that $k^H$ has no $l = 0$ term. In the context of perturbation theory this stands as a specific choice of spacetime gauge and we will refer to it as the (spacetime) $k$-gauge. However, the determination of the matching hypersurface is made in [57] (and most other works in the literature, see e.g. [16]) by resorting to another spacetime gauge, prescribed through the surfaces of constant energy density. Since we also want to examine the use of these different spacetime gauges in the literature, we consider a family of metrics $g_\varepsilon$ that can accommodate both spacetime gauges. To do that a crossed term in $(r, \theta)$ is needed.

Let us thus define the following one-parameter family $g_\varepsilon$ of stationary and axisymmetric metrics on $(\mathcal{V}, g)$, where $g = g_{\varepsilon=0}$, taken up to order $\varepsilon^2$

$$\begin{aligned} g_\varepsilon &= -e^{\nu(r)}\left(1 + 2\varepsilon^2 h(r,\theta)\right)dt^2 + e^{\lambda(r)}\left(1 + 2\varepsilon^2 m(r,\theta)\right)dr^2 + 2re^{\lambda(r)}\varepsilon^2\partial_\theta f(r,\theta)drd\theta \\ &\quad + r^2(1 + 2\varepsilon^2 k(r,\theta))\left[d\theta^2 + \sin^2\theta(d\varphi - \varepsilon\omega(r,\theta)dt)^2\right] + \mathcal{O}(\varepsilon^3), \end{aligned} \tag{5.1}$$

where $t \in (-\infty, \infty)$, $r > 0$, $\theta \in (0, \pi)$ and $\varphi \in [0, 2\pi)$. Clearly, an arbitrary function of $r$ can be added to $f(r, \theta)$ with no consequences. The appearance of $f$ differentiated is just a mere convenience. The (unique) axial Killing vector field [81] will be denoted by $\vec{\eta} = \partial_\varphi$, and we will single out the timelike Killing $\vec{\xi} = \partial_t$. The first and second order metric perturbation tensors, $K_1 = \partial_\varepsilon g_\varepsilon|_{\varepsilon=0}$ and $K_2 = \partial_\varepsilon^2 g_\varepsilon|_{\varepsilon=0}$ respectively, take thus the form

$$K_1 = -2r^2\,\omega(r,\theta)\sin^2\theta dt d\varphi, \tag{5.2}$$

$$\begin{aligned} K_2 &= \left(-4e^{\nu(r)}h(r,\theta) + 2r^2\sin^2\theta\omega^2(r,\theta)\right)dt^2 + 4e^{\lambda(r)}m(r,\theta)dr^2 \\ &\quad + 4r^2k(r,\theta)(d\theta^2 + \sin^2\theta d\varphi^2) + 4re^{\lambda(r)}\partial_\theta f(r,\theta)drd\theta, \end{aligned} \tag{5.3}$$

defined on the spherically symmetric and static spacetime background $(\mathcal{V}, g)$ with

$$g = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{5.4}$$

The (spacetime) gauge transformations described by $\vec{S}_1 = Ct\partial_\varphi$, with arbitrary constant $C$, at first order and $\vec{V}_2 = 2Y(r,\theta)\partial_r$, for an arbitrary $Y(r,\theta)$, are contained within the family $g_\varepsilon$. Under the gauge $\vec{S}_1 = Ct\partial_\varphi$, the perturbation tensor $K_1$ transforms as (3.23)

$$K_1{}^{(g)} = -2r^2\left(\omega - C\right)\sin^2\theta dt d\varphi, \tag{5.5}$$

while under a change $\vec{V}_2 = 2Y(r,\theta)\partial_r$, with $\vec{S}_1 = Ct\partial_\varphi$, $K_2$ transforms as (3.24)

$$\begin{aligned} K_2{}^{(g)} \;=\;& \left\{-4e^{\nu}\left(h + \frac{\nu'}{2}Y\right) + 2r^2\sin^2\theta(\omega - C)^2\right\}dt^2 + 4e^{\lambda}\left(m + e^{-\frac{\lambda}{2}}\left(Ye^{\lambda/2}\right)'\right)dr^2 \\ & +4r^2\left(k + \frac{Y}{r}\right)(d\theta^2 + \sin^2\theta d\varphi^2) + 4re^{\lambda}\partial_\theta\left(f + \frac{Y}{r}\right)drd\theta. \end{aligned} \tag{5.6}$$

We will refer to this class of second order gauge transformations as "radial" gauges.

A (spacetime) gauge whitin the set of these "radial" gauges will be fixed, partially or completely, whenever the functions appearing in $K_1$, (5.2), and/or $K_2$, (5.3), are restricted in any way. The remaining freedom would consist on the possible $C$ and $Y(r,\theta)$ that make the changes to the components of (5.5) and (5.6) fit, component-wise, within that restriction. The $k$-gauge, as mentioned, consists of imposing that the function $k(r,\theta)$ in (5.3) has no $l = 0$ part, and that $f = 0$. In that case, the restriction on the $K_{2\theta\theta}$ component implies that $Y(r,\theta)$ cannot have $l = 0$ part, while the restriction on the $K_{2r\theta}$ component needs that $Y(r,\theta)$ does not depend on $\theta$. The only possibility is thus $Y(r,\theta) = 0$, so that there is no freedom left. We thus say that the $k$-gauge fixes completely the "radial" gauge.

A further second order gauge $\vec{V}_2 = 2\beta t\partial_t$, for a constant $\beta$, transforms $h$ in (5.6) to $h + \beta$. This change reflects onto the freedom in shifting the gravitational potential in Newtonian theory, and can be used to fix, for instance, $h$ at infinity.

Let us now consider the background spacetimes $(\mathcal{V}^{\pm}, g^{\pm})$, with corresponding coordinates $\{t_{\pm}, r_{\pm}, \theta_{\pm}, \varphi_{\pm}\}$ and families of metrics $g_\varepsilon^{\pm}$ as given in (5.1). In what follows we present the perturbed matching over a spherically symmetric (and static) background configuration composed by the matching of $(\mathcal{V}^{+}, g^{+})$ and $(\mathcal{V}^{-}, g^{-})$ across a spherically symmetric hypersurface $\Sigma_0$.

The structure of the original metric (4.1) can be clearly recovered by taking $f = 0$ and noting that the choice of perturbation parameter $\varepsilon$ is not relevant, since families of solutions are obtained by scaling. The physics of the model will restrict the scalability (see Eq. (1) in [57]). Note, however, that the relation between the radial coordinates in (4.1) and (5.1) (either $r_{\pm}$) must still be determined in order to be able to compare the functions in (4.1) with those in (5.1).

## 5.2 Background configuration

The background configuration is chosen to be globally spherically symmetric and static. This translates to the fact that the matching of $(\mathcal{V}^+, g^+)$ and $(\mathcal{V}^-, g^-)$, through respective boundaries $\Sigma_0^+$ and $\Sigma_0^-$, is asked to preserve the symmetries (see Section 3 in [108]), both the spherical symmetry and staticity. Under that condition the hypersurfaces $\Sigma_0^+$ and $\Sigma_0^-$ to be matched can be eventually cast, without loss of generality, as (see e.g. [80])

$$\Sigma_0^+ \;=\; \{t_+ = \tau,\; r_+ = a_+,\; \theta_+ = \vartheta,\; \varphi_+ = \phi\}, \tag{5.7}$$
$$\Sigma_0^- \;=\; \{t_- = \tau,\; r_- = a_-,\; \theta_- = \vartheta,\; \varphi_- = \phi\}, \tag{5.8}$$

for constants $a_\pm > 0$. The coordinates $\{\tau, \vartheta, \phi\}$ parametrize the manifold $\Sigma$, an abstract copy of any $\Sigma_0^\pm$ (recall the construction in Section 3.2), so that $\Sigma \equiv \Sigma_0^+ = \Sigma_0^-$.

Note that in this point we are identifying the timelike Killing vectors $\vec{\xi}^\pm|_{\Sigma_0^\pm}$, so that $\partial_{t^+} \stackrel{\Sigma}{=} \partial_{t^-}$ (where $\partial_{t^\pm}$ leave $g^\pm$ (5.4) diagonal). This may seem an assumption, but in this background configuration is not. In the exterior we single out the integrable timelike Killing vector $\vec{\xi}^- = \partial_{t^-}$ which is unit at infinity. In the matching procedure, $\vec{\xi}^-|_\Sigma$ should be identified, in principle, with any timelike vector field of the + side, i.e. with any appropriate combination $\vec{\xi}^+|_\Sigma = a\partial_{t^+}|_\Sigma + b\partial_{\varphi^+}|_\Sigma + c\partial_{\theta^+}|_\Sigma$, for any constants $a$, $b$, $c$. However, Lemma 4 ensures that the integrability is preserved in the matching hypersurface, and therefore the only possibility left is that $\vec{\xi}^+ = a\partial_{t^+}$. Now, a trivial change in $t^+$ is used to absorb the constant $a$. Therefore we can choose $\partial_{t^+} \stackrel{\Sigma}{=} \partial_{t^-}$ without loss of generality.

The tangent vectors to $\Sigma_0^+$ and $\Sigma_0^-$ thus read

$$\vec{e}_a^{\,\pm}: \qquad \vec{e}_1^{\,\pm} = \partial_{t_\pm}|_{\Sigma_0^\pm}, \quad \vec{e}_3^{\,\pm} = \partial_{\theta_\pm}|_{\Sigma_0^\pm}, \quad \vec{e}_2^{\,\pm} = \partial_{\varphi_\pm}|_{\Sigma_0^\pm}, \tag{5.9}$$

and the corresponding unit normals are

$$\vec{n}^+ = -e^{-\frac{\lambda_+(a_+)}{2}}\partial_{r_+}|_{\Sigma_0^+}, \quad \vec{n}^- = -e^{-\frac{\lambda_-(a_-)}{2}}\partial_{r_-}|_{\Sigma_0^-}, \tag{5.10}$$

under the condition that $\vec{n}^+$ points $\mathcal{V}^+$ inwards and $\vec{n}^-$ points $\mathcal{V}^-$ outwards, so that as $r_+$ increases one reaches $\mathcal{V}^-$, and as $r_-$ increases one gets away of $\mathcal{V}^+$. This convention will be used in what follows in order to call $\mathcal{V}^+$ the *interior* and $\mathcal{V}^-$ the *exterior*. The hypersurfaces $\Sigma_0^\pm$ are timelike everywhere, and they are (equally) oriented by construction.

The first and second fundamental forms read

$$h^\pm_{ab}dx^a dx^b \;=\; -e^{\nu_\pm(a_\pm)}d\tau^2 + a_\pm^2(d\vartheta^2 + \sin^2\vartheta d\phi^2), \tag{5.11}$$
$$\kappa^\pm_{ab}dx^a dx^b \;=\; e^{-\frac{\lambda_\pm(a_\pm)}{2}}\left(\frac{1}{2}e^{\nu_\pm(a_\pm)}\nu'_\pm(a_\pm)d\tau^2 - a_\pm(d\vartheta^2 + \sin^2\vartheta d\phi^2)\right), \tag{5.12}$$

where a prime denotes differentiation with respect to the corresponding argument, i.e. the radial coordinate $r_+$ or $r_-$ accordingly. The matching conditions $h^+ = h^-$ and $\kappa^+ = \kappa^-$ are thus equivalent to

$$[\nu] = 0, \quad [\nu'] = 0, \quad [\lambda] = 0, \quad a := a_+ = a_-. \tag{5.13}$$

Recall we use the notation $[f] = f^+|_{\Sigma_0^+} - f^-|_{\Sigma_0^-}$ for objects $f^\pm$ defined at either side. For the sake of brevity, given a pair $f^\pm$ satisfying $[f] = 0$, we will simply denote by $f|_{\Sigma_0}$ either of the equivalent $f^+|_{\Sigma_0^+}$ or $f^-|_{\Sigma_0^+}$. The background matching hypersurface $\Sigma$ is endowed with the metric $h = -e^{\nu(a)}d\tau^2 + a^2(d\vartheta^2 + \sin^2\vartheta d\phi^2)$.

## 5.3 First order matching

Once the static and spherically symmetric background configuration has been constructed we proceed to study the perturbed matching to first order. As discussed above, the ingredients needed are the tensors that describe the perturbations at either side, i.e. the first and second metric perturbation tensors ${K_1}^\pm$ as defined above (5.2), plus the two (so far unknown) perturbation vectors $\vec{Z}_1^\pm$ given in the form (3.28). To ease the notation we will denote by $Q^\pm$ and $\vec{T}^\pm = T^\tau_\pm(\tau,\vartheta,\phi)\partial_\tau + T^\vartheta_\pm(\tau,\vartheta,\phi)\partial_\vartheta + T^\phi_\pm(\tau,\vartheta,\phi)\partial_\phi$ both the objects defined on each $\mathcal{V}^\pm$ and the corresponding pullback and pushforward quantities that live on $\Sigma$. The same applies for the functions $\omega^\pm$ in (5.2), which will be denoted equivalently as functions restricted to points on $\Sigma_0^\pm \subset \mathcal{V}^\pm$ and functions on $\Sigma$ whenever that does not lead to confusion. Since the final perturbed matched spacetime is assumed to preserve the axial symmetry, it seems natural to think that the functions $Q$ and the components of $\vec{T}$ will not depend on $\phi$. Nevertheless, we will take that as an assumption. The first and second order perturbed matchings are ruled by the particularisation of Theorem 4 together with Proposition 5 in Chapter 3 to the present setting with the above ingredients. For completeness, the explicit expressions of the first and second order first and second fundamental forms are included.

We start by calculating $h^{(1)}$ and $\kappa^{(1)}$ through expressions (3.38) and (3.39). Let us recall these are objects defined on $\Sigma$, which is timelike. The ingredients needed are the background embeddings (5.7), (5.8), with tangent basis (5.9) and unit normals (5.10), plus the first and second fundamental forms of $\Sigma$ (5.11) and (5.12), given that (5.13) holds, together with the first order perturbation tensors ${K_1}^\pm$ (5.2) restricted to $\Sigma_0^\pm$. The functions $Q_1^\pm(\tau,\vartheta)$ and vectors $\vec{T}_1^\pm = {T_1^\tau}^\pm(\tau,\vartheta)\partial_\tau + {T_1^\phi}^\pm(\tau,\vartheta)\partial_\phi + {T_1^\vartheta}^\pm(\tau,\vartheta)\partial_\vartheta$ on $\Sigma$ inherited from each side are left as unknowns. The explicit expressions of ${h^{(1)}}^\pm$ and ${\kappa^{(1)}}^\pm$

read

$$
\begin{aligned}
h^{(1)\pm}_{ij}dx^i dx^j \;\; = \;\; & e^{\nu(a)}\left(-2T_1^{\pm\tau}+\nu'(a)e^{-\frac{\lambda(a)}{2}}Q_1^{\pm}\right)d\tau^2+2\left(-e^{\nu(a)}T_1^{\pm\tau}{}_{,\vartheta}+a^2T_1^{\pm\vartheta}{}_{,\tau}\right)d\tau d\vartheta \\
&+2a^2\left(T_1^{\pm\phi}{}_{,\tau}-\omega^{\pm}(a,\vartheta)\right)\sin^2\vartheta d\tau d\phi \\
&+2a(aT_1^{\pm\vartheta}{}_{,\vartheta}-e^{-\frac{\lambda(a)}{2}}Q_1^{\pm})d\vartheta^2+2a^2T_1^{\pm\phi}{}_{,\vartheta}\sin^2\vartheta d\vartheta d\phi \\
&+2a\left(aT_1^{\pm\vartheta}\cos\vartheta-e^{-\frac{\lambda(a)}{2}}Q_1^{\pm}\sin\vartheta\right)\sin\vartheta d\phi^2,
\end{aligned}
$$

$$
\begin{aligned}
&\kappa^{(1)\pm}_{ij}dx^i dx^j = \\
&\left\{-Q^{\pm}_{1\,,\tau\tau}+e^{-\frac{\lambda(a)}{2}}e^{\nu(a)}\left(T_1^{\pm\tau}{}_{,\tau}+e^{-\frac{\lambda(a)}{2}}\frac{Q_1^{\pm}}{4}\left(\lambda'_{\pm}(a)\nu'(a)-2\nu''_{\pm}(a)-2\nu'^2(a)\right)\right)\right\}d\tau^2 \\
&-2\left\{Q^{\pm}_{1\,,\tau\vartheta}+e^{-\frac{\lambda(a)}{2}}\left(aT_1^{\pm\vartheta}{}_{,\tau}-\frac{1}{2}e^{\nu(a)}\nu'(a)T_1^{\pm\tau}{}_{,\vartheta}\right)\right\}d\tau d\vartheta \\
&+2ae^{-\frac{\lambda(a)}{2}}\left(-T_1^{\pm\phi}{}_{,\tau}+\frac{1}{2}a\omega'^{\pm}(a,\vartheta)+\omega^{\pm}(a,\vartheta)\right)\sin^2\vartheta d\tau d\phi \\
&-\left\{Q^{\pm}_{1\,,\vartheta\vartheta}+2aT_1^{\pm\vartheta}{}_{,\vartheta}+e^{-\lambda(a)}Q_1^{\pm}\left(\frac{1}{2}a\lambda'_{\pm}(a)-1\right)\right\}d\vartheta^2 \\
&-2ae^{-\frac{\lambda(a)}{2}}T_1^{\pm\phi}{}_{,\vartheta}\sin^2\vartheta d\vartheta d\phi \\
&-\left\{\left(Q^{\pm}_{1\,,\vartheta}+2ae^{-\frac{\lambda(a)}{2}}T_1^{\pm\vartheta}\right)\cos\vartheta+e^{-\lambda(a)}Q_1^{\pm}\left(\frac{1}{2}a\lambda'_{\pm}(a)-1\right)\sin\vartheta\right\}\sin\vartheta d\phi^2,
\end{aligned}
$$

where the background matching conditions (5.13) have been used to set $\nu_{\pm}(a)=\nu(a)$, $\nu'_{\pm}(a)=\nu'(a)$ and $\lambda_{\pm}(a)=\lambda(a)$.

The ordered procedure used to obtain and integrate the differences $[h^{(1)\pm}]$ and $[\kappa^{(1)\pm}]$ is the following. First, from $[h^{(1)}{}_{\vartheta\phi}]=0$ we obtain $[T_1{}^{\phi}]_{,\vartheta}=0$. On the other hand, the derivative $[h^{(1)}{}_{\tau\phi}]_{,\tau}=0$ yields $[T_1{}^{\phi}]_{,\tau\tau}=0$, and therefore $[T_1{}^{\phi}]=b_1\tau+C_2$ for arbitrary constants $b_1$ and $C_2$. As a result, $[h^{(1)}{}_{\tau\phi}]=0$ reads $[\omega]=b_1$.

Now, equation $[h^{(1)}{}_{\vartheta\vartheta}]\sin^2\vartheta-[h^{(1)}{}_{\phi\phi}]=0$ yields $[T_1{}^{\vartheta}]\cos\vartheta-[T_1{}^{\vartheta}]_{,\vartheta}\sin\vartheta=0$, which is integrated into $[T_1{}^{\vartheta}]=F(\tau)\sin\vartheta$ for some function $F(\tau)$. Equation $[h^{(1)}{}_{\vartheta\vartheta}]=0$ now reads $[Q_1]=e^{\lambda(a)/2}aF\cos\vartheta$. On the other hand, the compatibility condition to integrate $[T_1{}^{\tau}]$ is given by $2[h^{(1)}{}_{\tau\vartheta}]_{,\tau}-[h^{(1)}{}_{\tau\tau}]_{,\vartheta}=0$, which yields $\ddot{F}=-F\nu'(a)e^{\nu(a)}/2a$, and thence $[T_1{}^{\tau}]=C_1-e^{-\nu(a)}a^2\dot{F}\cos\vartheta$ for some arbitrary constant $C_1$. We have so far exhausted the conditions $[h^{(1)}{}_{ij}]=0$.

Given the above conditions, equation $[\kappa^{(1)}{}_{\tau\phi}]=0$ is now equivalent to $[\omega']=0$. The conditions on the metric perturbations have thus been obtained.

Consider the equation $[\kappa^{(1)}{}_{\tau\vartheta}]=0$, which now reads $\dot{F}a\sin\vartheta a(2e^{\lambda(a)}-2+a\nu'(a))=0$. If $2e^{\lambda(a)}-2+a\nu'(a)\neq 0$ we then have $\dot{F}=0$, which due to its previous equation can only be satisfied in the trivial case $F=0$. From the above, in particular, $[Q_1]=0$. Then, equations $[\kappa^{(1)}{}_{\phi\phi}] = 0$ and $[\kappa^{(1)}{}_{\vartheta\vartheta}] = 0$ just provide $Q_1[\lambda'] = 0$, from which $[\kappa^{(1)}{}_{\tau\tau}] = 0$ thus reads $Q_1[\nu''] = 0$.

The appearance of the constants $C_1$ and $C_2$ is a consequence of the isometries present in the background configuration, and cannot be determined [80] (see Section 3.5). Nevertheless, they can be safely absorbed by using a isomorphic *spacetime* gauge at one (any) side, say $\vec{S}_1^+ = C_1\partial_{t_+} + C_2\partial_{\varphi_+}$, which, by (3.34) leads to $\vec{T}_1^+ \to \vec{T}_1^+ - \vec{S}_1^+$ and obviously leaves the metric perturbation tensor ${K_1}^+$ unchanged. We can thus set $C_1 = C_2 = 0$ without loss of generality.

**Proposition 7** *Let $(\mathcal{V}, g)$ be the static and spherically symmetric spacetime resulting from the matching of $(\mathcal{V}^+, g^+)$ and $(\mathcal{V}^-, g^-)$, with $g^\pm$ given by (5.4) with respective $\pm$ in functions and coordinates, across $\Sigma_0^\pm$, defined by (5.7), (5.8), with $a(= a_+ = a_-) > 0$, so that the matching conditions (5.13) hold and the unit normals (5.10) are chosen following the above interior/exterior convention. Consider the metric perturbation tensors ${K_1}^\pm$ as defined in (5.2) at either side $\mathcal{V}^\pm$, plus two unknown functions $Q_1^\pm(\tau,\vartheta)$ and two unknown vectors $\vec{T}_1^\pm = T_1^{\pm\tau}(\tau,\vartheta)\partial_\tau + T_1^{\pm\vartheta}(\tau,\vartheta)\partial_\vartheta + T_1^{\pm\phi}(\tau,\vartheta)\partial_\phi$ on $\Sigma_0$.*

*The necessary and sufficient conditions that ${K_1}^\pm$ must satisfy to fulfil the first order matching conditions are*

$$[\omega] = b_1, \tag{5.14}$$

$$[\omega'] = 0, \tag{5.15}$$

*where $b_1$ is an arbitrary constant. Regarding the perturbed matching hypersurface, if*

$$2e^{\lambda(a)} - 2 + a\nu'(a) \neq 0 \tag{5.16}$$

*the remaining first order matching conditions read*

$$[\vec{T}_1] = b_1\tau\partial_\phi, \tag{5.17}$$

$$[Q_1] = 0, \qquad Q_1[\lambda'] = 0 \qquad Q_1[\nu''] = 0. \tag{5.18}$$

Note that, whenever (5.16) holds, although $[Q_1] = 0$ is always a necessary condition, so that $Q_1^+ = Q_1^- \equiv Q_1$, $Q_1 = 0$ is not. Indeed, if the background configuration satisfies $[\lambda'] = 0$ and $[\nu''] = 0$, $Q_1$ can be any arbitrary function of $(\tau, \vartheta)$. Let us remark that condition (5.16) will be satisfied in all cases we will be interested in (see Section 7.1).

## 5.4 Second order matching

We proceed to the perturbed matching to second order. As in the previous procedure for the first order matching, the ingredients needed are the perturbation tensors at either side, i.e. ${K_2}^{\pm}$, given by (5.3) and the two perturbation vectors $\vec{Z}_2^{\pm}$. The notation concerning these objects (both the vectors $\vec{Z}_2^{\pm}$ and the perturbation tensors ${K_2}^{\pm}$) is common to that used in the first order matching. Again, we assume that the functions $Q_2$ and the components of $\vec{T}_2$ do not depend on the coordinate $\phi$.

We shall make use of the results given in Proposition 7 for the first order configuration. The first and second order perturbed matchings are ruled by the particularisation of Theorem 4 together with Propositions 5 and 6 in Chapter 3 to the present setting with the above ingredients. For completeness, the explicit expressions of the first and second order first and second fundamental forms are included.

Let us first compute explicitly the scalar $\hat{Q}_2$ defined in (3.33), at each side $\Sigma_0^{\pm}$

$$\begin{aligned}\hat{Q}_2^{\pm} &= Q_2^{\pm} + a_{\pm} e^{-\lambda_{\pm}(a_{\pm})/2} \left\{ \frac{\nu'_{\pm}(a_{\pm}) e^{\nu_{\pm}(a_{\pm})}}{2a_{\pm}} (T_1^{\pm\tau})^2 - \sin^2\vartheta (T_1^{\pm\phi})^2 - (T_1^{\pm\vartheta})^2 \right\} \\ &\quad -2(T_1^{\pm\tau}\partial_\tau Q_1^{\pm} + T_1^{\pm\vartheta}\partial_\vartheta Q_1^{\pm}).\end{aligned}$$

The above first order matching conditions (5.17), thus lead to

$$[\hat{Q}_2] = [Q_2] + a e^{-\lambda(a)/2} \sin^2\vartheta\, b_1\tau \left(b_1\tau - 2T_1^{+\phi}\right) - 2(T_1^{\tau}\partial_\tau[Q_1] + T_1^{\vartheta}\partial_\vartheta[Q_1]).$$

This new $\hat{Q}_2$ will substitute the original $Q_2$ in this section.

The procedure is analogous to that of the previous proof. However, we first consider the case $[\lambda'] \neq 0$ or $[\nu''] \neq 0$, so that $Q_1 = 0$ necessarily. The explicit expression of $h^{(2)\pm}$ reads

$$\begin{aligned}
h^{(2)\pm}_{ab} dx^a dx^b &= \Big\{ -2e^{\nu(a)} \left( T^{\pm\tau}_{2\ ,\tau} + (T^{\tau}_{1\ ,\tau})^2 \right) + 2a^2 \left( T^{\pm\phi}_{1\ ,\tau} - \omega^{\pm}(a,\vartheta) \right)^2 \sin^2\vartheta \\
&+2a^2 (T^{\vartheta}_{1\ ,\tau})^2 - 4e^{\nu(a)} h(a,\vartheta) + e^{-\frac{\lambda(a)}{2}} e^{\nu(a)} \nu'(a) \hat{Q}_2 \Big\} d\tau^2 \\
&+2 \left\{ 2a^2 T^{\pm\phi}_{1\ ,\tau} T^{\pm\phi}_1 \cos\vartheta \sin\vartheta - e^{\nu(a)} T^{\pm\tau}_{2\ ,\vartheta} + a^2 T^{\pm\vartheta}_{2\ ,\tau} + 2a^2 T^{\vartheta}_{1\ ,\tau} T^{\vartheta}_{1\ ,\vartheta} \right. \\
&\left. +2a^2 T^{\pm\phi}_{1\ ,\vartheta} \left( T^{\pm\phi}_{1\ ,\tau} - \omega^{\pm}(a,\vartheta) \right) \sin^2\vartheta - 2e^{\nu(a)} T^{\tau}_{1\ ,\tau} T^{\tau}_{1\ ,\vartheta} \right\} d\tau d\vartheta \\
&+2a^2 \left\{ 2 \left( T^{\pm\phi}_{1\ ,\tau} T^{\vartheta}_1 - T^{\pm\phi}_1 T^{\vartheta}_{1\ ,\tau} - 2\omega^{\pm}(a,\vartheta) T^{\vartheta}_1 \right) \cos\vartheta \right. \\
&\left. + \left( T^{\pm\phi}_{2\ ,\tau} - 2T^{\tau}_{1\ ,\tau} \omega^{\pm}(a,\vartheta) - 2T^{\vartheta}_1 \omega^{\pm}_{,\vartheta}(a,\vartheta) \right) \sin\vartheta \right\} \sin\vartheta d\tau d\phi \\
&+2 \Big\{ a^2 \left( T^{\pm\phi}_1 \cos\vartheta + T^{\pm\phi}_{1\ ,\vartheta} \sin\vartheta \right)^2 - a^2 \sin^2\vartheta (T^{\pm\phi}_1)^2 + a^2 (T^{\vartheta}_{1\ ,\vartheta})^2 + a^2 T^{\pm\vartheta}_{2\ ,\vartheta} \\
&-e^{\nu(a)} (T^{\tau}_{1\ ,\vartheta})^2 + 2a^2 k(a,\vartheta) - e^{-\frac{\lambda(a)}{2}} a \hat{Q}_2^{\pm} \Big\} d\vartheta^2 \\
&+2a^2 \Big\{ 2T^{\vartheta}_1 T^{\pm\phi}_1 + \left( T^{\pm\phi}_{2\ ,\vartheta} - 2T^{\tau}_{1\ ,\vartheta} \omega^{\pm}(a,\vartheta) \right) \sin^2\vartheta \\
&+2 \left( T^{\pm\phi}_{1\ ,\vartheta} T^{\vartheta}_1 - T^{\pm\phi}_1 T^{\vartheta}_{1\ ,\vartheta} \right) \cos\vartheta \sin\vartheta \Big\} d\phi d\vartheta \\
&+2 \left\{ a^2 (T^{\pm\phi}_1)^2 \cos^2\vartheta \sin^2\vartheta + a^2 (T^{\vartheta}_1)^2 (1 - 2\sin^2\vartheta) + a^2 T^{\pm\vartheta}_2 \cos\vartheta \sin\vartheta \right. \\
&\left. + \left( 2a^2 k(a,\vartheta) - e^{-\frac{\lambda(a)}{2}} a \hat{Q}_2^{\pm} \right) \sin^2\vartheta \right\} d\phi^2,
\end{aligned}$$

where we have avoided the use of $\pm$ for quantities which already coincide at both sides. Apart from the background quantities $\lambda(a)$, $\nu(a)$ and $\nu'(a)$ this also applies to the first order objects that, in virtue of the matching conditions in Proposition 7, agree on $\Sigma_0$, i.e. $T_1^{\tau\pm}(\tau,\vartheta) = T_1^{\tau}(\tau,\vartheta)$, $T_1^{\vartheta\pm}(\tau,\vartheta) = T_1^{\vartheta}(\tau,\vartheta)$ and $\omega'_\pm(a,\vartheta) = \omega'(a,\vartheta)$.

From equations $[h^{(2)}{}_{\tau\phi}] = 0$ and $[h^{(2)}{}_{\vartheta\phi}] = 0$ we obtain expressions for $[T_2^\phi]_{,\tau}$ and $[T_2^\phi]_{,\vartheta}$ respectively. The integrability conditions are found to be automatically satisfied. The integration leads to

$$[T_2^\phi] = 2b_1(T_1^\tau + \tau T_1^\vartheta \cot\vartheta) + D_2 \tag{5.19}$$

for some constant $D_2$. Likewise, from $[h^{(2)}{}_{\tau\vartheta}] = 0$ and $[h^{(2)}{}_{\vartheta\vartheta}]\sin^2\vartheta - [h^{(2)}{}_{\phi\phi}] = 0$ we obtain, respectively, $[T_2^\vartheta]_{,\tau}$ and $[T_2^\vartheta]_{,\vartheta}$. However, since the equation $[h^{(2)}{}_{\tau\vartheta}] = 0$ also involves $[T_2^\tau]_{,\vartheta}$, this time the integrability condition provides a second order PDE for $[T_2^\tau]$ that contains only derivatives on $\vartheta$. This is integrated to yield

$$[T_2^\tau] = -a^2 \dot{F}(\tau) e^{-\nu(a)} \cos\vartheta + G(\tau) \tag{5.20}$$

for some functions $F(\tau)$, conveniently arranged, and $G(\tau)$. Making use of the expression (5.20) for $[T_2^\tau]$, $[T_2^\vartheta]$ can now be integrated in the form

$$[T_2^\vartheta] = \big(b_1\tau\cos\vartheta(b_1\tau - 2T_1^{+\phi}) + F(\tau) + C_3\big)\sin\vartheta, \tag{5.21}$$

for some constant $C_3$. Now, using (5.21) upon $[h^{(2)}{}_{\vartheta\vartheta}] = 0$ provides an equation for $[\hat{Q}_2]$, explicitly

$$[\hat{Q}_2] = a e^{\lambda(a)/2}\{2\,[k] + (F(\tau) + C_3)\cos\vartheta\}. \tag{5.22}$$

The remaining equation from the equality of the second order first fundamental forms is $[h^{(2)}{}_{\tau\tau}] = 0$. From its second derivative $[h^{(2)}{}_{\tau\tau}]_{,\tau\vartheta} = 0$ we first obtain a third order differential equation for $F(\tau)$ which can be integrated once in order to obtain

$$\ddot{F} = e^{\nu(a)}\nu'(a)(-F + H_1 - C_3)/2a, \tag{5.23}$$

where the constant of integration $H_1$ has been conveniently arranged. Using this relation back into the equation $[h^{(2)}{}_{\tau\tau}]_{,\tau} = 0$ we obtain $\ddot{G} = 0$, and therefore $G(\tau) = -H_0\tau + D_1$ for some constants $H_0$ and $D_1$. Finally, $[h^{(2)}{}_{\tau\tau}] = 0$ provides a relation between $[h]$ and $[k]$, namely $[h] = \frac{1}{2}H_0 + \frac{1}{4}a\nu'(a)\,\{2\,[k] + H_1\cos\vartheta\}$.

We have to impose now the equations for the perturbed second fundamental form,

$[\kappa^{(2)}{}_{ab}] = 0$. The explicit expression of $\kappa^{(2)}{}^{\pm}_{ab}$, included for completeness, is

$$\kappa^{(2)}{}^{\pm}_{ij}dx^i dx^j = \Big\{ -\hat{Q}^{\pm}_{2\,,\tau\tau} + \frac{1}{4}e^{\nu(a)-\lambda(a)}(\lambda_{\pm}{}'(a)\nu'(a) - 2(\nu_{\pm}{}''(a) + \nu'^2(a)))\hat{Q}^{\pm}_2$$
$$+e^{\nu(a)-\lambda(a)/2}\left(2h'^{\pm}(a,\vartheta) + \nu'(a)(2h^{\pm}(a,\vartheta) - m^{\pm}(a,\vartheta) + T_2^{\pm\tau}{}_{,\tau} + (T_1{}^{\tau}{}_{,\tau})^2\right)$$
$$-2e^{-\nu(a)}a(T_1{}^{\vartheta}{}_{,\tau})^2\big) - 2e^{-\lambda(a)/2}a\sin^2\vartheta\left(\omega^{\pm}(a,\vartheta) - T^{\pm\phi}_{1\ ,\tau}\right)\left(a\omega'(a,\vartheta) + \omega^{\pm}(a,\vartheta) - T^{\pm\phi}_{1\ ,\tau}\right)\Big\}\, d\tau^2$$
$$+2\left\{e^{\lambda(a)/2}a\sin^2\vartheta\,(-T_2^{\pm\phi}{}_{,\tau} + T_1{}^{\tau}{}_{,\tau}(a\omega'(a,\vartheta) + 2\omega^{\pm}(a,\vartheta)) + 2\cos\vartheta T_1^{\pm\phi}T_1{}^{\vartheta}{}_{,\tau}\right.$$
$$\left.+(2\cos\vartheta(2\omega^{\pm}(a,\vartheta) - T^{\pm\phi}_{1\ ,\tau} + a\omega'(a,\vartheta)) + \sin\vartheta(a\omega'(a,\vartheta) + 2\omega^{\pm}(a,\vartheta))_{,\vartheta})T_1{}^{\vartheta}\big)\right\} d\tau d\phi$$
$$+2\Big\{ -\hat{Q}^{\pm}_{2\,,\tau\vartheta} + e^{\lambda(a)/2}\left(-aT_2^{\pm\vartheta}{}_{,\tau} + \frac{\nu'(a)e^{\nu(a)}}{2}T_2^{\pm\tau}{}_{,\vartheta}\right) - 2T_1{}^{\vartheta}Q_{1,\tau\vartheta\vartheta} - 2e^{-\lambda(a)/2}aT_1{}^{\vartheta}{}_{,\tau}T_1{}^{\vartheta}{}_{,\vartheta}$$
$$+e^{\nu(a)+\lambda(a)/2}\nu'(a)T_1{}^{\tau}{}_{,\tau}T_1{}^{\tau}{}_{,\vartheta}$$
$$+e^{-\lambda(a)/2}a\sin\vartheta(-\cos\vartheta(T_1{}^{\vartheta 2})_{,\tau} + \sin\vartheta T^{\pm\phi}_{1\ ,\vartheta}(a\omega'(a,\vartheta) + 2\omega^{\pm}(a,\vartheta) - 2T^{\pm\phi}_{1\ ,\tau}))\Big\}\, d\tau d\vartheta$$
$$+\left\{2e^{-\lambda(a)/2}a\sin^2\vartheta(m^{\pm}(a,\vartheta) - 2k^{\pm}(a,\vartheta) - ak^{\pm}{}'(a,\vartheta)) + 4\sin\vartheta\cos\vartheta a e^{\lambda/2}\partial_\vartheta f^{\pm}(a,\vartheta)\right.$$
$$-\frac{1}{2}e^{-\lambda(a)}\sin^2\vartheta(-2 + a\lambda'_{\pm}(a))\hat{Q}^{\pm}_2 - \cos\vartheta\sin\vartheta\hat{Q}^{\pm}_{2\,,\vartheta} - 2e^{-\lambda(a)/2}a\sin\vartheta\cos\vartheta T_2^{\pm\vartheta}(\tau,\vartheta)$$
$$\left.+2a\sin^2\vartheta\cos^2\vartheta(T_1^{\pm\phi})^2 + 4a(\cos^2\vartheta - \sin^2\vartheta)(T_1^{\vartheta})^2\right\} d\phi^2$$
$$+2e^{\lambda/2}a\left\{-\sin^2\vartheta T_2^{\pm\phi}{}_{,\vartheta} + \sin^2\vartheta(2\omega^{\pm}(a,\vartheta) + a\omega'(a,\vartheta))T^{\tau}_{1\,,\vartheta} + (-2T_1^{\vartheta} + \sin 2\vartheta T^{\vartheta}_{1\,,\vartheta})T_1^{\pm\phi}\right.$$
$$\left.-\sin 2\vartheta T_1^{\vartheta}T_1^{\pm\phi}{}_{,\vartheta}\right\} d\phi d\vartheta$$
$$+\left\{2e^{-\lambda(a)/2}a(m^{\pm}(a,\vartheta) - 2k^{\pm}(a,\vartheta) - ak^{\pm}{}'(a,\vartheta) + 2e^{\lambda(a)}\partial^2_\vartheta f^{\pm}(a,\vartheta))\right.$$
$$-\hat{Q}^{\pm}_{2\,,\vartheta\vartheta} - \frac{e^{-\lambda(a)}}{2}(-2 + a\lambda'_{\pm}(a))\hat{Q}^{\pm}_2 - 2e^{-\lambda(a)/2}aT_2^{\pm\vartheta}{}_{,\vartheta}$$
$$+2a(2\cos 2\vartheta(T_1^{\pm\phi})^2 + \sin 2\vartheta(T_1^{\pm\phi})^2{}_{,\vartheta} + (1 - \cos 2\vartheta)(T_1^{\pm\phi}{}_{,\vartheta})^2)$$
$$\left.+e^{\nu(a)-\lambda(a)/2}\nu'(a)(T^{\tau}_{1\,,\vartheta})^2 - 2Q_{1,\tau\vartheta\vartheta}T_1^{\tau} - 2e^{-\lambda(a)/2}a(T^{\vartheta}_{1\,,\vartheta})^2 - 2Q_{1,\vartheta\vartheta\vartheta}T_1^{\vartheta}\right\} d\vartheta^2. \qquad (5.24)$$

Firstly, given that $[\omega'] = 0$, the equations $[\kappa^{(2)}{}_{\vartheta\phi}] = 0$ and $[\kappa^{(2)}{}_{\tau\phi}] = 0$ are automatically satisfied when (5.19) is taken into account. We start with the equation $[\kappa^{(2)}{}_{\tau\vartheta}] = 0$, which after using the explicit form of $[\vec{Z}_2]$ obtained from the equations for the first fundamental form, yields $\dot{F}\left(2 - 2e^{\lambda(a)} - a\nu'(a)\right) = 0$. If $2 - 2e^{\lambda(a)} - a\nu'(a) \neq 0$, we need $\dot{F} = 0$, and therefore, from (5.23) we obtain $F + C_3 = H_1$, which substituted on the above expressions for $[T_2^{\tau}]$, $[T_2^{\vartheta}]$ and $[\hat{Q}_2]$ leads to

$$[T_2^{\tau}] = -H_0\tau + D_1, \qquad (5.25)$$
$$[T_2^{\vartheta}] = \left(b_1\tau\cos\vartheta(b_1\tau - 2T_1^{+\phi}) + H_1\right)\sin\vartheta, \qquad (5.26)$$
$$[\hat{Q}_2] = ae^{\lambda(a)/2}\{(2\,[k] + H_1\cos\vartheta\}. \qquad (5.27)$$

On the other hand, the combination of equations $[\kappa^{(2)}{}_{\vartheta\vartheta}]\sin^2\vartheta - [\kappa^{(2)}{}_{\phi\phi}] = 0$, with $[T_2^{\vartheta}]$ and $[\hat{Q}_2]$ given by (5.26) and (5.27) respectively, yields a second order PDE involving $[k]-$

$[f]$, with derivatives on $\vartheta$ only, which is integrated to obtain $[k] = c_1(\tau)\cos\vartheta + c_2(\tau) + [f]$ for some functions $c_1(\tau)$ and $c_2(\tau)$. However, since $[k]_{,\tau} = [f]_{,\tau} = 0$, we readily have that $c_1(\tau) = c_1$ and $c_2(\tau) = c_2$ must be constants. Now, the equation $[\kappa^{(2)}{}_{\vartheta\vartheta}] = 0$ provides an expression for $[m]$, which left in terms of $[k]$, in particular, can be arranged as equation (5.32) below.

The only remaining equation is given by $[\kappa^{(2)}{}_{\tau\tau}] = 0$. Using (5.32) to substitute $[m]$ in $[\kappa^{(2)}{}_{\tau\tau}] = 0$ we obtain a relation between $[h']$, $[k']$ and $[k]$ (and $\hat{Q}_2^+$). That relation is given explicitly by equation (5.33).

Furthermore, from the above expression for $[\hat{Q}_2]$ we clearly also obtain that the difference $[\hat{Q}_2]$ cannot depend on $\tau$. For the same reason, using the above equations for $[m]$ and $[h']$, and since either $[\lambda'] \neq 0$ or $[\nu''] \neq 0$, then $\hat{Q}_2^+$ (and thus neither $\hat{Q}_2^-$) cannot depend on $\tau$.

In the case $[\lambda'] = [\nu''] = 0$ we can have, in principle, a non vanishing $Q_1(\tau,\vartheta)$. The appearance of $Q_1(\tau,\vartheta)$ in the expressions for $h^{(2)}{}_{ij}$ does not change the procedure to integrate the differences. For that reason, and due to their length, we avoid including the explicit expressions of $h^{(2)}{}_{ij}$ with $Q_1(\tau,\vartheta) \neq 0$. Equations $[h^{(2)}{}_{\tau\phi}] = 0$ and $[h^{(2)}{}_{\vartheta\phi}] = 0$ provide expressions for $[T_2^\phi]_{,\tau}$ and $[T_2^\phi]_{,\vartheta}$, the integrability conditions are automatically satisfied, and the integration leads to the expression

$$\left[T_2^\phi\right] = 2b_1(T_1^\tau + \tau T_1^\vartheta \cot\vartheta) + D_2 - \frac{2}{a}e^{-\lambda(a)/2}b_1\tau Q_1^+,$$

for some constant $D_2$. Note that this is exactly (5.19) plus the term in $Q_1^+$. Now, the remaining equations in the set $[h^{(2)}{}_{ab}] = 0$ show no terms involving $Q_1$. Therefore we obtain the same set of equations (5.20), (5.21), (5.22), (5.23), $G(\tau) = -H_0\tau + D_1$ for some constants $H_0$ and $C_1$, and $[h]$ is given by $[h] = \frac{1}{2}H_0 + \frac{1}{4}a\nu'(a)\{2[k] + H_1\cos\vartheta\}$, as in the case $Q_1 = 0$.

The equation $[\kappa^{(2)}{}_{\tau\vartheta}] = 0$ reads the same as in the $Q_1 = 0$ case, and therefore the condition $\dot{F}(\tau) = 0$, assuming that $2 - 2e^{\lambda(a)} - a\nu'(a) \neq 0$, is just recovered. That again leads to $F + C_3 = H_1$. As a result $[T_2^\tau]$, $[T_2^\vartheta]$ and $[\hat{Q}_2]$, are also given by (5.25), (5.26) and (5.27).

Likewise, the combination of equations $[\kappa^{(2)}{}_{\vartheta\vartheta}]\sin^2\vartheta - [\kappa^{(2)}{}_{\phi\phi}] = 0$ does not depend on $Q_1$ either, and therefore $[k] = c_1\cos\vartheta + c_2 + [f]$ all the same, for some constants $c_1$ and $c_2$. However, the equation $[\kappa^{(2)}{}_{\vartheta\vartheta}] = 0$ does contain a term involving $Q_1$. The expression for $[m]$ in this case is given by

$$\begin{aligned}[m] = a\,[k'] - \frac{1}{4}e^{-\lambda(a)}\,[\lambda'']\,(Q_1)^2 + \frac{1}{4}\,(a\lambda'(a)+2)\,\{2\,[k] + H_1\cos\vartheta\} \\ -\frac{1}{2}(H_1 + 2c_1)e^{\lambda(a)}\cos\vartheta, \qquad (5.28)\end{aligned}$$

which used in $[\kappa^{(2)}{}_{\tau\tau}] = 0$ provides the following expression of $[h']$

$$[h'] = \frac{1}{2}a\nu'(a)\,[k'] - \frac{1}{4}e^{-\lambda(a)}\,[\nu''']\,(Q_1)^2 + \frac{1}{4}\left(a\nu''(a) + \nu'(a)\right)\{2\,[k] + H_1\cos\vartheta\}$$
$$-\frac{1}{4}(H_1 + 2c_1)\nu'(a)e^{\lambda(a)}\cos\vartheta. \tag{5.29}$$

Finally, although equation $[\kappa^{(2)}{}_{\vartheta\phi}] = 0$ is automatically satisfied, in this case the equation $[\kappa^{(2)}{}_{\tau\phi}] = 0$ provides the condition $[\omega'']Q_1 = 0$.

As in the first order case, the constants $D_1$ and $D_2$ can be safely absorbed by using a isomorphic *spacetime* gauge at one (any) side, say $\vec{V}_2^+ = D_1\partial_{t_+} + D_2\partial_{\varphi_+}$, keeping $\vec{S}_1 = 0$. Clearly $\vec{S}_2^+ = \vec{V}_2^+$ and therefore by (3.35) that leads to $\vec{T}_2^+ \to \vec{T}_2^+ - \vec{S}_2^+$ and the second order metric perturbation tensor $K_2{}^+$ is unchanged. We can thus set $D_1 = D_2 = 0$ without loss of generality.

We have thus shown the following result.

**Proposition 8** *Let $(\mathcal{V}, g)$ with $\Sigma_0$ be the static and spherically symmetric background matched spacetime as described in Proposition 7, and assume that (5.16) is satisfied. Let it be perturbed to first order by $K_1{}^\pm$ plus $Q_1^\pm$ and $\vec{T}_1^\pm$ so that (5.14), (5.15), (5.17), (5.18) hold. Consider the second order metric perturbation tensor $K_2{}^\pm$ as defined in (5.3) at either side, plus two unknown functions $\hat{Q}_2^\pm(\tau,\vartheta)$ and two unknown vectors $\vec{T}_2^\pm = T_2^{\pm\tau}(\tau,\vartheta)\partial_\tau + T_2^{\pm\vartheta}(\tau,\vartheta)\partial_\vartheta + T_2^{\pm\phi}(\tau,\vartheta)\partial_\phi$ on $\Sigma_0$.*

*If either $[\lambda'] \neq 0$ or $[\nu''] \neq 0$, so that $(Q_1^\pm =)Q_1 = 0$, the necessary and sufficient conditions that $K_2{}^\pm$ must satisfy to fulfil the second order matching conditions are*

$$[k] = c_1\cos\vartheta + c_2 + [f] \tag{5.30}$$
$$[h] = \frac{1}{2}H_0 + \frac{1}{4}a\nu'(a)\{2\,[k] + H_1\cos\vartheta\} \tag{5.31}$$
$$[m] = a\,[k'] + \frac{1}{4}e^{-\lambda(a)/2}\,[\lambda']\,\hat{Q}_2^+ + \frac{1}{4}\left(a\lambda'_-(a) + 2\right)\{2\,[k] + H_1\cos\vartheta\}$$
$$-\frac{1}{2}(H_1 + 2c_1)e^{\lambda(a)}\cos\vartheta \tag{5.32}$$
$$[h'] = \frac{1}{2}a\nu'(a)\,[k'] + \frac{1}{4}e^{-\lambda(a)/2}\,[\nu'']\,\hat{Q}_2^+ + \frac{1}{4}\left(a\nu''_-(a) + \nu'(a)\right)\{2\,[k] + H_1\cos\vartheta\}$$
$$-\frac{1}{4}(H_1 + 2c_1)\nu'(a)e^{\lambda(a)}\cos\vartheta, \tag{5.33}$$

*for arbitrary constants $c_1$, $c_2$, $H_0$ and $H_1$ and function $\hat{Q}_2^+(\vartheta)$.*

*If $[\lambda'] = 0$ and $[\nu''] = 0$, then $[\omega'']Q_1 = 0$ and the above equations are the same except for two changes in (5.32) and (5.33) given respectively by*

$$[\lambda']\,\hat{Q}_2^+ \to -e^{-\lambda/2}\,[\lambda'']\,(Q_1)^2, \qquad [\nu'']\,\hat{Q}_2^+ \to -e^{-\lambda/2}\,[\nu''']\,(Q_1)^2. \tag{5.34}$$

*In all cases, the relation*

$$\left[\hat{Q}_2\right] = ae^{\lambda(a)/2}\left\{2\left[k\right] + H_1\cos\vartheta\right\} \tag{5.35}$$

*must hold, hence* $\left[\hat{Q}_2\right]$ *cannot depend on* $\tau$*, and the differences* $[\vec{T}_2^{\pm}]$ *satisfy*

$$\begin{aligned} [T_2^{\tau}] &= -H_0\tau, \\ \left[T_2^{\phi}\right] &= 2b_1(T_1^{\tau} + \tau T_1^{\vartheta}\cot\vartheta) - \frac{2}{a}e^{-\lambda(a)/2}b_1\tau Q_1^{+}, \\ \left[T_2^{\vartheta}\right] &= \big(b_1\tau\cos\vartheta(b_1\tau - 2T_1^{+\phi}) + H_1\big)\sin\vartheta. \end{aligned} \tag{5.36}$$

Let us remark that in the $Q_1 \neq 0$ case the corresponding equations for $[m]$ and $[h]$, (5.32) and (5.33) with the corresponding changes (5.34) (see (5.28) and (5.29)) imply that if $[\lambda''] \neq 0$ or $[\nu'''] \neq 0$ then $Q_1$ cannot depend on $\tau$. On the other hand, the condition $[\omega'']Q_1 = 0$ will be automatically satisfied in all cases of interest, once the field equations are imposed, as shown in Chapter 7.

# 6

# Field equations up to second order

In this chapter we present the field equations, in perturbation theory up to second order, corresponding to an axially symmetric, stationary and rigidly rotating perfect fluid ball with no convective motions. We also add the assumption of equatorial symmetry. We indicate, in explicit form, how the field equations in [57] are obtained. However, the set of functions used here to describe the second order perturbations is different from the set used in [57], although they can be put in correspondence by fixing some gauge freedom left.

The equations for the fluid depend on the equation of state, and cannot be analitically solved in general. A further chapter will be devoted to solve some typical models such as polytropes, realistic stars or constant density stars. The vacuum equations can be recovered by setting the energy density and pressure equal to zero. The Einstein's field equations (EFE's from now on) for vacuum do admit analytical solutions, not only for the background which is obviously given by the Schwarzschild solution, but also for the first and second order perturbations. However, in this chapter we will not relate the interior (fluid) and exterior (vacuum) solutions. This will be the purpose of the next chapter, where the combination of the geometrical matching conditions obtained in Chapter 5 and the information provided by the field equations result in a new set of matching conditions particularized to the explicit setting of an isolated fluid ball rotating in equilibrium.

To present the equations we will drop the $+$ and $-$ symbols in most places if they are not necessary. Both regions can be considered to be of perfect fluid type, from which the vacuum case is recovered trivially. To obtain the field equations, we impose that the metrics $\hat{g}_\varepsilon$ satisfy $\hat{G}(\hat{g}_\varepsilon)_{\alpha\beta} = 8\pi\hat{T}_{\varepsilon\alpha\beta}$ for an energy momentum tensor of the form

$$\hat{T}_\varepsilon = (\hat{E}_\varepsilon + \hat{P}_\varepsilon)\hat{\boldsymbol{u}}_\varepsilon \otimes \hat{\boldsymbol{u}}_\varepsilon + \hat{P}_\varepsilon \hat{g}_\varepsilon, \tag{6.1}$$

where $\hat{\boldsymbol{u}}_\varepsilon$ is the (unit) fluid flow, and $\hat{E}_\varepsilon$ and $\hat{P}_\varepsilon$, eigenvalues of $\hat{T}_\varepsilon$, the corresponding mass-energy density and pressure. Note that the fluid vector $\hat{\boldsymbol{u}}_\varepsilon$ and corresponding "hatted" scalars are objects defined, still, on each $(\mathcal{V}_\varepsilon, \hat{g}_\varepsilon)$. All these objects, in *covariant form*, are now pulled back through $\psi_\varepsilon$ down onto $(\mathcal{V}, g)$ (see Chapter 3). That defines the associated families of (tensorial) objects $\boldsymbol{U}_\varepsilon := \psi_\varepsilon^*(\hat{\boldsymbol{u}}_\varepsilon)$ and analogously for $G_\varepsilon$, $T_\varepsilon$, $E_\varepsilon$ and $P_\varepsilon$ on $(\mathcal{V}, g)$, which therefore satisfy

$$G(g_\varepsilon)_{\alpha\beta} = 8\pi T_{\varepsilon\alpha\beta} \tag{6.2}$$

with

$$T_\varepsilon = (E_\varepsilon + P_\varepsilon)\boldsymbol{U}_\varepsilon \otimes \boldsymbol{U}_\varepsilon + P_\varepsilon g_\varepsilon, \tag{6.3}$$

by construction. The $\varepsilon-$derivatives of (6.2) evaluated at $\varepsilon = 0$ provide the first and second order perturbed equations

$$G^{(1)} = 8\pi T^{(1)}, \tag{6.4}$$

$$G^{(2)} = 8\pi T^{(2)}, \tag{6.5}$$

where $G^{(1)}$ and $G^{(2)}$ are described in (3.18) and (3.19) and the perturbations of the energy momentum tensor (6.3), $T^{(1)}$ and $T^{(2)}$ will be specified explicitly in the corresponding sections for the first and second order (see (6.21) and (6.27)). It is worth mentioning that the families of objects do depend on the gauge defined by $\psi_\varepsilon$, and thus also the right and left hand sides of (6.2). However, the equations (6.2) themselves do not depend on the gauge, in the sense that if (6.2) are fulfilled in one gauge, they will be satisfied in any other gauge. Therefore, equations (6.4) and (6.5) do not depend on the choice of gauge. For completeness we include here the transformation of the first and second order perturbations of the energy momentum tensor in covariant form under a spacetime gauge (see Lemma 1 in [79])

$$T^{(1)(g)} = T^{(1)} + \mathcal{L}_{\vec{S}_1} T, \tag{6.6}$$

$$T^{(2)(g)} = T^{(2)} + \mathcal{L}_{\vec{V}_2} T + \mathcal{L}_{\vec{S}_1}\mathcal{L}_{\vec{S}_1} T + 2\mathcal{L}_{\vec{S}_1} T^{(1)}. \tag{6.7}$$

In fact, these apply to any 2-covariant tensor and in particular to the Einstein tensor. Using this, it is straightforward to check that the field equations (6.4), (6.5) are gauge invariant order by order.

On the other hand, the fluid vector in contravariant form can also be pushforwarded through $\psi_\varepsilon^{-1}$, and thus yet obtain another family of vectors $\vec{u}_\varepsilon := d\psi_\varepsilon^{-1}(\hat{\vec{u}}_\varepsilon)$ on $(\mathcal{V}, g)$. Since the normalization $\hat{u}_{\varepsilon\alpha}\hat{u}_\varepsilon{}^\alpha = -1$ holds at each $(\mathcal{V}_\varepsilon, \hat{g}_\varepsilon)$, we obtain $U_{\varepsilon\alpha}u_\varepsilon{}^\alpha = -1$ on $(\mathcal{V}, g)$. This can be shown to be equivalent to $g_{\varepsilon\alpha\beta}u_\varepsilon{}^\alpha u_\varepsilon{}^\beta = -1$, and corresponds to the normalisation condition that $\vec{u}_\varepsilon$ must satisfy. We can take now $\varepsilon$-derivatives and construct the expansion of $\vec{u}_\varepsilon$ as $\vec{u}_\varepsilon = \vec{u} + \varepsilon\vec{u}^{(1)} + \frac{1}{2}\varepsilon^2\vec{u}^{(2)} + \mathcal{O}(\varepsilon^3)$, where all the vector components depend on $r$ and $\theta$.

The absence of convective motions translates onto the condition that $\vec{u}_\varepsilon$ lies on the orbits of the group generated by $\{\vec{\eta}, \vec{\xi}\}$, this is $\vec{u}_\varepsilon \propto \vec{\xi} + \kappa(\varepsilon, r, \theta)\vec{\eta}$ for some function $\kappa$. Rigid rotation demands that $\kappa(\varepsilon, r, \theta)$ does not depend on $\{r, \theta\}$, so that $\vec{u}_\varepsilon$ are proportional to (timelike) Killing vector fields [103], i.e. $\vec{u}_\varepsilon = N(r, \theta, \varepsilon)(\vec{\xi} + \kappa(\varepsilon)\vec{\eta})$ for some function $\kappa(\varepsilon)$, with $N(r, \theta, \varepsilon)$ fixed by the above normalisation. A static background configuration forces $\kappa(0) = 0$, since $\vec{u}$ is parallel to the static Killing vector field $\partial_{t^+}$. Therefore $\kappa(\varepsilon) = \varepsilon\Omega + O(\varepsilon^2)$ for some constant $\Omega$. This constant $\Omega$ is gauge dependent (see below, in Section 6.2). Following [57] we assume that $\varepsilon$ drives a rotational peturbation, so that only odd powers enter $\kappa(\varepsilon)$, so that $\kappa(\varepsilon) = \varepsilon\Omega + O(\varepsilon^3)$. In components this is equivalent to

$$u_\varepsilon^\varphi = \varepsilon\Omega u_\varepsilon^t, \qquad u_\varepsilon^r = u_\varepsilon^\theta = 0, \tag{6.8}$$

up to second order. This (gauge-dependent) constant $\Omega$ differs from the perturbation parameter (which we denote by $\Omega^H$) as defined in [57]. In the present scheme the perturbation parameter $\varepsilon$ has been defined abstractly, a priori independently of the rotation parameter $\Omega$.

The energy density and pressure are expanded as

$$E_\varepsilon = E + \varepsilon E^{(1)} + \frac{1}{2}\varepsilon^2 E^{(2)} + \mathcal{O}(\varepsilon^3), \tag{6.9}$$

$$P_\varepsilon = P + \varepsilon P^{(1)} + \frac{1}{2}\varepsilon^2 P^{(2)} + \mathcal{O}(\varepsilon^3). \tag{6.10}$$

All functions in (6.9) and (6.10) depend on $r$ and $\theta$. We consider the existence of a barotropic equation of state for the $\varepsilon$-family, independent of $\varepsilon$, so that $P_\varepsilon$ is a function of $E_\varepsilon$ alone. Taking $\varepsilon$-derivatives at $\varepsilon = 0$, such relations yield a constraint for the first and second order expansions, which must satisfy, respectively

$$P^{(1)} - \frac{\partial P}{\partial E}E^{(1)} = 0,$$

$$P^{(2)} - \frac{\partial P}{\partial E}E^{(2)} - \frac{\partial^2 P}{\partial E^2}{E^{(1)}}^2 = 0. \tag{6.11}$$

The vacuum equations are obtained by simply setting $E_\varepsilon = P_\varepsilon = 0$.

## 6.1 Field equations of the background

The matter content of the interior region of the background configuration is a perfect fluid, static and spherically symmetric. Its normalized 4-velocity is $\vec{u} = e^{-\nu/2}\partial_t$. The two field equations providing $E$ and $P$ in terms of the metric functions are

$$\lambda' = \frac{1}{r}(1 - e^\lambda) + re^\lambda 8\pi E, \tag{6.12}$$

$$\nu' = \frac{1}{r}(e^\lambda - 1) + re^\lambda 8\pi P, \tag{6.13}$$

while the pressure isotropy condition, that in terms of the covariant tensors is $e^{\lambda(r)}T_{\varphi\varphi} - r^2\sin^2\theta T_{rr} = 0$, yields the equation

$$2r\nu'' + \nu'(r\nu' - 2) - \lambda'(2 + r\nu') + \frac{4}{r}\left(e^{\lambda} - 1\right) = 0. \tag{6.14}$$

Let us recall the definitions of Section 4.2 (which will be useful for the comparison of the expressions here with those in [57]),

$$j(r) := e^{-(\lambda+\nu)/2}, \tag{6.15}$$
$$1 - \frac{2M(r)}{r} := e^{-\lambda}. \tag{6.16}$$

The standard way to solve the background configuration consists of changing from the metric potentials $\{\lambda, \nu\}$ to the functions $\{M, P\}$. The system given by (6.12) and (6.14) is now written in the standard form (see (9) and (10) in [90])

$$\frac{dM(r)}{dr} = 4\pi r^2 E(r), \tag{6.17}$$
$$\frac{dP(r)}{dr} = -\frac{(E(r) + P(r))(M(r) + 4\pi r^3 P(r))}{r(r - 2M(r))}. \tag{6.18}$$

These are the well known TOV equations [90]. They determine the interior configuration provided a barotropic equation of state, that closes the system, and a value of, say, the central energy density. The metric potential $\nu$ is then obtained, up to an additive constant, integrating equation (6.13). Another useful expression for $\nu'$ is obtained by writing (6.14) in terms of $P$ using (6.12)-(6.13), which reads

$$P' = -\frac{\nu'}{2}(E + P). \tag{6.19}$$

In the **vacuum exterior** $(-)$ the field equations (6.14) imply that $M(r_-)$ is a constant, which will be denoted by $M$ as usual, and that

$$e^{-\lambda_-(r_-)} = e^{\nu_-(r_-)} = 1 - \frac{2M}{r_-} \qquad \Rightarrow j(r_-) = 1. \tag{6.20}$$

We will assume $M > 0$.

## 6.2 First order

The first order perturbations are ruled by the field equations (6.4). We proceed first to compute the l.h.s., i.e. the perturbation of the Einstein tensor (3.18). To this aim, we need the perturbation of the Ricci tensor ${R^{(1)}}_{\alpha\beta}$, computed using the formula (3.14) that involves only second derivatives of the $K_1$. The only non vanishing component of ${R^{(1)}}_{\alpha\beta}$ is found to be the $\{t, \varphi\}$. In addition, it is traceless, and taking into account the structure of $K_1$, it is easy to see that the first order perturbation of the Ricci scalar given by (3.16) vanishes. Hence, the only non vanishing component of the linearized Einstein tensor $G^{(1)}$, given in (3.18), is found to be the $\{t, \varphi\}$.

The r.h.s. of (6.4) consist of the first order perturbation of the energy momentum tensor (6.3), which reads

$$ {T^{(1)}}_{\alpha\beta} = (E^{(1)} + P^{(1)})u_\alpha u_\beta + 2(E+P)\left(u^\mu K_{1\mu(\alpha}u_{\beta)} + {u^{(1)}}_{(\alpha}u_{\beta)}\right) + P^{(1)}g_{\alpha\beta} + PK_{1\alpha\beta}. \tag{6.21} $$

Absence of convective motions and rigid rotation (6.8), together with the normalisation condition to first order

$$ 2u_\alpha u^{(1)\alpha} + K_{1\alpha\beta}u^\alpha u^\beta = 0, \tag{6.22} $$

imply $\vec{u}^{(1)} = \Omega e^{-\nu/2}\partial_\varphi$. Therefore, the explicit form of $T_1$ is found to be

$$ \begin{aligned} &{T^{(1)}}_{tt} = -g_{tt}E^{(1)}, \quad {T^{(1)}}_{rr} = g_{rr}P^{(1)}, \quad {T^{(1)}}_{\theta\theta} = g_{\theta\theta}P^{(1)}, \quad {T^{(1)}}_{\varphi\varphi} = g_{\varphi\varphi}P^{(1)}, \\ &{T^{(1)}}_{t\varphi} = -r^2\sin^2\theta(\Omega P + (\Omega - \omega)E). \end{aligned} \tag{6.23} $$

The imposition of the perturbed field equations (6.4) provides in the $\{t, t\}$ component $E^{(1)}(r, \theta) = 0$ and from any of the remaining diagonal components we obtain that $P^{(1)}(r, \theta) = 0$. The component $\{t, \varphi\}$ of (6.4) provides the following equation for the function $\omega$ [57]

$$ \frac{\partial}{\partial r}\left(r^4 j\frac{\partial\omega}{\partial r}\right) + \frac{r^2 je^\lambda}{\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial\omega}{\partial\theta}\right) + 4r^3j'(\omega - \Omega) = 0. \tag{6.24} $$

The equation for the exterior vacuum region $(-)$ is recovered by just setting $j = 1$ in the above.

Given the regularity condition at the origin, the asymptotic behaviour at infinity and the matching conditions (5.14)-(5.15), the functions $\omega^\pm(r_\pm, \theta_\pm)$ can be shown to be functions only of the corresponding radial coordinates. This is in agreement with Hartle's argument in [57]. We provide a full proof for this in Chapter 8, where we show that $\omega^\pm$ are, indeed, functions of $r^\pm$ alone. Hence, in the remaining part of this Chapter we will restrict ourselves to the study of $\omega = \omega(r)$ for the fluid and vacuum regions, so that equation (6.24) becomes

$$ \frac{1}{r^3}\frac{d}{dr}\left(r^4 j\frac{d\omega}{dr}\right) + 4j'(r)(\omega - \Omega) = 0. \tag{6.25} $$

This equation is integrated from the origin outwards, with conditions that ensure that the solution is regular at the origin.

Equation (6.25) in the **vacuum exterior** holds for $j = 1$, and the regular exterior solution is thus

$$\omega^- = \frac{2J}{r_-^3} + \omega_\infty^-,$$

for some constants $J$ [57] and $\omega_\infty^-$. A spacetime gauge driven by $\vec{S}_1{}^- = \omega_\infty^- t\partial_\varphi$ can now be used to remove $\omega_\infty^-$. We thus fix the first order spacetime gauge in the exterior region in order to set

$$\omega^- = \frac{2J}{r_-^3}. \tag{6.26}$$

## 6.3 Second order

The procedure is analogous to the first order. The perturbed field equations to second order are (6.5). The second order perturbation of the Ricci tensor and scalar are computed using the explicit form of $K_2$ given in (5.3) into the formulas (3.15) and (3.17) respectively. With these ingredients, the expression (3.19) provides the second perturbation of the Einstein tensor. It contains five nonvanishing components: the diagonal terms plus the crossed term $\{r, \theta\}$.

The conditions on the fluid flow (6.8) together with the normalisation condition now lead to $\vec{u}^{(2)} = \vec{u}^{(2)t}\partial_t$, where $\vec{u}^{(2)t} = e^{-3\nu/2}\left\{\Omega^2 g_{\varphi\varphi} + 2\Omega K_{1t\varphi} + K_{2tt}/2\right\}$.

The second order perturbation of the energy momentum tensor, obtained as usual from the second derivative $T^{(2)} = \partial_\varepsilon^2 T_\varepsilon|_{\varepsilon=0}$, is

$$\begin{aligned} T^{(2)}{}_{\alpha\beta} &= (E^{(2)} + P^{(2)})u_\alpha u_\beta + 2(E+P)\Big\{\big(K_{1\alpha\mu}K_{1\beta\nu}u^\mu u^\nu + 2u^\mu K_{1\mu(\alpha}u^{(1)}{}_{\beta)} + u^{(1)}{}_\alpha u^{(1)}{}_\beta\big) \\ &+ u^\mu K_{2\mu(\alpha}u_{\beta)} + 2u^\mu_{(1)}K_{1\mu(\alpha}u_{\beta)} + u^{(2)}{}_{(\alpha}u_{\beta)}\Big\} + P^{(2)}g_{\alpha\beta} + PK_{2\alpha\beta}, \end{aligned} \tag{6.27}$$

where we have made use of $E^{(1)} = P^{(1)} = 0$. We have all the ingredients to compute the field equations that to second order (6.5).

A key point in Hartle's model [57] is that the functions in $K_2{}^\pm$ contain only $l = 0, 2$ terms in an angular Legendre polynomial expansion (4.18). We leave the analysis of this assumption for Chapter 8, where we will prove that this is, indeed, the only possible angular structure in $K_2{}^\pm$ given equatorial symmetry and the field equations with the corresponding boundary conditions (ensuring the regularity of the solutions and satisfying the matching conditions) for the second order perturbations. However, in order to carry out the study of the whole angular structure of the perturbations in Chapter 8 we need to compute the field equations that the functions in $K_2$ satisfy in full. To this aim we expand each function in $K_2$ in Legendre polynomials, those corresponding to $l=0,2$ plus the remainder, *orthogonal* ($\perp$) to those two, so that

$$\begin{aligned} h(r,\theta) &= h_0(r) + h_2(r)P_2(\cos\theta) + h^{\perp}(r,\theta), \\ m(r,\theta) &= m_0(r) + m_2(r)P_2(\cos\theta) + m^{\perp}(r,\theta), \\ k(r,\theta) &= k_0(r) + k_2(r)P_2(\cos\theta) + k^{\perp}(r,\theta), \\ f(r,\theta) &= f_2(r)P_2(\cos\theta) + f^{\perp}(r,\theta), \end{aligned} \tag{6.28}$$

where we have defined

$$h_0(r) := \frac{1}{2}\int_{S^2} h\eta_{S^2}, \quad h_2(r) := \frac{5}{2}\int_{S^2} h\eta_{S^2}, \quad h^{\perp}(r,\theta) := h(r,\theta) - h_0(r) - h_2(r)P_2(\cos\theta),$$

so that the orthogonal components clearly satisfy

$$\int_{S^2} h^{\perp}\eta_{S^2} = 0, \quad \int_{S^2} h^{\perp}P_2\eta_{S^2} = 0, \tag{6.29}$$

and analogously for the rest of the functions. Equatorial symmetry is used only to get rid of $l=1$. No further restrictions will be imposed on the $\perp$ functions.

A straightforward calculation shows that the above angular structure assumed on the functions in $K_2$ is inherited, via the field equations (6.5), by the second order energy momentum tensor. In particular

$$\begin{aligned} E^{(2)}(r,\theta) &= E_0^{(2)}(r) + E_2^{(2)}(r)P_2(\cos\theta) + E^{(2)\perp}(r,\theta), \\ P^{(2)}(r,\theta) &= P_0^{(2)}(r) + P_2^{(2)}(r)P_2(\cos\theta) + P^{(2)\perp}(r,\theta). \end{aligned} \tag{6.30}$$

Given that $E^{(1)} = P^{(1)} = 0$, the barotropic character of the equation of state to second order (6.11) translates onto the condition

$$E^{(2)}P' - P^{(2)}E' = 0. \tag{6.31}$$

This relation is automatically satisfied by the $l=2$ and $\perp$ sectors, but it provides an independent equation for the $l=0$ sector.

In order to write down the second order field equations in a convenient and compact form, let us first define the following auxiliary "tilded" functions

$$\tilde{h}_0 := h_0 - \frac{1}{2}r\nu' k_0, \quad \tilde{m}_0 := m_0 - e^{-\lambda/2}\left(e^{\lambda/2}rk_0\right)', \tag{6.32}$$

$$\tilde{h}_2 := h_2 - \frac{1}{2}r\nu' f_2, \quad \tilde{m}_2 := m_2 - e^{-\lambda/2}\left(e^{\lambda/2}rf_2\right)', \quad \tilde{k}_2 := k_2 - f_2, \tag{6.33}$$

$$\tilde{h}^{\perp} := h^{\perp} - \frac{1}{2}r\nu' f^{\perp}, \quad \tilde{m}^{\perp} := m^{\perp} - e^{-\lambda/2}\left(e^{\lambda/2}rf^{\perp}\right)', \quad \tilde{k}^{\perp} := k^{\perp} - f^{\perp}. \tag{6.34}$$

These quantities are clearly invariant under the “radial” gauges class of transformations (5.6) since e.g. both $h - \frac{1}{2}r\nu' k$ and $h - \frac{1}{2}r\nu' f$ are.

We introduce now the above decomposed expressions of the relevant quantities into the field equations (6.5). By construction, the complete set of equations gets decomposed into the $l = 0, 2$ and $\perp$ sectors, which are independent and can thus be considered separately. Our purpose in the next two subsections is to recover and write down the field equations as closely as possible to the expressions presented in Chapter 4, or in Sections VII and VIII in the original reference [57].

## The EFEs in the $l = 0$ sector

The $l = 0$ sector of the field equations (6.5) can be shown to provide the following expressions for the second order energy density and pressure[1]

$$8\pi E_0^{(2)} = \frac{4}{r^2}\left(re^{-\lambda}\tilde{m}_0\right)' + \frac{8}{3}rjj'(\omega-\Omega)^2 - \frac{1}{3}j^2r^2\omega'^2 + 16\pi rE'k_0, \tag{6.35}$$

$$8\pi P_0^{(2)} = \frac{4}{r^2}\left\{e^{-\lambda}r\tilde{h}_0' - \tilde{m}_0\left(8\pi r^2 P + 1\right)\right\} + \frac{1}{3}r^2j^2\omega'^2 + 16\pi rP'k_0, \tag{6.36}$$

plus an equation for $\tilde{h}_0''$ of the form $\tilde{h}_0'' = F_1(\tilde{h}_0', \tilde{m}_0', \tilde{m}_0)$. A convenient auxiliary definition of the second order pressure is given by

$$\tilde{\mathcal{P}}_0 := \frac{P_0^{(2)} - 2rP'k_0}{2(E+P)} = \frac{P_0^{(2)}}{2(E+P)} + \frac{k_0}{r-2M(r)}\left(M(r) + 4\pi r^3 P\right), \tag{6.37}$$

where (6.18) has been used in the equality. This function is well defined at points where $E + P = 0$ (see below), and corresponds to the ($l = 0$ part of the) “pressure perturbation factor” as defined in (4.26) in Chapter 4, or equation (87) in [57].

On the other hand, the $l = 0$ part of equation (6.31), i.e. $E_0^{(2)}P' - P_0^{(2)}E' = 0$, combined with (6.35), yields a direct relation between $P_0^{(2)}$ and $m_0'$, which written in terms of “tilded” quantities reads

$$\left(re^{-\lambda}\tilde{m}_0\right)' = 4\pi r^2\frac{E'}{P'}(E+P)\tilde{\mathcal{P}}_0 + \frac{1}{12}j^2r^4\omega'^2 - \frac{2}{3}r^3jj'(\omega-\Omega)^2. \tag{6.38}$$

[1]These two equations correspond to (93) together with (95) and (94) along with (96) in [57], that is equations (4.24) and (4.25) in Chapter 4. Note that a global 2 factor on the right hand side here comes from the definitions (6.9) and (6.10) as compared with the definition of $\Delta G$ in Hartle's model [57] and Chapter 4, which already contains the $\varepsilon^2$ and 1/2 factors.

Now, the aforementioned equation for $\tilde{h}_0''$ can be rewritten, using (6.36) and (6.38) —also (6.16) to substitute the function $\lambda$ by $M$—, as a first order ODE for $\tilde{\mathcal{P}}_0$, that reads

$$\begin{aligned}\tilde{\mathcal{P}}_0{}' &= -\frac{4\pi(E+P)r^2}{r-2M(r)}\tilde{\mathcal{P}}_0 - \frac{(re^{-\lambda}\tilde{m}_0)r^2}{(r-2M(r))^2}\left(8\pi P+\frac{1}{r^2}\right) \\ &\quad + \frac{r^4 j^2}{12(r-2M(r))}\omega'^2 + \frac{1}{3}\left(\frac{r^3 j^2(\omega-\Omega)^2}{r-2M(r)}\right)'. \end{aligned} \qquad (6.39)$$

The set of functions that determines the $l=0$ sector can thus be taken to be $\{\tilde{\mathcal{P}}_0, \tilde{m}_0\}$, which satisfies the system (6.38), (6.39) given regularity conditions at the origin $r=0$. Equation (6.36) can be rewritten as

$$\tilde{h}_0' - re^{\lambda}\tilde{m}_0\left(8\pi P+\frac{1}{r^2}\right) = 4\pi r e^{\lambda}(E+P)\tilde{\mathcal{P}}_0 - \frac{1}{12}e^{\lambda}r^3 j^2\omega'^2. \qquad (6.40)$$

It is now trivial to check that (see (4.26) in Chapter 4, or (90) in [57])

$$\tilde{\mathcal{P}}_0 + \tilde{h}_0 - \frac{1}{3}r^2 e^{-\nu}(\omega-\Omega)^2 = \mu, \qquad (6.41)$$

for some constant $\mu$, is a first integral of (6.39) and (6.40). This relation shows, in particular, that $\tilde{\mathcal{P}}_0$ is well defined in $r_+ \in [0,a]$. The constant $\mu$ is identified in [57] as the second order to background ratio of the constant injection energy. In analogy with the Newtonian potential, $\tilde{h}_0$ (and thus $h_0$) is determined up to an arbitrary additive constant. This constant will be determined once a condition at infinity plus some continuity across the boundary of the body are imposed. We will discuss that below. Once that is fixed, the value of $\mu$ still depends on one factor, that is, the conditions one may impose on $\tilde{\mathcal{P}}_0$ at the origin. The latter depends on how one sets the value of the pressure (and thus of the energy density) of the rotating configuration at the origin with respect to that of the static configuration. We are interested in computing the perturbations in terms of the central energy density and according to the discussion in Chapter 4 regarding the computation of the change in mass, we impose $\tilde{\mathcal{P}}_0(0)=0$.

The fact that $k_0$ is "pure gauge" translates onto the fact that it does not enter the set of equations, and it is therefore not determined. The quantitites $E_0^{(2)}$ and $P_0^{(2)}$ are gauge dependent, and can only be computed, from (6.35) and (6.36) respectively, once $k_0$ is specified, i.e. by fixing the 'radial' gauge. Due to (6.7), under a second order gauge transformation driven by $\vec{V}_2 = 2\xi\partial_r$ and $\vec{S}_1 = 0$, we have $P^{(2)(g)} = P^{(2)} + 2\xi P'$ and analogously for $E^{(2)}$. Given that under the same change, we have $k_0{}^{(g)} = k_0 + \xi/r$ (see (5.6)). Therefore, the quantities independent of that choice, and thus the relevant ones, correspond to $E_0^{(2)} - 2E'rk_0$ and $P_0^{(2)} - 2P'rk_0$. This is the motivation for the introduction of the auxiliary function $\tilde{\mathcal{P}}_0$.

The equations for $\{\tilde{h}_0^-, \tilde{m}_0^-\}$ in the **vacuum exterior** are obtained by using (6.20) and the first order solution (6.26) in equations (6.35), (6.36) with their left hand sides and $P$ and $E$ set to zero. The general solutions are given by

$$r_- e^{-\lambda(r_-)} \tilde{m}_0^-(r_-) = \delta M - \frac{J^2}{r_-^3}, \tag{6.42}$$

$$\tilde{h}_0^-(r_-) - \tilde{h}_{0\infty} = -\frac{\delta M}{r_- - 2M} + \frac{J^2}{r_-^3(r_- - 2M)}, \tag{6.43}$$

where $\delta M$ is an arbitrary constant and $\tilde{h}_{0\infty}$ corresponds to the freedom of shifting the gravitational potential. It is common to choose $\tilde{h}_{0\infty} = 0$, so that $\tilde{h}_0$ vanishes at infinity. This is equivalent to a spacetime gauge driven by $\vec{V}_2 = 2\tilde{h}_{0\infty} t \partial_t$ as shown in Section 5.1. As mentioned above, the function $k_0^-$ remains undetermined.

## The EFEs in the $l = 2$ and $\perp$ sectors

The Einstein tensor to second order contains five nontrivial components. Hence, apart from the two field equations that provide the energy density and pressure to second order, the $l = 2$ and $\perp$ sectors provide three field equations. We take one of them directly as ${G^{(2)}}_{r\theta} = 8\pi {T^{(2)}}_{r\theta}$ and for the other two we use the combinations $e^{\lambda(r)}({G^{(2)}}_{\varphi\varphi} - {T^{(2)}}_{\varphi\varphi}) - r^2 \sin^2\theta({G^{(2)}}_{rr} - {T^{(2)}}_{rr}) = 0$ and $({G^{(2)}}_{\varphi\varphi} - {T^{(2)}}_{\varphi\varphi}) - \sin^2\theta({G^{(2)}}_{\theta\theta} - {T^{(2)}}_{\theta\theta}) = 0$. These result convenient because the perturbation of the pressure does not enter them.

Let us start considering the last of the three equations, which explicitly reads

$$\sin\theta \frac{\partial}{\partial\theta}\left(\frac{\partial_\theta\left(\tilde{h}^\perp + \tilde{m}^\perp + (\tilde{h}_2(r) + \tilde{m}_2(r) + f_\omega(r))P_2(\cos\theta)\right)}{\sin\theta}\right) = 0,$$

where we have collected the first order terms in the function

$$f_\omega(r) := -\frac{r^4 j^2}{6}\omega'^2 + \frac{r^3\,(j^2)'}{3}(\Omega - \omega)^2. \tag{6.44}$$

After taking into account that $\tilde{h}^\perp$ and $\tilde{m}^\perp$ are orthogonal to $P_0$ and $P_2$, this provides the following two equations

$$\tilde{h}_2(r) + \tilde{m}_2(r) + f_\omega(r) = 0. \tag{6.45}$$

$$\tilde{h}^\perp + \tilde{m}^\perp = 0. \tag{6.46}$$

We consider now the equation ${G^{(2)}}_{r\theta} = 0$. After an integration with respect to the angular coordinate it reads

$$\begin{aligned}&(2 - r\nu')(\tilde{h}_2(r)P_2(\cos\theta) + \tilde{h}^\perp) + (2 + r\nu')\left(\tilde{m}_2(r)P_2(\cos\theta) + \tilde{m}^\perp\right)\\ &-2r\left((\tilde{h}^{\perp\prime} + \tilde{k}^{\perp\prime}) + (\tilde{h}_2' + \tilde{k}_2')P_2(\cos\theta)\right) = 0,\end{aligned} \tag{6.47}$$

where an arbitrary function of integration is found to vanish identically after a projection to $P_0$. After having used equations (6.45) and (6.46) in order to get rid of the functions $\tilde{m}_2$ and $\tilde{m}^\perp$, (6.47) provides the following pair of equations

$$\begin{aligned} \tilde{h}_2' + \tilde{k}_2' + \nu'\tilde{h}_2 + \frac{r\nu'}{2} f_\omega &= 0, \qquad (6.48) \\ \partial_r(\tilde{h}^\perp + \tilde{k}^\perp) + \nu'\tilde{h}^\perp &= 0. \qquad (6.49) \end{aligned}$$

Finally, we consider the last independent equation, corresponding to $e^{\lambda(r)}({G^{(2)}}_{\varphi\varphi} - {T^{(2)}}_{\varphi\varphi}) - r^2\sin^2\theta({G^{(2)}}_{rr} - {T^{(2)}}_{rr}) = 0$. This is also separated into three contributions corresponding to the $l = 0, 2$ and orthogonal $\perp$ sectors and thus, it results in three independent equations, one for each sector. The equation for $l = 0$ yields the relation $\tilde{h}_0'' = F_1(\tilde{h}_0', \tilde{m}_0', \tilde{m}_0)$, mentioned in the previous section. The corresponding equations for $l = 2$ and orthogonal $\perp$ sectors are

$$\begin{aligned} 2\left\{r^2\nu'\tilde{h}_2' + 4e^\lambda\tilde{k}_2 + (r\nu'(-1 + r\nu') - r\lambda' + 6e^\lambda - 2)\tilde{h}_2\right\} + (2e^\lambda - (r\nu')^2)f_\omega &= 0, \\ r^2\nu'\left(\partial_r\tilde{h}^\perp - \left(\frac{j'}{j} + \frac{\nu''}{\nu'}\right)\tilde{h}^\perp\right) - e^\lambda\left(\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta(\tilde{h}^\perp + \tilde{k}^\perp)\right) + 2(\tilde{h}^\perp + \tilde{k}^\perp)\right) &= 0. \end{aligned} \qquad (6.50)$$

In order to derive the first of the equations we have used (6.45) to get rid of $\tilde{m}_2$ and (6.48) and its first derivative to substitute $\tilde{k}_2'$ and $\tilde{k}_2''$.

Finally, we check that the barotropic EOS does not provide any other independent equation here, contrary to what happens in $l = 0$. The expressions for the energy density and pressure in the sector $l = 2$ can be written as

$$\begin{aligned} E_2^{(2)} - 2E'rf_2 &= \frac{4E'}{3\nu'}\left(3\tilde{h}_2 + e^{-\nu}r^2(\omega - \Omega)^2\right), \qquad (6.51) \\ P_2^{(2)} - 2P'rf_2 &= -\frac{2}{3}(E + P)\left(3\tilde{h}_2 + e^{-\nu}r^2(\omega - \Omega)^2\right) \\ &= \frac{4P'}{3\nu'}\left(3\tilde{h}_2 + e^{-\nu}r^2(\omega - \Omega)^2\right), \qquad (6.52) \end{aligned}$$

where in the last equality identities from the background have been used. Note that the relation (6.31) holds for $E_2^{(2)}$ and $P_2^{(2)}$ as well as for the combinations $E_2^{(2)} - 2E'rf_2$ and $P_2^{(2)} - 2P'rf_2$.

The $\perp$ part of the pressure and the energy density, after some manipulations using the three field equations and background identities, are found to be related to the function $h^\perp$ by

$$E^{(2)\perp}(r,\theta) - 2E'rf^\perp = \frac{4E'}{\nu'}\tilde{h}^\perp, \quad P^{(2)\perp}(r,\theta) - 2P'rf^\perp = \frac{4P'}{\nu'}\tilde{h}^\perp. \qquad (6.53)$$

Hence, it is straightforward to check that $P^{(2)\perp}(r,\theta)E'(r) - E^{(2)\perp}(r,\theta)P'(r) = 0$ holds identically and we verify that the barotropic EOS equation does not add any new information. The expression of the pressure can be written as a hydrostatic equilibrium first integral for the orthogonal $\perp$ sector using the background relation (6.19), that provides

$$\frac{P^{(2)\perp}(r,\theta) - 2P'rf^{\perp}}{2(E+P)} + \tilde{h}^{\perp} = 0. \tag{6.54}$$

Let us conclude this section with a brief summary of the field equations for the $l = 2$ and $\perp$ sectors.

### The EFEs in the $l = 2$ sector

We include this summary in order to present the equations for $l = 2$ in the same fashion as in Chapter 4. After the definitions

$$v := h_2 + k_2, \qquad \tilde{v} = v - \left(1 + \frac{r\nu'}{2}\right) f_2, \tag{6.55}$$

the whole set of equations (6.45), (6.48) plus the first equation in (6.49) are equivalent to the system

$$\begin{aligned}
\tilde{v}' &= -\nu'\tilde{h}_2 + \left(\frac{1}{r} + \frac{\nu'}{2}\right)\left(-\frac{2}{3}r^3 jj'(\omega-\Omega)^2 + \frac{1}{6}j^2 r^4 \omega'^2\right), && (6.56)\\
\tilde{h}_2' &= \left\{-\nu' + \frac{r}{(r-2M(r))\nu'}\left(8\pi(E+P) - \frac{4M(r)}{r^3}\right)\right\}\tilde{h}_2 - \frac{4\tilde{v}}{r\nu'(r-2M(r))} \\
&\quad + \frac{1}{6}\left(\frac{1}{2}r\nu' - \frac{1}{(r-2M(r))\nu'}\right) r^3 j^2 \omega'^2 \\
&\quad - \frac{1}{3}\left(\frac{1}{2}r\nu' + \frac{1}{(r-2M(r))\nu'}\right) r^2 (j^2)'(\omega-\Omega)^2 && (6.57)
\end{aligned}$$

plus the algebraic equation for $\tilde{m}_2$

$$\tilde{m}_2 = \frac{1}{6}r^4 j^2 \omega'^2 - \frac{1}{3}r^3 (j^2)'(\omega-\Omega)^2 - \tilde{h}_2. \tag{6.58}$$

The convenient "pressure perturbation factor" in this case (see (6.37) for $l = 0$ to compare) corresponds to the following definition

$$\tilde{\mathcal{P}}_2 := \frac{P_2^{(2)} - 2P'rf_2}{2(E+P)}, \tag{6.59}$$

so that (6.52) just reads

$$\tilde{\mathcal{P}}_2 + \tilde{h}_2 + \frac{1}{3}e^{-\nu} r^2 (\omega-\Omega)^2 = 0. \tag{6.60}$$

This corresponds to (91) in [57], and, together with the above (6.41) form the $l = 0$ and $l = 2$ parts of the first integral $\mu$, (86) in [57] (named $\gamma$ there).

The interior region is thus determined by the solution of the pair $\{\tilde{h}_2^+, \tilde{k}_2^+\}$ to the system (6.56), (6.57) given regularity conditions at the origin $r_+ \to 0$, up to an arbitrary constant, say $A'$. Then, $\tilde{m}_2^+$ is directly obtained from (6.58). The function $f_2(r)$ does not enter the equations, and thus it is, as expected, pure gauge.

In the **vacuum exterior** region only equations (6.56)-(6.58) apply. Using (6.20), so that in particular $P = 0$, and (6.26), and imposing regularity at $r_- \to \infty$, the whole set of exterior functions $\{\tilde{h}_2^-, \tilde{k}_2^-, \tilde{m}_2^-\}$ is integrated and read

$$\tilde{h}_2^- = AQ_2^2\left(\frac{r_-}{M} - 1\right) + \frac{J^2}{r_-^3}\left(\frac{1}{M} + \frac{1}{r_-}\right), \tag{6.61}$$

$$\tilde{k}_2^- + \tilde{h}_2^- = A\left\{\frac{2M}{\sqrt{r_-(r_- - 2M)}}Q_2^1\left(\frac{r_-}{M} - 1\right)\right\} - \frac{J^2}{r_-^4}, \tag{6.62}$$

$$\tilde{m}_2^- = -AQ_2^2\left(\frac{r_-}{M} - 1\right) + \frac{J^2}{r_-^3}\left(\frac{1}{M} - \frac{5}{r_-}\right), \tag{6.63}$$

where $Q_l^m(x)$ stand for the associated Legendre functions of the second kind, and $A$ is an arbitrary constant. The constants $A'$ and $A$ are to be determined once the relations between $\{\tilde{h}_2^+, \tilde{k}_2^+\}$ and $\{\tilde{h}_2^-, \tilde{k}_2^-\}$ on the matching hypersurface $\Sigma$ are imposed.

### The EFEs in the orthogonal $\perp$ sector

The radial derivative of equation (6.49) can be used with equation (6.50) to obtain a PDE for the single function $\tilde{v}^\perp := \tilde{h}^\perp + \tilde{k}^\perp$. We define the auxiliary function

$$g(r) := 1/(j\nu'^2) \tag{6.64}$$

in order to write it as

$$\frac{1}{g(r)}\partial_r(g(r)\partial_r\tilde{v}^\perp(r,z)) + \frac{e^\lambda}{r^2}\partial_z\left((1-z^2)\partial_z\tilde{v}^\perp(r,z)\right) + \frac{2e^\lambda}{r^2}\tilde{v}^\perp(r,z) = 0.$$

An alternative form of this equation is

$$\Delta_\gamma\tilde{v}^\perp(r,z) = -\frac{2e^\lambda}{r^2}\left(\frac{e^\lambda g(r)}{r^2}\right)^{-2}\tilde{v}^\perp(r,z), \tag{6.65}$$

where $\Delta_\gamma$ is the Laplacian operator associated to the auxiliary metric $\gamma$

$$\gamma = \left(\frac{e^{\lambda/2}g(r)}{r^2}\right)^2(e^\lambda dr^2 + r^2 d\Omega^2). \tag{6.66}$$

After defining

$$V(r) := \frac{2e^{\lambda}}{r^2}\left(\frac{e^{\lambda}g(r)}{r^2}\right)^{-2},$$

equation (6.65) reads

$$(\Delta_\gamma + V(r))\tilde{v}^{\perp}(r,z) = 0. \tag{6.67}$$

The two remaining functions $\tilde{h}^{\perp}$ and $\tilde{m}^{\perp}$ are related to the radial derivative of $\tilde{v}^{\perp}$ by

$$\tilde{h}^{\perp} = -\tilde{m}^{\perp} = -\frac{1}{\nu'}\partial_r \tilde{v}^{\perp}. \tag{6.68}$$

In the **vacuum exterior**, the background solutions (6.20) provide $g = r_-(r_- - 2M)/2M$ and the function $\tilde{v}^{\perp}$ satisfies equation (6.67) for the following metric $\gamma^-$ and potential $V^-$

$$\gamma^- = \frac{r_-(r_- - 2M)^3}{16M^4}\left(\frac{1}{1-\frac{2M}{r_-}}dr_-^2 + r_-^2 d\Omega^2\right), \quad V^- = \frac{8M^2}{r_-(r_- - 2M)}. \tag{6.69}$$

The algebraic equations (6.68) read, after using the vacuum solution for $\nu$ (6.20)

$$\tilde{h}^{\perp -} = -\tilde{m}^{\perp -} = -\frac{r_-(r_- - 2M)}{2M}\partial_{r_-}\tilde{v}^{\perp -}. \tag{6.70}$$

# 7

# Matching of a perfect fluid with vacuum to second order

In this Chapter we combine the matching conditions obtained in Chapter 5 with the field equations for the perfect fluid and vacuum of Chapter 6.

Apart from the boundary conditions for the functions in the metric $g$ and in the perturbation tensors $K_1$ and $K_2$, the shape of the boundary of the star is determined.

In the first order perturbations, the field equations do no provide much more information. In fact, the only field equation provides just the jump in the second derivative of $\omega$. However, the knowledge of the explicit exterior solution helps us to clarify some aspects about the spacetime gauge to this order. Also, we obtain that the hypersurface remains unchanged, unless the energy density vanishes at the boundary. In this case the first order deformation is determined by the second order matching.

Regarding the second order perturbations, the situation becomes more involved. In first place, we formulate the matching conditions given in Chapter 5 for the *tilded* functions introduced in Chapter 6. This is convenient because it allows us to present a set of matching conditions in which the spurious degrees of freedom from the "radial" family of spacetime gauges do not appear (they have been absorbed by the *tilded* functions). Secondly, we determine the jumps for the relevant functions, i.e. $\tilde{h}_0$ (and $\tilde{h}'_0$), $\tilde{m}_0$, $\tilde{h}_2$ (and $\tilde{h}'_2$), $\tilde{k}_2$ (and $\tilde{k}'_2$) and $\tilde{m}_2$. Finally we find that the deformation of the surface of the star presents a behaviour similar to the one found at first order. Its determination order by order is only possible if the energy density does not vanish at the surface $\Sigma_0$. We present these results at the end of this Chapter in the form of a Theorem and afterwards we discuss the way to determine the surface. Let us stress that in the formulation of Theorem 5 we assume that the functions $\omega^{\pm}$ do not depend on the respective angular coordinate and that the $\perp$ sector in the second order perturbation vanishes. Nevertheless, we will deal with these assumptions in Chapter 8, where we show that these restrictions are not needed, since they arise as a consequence of the matching.

## 7.1 Background

The matching conditions of the background (5.13) involve the functions $\nu$, $\lambda$ and $\nu'$. The condition $[\lambda'] = 0$ translates, via (6.16) and (6.20), into the equality of the interior mass $M(a)$ and the exterior constant $M$. Precisely, the exterior solutions (6.20) and the matching of the background (5.13) imply that having fixed the exterior potential $\nu_-(r_-)$ by (6.20), the interior potential loses the freedom of a constant shift since it is fixed by the condition

$$\nu_+(a) = -\lambda_+(a) = \log\left(1 - \frac{2M}{a}\right). \tag{7.1}$$

The remaining condition in (5.13) enforces the normal derivatives of the potentials $\nu^\pm$ to agree on $\Sigma_0$

$$\nu'_+(a) = \nu'_-(a) = \frac{2M}{a(a-2M)} =: \nu'(a). \tag{7.2}$$

Equation (6.13) relates this (vanishing) jump with the pressure, so that we immediately find

$$[\nu'] = 8\pi a e^{\lambda(a)}[P] = 0. \tag{7.3}$$

Finally, the two remaining independent field equations (6.12), (6.14) combined with the matching conditions (5.13) allow us to express the differences of the derivative of the functions of the metric in terms of the fluid variables

$$[\lambda'] = 8\pi a e^{\lambda(a)}[E] = 8\pi \frac{a^2}{a-2M}[E], \tag{7.4}$$

$$\begin{aligned}[\nu''] &= \left(1 + \frac{a\nu'(a)}{2}\right)\frac{[\lambda']}{a} = \left(1 + \frac{a\nu'(a)}{2}\right) e^{\lambda(a)} 8\pi [E] \\ &= 8\pi \frac{a(a-M)}{(a-2M)^2}[E].\end{aligned} \tag{7.5}$$

Note first that the jumps in $[\lambda']$ and $[\nu'']$ are not independent. On the other hand observe that for a vacuum exterior, the difference $[E]$ corresponds to the value of the interior energy density $E_+$ on $\Sigma_0$, this is $[E] = E_+(a)$. We just prefer to keep $[E]$ in some expressions for the sake of generality, since they apply in the matching of two fluids, and the notation is, in fact, more compact.

It must be stressed that whereas the matching condition (7.3) implies, for a vacuum exterior, that $P(r_+)$ must vanish on the embedded $\Sigma_0$, the energy density $E(a)$ at the boundary stays free, *a priori*. Its value will be determined, if any, by the equation of state.

For later use, it is easy to show that (5.13) imply that $[j] = 0$, so that $j(a) = 1$, and

$$[j'] = -1/2[\lambda'] \tag{7.6}$$

by construction.

Finally, let us consider the particular case for which $[E] = 0$, so that $E_+(a) = 0$. The matching conditions are obtained directly from (7.3)-(7.5) above, but the next order derivatives of the metric potentials are relevant in order to compute the perturbative matching conditions. Hence, we include them here for completeness

$$[\nu'] = ae^{\lambda(a)} 8\pi [P] = 0, \tag{7.7}$$

$$[\lambda'] = [\nu''] = 0, \tag{7.8}$$

$$[\lambda''] = 8\pi a e^{\lambda}[E'] = 8\pi \frac{a^2}{a-2M}[E'], \tag{7.9}$$

$$[\nu'''] = \frac{1}{a}\left(1 + \frac{a\lambda'(a)}{2}\right)[\lambda''] = 8\pi \frac{a(a-3M)}{(a-2M)^2}[E']. \tag{7.10}$$

## 7.2 First order

Recall that from Proposition 7 we already have $[\omega] = b_1$ and $[\omega'] = 0$. Using these matching conditions and the field equation (6.25) and (7.6) we obtain that

$$[\omega''] = [\lambda']\left(\frac{1}{2}\omega'(a) + \frac{2}{a}(\omega^+(a) - \Omega)\right). \tag{7.11}$$

Regarding the determination of the deformation, the condition (5.16) of Proposition 7 is now equivalent to $M \neq 0$. The remark made after Proposition 7 can be now stated in terms of a physical property of the interior and exterior background configuration: whenever there is a jump in the energy density at the surface, $Q_1^-(= Q_1^+)$ must vanish necessarily by (5.18). However, if $[E] = 0$ the function $Q_1(\tau, \vartheta)$ is not determined, in principle, and enters the second order. Nevertheless, as shown in Section 7.3 when analysing the determination of the surface of the rotating star at second order, $Q_1$ will necessarily vanish if $[E'] \neq 0$. In Section 7.5 the whole case $[E] = 0$ is discussed.

### On gauges at first order

We discuss next the meaning of the constant $b_1$ in (5.14), how it is related with gauges, and its role on the determination of the rotation of the perfect fluid star. Consider a spacetime gauge change in either $(\mathcal{V}^\pm, g^\pm)$ defined by $\vec{S}_1 = Ct\partial_\varphi$ (we drop the $\pm$ for clarity, the two $C^\pm$ being independent). The rules of transformation of the first order metric perturbation tensor (3.23), the energy momentum tensor (6.6), and of the first order deformation vector (3.34) imply, respectively, $\omega^g = \omega - C$, $\Omega^g = \Omega - C$ and $b_1^g = b_1 - C$. First, note that $\omega^+ - \Omega$ is independent with respect to that gauge. This quantity is essentially the $\tilde{\omega}$ (up to a sign) defined by Hartle in [57] (see Chapter 4).

As discussed, the first order matching conditions are invariant under such spacetime gauges (at either or both sides, with corresponding $C^+$ and $C^-$), that is, the first order matching conditions (5.14), (5.15), (5.17) and (5.18) transform to just the same expressions with $^g$ superscripts.

This first order gauge at either side $\pm$ is fixed (and completely fixed) once the value of the respective function $\omega^\pm$ is fixed at some point (or infinity). The equation for $\omega^-$ is usually integrated in the exterior vacuum region assuming that $\omega^-$ vanishes at $r_- \to \infty$. By doing that $\partial_{t^-}$ is chosen to represent the "right" observer at infinity. At infinity, the vector $\partial_{t^-}$ is thus assumed to be both unit and orthogonal, with respect to $g_\varepsilon$ to first order, to the axial Killing vector $\partial_{\varphi^-}$. The exterior choice of gauge thus fixes $\omega^-$, and it is given by (6.26).

Regarding the interior region, the above spacetime gauge for some $C^+$ can then be used to get rid of one of the two constants that describe the configuration at first order, either $b_1$ or $\Omega$, but clearly not both. The transformations of $b_1$ and $\Omega$ suggest building a quantity defined on $\Sigma_0$ as

$$\Omega_\infty = \Omega - b_1, \tag{7.12}$$

invariant under the gauge $\vec{S}_1$. The meaning of this constant is the following. $\Omega$ defines the rotation of the fluid flow with respect to the interior observer $\partial_{t^+}$, and $b_1$ determines the tilt on $\Sigma_0$ between that interior observer $\partial_{t^+}$ and the (already fixed) exterior observer $\partial_{t^-}$, explicitly $\partial_{t^+}|_{\Sigma_0} = \partial_{t^-}|_{\Sigma_0} - \varepsilon b_1 \partial_\varphi|_{\Sigma_0}$. The difference $\Omega_\infty$ thus describes the tilt of the fluid flow with respect to the continuous extension of the exterior observer to the interior, and thence, measures the rotation of the fluid with respect to the unit non-rotating observer at infinity.

The value of the "invariant" quantity $\tilde{\omega}(r) := \omega^+(r) - \Omega$ at the boundary can then be expressed thanks to the condition (5.14) as $\omega^+(a) - \Omega = \omega^-(a) - \Omega_\infty$, i.e.

$$\tilde{\omega}^+(a) = 2J/a^3 - \Omega_\infty. \tag{7.13}$$

This yields the desired relation between the $\tilde{\omega}^+(a)$, integrated via (6.24) from the origin, the constant $J$ and the rotation of the star, thus described by $\Omega_\infty$.

In [57] the function $\omega$ is assumed to be "continuous" by construction. In the present general setting that corresponds to a choice of gauge in the interior region for which $b_1 = 0$, and therefore $\Omega(= \Omega_\infty)$ corresponds indeed to the rotation of the fluid as measured by the unit exterior observer. The relation between $\Omega$ and $\Omega^H$ is thus explicitly given by $\Omega^H = \varepsilon\Omega_\infty$.

In contrast, in [16] the gauge in the interior is chosen so that the interior observer $\partial_{t^+}$ moves with the fluid, i.e. $\Omega = 0$ (comoving gauge). Thereby, since the freedom one may have in the interior driven by $\vec{S}_1$ has been already fixed, the price to pay is a rotation in the matching hypersurface given by the constant $b_1$, which corresponds to the parameter $-c_4\Omega$ in [16], so that $\Omega_\infty$ corresponds to "$c_4\Omega$" there.

## 7.3 Second order

We particularize first the matching conditions given in Proposition 8 for the $l = 0, 2$ and orthogonal $\perp$ sectors of the angular expansion of the perturbation functions (6.28) at both sides. The field equations in the background allow us to express the differences $[\lambda']$ and $[\nu'']$ in terms of $[E]$ by direct use of (7.4) and (7.5). However, we will not use those relations in some places, nor the explicit form of $\nu_-(r_-)$ in the exterior, to keep more compact expressions. Let us recall that condition (5.16) now just reads $M \neq 0$ given the exterior is vacuum.

Clearly, for all pairs $f^\pm(r_\pm, \theta_\pm)$ such that $f = f_0(r) + f_2(r)P_2(\cos\theta) + f^\perp(r, \theta)$ we have $[f] = [f_0] + [f_2]P_2(\cos\vartheta) + [f^\perp](\vartheta)$. Note that $[f_0]$ and $[f_2]$ are constants.

Equation (5.30) is satisfied if and only if $c_1 = 0$ plus

$$[k_2] = [f_2]\,, \qquad [k^\perp](\vartheta) = [f^\perp](\vartheta). \tag{7.14}$$

The constant $c_2$ just corresponds to the difference $[k_0]$, i.e. $[k_0] = c_2$.

Likewise, equation (5.31) is satisfied if and only if $H_1 = 0$ plus

$$[h_0] = \frac{H_0}{2} + \frac{1}{2}a\nu'(a)\,[k_0] \tag{7.15}$$

$$[h_2] = \frac{1}{2}a\nu'(a)\,[f_2]\,, \tag{7.16}$$

$$[h^\perp](\vartheta) = \frac{1}{2}a\nu'(a)[f^\perp](\vartheta). \tag{7.17}$$

Equation (5.35), since $c_1$ and $H_1$ vanish, imposes the following expansion of $[\hat{Q}_2](\vartheta)$

$$[\hat{Q}_2](\vartheta) = [\hat{Q}_{2(0)}] + [\hat{Q}_{2(2)}]P_2(\cos\vartheta) + [\hat{Q}_2^\perp](\vartheta)$$

for some constants $[\hat{Q}_{2(0)}]$ and $[\hat{Q}_{2(2)}]$ and an arbirary function $[\hat{Q}_2^\perp]$ of $\vartheta$, orthogonal to $P_0(\cos\vartheta)$ and $P_2(\cos\vartheta)$. Equation (5.35) is thus equivalent to

$$[\hat{Q}_{2(0)}] = 2ae^{-\nu(a)/2}\,[k_0] \tag{7.18}$$

$$[\hat{Q}_{2(2)}] = 2ae^{-\nu(a)/2}\,[f_2]\,, \tag{7.19}$$

$$[\hat{Q}_2^\perp](\vartheta) = 2ae^{-\nu(a)/2}[f^\perp](\vartheta), \tag{7.20}$$

where here, and in the following expressions, equation (7.1) is used to set $\lambda(a) = -\nu(a)$. Take now the equations (5.28) and (5.29) for the differences $[m]$ and $[h']$. In the case $[E] \neq 0$ ($[\lambda'] \neq 0$ and $[\nu''] \neq 0$), for which $Q_1 = 0$ necessarily, we recall we have $\hat{Q}_2^+ = \hat{Q}_2^+(\vartheta)$ and therefore both $\hat{Q}_2^\pm$ due to the above, so that

$$\hat{Q}_2^\pm(\vartheta) = \hat{Q}_2{}^\pm_{(0)} + \hat{Q}_2{}^\pm_{(2)} P_2(\cos\vartheta) + \hat{Q}_2^{\perp\pm}(\vartheta), \tag{7.21}$$

with constants $\hat{Q}_2{}^\pm_{(0)}$ and $\hat{Q}_2{}^\pm_{(2)}$. Thence, equation (5.32) holds iff

$$[m_0] = a\,[k_0'] + \frac{1}{4}e^{\nu(a)/2}[\lambda']\hat{Q}_2{}^+_{(0)} + \frac{1}{2}(a\lambda'_-(a) + 2)\,[k_0] \tag{7.22}$$

$$[m_2] = a\,[k_2'] + \frac{1}{4}e^{\nu(a)/2}[\lambda']\hat{Q}_2{}^+_{(2)} + \frac{1}{2}(a\lambda'_-(a) + 2)\,[f_2]\,, \tag{7.23}$$

$$[m^\perp](\vartheta) = a[k^\perp](\vartheta) + \frac{1}{4}e^{\nu(a)/2}[\lambda']\hat{Q}_2^{\perp+}(\vartheta) + \frac{1}{2}(a\lambda'_-(a) + 2)[f^\perp](\vartheta), \tag{7.24}$$

while equation (5.33) does whenever

$$[h_0'] = \frac{1}{2}a\nu'(a)\,[k_0'] + \frac{1}{4}e^{\nu(a)/2}[\nu'']\hat{Q}_2{}^+_{(0)} + \frac{1}{2}(a\nu''_-(a) + \nu'(a))\,[k_0]\,, \tag{7.25}$$

$$[h_2'] = \frac{1}{2}a\nu'(a)\,[k_2'] + \frac{1}{4}e^{\nu(a)/2}[\nu'']\hat{Q}_2{}^+_{(2)} + \frac{1}{2}(a\nu''_-(a) + \nu'(a))\,[f_2]\,, \tag{7.26}$$

$$\left[h^{\perp\prime}\right](\vartheta) = \frac{1}{2}a\nu'(a)[k^{\perp\prime}](\vartheta) + \frac{1}{4}e^{\nu(a)/2}[\nu'']\hat{Q}_2^{\perp+}(\vartheta) + \frac{1}{2}(a\nu''_-(a) + \nu'(a))[f^\perp](\vartheta). \tag{7.27}$$

In the case $[E] = 0$, condition $[\omega'']Q_1 = 0$ must be satisfied (Proposition 8), but it provides no information, since $[\omega''] = 0$ as follows from (7.11) and (7.4). On the other hand, the equations corresponding to (5.32) and (5.33) with the changed terms (5.34) contain a term proportional to $[E'](Q_1)^2$. If $[E'] = 0$ we recover the above equations (with $[\lambda'] = [\nu'] = [E] = 0$) and therefore one only needs considering the case $[E'] \neq 0$. In that case the equations imply, analogously, that $Q_1$ does not depend on $\tau$ and that it must satisfy $(Q_1)^2 = q_0 + q_2 P_2(\cos\vartheta) + q^\perp(\vartheta)$ for some constants $q_0$ and $q_2$ and a function $q^\perp(\vartheta)$ (orthogonal to $P_0(\cos\vartheta)$ and $P_2(\cos\vartheta)$).

Some remarks are in order now, which will lead us eventually to the determination of the deformation of the matching hypersurface at second order in any "radial" gauge –recall that the deformation vectors $\vec{Z}$ are gauge dependent, and therefore the functions $Q$ describe the deformation with respect to the gauge chosen. The appropriate quantities are constructed as follows

$$\Xi_0 := \hat{Q}_{2(0)} - 2ae^{-\nu(a)/2}k_0(a), \quad \Xi_2 := \hat{Q}_{2(2)} - 2ae^{-\nu(a)/2}f_2(a), \tag{7.28}$$
$$\Xi^\perp(\vartheta) := \hat{Q}_2^\perp(\vartheta) - 2ae^{-\nu(a)/2}f^\perp(a,\vartheta),$$

on $\Sigma_0$ from either side + and −. These three quantities are "radial"-gauge independent, since the gauge defined by $\vec{V}_2 = 2Y(r,\theta)\partial_r$ (and $\vec{S}_1 = Ct\partial_\phi$) induces via (3.35) the transformation $\hat{Q}_2^g = \hat{Q}_2 + 2Ye^{\lambda(a)/2}$, while $k^g = k + Y/r$ and $f^g = f + Y/r$, see (5.6). On the other hand, the relations (7.18)-(7.20) just read

$$[\Xi_0] = 0, \qquad [\Xi_2] = 0, \qquad [\Xi^\perp](\vartheta) = 0, \tag{7.29}$$

meaning that the quantities coincide as computed from either side. How the actual deformation $\Sigma_\varepsilon^+$ out from the spherical $\Sigma_0$ is encoded in terms of $\Xi_0$ and $\Xi_2$ is described in Section 7.5.

The above matching conditions to second order have yet to be combined with the constraints provided by the field equations at either side. We obtain the final expressions of the matching conditions to second order using the second order field equations for the perfect fluid interior and the vacuum exterior next.

Regarding the $l = 0$ sector, the differences of the field equations do not provide any constraints to the matching conditions in the sense that the differences $[k_0]$ and $[k_0']$ remain arbitrary (constants). This, as expected, is related to the fact that $k_0$ is pure gauge. The $l = 0$ matching conditions (7.15), (7.22) and (7.25) can be written in terms of the "tilded" functions (6.32) and the deformation functions (7.28) in the case $[E] \neq 0$ as follows,

$$[\tilde{h}_0] = \frac{H_0}{2}, \tag{7.30}$$

$$[\tilde{h}_0'] = \frac{a - M}{a(a - 2M)}[\tilde{m}_0], \tag{7.31}$$

$$[\tilde{m}_0] = 2\pi[E]e^{-\nu(a)/2}a\Xi_0, \tag{7.32}$$

while if $[E] = 0$ equation (7.32) is replaced by

$$[\tilde{m}_0] = -2\pi[E']e^{-\nu(a)/2}aq_0^2. \tag{7.33}$$

The background matching configuration relations (7.4) and (7.5) have been used to express the background difference functions in terms of $[E]$, together with (6.20) to write

$$a\nu'(a) = \frac{2M}{a - 2M} = e^{-\nu(a)}\frac{2M}{a}. \tag{7.34}$$

Recall, from the discussion in Section 6.3, that the arbitrary shift in the function $\tilde{h}_0^-(r_-)$ was fixed at infinity, demanding that $\tilde{h}_0^-(r_-)$ vanishes there. The arbitrariness in shifting $\tilde{h}_0^+(r_+)$ corresponds here to the appearance of the free constant $H_0$. One can always fix the shift in $\tilde{h}_0^+(r_+)$ in the interior simply by choosing $H_0$. This just mirrors the fact that in Newtonian theory the potential is fixed at infinity and then taken to the interior of the body simply by imposing continuity across the boundary.

It must stressed, however, that the argument about the "continuity" of $\tilde{h}_0$ does not stand for the other function $\tilde{m}_0$ in general. Consider first the difference of equation (6.36) for a vacuum exterior combined with the two matching conditions (7.30), (7.31) at hand, which leads to the relation

$$[\tilde{m}_0] = -4\pi \frac{a^3}{M}[E]\tilde{\mathcal{P}}_0(a), \tag{7.35}$$

after using the definition (6.37). Note that this equation holds always, irrespective of the vanishing (or not) of $[E]$. Now, in the case $[E] \neq 0$, (7.32) can be finally rewritten as

$$\left(2[E]\tilde{\mathcal{P}}_0(a) =\right) \quad P_0^{(2)}(a) - 2aP'(a)k_0^+(a) = -\frac{M}{a^2}e^{-\nu(a)/2}[E]\Xi_0. \tag{7.36}$$

In the $[E] = 0$ case equation (7.35) clearly implies $[\tilde{m}_0] = 0$ and therefore (7.33) yields

$$[E']q_0 = 0. \tag{7.37}$$

The implication of (7.35) is that *the values of the functions $\tilde{m}_0^+(a)$ and $\tilde{m}_0^-(a)$ coincide if and only if $[E]\tilde{\mathcal{P}}_0(a) = 0$.* This fact turns out to be in contradiction with the assumption made in [57] stating that $m_0^H$ is "continuous" at the boundary, with consequences on the determination of $\delta M$ (see Section 7.5).

Finally, the field equation (6.36) at both sides ($\pm$) can be used to replace the condition (7.31) by (7.36). To sum up, given the Einstein's field equations hold, in the $l = 0$ sector the set of matching conditions can be given by the two conditions (7.30) and either (7.32) or (7.35), plus the relation (7.36).

In the $l = 2$ and orthogonal $\perp$ sectors things are different, in the sense that the field equations provide, in principle, further constraints to the matching conditions. We present the two sectors separately, starting with the $l = 2$. Taking the differences of the field equations (6.56), (6.57) and (6.58) we obtain three equations for the differences $[\tilde{m}_2]$, $[\tilde{k}_2']$, $[\tilde{h}_2']$ which have to be added to the relations in (7.23) and (7.26) and the relations (7.14) and (7.16) that already determine $[\tilde{k}_2]$ and $[\tilde{h}_2]$ trivially. The number of independent equations turns out to be four plus these two trivial ones, and can be finally cast, when $[E] \neq 0$ ($\Rightarrow Q_1 = 0$), as

$$[\tilde{k}_2] = 0, \qquad [\tilde{h}_2] = 0, \tag{7.38}$$

$$[E]\left\{\tilde{h}_2(a) - \frac{1}{4}\nu'(a)e^{\nu(a)/2}\Xi_2 + \frac{1}{3}a^2 e^{-\nu(a)}\left(\frac{2J}{a^3} - \Omega_\infty\right)^2\right\} = 0, \tag{7.39}$$

plus

$$[\tilde{h}_2'] = 4\pi[E]\frac{a^2}{M}\tilde{h}_2(a) + \frac{4}{3}\pi[E]\frac{a^2}{M}e^{-2\nu(a)}\left((a-M)^2+M^2\right)\left(\frac{2J}{a^3}-\Omega_\infty\right)^2, \tag{7.40}$$

$$[\tilde{k}_2'] = -4\pi[E]\frac{a^2}{M}\tilde{h}_2(a) - \frac{4}{3}\pi[E]\frac{a^3}{M}(a-2M)e^{-\nu(a)}\left(\frac{2J}{a^3}-\Omega_\infty\right)^2, \tag{7.41}$$

$$[\tilde{m}_2] = \frac{8}{3}\pi a^4[E]e^{-\nu(a)}\left(\frac{2J}{a^3}-\Omega_\infty\right)^2, \tag{7.42}$$

where we have used, in particular, that

$$[j'(\omega-\Omega)^2] = -\frac{1}{2}[\lambda']\left(\frac{2J}{a^3}-\Omega_\infty\right)^2 = -4\pi a[E]e^{-\nu(a)}\left(\frac{2J}{a^3}-\Omega_\infty\right)^2$$

given the exterior region is vacuum. Therefore, for $[E]\neq 0$ the set of matching conditions for the $l=2$ sector is composed by only three equations, given by the two in (7.38), and (7.39). The three relations (7.40), (7.41) and (7.42) are now a consequence of (7.38) and (7.39) and the field equations (6.56), (6.57) and (6.58) at both sides. Regarding the $[E]=0$ case, the above equations for the $l=2$ sector (7.38)-(7.42) hold. However, (7.39) has to be substituted by $[E']q_2=0$.

For convenience, we present the matching conditions of the $\perp$ sector in terms of $\tilde{v}^\perp$. First, the matching conditions (7.14) and (7.17) translate into

$$[\tilde{h}^\perp]=0, \qquad [\tilde{k}^\perp]=0, \tag{7.43}$$

so that, by the definition of $v^\perp$ we have

$$[\tilde{v}^\perp]=0. \tag{7.44}$$

The field equation (6.49) provides, using the previous matching conditions,

$$[\tilde{v}^{\perp\prime}](\vartheta)=0. \tag{7.45}$$

The matching condition (7.27) with the field equation (6.50) results in

$$[E]\left\{\tilde{h}^\perp(\vartheta)+\frac{\nu'(a)e^{\nu(a)/2}}{4}\Xi^\perp(\vartheta)\right\}=0. \tag{7.46}$$

Finally, the two remaining conditions result from the field equations (6.46) and (6.50), and they read respectively

$$[\tilde{m}^\perp](\vartheta)=0, \tag{7.47}$$

$$[\tilde{h}^{\perp\prime}](\vartheta)=\frac{4\pi a^2}{M}[E]\tilde{h}^\perp(\vartheta). \tag{7.48}$$

If $[E] = 0$, (7.43), (7.44), (7.45) and (7.47) hold, but (7.46) must be replaced by $[E']q^{\perp} = 0$ and (7.48) by $[h^{\perp\prime}] = 0$. We thus have $[E']Q_1 = 0$. As a first consequence, some of the matching conditions above, (7.38)-(7.42) for $l = 2$ and (7.43)-(7.45) plus (7.47) and (7.48) for the $\perp$ sector always hold true, irrespective of whether or not $[E]$ vanishes. Finally, if $[E'] \neq 0$ then

$$Q_1 = 0. \tag{7.49}$$

## 7.4 The matching of the $l = 0, 2$ sectors

We devote this section to the analysis of the matching of the $l = 0, 2$ sectors. To this aim we will take as an assumption that $\omega$ is a function of $r$ alone, so that we are able to compare our results with the development in Chapter 4 ([57]). We start with the formulation of a theorem for the perturbed matching to second order of the perfect fluid and vacuum.

**Theorem 5** *Let $(\mathcal{V}, g)$ with $\Sigma_0$ be the static and spherically symmetric background matched spacetime configuration, perturbed at either side to first order by the functions $\omega^{\pm}(r_{\pm}, \theta_{\pm})$ through ${K_1}^{\pm}$ as defined in (5.2) plus the unknowns $Q_1^{\pm}(\tau, \vartheta)$ and $\vec{T}_1^{\pm}(\tau, \vartheta)$, as described in Proposition 7, so that the first order matching conditions (5.14) and (5.15) plus (5.17) and (5.18) hold. Let the configuration be perturbed to second order by ${K_2}^{\pm}$ as defined in (5.3), plus the unknowns $\hat{Q}_2^{\pm}(\tau, \vartheta)$ and $\vec{T}_2^{\pm}(\tau, \vartheta)$ on $\Sigma_0$, and assume that the interior region (+) satisfies the field equations for a perfect fluid with barotropic equation of state and that the exterior (−) region is asymptotically flat and satisfies the vacuum field equations up to second order.*

*The energy density $E(r_+)$ and pressure $P(r_+)$ of the interior background configuration are given by (6.12) and (6.13) and must satisfy (6.19). The background exterior vacuum solution is given by (6.20), and we assume $0 < 2M < a$. Consider the convenient background quantities defined in (6.16).*

*Let $\vec{u}_\varepsilon$ be the unit vector fluid corresponding to the interior family of metric tensors $g_\varepsilon^+ = g^+ + \varepsilon {K_1}^+ + \frac{1}{2}\varepsilon^2 {K_2}^+ + \mathcal{O}(\varepsilon^3)$. Assume that $\vec{u}_\varepsilon$ satisfies (6.8) for some constant $\Omega$. Let $J$ be defined by the first order exterior solution (6.26).*

*Assume finally at both sides (±) that the first order function $\omega$ depends only on the radial coordinate, and that the second order functions are decomposed in Legendre polynomials in terms of $\{h_0, h_2, m_0, m_2, k_0, k_2, f_2\}$ by (6.28).*

*Then*

1. *The second order pressure $P^{(2)}$ and energy density $E^{(2)}$ of the fluid inherit the same angular dependency, that is, (6.30) hold for some $E_0^{(2)}(r), E_2^{(2)}(r), P_0^{(2)}(r)$ and*

*$P_2^{(2)}(r)$. With the help of convenient alternative "tilded" counterparts, defined in (6.32)-(6.33) plus (6.37) and (6.59), the Einstein's field equations in the interior can be expressed as the system (6.38), (6.39) and (6.41) for some constant $\gamma$ for the set $\{\tilde{\mathcal{P}}_0^+, \tilde{m}_0^+, \tilde{h}_0^+\}$ plus the system (6.56), (6.57), (6.58) for the set $\{\tilde{h}_2^+, \tilde{k}_2^+, \tilde{m}_2^+\}$. The vacuum solution at second order is given by (6.42), (6.43), (6.61), (6.62) and (6.63) where $\delta M$ and $A$ are arbitrary constants.*

2. *Given the Einstein's field equations of the previous point are satisfied, the necessary and sufficient conditions that the metric perturbation tensors $K_2{}^{\pm}$ must satisfy to fulfil the second order matching conditions are given by (7.30) and (7.35) for the sets $\{\tilde{\mathcal{P}}_0^\pm, \tilde{m}_0^\pm, \tilde{h}_0^\pm\}$, with arbitrary constant $H_0$, and the two equations in (7.38) for the sets $\{\tilde{h}_2^\pm, \tilde{k}_2^\pm, \tilde{m}_2^\pm\}$.*

Regarding the deformation of $\Sigma_0$, expressions (7.36) and (7.39) show explicitly how the quantities $\Xi_0$ and $\Xi_2$ are linked in a 'radial'-gauge invariant manner to the jump in the pressure at second order across the boundary of the star through the value of the energy density of the background configuration at $\Sigma_0$. Whenever $[E] \neq 0$, equations (7.36) and (7.39) directly determine $\Xi_0$ and $\Xi_2$ in terms of $\tilde{\mathcal{P}}_0(a)$ and $\tilde{h}_2(a)$ respectively, which are quantities that are obtained by integration from the origin. Equations (7.36) and (7.39) can then be cast as

$$\Xi_0 = -\frac{2a^2}{M} e^{\nu(a)/2} \tilde{\mathcal{P}}_0(a), \tag{7.50}$$

$$\begin{aligned}\Xi_2 &= e^{-\nu(a)/2} \frac{2a(a-2M)}{M} \left( \tilde{h}_2(a) + \frac{1}{3} \frac{a^3}{a-2M} \left( \frac{2J}{a^3} - \Omega_\infty \right)^2 \right) \\ &= -\frac{2a^2}{M} e^{\nu(a)/2} \tilde{\mathcal{P}}_2(a),\end{aligned} \tag{7.51}$$

after using (7.34) and (6.60) in the first and second equalities in the latter, respectively.

However, if $[E] = 0$, since $\hat{Q}_2^\pm$ are only defined on $\Sigma_0$ we cannot determine the deformation directly from the above, in the same way $Q_1$ is undetermined in the first order problem in that case.

This is to be expected. In fact, as an extreme case, when matching two vacuum regions the matching hypersurface is not determined in general. The idea is that in order to have a boundary determined by the matching, the energy density must depart from zero as one moves to the interior, so that the star indeed extends no further than, and up to, that surface. A sufficient condition is that $[E'] \neq 0$. In that case it can be shown that one can make use of the gauge that follows the surfaces of constant energy density, which has been used so extensively in the literature, specially in [57]. In order to determine the deformation one can then extend $\Xi_0$ and $\Xi_2$ to the interior, say using some functions

$\xi_0(r_+)$ and $\xi_2(r_+)$ in a convenient way, using that gauge, to finally obtain the deformation by continuity. This is discussed in the final part of this section, where it is shown, in particular, that (7.50) and (7.51) will hold also when $E(a) = 0$, under the condition that the gauge that follows the surfaces of constant energy density exists. This suggests the fact that equations (7.36) and (7.39) are expected to appear again at higher orders, in the same way the condition $[E]Q_1 = 0$ of the first order problem appears as $[E'](Q_1)^2 = 0$ at second order.

Before performing an exhaustive discussion on the determination of the deformation of the star to second order, we acommodate Hartle's model in our setting, establishing the explicit correspondences between coordinates and gauges used in both methods. Apart from the obvious interest of comparing our results with those in [57], this will help us to determine the deformation of $\Sigma_0$ even when $[E] = 0$. This is accomplished inside a suitable (surface) gauge, that we denote as the $E-$gauge. First we discuss its existence and how to construct it and finally we point out how the deformation of the star is encoded in this gauge.

## 7.5 Comparison with Hartle's results: amending the mass

The spacetime gauge used in [57] at first order corresponds to setting $b_1 = 0$ here (since $\omega$ is assumed to be continuous in [57]), while at second order the starting point is the choice of gauge that corresponds here to setting $k_0^\pm = 0$ and $f_2^\pm = 0$. We refer to this choice as the $k$-gauge. At some point another spacetime gauge comoving with the deformation is introduced in [57]. A discussion of the use of that gauge in [57] (also in [16]) can be found in Section 7.5.

In the $k$-gauge all the "tilded" functions (6.32) and (6.33) equal the non-"tilded" counterparts, and in the interior region (+), the functions $\tilde{\mathcal{P}}_0$ and $\tilde{\mathcal{P}}_2$ are just rescalings of their respective $P_0^{(2)}$ and $P_2^{(2)}$, that is, $\tilde{\mathcal{P}}_{0/2} = P_{0/2}^{(2)}/(2(E+P)) := \mathcal{P}_{0/2}$. To avoid having to rewrite all the previous equations without tildes we will simply use the "tilded" functions in what follows.

Let us first concentrate on the $l = 0$ sector. Regarding the interior region, the system (6.38)-(6.39) plus equation (6.40) for the set $\{re^{-\lambda}\tilde{m}_0^+, \tilde{\mathcal{P}}_0, \tilde{h}_0^+\}$, as functions of $r$ ($r_+$ in fact) coincide one by one with the coupled equations (4.24) and (4.25), plus (4.26) in Chapter 4 (or (90), (97) and (100) in [57]) for $\{m_0^H, p_0^{H*}, h_0^H\}$ as functions of $R$, which has the same range as $r_+$. To be precise, one can forget about $r_+$ and $R$ and just establish a common variable $s$, so that the sets of equations in Chapter 6 and in Chapter

4 or [57] hold in the range $s \in (0, a]$. Given common conditions at $s \to 0$ the problem for $\{re^{-\lambda}\tilde{m}_0^+, \tilde{\mathcal{P}}_0, \tilde{h}_0\}$ coincides with the problem for $\{m_0^H, p_0^{H*}, h_0^H\}$ and therefore $m_0^H(s) = se^{-\lambda(s)}\tilde{m}_0(s)$, $p_0^{H*}(s) = \tilde{\mathcal{P}}_0(s)$ and $h_0^H(s) = \tilde{h}_0(s)$ (up to a free additive constant) necessarily for $s \in (0, a]$, i.e. *in the interior region*.

In the vacuum exterior region $m_0 = \tilde{m}_0$ and $h_0 = \tilde{h}_0$ are given by (6.42) and (6.43) respectively. Again, these two expressions correspond to (4.27) and (4.28) in Chapter 4 ((105) and (106) in [57]) for $m_0^H$ and $h_0^H$ respectively, in terms of a variable $r$ in the range $r \in [a, \infty)$.

Therefore, the matching conditions for the function $\tilde{h}_0$ given by (7.15) and (7.30), and for the function $\tilde{m}_0$ given by (7.32), translate directly to matching conditions on $h_0^H$ and $m_0^H$. As discussed previously, the free additive constant in $\tilde{h}_0^+$ (and so in $h_0^H$) can be used to set $H_0 = 0$. In an abuse of terminology, the assumption of a "continuous" $h_0^H$ is thus consistent.

The function $m_0^H$ is also assumed to be "continuous" in [57] Section VII, when the value of $m_0^H(a)$ as computed from the interior is equated to the expression of $m_0^H(r)$ in the exterior at $r = a$ in order to obtain the constant $\delta M$ in (4.30), or (107) in [57]. However, the correct matching condition is given by (7.35), which in the $k$-gauge, and since $[\lambda] = 0$, can be expressed as

$$[m_0^H] = -4\pi \frac{a^3}{M}(a - 2M)E(a)p_0^{H*}(a) \tag{7.52}$$

using the notation in Chapter 4, [57]. As a result, given the value $m_0^H(a)$ as computed from the interior, the value of the change in mass in (6.42) is given by

$$\delta M = m_0^H(a) + \frac{J^2}{a^3} + 4\pi \frac{a^3}{M}(a - 2M)E(a)p_0^{H*}(a). \tag{7.53}$$

The last term corresponds to the jump of the values of $\tilde{m}_0$ at the boundary, and it is not present in the expression for the change of mass (4.30), this is (107) in [57] and in the subsequent works, e.g. [64, 65]. Of course, whenever the density of mass-energy vanishes at the surface of the star, $E(a) = 0$, this term has no consequences. This will happen in many situations, as in the cases of equations of state that imply the vanishing of the energy density at points where the pressure vanishes, polytropes for instance. In fact, in the series of papers started by [64, 65] all the equations of state considered satisfy that condition, and therefore the computation of the change of mass is not affected by the correction in (7.53).

However, in more general situations that is not going to be the case. As an example, models for quark stars that rely on a non-zero value of $E$ at the surface have been considered in the literature (see e.g. [34]). In particular, models of stars based on a constant background $E$ in the interior are affected by that term and the computation of the change in mass should be corrected.

Let us now jump to the $l = 2$ sector. In the interior region the equation (6.58) plus the system (6.56)-(6.57) for the set $\{re^{-\lambda}\tilde{m}_2^+, \tilde{h}_2^+ + \tilde{k}_2^+, \tilde{h}_2^+\}$ as functions of $r_+$ coincide one by one with equation (4.35) plus the coupled equations (4.33)-(4.34) in Chapter 4, (120), (125) and (126) in [57], for $\{m_2^H, v^H := h_2^H + k_2^H, h_2^H\}$ as functions of $r$, which has the same range as $r_+$. The same argument as in the $l = 0$ sector shows that the problems coincide and therefore we can set $m_2^H(s) = se^{-\lambda(s)}\tilde{m}_2(s)$, $h_2^H(s) = \tilde{h}_2(s)$ and $k_2^H(s) = \tilde{k}_2(s)$ for $s \in (0, a]$. In the vacuum exterior region $h_2 = \tilde{h}_2$ and $k_2 = \tilde{k}_2$ are given by (6.61) and (6.62), which correspond to (4.37) and (4.38) respectively in terms of a variable $r$ in the range $r \in [a, \infty)$. The comparison of (6.60) with (4.36) ((91) in [57]) implies the correspondence $p_2^{H*}(s) = \tilde{\mathcal{P}}_2(s)$. The two matching conditions in (7.38) simply state that $h_2^H$ and $k_2^H$ are "continuous" on the boundary. The assumption made in [57] regarding the $l = 2$ sector is thus consistent. This "continuity" of $h_2^H$ and $k_2^H$ is finally used in order to fix the free constants $A'$ and $A$ in the interior and exterior regions respectively, thus fixing completely the global problem in the $l = 2$ sector.

We discuss finally the deformation of the boundary. In [57] the analysis of the deformation needs the introduction of a function $\xi^H(r, \theta) = \xi_0^H(r) + \xi_2^H(r)P_2(\cos\theta)$ defined in the whole interior region by imposing $P_\varepsilon(R + \varepsilon^2\xi^H(R, \theta), \theta) = P(R)$ for $R \in [0, a]$ (see also the discussion in [16]). The deformation is then determined by the values $\xi_0^H(a)$ and $\xi_2^H(a)$.

Let us recall that in the present treatment the deformation is described by $\Xi_0$ and $\Xi_2$, which are determined by equations (7.50) and (7.51) whenever $E(a) \neq 0$. In the case $E(a) = 0$ the deformation can be determined by relying on a particular gauge in order to define extensions for both $\Xi_0$ and $\Xi_2$. The correspondence of $\xi_0^H(r)$ and $\xi_2^H(r)$ as *functions* defined in the interior region with quantities in the treatment presented here rely, in fact, on the construction of those extensions. This is discussed in the next section, where it is shown how equations (7.50) and (7.51) hold in all cases, and that the values $\xi_0^H(a)$ and $\xi_2^H(a)$ correspond to

$$\xi_0^H(a) = -\frac{1}{2}e^{\nu(a)/2}\Xi_0, \qquad \xi_2^H(a) = -\frac{1}{2}e^{\nu(a)/2}\Xi_2. \tag{7.54}$$

(The relative minus sign comes from the orientation of the normal chosen in (5.10), which goes as $-\partial_r$.) Indeed, the former translates, via (7.50), to equation (4.31), (117) in [57], which should in fact be corrected to $\xi_0^H(a) = p_0^{H*}(a)a(a - 2M)/M$, whose value describes the average expansion of the shape of the star [64, 16]. The combination of the latter with (7.51) enters the different definitions of the ellipticity of the star found in the literature (see e.g. [64], [16]) accordingly. In particular, it provides the expression for the ellipticity as defined in [64] by (4.40), which after using (7.54) and (7.51) thus reads

$$e = \sqrt{-3\left\{\tilde{k}_2(a) - \frac{a-2M}{M}\left(\tilde{h}_2(a) + \frac{1}{3}\frac{a^3}{a-2M}\left(\frac{2J}{a^3} - \Omega_\infty\right)^2\right)\right\}}.$$

## On the deformation of the star

We devote this section to discuss the deformation of the surface, and at the same time, study the relationship of the two gauges used in [57] (also in [16]).

In order to describe the deformation of the surface, motivated by the approaches taken in Newtonian theory, it has been common in the literature to focus on the surfaces of constant energy density or, equivalently, of constant pressure given a barotropic equation of state. This consists after all of a choice of gauge in which the surfaces of constant energy density (or pressure) in the interior region of the perturbed configuration are those of constant radial coordinate. This is described in [57] (see also [16]) as a change from the original coordinate $r^H$ (the initial gauge corresponds to the $k$-gauge) to another $R$ defined by (the inverse of)

$$\{R, \theta\} \to \{r^H = r^H_\varepsilon(R, \theta), \theta\} \tag{7.55}$$

for some function $r^H_\varepsilon(R,\theta)$ satisfying $r^H_0(R,\theta) = R$ and

$$E_\varepsilon(r^H_\varepsilon(R,\theta),\theta) = E(R), \tag{7.56}$$

where $E_\varepsilon$ is the energy density corresponding to $g_\varepsilon$ (see (6.3)) in the $k$-gauge. The surfaces of constant energy density in the perturbed configuration, $E_\varepsilon$, are then those of constant $R$, and their values correspond to the values the pressure of the background configuration $E$ take at those $R \in (0, a]$. In the present terminology that corresponds to moving to another gauge, to which we refer to as the $E$-gauge. Note that (7.56) is imposed for all $\varepsilon$ in some neighbourhood around 0, and therefore for all orders. To second order $r^H_\varepsilon(R,\theta)$ is specified in [57] as

$$r^H_\varepsilon(R,\theta) = R + \varepsilon^2 \xi^H(R,\theta) + O(\varepsilon^3), \tag{7.57}$$

where for clarity we write explicitly the perturbation parameter at this point. We do not comment yet on the existence nor uniqueness of the $E$-gauge.

In [57] the perturbed surface is then *defined* as the surface of constant energy density that equals the value of the energy density at the surface of the static configuration. Explicitly, $\Sigma_\varepsilon$ is defined to have the form $\Sigma_\varepsilon : r^H = r^H_\varepsilon(a,\theta)$, which is equivalent to $R = a$ by construction.

Let us formulate that condition in the present treatment. Indicating with a $^{(E)}$ when a (gauge-dependent) quantity or object refers to the $E$-gauge, the expression (7.56) can be cast just as

$$E^{(n)(E)} = 0$$

for all orders $n \geq 1$ (note $E^{(0)(E)} = E$). At each order $n$ that condition would determine, in principle, the $E$-gauge at the corresponding order. The perturbed matching hypersurface $\Sigma_\varepsilon$ would then be *defined* by imposing $\Sigma_\varepsilon^{(E)} = \Sigma_0$ pointwise. In other words, the perturbed matching hypersurface is *defined* by imposing that the $E$-gauge is, at the same time, a "surface-comoving" gauge [1].

Given a barotropic equation of state all the above can be stated in terms of the pressure. The $E$-gauge is then also determined by

$$P^{(n)(E)} = 0 \tag{7.58}$$

for all $n \geq 1$. Since the interior pressure necessarily vanishes at the boundary in the background configuration, imposing that the $E$-gauge is also a "surface-comoving" gauge implies that the whole perturbed pressure computed in the $E$-gauge vanishes at the perturbed boundary. This is the view taken in [16] and many other works (see e.g. [21, 84]).

Clearly, given a barotropic equation of state, the approach taken in terms of $E$ (say, approach "E") and that in terms of $P$ (approach "P") lead to the same conclusion. However, their justifications are of different nature, apart from the possible problems of existence.

Regarding the approach "E", if $E(a) \neq 0$ the fact that the perturbed energy density attains that value $E(a)$ at the boundary may, in principle and in general, seem to constitute an assumption. Probably due to this difficulty the approach "P" has seemed to be preferred in many works since the vanishing of the (perturbed) "pressure" on the surface is what one would expect on physical grounds. However, that would be an erroneous statement as such, and in general, since $P_\varepsilon$ is gauge dependent (see Chapter 7). One should, at least, prove in which gauge that should happen. Indeed, the matching conditions in the exact case restrict the possible jumps of the Einstein tensor across the surface. However, it remains to be shown how this fact translates to the perturbative matching scheme in the general case. A general consistent approach should not rely, in principle, on the use of a result (the vanishing of a "pressure" in a certain gauge) that has to be proven, in fact, as a consequence of the procedure.

Finally, the definition of the deformation of the star in terms of the $E$-gauge should control and take care of the existence (and uniqueness, if needed) of the gauge. For

[1] The surface comoving gauges and the surface gauges are defined in Chapter 3 just after Proposition 4.

instance, in the simplest case of a constant energy density interior background $E(r) = E(a) = const.$ the $E$-gauge cannot be determined using (7.56), and thus, neither the deformation. Instead, the "P" approach has to be used, for which the $E$-gauge can be constructed. This is implicitly done in works focused on stars of constant energy density, such as [30].

Nevertheless, the determination of $\Sigma_\varepsilon$ using the $E$-gauge is well justified if $E(a) = 0$ but $E(r) \neq 0$ ($> 0$ in fact) for $r \in (0, a)$, since then the perturbed star (perfect-fluid region) extends up to where $E_\varepsilon$ vanishes, and no further. By the local nature of the matching, one could relax this condition to $E(a - \delta) \neq 0$ for all $\delta > 0$ in some neighbourhood of $a$. This condition (and analyticity of $E(r)$) demand that there exists $n$ such that $n$-th derivative $d^n E/dr^n(a)$ at $r = a$ is non-zero. The implicit function theorem can then be applied to every differentiation of (7.56) with respect to $\varepsilon$ evaluated at $\varepsilon = 0$ in order to show that $r^H_\varepsilon(a, \theta)$ can be obtained order by order from (7.56). The full proof is out of the scope of this thesis and will be presented elsewhere. When needed, we will simply assume that the $E$-gauge can be constructed from $r = a$ inwards.

As stressed, in the present treatment no argument about the vanishing of the pressure of the perturbed configuration $P_\varepsilon$ has been made, nor any specific gauge has been used. In Sections 7.2 and 7.3 it has been shown how the deformation of the boundary, described by the quantities $Q_1$ of the first order and $\Xi_0$ and $\Xi_2$ of the second order, are determined by $Q_1 = 0$ when $E(a) \neq 0$ or $E'(a) \neq 0$, and (7.50) and (7.51) when $E(a) \neq 0$, respectively, and how that agrees with the results in [57].

In what follows we first show explicitly that the $E$-gauge is indeed a "surface gauge" when $E(a) \neq 0$, at least to second order. This shows, at the same time, that the usual "vanishing of the pressure at the boundary" in the exact case translates in this perturbative scenario to $P^{(E)}_\varepsilon|_{\Sigma^{(E)}_\varepsilon} = 0$, i.e. that the perturbed pressure in the $E$-gauge must vanish at the perturbed surface (at least to second order). Secondly, we use the definition of the perturbed surface when $E(a) = 0$ by means of the $E$-gauge (approach "E") to show that, given the $E$-gauge exists (and is unique), then $Q_1 = 0$ and equations (7.50) and (7.51) hold even when $E(a) = 0$.

Not to overwhelm the notation let us drop the interior + superscripts in the following when not needed.

As shown in Section 6.2, at first order we have $E^{(1)} = P^{(1)} = 0$, and the condition $E(a) \neq 0$ already implies $Q_1 = 0$. Therefore, the family of gauges chosen for the family (5.1) satisfies the $E$-gauge condition to first order. Since $Q_1 = 0$, $\Sigma_\varepsilon$ coincides at first order with $\Sigma_0$ as a set of points. The $E$-gauge is therefore a "surface-comoving" gauge up to first order. A *hypersurface* gauge can be used to fix $\vec{T}^+_1 = 0$, so that the perturbed $\Sigma_\varepsilon$ coincides at first order with $\Sigma_0$ pointwise, so that the $E$-gauge is, moreover, a "surface" gauge up to first order.

Regarding the second order, let us recall that given conditions at the origin (such that $\tilde{\mathcal{P}}_0(0)$ vanishes) $\tilde{\mathcal{P}}_0(r)$ is fully determined by the $l=0$ field equations, and $\tilde{\mathcal{P}}_2(r)$ is obtained from (6.60), once $\tilde{h}_2(r)$ is fully determined, in turn, by the $l=2$ field equations and the condition at the origin and at the boundary $r=a$ coming from the "continuity" of the functions $\tilde{h}_2$ and $\tilde{k}_2$. Now, the $E$-gauge is selected by fixing $k_0(r)$ and $f_2(r)$ so that $P_0^{(2)(E)}(r)$ and $P_2^{(2)(E)}(r)$ vanish. From (6.37) and (6.59) this is accomplished by imposing

$$k_0^{(E)} = -\frac{E+P}{rP'}\tilde{\mathcal{P}}_0, \qquad f_2^{(E)} = -\frac{E+P}{rP'}\tilde{\mathcal{P}}_2. \tag{7.59}$$

We are ready to show that if (7.50) and (7.51) hold then $Q_2^{(E)} = 0$. This follows directly from the definitions (7.28), which in the $E$-gauge read

$$\hat{Q}_{2}{}^{(E)}_{(0)} = \Xi_0 + 2ae^{-\nu(a)/2}k_0^{(E)}(a), \qquad \hat{Q}_{2}{}^{(E)}_{(2)} = \Xi_2 + 2ae^{-\nu(a)/2}f_2^{(E)}(a).$$

Equations (7.50) and (7.51) together with (7.59) evaluated on $r=a$ readily imply $\hat{Q}_{2}{}^{(E)}_{(0)} = \hat{Q}_{2}{}^{(E)}_{(2)} = 0$. Finally, since we have chosen $\vec{T}_1 = 0$ at first order, then $Q_{2}{}^{(E)}_{(0)} = Q_{2}{}^{(E)}_{(2)} = 0$ as follow from the definitions (7.28). It only remains, again, to choose a convenient *hypersurface* gauge to second order to fix $\vec{T}_2^+ = 0$ so that the perturbed $\Sigma_\varepsilon$ coincides with $\Sigma_0$ at second order, not only as a set of points, but pointwise. We have thus shown that *the $E$-gauge is indeed a "surface gauge" whenever $E(a) \neq 0$, at least to second order*, as expected.

Let us consider now the case $E(a)=0$ under the conditions that ensure the existence and construction of the $E$-gauge. The matching hypersurface $\Sigma_\varepsilon$ is then determined by the coincidence of $\Sigma_\varepsilon^{(E)}$ and $\Sigma_0$ pointwise (in $\mathcal{V}_0^+$, mind the $+$ superscript). This condition is equivalent, up to second order, to $Q_1^{+(E)} = Q_2^{+(E)} = 0$ together with a hypersurface gauge choice such that $\vec{T}_1^+ = \vec{T}_2^+ = 0$ at each order. At first order we thus have the required result by construction. At second order, the equations defining $\Xi_{0/2}$ (7.28) in the interior read then

$$\Xi_0 = -2ae^{-\nu(a)/2}k_0^{(E)}(a), \qquad \Xi_2 = -2ae^{-\nu(a)/2}f_2^{(E)}(a),$$

which combined with (7.59), yield (7.50) and (7.51).

We must finally address the issue of how $\Xi_{0/2}$, given by (7.50) and (7.51), describe the deformation of the surface. The key is to show how the deformation quantities $\Xi_{0/2}$, defined on $\Sigma_0$, can be extended to the interior region and how that relates to the change from the $k$-gauge to the $E$-gauge. We start by defining that change in terms of $\vec{V}_2$. Let us, for simplicity, set $\vec{S}_1 = 0$ so that $\vec{S}_2 = \vec{V}_2$. Including $\vec{S}_1 = Ct\partial_\varphi$ does not add anything relevant to the analysis. Recall that the $k$-gauge is defined by $k_0^{(k)} = 0$ and $f_2^{(k)} = 0$. Given that the second order change $\vec{V}_2 = 2Y(r,\theta)\partial_r$ induces (5.6) (with $C = 0$), it is immediate to check (recall the freedom in defining $f(r,\theta)$) that the change from the $k$-gauge to the $E$-gauge is accomplished by setting

$$\vec{V}_2 = 2r\left(k_0^{(E)} + f_2^{(E)} P_2(\cos\theta)\right)\partial_r = -2\frac{E+P}{P'}\left(\tilde{\mathcal{P}}_0 + \tilde{\mathcal{P}}_2 P_2(\cos\theta\right)\partial_r, \tag{7.60}$$

where the second equality follows from (7.59). Note that the relation $k_2^{(k)} = k_2^{(E)} - f_2^{(E)}$ holds (recall the radial gauge transformations (5.6)). Also, from relations (6.19) and (7.2) we can see that the vector field $\vec{V}_2$ does not vanish at $r = a$ unless $\tilde{\mathcal{P}} = 0$ there.

On the other hand, given the definition of the second order gauge vectors in (3.22), the second order gauge $\vec{V}_2 = 2Y(r,\theta)\partial_r$ with $\vec{S}_1 = 0$ corresponds to a diffeomorphism $\Omega_\varepsilon : \mathcal{V}_0 \to \mathcal{V}_0$ of the form $(s,\theta) \to (\mathcal{R}_\varepsilon(s,\theta),\theta)$ for $s \in [0,a]$ defined by $\mathcal{R}_\varepsilon(s,\theta) = s + \varepsilon^2 Y(s,\theta)$. Given (7.60), we thus have

$$\mathcal{R}_\varepsilon(s,\theta) = s - \varepsilon^2 \frac{E(s)+P(s)}{P'(s)}\left(\tilde{\mathcal{P}}_0(s) + \tilde{\mathcal{P}}_2(s) P_2(\cos\theta)\right). \tag{7.61}$$

Let us recall again (see Section 7.5) that the coordinate $R$ used in [57] ranges from 0 to $a$, and therefore (7.61) can be compared with the expression (7.57) in the form $r_\varepsilon^H(s,\theta) = s + \varepsilon^2 \xi^H(s,\theta) + O(\varepsilon^3)$ to obtain

$$\xi^H = -\frac{E+P}{P'}\left(\tilde{\mathcal{P}}_0 + \tilde{\mathcal{P}}_2 P_2(\cos\theta)\right).$$

Now, this is in agreement with $\xi^H = \xi_0^H + \xi_2^H P_2(\cos\theta)$ for $\xi_{0/2}^H = -\frac{E+P}{P'} p_{0/2}^H{}^*$, as follows from (90) and (91) in [57] and the correpondences $p_{0/2}^H{}^*(s) = \tilde{\mathcal{P}}_{0/2}(s)$ found in Section 7.5.

Expression (7.61) suggests the construction of two functions in the interior

$$\xi_{0/2} := 2\frac{E+P}{P'} e^{-\nu/2} \tilde{\mathcal{P}}_{0/2}. \tag{7.62}$$

These, evaluated at $r = a$, and given that (7.50) and (7.51) hold, lead to

$$\xi_{0/2}(a) = \Xi_{0/2}.$$

The functions $\xi_{0/2}$ (7.62) are therefore extensions of $\Xi_{0/2}$, as *defined* in (7.50) and (7.51), to all the interior region, and are 'radial'-gauge independent by construction. The information of the deformation of the star in the $k$-gauge is therefore encoded in the functions $\xi_{0/2}$, whereas in the $E$-gauge that information lies in the functions $k_0^{(E)}$ and $f_2^{(E)}$.

Using the correspondence $\xi_{0/2}(s) = -2e^{-\nu/2}\xi_{0/2}^H(s)$, so that $\Xi_{0/2} = -2e^{-\nu/2}\xi_{0/2}^H(a)$, the analysis of the deformation of the star in terms of $\Xi_0$ and $\Xi_2$ follows then from the discussions in [57] (see also [16]). Note that the minus sign in the correspondence comes from the choice of the normals as defined in (5.10), which point towards the origin.

# 8

# On the angular structure of the perturbations

This chapter is aimed at showing that the only possible angular structure of the functions in the perturbation tensors (5.2) and (5.3) is such that $\omega$ is a function of the radial coordinate alone and the expansions (6.28) for the second order functions contain only the terms $l = 0$ and $l = 2$, i.e. the orthogonal $\perp$ sector must vanish. We rely on the construction made in Chapters 5 to 7 in which we have two problems, the interior and the exterior, each one characterized by an elliptic operator in the corresponding domain plus some boundary conditions that arise from the matching procedure. We show that the only differentiable and regular solutions for the first and second order perturbation problems are those for which the $\perp$ sector vanishes.

In [57] the behaviour of the function $\omega(r)$ was somehow adressed. The argument given there relies on the assumption of the continuity of this function (and its derivative) everywhere, including at the boundary that separates fluid and vacuum, in the coordinates used to write the metric as (4.1). Under this assumption, the global problem, i.e. in the domain $r \in (0, \infty)$, is considered and it is argued that fulfilling the conditions of regularity at the origin plus "asymptotic flatness" at infinity requires that the function $\omega$ cannot depend on the angular coordinate $\theta$. The angular structure of the second order perturbations is also discussed in [57]. First, the non-equatorially symmetric part in the second order functions in (4.1), that corresponds to the odd $l$'s in an expansion in Legendre polynomials of the functions involved, is ruled out. After this, the second order field equations are obtained to find that they contain inhomogeneous terms proportional to $(\Omega - \omega)^2$ or to $\omega'^2$ acting as sources only for the $l = 0$ and $l = 2$ modes of the second order functions in (4.1). The homogeneous problem for the rest of the $l$'s is circumvented arguing that in absence of rotation no contributions of such type are found.

The analogous problem in the Newtonian approach for polytropic equations of state was analyzed by Kovetz in [72]. He revisited the paper by Chandrasekhar [23] to show that the rotational perturbations to the Emden's function, that in the politropic model are translated into the perturbations of the Newtonian potential and the density profile (see Chapter 9 for a brief description of polytropes), contain only $l = 0, 2$ modes. The proof is given for a polytropic index $n$ in the interval $0 \leq n \leq 5$, although it can be checked that it holds also for $n \geq 5$ [1]. The present work generalizes the work by Kovetz in the Newtonian case.

## 8.1 Notation and considerations

For notational convenience, let us start with a definition of limits of functions we will use later.

### Limit for functions defined on subsets of the real line

Let $(a, b)$ be an open interval in $\mathbb{R}$ and $p$ a point in $(a, b)$. Let $f$ be a real valued function defined on all of $(a, b)$ except possibly at $p$. It is then said that the limit of $f$ as $x$ approaches $p$ is $L$ if, for every real $\varepsilon > 0$, there exists a real $\delta > 0$ such that $0 < |x - p| < \delta$ and $x \in (a, b)$ implies $|f(x) - L| < \varepsilon$.

### Strong maximum principle and boundary point lemma

We stick to the notation, conventions and definitions regarding elliptic operators from [54]:

- $D$ denotes a domain (a proper open connected subset in $\mathbb{R}^n$). It is not necessarily bounded.
- $L = a^{ij}(x)\frac{\partial^2}{\partial x_i x_j} + b^i(x)\frac{\partial}{\partial x_i} + c(x)$ is elliptic at a point $x \in D$ if the coefficient matrix $a^{ij}(x)$ is positive, i.e. if $\lambda(x)$ and $\Lambda(x)$ denote the minimum and maximum eigenvalues of $a^{ij}(x)$ then $0 < \lambda(x)|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2$ for all $\xi \in \mathbb{R}^n - \{0\}$.
- $L$ is uniformly elliptic in $D$ if $\Lambda/\lambda$ is bounded everywhere in $D$.
- The condition $|b^i(x)|/\lambda(x) \leq const < \infty$, $x \in D$ will be assumed.
- $\partial_\nu$ denotes the outward unit normal derivative to $\partial D$.

From the same reference, we include for completeness the boundary point lemma and the strong maximum principle.

[1] One of the steps of the proof given in [72] consist of showing that the minimum of a function that depends on the polytropic index $n$ is positive. The computation must be performed numerically and it is done in the original reference [72] for $1 \leq n \leq 5$. However, we have not found any $n$ greater than 5 for which the minimum becomes negative. It does for $0 < n < 1$, which in [72] is treated separately.

**Lemma 5** ***Boundary point lemma ([54])***

*Suppose that $L$ is uniformly elliptic, $c = 0$ and $Lu \geq 0$ in $D$. Let $x_0 \in \partial D$ be such that*

1. *$u$ is continuous at $x_0$,*
2. *$u(x_0) > u(x)$ for all $x \in D$,*
3. *$\partial D$ satisfies an interior sphere condition at $x_0$.*

*Then the outer normal derivative of $u$ at $x_0$, if it exists, satisfies the strict inequality*

$$\frac{\partial u}{\partial \nu}(x_0) > 0. \tag{8.1}$$

*If $c \leq 0$ and $c/\lambda$ is bounded, the same conclusion holds provided $u(x_0) \geq 0$, and if $u(x_0) = 0$ the same conclusion holds irrespective of the sign of $c$.*

**Theorem 6** ***Strong maximum principle ([54])***

*Let $L$ be uniformly elliptic, $c = 0$ and $Lu \geq 0 (\leq 0)$ in a domain $D$ (not necessarily bounded). Then if $u$ achieves its maximum (minimum) in the interior of $D$, $u$ is a constant. If $c \leq 0$ and $c/\lambda$ is bounded, then $u$ cannot achieve a non-negative maximum (non-positive minimum) in the interior of $D$ unless it is constant. If $c < 0$ at some point, then the constant of the theorem is obviously zero.*

## 8.2 Lemmas

We define the intervals $I^+ = (0, A)$ and $I^- = (A, \infty)$ of the real line for some $A > 0$. Also, we will use the notation $I^+_\alpha = (\alpha, A)$. The boundaries at $A$, and $\alpha$, satisfy the interior sphere condition trivially.

**Lemma 6** *In $I^+$, let the uniformly elliptic operator $L^+$ be*

$$L^+ := \frac{d^2}{dR^2} + b^+(R)\frac{d}{dR} + c^+(R), \tag{8.2}$$

*where $b^+(R)$ and $c^+(R)$ are bounded functions in $I^+_\alpha$, for all $0 < \alpha < A$. Also, $c^+(R) < 0$ in $I^+$. Let $f \in C^2(I^+) \cap C^0(\overline{I^+}) \cap C^1(I^+ \cup \{A\})$ that satisfies $L^+ f = 0$ and $f(0) = 0$.*

*Then the following holds*

- $f(A) > 0 \Rightarrow \partial_R f(A) > 0.$
- $f(A) < 0 \Rightarrow \partial_R f(A) < 0.$

- $f(A) = 0 \Rightarrow f(R) = 0 \quad \forall R \in \overline{I^+}$.

**Proof:** Consider first $f(A) > 0$. $f(0) = 0$ implies that for all $\delta > 0$ there exists $\varepsilon > 0$ such that $|f(R)| < \delta$ whenever $R < \varepsilon$. Set $\delta = f(A) > 0$. Clearly $\varepsilon < A$. Define the domain $\tilde{D}^+ \subset I^+$ as $\tilde{D}^+ = (\varepsilon, A)$, non empty by construction, and consider the operator $L^+$ in $\tilde{D}^+$. Since $f(R) < f(A)$ and $f \in C^2(\tilde{D}^+)$, $f$ is nonconstant in $\tilde{D}^+$, and thus, since $b^+$ and $c^+$ are bounded in $\tilde{D}^+$, the strong maximum principle ensures that the function $f$ does not attain a non-negative maximum in $\tilde{D}^+$. Hence the function $f$ in $\tilde{D}^+$ attains its maximum at $R = A$. Now, given that $f(R) \in C^0(\overline{I^+})$, with $f(A) > f(R)$ for all $R \in \tilde{D}^+$, the boundary point lemma states that $\partial_R f|_{R=A} > 0$.

Consider now $f(A) < 0$. The change $f(R) \to -f(R)$ leads to the previous case, since $-f(A) > 0$ and thus $\partial_R f|_{R=A} < 0$.

Assume $f(A) = 0$ and that $f$ is nonconstant. By the strong maximum principle $f$ cannot attain a non-negative maximum nor a non-positive minimum in $I^+_\alpha$, for some fixed value of $\alpha$. Since the boundary is composed by two points, either $f(A) = 0$ is the non negative maximum and $f(\alpha) \leq 0$ is the non positive minimum, or $f(A) = 0$ is the non positive minimum and $f(\alpha) \geq 0$ is the non negative maximum.

Let us consider first $f(\alpha) = 0$. Since $L^+ f = 0$ in a bounded domain $I^+_\alpha$ and $f = 0$ in $\partial I^+_\alpha$, the function $f$ is zero in $I^+_\alpha$. Take $f(\alpha) > 0$. Given $f(0) = 0$ there exists $\varepsilon < \alpha$ for which $|f(R < \varepsilon)| < f(\alpha)$. Now choose $B$ satisfying $0 < B < \varepsilon$ to define still another $I^+_B$. Since $f(B) < f(\alpha)$ by construction, $f$ does not attain its maximum on $\partial I^+_B$. Hence $f$ must be constant in $I^+_B$, and zero because $f(A) = 0$. The same reasoning works for $f(\alpha) < 0$, setting $f \to -f$. This proves $f = 0$ in $I^+_\alpha$ for all $\alpha \in (0, A)$. Therefore, $f = 0$ in $\overline{I^+}$ follows by continuity.

■

**Lemma 7** *In $I^-$, let the uniformly elliptic operator $L^-$ be*

$$L^- := \frac{d^2}{dR^2} + b^-(R)\frac{d}{dR} + c^-(R), \tag{8.3}$$

*where $b^-(R)$ and $c^-(R)$ are bounded functions in $I^-$. Also, $c^-(R) < 0$ in $I^-$. Let $f \in C^2(I^-) \cap C^1(\overline{I^-})$ that satisfies $L^- f = 0$ and*

$$\lim_{R\to\infty} f(R) = 0. \tag{8.4}$$

*Then the following holds*

- $f(A) > 0 \Rightarrow \partial_R f(A) < 0$.

- $f(A) < 0 \Rightarrow \partial_R f(A) > 0$.
- $f(A) = 0 \Rightarrow f(R) = 0 \quad \forall R \in \overline{I^-}$.

**Proof:** Assume that $|f(A)| > 0$. A direct application of the strong maximum principle and the boundary point lemma gives the stated result.

Suppose now $f(A) = 0$ and that $f(R_0) > 0$ for some $R_0 > A$. Given the limit condition (8.4), fix the constant $\delta = f(R_0)$ and $\varepsilon > R_0$ such that $|f(R > \varepsilon)| < \delta$. Take now the interval $(A, \varepsilon)$. By construction $0 = f(A) \leq |f(\varepsilon)| \leq f(R_0)$. Since the function is achieving a non negative maximum equal or greater than $f(R_0)$ in $(A, \varepsilon)$, $f$ must be constant in this interval, and zero, because $f(A) = 0$. If $f(R_0) < 0$ the same applies to $-f$. Then $f = 0$ in $(A, R_0)$. This result holds for any $R_0 > A$ and therefore for the whole $I^-$. ∎

**Lemma 8** *Consider the problems for $\{L^+, f^+\}$ in $I^+$ and for $\{L^-, f^-\}$ in $I^-$ that satisfy the conditions of Lemmas 6 and 7 respectively. Impose the boundary conditions*

$$f^+(A^+) - f^-(A^-) = 0, \quad (\partial_R f)^+|_{R^+=A^+} - (\partial_R f)^-|_{R^-=A^-} = 0. \tag{8.5}$$

*Then $f^+ = 0$ in $\overline{I^+}$ and $f^- = 0$ in $\overline{I^-}$.*

**Proof:** Suppose $f^+(A^+) = f^-(A^-) > 0$. Then by Lemma 6, $(\partial_R f)^+|_{R^+=A^+} > 0$. On the other hand, by Lemma 7, $(\partial_R f)^-|_{R^-=A^-} < 0$. Thus, the second matching condition in (8.5) cannot be fulfilled. The same result follows for $f^+(A^+) = f^-(A^-) < 0$. For $f^+(A^+) = f^-(A^-) = 0$ Lemmas 6 and 7 lead to the result. ∎

**Lemma 9** *Consider the problems for $\{L_1^-, f_1^-\}$ in $I_1^- = (A_1, \infty)$ and for $\{L_2^-, f_2^-\}$ in $I_2^- = (A_2, \infty)$ that satisfy the conditions of Lemma 7. Impose the boundary conditions*

$$f_1^-(A_1) - f_2^-(A_2) = 0, \quad (\partial_R f)_1^-|_{R_1=A_1} + \beta^2 (\partial_R f)_2^-|_{R_2=A_2} = 0 \tag{8.6}$$

*for some (nonzero) constants $\beta$, $A_1$, $A_2$. Then $f_1^- = 0$ in $\overline{I_1^-}$ and $f_2^- = 0$ in $\overline{I_2^-}$.*

**Proof:** Suppose $f_1^-(A_1) = f_2^-(A_2) > 0$. Following Lemma 7, the derivatives $(\partial_R f)_1^-|_{R_1=A_1}$ and $(\partial_R f)_2^-|_{R_2=A_2}$ take the same signs. Hence, the second matching condition in (8.6) cannot be fulfilled. The same conclusion is reached if $f_1^-(A_1) = f_2^-(A_2) < 0$. The remaining possibility $f_1^-(A_1) = f_2^-(A_2) = 0$, implies that $f_1^- = 0$ in $\overline{I_1^-}$ and $f_2^- = 0$ in $\overline{I_2^-}$, by Lemma 7. ∎

## 8.3 First order problem

Let us define the domains $D^+ : \{r_+ \in (0,a) \times S^2\}$ and $D^- : \{r_- \in (a,\infty) \times S^2\}$, where $S^2$ are the unit round spheres. $D^+$ and $D^-$ will correspond to the interior and exterior problems respectively. We will simply refer by $D$ either one, and denote coordinates without $\pm$ generically, if this leads not to confusion.

We define the spherical potential functions $\mathcal{G}^\pm(r,\theta)$ relative to $\omega^-(r,\theta)-\Omega$ and $\omega^+(r,\theta)$ by

$$\omega^- - \Omega = \frac{1}{\sin\theta}\partial_\theta \mathcal{G}^-, \quad \omega^+ = \frac{1}{\sin\theta}\partial_\theta \mathcal{G}^+ \tag{8.7}$$

The freedom in the addition of an arbitrary function of $r$ is used to fix $\mathcal{G}$ so that $\mathcal{G}$ is orthogonal to the $l=0$ Legendre polynomial $P_0(\cos\theta)$ on the unit round sphere, that is,

$$\int_{S^2} \mathcal{G}\eta_{S^2} = 0, \tag{8.8}$$

where $\eta_{S^2}$ denotes the volume element on $S^2$. The equation (6.24) for $\omega$ translates to the potential $\mathcal{G}$, for either $\pm$, as

$$\triangle_\gamma \mathcal{G} + V\mathcal{G} = 0 \tag{8.9}$$

for the metric $\gamma = r^4 e^{-\nu}\left(e^\lambda dr^2 + r^2 d\Omega^2\right)$, where $d\Omega^2$ denotes the metric on $S^2$, and with $V(r) = \frac{2}{r^6}e^\nu\left(1+2re^{-\lambda}j'/j\right)$. The change $r^3 = 3R$ renders the metric $\gamma$ and potential $V$ as

$$\gamma = e^{-\nu}\left(e^\lambda dR^2 + 9R^2 d\Omega^2\right), \qquad V(R) = \frac{2}{9R^2}e^\nu\left(1+6Re^{-\lambda}\frac{j'(R)}{j(R)}\right). \tag{8.10}$$

Let us define $A := a^3/3$. The problems for $\mathcal{G}^\pm$ given by (8.9) are thus defined on the spaces $D^\pm$, now endowed with the metrics $\gamma^\pm$. Each space $(D^\pm,\gamma^\pm)$ can now be decomposed as $I^\pm \times S_R$ where $S_R$ are round spheres of radius $\rho(R) = 3e^{-\nu/2}R$ and $I^+ = (0,A)$, $I^- = (A,\infty)$. The important feature regarding the space $(D^+,\gamma^+)$ is that its closure $\overline{D^+}$ is completed, apart from the sphere of radius $\rho(A)$, with a *point* attached to $R=0$.

On the unit sphere $S^2$ we define the quantities $\mathcal{G}_l$ at each $R$ by

$$\mathcal{G}_l(R) := \int_{S^2} \mathcal{G}P_l(\cos\theta)\eta_{S^2}. \tag{8.11}$$

The operator $\triangle_\gamma$ separates into $\triangle_\gamma = L_R + \rho^{-2}\Delta_{S^2}$, where $L_R$ is a purely radial second order differential operator and $\Delta_{S^2}$ is the Laplacian on the (unit) sphere. The equations for the radial functions $\mathcal{G}_l$ are obtained by the integration of equation (8.9) around $S^2$ and read

$$\int_{S^2} (\triangle_\gamma + V)\mathcal{G}P_l\eta_{S^2} = 0 \tag{8.12a}$$

$$\Rightarrow \left(L_R - \frac{l(l+1)}{\rho^2} + V\right)\mathcal{G}_l = 0. \tag{8.12b}$$

The first and third terms in (8.12b) follow from the nondependency of $L_R$ and $V$ on the angular coordinates, while the second term arises integrating by parts and using $(\Delta_{S^2} + l(l+1))P_l = 0$. The equations for the functions $\mathcal{G}_l$ explicitly read

$$L\mathcal{G}_l := \frac{1}{R^2 j}\frac{d}{dR}\left(R^2 j \frac{d\mathcal{G}_l}{dR}\right) + \frac{1}{9R^2}\left(e^{\lambda}(2 - l(l+1)) + 12R\frac{j'(R)}{j(R)}\right)\mathcal{G}_l = 0, \qquad (8.13)$$

which hold in the respective $I^{\pm}$. The matching conditions for the functions $\omega^{\pm}$ (5.14) and (5.15) imply, for their respective $\mathcal{G}_l^{\pm}$, that

$$\mathcal{G}_l^{+}|_{R_+=A} - \mathcal{G}_l^{-}|_{R_-=A} = 0, \quad (\partial_{R^+}\mathcal{G}_l)|_{R_+=A} - (\partial_{R^-}\mathcal{G}_l)|_{R_-=A} = 0, \quad l \geq 2, \qquad (8.14)$$

where we have used $\vec{n}^{\pm} = -(3A)^{2/3}e^{\lambda/2}\partial_{R^{\pm}}|_{R^{\pm}=A}$.

From here on, we restrict ourselves to $l \geq 2$. Let us study first the interior problem. The equation (8.13) for $\mathcal{G}_l^+$ holds in $I^+$ and can be rewritten as

$$L^+\mathcal{G}_l^+ = \frac{d^2\mathcal{G}_l^+}{dR_+{}^2} + \left(\frac{2}{R_+} + \frac{j^{+\prime}}{j^+}\right)\frac{d\mathcal{G}_l^+}{dR_+} + \frac{1}{9R_+{}^2}\left(e^{\lambda^+}(2 - l(l+1)) + 12R_+\frac{j^{+\prime}}{j^+}\right)\mathcal{G}_l^+ = 0, \qquad (8.15)$$

which adapts to the notation of Lemma 6 by identifying

$$b^+(R_+) = \frac{2}{R_+} + \frac{j^{+\prime}}{j^+}, \qquad (8.16)$$

$$c^+(R_+) = \frac{1}{9R_+{}^2}\left(e^{\lambda^+}(2 - l(l+1)) + 12R_+\frac{j^{+\prime}}{j^+}\right). \qquad (8.17)$$

In order to analyse the behaviour of the functions $b^+(R_+)$ and $c^+(R_+)$ in the operator $L^+$, let us restrict ourselves to regular interior background configurations in which neither the pressure nor the energy density diverge and the sum $E^+ + P^+ \geq 0$. Recall also that equations (6.12), (6.13) and (6.15) of the background configuration lead to

$$\frac{j^{+\prime}}{j^+} = -\frac{4\pi e^{\lambda^+}}{(3R_+)^{1/3}}(E^+ + P^+) \leq 0. \qquad (8.18)$$

The limiting value of the function $e^{\lambda^+}$ near the origin is found to be $(1 - 8\pi(3R_+)^{2/3}E_c^+/3)^{-1}$, and therefore it remains bounded. Hence $b^+$ and $c^+$ are bounded functions, and $c^+(R_+) < 0$, in $I_\alpha^+$.

We require that $\mathcal{G}^+ \in C^2(D^+)$, and to be once differentiable on the sphere at $A$ and bounded at the origin. Now, since $R_+ = 0$ in $\overline{D^+}$ is a point, in order to have $\mathcal{G}^+$ defined there, the limit $\lim_{R^+\to 0}\mathcal{G}^+$ cannot depend on the angular coordinate $\theta$, that is, $\lim_{R^+\to 0}\mathcal{G}^+$ must be a constant, and because of the orthogonality condition (8.8) that constant is zero. All in all, regularity of $\mathcal{G}^+$ at $D^+$ implies

$$\lim_{R_+\to 0}\mathcal{G}^+ = 0. \qquad (8.19)$$

The function $\mathcal{G}_l^+(R_+)$ in $I^+$ is thus extended to the origin ($R_+ = 0$) by continuity imposing $\mathcal{G}_l^+(0) = 0$, and therefore $\mathcal{G}_l^+ \in C^2(I^+) \cap C^0(\overline{I^+}) \cap C^1(I^+ \cup \{A\})$. Hence, the interior problem for $\{L^+, \mathcal{G}_l^+\}$ in $I^+$ satisfies the conditions addressed in Lemma 6.

In $D^-$, $j^-(R_-) = 1$, and equation (8.13) just reads

$$L^-\mathcal{G}_l^- = \frac{d^2\mathcal{G}_l^-}{dR_-{}^2} + \frac{2}{R_-}\frac{d\mathcal{G}_l^-}{dR_-} + \frac{e^{\lambda_-}}{9R_-{}^2}(2 - l(l+1))\mathcal{G}_l^- = 0, \tag{8.20}$$

in the domain $I^- = (A, \infty)$. Thus, we identify

$$b^-(R_-) = \frac{2}{R_-}, \tag{8.21}$$

$$c^-(R_-) = \frac{e^{\lambda^-}}{9R_-{}^2}(2 - l(l+1)) = \frac{1}{9R_-{}^2}\left(1 - \frac{2M}{(3R_-)^{1/3}}\right)^{-1}(2 - l(l+1)). \tag{8.22}$$

We consider background vacuum configurations for which $0 < 2M < a \le r_-$, so that $0 < 2M < (3A)^{1/3} \le (3R_-)^{1/3}$, for which $b^-$ and $c^-$ are bounded and $c^- < 0$ in $I^-$.

We demand that $\mathcal{G}^- \in C^2(D^-) \cap C^1(\overline{D^-})$. Hence $\mathcal{G}_l^- \in C^2(I^-) \cap C^1(\overline{I^-})$. Apart from this, we also ask for regularity at infinity. An observation of equation (8.20) in $u \equiv R_-{}^{-1}$ reveals that in the limit $R_- \to \infty \Leftrightarrow u \to 0$, regular solutions must vanish for $l \neq 1$, this is

$$\lim_{R_- \to \infty} \mathcal{G}_l^-(R_-) = 0 \qquad l \neq 1. \tag{8.23}$$

Hence, the exterior problem for $\{L^-, \mathcal{G}_l^-\}$ in $I^-$ fulfils the conditions of Lemma 7.

Recall now the matching conditions (8.14). By Lemma 8, the only possibility is that $\mathcal{G}_l^+(R_+) = 0$ in $\overline{I^+}$ and $\mathcal{G}_l^-(R_-) = 0$ in $\overline{I^-}$ for $l \ge 2$. It follows from (8.11) that $\mathcal{G}(R, \theta)$ for each $R$ is orthogonal to every Legendre polynomial in the unit sphere except to $P_1$. This implies, via (8.7), that $\tilde{\omega}$ is a function of $R$ alone.

**Proposition 9** *Consider the functions $\omega^\pm(r, \theta)$, defined respectively in the interior domain $D^+ : I^+ = (0, a) \times S_{r_+}$, where $\omega^+$ is a $C^2(D^+)$, differentiable once on $S_a$ and bounded in $\overline{D^+}$ function, and in the exterior domain $D^- : I^- = (a, \infty) \times S_{r_-}$, where $\omega^-$ is a $C^2(D^-) \cap C^1(\overline{D^-})$ function regular at infinity. Let $\omega^\pm$ satisfy equation (6.24), particularized adequately for each domain, and let the two problems be related in $r_\pm = a$ by the boundary conditions (5.14) and (5.15). Assume that $E + P \ge 0$ in $D^+$ and that $0 < 2M < a$. Then, neither of the $\omega^\pm$ can depend on the corresponding angular coordinate, that is,*

$$\omega^+ = \omega^+(r_+), \quad \omega^- = \omega^-(r_-). \tag{8.24}$$

## 8.4 Second order problem

The second order problem reduces to the study of the functions $\tilde{v}^{\perp\pm}$, defined in Section 6.3 by $\tilde{v}^{\perp} := \tilde{h}^{\perp} + \tilde{k}^{\perp}$. These functions satisfy, in the corresponding regions (we drop the $\pm$ here), the equation (6.67)

$$(\Delta_\gamma + V(r))\tilde{v}^{\perp}(r,z) = 0, \tag{8.25}$$

written in terms of the auxiliary metric (6.66) and the radial potential $V$ that in terms of the function $g(r)$ (6.64) read

$$\gamma = \left(\frac{e^{\lambda/2}g(r)}{r^2}\right)^2 (e^{\lambda}dr^2 + r^2 d\Omega^2), \tag{8.26}$$

$$V(r) = \frac{2e^{\lambda}}{r^2}\left(\frac{e^{\lambda}g(r)}{r^2}\right)^{-2}. \tag{8.27}$$

Let us remark that the potential $V$ is positive everywhere in both (interior and exterior) regions. We need to demand that $\tilde{v}^{\perp\pm} \in C^2(D^{\pm})$ respectively and that they have well defined normal derivatives at the boundaries.

We consider the interior problem first. In the limit $r_+ \to 0$, making use of

$$\lim_{r_+\to 0} g^+(r_+) = \lim_{r_+\to 0} \frac{1}{j_c}\left(\frac{1}{8\pi(P_c + E_c/3)}\right)^2 \frac{1}{r_+^2}, \quad \lim_{r_+\to 0} e^{\lambda(r_+)} = 1, \tag{8.28}$$

the metric $\gamma^+$ (8.26) and the potential $V^+$ (8.27) show the following behaviour

$$\gamma^+_{r_+\to 0} = j_c^{-2}\left(8\pi(P_c + E_c/3)\right)^{-4}\left(\frac{dr_+{}^2}{r_+{}^8} + \frac{d\Omega^2}{r_+{}^6}\right), \tag{8.29}$$

$$V^+(r_+\to 0) = 2(8\pi)^4 j_c^2 \left(P_c + \frac{E_c}{3}\right)^4 r_+{}^6. \tag{8.30}$$

Hence the radial coordinate $R_+ := r_+^{-3}/3$, $R_+ \in (\tilde{A} := a^{-3}/3, \infty)$, is adapted to the area of the spheres in this limit since

$$\gamma^+_{R_+\to\infty} = j_c^{-2}\left(8\pi(P_c + E_c/3)\right)^{-4}\left(dR_+{}^2 + 9R_+{}^2 d\Omega^2\right), \tag{8.31}$$

$$V^+(R_+\to\infty) = \frac{2j_c^2}{9}\left(P_c + \frac{E_c}{3}\right)^4 \frac{1}{R_+{}^2}. \tag{8.32}$$

Keeping this radial coordinate for the whole interior renders the metric $\gamma^+$ (8.26) as

$$\gamma^+ = \psi_+^2(R_+)\left(e^{\lambda(R_+)}dR_+{}^2 + 9R_+{}^2 d\Omega^2\right), \tag{8.33}$$

with $\psi_+^2(R_+) := (3R_+)^{-4/3}e^{\lambda(R_+)}g_+{}^2(R_+)$, and the potential reads

$$V(R_+) = \frac{2e^{\lambda^+(R_+)}}{(3R_+)^{-2/3}g^{+2}(R_+)}. \tag{8.34}$$

The space $(D^+, \gamma^+)$ is decomposed as $I^+ \times S_{R_+}$ where $S_{R_+}$ are round spheres of radius $3R_+\psi_+$ and $I^+ = (\tilde{A}, \infty)$. On the other hand, the operator $\triangle_{\gamma^+}$ gets decomposed in the radial and angular differential operators as

$$\triangle_{\gamma^+} = \frac{3^{4/3}e^{-2\lambda^+}}{g^{+3}} \frac{\partial}{\partial R_+}\left({R_+}^{4/3} g^+ \frac{\partial}{\partial R_+}\right) + \left(\frac{e^{-\lambda^+/2}}{(3R_+)^{1/3} g^+}\right)^2 \Delta^+_{S_2}. \tag{8.35}$$

On the unit sphere $S^2$ we define the quantities $v_l^+$ by

$$v_l^+(R_+) := \int_{S^2} \tilde{v}^{\perp+} P_l(\cos\theta_+)\eta^+_{S^2}. \tag{8.36}$$

Integrating equation (8.25), with the potential given explicitly by (8.34) and the differential operator by (8.35), around the unit sphere via the formula (8.12a), the equation for the $v_l^+$ results to be

$$\frac{d^2 v_l^+}{d{R_+}^2} + \left(\frac{4}{3R_+} + \frac{g^{+\prime}}{g^+}\right)\frac{dv_l^+}{dR_+} + \frac{e^\lambda}{9{R_+}^2}(2 - l(l+1))v_l^+ = 0. \tag{8.37}$$

The coefficient of the first derivative is expanded in terms of the explicit value of $g(R_+)$ as

$$b^+(R_+) = \frac{4}{3R_+} - \frac{j^{+\prime}}{j^+} - 2\frac{\nu^{+\prime\prime}}{\nu^{+\prime}}, \tag{8.38}$$

where

$$\begin{aligned} \frac{j^{+\prime}}{j^+} &= \frac{4\pi e^{\lambda^+}}{(3R_+)^{5/3}}(E^+ + P^+), \\ \frac{\nu^{+\prime\prime}}{\nu^{+\prime}} &= \frac{4(1-e^{\lambda^+}) + 3R_+(\lambda^{+\prime}(-2+3\nu^{+\prime}R_+) - \nu^{+\prime}(10+3\nu^{+\prime}R_+))}{18{R_+}^2\nu^{+\prime}}. \end{aligned}$$

Using the equations for the background (6.12), (6.13) and (6.17), the asymptotic limits of the functions $M$, $E$, $P$ allow us to determine that $\nu^{+\prime\prime}/\nu^{+\prime}$ decays to infinity as $-5/3R_+$. Finally, putting together the behaviour of $j^{+\prime}/j^+$ and $\nu^{+\prime\prime}/\nu^{+\prime}$ at infinity, we see that $\lim_{R_+\to\infty} b^+ = -1/3R_+$. Exploring equation (8.37) in this limit we can readily check that for $l > 1$

$$\lim_{R_+\to\infty} {v_l}^+ = 0. \tag{8.39}$$

Thus, the interior problem in $I^+$ for $v_l^+ \in C^2(I^+) \cap C^1(\overline{I^+})$ satisfies the assumptions in Lemma 7.

We follow an analogous procedure to treat the exterior problem. First, we perform the change of radial coordinate $R_- := (r_-)^3/3$ and define

$$\psi_-^2(R_-) := \frac{1}{16M^4}\left(1 - \frac{2M}{(3R_-)^{1/3}}\right)^3,$$

so that the metric and the potential become

$$\gamma^- = \psi_-^2(R_-)\left[\left(1-\frac{2M}{(3R_-)^{1/3}}\right)^{-1} dR_-{}^2 + 9(R_-)^2 d\Omega^2)\right], \tag{8.40}$$

$$V^-(R_-) = \frac{32M^4}{9R_-{}^2}\left(1-\frac{2M}{(3R_-)^{1/3}}\right)^{-3}. \tag{8.41}$$

Now $(D^-,\gamma^-)$ is decomposed as $I^- \times S_{R_-}$ where $S_{R_-}$ are round spheres of radius $3R_-\psi_-$ and $I^- = (A,\infty)$. We define the functions $v_l^-$ integrating the product of $\tilde{v}^{\perp -}$ with the legendre polynomials over the unit sphere, just as the $v_l^+$ in (8.36).

The integration of equation (8.25) around the unit sphere results now in

$$\frac{d^2 v_l^-}{dR_-^2} + \underbrace{\frac{2}{3R_-}\left(2+\left(1-\frac{2M}{(3R_-)^{1/3}}\right)^{-1}\right)}_{b^-(R_-)}\frac{dv_l^-}{dR_-} + \underbrace{\frac{1}{(3R_-)^2}\frac{2-l(l+1)}{1-\frac{2M}{(3R_-)^{1/3}}}}_{c^-(R_-)} v_l^- = 0. \tag{8.42}$$

Clearly the functions $b^-$ and $c^-$ are bounded in $I^-$, since $2M < r_- = (3R_-)^{1/3}$. In addition to this, the function $c^-$ is negative for $l > 2$. Finally, in the limit $R_- \to \infty$ equation (8.42) agrees with (8.20). Thus, regular solutions must vanish at infinity,

$$\lim_{R_-\to\infty} v_l{}^- = 0. \tag{8.43}$$

Hence the problem for $v_l^+ \in C^2(I^+) \cap C^1(\overline{I^+})$ fits in Lemma 7.

Recall now the matching conditions (7.44) and (7.45) at $\Sigma_0$. The latter reads

$$(\vec{n} v_l)^+|_{r^+=a} - (\vec{n} v_l)^-|_{r^-=a} = 0 \Rightarrow -(\partial_r v_l)^+|_{r^+=a} + (\partial_r v_l)^-|_{r^-=a} = 0$$
$$\Rightarrow \partial_{R^+} v_l^+|_{R^+=\tilde{A}} + a^6 \partial_{R^-} v_l^-|_{R^-=A} = 0,$$

where we have used the explicit form of the normal vectors $\vec{n}^\pm$ and expressed them in terms of $R^+$ and $R^-$ respectively. These result in the matching of the two problems as described in Lemma 9, with $\beta^2 = a^6$. Thus, a direct application of Lemma 9 leads to the result that the functions $v_l^\pm$ must vanish in $\overline{I^\pm}$ for all $l > 2$. This clearly implies that $\tilde{v}^{\perp\pm} = 0$ in their respective $D^\pm$.

**Proposition 10** *Consider the functions $\tilde{v}^{\perp\pm}(r,\theta)$, defined respectively in the interior domain $D^+ : I^+ = (0,a) \times S_{r_+}$, where $\tilde{v}^{\perp +}$ is a $C^2(D^-) \cap C^1(\overline{D^-})$ function regular at the origin, and in the exterior domain $D^- : I^- = (a,\infty) \times S_{r_-}$, where $\tilde{v}^{\perp -}$ is a $C^2(D^-) \cap C^1(\overline{D^-})$ function regular at infinity. Let $\tilde{v}^{\perp\pm}$ satisfy equation (8.25), written in terms of the auxiliary metric (8.26) and the potential (8.27), particularized accordingly for each domain. Assume that $E + P \geq 0$ in $D^+$ and that $0 < 2M < a$. Let the two problems be related in $r_\pm = a > 2M > 0$ by the boundary conditions (7.44) and (7.45). Then, $\tilde{v}^{\perp +} = 0$ in $\overline{D^+}$ and $\tilde{v}^{\perp -} = 0$ in $\overline{D^-}$.*

A direct application of the field equations lead to the vanishing of the $\perp$ sector in every function of the second order perturbation tensor. The field equation (6.49) now provides $\tilde{h}^{\perp} = 0$, so that $\tilde{k}^{\perp}$ also has to vanish, by the definition of $\tilde{v}^{\perp}$. Finally the algebraic relation (6.46) implies that $\tilde{m}^{\perp} = 0$, so that the full orthogonal sector is ruled out. Then we conclude

**Corollary 10.1** *Given the problem set in Chapters 6 and 7, the angular structure of the functions in the perturbation tensor $K_2$ (5.3) is given by*

$$\begin{aligned} h(r,\theta) &= h_0(r) + h_2(r)P_2(\cos\theta), \\ k(r,\theta) &= k_0(r) + k_2(r)P_2(\cos\theta), \\ m(r,\theta) &= m_0(r) + m_2(r)P_2(\cos\theta), \\ f(r,\theta) &= f_2(r)P_2(\cos\theta), \end{aligned} \tag{8.44}$$

*where equatorial symmetry has been imposed to rule out the contribution in $l = 1$ in the expansions above.*

We have showed in Propositions 9 and 10 that the matching itself, supported with regularity conditions at the origin/infinity on the functions involved, determines their angular behaviour. In view of these results, we can reformulate Theorem 5 without the assumptions regarding the angular behaviour of the perturbations.

**Theorem 7** *Let $(\mathcal{V}, g)$ with $\Sigma_0$ be the static and spherically symmetric background matched spacetime configuration, perturbed at either side to first order by the functions $\omega^{\pm}(r_{\pm}, \theta_{\pm})$ through $K_1{}^{\pm}$ as defined in (5.2) plus the unknowns $Q_1^{\pm}(\tau, \vartheta)$ and $\vec{T}_1^{\pm}(\tau, \vartheta)$, as described in Proposition 7, so that the first order matching conditions (5.14) and (5.15) plus (5.17) and (5.18) hold. Let the configuration be perturbed to second order by $K_2{}^{\pm}$ as defined in (5.3), plus the unknowns $\hat{Q}_2^{\pm}(\tau, \vartheta)$ and $\vec{T}_2^{\pm}(\tau, \vartheta)$ on $\Sigma_0$, and assume that the interior region (+) satisfies the field equations for a perfect fluid with barotropic equation of state and that the exterior (−) region is asymptotically flat and satisfies the vacuum field equations up to second order.*

*The energy density $E(r_+)$ and pressure $P(r_+)$ of the interior background configuration are given by (6.12) and (6.13) and must satisfy (6.19). Assume $E + P \geq 0$ in all the fluid. The background exterior vacuum solution is given by (6.20), and we assume $0 < 2M < a$. Consider the convenient background quantities defined in (6.16).*

*Let $\vec{u}_\varepsilon$ be the unit vector fluid corresponding to the interior family of metric tensors $g_\varepsilon^+ = g^+ + \varepsilon K_1{}^+ + \frac{1}{2}\varepsilon^2 K_2{}^+ + \mathcal{O}(\varepsilon^3)$. Assume that $\vec{u}_\varepsilon$ satisfies (6.8) for some constant $\Omega$. Let $J$ be defined by the first order exterior solution (6.26).*

*Then*

1. *At both sides (±) that the first order function* $\omega$ *depends only on the radial coordinate, and given equatorial symmetry, the second order functions are decomposed in Legendre polynomials in terms of* $\{h_0, h_2, m_0, m_2, k_0, k_2, f_2\}$ *by (8.44).*

2. *The second order pressure* $P^{(2)}$ *and energy density* $E^{(2)}$ *of the fluid inherit the same angular dependency, that is, (6.30) hold for some* $E_0^{(2)}(r), E_2^{(2)}(r), P_0^{(2)}(r)$ *and* $P_2^{(2)}(r)$. *With the help of convenient alternative "tilded" counterparts, defined in (6.32)-(6.33) plus (6.37) and (6.59), the Einstein's field equations in the interior can be expressed as the system (6.38), (6.39) and (6.41) for some constant* $\gamma$ *for the set* $\{\tilde{\mathcal{P}}_0^+, \tilde{m}_0^+, \tilde{h}_0^+\}$ *plus the system (6.56), (6.57), (6.58) for the set* $\{\tilde{h}_2^+, \tilde{k}_2^+, \tilde{m}_2^+\}$. *The vacuum solution at second order is given by (6.42), (6.43), (6.61), (6.62) and (6.63) where* $\delta M$ *and* $A$ *are arbitrary constants.*

3. *Given the Einstein's field equations of the previous point are satisfied, the necessary and sufficient conditions that the metric perturbation tensors* $K_2{}^{\pm}$ *must satisfy to fulfil the second order matching conditions are given by (7.30) and (7.35) for the sets* $\{\tilde{\mathcal{P}}_0^\pm, \tilde{m}_0^\pm, \tilde{h}_0^\pm\}$, *with arbitrary constant* $H_0$, *and the two equations in (7.38) for the sets* $\{\tilde{h}_2^\pm, \tilde{k}_2^\pm, \tilde{m}_2^\pm\}$.

9

# The mass in Newtonian gravity and GR

The original treatment aimed at the study of (rigidly) rotating stars in a perturbative scheme is due to Chandrasekhar for Newtonian gravity, back in 1933 [23]. It was only in 1967 that Hartle put forward the model within the realm of General Relativity [57]. Although the study in [57] covers any barotropic equation of state, the work in [23] focuses, from some point onwards, only on polytropic equations of state, i.e. of the form $p = K\rho^{1+\frac{1}{n}}$ for some constants $K$ and $n$, where $p$ and $\rho$ denote the pressure and the mass density of the star. The relationship between the Newtonian and the GR approaches was presented in [57], and the GR procedure was found to be consistent with the Newtonian case by taking care of the suitable limit.

However, the computation of the total mass of the rotating configuration as a function of the central density shown in [57] has to be amended by a term proportional to the value of the background energy density at the surface of the star, explicitly given by (7.53). That value is zero for certain equations of state (including polytropic EOS), but it does not vanish necessarily (for instance in models of strange quark stars [34]). As we show next, that term contributes to the Newtonian limit, and appears indeed, although implicitly, in the original work by Chandrasekhar [23]. Since most of the models for stars rely on a polytropic EOS, the appearance of that term had been somehow forgotten, even in the review of the Newtonian approach in [57].

## 9.1 The Newtonian star

We first concentrate on the computation of the mass as stated in [23], expand that for a general case (for any equation of state, so that the density at the interior does not necessarily vanishes at the boundary of the star), and show how the expressions in [23] for polytropic equations of state follow indeed. In p.396 of [23] the mass is claimed to be given by

$$M = 2\pi \int \int \rho r^2 dr d\mu, \tag{9.1}$$

where $\mu := \cos\theta$, and $\{r, \theta, \phi\}$ are spherical coordinates, so that $r$ and $\theta$ are the radial coordinate and azimutal angle on the sphere respectively.

In agreement with the next equation in [23], as we will show later, (9.1) stands for the integral over the deformed volume. Indeed, the shape of the star is described in [23] to be the sphere of the background configuration plus a deformation at first order in a perturbation parameter $v$, which corresponds to a second order in the angular velocity $\omega^2$ over the value of the central density, this last denoted by $\lambda$ in [23], but we will use $\rho_c$ here. Thus the perturbation parameter $v$, defined in (10) in [23] is $v := \omega^2/2\pi G\rho_c$.

Let us first review just the necessary of the Newtonian treatment in order to obtain the expression of the total mass of the rotating star, suitable to be computed by solving the relevant problems at different orders in the perturbation. We follow essentially the description of the Newtonian approach as made in [57], and will compare with that in [23] when necessary.

## Preliminaries on the Newtonian general approach

The **non-rotating** static spherically symmetric configuration is described by the mass density $\rho^{(0)}(r)$, pressure $p^{(0)}(r)$ and Newtonian potential $U^{(0)}(r)$. The radial variable $r$ runs from 0 to $a$ in the interior and $r > a$ corresponds to the vacuum exterior. We will only assume that $\rho^{(0)}$ is smooth in $(0, a)$ and vanishes for all $r > a$. In other words, $\rho^{(0)}$ is piecewise differentiable in $(0, \infty)$, smooth except at $a$, where $\rho^{(0)}$ is allowed to have a jump. The same applies to $p^{(0)}(r)$, except that $p^{(0)}$ vanishes at $a$ necessarily, and it is therefore continuous. From now onwards, the values of $\rho^{(0)}(a)$ and $\rho^{(0)\prime}(a)$ (or any function explicitly defined in $(0, a)$) must be understood as the limits of $\rho^{(0)}(s)$ and $\rho^{(0)\prime}(s)$ as $s \to a$, i.e. their limits from the interior.

The three equations of structure that govern the configuration are a barotropic equation of state $p^{(0)} = p^{(0)}(\rho^{(0)})$, the hydrostatic equilibrium first integral and the Poisson equation, i.e.

$$\gamma = \int_0^r \frac{1}{\rho^{(0)}(s)} \frac{dp^{(0)}(s)}{ds} ds + U^{(0)}(r), \quad \nabla^2 U^{(0)}(r) = 4\pi G \rho^{(0)}(r), \tag{9.2}$$

where the constant $\gamma$ is identified as the chemical potential. We use $\nabla^2$ for the flat Laplacian in spherical coordinates $\{r, \theta, \phi\}$. The two equations above combined yield

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{(0)}}\frac{dp^{(0)}}{dr}\right) = -4\pi G \rho^{(0)}, \tag{9.3}$$

which is recognized as the *fundamental equation* in [23]. In spherically symmetric configurations the mass function is given by

$$M^{(0)}(r) = 4\pi \int_0^r \rho^{(0)}(s) s^2 ds. \tag{9.4}$$

Given that $M^{(0)}(0) = 0$ and that regularity at the origin $r = 0$ implies $dU^{(0)}/dr(0) = 0$, the mass and the potential are related by

$$\frac{dU^{(0)}(r)}{dr} = \frac{G}{r^2} M^{(0)}(r).$$

The system of three equations can be integrated in terms of boundary conditions at the origin and thus provide, e.g. , the total mass of the star, $M_S^{(0)} := M^{(0)}(a)$, in terms of the central density $\rho_c = \rho^{(0)}(0)$. We denote that function by $M_S^{(0)}(\rho_c)$.

Consider now the (perturbed) **rotating** configuration. The mass density of the rotating configuration $\rho(r, \mu)$ and the gravitational potential $U(r, \mu)$ are expanded perturbatively to first order in $v$ as

$$\rho(r, \mu) = \rho^{(0)}(r) + v\rho^{(2)}(r, \mu) + \mathcal{O}(v^2). \tag{9.5}$$

$$U(r, \theta) = U^{(0)}(r) + vU^{(2)}(r, \theta) + O(v^2). \tag{9.6}$$

A new radial coordinate $R$ is now chosen so that it labels surfaces of constant density in the rotating configuration by [57] (see Chapter 4)

$$\rho(r(R, \mu), \mu) = \rho^{(0)}(R). \tag{9.7}$$

The interior of the rotating star is therefore defined by $R \in (0, a)$ by construction, and its surface located at $R = a$. The change between $R$ and $r$ must thus have the form

$$r(R, \mu) = R + v\zeta(R, \mu) + \mathcal{O}(v^2) \tag{9.8}$$

for some (differentiable) function $\zeta(R, \mu)$, which thus describes the deformation of the surface [23, 57]. Given that for any $f(r, \mu)$ differentiable in $r \in (0, a)$ we have

$$f(r(R, \mu), \mu) = f(R, \mu) + vf'(R, \mu)\zeta + \mathcal{O}(v^2), \tag{9.9}$$

where the prime denotes differentiation with respect to the first (or only) argument, the relation (9.7) provides, in particular,

$$\rho^{(2)}(R, \mu) = -\zeta(R, \mu)\frac{d\rho^{(0)}(R)}{dR}. \tag{9.10}$$

## Polytrope in [23]

We have described the relevant equations to describe a Newtonian static star. Let us now specify a polytropic equation of state, i.e. of the form

$$p = K\rho^{1+\frac{1}{n}}, \tag{9.11}$$

where $K$ is a constant, closely related to the speed of sound, and $n$ is the polytropic index. It is useful to replace the density $\rho$ by an adimensional function $\theta$, known as the Emden's function, for reasons that will be clear soon. In [23] the radial coordinate $r$ is, instead, conveniently rescaled to a new radial coordinate $\xi$ (eq. (9) in [23]). These substitutions read explicitly

$$\rho = \rho_c\theta^n, \quad p = K\rho_c^{1+1/n}\theta^{1+n}, \quad r = \left(\frac{(1+n)K}{4\pi G}\rho_c^{1/n-1}\right)^{1/2}\xi. \tag{9.12}$$

Now the *fundamental equation* (9.3) is written in the form

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n. \tag{9.13}$$

This is known as the Lane-Emden equation of index $n$. It is integrated from the origin outwards with the conditions $\theta(0) = 1$ and $\theta'(0) = 0$. The solution $\theta(\xi)$ is named the Lane-Emden function of index $n$. The shape of the star is then described in [23] to be the sphere of the background configuration $\xi_1$. Note that both the density and pressure vanish there.

The perturbation method in [23] differs a bit in its approach, since the starting point consists in taking perturbations directly in the Emden's funcion, generalizing it to

$$\Theta = \theta + v\Psi + O(v^2), \quad \Psi = \psi_0(\xi) + \sum_{j=1}^{\infty} A_j\psi_j(\xi)P_l(\mu) \tag{9.14}$$

and then explore how it induces perturbations in the rest of the relevant quantities of the model, such as the potential, the pressure and the density. We focus in this last one, that, in analogy with the corresponding expression for the non rotating case in (9.12), reads

$$\rho = \rho_c\Theta^n = \rho_c(\theta^n + vn\theta^{n-1}\Psi) + O(v^2). \tag{9.15}$$

The model is built generalizing the *fundamental equation* (9.3) for the rotating case and imposing continuity of the perturbed potential and its first normal derivative to the background surface of the star, given by the sphere of radius $\xi_1$. The functions $\psi_0$ and $\psi_2$ are found to satisfy the problems [23]

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right) = -n\theta^{n-1}\psi_0 + 1, \quad \psi_0(0) = 0, \;\; \psi_0'(0) = 0, \tag{9.16}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_2}{d\xi}\right) = \left(-n\theta^{n-1} + \frac{6}{\xi^2}\right)\psi_2, \quad \psi_2(0) = 0, \;\; \psi_2'(0) = 0. \tag{9.17}$$

The deformation in the polytropic setting corresponds to $d\xi(\xi_1,\mu)$ for some function $d\xi(\xi,\mu)$ that can be extracted from the terms in the $v$ factors in equations (36) and (38) in [23]. Obviously $v\zeta(R,\mu)$ scales to $d\xi(\xi,\mu)$ as $r$ scales to $\xi$ (9) in [23]. Note that $d\xi$ contains $v$. A direct application of formula (9.10) results in the relation

$$\theta^{n-1}\left(d\xi + v\frac{\Psi}{\theta'}\right) = 0. \tag{9.18}$$

This relation is used in [23], evaluated at $\xi_1$, to determine the deformation once the function $\Psi$ is known. The final expression provided in [23] results to be

$$d\xi(\xi_1,\mu) = -\frac{v}{\theta'(\xi_1)}\left(\psi_0(\xi_1) + \frac{5}{6}\frac{\xi_1^2}{3\psi_2(\xi_1)+\xi_1\psi_2'(\xi_1)}\psi_2(\xi_1)P_2(\mu)\right). \tag{9.19}$$

## The change in mass

Let us expand the integral (9.1) in the rotational parameter $v$, which reads explicitly

$$M = 2\pi\int_{-1}^{1}\int_{0}^{a+v\zeta(a,\mu)}\left(\rho^{(0)}(r) + v\rho^{(2)}(r,\mu)\right)r^2 dr d\mu + \mathcal{O}(v^2). \tag{9.20}$$

For a polytropic equation of state recall that $a$ is in correspondence with $\xi_1$, the first zero of Emden's function with index $n$ and the deformation corresponds to $d\xi(\xi_1,\mu)$. After using (9.12), (9.15) and (9.19) for the polytropic equation of state, (9.20) can be shown (see below) to translate, up to order $v$, to

$$M = 4\pi\left[\frac{(n+1)K}{4\pi G}\rho_c^{\frac{1}{n}-1}\right]^{3/2}\rho_c\int_0^{\xi_1+d\xi_1}(\theta^n + vn\theta^{n-1}\psi_0)\xi^2 d\xi, \tag{9.21}$$

as it stands in p.396 in [23], where $d\xi_1$ denotes the $l=0$ part of $d\xi(\xi_1,\mu)$, which equals $d\xi_1 = -v\psi_0(\xi_1)/\theta'(\xi_1)$[1], see (9.19). $d\xi_1$ is the expansion of the star, as noted in [23]. Only the $l=0$ sector contributes to the integral.

In order to obtain (9.21) and go further let us develop (9.20). Since the Jacobian of

[1]Since $\theta'(\xi_1) < 0$ [23], $-\theta'(\xi_1)$ always appears as $|\theta'(\xi_1)|$ in [23].

the change (9.8) is $1 + v\partial\zeta/\partial R$, the integral (9.20) expands as

$$
\begin{aligned}
M &= 2\pi \int_{-1}^{1}\int_{0}^{a} \rho(r(R,\mu),\mu)(R^2 + 2v\zeta R)(1 + v\frac{\partial\zeta}{\partial R})dRd\mu + \mathcal{O}(v^2) \\
&= 2\pi \int_{-1}^{1}\int_{0}^{a} \rho^{(0)}(R)(R^2 + 2v\zeta R)(1 + v\frac{\partial\zeta}{\partial R})dRd\mu + \mathcal{O}(v^2) \\
&= 2\pi \int_{-1}^{1}\int_{0}^{a} \left[\rho^{(0)}(R)R^2 + v\left(2\rho^{(0)}(R)\zeta R + R^2\rho^{(0)}(R)\frac{\partial\zeta}{\partial R}\right)\right] dRd\mu + \mathcal{O}(v^2) \\
&= 2\pi \int_{-1}^{1}\int_{0}^{a} \left[\rho^{(0)}(R)R^2 + v\left(-R^2\zeta\frac{d\rho^{(0)}}{dR} + \frac{\partial}{\partial R}\left(R^2\rho^{(0)}(R)\zeta\right)\right)\right] dRd\mu + \mathcal{O}(v^2) \\
&= 4\pi \int_{0}^{a} \rho^{(0)}(R)R^2 dR - 2\pi v \int_{-1}^{1}\int_{0}^{a} R^2\zeta\frac{d\rho^{(0)}}{dR}dRd\mu + 2\pi v\int_{-1}^{1} a^2\rho^{(0)}(a)\zeta(a,\mu)d\mu \\
&+\mathcal{O}(v^2), \qquad (9.22)
\end{aligned}
$$

where the relation (9.7) that defines $R$ has been used in the second equality. The first term in the final expression (9.22) corresponds to $M^{(0)}(a)$ by (9.4). The second term is more easily recognised by using (9.10), which allows us to write the expression (9.22) as

$$
\begin{aligned}
M &= 4\pi \int_{0}^{a} \rho^{(0)}(R)R^2 dR + 2\pi v \int_{-1}^{1}\int_{0}^{a} \rho^{(2)}(R,\mu)R^2 dRd\mu \\
&\quad +2\pi v\int_{-1}^{1} a^2\rho^{(0)}(a)\zeta(a,\mu)d\mu + \mathcal{O}(v^2). \qquad (9.23)
\end{aligned}
$$

From now onwards let us denote by a $f_0$ (subindex $_0$) the part of any function $f$ parallel to the Legendre polynomial $P_0(\mu)(=1)$. In other words, $f_0(\cdot) := \frac{1}{2}\int f(\cdot,\mu)P_0(\mu)d\mu$. We will also refer to $f_0$ as the $l = 0$ sector of $f$. The mass (9.23) thus reads

$$
M = 4\pi \int_{0}^{a} \rho^{(0)}(s)s^2 ds + 4\pi v \int_{0}^{a} \rho_0^{(2)}(s)s^2 ds + 4\pi v a^2\rho^{(0)}(a)\zeta_0(a) + \mathcal{O}(v^2). \qquad (9.24)
$$

The fact that only the $l = 0$ sector contributes to the integral is now explicit.

For polytropic equations of state, after using (9.12), (9.15), (9.14) and (9.19), equation (9.24) directly translates, up to order $v$, to

$$
M = 4\pi \left[\frac{(n+1)K}{4\pi G}\rho_c^{\frac{1}{n}-\frac{1}{3}}\right]^{3/2} \left\{\int_0^{\xi_1} \theta^n\xi^2 d\xi + v\int_0^{\xi_1} n\theta^{n-1}v\psi_0\xi^2 d\xi - v\xi_1^2\theta^n(\xi_1)\frac{\psi_0(\xi_1)}{\theta'(\xi_1)}\right\}, \qquad (9.25)
$$

which is not difficult to show to be equivalent to (9.21) irrespective of the equation that the function $\theta(\xi)$ satisfies.

The crucial point here is, let us recall, that the function $\theta(r)$ is Emden's function, for which $\theta(\xi_1) = 0$ by construction, which is equivalent to $\rho^{(0)}(a) = 0$. The above expression (9.25) for the total mass is obviously presented in [23] without the last term, which vanishes (see above (40) in [23]). However, in general, the mass density $\rho^{(0)}(R)$ of the background spherical configuration does not have to vanish necessarily at the boundary $R \to a$. The expression of the total mass in [23], made explicit for a class of equations of state for which the mass density vanishes at the surface of the star, seems to have misled many authors to forget the third term in (9.24). Even the author himself forgot, many years later, to include that term when exploring homogeneous (constant $\rho$) stars in GR [30]. The correction to the calculation of the mass of homogeneous stars can now be found in Chapter 10 (or [93]).

The third term in (9.24), proportional to $\rho^{(0)}(a)$, corresponds, precisely, to the Newtonian limit of the term in (7.53) that amends the "change in mass" computed in [57]. That is shown in the following section, where we very briefly review the equations for the perturbed configuration needed in both Newtonian gravity and GR.

## 9.2 The mass in Newtonian gravity and GR

### Newtonian gravity

Let us consider the $l = 0$ sector of the perturbation of the Newtonian potential (9.6). As in the background configuration, apart from the given barotropic equation of state, the perturbation at first order in $v$ is governed by a hydrostatic equilibrium first integral and a Poisson equation

$$U_0^{(2)}(R) + \zeta_0(R)\frac{dU^{(0)}(R)}{dR} - \frac{2\pi G\rho_c R^2}{3} = 0, \tag{9.26}$$

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{dU_0^{(2)}(R)}{dR}\right) = -4\pi G\zeta_0(R)\frac{d\rho^{(0)}(R)}{dR}, \tag{9.27}$$

where the Poisson equation for the nonrotating potential has been used in the second equality. Note that from (9.10) we have $\rho_0^{(2)}(R) = -\zeta_0(R)d\rho^{(0)}(R)/dR$, so that the right hand side of (9.27) can be also expressed as $4\pi G\rho_0^{(2)}(R)$. It is important to note that the domains of definition of these equations are given by $R \in (0, a)$ for the interior and $R > a$ for vacuum, and suitable boundary conditions (including regularity at the origin and at infinity) are imposed accordingly.

It is convenient to change the functions $\{U_0^{(2)}, \zeta_0\}$ that describe the configuration to a new set $\{M^{(2)}, p_0^*\}$, suitable to be compared with the relativistic model, defined as follows,

$$M^{(2)}(R) := 4\pi\int_0^R \rho_0^{(2)}(s)s^2 ds = -4\pi\int_0^R \zeta_0(s)\frac{d\rho^{(0)}(s)}{ds}s^2 ds, \tag{9.28}$$

$$p_0^*(R) := \frac{GM^{(0)}(R)}{R^2}\zeta_0(R). \tag{9.29}$$

The definition (9.29) can be expressed in terms of the pressure and density of the background configuration by differentiating the hydrostatic equilibrium first integral for the static configuration (first equation in (9.2)), which provides

$$\frac{dU^{(0)}(R)}{dR}+\frac{1}{\rho^{(0)}(R)}\frac{dp^{(0)}(R)}{dR}=0,$$

so that

$$\zeta_0(R)=-\rho^{(0)}(R)\left(\frac{dp^{(0)}(R)}{dR}\right)^{-1}p_0^*(R). \tag{9.30}$$

On the other hand, the second order Poisson equation (9.27) can be expressed in terms of the pressure perturbation factor by using (9.30) to get

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{dU_0^{(2)}(R)}{dR}\right)=4\pi G\frac{d\rho^{(0)}}{dp^{(0)}}\rho^{(0)}p_0^*(R). \tag{9.31}$$

We can also rewrite the expression for $M^{(2)}$, (9.28), using (9.30) and (9.29), which in differential form reads (see (15) in [57])

$$\frac{dM^{(2)}(R)}{dR}=4\pi R^2\frac{d\rho^{(0)}}{dp^{(0)}}\rho^{(0)}p_0^*(R). \tag{9.32}$$

The equation for $p_0^*$ is obtained as follows. Combine (9.31) with (9.32) to get rid of $p_0^*$ and integrate once taking into account that $M^{(2)}(0)=0$ by construction, and $dU_0^{(2)}/dR|_{R=0}=0$ for a regular origin. We thus obtain

$$\frac{dU_0^{(2)}(R)}{dR}=\frac{G}{R^2}M^{(2)}(R), \tag{9.33}$$

in analogy with the background configuration. Finally, take the derivative of the hydrostatic equilibrium first integral (9.26)

$$\frac{dU_0^{(2)}(R)}{dR}+\frac{d}{dR}\left(\zeta_0(R)\frac{dU^{(0)}(R)}{dR}\right)-\frac{4\pi G\rho_c}{3}R=0, \tag{9.34}$$

and use (9.33) and (9.29) to obtain (see (15) in [57])

$$\frac{dp_0^*(R)}{dR}=-\frac{G}{R^2}M^{(2)}(R)+\frac{4\pi G\rho_c}{3}R. \tag{9.35}$$

The system of equations for the functions $\{M^{(2)},p_0^*\}$ is formed by (9.32) and (9.35) on the domain $R\in(0,a)$. As in the background configuration system, this problem allows us to integrate $\{M^{(2)},p_0^*\}$ given boundary conditions at the origin. In particular one can compute $M_S^{(2)}:=M^{(2)}(a)$ as a function of the (total) central density $\hat{\rho}_c$, and thus construct a function $M_S^{(2)}(\hat{\rho}_c)$. Let us recall that to add this function to the contribution from the background configuration $M_S^{(0)}(\rho_c)$ it is, of course, necessary to choose $\hat{\rho}_c = \rho_c$, so that $\rho_c$ becomes a parameter of the whole perturbed configuration. That implies choosing $p_0^*(0) = 0$.

The total mass of the rotating configuration (9.24), taking into account (9.4), (9.28) and (9.29), can be expressed as

$$M = M^{(0)}(a) + vM^{(2)}(a) + 4\pi v \frac{a^4}{GM^{(0)}(a)} \rho^{(0)}(a) p_0^*(a) + \mathcal{O}(v^2). \tag{9.36}$$

Note, again, that this sum makes sense once the functions involved are computed given common boundary data, in terms of a common set of parameters, as for instance $\rho_c$. Nevertheless, the choice of parameter used to compute those functions is irrelevant for our purposes. The contribution of the perturbation to the total mass in Newtonian gravity is given by

$$M_T^{(2)} = M^{(2)}(a) + 4\pi \frac{a^4}{GM^{(0)}(a)} \rho^{(0)}(a) p_0^*(a). \tag{9.37}$$

The second term in the above expression is missing in the first equality of equation (18) in [57].

## General Relativity

The general relativistic treatment of the problem has been extensively treated in Chapters 5 to 7. However, we include here the most relevant equations to discuss the change in mass and compare it with the Newtonian model, even when these have already been presented in the previous chapters.

We include the metric up to second order in some parameter $\varepsilon$, which is esentially (5.1) but with the function $m$ scaled to agree with this same function as defined in (4.1). To ease the reading we include it here. It reads

$$\begin{aligned} g_\varepsilon \;\; = \;\; & -e^{\nu(r)}(1 + 2\varepsilon^2 h(r,\theta))dt^2 + e^{\lambda(r)} \left(1 + 2\varepsilon^2 \frac{m(r,\theta)}{r - 2M}\right) dr^2 \\ & + r^2(1 + 2\varepsilon^2 k(r,\theta)) \left(d\theta^2 + \sin^2\theta (d\varphi - \varepsilon\omega(r)dt)^2\right). \end{aligned}$$

We use geometrized units for convenience, so that $G = c = 1$ unless otherwise stated. We can fix the (dimensionless) perturbation parameter $\varepsilon$ in analogy with the formalism developed in [23] for the Newtonian model. To this aim we set $\varepsilon^2 = v = \omega^2/2\pi E(0)$, where $E(0)$ is the energy density of the background configuration at the origin, and $\omega$ is the constant angular velocity of the fluid, as in the Newtonian treatment. Therefore, the quantity that drives the perturbations in [57] is expressed here by $\Omega^H = \sqrt{2\pi E(0)v}$, whereas the constant $\Omega$ used in Chapter 6 is identified with $\Omega = \sqrt{2\pi E(0)}$ in this convention.[2]

We shall keep the perturbation parameter $\varepsilon$ and the constant $\Omega$ in this section in order to ease the comparison with Chapters 6 and 7, although the identifications will be made explicit when the Newtonian limit is taken.

As in the Newtonian case, we only need focusing on the $l = 0$ sector of the solution for our purposes. The coordinate $r$ is fixed by choosing $k_0(r) = 0$ [57] (see also Chapter 6 for a discussion on the choice of gauges). The asymptotically flat vacuum solution is given by (4.13), (4.16), (4.27) and (4.28) [57]

$$
\begin{aligned}
e^{\nu_{vac}(r)} = 1 - \frac{2M}{r} = e^{-\lambda_{vac}(r)}, \quad \omega^{vac}(r) = \frac{2J}{r^3},\\
h_0^{vac}(r) = -\frac{\delta M}{r-2M} + \frac{J^2}{r^3(r-2M)}, \quad {m_0}^{vac}(r) = \delta M - \frac{J^2}{r^3},
\end{aligned} \tag{9.38}
$$

where $M$, $J$ and $\delta M$ are constants. In the analysis of the background and first order configurations, $M$ and $J$ are identified as the background mass and the angular momentum, respectively. The equations governing the background and first order configurations are used to compute $M$ and $J$ given suitable data at the origin. We refer to the summary in Chapter 4 for a full account (see also [16, 95]). The constant $\delta M$, still to be determined, is identified with the "change in mass" due to the second order perturbation, or simply the contribution to the mass at second order, due to the asymptotic behaviour of the angular independent part of $g_{rr}$ (recall (4.29)). The $l = 0$ sector of the (second order) perturbation interior configuration is completely determined by the pair of functions $\{m_0(r), \tilde{\mathcal{P}}_0(r)\}$, with (recall (6.37))

$$
\tilde{\mathcal{P}}_0 := \frac{P_0^{(2)}}{2(E+P)}, \tag{9.39}
$$

where $E$ and $P$ are the energy density and pressure of the static background interior, respectively, and $P^{(2)}(r,\theta)$ the perturbation to the pressure (see (4.23) for this alternative definition of the same function $p_0^{H*}$ in [57]). The system of equations that $\{m_0, \tilde{\mathcal{P}}_0\}$ satisfy is to be fulfilled in the domain $r \in (0, a)$, with suitable boundary conditions given by [3]

[2] This value of the angular velocity is set as a standard scale in numerical works (see e.g. [34], where $\Omega^* = \sqrt{M/a^3}$ is chosen). For a constant density star it is easy to check that $\sqrt{2\pi E(0)} = \sqrt{3M/2a^3} = \sqrt{3/2}\,\Omega^*$.

[3] In these two equations from Chapter 6, the substitution $re^{-\lambda}m_0 \to m_0$ must be made to follow this section.

(6.38) and (6.39), and read

$$\frac{dm_0}{dr} = 4\pi r^2(E+P)\frac{dE}{dP}\tilde{\mathcal{P}}_0 + \frac{1}{12}j^2r^4\left(\frac{d\tilde{\omega}}{dr}\right)^2 - \frac{2}{3}r^3j\frac{dj}{dr}\tilde{\omega}^2, \tag{9.40}$$

$$\frac{d\tilde{\mathcal{P}}_0}{dr} = -4\pi\frac{(E+P)r^2}{r-2M}\tilde{\mathcal{P}}_0 - \frac{r^2m_0}{(r-2M)^2} + \frac{1}{12}\frac{r^4j^2}{r-2M}\left(\frac{d\tilde{\omega}}{dr}\right)^2 + \frac{1}{3}\frac{d}{dr}\left(\frac{r^3j^2\tilde{\omega}^2}{r-2M}\right), \tag{9.41}$$

where $\tilde{\omega}(r) := \omega(r) - \Omega$ and $j(r) := \exp[-(\nu+\lambda)/2]$.

The value of $\delta M$ is determined in terms of interior quantities using the matching conditions for the exterior and interior problems to second order provided in Chapter 7. In particular, a function $m_0(s)$ for $s \in (0,\infty)$ constructed by joining $m_0(s)$ and $m_0{}^{vac}(s)$ across $s = a$ *is not continuous in general*, since it presents a jump proportional to $E(a)$. The result is given in (7.53)

$$\delta M = m_0(a) + \frac{J^2}{a^3} + 4\pi a^3\frac{a-2M}{M}E(a)\tilde{\mathcal{P}}_0(a). \tag{9.42}$$

As in the Newtonian case, the background quantities $E(a)$, $M$ and $J$, and the perturbation ones, $m_0(a)$ and $\tilde{\mathcal{P}}_0(a)$ are to be computed by solving the corresponding system of equations given the (common) relevant data at the origin. In [57] the parameter chosen is the central density $\rho_c$, but, as mentioned above, that choice is not relevant for this discussion.

## Newtonian limit

Our purpose now is to obtain the Newtonian limit of $\delta M$ in (9.42) and compare it with the contribution to the mass of the perturbation in the Newtonian approach, $M_T^{(2)}$, given by (9.37). First, though, it is convenient to find the Newtonian limit for the system (9.40) and (9.41) in order to relate $\{m_0, \tilde{\mathcal{P}}_0\}$ with the pair $\{M^{(2)}, p_0^*\}$ from the Newtonian approach. This is achieved by performing an expansion in powers of $1/c$ as (see [57])

$$M = \frac{G}{c^2}M_S^{(0)} + \mathcal{O}\left(\frac{1}{c^4}\right), \quad E(r) = \frac{G}{c^2}\rho^{(0)}(r) + \mathcal{O}\left(\frac{1}{c^4}\right), \quad P(r) = \frac{G}{c^4}p^{(0)}(r) + \mathcal{O}\left(\frac{1}{c^6}\right),$$

$$\tilde{\omega}(r) = -\frac{\sqrt{2\pi G\rho_c}}{c} + \mathcal{O}\left(\frac{1}{c^3}\right),$$

$$\tilde{\mathcal{P}}_0(r) = \frac{1}{c^2}\tilde{p}_0^*(r) + \mathcal{O}\left(\frac{1}{c^4}\right), \quad m_0(r) = \frac{G}{c^2}\tilde{m}_0(r) + \mathcal{O}\left(\frac{1}{c^4}\right),$$

for some functions $\tilde{m}_0$ and $\tilde{p}_0^*$, where $\rho^{(0)}$ (and $\rho_c$), $p^{(0)}$ and $M^{(0)}$ correspond to the functions describing the Newtonian background configuration. Note that, concerning the first order (in $\varepsilon$), the function $\tilde{\omega}(r)$ is constant at lowest order in $1/c$ [57]. Given the system (9.40) and (9.41), the pair $\{\tilde{m}_0, \tilde{p}_0^*\}$ thus satisfies (see (102)-(103) in [57])

$$\frac{d\tilde{m}_0}{dr} = 4\pi r^2 \frac{d\rho^{(0)}}{dp^{(0)}} \rho \tilde{p}_0^*, \tag{9.43}$$

$$\frac{d\tilde{p}_0^*}{dr} = -\frac{G\tilde{m}_0}{r^2} + \frac{4\pi G\rho_c}{3} r. \tag{9.44}$$

Compare this system of equations with (9.32) and (9.35). The functions arising from the Newtonian limit $\{\tilde{m}_0, \tilde{p}_0^*\}$ and the functions in the perturbed Newtonian model $\{M^{(2)}, p_0^*\}$ satisfy the same equations *in the same domain* $r, R \in (0, a)$. Therefore, the pair $\{\tilde{m}_0, \tilde{p}_0^*\}$ is equivalent to $\{M^{(2)}, p_0^*\}$ for $r, R < a$. We can now substitute $\{\tilde{m}_0, \tilde{p}_0^*\}$ by $\{M^{(2)}, p_0^*\}$ in the following.

The Newtonian limit for (9.42) is obtained following procedure above together with

$$\delta M = \frac{G}{c^2} \widetilde{\delta M} + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{9.45}$$

for some $\widetilde{\delta M}$, from where (9.42) becomes

$$\widetilde{\delta M} = M^{(2)}(a) + 4\pi \frac{a^4}{G M_S^{(0)}} \rho^{(0)}(a) p_0^*(a) + \mathcal{O}\left(\frac{1}{c^4}\right). \tag{9.46}$$

Comparing this expression with (9.37) we finally find

$$\widetilde{\delta M} = M_T^{(2)},$$

that is, the Newtonian limit of the contribution (to second order) of the perturbation to the mass in GR is non zero, and agrees with the same quantity computed in Newtonian gravity.

## 9.3 The Newtonian matching conditions

As a final remark, let us comment on the boundary conditions at the surface of the star, the matching between the interior and exterior problems at each order, involved in the Newtonian approach. Some objections to the Newtonian matching problem stated in [23] were raised in [69]. Those were finally solved by Chandrasekhar and Lebovitz in [29] by properly formulating the matching and producing the same results. However, [29] concerns, again, only polytropic equations of state, and the matching conditions are obtained only for that case, which in particular satisfies $\rho(a) = 0$.

Let us, for completeness, deduce here the matching conditions for the perturbed Newtonian potential in the general case, which, as expected, turns out to be compatible with the obtaining of the perturbed mass (9.36). The only assumption to be made is that the quantities describing the spherically symmetric static background configuration, $\rho^{(0)}$ and $p^{(0)}$, are piecewise differentiable. No particular equation of state is prescribed, and $\rho^{(0)}$ is allowed to have a jump (at least) at some value of the radius $a$. Note that by the local nature of the problem (the matching), we only need to demand differentiability in a neighbourhood of $a$ except at $a$, so that $\rho^{(0)}$ could present jumps at other values of the radius, for which analogous corresponding matching conditions would apply. In order to deal with a general jump we will not assume the "outer region" (or "exterior"), defined by a radius bigger than $a$, to be vacuum.

The background (static) configuration can be solved by considering two problems, the interior ($r < a$) and the exterior ($r > a$), and deducing the boundary conditions at the common boundary, i.e. the matching conditions. In fact, only one relevant function needs to be considered, and we are going to focus on the Newtonian potential. The matching of the background problem for the Newtonian potentials $U^{(0)}_{int}(r)$ and $U^{(0)}_{ext}(r)$ requires the equality of the radial derivative of the potentials at $r = a$. Clearly the potential itself can present a jump, but it is customary to take it continuous and fix the value at infinity to determine its value everywhere. The matching conditions in the background configuration are thus $U^{(0)\prime}{}_{int}(a) = U^{(0)\prime}{}_{ext}(a)$, plus the convenient $U^{(0)}_{int}(a) = U^{(0)}_{ext}(a)$.

To deal with a perturbative setting one has to resort to respective families of interior and exterior problems, defined by some parameter $v$, with corresponding Newtonian potentials $U_{int}(r, \mu, v)$ and $U_{ext}(r, \mu, v)$, that match at each $v$. We fix in $v = 0$ the background static configuration. The family of interior problems is set to live, for each $v$, in a (connected) surface $\{r, \mu\}$ bounded by the curve $\Sigma_v$ determined by

$$\Sigma_v : \{r = \gamma(a, \mu, v)\} \qquad \text{with} \qquad \gamma(a, \mu, 0) = a. \tag{9.47}$$

We are implicitly assuming that $\gamma(a, \mu, v)$ is smooth in all its arguments, and that it splits the strip $\{r, \mu\}$ into two regions for each $v$. The surfaces for the exterior problems are taken to be bounded by $\Sigma_v$ and lie in the other region. Let us define $\zeta(a, \mu)$ as $\zeta(a, \mu) := \partial_v \gamma(a, \mu, v)|_{v=0}$. Thus, to first order the tangent vector and normal vectors to $\Sigma_v$ can be chosen as

$$\vec{e}_v = \frac{\partial \gamma}{\partial \mu} \frac{\partial}{\partial r}\bigg|_{\Sigma_v} + \frac{\partial}{\partial \theta}\bigg|_{\Sigma_v}, \quad \vec{n}_v = -\gamma \frac{\partial}{\partial r}\bigg|_{\Sigma_v} + \frac{1}{\gamma}\frac{\partial \gamma}{\partial \mu} \frac{\partial}{\partial \theta}\bigg|_{\Sigma_v}, \tag{9.48}$$

where $\vec{n}$ points towards the interior of the body and they are normalized so that $(\vec{n}, \vec{n}) = (\vec{e}, \vec{e})$.

We start with some preliminaries. Consider any function $f$ that depends on $v$ on two arguments by $f(\gamma(a, \mu, v), \mu, v)$, and that it is differentiable with respect to the three arguments. Let us use $\partial_v$ to denote a derivative with respect to the third argument, and a prime $'$ with respect to the first, and define $f^{(0)}(a,\mu) := f(\gamma(a,\mu,v),\mu,v)|_{v=0} = f(a,\mu,0)$, and $f^{(2)}(a,\mu) := \partial_v f(\gamma(a,\mu,v),\mu,v)|_{v=0}$. Assume now that $f$ satisfies the equation

$$f(\gamma(a,\mu,v),\mu,v) = 0.$$

Evaluating the equation at $v=0$ we obtain

$$f^{(0)}(a,\mu) = 0, \tag{9.49}$$

while differentiating with respect to $v$, and then evaluating at $v=0$ we get

$$f^{(2)}(a,\mu) + f^{(0)\prime}(a,\mu)\zeta(a,\mu) = 0. \tag{9.50}$$

The matching of the problems for $U_{int}(r,\mu,v)$ and $U_{ext}(r,\mu,v)$ at each $v$ accounts now, as in the background configuration, for the equality of the normal derivative of the potentials at the common boundary $\Sigma_v$. Again, the potential itself can present a jump (at each $v$), but, as customary, we take it continuous. Note that we include this condition for completeness, but it does not affect the result. We thus take $U_{int}|_{\Sigma_v} = U_{ext}|_{\Sigma_v}$ and $\vec{n}(U_{int})|_{\Sigma_v} = \vec{n}(U_{ext})|_{\Sigma_v}$, which explicitly read

$$U_{int}(\gamma(a,\mu,v),\mu,v) = U_{ext}(\gamma(a,\mu,v),\mu,v), \tag{9.51}$$

$$\vec{n}(U_{int})(\gamma(a,\mu,v),\mu,v) = \vec{n}(U_{ext})(\gamma(a,\mu,v),\mu,v), \tag{9.52}$$

due to (9.47). Let us finally use the notation $[g] := g_{int}|_{\Sigma_v} - g_{ext}|_{\Sigma_v}$ for any object $g$ with limits at $\Sigma_v$ from the interior, $g_{int}$, and the exterior, $g_{ext}$, so that the matching conditions read

$$[U](\gamma(a,\mu,v),\mu,v) = 0, \tag{9.53}$$

$$[\vec{n}(U)](\gamma(a,\mu,v),\mu,v) = 0. \tag{9.54}$$

Observe now that, by (9.48)

$$[\vec{n}(U)](\gamma(a,\mu,v),\mu,v) = -\gamma[U'](\gamma(a,\mu,v),\mu,v) + \frac{1}{\gamma}\frac{\partial\gamma}{\partial\mu}\left[\frac{\partial U}{\partial\theta}\right](\gamma(a,\mu,v),\mu,v), \tag{9.55}$$

while the derivative with respect to $\theta$ satisfies

$$\left[\frac{\partial}{\partial\mu}U(\gamma(a,\mu,v),\mu,v)\right] = \frac{\partial\gamma(a,\mu,v)}{\partial\mu}\left[U'\right](\gamma(a,\mu,v),\mu,v) + \left[\frac{\partial U}{\partial\theta}\right](\gamma(a,\mu,v),\mu,v). \tag{9.56}$$

The left hand side of (9.56) vanishes due to (9.53) because

$$[\partial_\mu U(\gamma(a,\mu,v),\mu,v)] = \partial_\mu[U](\gamma(a,\mu,v),\mu,v) = 0.$$

Note that $[U] = \text{const.}$ leads to the same conclusion. Therefore, (9.55) reads

$$[\vec{n}U](\gamma(a,\mu,v),\mu,v) = [\vec{n}(U)] = -\left(\gamma + \frac{1}{\gamma}\left(\frac{\partial\gamma}{\partial\mu}\right)^2\right)[U'](\gamma(a,\mu,v),\mu,v)$$

once (9.53) holds. The two conditions (9.53) and (9.54) are thus equivalent to the couple $[U](\gamma(a,\mu,v),\mu,v) = 0$ and $[U'](\gamma(a,\mu,v),\mu,v) = 0$. We have written the matching conditions as two functions that satisfy the requirements for $f$ above, so it is now just a matter of applying equations (9.49) and (9.50) to both $[U]$ and $[U']$. The four equations thus obtained read

$$\begin{aligned} [U] = 0 \Rightarrow \quad & [U^{(0)}](a) = 0, \qquad [U^{(2)}](a,\mu) = -[U^{(0)\prime}](a)\zeta(a,\mu) = 0, \\ [U'] = 0 \Rightarrow \quad & [U^{(0)\prime}](a) = 0, \qquad [U^{(2)\prime}](a,\mu) = -[U^{(0)\prime\prime}](a)\zeta(a,\mu), \end{aligned} \tag{9.57}$$

where we have used that the background potentials $U^{(0)}$'s do not depend on $\mu$. Equation (9.57) yields, after using the Poisson equation at each side,

$$[U^{(2)\prime}](a,\mu) = -4\pi G[\rho^{(0)}](a)\zeta(a,\mu). \tag{9.58}$$

Given only a piecewise differentiability condition on the background configuration, as described above, *the Newtonian perturbed matching conditions (up to first order in $v$) around any point $s = a$ are given by the coincidence of $U^{(0)\prime}(a)$ at both sides (interior and exterior), which corresponds to the matching of the background configuration, and (9.58), the condition at first order.*

In particular, if we demand a vacuum exterior, so that $\rho^{(0)}_{ext} = 0$, we have $[\rho^{(0)}](a) = \rho^{(0)}_{int}(a)$. Furthermore, if $\rho^{(0)}_{int}(r)$ has no other jumps (and is smooth in $r \in (0,a)$), then $[\rho^{(0)}](a) = \rho^{(0)}_{int}(a)$ is simply our $\rho^{(0)}(a)$ of the previous sections. Therefore, as expected, the radial derivative of the Newtonial potential at first order in $v$ suffers a jump at $a$, which is proportional to $\rho^{(0)}(a)$ for a vacuum exterior. The perturbed mass can now be computed from the Newtonian potential, and it is straightforward to show that this jump generates the term proportional to $\rho^{(0)}(a)$ in (9.24) (or (9.36)). Let us stress that if $\rho^{(0)}(r)$ is allowed to have more jumps, say $a_i$, in the interior of the star, the expression of the total mass would simply contain a term contributing from each corresponding jump discontinuity $[\rho^{(0)}](a_i)$.

Finally, it can be shown that the matching condition for $h'$ (7.31), which suffers a jump proportional to the jump of $m$, agrees with (9.58) after taking the Newtonian limit.

10

# The mass of homogeneous stars and strange stars

The main conclusion derived from our treatment of the problem of an isolated slowly rotating compact body in Chapters 5 to 7 in practical terms is that the perturbation functions to first and second order in Hartle's setting can, indeed, be taken as *continuous* at the surface of the star except when the energy density is *discontinuous* there, in which case a corresponding discontinuity appears in the radial function $m_0$ (the other perturbation functions remaining continuous). The discontinuity in $m_0$ is proportional to the value of the energy density at the surface of the star, i.e. to the discontinuity of the energy density there. In Chapter 7 an explicit correspondence between Hartle's and our settings has been presented, putting emphasis in the outcomes of the model such as the change in mass and the shape of the star. The single outcome of the model directly affected by the discontinuity of $m_0$ is the change in mass $\delta M$, defined as the contribution to the mass due to the rotation. The rest of the outcomes of the model regarding the frame dragging and the shape of the star are not affected by the discontinuity of $m_0$.

The correction to $\delta M$ given in (7.53), proportional to the energy density evaluated at the surface of the star, is negligible or just zero when the usual equations of state for neutron stars are considered, since the pressure and the energy density typically decrease together and vanish simultaneously at the surface of the star. This is the behaviour shown by, e.g. polytropic EOS's. Nevertheless, the correction to the mass may be relevant in other EOS's for which the energy density takes a finite value at the boundary, which is precisely the case of linear equations of state, for instance those used to describe strange quark stars [34], or constant density (homogeneous) stars.

In this chapter we compute the mass of rotating stars for two equations of state: constant density and a particular linear EOS, because the mass correcting term will be relevant there. The results regarding homogeneous stars were published in [93].

In order to explore other EOS's, we use a code written in Fortran to solve the model numerically in three steps, one for the static configuration (TOV) and the other two for the perturbations. In every step, the corresponding ODE's are integrated making use of a classical Runge Kutta method (RK4). Hence, provided the two relevant parameters, i.e. the central density of the star and an angular velocity, we are able to solve the whole model and obtain the values of the functions $M$, $P$, $\omega$, $m_0$, $h_0$, $v$, $h_2$ and $m_2$ at any point of the star, labeled by the coordinate $r$. Furthermore, the values of these functions at $r = a$ allow us to determine the physical properties of the star, such as $M$, $J$, $\delta M$, the quadrupole moment $Q$ or the ellipticity $e$. However, we are interested, more than in computing single stars, in the physical properties of the families of stars that arise from letting the central energy density vary in some range, say $(E_c^{(i)}, E_c^{(f)})$, this last given by some physical criteria adequated to the particular case under study. This is achieved by an iterative process. We solve the star corresponding to a certain value of the energy density, save the physical properties or the values of interest of the star and repeat the process until we cover the whole range of central densities. In this form, we can visualize how the mass of the star varies with respect to its radius, or its central density. Some examples will be shown along this chapter. We included the possibility of working with different EOS's, such as polytropes (based on [61]), constant density as in [30], linear EOS of the type in [34] and tabulated EOS (see for instance [5] or [105]), which has been useful in order to check the validity of the results, for instance comparing with [13] or [61]. However, in this thesis we will restrict ourselves to the study of homogeneous stars and strange quark matter stars. The numerical code has been developed in collaboration with Nicolas Sanchis-Gual and José A. Font, from the Universitat de València.

Let us remark that we compute sequences of stars varying the central energy density with the velocity of rotation fixed to $\Omega = \Omega^* = \sqrt{M/a^3}$, as it is usually done in the literature and it has been discussed in the Introduction. This may seem to contradict the statement done after introducing the first order equation (6.25), when we say that we specify a value of $\tilde{\omega}$ at the origin to ensure regularity there. The first order perturbations are fully determined by one parameter, that we have chosen in Chapter 6 to be, precisely $\tilde{\omega}_c := \tilde{\omega}(r \to 0) = const = 1$. In other words, we work in units of $\tilde{\omega}_c$. Yet, the matching conditions provide the value of the angular velocity $\Omega$ corresponding to this choice of $\tilde{\omega}_c$ by means of (7.13). The key point here is that other models with different velocities of rotation can be obtained simply by scaling. For example, the model associated to the critical angular velocity $\Omega^*$ can be obtained by scaling

$$f^{f.o.}_{new} = \frac{\Omega^*}{\Omega} f^{f.o.}_{old}, \quad f^{s.o.}_{new} = \left(\frac{\Omega^*}{\Omega}\right)^2 f^{s.o.}_{old}, \tag{10.1}$$

where $f^{f.o.}_{new}$ is any first order quantity associated to a model with an angular velocity $\Omega^*$

and $f_{old}^{f.o.}$ is the corresponding quantity computed with a velocity $\Omega$. The second order works analogously.

## 10.1 Homogeneous stars

In this section the particular case of homogeneous stars is studied. This equation of state may not be realistic in physical terms, but its study is interesting for several reasons. First of all the spherically symmetric and static configuration can be solved analytically and hence, the perturbed field equations become simpler. Secondly, it is interesting to know how noticeable the correction for the change in mass might be in numerical terms. There are factors in the change in mass that are easy to estimate and, in fact, can be taken as inputs for the model, as the mass and the size of the static and spherically symmetric star. In contrast, the perturbation to the pressure is not easily estimated and the model must be solved to second order. Another important factor is the value of the energy density at the surface of the star, in the non-rotating configuration, and this is precisely the reason for having chosen this particular EOS. It is probably one of the most favourable cases for the correction. Thus, it is reasonable to think that the constant energy EOS may constitute a numerical bound for the amended change in mass, since for any other of the usual EOS the value of the energy density at the surface will not be as important as in the present case.

Homogeneous stars drew the attention of Chandrasekhar and Miller [30], back in 1974. In that work, they use Hartle's formalism to solve the homogeneous star in the slow rotating approximation. To this aim, they present the perturbed field equations up to second order in terms of a suitable *radial* coordinate adapted to the background solution. They also provide the boundary conditions that ensure regularity at the center of the fluid ball order by order. Finally, they solve numerically the perturbed equations and use Hartle's formalism in order to determine the value of the constants that characterize the vacuum solution, such as the angular momentum $J$ (or equivalently the moment of inertia $I$), the change in mass $\delta M$ or the ellipticity of the surface of the star $e$, in their notation. They present first the numerical solution of the first order problem and show the behaviour of the function $\tilde{\omega}$ (see Figures 1 and 7 in [30]) that drives the frame dragging effect, and the momentum of inertia of the star (Figure 2 therein). Regarding the second order perturbations, they focus on the calculation of the deformation of the star due to the rotation. This is described in terms of two components, an homogeneous enlargement (or contraction) arising from the $l = 0$ sector in a Legendre polynomial expansion of the perturbations and the ellipticity of the surface of the star, originated by the $l = 2$ sector in the aforementioned expansion. The homogeneous component is presented in Figure 3 (in their paper) as a function of the ratio of the radius of the star to its Schwarzschild radius (hereafter $a/R_S$). The ellipticity of the configuration is studied in detail to show, remarkably, that as a function of $a/R_S$ it is not monotonic and it presents a maximum at $a/R_S = 2.4$ (see Figure 5 in [30]).

In this Section we recalculate the value of $\delta M$, as described in Chapter 7, for homogeneous stars. Let us remark that this correction does not affect the results regarding the frame dragging effect nor the shape of the rotating stars, fully studied in [30] as mentioned above. In the present work the results are displayed in an analogous way to [30] to ease the comparison of tables and figures.

When the energy density $E$ is constant, the equations of structure (6.17), (6.18) that govern the (static, spherically symmetric) background configuration admit an analytical solution. In terms of the constant density $E$ and the central pressure $P_c$, that solution is given by

$$\frac{P+\frac{E}{3}}{P+E} = \frac{P_c+\frac{E}{3}}{P_c+E}\sqrt{1-\frac{8\pi E {r_+}^2}{3}}, \tag{10.2}$$

$$e^{-\lambda^+(r_+)} = 1-\frac{2M^{(0)}(r_+)}{r_+}, \qquad M^{(0)}(r_+) = \frac{4\pi}{3}E{r_+}^3\,, \tag{10.3}$$

$$e^{\nu^+(r_+)/2} = e^{\nu^+(0)/2}\left(-1+\frac{P_c+\frac{E}{3}}{P_c+E}\sqrt{1-\frac{8\pi E{r_+}^2}{3}}\right)\left(-1+\frac{P_c+\frac{E}{3}}{P_c+E}\right)^{-1}. \tag{10.4}$$

This solution stands for the whole interior region, i.e. from $r_+ = 0$ to $r_+ = a$. The vacuum solution is given by (6.20) and extends from $r_- = a$ to infinity. The two solutions are related by means of the matching conditions for the background configuration (5.13). The continuity of $\lambda$ and $\nu'$ implies that $M = 4\pi E a^3/3$ and $P(a) = 0$.

With the background configuration already matched, it is convenient to change from the *interior* parameters $\{E, P_c\}$ to the *exterior* parameters $\{M, a\}$. Thence the solution (10.2) takes the form as given in [86, 111]. In order to present the results as in [30], the exterior parameters still have to be scaled with the Schwarzschild radius $R_S := 2M$ so that they become $\{R_S, a/R_S\}$. Inverting the relation between the parameters one finds

$$E = \frac{3}{8\pi R_S^2}\left(\frac{a}{R_S}\right)^{-3}, \quad P_c = \frac{3}{8\pi R_S^2}\left(\frac{a}{R_S}\right)^{-3}\frac{1-\sqrt{1-\left(\frac{a}{R_S}\right)^{-1}}}{3\sqrt{1-\left(\frac{a}{R_S}\right)^{-1}}-1}. \tag{10.5}$$

Equation (10.5) implies a constraint on the background exterior parameters. In order to keep the central pressure finite, the following inequality must hold [30]

$$0 \le \frac{9}{4}M < a \Leftrightarrow 0 \le a < \sqrt{\frac{1}{3\pi E}}. \tag{10.6}$$

We want to write the field equations for the fluid as in [30]. For this, two constants that replace $E$ and $a$ must be introduced as

$$\alpha := \sqrt{\frac{3}{8\pi E}}, \qquad \kappa := 3\sqrt{1-\frac{a^2}{\alpha^2}} - 1. \tag{10.7}$$

The constant $\alpha$ is related to the Schwarzschild radius $R_S$ by

$$\alpha = R_S \left(\frac{a}{R_S}\right)^{3/2}. \tag{10.8}$$

Hence, given any function in units of $\alpha$, it can be easily converted to units of the Schwarzschild radius by simply scaling it with the proper factor $a/R_S$. Following the conventions in [30], the radial coordinate in the interior region is finally substituted by

$$x := 1 - \sqrt{1 - \left(\frac{r_+}{\alpha}\right)^2}. \tag{10.9}$$

This change is well defined once the inequality (10.6) holds. The domain of definition of this coordinate is $x \in (0, 2/3)$, and the origin corresponds to $x \to 0$. The radius of the star is denoted by $X$, which in terms of $\kappa$ is expressed as $X = (2-\kappa)/3$. Finally, in terms of $x$ the auxiliary function $j^+$ reads $j^+ = 2(1-x)/(\kappa + x)$.

## First order

Considering the background solution (10.2)-(10.4) and the definitions introduced so far, the first order field equation (6.25) casted for the function $\tilde{\omega}^+ := \Omega - \omega^+$ for the interior region is written in terms of the radial coordinate $x$ as

$$-x(2-x)(x+\kappa)\frac{d^2\tilde{\omega}^+}{dx^2} + (4x^2 - x(3-5\kappa) - 5\kappa)\frac{d\tilde{\omega}^+}{dx} + 4(1+\kappa)\tilde{\omega}^+ = 0. \tag{10.10}$$

The behaviour of $\tilde{\omega}^+$ near the origin is fixed by

$$\tilde{\omega}^+ = \tilde{\omega}_c^+ \left(1 + \frac{4(1+\kappa)}{5\kappa}x\right) + O(x^2). \tag{10.11}$$

The values of $\tilde{\omega}^+(a)$ and its first derivative $\tilde{\omega}^{+\prime}(a)$ allow us to determine the angular momentum $J$ and the angular velocity $\Omega$ by means of (4.17). In terms of the coordinate $x$ these relations are

$$\frac{J}{\tilde{\omega}_c R_S^3} = (a/R_S)^{5/2}\frac{\sqrt{X(2-X)}}{6(1-X)}\left(\frac{d(\tilde{\omega}/\tilde{\omega}_c)}{dx}\right)_{x=X} \tag{10.12}$$

$$\frac{\Omega}{\tilde{\omega}_c} = \frac{\tilde{\omega}(X)}{\tilde{\omega}_c} + \frac{2}{(a/R_S)^3}\frac{J}{\tilde{\omega}_c R_S^3} \tag{10.13}$$

The quotient of these rescaled $J$ and $\Omega$ is used in [30] to compute the momentum of inertia and the normalized momentum of inertia as

$$I/R_S^3 = \frac{J/\tilde{\omega}_c R_S^3}{\Omega/\tilde{\omega}_c}, \quad i/R_S^3 := \frac{I/R_S^3}{Ma^2}. \tag{10.14}$$

Note that the field equation for the first order (10.10) and the condition at the origin (10.11) are formulated for $\tilde{\omega}^+/\tilde{\omega}_c^+$, and thus, depend only on one free parameter, $\kappa$, from the background configuration. This parameter $\kappa$ is in fact determined by the ratio of the radius of the spherical star to the Schwarzschild radius $a/R_S$ by means of (10.7) and (10.8). Hence the model is solved just specifying a value of $a/R_S$. A sequence of models with different values of $a/R_S$ is explored and the results for the first order are presented below.

The first order results are shown in Table 10.1, which includes the moment of inertia and the value $\tilde{\omega}^+/(J/R_S^3)|_{\Sigma_0}$ for different values of $a/R_S$. For a solid sphere in the Newtonian regime, the normalized momentum of inertia $i := I/Ma^2$ takes the value 2/5, which is achieved asymptotically (see Table 10.1). The comparison with a sphere makes sense because the first order perturbations do not change the shape of the star. Hence, the deviation between this value and the values shown in Table 10.1 is an effect of the twisted geometry. These results [1] fully agree with those presented in [30].

[1]Values for $a/R_S = 9/8$ are not shown because although the perturbations can still be solved [30], the background solution is not regular.

Table 10.1: $I$ and $i$ in units of $R_S^3$, and $\tilde{\omega}(a)/(J/R_S^3)$ for some values of $a/R_S$.

| $a/R_S$ | $I$ | $i$ | $\tilde{\omega}(a)$ |
|---|---|---|---|
| 1.15 | 0.5105 | 0.7720 | 0.64391 |
| 1.2 | 0.5248 | 0.7289 | 0.74818 |
| 1.3 | 0.5657 | 0.6695 | 0.85741 |
| 1.4 | 0.6171 | 0.6296 | 0.89174 |
| 1.5 | 0.6758 | 0.6007 | 0.88714 |
| 1.6 | 0.7406 | 0.5786 | 0.86205 |
| 1.7 | 0.8106 | 0.5610 | 0.82652 |
| 1.8 | 0.8856 | 0.5467 | 0.78622 |
| 1.9 | 0.9653 | 0.5348 | 0.74441 |
| 2.0 | 1.049 | 0.5247 | 0.70294 |
| 2.5 | 1.534 | 0.4910 | 0.52376 |
| 3.0 | 2.123 | 0.4717 | 0.39700 |
| 4.0 | 3.604 | 0.4505 | 0.24623 |
| 5.0 | 5.487 | 0.4390 | 0.16625 |
| 10.0 | 20.91 | 0.4182 | 0.045821 |
| 20.0 | 81.77 | 0.4088 | 0.011980 |
| 35.0 | 248.1 | 0.4050 | 0.0039848 |
| 50.0 | 504.3 | 0.4035 | 0.0019668 |
| 100.0 | 2009 | 0.4017 | 0.00049585 |

## Second order

The second order field equations [2] for the pair $\{m_0^+, \tilde{P}_0^{(2)}\}$, (6.38) and (6.39), are written in terms of the radial coordinate $x$ as [30]

$$\begin{aligned}\alpha^{-3}\frac{dm_0^+}{dx} &= \frac{(1-x)((2-x)x)^{3/2}}{(\kappa+x)^2}\\ &\quad\left(\frac{1}{3}(2-x)x\left(\frac{d\tilde{\omega}^+}{dx}\right)^2+\frac{8(\kappa+1)}{3(\kappa+x)}\tilde{\omega}^{+2}\right), \qquad (10.15)\\ \alpha^{-2}\frac{d\tilde{P}_0^{(2)}}{dx} &= -\frac{\kappa+1}{(1-x)(\kappa+x)}\alpha^{-2}\tilde{P}_0^{(2)}\\ &\quad-\frac{2+(\kappa+1)(1-x)-3(1-x)^2}{(\kappa+x)(1-x)^2x^{3/2}(2-x)^{3/2}}\alpha^{-3}m_0^+\\ &\quad+\frac{8(2-x)x}{3(\kappa+x)^2}\tilde{\omega}^+\frac{d\tilde{\omega}^+}{dx}-\frac{8(\kappa(x-1)+x)}{3(\kappa+x)^3}\tilde{\omega}^{+2}\\ &\quad+\frac{(2-x)^2x^2}{3(1-x)(\kappa+x)^2}\left(\frac{d\tilde{\omega}^+}{dx}\right)^2. \qquad (10.16)\end{aligned}$$

[2]In this chapter we rescale the function $m_0$ of Chapters 5, to 7 by $re^{-\lambda}m_0 \to m_0$. This allows us to compare directly with [30].

Regularity at the origin and the preservation of the central density demand [30]

$$\frac{m_0^+}{\alpha^3\tilde{\omega}_c^{+2}}(x \to 0) = \frac{32\sqrt{2}(\kappa+1)x^{5/2}}{15\kappa^3} + O(x^{7/2}), \tag{10.17}$$

$$\frac{\tilde{P}_0^{(2)}}{\alpha^2\tilde{\omega}_c^{+2}}(x \to 0) = \frac{8x}{3\kappa^2} + O(x^2). \tag{10.18}$$

Once the values of the functions $m_0^+$ and $\tilde{P}_0^{(2)}$ in $x = X$ are found, the change in mass is calculated using expression (7.53). In order to present the numerical results, it is convenient to express $\delta M$ divided by the mass of the background configuration $M$, and in units of $J^2/R_S^4$. We also split it into two components, the $\delta M^{(O)}$ referring to the change in mass (4.30) and $\delta M^{(C)}$ as the amending term in (7.53). These two components, written in the covenient units read

$$\frac{\delta M}{M}^{(O)} = \frac{J^2}{R_S^4}\left(2\frac{m_0^+}{(J^2/R_S^3)}\bigg|_{\Sigma_0} + 2\left(\frac{a}{R_S}\right)^{-3}\right), \tag{10.19}$$

$$\frac{\delta M}{M}^{(C)} = \frac{J^2}{R_S^4}\left(6\left(\frac{a}{R_S}-1\right)\frac{\tilde{P}_0^{(2)}}{(J^2/R_S^4)}\bigg|_{\Sigma_0}\right), \tag{10.20}$$

so that

$$\frac{\delta M}{M} = \frac{\delta M}{M}^{(O)} + \frac{\delta M}{M}^{(C)} \tag{10.21}$$

The field equations for the second order (10.15), (10.16) and the conditions at the origin (10.17), (10.18) are formulated for $m_0^+/\alpha^3\tilde{\omega}_c^{+2}$ and $\tilde{P}_0^{(2)}/\alpha^2\tilde{\omega}_c^{+2}$. Thence, they depend only on the background parameter $\kappa$, which as discussed for the first order, can be expressed in terms of the ratio $a/R_S$. Hence a sequence of models to second order with different values of $a/R_S$ is computed below.

The numerical results for the second order are summarized in Table 10.2 and Figures 10.1 and 10.2. In Figure 10.1 $m_0^+$ in units of $J^2/R_S^3$ and $\tilde{P}_0^{(2)}$ in units of $J^2/R_S^4$ at $r_+ = a$ are shown as functions of $a/R_S$, i.e. $m_0^+/(J^2/R_S^3)|_{\Sigma_0}(a/R_S)$ and $\tilde{P}_0^{(2)}/(J^2/R_S^4)|_{\Sigma_0}(a/R_S)$ respectively. In order not to overwhelm the notation in the subsequent discussion, let us refer to these two previous functions as $m_{0\natural}(a/R_S)$ and $\tilde{P}_{0\natural}(a/R_S)$ respectively. As mentioned in [30] these are not monotonic functions and they both present a maximum, the first one at $a/R_S \sim 1.29$ and the second at $a/R_S \sim 1.82$. Note that the function $\tilde{P}_{0\natural}$ is negative for small values of $a/R_S$ and this implies that the average deformation of the star to second order [3] can be either negative or positive, so that the star may show either contraction or expansion to second order depending on the background parameters. It is worth noting that $m_{0\natural}$ and $\tilde{P}_{0\natural}$ attain values of the same order and the ratio is about 1.63

[3] The relation of the pressure and the shape of the star is addressed in [57].

Table 10.2: The change in mass $\delta M^{(O)}/M$ typical from the literature, the amended change in mass $\delta M/M$ and the fraction of the correction with respect to the total change in mass are presented for different values of $a/R_S$.

| $R/R_S$ | $\delta M^{(O)}/M$ | $\delta M/M$ | $\|\delta M^{(C)}\|/\delta M$ |
|---|---|---|---|
| 1.15 | 3.454 | 2.348 | 0.4711 |
| 1.2 | 3.412 | 2.725 | 0.2524 |
| 1.3 | 3.225 | 3.506 | 0.0803 |
| 1.4 | 2.993 | 4.246 | 0.2952 |
| 1.5 | 2.757 | 4.904 | 0.4379 |
| 1.6 | 2.533 | 5.474 | 0.5372 |
| 1.7 | 2.327 | 5.954 | 0.6091 |
| 1.8 | 2.140 | 6.355 | 0.6632 |
| 1.9 | 1.971 | 6.684 | 0.7051 |
| 2.0 | 1.819 | 6.951 | 0.7383 |
| 2.5 | 1.259 | 7.631 | 0.8350 |
| 3.0 | 0.9163 | 7.690 | 0.8808 |
| 4.0 | 0.5440 | 7.173 | 0.9242 |
| 5.0 | 0.3588 | 6.482 | 0.9446 |
| 10.0 | 0.09493 | 4.065 | 0.9766 |
| 20.0 | 0.02437 | 2.259 | 0.9892 |
| 35.0 | 0.008049 | 1.349 | 0.9940 |
| 50.0 | 0.003966 | 0.9608 | 0.9959 |
| 100.0 | 0.001008 | 0.4901 | 0.9979 |

for big values of $a/R_S$. For a more detailed discussion we refer the reader to the original work [30].

Finally, the results including the corrected change in mass and their comparison with those presented in [30] are shown in Table 10.2 and Figure 10.2. In Table 10.2 some values of the change in mass as a function of $a/R_S$ are presented. In the second column the value of $\delta M^{(O)}/M$, which corresponds to the $\delta M/M$ given in[30], is shown, whereas the third column includes the correct change in mass (7.53) and, lastly, the fourth column shows the fraction of the change in mass that corresponds to the correction. In fact, the correction becomes the dominant contribution in $\delta M$ as the quotient $a/R_S$ increases. The behaviour of $\delta M/M$ is shown for a wide range of the variable $a/R_S$ in Figure 10.2. $\delta M/M$ presents a maximum at $a/R_S \sim 2.81$ and then decreases more slowly than the $\delta M^{(O)}/M$ presented in [30], which decays monotonically. The original $\delta M^{(O)}/M$ and the amended $\delta M/M$ agree for $a/R_S \sim 1.27$, where $\tilde{P}_{0\natural}$ vanishes. Below this point, the star contracts (in average), and above it, the star expands.

There is a combination of two facts that makes the correction $\delta M^{(C)}$ not only noticeable, but also dominant in homogeneous stars. On the one hand, as shown in Figure 10.1,

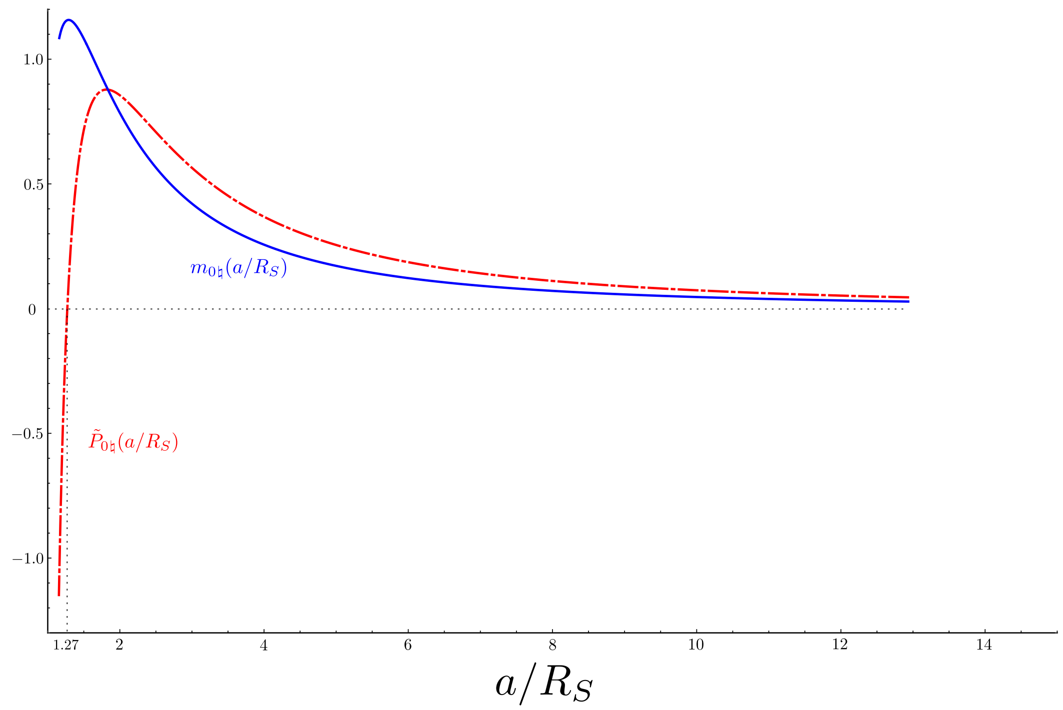


Figure 10.1: The perturbation to the pressure to second order $\tilde{P}_{0\natural}(a/R_S)$ and $m_{0\natural}(a/R_S)$.

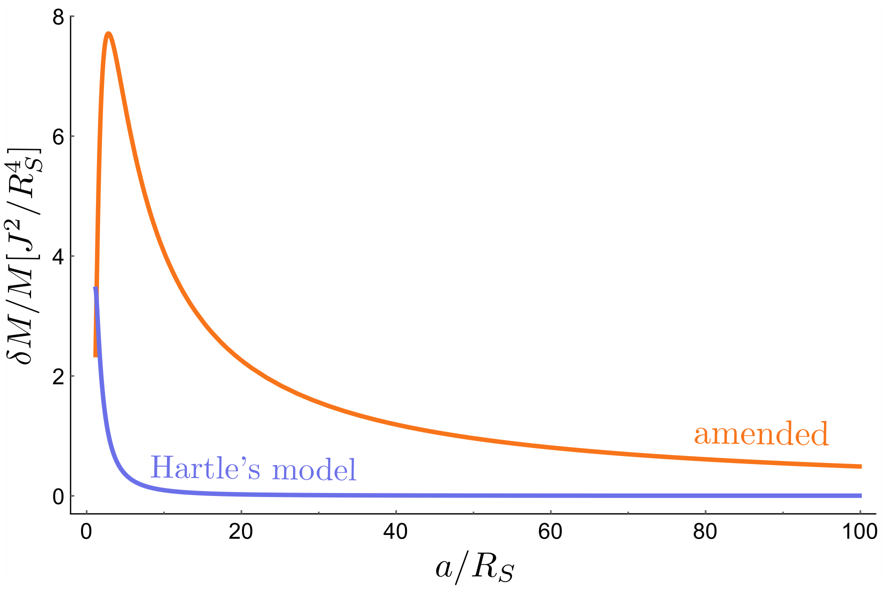


Figure 10.2: The original and the amended changes in mass versus the normalized radius of the static star $a/R_S$.

$m_{0\natural}$ and $\tilde{P}_{0\natural}$ are quantities of the same order. On the other hand, in formula (10.21), the coefficient with $\tilde{P}_{0\natural}$ scales linearly with $a/R_S$. These results definitely reveal that the correction to the change in mass is important in homogeneous stars and it can not be neglected at all.

## 10.2 Strange stars

Brecher [17] and Fechner and Joss [48] considered the possibility of stars composed of quark matter. They computed the macroscopic properties of such stars, using several equations of state based on models for low mass quarks in quantum chromodynamics. The two important conclusions derived from [48] are that these quark stars can be stable and that their macroscopic properties, for instance the mass or the moment of inertia, are very close to the values shown by standard neutron stars.

Witten explored the possibility of astrophysical objects composed of a quark matter in [113]. He proposed that in the core of a neutron star, the pressure is high enough to allow the formation of quark matter. Once there is presence of quarks in the nucleus, the equilibrium in strangeness is achieved by weak interaction mechanisms. Then the quark matter absorbs free neutrons and the final picture ends as a core composed of quark matter shrouded by an outer layer or crust made of nuclei and electrons (for a description of the crust see for instance [114]). Even more, Witten suggests the possibility of a pure quark star, without a crust. Colpi and Miller studied a model of this pure quark star using Hartle's model for slow rotation in [34]. As in [113], they proposed the use of the MIT bag model to account for the microphysics of the star. The MIT bag model supplies a linear equation of state of the type

$$P = \frac{1}{3}(E - 4B), \quad E \geq 4B. \tag{10.22}$$

Here, $P$ and $E$ are the pressure and energy respectively and $B$ is the *bag constant*. It is a phenomenological constant and is usually taken as $B \approx 56\text{MeV}\,\text{fm}^{-3}$, suggested by hadronic models [42]. This model represents matter with quarks of the type up, down and strange almost in the same quantity, mixed with the necessary amount of electrons to guarantee the neutrality of the charge. Equations of state of this kind have also been studied under the CMMR formalism [21], a treatment based on the post-Minkowskian and small deformation approximations in [36, 37], as a particular case of the linear EOS.

Although a star fully governed by this EOS might be quite unrealistic, we consider it in order to compare our results with [34], especially those regarding the change in mass. Note that a more realistic bilayer interior, or a single combination fluid-crust, both of them computed via Hartle's model will suffer from the same problems that we address in this Section for Colpi and Miller's model [34]. This happens because the (background) hypersurface separating the two fluids will be determined by the equality of the corresponding pressures. However, the energy densities are not force to agree there, and thus, the functions $m_0$ will still present a jump. This jump provides the initial value to integrate the function $m_0$ of the enveloping fluid from the matching hypersurface with the core outwards and it propagates to the computation of the change in mass when matching this outer fluid with vacuum.

According to the paper by Witten [113], the maximum mass for a stable static configuration is given by the expression

$$\frac{M_{max}}{M_\odot} = 2.00\sqrt{\frac{56}{B(\mathrm{MeV\,fm^{-3}})}}, \tag{10.23}$$

that for our choice of $B = 56.25\mathrm{MeV\,fm^{-3}}$ [4] returns a value of $M_{max}/M_\odot \approx 2$. The simulations comprise the range of central energies $(4.10 \cdot 10^{14}\mathrm{g\,cm^{-3}}, 3.01 \cdot 10^{15}\mathrm{g\,cm^{-3}})$. The smallest value of the central energy density in the interval generates a non-rotating model with almost no mass, of approximately 0.04 solar masses. Nonrotating models with a central density of about $E_c \approx 1.92 \cdot 10^{15}\ \mathrm{g\,cm^{-3}}$ or greater surpass the maximum mass limit and start to suffer instabilities due to radial perturbations. However, we end the sequence with a value of the central energy density close to the critical one, but slightly bigger to observe simply the behaviour of the correcting term. Anyhow, the stability limit is highlighted in all the figures as well as in the table showing the data. In the figures, the area in gray delimits the (non-rotating) stability limit.

In order to compare our results with those in [34] we define the fractional change in mass and the total mass

$$f := \frac{\delta M}{M}, \quad M_{total} := M + \delta M = M(1+f). \tag{10.24}$$

In the following, we attach three figures from [34] and one table showing the relevant numerical values with the corrected behaviours of the mass. Figure 10.3 shows the mass of the configuration against the central energy density, Figure 10.4 adresses the relation between the mass and the mean radius and in Figure 10.5 we display the fractional increase in mass computed via Hartle's model and amended. Let us note that we recover the results from [34] if we restrict ourselves to Hartle's model. However, our results show that the correction to the change in mass is the dominant contribution to it, leading to higher values of the total mass of the rotating star than those computed in [34]. The correction is weighted by the value of the energy density at the boundary, whose value is

[4] We choose this value to match the choice in [34].

| $E_c$ ($10^{14}\,\mathrm{g\,cm^{-3}}$) | $a/R_S$ | $M/M_\odot$ | $f^H$ | $f$ | $M_{total}/M_\odot$ | $R^{(2)}/R_S$ |
|---|---|---|---|---|---|---|
| 4.1 | 31.6 | $3.8 \cdot 10^{-2}$ | $1.58 \cdot 10^{-2}$ | 0.974 | $7.5 \cdot 10^{-2}$ | 41.8 |
| 5.72 | 3.36 | 1.03 | 0.162 | 0.843 | 1.89 | 4.23 |
| 6.66 | 2.74 | 1.35 | 0.208 | 0.831 | 2.48 | 3.44 |
| 6.79 | 2.69 | 1.39 | 0.213 | 0.830 | 2.54 | 3.37 |
| 9.35 | 2.18 | 1.79 | 0.286 | 0.820 | 3.26 | 2.72 |
| 12.0 | 2.00 | 1.94 | 0.329 | 0.811 | 3.52 | 2.49 |
| 14.7 | 1.92 | 2.00 | 0.354 | 0.801 | 3.61 | 2.38 |
| 17.4 | 1.87 | 2.03 (2.027) | 0.370 | 0.789 | 3.63 | 2.32 |
| 19.2 | 1.85 | 2.03 | 0.377 | 0.782 | 3.62 | 2.29 |
| 20.1 | 1.84 | 2.03 (2.029) | 0.380 | 0.778 | 3.61 | 2.28 |
| 22.8 | 1.82 | 2.02 | 0.386 | 0.767 | 3.57 | 2.25 |
| 25.5 | 1.81 | 2.01 | 0.390 | 0.756 | 3.53 | 2.23 |
| 28.2 | 1.80 | 2.00 | 0.393 | 0.746 | 3.48 | 2.22 |
| 30.9 | 1.79 | 1.98 | 0.394 | 0.736 | 3.44 | 2.21 |

Table 10.3: $E_c$ is the central density common to the static and rotating configurations. $a$ and $R^{(2)}$ are the static and average perturbed radius, both of them measured in Scharzschild radius (computed with $M$). $M$ is the static mass and $M_{total}$ the perturbed mass. Finally, $f^H$ stands for the fractional change in mass computed using Hartle's model and $f$ is the same quantity with the corrective term taken into account. $M_{total}$ is computed with $f$.

proportional to the bag constant. But this last is comparable to the physically reasonable central densities, so that the high values of $\delta M$ obtained are well justified.

We can observe from either of the three figures that when we take into account the corrected change in mass, the total mass is increased drastically due to rotation, if compared with the results in [34] or even with respect to standard neutron stars. In fact, a quick comparison of the fractional increase of mass with and without the correction reveals that these differ by factor greater than 2, even inside the range of (non-rotating) stable configurations. The maximum difference in the predicted total masses is achieved for a density of $E_c = 9.85 \cdot 10^{14}\ \mathrm{g\,cm^{-3}}$, as seen from Fig. 10.3 and it is almost of one Solar mass. As a matter of fact, our results could enter in contradiction with the claim by Fechner and Joss about the indistinguisability of neutrons and quark stars. Note that most of the stable non-rotating models considered give rise to rotating configurations with masses greater than $3.3M_\odot$ which exceeds the maximum mass considered for neutron stars (see [22] or [104]).

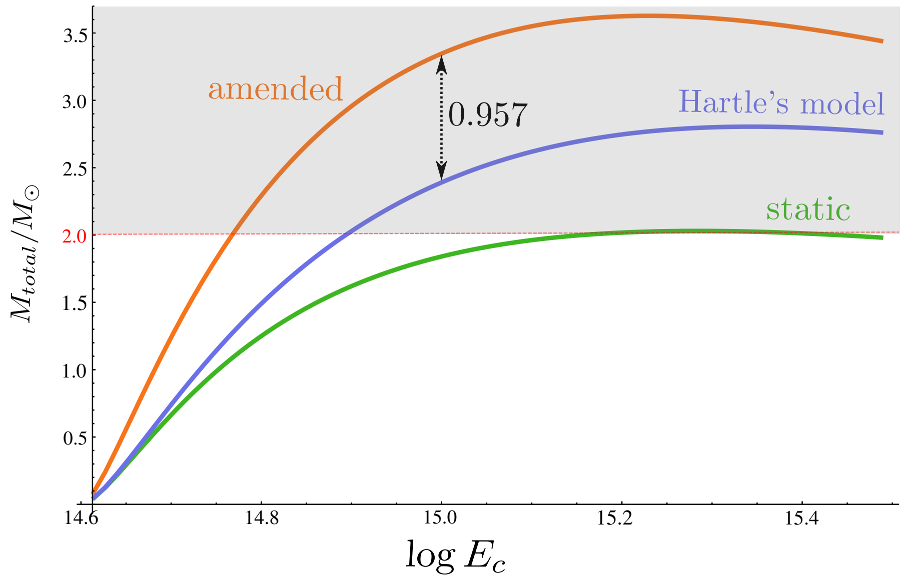


Figure 10.3: This figure corresponds to Fig. 1 in [34]. The ratio of the total mass of the star measured in Solar masses versus the central density of the model, common to the rotating and static configurations, is shown. The density is measured in units of $\mathrm{g\,cm^{-3}}$, and in logaritmic scale (in base 10). The maximum difference between the change in mass from [34] and the amended change in mass is indicated and corresponds to $E_c = 9.85 \cdot 10^{14}\ \mathrm{g\,cm^{-3}}$.

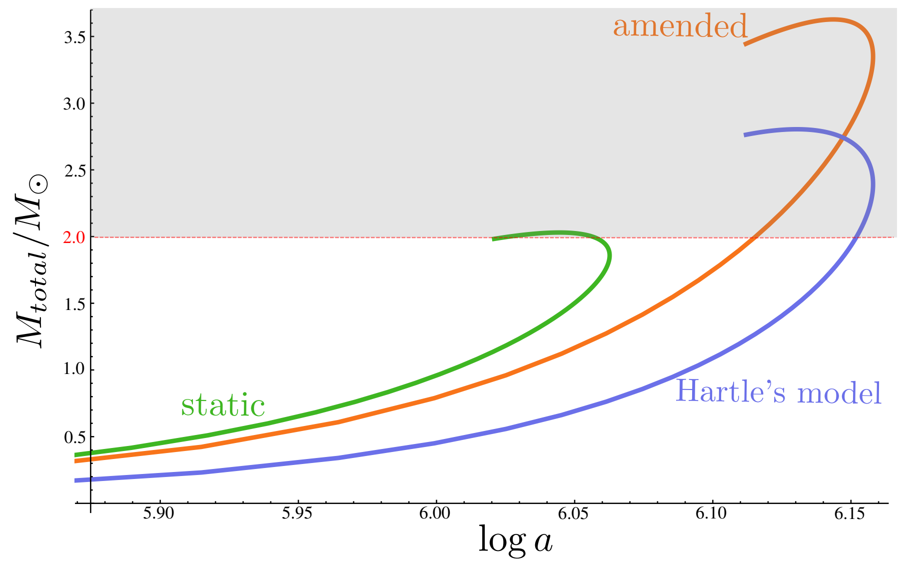


Figure 10.4: This figure corresponds to Fig. 2 in [34]. In the vertical axis we show again the ratio of the total mass of the star with respect to the Solar mass versus the radius of the star. In this case, the static radius is used for the corresponding model, whereas the mean perturbed radius is taken for the perturbations. In both cases, the radius is measured in units of cm, and in logaritmic scale.

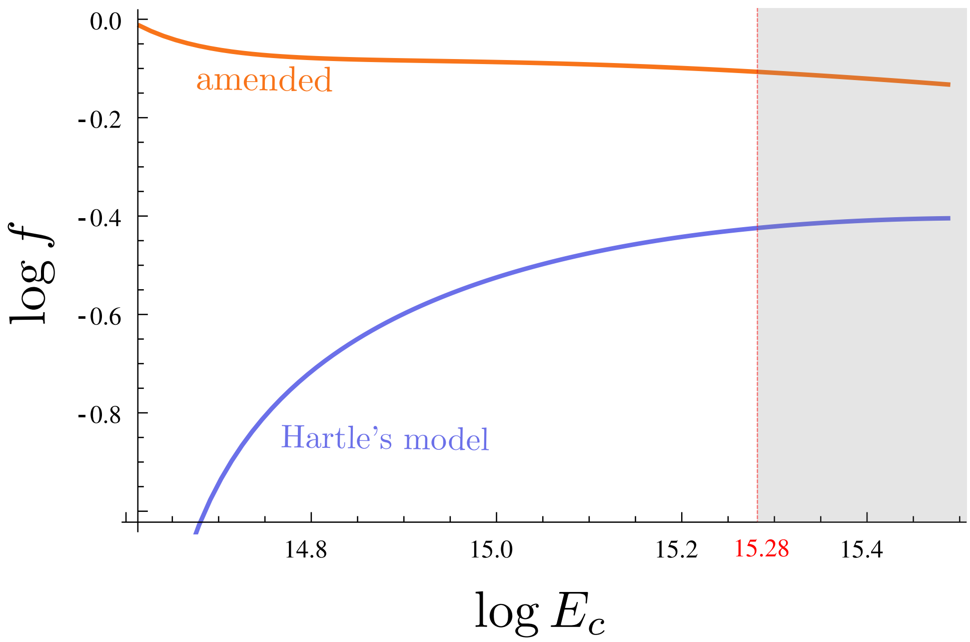


Figure 10.5: This figure corresponds to Fig. 3 in [34].We compare the fractional changes in the mass, for the result given using Hartle's model and formula (7.53). These quantities have no dimensions. The $x-$axis measures the central density of the model, with units of $\mathrm{g\,cm^{-3}}$ and plotted in logaritmic scale.

11

# Junction conditions in quadratic gravity

*Quadratic gravity* refers to theories generalizing General Relativity (GR) by adding terms quadratic in the curvature to the Lagrangian density. The motivations for such modifications go back several decades ago (see the critic paper [97]), and today there is a general consensus that modern string theory (see e.g. [4]) and other approaches to quantum gravity (see e.g. [89]) present that structure, even with higher powers of the curvature tensor, in their effective actions.

On the other hand, many times it is convenient to have a description of concentrated sources, that is, of concentrated matter and energy in gravity theories. These concentrated sources represent for instance thin shells of matter (or braneworlds, or domain walls) and impulsive matter or gravitational waves. They can mathematically be modelled by using distributions, such as Dirac deltas or the like, hence, one has to resort to using *tensor distributions*. However, one cannot simply assume that the metric is a distribution because products of distributions are not well defined in general, and therefore the curvature (and Einstein) tensor will not be defined. Thus, one must identify the class of metrics whose curvature is defined as a distribution, and such that the field equations make sense. For sources on thin shells, the appropriate class of metrics were identified in [67, 75, 106] in GR, further discussed in [53]. Essentially, these are the metrics which are smooth except on localized hypersurfaces where the metric is only continuous.

We carry on a similar program in the most general quadratic theory of gravity, where extra care must be taken: the field equations, as well as the Lagrangian density, contain products of Riemann tensors, and, moreover, their second derivatives. Therefore, the *singular distributional part* —such as the Dirac deltas— cannot arise in the Riemann tensor itself, which can have at most finite jumps except in some very excepctional situations. We identify these and then concentrate on the generic, and more relevant, situation performing a detailed calculation using the rigorous calculus of tensor distributions to obtain the energy-momentum quantities on the shells. They depend on the extrinsic geometrical properties of the hypersurface supporting it, as well as on the possible discontinuities of the curvature and their derivatives.

Surprisingly, and as already demonstrated in [98, 99, 100], a contribution of "dipole" type also appears in the energy-momentum content supported on the shell. This is what we call a double layer, in analogy with the terminology used in classical electrodynamics [68] for the case of electrodipole surface distributions. This analogy make the interpretation of these double layers somewhat misterious, as there are no negative masses —and thus no mass dipoles— in gravitation. One of our purposes is to shed some light into this new mystery. From our results and those in [98, 99, 100], these double layers seem to arise when abrupt changes in the Einstein tensor occur.

We also find the field equations obeyed by all these energy-momentum quantities, which generalize the traditional Israel equations [67], and describe the conservation of energy and momentum. Actually, we explicitly prove that the full energy-momentum tensor is divergence-free (in the distributional sense) by virtue of the mentioned field equations.

Previous works on junction conditions in quadratic gravity include [8, 41, 44, 110] —see also [43, 56] for the Gauss-Bonnet case—, but none of them provided the correct full field equations with matter outside the shell, and they all missed the double-layer contributions, which are fundamental for the energy-momentum conservation. Maybe this is due to the extended use of Gaussian coordinates based on the thin shell: this prevents from making a mathematically sound analysis of the distributional part of the energy-momentum tensor, as the derivatives of the Dirac delta supported on the shell seem to be ill-defined in those coordinates.

This chapter is structured as follows. The quadratic gravity field equations are introduced in Section 11.2, where the proper junction conditions for the description of thin shells (layers) are found. This is achieved by using distributional calculus, briefly reviewed in Chapter 2. In Section 11.3, the matter content supported on the layer, i.e. the distributional part of the global energy momentum tensor, is found to contain a "usual" Dirac-delta term $\widetilde{T}_{\mu\nu}\delta^{\Sigma}$ together with another contribution of double-layer type as mentioned above; the latter is denoted by $\underline{t}_{\mu\nu}$. Then, both $\widetilde{T}_{\mu\nu}$ and $\underline{t}_{\mu\nu}$ are computed in terms of geometrical quantities: the curvatures at either side of the layer and the extrinsic and intrinsic geometry of the hypersurface supporting it. The tensor $\widetilde{T}_{\mu\nu}$ is decomposed into the proper energy momentum of the shell $\tau_{\alpha\beta}$, external flux momentum $\tau_{\alpha}$ and external pressure (or tension) $\tau$ corresponding to the completely tangent, tangent-normal and normal parts respectively. The double layer energy-momentum tensor distribution is found to resemble the energy-momentum content of a dipole surface charge distribution with strength $\mu_{\alpha\beta}$. This strength depends on the jump of the Einstein –or equivalently the

Ricci— tensor at the layer. The allowed jumps of the curvature (and its derivatives up to second order) at the layer are determined in Section 11.4, again from a purely geometrical perspective.

The general quadratic gravity field equations are obtained in Section 11.5. These are the inherited field equations on the layer, and they involve $\tau_{\alpha\beta}$, $\tau_\alpha$, $\tau$ and $\mu_{\alpha\beta}$ together with jumps on the layer of the spacetime energy-momentum tensor. These fundamental equations are the generalization of the Israel equations in GR to the general quadratic gravity theories. The covariant conservation of the full energy-momentum tensor with its distributional parts is explicitly demonstrated in Section 11.6, where we discuss how the double layer term is necessary for that. The field equations on the layer are analysed and further discussed in Section 11.7, where a classification of the junction conditions in the following cases are presented: proper matching, thin shells with no double layers, and pure double layers. In particular we find that if there is no double layer, then no external flux momentum $\tau_\alpha$ nor external tension $\tau$ can exist. Finally, in Section 11.8 some comparisons with the general GR case, and particular matchings of spacetimes, are provided. It is found that any GR solution containing a proper matching hypersurface will contain a double layer and/or a thin shell at the matching hypersurface if the true theory is quadratic. Therefore, if any quantum regimes require, excite or switch on quadratic terms in the Lagrangian density, then GR solutions modelling two regions with different matter contents will develop thin shells and double layers on their interfaces.

On the other hand, we include, in Appendix B, a discussion about the difficulty, and in fact inconvenience, of using Gaussian coordinates for dealing with layers in quadratic Lagrangian theories, as it has been often done in the literature.

## 11.1 Motivation

A general result proven in [82] is that the second Bianchi identity holds in the distributional sense:

$$\nabla_\rho \underline{R}^\alpha{}_{\beta\mu\nu} + \nabla_\mu \underline{R}^\alpha{}_{\beta\nu\rho} + \nabla_\nu \underline{R}^\alpha{}_{\beta\rho\mu} = 0$$

from where one deduces by contraction

$$\nabla^\beta \underline{G}_{\beta\mu} = 0$$

for the Einstein tensor distribution. By using (2.37) and the general formula (2.27) this implies

$$0 = \nabla^\beta \underline{G}_{\beta\mu} = n^\beta \left[G_{\beta\mu}\right] \delta^\Sigma + \nabla^\beta \left(\mathcal{G}_{\beta\mu} \delta^\Sigma\right) . \tag{11.1}$$

The second summand on the righthand side is computed according to the general formula (A.21) in Appendix A.1

$$\nabla^\beta\left(\mathcal{G}_{\beta\mu}\delta^\Sigma\right)=g^{\beta\rho}\nabla_\rho\left(\mathcal{G}_{\beta\mu}\delta^\Sigma\right)=g^{\beta\rho}\nabla_\sigma\left(\mathcal{G}_{\beta\mu}n_\rho n^\sigma\delta^\Sigma\right)+g^{\beta\rho}h^\lambda_\rho\nabla_\lambda\mathcal{G}_{\beta\mu}\delta^\Sigma=h^{\rho\lambda}\nabla_\lambda\mathcal{G}_{\rho\mu}\,\delta^\Sigma$$

which, via (A.6) finally gives

$$\nabla^\beta\left(\mathcal{G}_{\beta\mu}\delta^\Sigma\right)=\left(\overline{\nabla}^\beta\mathcal{G}_{\beta\mu}-K^\Sigma_{\rho\sigma}\mathcal{G}^{\rho\sigma}n_\mu\right)\delta^\Sigma\,.$$

Introducing this into (11.1) we arrive at

$$0=\delta^\Sigma\left(n^\beta\left[G_{\beta\mu}\right]+\overline{\nabla}^\beta\mathcal{G}_{\beta\mu}-\frac{1}{2}n_\mu\mathcal{G}^{\rho\sigma}(K^+_{\rho\sigma}+K^-_{\rho\sigma})\right)$$

which implies, by taking the normal and tangent components, the following relations

$$(K^+_{\rho\sigma}+K^-_{\rho\sigma})\mathcal{G}^{\rho\sigma}=2n^\beta n^\mu\left[G_{\beta\mu}\right]=2n^\beta n^\mu\left[R_{\beta\mu}\right]-[R], \tag{11.2}$$

$$\overline{\nabla}^\beta\mathcal{G}_{\beta\mu}=-n^\rho h^\sigma{}_\mu\left[G_{\rho\sigma}\right]=-n^\rho h^\sigma{}_\mu\left[R_{\rho\sigma}\right]. \tag{11.3}$$

(These equations can also be obtained [67] by using part of the Gauss and Codazzi equations for $\Sigma$ on both sides, specifically (2.66) and (2.67) in Chapter 2).

A very important remark is that all formulae in this section are *purely geometric*, independent of any field equations, and therefore valid in any theory of gravity based on a Lorentzian manifold. The translation of equations (11.2) and (11.3) to quantities related to the energy momentum tensor in General Relativity (2.55) is straigthforward and we find

$$(K^+_{\rho\sigma}+K^-_{\rho\sigma})\tau^{\rho\sigma}=2n^\beta n^\mu\left[T_{\beta\mu}\right], \tag{11.4}$$

$$\overline{\nabla}^\beta\tau_{\beta\mu}=-n^\rho h^\sigma{}_\mu\left[T_{\rho\sigma}\right]. \tag{11.5}$$

Let us comment now about other theories of gravity, such as the $F(R)$ theories. In these, the field equations for a generic theory, without specifying $F$ explicitly, read

$$\begin{aligned}&F'(R)R_{\mu\nu}-\frac{1}{2}F(R)g_{\mu\nu}-F''(R)(\nabla_\mu\nabla_\nu R-g_{\mu\nu}\nabla_\rho\nabla^\rho R)\\&-F'''(R)(\nabla_\mu R\nabla_\nu R-g_{\mu\nu}\nabla_\rho R\nabla^\rho R)=\kappa T_{\mu\nu},\end{aligned} \tag{11.6}$$

where $\kappa=8\pi G/c^4$ is the gravitational coupling constant and a prime denotes differentiation with respect to the only argument. Note then that the translation from (11.2), (11.3) to (11.4), (11.5) is not trivial. However, a remarkable result found in [98] is that the relations regarding the energy momentum tensor (11.4) and (11.5) hold for theories with $F''(R)\neq 0$ (see the first Section of the Appendix in [98]). Theories with $F''(R)=0$

are studied in depth in [99], where it is found that (11.4) and (11.5) do not hold in general (these become (9) in [99]), but they do if one requires a matching without double layers. In [98] the fact that (11.4) and (11.5) may hold for any (diffeomorphism invariant) theory of gravity is left as an open question. Thus, one of the main motivations for the work developed in this chapter is to go an step further in this conjecture and explore the quadratic theories of gravity.

## 11.2 Quadratic gravity

We are going to concentrate on the case of quadratic theories of gravity because, apart from its own intrinsic interest and as we are going to discuss, they allow for cases where gravitational double layers arise. Let us consider a quadratic theory of gravity in $n+1$ dimensions described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2\kappa}\left(R - 2\Lambda + a_1 R^2 + a_2 R_{\mu\nu}R^{\mu\nu} + a_3 R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}\right) + \mathcal{L}_{matter}, \tag{11.7}$$

where $\Lambda$ is the cosmological constant, $a_1, a_2, a_3$ are three constants selecting the particular theory, and $\mathcal{L}_{matter}$ is the Lagrangian density describing the matter fields. $\Lambda^{-1}$ and $a_1, a_2, a_3$ have physical units of $L^2$. The field equations derived from this Lagrangian read (see e.g. [47] and references therein)

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} + G^{\square}_{\alpha\beta} = \kappa T_{\alpha\beta}, \tag{11.8}$$

where $T_{\alpha\beta}$ is the energy-momentum tensor of the matter fields derived from $\mathcal{L}_{matter}$, $G_{\alpha\beta}$ is the Einstein tensor and $G^{\square}_{\alpha\beta}$ encodes the part that comes from the quadratic terms:

$$\begin{aligned} G^{\square}_{\alpha\beta} \;=\; & 2\Big\{ a_1 R R_{\alpha\beta} - 2a_3 R_{\alpha\mu}R^{\mu}_{\beta} + a_3 R_{\alpha\rho\mu\nu}R_{\beta}{}^{\rho\mu\nu} + (a_2 + 2a_3)R_{\alpha\mu\beta\nu}R^{\mu\nu} \\ & - \left(a_1 + \frac{1}{2}a_2 + a_3\right)\nabla_\alpha\nabla_\beta R + \left(\frac{1}{2}a_2 + 2a_3\right)\square R_{\alpha\beta}\Big\} \\ & -\frac{1}{2}g_{\alpha\beta}\left\{(a_1 R^2 + a_2 R_{\mu\nu}R^{\mu\nu} + a_3 R_{\rho\gamma\mu\nu}R^{\rho\gamma\mu\nu}) - (4a_1 + a_2)\square R\right\}, \end{aligned} \tag{11.9}$$

where $\square := g^{\mu\nu}\nabla_\mu\nabla_\nu$ is notation for the D'Alembertian in $(\mathcal{V}, g)$.

If we want to find the proper junction conditions, or a description of thin shells or braneworlds in these theories, we have to resort to the distributional calculus (see Chapter 2 and Appendix A). Then, in order to have the Lagrangian density as well as the tensor $G^{\square}_{\alpha\beta}$ well defined in a distributional sense —so that the field equations (11.8) are sensible mathematically—, one has to avoid any multiplication of singular distributions (such as "$\delta^\Sigma\delta^\Sigma$"). One could also hope for some cancellation of such terms between different parts of the Lagrangian, and of $G^{\Box}_{\alpha\beta}$, and this is discussed in the following subsection for completeness, but one has to bear in mind that these cancellations are probably ill defined anyway, and thus not relevant. In order to properly deal with products of distributions we would need a more general calculus, based e.g. on Colombeau algebras [33, 102], and hope that those cancellations certainly occur and are well defined.

## Dubious possible cancellation of non-linear $\delta^{\Sigma}\delta^{\Sigma}$ terms

Let us start by examining the Lagrangian (11.7) recalling that the different curvature terms possess now singular parts proportional to $\delta^{\Sigma}$, as given in (2.61) and its contractions (2.62) and (2.63). One could naively compute the products of these singular parts arising from the quadratic terms in (11.7) and collect them in a common-factor fashion. The result would be a term of type

$$\delta^{\Sigma}\delta^{\Sigma}\left(2\kappa_1[K^{\rho}_{\rho}]^2 + 2\kappa_2[K_{\alpha\beta}][K^{\alpha\beta}]\right)$$

where we have introduced the abbreviations

$$\kappa_1 := 2a_1 + a_2/2, \qquad \kappa_2 := 2a_3 + a_2/2 \tag{11.10}$$

to be used repeatedly in what follows. Then, one should require the vanishing of the term in brackets. A similar naive compilation should be performed with the non-linear distributions arising from the quadratic terms in the field equations (11.9). Imposing again that the full combination must vanish, and separating the resulting condition into its normal and tangent parts to $\Sigma$ we would find

$$\left\{\kappa_1[K^{\rho}_{\rho}]^2 + \kappa_2(3[K^{\mu\nu}][K_{\mu\nu}] - 2[K^{\rho}_{\rho}]^2)\right\} n_\alpha n_\beta \tag{11.11a}$$

$$+ \kappa_1[K^{\rho}_{\rho}](2[K_{\alpha\beta}] - [K^{\rho}_{\rho}]h_{\alpha\beta}) + \kappa_2(2[K^{\rho}_{\rho}][K_{\alpha\beta}] - [K_{\mu\nu}][K^{\mu\nu}]h_{\alpha\beta}) = 0. \tag{11.11b}$$

The normal (11.11a) and tangent (11.11b) parts must vanish separately. In particular the trace of the tangent part reads

$$\kappa_1[K^{\rho}_{\rho}]^2(2-n) + \kappa_2(2[K^{\rho}_{\rho}]^2 - n[K_{\mu\nu}][K^{\mu\nu}]) = 0. \tag{11.12}$$

We see directly that $\kappa_1 = \kappa_2 = 0$ solves (11.11), but in order to find all solutions we compute the determinant of the system (11.11a) and (11.12). This yields

$$(3-n)[K^{\rho}_{\rho}]^2([K^{\rho}_{\rho}]^2 - [K^{\mu\nu}][K_{\mu\nu}]) = 0. \tag{11.13}$$

Take first $[K^{\rho}_{\rho}] = 0$. Then, (11.11a) $= 0$ and (11.12) reduce to $\kappa_2[K_{\mu\nu}][K^{\mu\nu}] = 0$. If $[K^{\rho}_{\rho}] \neq 0$ but $[K^{\rho}_{\rho}]^2 = [K^{\mu\nu}][K_{\mu\nu}]$, (11.11a) $= 0$ reads $(\kappa_1 + \kappa_2)[K^{\rho}_{\rho}]^2 = 0$ and (11.12) is redundant since it becomes $(\kappa_1+\kappa_2)[K^\rho_\rho]^2(2[K_{\alpha\beta}]-[K^\rho_\rho]h_{\alpha\beta}) = 0$. Thus, $\kappa_1+\kappa_2 = 0$ would follow. Finally, if $n = 3$ (and $[K^\rho_\rho]^2 \neq 0$), (11.11) yields a new possibility not considered so far, summarized in

$$[K_{\alpha\beta}] = \frac{1}{3}h_{\alpha\beta} \Rightarrow [K^\rho_\rho] = 1, \quad [K_{\alpha\beta}][K^{\alpha\beta}] = \frac{1}{3}, \qquad \kappa_1 - \kappa_2 = 0. \tag{11.14}$$

In short, each of the following possibilities would seem to allow for the mutual annihilation of "$\delta^\Sigma\delta^\Sigma$" terms in (11.9) —and in (11.7)—:

1. $\kappa_1 = \kappa_2 = 0$.

2. $[K^\rho_\rho] = 0$ and $\kappa_2 = 0$.

3. $[K^\rho_\rho]^2 = [K_{\mu\nu}][K^{\mu\nu}] = 0$.

4. $[K^\rho_\rho]^2 = [K_{\mu\nu}][K^{\mu\nu}] \neq 0$ and $\kappa_1 + \kappa_2 = 0$.

5. If the spacetime is 4-dimensional, $\kappa_1 - \kappa_2 = 0$ and $[K_{\alpha\beta}] = h_{\alpha\beta}/3$.

Despite we have included this analysis here for completeness, we should not forget that these cases are not mathematically correct, and therefore they should not be fully admitted unless a more rigorous study is performed showing its feasibility. To understand the problems behind these naive calculations, we want to emphasize that there is no known way to give a sensible meaning to $\delta^\Sigma\delta^\Sigma$, let alone to things such as $f\delta^\Sigma\delta^\Sigma$. Thus, taking for granted that combinations of type $f_1\delta^\Sigma\delta^\Sigma + f_2\delta^\Sigma\delta^\Sigma$ are related to $(f_1 + f_2)\delta^\Sigma\delta^\Sigma$ is, at least, dubious. Such difficulties were, for instance, noted in [43] for the Gauss-Bonnet case —corresponding to the possibility 1 above—, and one has to resort to analyzing thick shells, that is, layers with a finite width, or to a setting more general than distributions, such as the theory of nonlinear generalized functions described in [33, 102] and references therein. The thin shell formalism is simply not available. Therefore, we will abandon this route for now, and we will concentrate on the generic and well-defined cases analyzed in the next subsection.

### Well defined possibilities: no $\delta^\Sigma\delta^\Sigma$ terms

The only mathematically well-defined possibilities in the available theory of distributions for the thin shell formalism, as just argued, are those where no $\delta^\Sigma\delta^\Sigma$ term ever arises, leading to two different possibilities if we let aside the case of GR (defined by $a_1 = a_2 = a_3 = 0$):

1. If either $a_2$ or $a_3$ is different from zero, then products of the Ricci tensor by itself, or by the Riemann tensor, appear in (11.9) and these are ill-defined if the singular parts (2.61) and (2.62) are non-zero. Thus, we must demand that the singular parts (2.61) and (2.62) vanish which happens, as proven in Chapter 2, if and only if the jump of the second fundamental form vanishes. Thus, in this situation it is indispensable to require
$$[K_{\mu\nu}] = 0. \tag{11.15}$$
In this case, all the curvature tensors are tensor distributions associated to tensor fields with possible discontinuities across the embdedded $\Sigma$. Observe that then the Lagrangian density (11.7) is also a well defined, locally integrable, function.

2. If on the other hand $a_2 = a_3 = 0$, then only products of $R$ by itself or by the Ricci tensor appear in (11.9), and thus it is enough to demand that $R$ is a locally integrable function without singular part. Hence, in this case it is enough to require that (2.63) vanishes, that is to say, that the trace of the second fundamental form has no jump: $[K^{\rho}_{\rho}] = 0$. Observe that, again, the Lagrangian density (11.7) is in this case a well-defined locally integrable function.

In any of the above two possibilities, expression (11.8) with (11.9) has a remarkable property: *there are no terms quadratic in derivatives of the curvature tensors.* Taking into account that tensor distributions can be covariantly differentiated according to the rules explained in Chapter 2 and Appendix A, the derivatives of the curvature tensors may have singular parts and still the field equations (11.8) are mathematically sound. This opens the door for the existence of matching hypersurfaces which represent *double layers.* Case 2 above was extensively treated in [98, 99, 100], where gravitational double layers were found for the first time. Therefore, we will here concentrate in the more general case 1, and thus we will assume hereafter that (11.15) holds. Notice that (11.15) coincide precisely with the matching conditions that are needed in General Relativity to avoid distributional matter contents, as follows from (2.64) together with the Einstein field equations.

Once (11.15) is enforced, the lefthand side of the field equations (11.8) can be computed in the distributional sense. From (2.33) and (11.15) we know that the Riemann tensor distribution
$$\underline{R}_{\alpha\beta\mu\nu} = R^{+}_{\alpha\beta\mu\nu}\underline{\theta} + R^{-}_{\alpha\beta\mu\nu}(\underline{1} - \underline{\theta}),$$
is actually associated to a locally integrable (and piecewise differentiable) tensor field. However, this tensor field may be discontinuous across $\Sigma$, and thus $[R_{\alpha\beta\mu\nu}]$ may be non-vanishing. This leads, when computing covariant derivatives of $\underline{R}_{\alpha\beta\mu\nu}$, to singular terms

proportional $\delta^\Sigma$ and its derivatives. And these are going to arise in $\underline{G}^{\Box}_{\alpha\beta}$. Thus, the energy-momentum tensor on the righthand side of (11.8) must be treated as a tensor distribution and contain such terms, localized on Σ, giving the energy-matter contents of the thin shell or double layer.

In order to compute this matter content supported on Σ we only have to calculate the singular part of $\underline{G}^{\Box}_{\alpha\beta}$, because $\mathcal{G}_{\alpha\beta}$ in (2.37) vanishes as follows from (11.15) with (2.64). But the only terms in (11.9) that are relevant for this singular part are $\nabla_\alpha\nabla_\beta R$ and $\Box R_{\alpha\beta}$ (and its contraction $\Box R$). More precisely, we need to obtain the singular part of the expression

$$\begin{aligned}&-(2a_1+a_2+2a_3)\,\nabla_\alpha\nabla_\beta\underline{R}+(a_2+4a_3)\,\Box\underline{R}_{\alpha\beta}+\left(2a_1+\frac{1}{2}a_2\right)\Box\underline{R}\,g_{\alpha\beta}\\&=-(\kappa_1+\kappa_2)\,\nabla_\alpha\nabla_\beta\underline{R}+2\kappa_2\Box\underline{R}_{\alpha\beta}+\kappa_1\Box\underline{R}\,g_{\alpha\beta}.\end{aligned} \tag{11.16}$$

This is the purpose of the next section.

## 11.3 Energy-momentum on the layer Σ

From (2.35) and the assumption (11.15) we know that

$$\underline{R}_{\alpha\beta}=R^+_{\alpha\beta}\,\underline{\theta}+R^-_{\alpha\beta}(\underline{1}-\underline{\theta})$$

from where, using the general formula (2.27) twice we deduce

$$\begin{aligned}\nabla_\nu\underline{R}_{\alpha\beta}&=\nabla_\nu R^+_{\alpha\beta}\,\underline{\theta}+\nabla_\nu R^-_{\alpha\beta}(\underline{1}-\underline{\theta})+[R_{\alpha\beta}]n_\nu\delta^\Sigma,\\\nabla_\mu\nabla_\nu\underline{R}_{\alpha\beta}&=\nabla_\mu\nabla_\nu R^+_{\alpha\beta}\,\underline{\theta}+\nabla_\mu\nabla_\nu R^-_{\alpha\beta}(\underline{1}-\underline{\theta})+[\nabla_\nu R_{\alpha\beta}]n_\mu\delta^\Sigma\\&\quad+\nabla_\mu\left([R_{\alpha\beta}]n_\nu\delta^\Sigma\right).\end{aligned} \tag{11.17}$$

Via contractions here, or directly from (2.36), we also obtain

$$\begin{aligned}\underline{R}&=R^+\underline{\theta}+R^-(\underline{1}-\underline{\theta}),\\\nabla_\nu\underline{R}&=\nabla_\nu R^+\underline{\theta}+\nabla_\nu R^-(\underline{1}-\underline{\theta})+[R]n_\nu\delta^\Sigma,\\\nabla_\mu\nabla_\nu\underline{R}&=\nabla_\mu\nabla_\nu R^+\underline{\theta}+\nabla_\mu\nabla_\nu R^-(\underline{1}-\underline{\theta})+[\nabla_\nu R]n_\mu\delta^\Sigma+\nabla_\mu\left([R]n_\nu\delta^\Sigma\right)\end{aligned} \tag{11.18}$$

as well as

$$\Box\underline{R}_{\alpha\beta}=\Box R^+_{\alpha\beta}\underline{\theta}+\Box R^-_{\alpha\beta}(\underline{1}-\underline{\theta})+n^\rho[\nabla_\rho R_{\alpha\beta}]\delta^\Sigma+g^{\mu\nu}\nabla_\mu\left([R_{\alpha\beta}]n_\nu\delta^\Sigma\right), \tag{11.19}$$

$$\Box\underline{R}=\Box R^+\underline{\theta}+\Box R^-(\underline{1}-\underline{\theta})+n^\rho[\nabla_\rho R]\delta^\Sigma+g^{\mu\nu}\nabla_\mu\left([R]n_\nu\delta^\Sigma\right). \tag{11.20}$$

Thus, we need to control the discontinuities of the Ricci tensor and the scalar curvature, and also to provide an expression for the singular distribution $\nabla_\mu\left([R_{\alpha\beta}]n_\nu\delta^\Sigma\right)$ supported on $\Sigma$. The general formula (A.21) provides

$$\nabla_\mu\left(n_\nu[R_{\alpha\beta}]\delta^\Sigma\right) = \nabla_\rho\left([R_{\alpha\beta}]n_\mu n_\nu n^\rho\delta^\Sigma\right) + \left\{h^\rho_\mu\nabla_\rho(n_\nu[R_{\alpha\beta}]) - K^\rho{}_\rho\,[R_{\alpha\beta}]n_\mu n_\nu\right\}\delta^\Sigma.$$

At this point we introduce a 4-covariant tensor distribution $\underline{\Delta}_{\mu\nu\alpha\beta}$ with support on $\Sigma$, which takes care of the first summand here and is defined by

$$\underline{\Delta}_{\mu\nu\alpha\beta} := \nabla_\rho\left([R_{\alpha\beta}]n_\mu n_\nu n^\rho\,\delta^\Sigma\right)$$

or equivalently by

$$\left\langle\underline{\Delta}_{\mu\nu\alpha\beta}, Y^{\mu\nu\alpha\beta}\right\rangle := -\int_\Sigma [R_{\alpha\beta}]n_\nu n_\mu n^\rho\nabla_\rho Y^{\mu\nu\alpha\beta}dv.$$

Note that $\underline{\Delta}_{\mu\nu\alpha\beta} = \underline{\Delta}_{\nu\mu\alpha\beta} = \underline{\Delta}_{\mu\nu\beta\alpha}$. In summary, we have

$$\nabla_\mu\left(n_\nu[R_{\alpha\beta}]\delta^\Sigma\right) = \underline{\Delta}_{\mu\nu\alpha\beta} + \left\{n_\nu h^\rho_\mu\nabla_\rho[R_{\alpha\beta}] + [R_{\alpha\beta}](K_{\mu\nu} - K^\rho{}_\rho\,n_\mu n_\nu)\right\}\delta^\Sigma$$

and therefore (11.17) becomes

$$\begin{aligned}\nabla_\mu\nabla_\nu\underline{R}_{\alpha\beta} &= \nabla_\mu\nabla_\nu R^+_{\alpha\beta}\,\underline{\theta} + \nabla_\mu\nabla_\nu R^-_{\alpha\beta}(\underline{1} - \underline{\theta}) + \underline{\Delta}_{\mu\nu\alpha\beta}\\ &+ \left\{[\nabla_\nu R_{\alpha\beta}]n_\mu + n_\nu h^\rho_\mu\nabla_\rho[R_{\alpha\beta}] + [R_{\alpha\beta}](K_{\mu\nu} - K^\rho{}_\rho\,n_\mu n_\nu)\right\}\delta^\Sigma.\end{aligned}$$

From the general formula (A.19), conveniently generalised, we have

$$[\nabla_\rho R_{\beta\mu}] = n_\rho r_{\beta\mu} + h^\sigma_\rho\nabla_\sigma[R_{\beta\mu}], \tag{11.21}$$

where

$$r_{\beta\mu} := n^\rho[\nabla_\rho R_{\beta\mu}], \qquad r_{\beta\mu} = r_{\mu\beta} \tag{11.22}$$

are the discontinuities of the normal derivatives of the Ricci tensor. Thus, we finally get

$$\begin{aligned}\nabla_\mu\nabla_\nu\underline{R}_{\alpha\beta} &= \nabla_\mu\nabla_\nu R^+_{\alpha\beta}\,\underline{\theta} + \nabla_\mu\nabla_\nu R^-_{\alpha\beta}(\underline{1} - \underline{\theta}) + \underline{\Delta}_{\mu\nu\alpha\beta}\\ &+ \left\{r_{\alpha\beta}\,n_\nu n_\mu + n_\mu h^\rho_\nu\nabla_\rho[R_{\alpha\beta}] + n_\nu h^\rho_\mu\nabla_\rho[R_{\alpha\beta}] + [R_{\alpha\beta}](K_{\mu\nu} - K^\rho{}_\rho\,n_\mu n_\nu)\right\}\delta^\Sigma.\end{aligned} \tag{11.23}$$

Observe that the entire singular part is symmetric in $(\alpha\beta)$ and in $(\mu\nu)$.

From (11.23) we immediately get all the sought terms. First, by contracting with $g^{\alpha\beta}$ we find [98, 99, 100]

$$\begin{aligned}\nabla_\mu\nabla_\nu\underline{R} &= \nabla_\mu\nabla_\nu R^+\,\underline{\theta} + \nabla_\mu\nabla_\nu R^-(\underline{1} - \underline{\theta}) + \underline{\Delta}_{\mu\nu}\\ &+ \left\{b n_\nu n_\mu + n_\mu\overline{\nabla}_\nu[R] + n_\nu\overline{\nabla}_\mu[R] + [R](K_{\mu\nu} - K^\rho{}_\rho\,n_\mu n_\nu)\right\}\delta^\Sigma\end{aligned} \tag{11.24}$$

where [99, 100]

$$b := r^{\rho}_{\rho} = n^{\rho}\nabla_{\rho}[R] \tag{11.25}$$

measures the discontinuity on the normal derivative of the scalar curvature, and [99]

$$\underline{\Delta}_{\mu\nu} := g^{\alpha\beta}\underline{\Delta}_{\mu\nu\alpha\beta}$$

is a 2-covariant symmetric tensor distribution with support on Σ acting as follows[1]

$$\left\langle \underline{\Delta}_{\mu\nu}, Y^{\mu\nu} \right\rangle := -\int_{\Sigma}[R]n_{\nu}n_{\mu}n^{\rho}\nabla_{\rho}Y^{\mu\nu}dv; \qquad \underline{\Delta}_{\mu\nu} = \nabla_{\rho}\left([R]n_{\mu}n_{\nu}n^{\rho}\delta^{\Sigma}\right). \tag{11.26}$$

Similarly, contracting (11.23) with $g^{\mu\nu}$ we readily get

$$\Box R_{\alpha\beta} = \Box R^{+}_{\alpha\beta}\underline{\theta} + \Box R^{-}_{\alpha\beta}(\underline{1}-\underline{\theta}) + r_{\alpha\beta}\delta^{\Sigma} + g^{\mu\nu}\underline{\Delta}_{\mu\nu\alpha\beta} \tag{11.27}$$

where the last distribution acts as follows

$$\begin{aligned}\left\langle g^{\mu\nu}\underline{\Delta}_{\mu\nu\alpha\beta}, Y^{\alpha\beta}\right\rangle &= \left\langle \underline{\Delta}_{\mu\nu\alpha\beta}, g^{\mu\nu}Y^{\alpha\beta}\right\rangle = -\int_{\Sigma}[R_{\alpha\beta}]n_{\nu}n_{\mu}n^{\rho}\nabla_{\rho}(Y^{\alpha\beta}g^{\mu\nu})dv \\ &= -\int_{\Sigma}[R_{\alpha\beta}]n^{\rho}\nabla_{\rho}Y^{\alpha\beta}dv; \qquad g^{\mu\nu}\underline{\Delta}_{\mu\nu\alpha\beta} = \nabla_{\rho}\left([R_{\alpha\beta}]n^{\rho}\delta^{\Sigma}\right).\end{aligned}$$

Finally, by tracing either of (11.24) or (11.27) we easily derive

$$\Box\underline{R} = \Box R^{+}\underline{\theta} + \Box R^{-}(\underline{1}-\underline{\theta}) + b\,\delta^{\Sigma} + \underline{\Delta}, \tag{11.28}$$

where we have introduced the notation $\underline{\Delta} := g^{\mu\nu}\underline{\Delta}_{\mu\nu}$. Note that [98]

$$\left\langle \underline{\Delta}, Y\right\rangle = \left\langle g^{\mu\nu}\underline{\Delta}_{\mu\nu}, Y\right\rangle = -\int_{\Sigma}[R]n^{\rho}\nabla_{\rho}Y\,dv; \qquad \underline{\Delta} = \nabla_{\rho}\left([R]n^{\rho}\delta^{\Sigma}\right).$$

What we have proven is that the distribution $\underline{G}^{\Box}_{\alpha\beta}$ takes the following form

$$\underline{G}^{\Box}_{\alpha\beta} = \underline{G^{\Box+}}_{\alpha\beta}\underline{\theta} + \underline{G^{\Box-}}_{\alpha\beta}(\underline{1}-\underline{\theta}) + \widetilde{G}_{\alpha\beta}\delta^{\Sigma} + \mathscr{G}_{\alpha\beta} \tag{11.29}$$

where

$$\widetilde{G}_{\alpha\beta} = 2\kappa_2 r_{\alpha\beta} + \kappa_1 b g_{\alpha\beta} - (\kappa_1+\kappa_2)\left\{b n_{\alpha}n_{\beta} + n_{\alpha}\overline{\nabla}_{\beta}[R] + n_{\beta}\overline{\nabla}_{\alpha}[R] + [R](K_{\alpha\beta} - K^{\rho}{}_{\rho}\, n_{\alpha}n_{\beta})\right\}, \tag{11.30}$$

and after a trivial rearrangement

$$\mathscr{G}_{\alpha\beta} = \kappa_1\left(g_{\alpha\beta}\underline{\Delta} - \underline{\Delta}_{\alpha\beta}\right) + \kappa_2\left(2g^{\mu\nu}\underline{\Delta}_{\mu\nu\alpha\beta} - \underline{\Delta}_{\alpha\beta}\right). \tag{11.31}$$

[1] There are some errata in the formulae for $\underline{\Delta}_{\mu\nu}$ and $\underline{\Omega}_{\mu\nu}$ in [98], and for $\underline{t}_{\mu\nu}$ in [99, 100]: in all cases $Y$ must be replaced by $Y^{\mu\nu}$.

From (11.31) we define two new 2-covariant tensor distributions with support on Σ [99]:

$$\underline{\Omega}_{\alpha\beta} := g_{\alpha\beta}\underline{\Delta} - \underline{\Delta}_{\alpha\beta} = \nabla_\rho\left([R]h_{\alpha\beta}n^\rho\delta^\Sigma\right); \qquad \left\langle \underline{\Omega}_{\alpha\beta}, Y^{\alpha\beta}\right\rangle = -\int_\Sigma [R]h_{\alpha\beta}\, n^\rho\nabla_\rho Y^{\alpha\beta} dv \tag{11.32}$$

and

$$\underline{\Phi}_{\alpha\beta} := g^{\mu\nu}\underline{\Delta}_{\mu\nu\alpha\beta} - \frac{1}{2}\underline{\Delta}_{\alpha\beta} - \frac{1}{2}\underline{\Omega}_{\alpha\beta} = \nabla_\rho\left([G_{\alpha\beta}]n^\rho\delta^\Sigma\right); \;\; \left\langle \underline{\Phi}_{\alpha\beta}, Y^{\alpha\beta}\right\rangle = -\int_\Sigma [G_{\alpha\beta}]\, n^\rho\nabla_\rho Y^{\alpha\beta} dv \tag{11.33}$$

(recall that $[G_{\alpha\beta}]$ is tangent to Σ, $n^\alpha[G_{\alpha\beta}] = 0$ (2.54)). With these definitions, (11.31) is rewritten simply as

$$\mathscr{G}_{\alpha\beta} = (\kappa_1 + \kappa_2)\underline{\Omega}_{\alpha\beta} + 2\kappa_2\underline{\Phi}_{\alpha\beta}; \qquad \mathscr{G}_{\alpha\beta} = \nabla_\rho\left(\{(\kappa_1+\kappa_2)[R]h_{\alpha\beta} + 2\kappa_2[G_{\alpha\beta}]\}\, n^\rho\delta^\Sigma\right). \tag{11.34}$$

Given the structure (11.29), the field equations (11.8) can only be satisfied if the energy-momentum tensor on the righthand side is a tensor distribution with the following terms

$$\underline{T}_{\mu\nu} = T^+_{\mu\nu}\underline{\theta} + T^-_{\mu\nu}(\underline{1} - \underline{\theta}) + \widetilde{T}_{\mu\nu}\delta^\Sigma + \underline{t}_{\mu\nu} \tag{11.35}$$

where $\widetilde{T}_{\mu\nu}$ is a symmetric tensor field defined only on Σ and $\underline{t}_{\mu\nu}$ is by definition the singular part of $\underline{T}_{\mu\nu}$ with support on Σ *not proportional* to $\delta^\Sigma$. We perform an orthogonal decomposition of $\widetilde{T}_{\mu\nu}$ into tangent, normal-tangent and normal parts with respect to Σ

$$\widetilde{T}_{\mu\nu} = \tau_{\mu\nu} + \tau_\mu n_\nu + \tau_\nu n_\mu + \tau n_\mu n_\nu \tag{11.36}$$

with

$$\tau_{\mu\nu} := h^\rho_\mu h^\sigma_\nu \widetilde{T}_{\rho\sigma}, \quad \tau_{\mu\nu} = \tau_{\nu\mu}, \quad n^\mu\tau_{\mu\nu} = 0; \qquad \tau_\mu := h^\rho_\mu\widetilde{T}_{\rho\nu}n^\nu, \quad n^\mu\tau_\mu = 0; \qquad \tau := n^\mu n^\nu\widetilde{T}_{\mu\nu}$$

so that

$$\underline{T}_{\mu\nu} = T^+_{\mu\nu}\underline{\theta} + T^-_{\mu\nu}(\underline{1} - \underline{\theta}) + \left(\tau_{\mu\nu} + \tau_\mu n_\nu + \tau_\nu n_\mu + \tau n_\mu n_\nu\right)\underline{\delta}^\Sigma + \underline{t}_{\mu\nu}. \tag{11.37}$$

Compare this expression with the form of the energy momentum tensor in GR given by (2.55). Following [99, 100] the proposed names for the objects in (11.37) supported on Σ, with their respective explicit expressions, are:

1. the *energy-momentum tensor* $\tau_{\alpha\beta}$ on Σ, given by

$$\kappa\tau_{\alpha\beta} = -(\kappa_1+\kappa_2)[R]K_{\alpha\beta} + \kappa_1 b h_{\alpha\beta} + 2\kappa_2 r_{\mu\nu}h^\mu_\alpha h^\nu_\beta. \tag{11.38}$$

   $\tau_{\alpha\beta}$ is the only quantity usually defined in standard shells (see Section 2.2).

2. the *external flux momentum* $\tau_\alpha$ defined by

$$\kappa\tau_\alpha = -(\kappa_1+\kappa_2)\overline{\nabla}_\alpha[R] + 2\kappa_2 r_{\mu\nu}n^\mu h^\nu_\alpha. \tag{11.39}$$

   This momentum vector describes normal-tangent components of $\underline{T}_{\mu\nu}$ supported on $\Sigma$. Nothing like that exists in GR. Let us stress that this "external" flux momentum should not be confused with the "flux momentum" defined in thin shells in GR (see e.g. [50]).

3. the *external pressure or tension* $\tau$

$$\kappa\tau = (\kappa_1+\kappa_2)[R]K^\rho_\rho + \kappa_2(2r_{\mu\nu}n^\mu n^\nu - b). \tag{11.40}$$

   Taking the trace of (11.38) one obtains a relation between $b$, $\tau$ and the trace of $\tau_{\mu\nu}$:

$$\kappa\left(\tau^\rho{}_\rho + \tau\right) = (\kappa_1 n + \kappa_2)b \tag{11.41}$$

   The scalar $\tau$ measures the total normal pressure/tension supported on $\Sigma$. Again, such a scalar does not exist in GR.

4. the *double-layer energy-momentum tensor distribution* $\underline{t}_{\alpha\beta}$, which is defined by

$$\kappa\underline{t}_{\alpha\beta} = \mathscr{G}_{\alpha\beta} = \nabla_\rho\left(\{(\kappa_1+\kappa_2)[R]h_{\alpha\beta} + 2\kappa_2[G_{\alpha\beta}]\}\, n^\rho\delta^\Sigma\right) \tag{11.42}$$

   or, equivalently, by acting on any test tensor field $Y^{\alpha\beta}$ as

$$\kappa\left\langle\underline{t}_{\alpha\beta}, Y^{\alpha\beta}\right\rangle = -\int_\Sigma \{(\kappa_1+\kappa_2)[R]h_{\alpha\beta} + 2\kappa_2[G_{\alpha\beta}]\}\, n^\rho\nabla_\rho Y^{\alpha\beta} dv\,. \tag{11.43}$$

   $\underline{t}_{\alpha\beta}$ is a symmetric tensor distribution of "delta-prime" type: it has support on $\Sigma$ but its product with objects intrinsic to $\Sigma$ is not defined unless their extensions off $\Sigma$ are known. As argued in [99, 100], $\underline{t}_{\alpha\beta}$ resembles the energy-momentum content of double-layer surface charge distributions, or "dipole distributions", with strength

$$\kappa\mu_{\alpha\beta} := (\kappa_1+\kappa_2)[R]h_{\alpha\beta} + 2\kappa_2[G_{\alpha\beta}], \qquad \mu_{\alpha\beta} = \mu_{\beta\alpha}, \quad n^\alpha\mu_{\alpha\beta} = 0. \tag{11.44}$$

   We note in passing that

$$\kappa\mu^\rho{}_\rho = (\kappa_1 n + \kappa_2)[R], \qquad \kappa\underline{t}^\rho{}_\rho = (\kappa_1 n + \kappa_2)\underline{\Delta} \tag{11.45}$$

   The appearance of such double layers is remarkable, as "massive dipoles" do not exist. However, in quadratic theories of gravity they arise, as we have just shown, in the generic situation when thin shells are considered. In this case, $\underline{t}_{\alpha\beta}$ seems to represent the idealization of abrupt changes, or jumps, in the curvature of the space-time.

## 11.4 Curvature discontinuities

In the next section, we are going to derive the field equations satisfied by the energy-momentum quantities (11.38), (11.39), (11.40) and (11.44) supported on $\Sigma$. To that end, we have to perform a detailed calculation of the discontinuities of the field equations (11.8): they obviously include the discontinuities of the energy-momentum tensor $T_{\mu\nu}$ which must be related to the energy-momentum content concentrated on $\Sigma$.

The discontinuity of the lefthand side of (11.8) contains $[G^{\square}_{\alpha\beta}]$ (actually, we will only need $n^\alpha[G^{\square}_{\alpha\beta}]$) and this involves discontinuities of quadratic terms in the Riemann tensor, such as $[R^2]$, $[R_{\alpha\beta}R^{\alpha\beta}]$, $[R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}]$, $[RR_{\alpha\beta}]$, $[R_{\alpha\mu}R^{\mu}_{\beta}]$, $[R_{\alpha\rho\mu\nu}R_{\beta}{}^{\rho\mu\nu}]$ and $[R_{\alpha\mu\beta\nu}R^{\mu\nu}]$, as well as discontinuities of derivatives of the curvature tensors, such as $[\nabla_\alpha\nabla_\beta R]$, $[\square R_{\alpha\beta}]$ or $[\square R]$. Thus, we have to use systematically the rules (A.15) and either of (A.19) or (A.20) supplemented with (11.15), and we also need to have some knowledge on the discontinuities of the Riemann tensor (and its derivatives).

### Discontinuities of the curvature tensors

Thus, let us start by controlling the allowed discontinuities of the Riemann tensor across $\Sigma$. Requirement (11.15) implies that the *matching conditions* (for timelike hypersurfaces) introduced in Chapter 2 hold. The implications are then that the jump in the Christoffel symbols vanishes $[\Gamma^\alpha_{\beta\mu}] = 0$ (recall the relation between the difference of the second fundamental forms and Christoffel symbols (2.59)).

The jump of the Riemann tensor is given by the standard formula (2.50). The independent $n(n+1)/2$ allowed discontinuities for the curvature tensor are encoded in the symmetric tensor $B_{\alpha\beta}$, that recall, can be chosen to be fully tangent to $\Sigma$. The discontinuities of the Ricci tensor, Ricci scalar and Einstein tensor are given by (2.68), (2.69) and (2.71). Equivalently to these, we can write

$$B_{\beta\mu} = [R_{\beta\mu}] - \frac{1}{2}[R]n_\beta n_\mu = [G_{\beta\mu}] + \frac{1}{2}h_{\beta\mu}[R], \quad B^\mu_\mu = \frac{[R]}{2}, \tag{11.46}$$

that tells us that the $n(n+1)/2$ allowed independent discontinuities of the Riemann tensor can be chosen to be the discontinuities of the $\Sigma$-tangent part of the Einstein tensor (or equivalently, of the Ricci tensor).

## Discontinuities of terms quadratic in the curvature

Now, let us concern ourselves with the many terms in (11.9) quadratic in curvature tensors. To start with, using (11.46) with (A.15) we readily obtain

$$[R_{\alpha\beta}R^{\alpha\beta}] = 2[R^{\alpha\beta}]R^{\Sigma}_{\alpha\beta} = \left(B^{\alpha\beta} + \frac{1}{2}[R]n^{\alpha}n^{\beta}\right) R^{\Sigma}_{\alpha\beta}, \tag{11.47}$$

$$[RR_{\alpha\beta}] = R^{\Sigma}[R_{\alpha\beta}] + [R]R^{\Sigma}_{\alpha\beta} = R^{\Sigma}\left(B_{\alpha\beta} + \frac{1}{2}n_{\alpha}n_{\beta}[R]\right) + RR^{\Sigma}_{\alpha\beta}, \tag{11.48}$$

$$[R^2] = 2[R]R^{\Sigma}. \tag{11.49}$$

Regarding $n^{\alpha}[R_{\alpha\mu}R^{\mu}_{\beta}]$, let us first consider the contraction $n^{\sigma}n^{\mu}[R^{\gamma}_{\sigma}R_{\gamma\mu}]$. The chain of equalities

$$n^{\sigma}n^{\mu}[R^{\gamma}_{\sigma}R_{\gamma\mu}] = 2n^{\sigma}n^{\mu}R^{\Sigma\gamma}{}_{\sigma}[R_{\gamma\mu}] = [R]n^{\gamma}n^{\mu}R^{\Sigma}{}_{\gamma\mu} \tag{11.50}$$

follows from (2.68) (or (11.46)) and (A.15). Half-adding the two $\pm$ equations (A.12) and using the result in (11.50) we derive

$$n^{\sigma}n^{\mu}[R^{\gamma}_{\sigma}R_{\gamma\mu}] = \frac{1}{2}[R](R^{\Sigma} - \overline{R} + (K^{\rho}_{\rho})^2 - K_{\rho\sigma}K^{\rho\sigma}). \tag{11.51}$$

Analogous procedures using the Gauss equation (A.7) accordingly yield

$$n^{\sigma}h^{\mu}_{\nu}[R^{\gamma}_{\sigma}R_{\gamma\mu}] = B_{\alpha\nu}(\overline{\nabla}_{\beta}K^{\beta\alpha} - \overline{\nabla}^{\alpha}K^{\rho}_{\rho}) + \frac{1}{2}[R](\overline{\nabla}_{\alpha}K^{\alpha}_{\nu} - \overline{\nabla}_{\nu}K^{\rho}_{\rho}), \tag{11.52}$$

$$n^{\alpha}n^{\beta}[R_{\alpha\mu\beta\nu}R^{\mu\nu}] = \left(2R^{\Sigma}_{\mu\nu} - \overline{R}_{\mu\nu} + K^{\rho}_{\rho}K_{\mu\nu} - K_{\mu\sigma}K^{\sigma}_{\nu}\right) B^{\mu\nu}, \tag{11.53}$$

$$n^{\alpha}h^{\beta}_{\lambda}[R_{\alpha\mu\beta\nu}R^{\mu\nu}] = (\overline{\nabla}_{\beta}K_{\lambda\alpha} - \overline{\nabla}_{\lambda}K_{\beta\alpha})B^{\alpha\beta} - (\overline{\nabla}_{\alpha}K^{\alpha\beta} - \overline{\nabla}^{\beta}K^{\rho}_{\rho})B_{\beta\lambda}. \tag{11.54}$$

$$n^{\alpha}n^{\beta}[R_{\alpha\rho\mu\nu}R_{\beta}{}^{\rho\mu\nu}] = 4n^{\alpha}n^{\beta}[R_{\alpha\mu\beta\nu}R^{\mu\nu}] - 4R^{\Sigma}_{\rho\sigma}B^{\rho\sigma}, \tag{11.55}$$

$$n^{\alpha}h^{\beta}_{\lambda}[R_{\alpha\rho\mu\nu}R_{\beta}{}^{\rho\mu\nu}] = 2B^{\alpha\beta}(\overline{\nabla}_{\alpha}K_{\lambda\beta} - \overline{\nabla}_{\lambda}K_{\alpha\beta}), \tag{11.56}$$

$$[R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}] = 2n^{\alpha}n^{\beta}[R_{\alpha\rho\mu\nu}R_{\beta}{}^{\rho\mu\nu}] = 8B^{\alpha\beta}(R^{\Sigma}_{\alpha\beta} - \overline{R}_{\alpha\beta} + K^{\rho}_{\rho}K_{\alpha\beta} - K_{\alpha\rho}K^{\rho}_{\beta}). \tag{11.57}$$

## Discontinuities of the first derivative of the curvature tensors

Concerning the covariant derivative of the Riemann tensor, the general formula (A.19) leads to

$$[\nabla_{\rho}R_{\alpha\beta\lambda\mu}] = n_{\rho}r_{\alpha\beta\lambda\mu} + h^{\sigma}_{\rho}\nabla_{\sigma}[R_{\alpha\beta\lambda\mu}], \tag{11.58}$$

where $r_{\alpha\beta\mu\nu}$ is a tensor field defined only on $\Sigma$ and with the symmetries of a Riemann tensor. Using the second Bianchi identity for the Riemann tensor the previous formula implies

$$n_{[\rho}r_{\alpha\beta]\lambda\mu} + h_{\sigma[\rho}\nabla^{\sigma}[R_{\alpha\beta]\lambda\mu}] = 0$$

which, on using (2.50) and after some calculations, implies the following structure for $r_{\alpha\beta\mu\nu}$:

$$\begin{aligned} r_{\alpha\beta\mu\nu} &= K_{\alpha\mu}B_{\nu\beta} - K_{\alpha\nu}B_{\mu\beta} + K_{\beta\nu}B_{\mu\alpha} - K_{\beta\mu}B_{\nu\alpha} \\ &+ \left(\overline{\nabla}_\mu B_{\rho\nu} - \overline{\nabla}_\nu B_{\rho\mu}\right)(n_\alpha h^\rho_\beta - n_\beta h^\rho_\alpha) + \left(\overline{\nabla}_\alpha B_{\rho\beta} - \overline{\nabla}_\beta B_{\rho\alpha}\right)(n_\mu h^\rho_\nu - n_\nu h^\rho_\mu) \\ &+ n_\alpha n_\mu \rho_{\beta\nu} - n_\alpha n_\nu \rho_{\beta\mu} - n_\beta n_\mu \rho_{\alpha\nu} + n_\beta n_\nu \rho_{\alpha\mu}, \end{aligned} \tag{11.59}$$

where $\rho_{\beta\mu}$ is a new symmetric tensor field, defined only on $\Sigma$ and tangent to $\Sigma$, $n^\beta \rho_{\beta\mu} = 0$, which encodes the *allowed new* independent discontinuities of the covariant derivative of the Riemann tensor. There are $n(n+1)/2$ of those again. As far as we know, relation (11.59) has only been derived in [82].

Contraction of (11.59) leads to the equation (11.21), but now with an explicit expression for the discontinuity of the normal derivative of the Ricci tensor which reads, on using (11.46)

$$\begin{aligned} r_{\beta\nu} &= \rho_{\beta\nu} + K^\rho_\rho B_{\beta\nu} + \frac{1}{2}[R]K_{\beta\nu} - K_{\rho\beta}B^\rho_\nu - B_{\rho\beta}K^\rho_\nu \\ &- n_\beta \overline{\nabla}_\rho [G^\rho_\nu] - n_\nu \overline{\nabla}_\rho [G^\rho_\beta] \\ &+ n_\beta n_\nu \rho^\alpha_\alpha, \end{aligned} \tag{11.60}$$

where a natural orthogonal decomposition of $r_{\beta\mu}$ appears: the first line is its complete tangent part which, given that $\rho_{\beta\nu}$ entails the allowed new independent discontinuities, is in itself a symmetric tensor field tangent to $\Sigma$ codifying those discontinuities. We are going to denote it by

$$\mathcal{R}_{\beta\mu} := h^\rho_\beta h^\sigma_\mu r_{\rho\sigma} = h^\rho_\beta h^\sigma_\mu n^\lambda [\nabla_\lambda R_{\rho\sigma}]; \tag{11.61}$$

the second line is its tangent-normal part, which is completely determined by the covariant derivative within $\Sigma$ of the discontinuity of the Einstein tensor

$$n^\beta h^\nu_\mu r_{\beta\nu} = -\overline{\nabla}^\rho [G_{\rho\mu}]; \tag{11.62}$$

and finally, the third line gives the total normal component of $r_{\beta\mu}$, which can be related to the discontinuity (11.25) of the normal derivative of $R$ by simply taking the trace $r^\rho_\rho = b$ leading to

$$r_{\beta\mu} n^\beta n^\mu = \frac{b}{2} + K^{\rho\sigma}[G_{\rho\sigma}]. \tag{11.63}$$

Using this we get a useful relation for the trace of $\mathcal{R}_{\alpha\beta}$, that does not depend on $\rho_{\alpha\beta}$

$$\mathcal{R}^\alpha_\alpha = \frac{b}{2} - K^{\rho\sigma}[G_{\rho\sigma}]. \tag{11.64}$$

## Discontinuities of the second-order derivatives

Let us now consider the jumps in the second derivatives of the Ricci tensor. The starting point is equation (11.21). We can find an expression for the second summand there by differentiating (2.68) along $\Sigma$ and using the general rule (A.6) (see Appendix A.1),

$$h^\sigma_\rho \nabla_\sigma [R_{\beta\mu}] = \frac{1}{2} n_\beta n_\mu \overline{\nabla}_\rho [R] + n_{(\mu} \left( K_{\beta)\rho} [R] - 2B_{\beta)\lambda} K^\lambda_\rho \right) + \overline{\nabla}_\rho B_{\beta\mu}. \tag{11.65}$$

The jumps of the second-order derivatives of the Ricci tensor, due to the general formula (A.19), can be written as

$$[\nabla_\lambda \nabla_\rho R_{\beta\mu}] = n_\lambda A_{\rho\beta\mu} + h^\sigma_\lambda \nabla_\sigma [\nabla_\rho R_{\beta\mu}] \tag{11.66}$$

where $A_{\rho\beta\mu} = A_{\rho(\beta\mu)}$ is a shorthand for

$$A_{\rho\beta\mu} = n^\lambda [\nabla_\lambda \nabla_\rho R_{\beta\mu}].$$

The last term $h^\sigma_\lambda \nabla_\sigma [\nabla_\rho R_{\beta\mu}]$ can be further expanded by first using (11.21) to obtain

$$h^\sigma_\lambda \nabla_\sigma [\nabla_\rho R_{\beta\mu}] = K_{\lambda\rho} r_{\beta\mu} + n_\rho h^\sigma_\lambda \nabla_\sigma r_{\beta\mu} + h^\sigma_\lambda \nabla_\sigma \left( h^\gamma_\rho \nabla_\gamma [R_{\beta\mu}] \right)$$

and then computing the last summand here, which leads to

$$\begin{aligned} h^\sigma_\lambda \nabla_\sigma [\nabla_\rho R_{\beta\mu}] &= K_{\lambda\rho} r_{\beta\mu} + 2n_{(\mu} K_{\beta)(\lambda} \overline{\nabla}_{\rho)} [R] + [R] K_{\rho(\beta} K_{\mu)\lambda} - 4K^\gamma_{(\rho} \overline{\nabla}_{\lambda)} B_{\gamma(\beta} n_{\mu)} \\ &+ n_\beta n_\mu \left( \frac{1}{2} \overline{\nabla}_\lambda \overline{\nabla}_\rho [R] - [R] K^\sigma_\lambda K_{\rho\sigma} + 2K^\gamma_\rho K^\sigma_\lambda B_{\sigma\gamma} \right) \\ &+ \left( \overline{\nabla}_\lambda K^\gamma_\rho - n_\rho K^\sigma_\lambda K^\gamma_\sigma \right) \left( [R] h_{\gamma(\beta} - 2B_{\gamma(\beta} \right) n_{\mu)} - \frac{1}{2} n_\mu n_\beta n_\rho K^\sigma_\lambda \overline{\nabla}_\sigma [R] \\ &+ \overline{\nabla}_\lambda \overline{\nabla}_\rho B_{\beta\mu} - 2K^\gamma_\rho B_{\gamma(\beta} K_{\mu)\lambda} - n_\rho K^\sigma_\lambda \overline{\nabla}_\sigma B_{\beta\mu} + n_\rho h^\sigma_\lambda \nabla_\sigma r_{\beta\mu}. \end{aligned} \tag{11.67}$$

Let us stress the fact that all the terms in the first two lines in the above expression are symmetric in $(\lambda\rho)$.

Concerning $A_{\rho\beta\mu}$, let us first decompose it into normal and tangential parts by

$$A_{\rho\beta\mu} = n_\rho A_{\beta\mu} + h^\gamma_\rho A_{\gamma\beta\mu}, \qquad A_{\beta\mu} := n^\rho A_{\rho\beta\mu}, \quad A_{\beta\mu} = A_{\mu\beta}.$$

In order to obtain an expression for $h^\gamma_\rho A_{\gamma\beta\mu}$ we take the antisymmetric part of (11.66) with respect to $[\lambda\rho]$, and contract with $n^\lambda$. For the left hand side of (11.66) we use the Ricci identity applied to the Ricci tensor at both sides $V^\pm$, and take the difference of the limits on $\Sigma$, so that

$$[(\nabla_\lambda \nabla_\rho - \nabla_\rho \nabla_\lambda) R_{\beta\mu}] = [R^\gamma{}_{\beta\rho\lambda} R_{\gamma\mu}] + [R^\gamma{}_{\mu\rho\lambda} R_{\beta\gamma}].$$

For the right hand side of (11.66), after the contraction with $n^\lambda$, we get

$$A_{\rho\beta\mu} - n^\lambda n_\rho A_{\lambda\beta\mu} - n^\lambda h^\sigma_\rho \nabla_\sigma [\nabla_\lambda R_{\beta\mu}] = h^\gamma_\rho A_{\gamma\beta\mu} - n^\lambda h^\sigma_\rho \nabla_\sigma [\nabla_\lambda R_{\beta\mu}].$$

Isolating $h^\gamma_\rho A_{\gamma\beta\mu}$ and using (11.67) for the last term of the above equation, it is then straightforward to obtain

$$\begin{aligned} A_{\rho\beta\mu} &= n_\rho A_{\beta\mu} + n^\nu [R^\gamma{}_{\beta\rho\nu} R_{\gamma\mu}] + n^\nu [R^\gamma{}_{\mu\rho\nu} R_{\beta\gamma}] + h^\sigma_\rho \nabla_\sigma r_{\beta\mu} \\ &- \frac{1}{2} n_\mu n_\beta K^\sigma_\rho \overline{\nabla}_\sigma [R] - K^\sigma_\rho \overline{\nabla}_\sigma B_{\beta\mu} - [R] K^\sigma_\rho K_{\sigma(\beta} n_{\mu)} + 2 K^\sigma_\rho K^\gamma_\sigma B_{\gamma(\beta} n_{\mu)}. \end{aligned} \tag{11.68}$$

The expression for $[\nabla_\lambda \nabla_\rho R_{\beta\mu}]$ now follows by combining (11.66) with (11.67) and (11.68). After little rearrangements, that reads

$$\begin{aligned} [\nabla_\lambda \nabla_\rho R_{\beta\mu}] &= n_\lambda n_\rho A_{\beta\mu} + n_\lambda n^\nu \left( [R^\gamma{}_{\beta\rho\nu} R_{\gamma\mu}] + [R^\gamma{}_{\mu\rho\nu} R_{\beta\gamma}] \right) + 2 n_{(\lambda} h_{\rho)}{}^\sigma \nabla_\sigma r_{\beta\mu} \\ &- n_\mu n_\beta n_{(\lambda} K_{\rho)}{}^\sigma \overline{\nabla}_\sigma [R] - 2 n_{(\lambda} K_{\rho)}{}^\sigma \overline{\nabla}_\sigma B_{\beta\mu} - 2 [R] n_{(\lambda} K_{\rho)}{}^\sigma K_{\sigma(\beta} n_{\mu)} \\ &+ 4 n_{(\lambda} K_{\rho)}{}^\sigma K^\gamma_\sigma B_{\gamma(\beta} n_{\mu)} + 2 n_{(\mu} K_{\beta)(\lambda} \overline{\nabla}_{\rho)} [R] + [R] K_{\rho(\beta} K_{\mu)\lambda} - 4 K^\gamma{}_{(\rho} \overline{\nabla}_{\lambda)} B_{\gamma(\beta} n_{\mu)} \\ &+ n_\beta n_\mu \left( \frac{1}{2} \overline{\nabla}_\lambda \overline{\nabla}_\rho [R] - [R] K^\sigma_\lambda K_{\rho\sigma} + 2 K^\gamma_\rho K^\sigma_\lambda B_{\sigma\gamma} \right) + K_{\lambda\rho} r_{\beta\mu} \\ &+ \overline{\nabla}_\lambda K^\gamma_\rho \left( [R] h_{\gamma(\beta} - 2 B_{\gamma(\beta} \right) n_{\mu)} + \overline{\nabla}_\lambda \overline{\nabla}_\rho B_{\beta\mu} - 2 K^\gamma_\rho B_{\gamma(\beta} K_{\mu)\lambda}. \end{aligned} \tag{11.69}$$

We must stress the fact that there are still terms in (11.69), i.e. $A_{\beta\mu}$ and $r_{\beta\mu}$, that are not completely independent.

The contraction of (11.69) with $g^{\rho\lambda}$ yields

$$\begin{aligned} [\Box R_{\beta\mu}] &= A_{\beta\mu} + K r_{\beta\mu} + n_\mu n_\beta \left( \frac{1}{2} \overline{\Box} [R] - [R] K^{\rho\sigma} K_{\rho\sigma} + 2 K^{\sigma\rho} K^\gamma_\rho B_{\sigma\gamma} \right) \\ &+ 2 n_{(\mu} h^\lambda_{\beta)} \left( \overline{\nabla}_\rho [R] K^\rho_\lambda - 2 K^{\gamma\rho} \overline{\nabla}_\rho B_{\gamma\lambda} + \frac{1}{2} \overline{\nabla}_\rho K^\rho_\lambda - \overline{\nabla}_\rho K^{\rho\gamma} B_{\gamma\lambda} \right) \\ &+ [R] K_{\rho\beta} K^\rho_\mu + \overline{\Box} B_{\beta\mu} - 2 K^\gamma_\rho K^\rho_{(\mu} B_{\beta)\gamma}, \end{aligned} \tag{11.70}$$

while contracting with $g^{\beta\mu}$ we obtain [98]

$$[\nabla_\nu \nabla_\mu R] = A^\rho_\rho n_\nu n_\mu + 2 n_{(\nu} \overline{\nabla}_{\mu)} b - 2 n_{(\nu} K^\lambda_{\mu)} \overline{\nabla}_\lambda [R] + b K_{\nu\mu} + \overline{\nabla}_\nu \overline{\nabla}_\mu [R]. \tag{11.71}$$

From any of the previous we readily have

$$[\Box R] = A^\rho_\rho + bK + \overline{\Box} [R]. \tag{11.72}$$

The energy-momentum quantities (11.38-11.40) will arise from the discontinuities of the *normal* components of the lefthand side of (11.8). In other words, we will only need to consider $n^\alpha [G^\Box_{\alpha\beta}]$. Observe then that $A_{\beta\mu}$ only appears in $[\Box R_{\beta\mu}]$, and since we only need the terms contracted with the normal once, in particular $n^\beta[\Box R_{\beta\mu}]$, we are only interested in controlling $n^\beta A_{\beta\mu}$. This can be done by using the identities $2\nabla^\rho R^\pm_{\rho\mu} = \nabla_\mu R^\pm$ at both sides of $\Sigma$, and taking the difference after one further differentiation:

$$n^\nu[\nabla_\nu\nabla^\rho R_{\rho\mu}] = \frac{1}{2}n^\nu[\nabla_\nu\nabla_\mu R]. \tag{11.73}$$

The lefthand side here comes from (11.66) combined with (11.68) after one contraction, whereas for the righthand side we simply have to contract (11.71) with $n^\nu$. Equation (11.73) is thus found to be equivalent to

$$\begin{aligned} n^\rho A_{\rho\mu} + n^\sigma\left(-[R^\gamma_\sigma R_{\gamma\mu}] + [R_{\gamma\mu\rho\sigma}R^{\gamma\rho}]\right) + h^{\beta\sigma}\nabla_\sigma r_{\beta\mu} - K^{\beta\sigma}\overline{\nabla}_\sigma B_{\beta\mu} \\ -n_\mu\left(\frac{1}{2}[R]K_{\rho\sigma}K^{\rho\sigma} - K^{\sigma\beta}K^\gamma_\sigma B_{\gamma\beta}\right) = \frac{1}{2}\left(A^\rho_\rho n_\mu + \overline{\nabla}_\mu b - K^\lambda_\mu\overline{\nabla}_\lambda[R]\right). \end{aligned} \tag{11.74}$$

## Discontinuities of the quadratic part $[G^\Box_{\alpha\beta}]$

We are now ready to compute the full $n^\alpha[G^\Box_{\alpha\beta}]$. To keep track of the different terms, we split the compilation of terms in three parts, corresponding to the terms multiplied by either of the three constants $a_1, a_2, a_3$ in (11.9).

- Terms with $a_1$:

  The terms in (11.9) that go with $a_1$ are

  $$G^{\Box a_1}_{\alpha\beta} := 2RR_{\alpha\beta} - 2\nabla_\beta\nabla_\alpha R - \frac{1}{2}g_{\alpha\beta}R^2 + 2g_{\alpha\beta}\Box R,$$

  and we can compute their jump using (11.48), (11.71) and (11.72) to obtain

  $$n^\alpha n^\beta[G^{\Box a_1}_{\alpha\beta}] = 2[R]R^\Sigma_{\alpha\beta}n^\alpha n^\beta + 2bK^\rho_\rho + 2\overline{\Box}[R] \tag{11.75}$$

  and

  $$n^\alpha h^\beta_\mu[G^{\Box a_1}_{\alpha\beta}] = 2[R]R^\Sigma_{\alpha\beta}n^\alpha h^\beta_\mu - 2\overline{\nabla}_\mu b + 2K^\alpha_\mu\overline{\nabla}_\alpha[R]. \tag{11.76}$$

- Terms with $a_2$:

  The terms in (11.9) relative to $a_2$ are

  $$G^{\Box a_2}_{\alpha\beta} := 2R_{\alpha\mu\beta\nu}R^{\mu\nu} - \nabla_\beta\nabla_\alpha R + \Box R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\left(R_{\mu\nu}R^{\mu\nu} - \Box R\right).$$

  Before using (11.53) and (11.54) it is convenient to write down $n^\alpha[\Box R_{\alpha\beta}]$ using (11.70) combined with (11.74), since some terms simplify. With the help of (11.53), (11.54), (11.71), (11.48) and (11.61-11.63) it is then easy to get

$$n^\alpha n^\beta [G^{\square a_2}_{\alpha\beta}] = \frac{b}{2}K^\rho_\rho + \overline{\square}[R] + \frac{1}{4}[R](R^\Sigma - \overline{R} + (K^\rho_\rho)^2) - \frac{3}{4}[R]K_{\rho\sigma}K^{\rho\sigma} + \mathcal{R}_{\rho\sigma}K^{\rho\sigma} + \overline{\nabla}_\rho\overline{\nabla}_\mu[G^{\rho\mu}] + B^{\mu\nu}(R^\Sigma_{\mu\nu} - \overline{R}_{\mu\nu} + K^\rho_\rho K_{\mu\nu}), \quad (11.77)$$

$$n^\alpha h^\beta_\mu [G^{\square a_2}_{\alpha\beta}] = -\frac{1}{2}\overline{\nabla}_\mu b + \frac{3}{2}K^\alpha_\mu\overline{\nabla}_\alpha[R] + [R]\left(\overline{\nabla}_\alpha K^\alpha_\mu - \frac{1}{2}\overline{\nabla}_\mu K\right) - \overline{\nabla}_\alpha\mathcal{R}^\alpha_\mu + K^\alpha_\mu\overline{\nabla}^\nu[G_{\nu\alpha}] + B^{\alpha\beta}(\overline{\nabla}_\beta K_{\alpha\mu} - \overline{\nabla}_\mu K_{\alpha\beta}) - B_{\alpha\mu}\overline{\nabla}_\beta K^{\alpha\beta} - K^{\alpha\beta}\overline{\nabla}_\beta B_{\alpha\mu}. \quad (11.78)$$

- Terms with $a_3$:

  Regarding $a_3$ we have

$$G^{\square a_3}_{\alpha\beta} := -4R_{\alpha\mu}R^\mu_\beta + 2R_{\alpha\rho\mu\nu}R_\beta^{\ \rho\mu\nu} + 4R_{\alpha\mu\beta\nu}R^{\mu\nu} - 2\nabla_\beta\nabla_\alpha R + 4\square R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R_{\rho\gamma\mu\nu}R^{\rho\gamma\mu\nu}.$$

  All terms have already appeared except for the last one, for which we use (11.57). Straightforward calculations lead to

$$n^\alpha n^\beta [G^{\square a_3}_{\alpha\beta}] = 4\mathcal{R}_{\alpha\beta}K^{\alpha\beta} + 4\overline{\nabla}_\alpha\overline{\nabla}_\beta[G^{\alpha\beta}] + 4[G_{\alpha\rho}]K^{\alpha\beta}K^\rho_\beta + 2\overline{\square}[R] + 4B^{\alpha\beta}(R^\Sigma_{\alpha\beta} - \overline{R}_{\alpha\beta} + K_{\alpha\beta}K^\rho_\rho - K_{\alpha\rho}K^\rho_\beta), \quad (11.79)$$

$$n^\alpha h^\beta_\mu [G^{\square a_3}_{\alpha\beta}] = +4K^\alpha_\mu\overline{\nabla}^\beta[G_{\beta\alpha}] - 4\overline{\nabla}_\alpha\mathcal{R}^\alpha_\mu + 4K^\alpha_\mu\overline{\nabla}_\alpha[R] - 4\overline{\nabla}_\beta(B_{\alpha\mu}K^{\alpha\beta}) + 2[R]\overline{\nabla}_\alpha K^\alpha_\mu - 4B_{\beta\mu}\overline{\nabla}_\alpha K^{\alpha\beta} + 4B^{\alpha\beta}(\overline{\nabla}_\beta K_{\alpha\mu} - \overline{\nabla}_\mu K_{\alpha\beta}) \quad (11.80)$$

Collecting all the above, we finally obtain

$$n^\alpha n^\beta [G^\square_{\alpha\beta}] = \kappa_1\left\{bK^\rho_\rho + \overline{\square}[R] + \frac{1}{2}\left(R^\Sigma - \overline{R} + (K^\rho_\rho)^2 - K_{\rho\sigma}K^{\rho\sigma}\right)\right\} + \kappa_2\left\{2\mathcal{R}_{\alpha\beta}K^{\alpha\beta} + 2\overline{\nabla}_\alpha\overline{\nabla}_\beta[G^{\alpha\beta}] + 2B^{\alpha\beta}(R^\Sigma_{\alpha\beta} - \overline{R}_{\alpha\beta} + K_{\alpha\beta}K^\rho_\rho - K_{\alpha\rho}K^\rho_\beta) + 2[G_{\alpha\mu}]K^{\alpha\beta}K^\mu_\beta + \overline{\square}[R]\right\} \quad (11.81)$$

$$n^\alpha h^\beta_\mu [G^\square_{\alpha\beta}] = \kappa_1\left\{[R](\overline{\nabla}_\alpha K^\alpha_\mu - \overline{\nabla}_\mu K^\rho_\rho) - \overline{\nabla}_\mu b + K^\alpha_\mu\overline{\nabla}_\alpha[R]\right\} + \kappa_2\left\{-2\overline{\nabla}_\alpha\mathcal{R}^\alpha_\mu + 2K^\alpha_\mu\overline{\nabla}^\beta[G_{\beta\alpha}] + 2B^{\alpha\beta}(\overline{\nabla}_\beta K_{\alpha\mu} - \overline{\nabla}_\mu K_{\alpha\beta}) + 2K^\alpha_\mu\overline{\nabla}_\alpha[R] + [R]\overline{\nabla}_\alpha K^\alpha_\mu - 2B_{\alpha\mu}\overline{\nabla}_\beta K^{\alpha\beta} - 2K^{\alpha\beta}\overline{\nabla}_\beta B_{\alpha\mu}\right\}. \quad (11.82)$$

**Remark:** As a final remark, we would like to stress that all the discontinuities computed in this section 11.4 are purely geometrical, and therefore valid in any theory based on a Lorentzian manifold whenever (11.15) holds.

## 11.5 Field equations on the layer $\Sigma$

Relations (11.81) and (11.82) are the equations we were looking for, but we wish to rewrite them in terms of the (derivatives of) the energy-momentum quantities supported on $\Sigma$ given in (11.38-11.40) and (11.44). Observe, first of all, that the three relations (11.61), (11.62) and (11.63) allow us to rewrite the energy-momentum contents supported on $\Sigma$ (11.38-11.40) as follows

$$\kappa\tau_{\alpha\beta} = -(\kappa_1+\kappa_2)[R]K_{\alpha\beta} + \kappa_1 b h_{\alpha\beta} + 2\kappa_2\mathcal{R}_{\alpha\beta}, \tag{11.83}$$
$$\kappa\tau_\alpha = -(\kappa_1+\kappa_2)\overline{\nabla}_\alpha[R] - 2\kappa_2\overline{\nabla}^\rho[G_{\rho\alpha}], \tag{11.84}$$
$$\kappa\tau = (\kappa_1+\kappa_2)[R]K^\rho_\rho + 2\kappa_2 K^{\rho\sigma}[G_{\rho\sigma}], \tag{11.85}$$

and using the definition of the double-layer strength (11.44) the last two here can be rewritten as

$$\tau_\alpha = -\overline{\nabla}^\rho\mu_{\rho\alpha}, \tag{11.86}$$
$$\tau = K^{\rho\sigma}\mu_{\rho\sigma}. \tag{11.87}$$

Now, a direct computation provides the following expressions for some combinations of derivatives of these objects:

$$\begin{aligned}\kappa\left(\overline{\nabla}^\beta\tau_{\alpha\beta} + K^\rho_\rho\tau_\alpha + \overline{\nabla}_\alpha\tau\right) &= -(\kappa_1+\kappa_2)\left(K_\alpha{}^\beta\overline{\nabla}_\beta[R] + [R](\overline{\nabla}^\beta K_{\alpha\beta} - \overline{\nabla}_\alpha K^\rho_\rho)\right)\\ &\quad+\kappa_1\overline{\nabla}_\alpha b + 2\kappa_2(\overline{\nabla}^\beta\mathcal{R}_{\alpha\beta} + \overline{\nabla}_\alpha(K^{\rho\sigma}[G_{\rho\sigma}]) + K^\rho_\rho\overline{\nabla}^\mu[G_{\mu\alpha}]),\end{aligned} \tag{11.88}$$

$$\kappa\left(\tau_{\alpha\beta}K^{\alpha\beta} - \overline{\nabla}^\alpha\tau_\alpha\right) = (\kappa_1+\kappa_2)(\overline{\Box}[R] - [R]K_{\rho\sigma}K^{\rho\sigma}) \tag{11.89}$$
$$+\kappa_1 b K^\rho_\rho + 2\kappa_2(K^{\rho\sigma}\mathcal{R}_{\rho\sigma} + \overline{\nabla}^\alpha\overline{\nabla}^\beta[G_{\alpha\beta}]). \tag{11.90}$$

Using these, equations (11.82) and (11.81) become respectively (after some rewriting using (A.8) and (A.9) and (11.8))

$$\begin{aligned}\kappa\left(n^\alpha h^\rho_\beta[T_{\alpha\rho}] + \overline{\nabla}^\alpha\tau_{\alpha\beta} + K^\rho_\rho\tau_\beta + \overline{\nabla}_\beta\tau\right) &= 2\kappa_2\left\{K^{\alpha\rho}\overline{\nabla}_\beta[G_{\alpha\rho}] - K^\rho_\rho\overline{\nabla}^\alpha[G_{\alpha\beta}]\right.\\ &\quad\left.+\overline{\nabla}_\rho([G^{\alpha\rho}]K_{\alpha\beta}) - \overline{\nabla}_\rho([G_{\alpha\beta}]K^{\alpha\rho})\right\},\\ \kappa\left(n^\alpha n^\beta[T_{\alpha\beta}] + \overline{\nabla}^\alpha\tau_\alpha - \tau_{\alpha\beta}K^{\alpha\beta}\right) &= (\kappa_1+\kappa_2)[R]\left(n^\alpha n^\beta R^\Sigma_{\alpha\beta} + K_{\alpha\beta}K^{\alpha\beta}\right)\\ &\quad+2\kappa_2[G^{\mu\nu}]\left(n^\alpha n^\gamma R^\Sigma_{\alpha\mu\gamma\nu} + K^\rho_\mu K_{\nu\rho}\right).\end{aligned}$$

Using now the definition of the strength (11.44) these become

$$\begin{aligned}n^\alpha h^\rho_\beta[T_{\alpha\rho}] + \overline{\nabla}^\alpha\tau_{\alpha\beta} + K^\rho_\rho\tau_\beta + \overline{\nabla}_\beta\tau &= K^{\alpha\rho}\overline{\nabla}_\beta\mu_{\alpha\rho} - K^\rho_\rho\overline{\nabla}^\alpha\mu_{\alpha\beta}\\ &\quad+\overline{\nabla}_\rho(\mu^{\alpha\rho}K_{\alpha\beta}) - \overline{\nabla}_\rho(\mu_{\alpha\beta}K^{\alpha\rho})\\ n^\alpha n^\beta[T_{\alpha\beta}] + \overline{\nabla}^\alpha\tau_\alpha - \tau_{\alpha\beta}K^{\alpha\beta} &= \mu^{\mu\nu}\left(n^\alpha n^\gamma R^\Sigma_{\alpha\mu\gamma\nu} + K^\rho_\mu K_{\nu\rho}\right).\end{aligned}$$

Recalling here the relations (11.86) and (11.87) between $\tau_\alpha$ and $\tau$ with the double-layer strength $\mu_{\alpha\beta}$, we finally obtain the following field equations

$$n^\alpha h^\rho_\beta [T_{\alpha\rho}] + \overline{\nabla}^\alpha \tau_{\alpha\beta} = -\mu_{\alpha\rho} \overline{\nabla}_\beta K^{\alpha\rho} + \overline{\nabla}_\rho (\mu^{\alpha\rho} K_{\alpha\beta}) - \overline{\nabla}_\rho (\mu_{\alpha\beta} K^{\alpha\rho}), \quad (11.91)$$

$$n^\alpha n^\beta [T_{\alpha\beta}] - \tau_{\alpha\beta} K^{\alpha\beta} = \overline{\nabla}^\alpha \overline{\nabla}^\beta \mu_{\alpha\beta} + \mu^{\mu\nu} \left( n^\alpha n^\gamma R^\Sigma_{\alpha\mu\gamma\nu} + K^\rho_\mu K_{\nu\rho} \right). \quad (11.92)$$

These are the fundamental field equations satisfied by the energy-momentum quantities (11.38) and (11.44) within $\Sigma$. They generalize the classical Israel equations of GR [67] and they are very satisfactory from a physical point of view. They possess an obvious structure with a clear interpretation as energy-momentum conservation relations. There are three type of terms in these relations. The first type is given by the corresponding first summands on the lefthand side. They simply describe the jump of the normal components of the energy-momentum tensor across $\Sigma$. Therefore, they are somehow the main source for the energy-momentum contents in $\Sigma$. The second type of terms are those on the lefthand side involving $\tau_{\alpha\beta}$, the energy-momentum tensor in the shell/layer $\Sigma$. We want to remark that the first equation (11.91) provides the divergence of $\tau_{\alpha\beta}$. Finally, the third type of terms are those on the righthand side, involving the strength $\mu_{\alpha\beta}$ of a double layer. These terms act also as sources of the energy-momentum contents within $\Sigma$, combined with extrinsic geometric properties of $\Sigma$ and curvature components in the space-time.

An alternative version of (11.91), after use of the Codazzi equation (A.10), reads

$$n^\alpha h^\rho_\beta [T_{\alpha\rho}] + \overline{\nabla}^\alpha \tau_{\alpha\beta} = \mu^{\alpha\rho} n^\sigma R^\Sigma_{\sigma\alpha\lambda\rho} h^\lambda_\beta + K_{\alpha\beta} \overline{\nabla}_\rho \mu^{\alpha\rho} - \overline{\nabla}_\rho (\mu_{\alpha\beta} K^{\alpha\rho}). \quad (11.93)$$

Note that the allowed jumps in the Riemann tensor (2.50) lead to $n^\sigma [R_{\sigma\alpha\lambda\rho}] h^\alpha_\gamma h^\lambda_\beta h^\rho_\xi = 0$ and therefore the term $\mu^{\alpha\rho} n^\sigma R^\Sigma_{\sigma\alpha\lambda\rho} h^\lambda_\beta$ in the last formula can be written simply as $\mu^{\alpha\rho} n^\sigma R_{\sigma\alpha\lambda\rho} h^\lambda_\beta$.

## 11.6 Energy-momentum conservation

The divergence of the lefthand side of the field equations (11.8) vanishes identically due to the Ricci and Bianchi identities, and therefore, the conservation equation for the energy-momentum tensor $\nabla_\mu T^{\mu\nu} = 0$ follows. In our situation, however, we are dealing with tensor distributions, and with (11.8) considered in a distributional sense. The question arises if whether or not the energy-momentum tensor distribution (11.37) is covariantly conserved. We know that the Bianchi and Ricci identities hold for distributions (see Appendices), hence it is expected that the divergence of the $\underline{T}_{\mu\nu}$ vanishes when distributions are considered. In this section we prove that this is the case, when taking into account the fundamental field equations (11.91) and (11.92). The following calculation can be alternatively seen, therefore, as an independent derivation of (11.91) and (11.92) —from the covariant conservation of $\underline{T}_{\mu\nu}$.

From (11.35) and (2.27) we directly get

$$\nabla^\alpha \underline{T}_{\alpha\beta} = n^\alpha [T_{\alpha\beta}]\delta^\Sigma + \nabla^\alpha(\widetilde{T}_{\alpha\beta}\delta^\Sigma) + \nabla^\alpha \underline{t}_{\alpha\beta}. \tag{11.94}$$

Let us first compute the middle term on the righthand side. From the orthogonal decomposition (11.36)

$$\nabla^\alpha(\widetilde{T}_{\alpha\beta}\delta^\Sigma) = \nabla^\alpha\left(\{\tau_\beta + \tau n_\beta\}\, n_\alpha \delta^\Sigma\right) + \nabla^\alpha\left(\{\tau_{\alpha\beta} + \tau_\alpha n_\beta\}\,\delta^\Sigma\right)$$

and using the general formula (A.21) the second summand can be expanded to get

$$\nabla^\alpha(\widetilde{T}_{\alpha\beta}\delta^\Sigma) = \nabla^\alpha\left(\{\tau_\beta + \tau n_\beta\}\, n_\alpha \delta^\Sigma\right) + \left(\overline{\nabla}^\alpha \tau_{\alpha\beta} - \tau_{\alpha\rho}K^{\alpha\rho}n_\beta + \tau^\alpha K_{\alpha\beta} + n_\beta \overline{\nabla}^\alpha \tau_\alpha\right)\delta^\Sigma$$

so that with the help of (11.86) we get

$$\begin{aligned}\nabla^\alpha(\widetilde{T}_{\alpha\beta}\delta^\Sigma) &= \nabla^\alpha\left(\{\tau_\beta + \tau n_\beta\}\, n_\alpha \delta^\Sigma\right)\\ &+ \left(\overline{\nabla}^\alpha \tau_{\alpha\beta} - \tau_{\alpha\rho}K^{\alpha\rho}n_\beta - K_{\alpha\beta}\overline{\nabla}^\rho \mu_{\rho\alpha} - n_\beta \overline{\nabla}^\alpha\overline{\nabla}^\rho \mu_{\alpha\rho}\right)\delta^\Sigma.\end{aligned} \tag{11.95}$$

With respect to the last term in (11.94), on using definitions (11.43) and (11.44) we can write for any test vector field $Y^\beta$ and using the Ricci identity

$$\begin{aligned}\left\langle \nabla^\alpha \underline{t}_{\alpha\beta}, Y^\beta\right\rangle = -\left\langle \underline{t}_{\alpha\beta}, \nabla^\alpha Y^\beta\right\rangle &= \int_\Sigma \mu_{\alpha\beta} n^\rho \nabla_\rho \nabla^\alpha Y^\beta dv\\ &= \int_\Sigma \left(\mu_{\alpha\beta} n^\rho \left\{\nabla^\alpha \nabla_\rho Y^\beta + R^\beta{}_{\sigma\rho}{}^\alpha Y^\sigma\right\}\right) dv\\ &= \int_\Sigma \mu_{\alpha\beta} n^\rho \nabla^\alpha \nabla_\rho Y^\beta dv - \left\langle n^\rho \mu^{\alpha\sigma} R^\Sigma_{\rho\alpha\beta\sigma}\delta^\Sigma, Y^\beta\right\rangle.\end{aligned}$$

The first integral here can be expanded as

$$\begin{aligned}\int_\Sigma \mu_{\alpha\beta} n^\rho \nabla^\alpha \nabla_\rho Y^\beta dv &= \int_\Sigma \mu_{\alpha\beta}\left\{\nabla^\alpha(n^\rho \nabla_\rho Y^\beta) - K^{\alpha\rho}\nabla_\rho Y^\beta\right\} dv\\ = \int_\Sigma n^\rho \nabla_\rho Y^\beta \left(\mu_{\alpha\sigma}K^{\alpha\sigma}n_\beta - \overline{\nabla}^\alpha \mu_{\alpha\beta}\right) dv &- \int_\Sigma \mu_{\alpha\beta}K^{\alpha\rho}\left(\overline{\nabla}_\rho \overline{Y}^\beta + (n_\sigma Y^\sigma)K_\rho{}^\beta\right) dv\\ = \int_\Sigma (\tau n_\beta + \tau_\beta) n^\rho \nabla_\rho Y^\beta dv &+ \int_\Sigma Y^\beta\left(\overline{\nabla}_\rho(\mu_{\alpha\beta}K^{\alpha\rho}) - n_\beta \mu_{\alpha\sigma}K^{\alpha\rho}K_\rho{}^\sigma\right) dv\\ = -\left\langle \nabla^\alpha\left(\{\tau_\beta + \tau n_\beta\}\, n_\alpha\delta^\Sigma\right), Y^\beta\right\rangle &+ \left\langle \left(\overline{\nabla}_\rho(\mu_{\alpha\beta}K^{\alpha\rho}) - n_\beta \mu_{\alpha\sigma}K^{\alpha\rho}K_\rho{}^\sigma\right)\delta^\Sigma, Y^\beta\right\rangle\end{aligned}$$

so that we arrive at

$$\nabla^\alpha \underline{t}_{\alpha\beta} = -\nabla^\alpha\left(\{\tau_\beta + \tau n_\beta\}\, n_\alpha\delta^\Sigma\right) + \left(\overline{\nabla}_\rho(\mu_{\alpha\beta}K^{\alpha\rho}) - n_\beta \mu_{\alpha\sigma}K^{\alpha\rho}K_\rho{}^\sigma - n^\rho \mu^{\alpha\sigma}R^\Sigma_{\rho\alpha\beta\sigma}\right)\delta^\Sigma. \tag{11.96}$$

Adding up (11.95) and (11.96) to (11.94) we finally obtain

$$\begin{aligned}\nabla^\alpha \underline{T}_{\alpha\beta} &= \left\{ n^\alpha [T_{\alpha\beta}] + \overline{\nabla}^\alpha \tau_{\alpha\beta} - \tau_{\alpha\rho} K^{\alpha\rho} n_\beta + \overline{\nabla}_\rho (\mu_{\alpha\beta} K^{\alpha\rho}) \right. \\ &\left. -n_\beta \mu_{\alpha\sigma} K^{\alpha\rho} K_\rho{}^\sigma - n^\rho \mu^{\alpha\sigma} R^\Sigma_{\rho\alpha\beta\sigma} - K_{\alpha\beta} \overline{\nabla}^\rho \mu_{\rho\alpha} - n_\beta \overline{\nabla}^\alpha \overline{\nabla}^\rho \mu_{\alpha\rho} \right\} \delta^\Sigma .\end{aligned}$$

The fundamental equations (11.92) and (11.93) prove the vanishing of this expression leading to

$$\nabla^\alpha \underline{T}_{\alpha\beta} = 0$$

as claimed. As remarked in [99, 100], this calculation shows that the double-layer energy-momentum distribution $\underline{t}_{\alpha\beta}$ is essential to keep energy-momentum conservation. Without the double-layer contribution the total energy-momentum tensor distribution $\underline{T}_{\alpha\beta}$ would not be covariantly conserved.

## 11.7 Matching hypersurfaces, thin shells and double layers

Once we have discussed the junction in the case of gravity theories with quadratic terms, and have obtained the corresponding field equations on $\Sigma$, we are in disposition to analyze their consequences. Before entering into this discussion, it is convenient to remark the following important result.

**Result 1** *If there is no double layer (that is $\mu_{\alpha\beta} = 0$), then there can be neither external flux momentum $\tau_\alpha$ nor external pressure/tension $\tau$.*

This follows directly from expressions (11.86) and (11.87). In other words, there exist non-vanishing external flux momentum and/or external pressure/tension *only if* there is a double layer.

Thus, there are three levels of junction depending on whether or not thin shells and/or double layers are allowed. We will term them as:

- *Proper matching*: this is the case where the matching hypersurface $\Sigma$ does not support any distributional matter content, describing simply an interface with jumps in the energy-momentum tensor, so that there are neither thin shells nor double layers. This situation models, for instance, the gravitational field of stars (non-empty interior) with a vacuum exterior. Or the case of vacuoles in cosmological surroundings.

- *Thin shells, but no double layer*: This is an idealized situation where an enormous quantity of matter is concentrated on a very thin region mathematically described by $\Sigma$ but no double layer is permitted to exist. Thus, delta-type terms proportional to $\delta^{\Sigma}$ are allowed, and the expression (11.38) provides the energy-momentum tensor of the thin shell. However, from Result 1 the other possible quantities (11.39) and (11.40) accompanying $\delta^{\Sigma}$ vanish identically. This situation is analogous to that in GR where only (11.38) appears. The main difference with a generic quadratic gravity arises in the explicit expression for (11.38), as the field equations turn out to adopt the same form.

- *Double layers*: this is the general case with no further assumptions, which describes a large concentration of matter on $\Sigma$, as in the previous case, but accompanied with a brusque jump in the curvature of the spacetime. Still, there are several sub-possibilities depending on the vanishing or not of any of (11.38), (11.39) or (11.40). There is also an extreme possibility, that we term a *pure double layer*, where the thin shell is not present but the double layer is: this is the case with all (11.38), (11.39) and (11.40) vanishing but a non-vanishing (11.43). Nothing like any of these different possibilities can be described in GR.

We classify the junction condition for these different cases in turn.

## Thin shells without double layer

From (11.43) follows that the strength of the double layer $\mu_{\alpha\beta}$ must be set to zero, and thus from (11.44) we have

$$(\kappa_1 + \kappa_2)[R]h_{\alpha\beta} + 2\kappa_2[G_{\alpha\beta}] = 0 \qquad \Longrightarrow \quad (\kappa_2 + n\kappa_1)[R] = 0, \tag{11.97}$$

which implies that $\tau$ and $\tau_\alpha$ both vanish (see Result 1). Hence, only the tangential part of the distributional energy momentum tensor on $\Sigma$ survives, given explicitly by (11.83). Its trace, upon using (11.64), reads

$$\kappa\tau^{\alpha}_{\alpha} = (n\kappa_1 + \kappa_2)b - K^{\alpha\beta}\mu_{\alpha\beta} = (n\kappa_1 + \kappa_2)b. \tag{11.98}$$

The equations (11.91) and (11.92) in this case read

$$n^{\alpha}h^{\rho}_{\beta}[T_{\alpha\rho}] = -\overline{\nabla}^{\alpha}\tau_{\alpha\beta}, \qquad n^{\alpha}n^{\beta}[T_{\alpha\beta}] = \tau_{\alpha\beta}K^{\alpha\beta}. \tag{11.99}$$

Observe that, remarkably, these are identical with the Israel conditions derived in GR.

We have to distinguish whether $\kappa_2 = 0$ or not.

- $\kappa_2 \neq 0$.

If $(n\kappa_1 + \kappa_2) \neq 0$ relations (11.97) imply that $[R] = 0$ and $[G_{\alpha\beta}] = 0$ in full. Direct consequences are $[R_{\alpha\beta}] = [R_{\alpha\beta\mu\nu}] = 0$, and the discontinuities in the derivatives are given by

$$[\nabla_\mu R_{\alpha\beta\lambda\nu}] = (n_\alpha n_\lambda \mathcal{R}_{\beta\nu} - n_\alpha n_\nu \mathcal{R}_{\beta\lambda} - n_\beta n_\lambda \mathcal{R}_{\alpha\nu} + n_\beta n_\nu \mathcal{R}_{\alpha\lambda}) n_\mu, \tag{11.100}$$

for some symmetric tensor $\mathcal{R}_{\alpha\beta}$ tangent to $\Sigma$. From (11.41) we get $b = 2\mathcal{R}^\rho_\rho$ and therefore the energy-momentum tensor (11.38) on $\Sigma$ just reads

$$\kappa\tau_{\alpha\beta} = \kappa_1 b h_{\alpha\beta} + 2\kappa_2 \mathcal{R}_{\alpha\beta}.$$

With regard to the exceptional possibility $n\kappa_1 + \kappa_2 = 0$, equation (11.97) implies in particular that the tensor $B_{\alpha\beta}$ is proportional to the first fundamental form. The explicit relation reads

$$B_{\alpha\beta} = \frac{1}{2n}[R] h_{\alpha\beta},$$

which for the discontinuity of the Riemann tensor produces

$$[R_{\alpha\beta\lambda\mu}] = \frac{[R]}{2n}\left(n_\alpha n_\lambda h_{\beta\mu} - n_\lambda n_\beta h_{\alpha\mu} - n_\mu n_\alpha h_{\beta\lambda} + n_\mu n_\beta h_{\alpha\lambda}\right). \tag{11.101}$$

Taking contractions in this last expression we find the allowed jumps in the Ricci and Einstein tensor

$$[R_{\alpha\beta}] = \frac{[R]}{2}\left(\frac{1}{n} h_{\alpha\beta} + n_\alpha n_\beta\right) \Rightarrow [G_{\alpha\beta}] = \frac{1-n}{2n}[R] h_{\alpha\beta}. \tag{11.102}$$

Note $[R]$ is the only degree of freedom allowed for the discontinuities of the curvature tensors.

The remaining allowed discontinuities of the derivative of the Ricci tensor are encoded in $r_{\alpha\beta} = n^\mu [\nabla_\mu R_{\alpha\beta}]$, so that

$$[\nabla_\mu R_{\alpha\beta}] = r_{\alpha\beta} n_\mu + \frac{1}{2}\left(n_\alpha n_\beta + \frac{1}{n} h_{\alpha\beta}\right)\overline{\nabla}_\mu [R] + \left(\frac{1-n}{2n}\right)[R]\left(n_\alpha K_{\beta\mu} + n_\beta K_{\alpha\mu}\right). \tag{11.103}$$

Recalling that $b = r^\alpha_\alpha = n^\rho [\nabla_\rho R]$ the explicit form of the energy momentum tensor on $\Sigma$ can be obtained from (11.83). Due to (11.98), $\tau_{\alpha\beta}$ is traceless. Nevertheless, the relevance of this exceptional case is probably marginal, as the coupling constants satisfy a dimensionally dependent condition.

- $\kappa_2 = 0$.

We have to assume then that $\kappa_1 \neq 0$, as otherwise all the terms (11.38), (11.39) and (11.40) vanish identically and thus there are no thin shells. Let us also recall that

$a_2$ and $a_3$ are assumed not to vanish simultaneously, as that case was fully analysed in [98, 99, 100], so it would be more precise to label this case as $a_2 = -4a_3$ with $a_1 \neq a_3$.

This case reduces to the condition $[R] = 0$ (see (11.97)). All the remaining jumps on the curvature tensor and its derivatives are allowed. The energy-momentum tensor on $\Sigma$ is just given by

$$\kappa \tau_{\alpha\beta} = \kappa_1 b h_{\alpha\beta}, \tag{11.104}$$

with $b = n^\alpha [\nabla_\alpha R]$, and therefore the thin shell $\Sigma$ only contains, at most, a "cosmological constant"-type of matter content.

## Proper matching hypersurface

In addition to the requirement imposed in the previous case of thin shells, we demand now that the full $\tilde{T}_{\alpha\beta}$ vanishes. Thus we have to add $\tau_{\alpha\beta} = 0$ to the conditions discussed in the previous Subsection 11.7. In general, from (11.99) we have

$$n^\alpha [T_{\alpha\beta}] = 0 \tag{11.105}$$

which adopt exactly the same form as in GR and we call the generalized Israel conditions. They imply the continuity of the normal components of the energy-momentum tensor across $\Sigma$.

Again, we have to distinguish two cases depending on whether $\kappa_2$ vanishes or not.

- $\kappa_2 \neq 0$.

  If $(n\kappa_1 + \kappa_2) \neq 0$, we already know from the previous section that $[R] = 0$ and $[G_{\alpha\beta}] = 0$. The trace relation (11.98) provides $b = 0$ and moreover $\tau_{\alpha\beta} = 0$ implies, via (11.83), $\mathcal{R}_{\alpha\beta} = 0$. Plugging this information into (11.100) it follows that the derivatives of the curvature tensors do not present discontinuities.

  **Result 2** *In the generic case with $4a_3 + a_2 \neq 0$ and $4a_3 + (1+n)a_2 + 4na_1 \neq 0$, the full set of matching conditions amount to those of GR (agreement of the first and second fundamental forms on $\Sigma$) plus the agreement of the Ricci tensor and its first derivative on $\Sigma$:*

  $$[R_{\alpha\beta}] = 0, \qquad [\nabla_\rho R_{\alpha\beta}] = 0. \tag{11.106}$$

  This actually implies that the full Riemann tensor and its first derivatives have no jumps across $\Sigma$:

  $$[R_{\alpha\beta\lambda\mu}] = 0, \qquad [\nabla_\rho R_{\alpha\beta\lambda\mu}] = 0.$$

With regard to the exceptional possibility $\kappa_2+n\kappa_1=0$, the curvature tensors satisfy (11.101) and (11.102). Now $\tau_{\alpha\beta}=0$ provides

$$\mathcal{R}_{\alpha\beta}=\frac{1}{2n}\left((n-1)[R]K_{\alpha\beta}+bh_{\alpha\beta}\right),$$

and thus $r_{\alpha\beta}=n^\rho[\nabla_\rho R_{\alpha\beta}]$ gets determined in terms of $[R]$ and $b$, so that (11.103) for $[\nabla_\mu R_{\alpha\beta}]$ reads

$$\begin{aligned}[\nabla_\mu R_{\alpha\beta}] &= \left(\frac{1}{2n}\left((n-1)[R]K_{\alpha\beta}+bh_{\alpha\beta}\right)-2n_{(\alpha}\overline{\nabla}_{\beta)}[R]+\left(\frac{b}{2}+\frac{1-n}{2n}[R]K^\rho_\rho\right)n_\alpha n_\beta\right)n_\mu\\ &+ \frac{1}{2}\left(n_\alpha n_\beta+\frac{1}{n}h_{\alpha\beta}\right)\overline{\nabla}_\mu[R]+\left(\frac{1-n}{2n}\right)[R]\left(n_\alpha K_{\beta\mu}+n_\beta K_{\alpha\mu}\right).\end{aligned}$$

Hence, the entire set of discontinuities of the Riemann tensor and its first derivative can be written just in terms of $[R]$ and $b=n^\rho[\nabla_\rho R]$, which remain as two free degrees of freedom. As mentioned before, this case is probably irrelevant due to its defining condition depending on the dimension $n$.

- $\kappa_2=0$ but $\kappa_1\neq 0$.

  From the results from the previous section we know that $[R]=0$ and the energy momentum on $\Sigma$ is given by (11.104). Thus, for a proper matching we find $b=0$. The discontinuity in the derivative is

$$\begin{aligned}[\nabla_\mu R_{\alpha\beta}] &= n_\mu\left([R_{\rho\nu}]K^{\rho\nu}n_\alpha n_\beta-2\overline{\nabla}^\rho[R_{\rho(\beta}]n_{\alpha)}+\mathcal{R}_{\alpha\beta}\right)\\ &+ \overline{\nabla}_\mu[R_{\alpha\beta}]-2K^\rho_\mu[R_{\rho(\alpha}]n_{\beta)},\end{aligned}$$

  where also $\mathcal{R}^\rho_\rho=-K^{\alpha\beta}[R_{\alpha\beta}]$.

- $\kappa_1=\kappa_2=0$.

  Or equivalently $a_1=a_3=-a_2/4$. In this case all the terms (11.38), (11.39) and (11.40) and (11.43) vanish identically and thus there are no further restrictions other than $[K_{ab}]=0$. The junction conditions are just the same as in GR. This is the case where the quadratic part of the Lagrangian (11.7) is the Gauss-Bonnet term [76].

## The double layer fauna; pure double layers

The generic occurrence in quadratic gravity, as shown above, is that any thin shell comes accompanied by a double layer, which in turn generically implies the existence of non-zero external pressure/tension and external flux momentum. However, there are several special possibilities in which one of these quantities, or all, disappear. This gives rise to a fauna of different kinds of double layers. There is also the possibility that the double layer term (11.43) is non-zero while the remaining distributional part in the energy-momentum tensor, that is $\tilde{T}_{\alpha\beta}\delta^{\Sigma}$, vanishes. In other words, a double layer without a classical thin shell. We call such a case a *pure double layer*. In the rest of this section we explore this novel possibility.

For pure double layers, the vanishing of the external pressure $\tau$ plus the energy flux $\tau_\alpha$ first imply, by virtue of (11.39) and (11.86)

$$\mu_{\alpha\beta}K^{\alpha\beta} = 0, \quad \overline{\nabla}^{\rho}\mu_{\rho\alpha} = 0. \tag{11.107}$$

In particular, then, the double layer strength is conserved.

The first equation in (11.107) yields

$$(\kappa_1 + \kappa_2)[R]K^{\sigma}_{\sigma} + 2\kappa_2 K^{\rho\sigma}[G_{\rho\sigma}] = 0 \tag{11.108}$$

while the second gives

$$(\kappa_1 + \kappa_2)\overline{\nabla}_{\alpha}[R] + 2\kappa_2\overline{\nabla}^{\rho}[G_{\rho\alpha}] = 0. \tag{11.109}$$

Equation (11.108) combined with the vanishing of the trace of $\tau_{\alpha\beta}$ provides

$$(\kappa_1 n + \kappa_2)b = 0 \tag{11.110}$$

so that, generically — $n\kappa_1 + \kappa_2 \neq 0$ — one has $b = 0$. A first consequence is that the jump in the derivative of the Ricci scalar is now tangent to $\Sigma$ and fully determined by the tangent derivative of $[R]$

$$[\nabla_{\alpha}R] = \overline{\nabla}_{\alpha}[R]. \tag{11.111}$$

The vanishing of $\tau_{\alpha\beta}$, using (11.38), is now equivalent to

$$\kappa_2\mathcal{R}_{\alpha\beta} = (\kappa_1 + \kappa_2)\frac{[R]}{2}K_{\alpha\beta}. \tag{11.112}$$

The expression for the discontinuity of the normal derivative of the Ricci tensor has to be studied depending on $\kappa_2$ vanishing or not.

- $\kappa_2 \neq 0$

  The relations above allow us to write the discontinuity of the normal derivative of the Ricci tensor as

$$r_{\alpha\beta} = \frac{1}{2}\left(1 + \frac{\kappa_1}{\kappa_2}\right)([R]K_{\alpha\beta} + n_{\beta}\overline{\nabla}_{\alpha}[R] + n_{\alpha}\overline{\nabla}_{\beta}[R] - K[R]n_{\alpha}n_{\beta}),$$

  whereas the tangent part of the derivative keeps its original form given in (11.65).

- $\kappa_2 = 0$ (and $\kappa_1 \neq 0$).

  Equations (11.109) and (11.112) read

$$\overline{\nabla}_\alpha [R] = 0, \quad [R] K_{\alpha\beta} = 0, \tag{11.113}$$

  and (11.108) is automatically satisfied. Thus, (11.111) implies $[\nabla_\alpha R] = 0$. Observe that since $\kappa_2 = 0$, (11.44) establishes that the strength of the double layer is proportional to $[R]$. Hence, in order to have a nonzero $\mu_{\alpha\beta}$, $[R]$ cannot vanish. Then $K_{\alpha\beta} = 0$ necessarily, and the allowed jumps are encoded in $[R_{\alpha\beta}]$ and $r_{\alpha\beta}$.

For completeness, we provide finally the formulas for the exceptional case $n\kappa_1 + \kappa_2 = 0$ —discarding $\kappa_1 = \kappa_2 = 0$ for which the double layer simply disappears. The equations $\tau = 0$, $\tau_\alpha = 0$ and $\tau_{\alpha\beta} = 0$ result, respectively, in

$$\begin{aligned}
(1-n)[R] K^\beta_\beta - 2n K^{\alpha\beta} [G_{\alpha\beta}] &= 0, \\
(1-n) \overline{\nabla}_\alpha [R] - 2n \overline{\nabla}^\rho [G_{\rho\alpha}] &= 0, \\
(1-n)[R] K_{\alpha\beta} - b h_{\alpha\beta} + 2n \mathcal{R}_{\alpha\beta} &= 0.
\end{aligned}$$

While the third equation provides $\mathcal{R}_{\alpha\beta}$, the first two constitute constraints on the allowed jumps of the Ricci tensor that should be analysed in each particular situation. In all cases, the allowed discontinuity in the derivative of the Ricci tensor can be written as

$$\begin{aligned}
r_{\alpha\beta} = & -\frac{1}{2n} \left( (1-n)[R] K_{\alpha\beta} - b h_{\alpha\beta} \right) - \frac{1-n}{2n} \left( n_\beta \overline{\nabla}_\alpha [R] + n_\alpha \overline{\nabla}_\beta [R] \right) \\
& + \frac{1}{2} \left( b + \frac{1-n}{n} [R] K^\rho_\rho \right) n_\alpha n_\beta .
\end{aligned}$$

Observe that now the strength of the double layer is traceless, $\mu^\rho_\rho = 0$ (see e.g.(11.45)).

## 11.8 Consequences

The proper matching conditions in GR are the agreement of the first and second fundamental forms on $\Sigma$. Therefore, any matching hypersurface in GR satisfies (11.15), and the allowed jumps in the energy-momentum tensor are equivalent to non-vanishing discontinuities of the Ricci (and Riemann) tensor. Thus, in GR properly matched space-times will generally have $[R_{\alpha\beta}] \neq 0$.

This simple known fact implies that any GR-solution containing a proper matching hypersurface will contain a double layer and/or a thin shell at the matching hypersurface if the true theory is quadratic. At least two relevant consequences follow from this fact: (i) generically, matched solutions in GR are no longer solutions in quadratic theories;

and (ii) if any quantum regimes require, excite or switch on quadratic terms in the Lagrangian density, then GR solutions modelling two regions with different matter contents will develop thin shells and double layers on their interfaces. Let us elaborate.

Consider, for instance, the case of a perfect fluid matched to a vacuum in GR. As is well known, the GR matching hypersurface is determined by the condition that

$$p^{GR}|_{\Sigma} = 0$$

where $p^{GR}$ is the isotropic pressure of the fluid in GR. It follows that the Ricci tensor has a discontinuity of the following type

$$[G_{\alpha\beta}] = \kappa\varrho^{GR} u_\alpha u_\beta\big|_{\Sigma}, \qquad [R_{\alpha\beta}] = \kappa\varrho^{GR}\left(u_\alpha u_\beta + \frac{1}{n-1}g_{\alpha\beta}\right)\Big|_{\Sigma}$$

$u^\alpha$ being the unit velocity vector of the perfect fluid. Therefore, using (11.83-11.85) and (11.43) we see that the very same space-time has, in any quadratic theory of gravity, an energy-momentum tensor distribution with all type of thin-shell and double-layer terms.

Imagine the situation of a collapsing perfect fluid (to form a black hole, say) with vacuum exterior. Then one can use any of the known solutions in GR to describe this situation —the reader may have in mind, for instance, the Oppenheimer-Snyder model. The GR solution describes this process accurately in the initial and intermediate stages, when the curvature of the space-time is moderate and the values of $a_1 R^2$, $a_2 R_{\alpha\beta}R^{\alpha\beta}$ and $a_3 R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ for instance, or other similar quantities, are small enough to render any quadratic terms in the Lagrangian totally negligible. However, as the collapse proceeds and one approaches the black hole regions —and later the classical singularity–, regimes with very high curvatures are reached. Then, the quadratic terms coming from any quantum corrections (be they from string theory counter-terms, or any other) to the Einstein-Hilbert Lagrangian start to be important, and actually to dominate, the curvature being enormous. In this regime, the original matching hypersurface becomes actually a thin double layer.

Of course, the description of a global space-time via a matching is an approximation, and also the use of tensor distributions is also just another approximation to a real situation where a gigantic quantity of matter can be concentrated around a very thin region of the space-time. Nevertheless, both approximations are satisfactory in the sense that they are believed to actually mimic a realistic situation where the layer is thick and the jumps in the energy variables are extremely big, but finite. If this is the case, then the above reasoning seems to imply that, *if quadratic theories of gravity are correct*, at least in some extreme regimes, then a huge concentration of matter will develop around the interface of the interior and the exterior of the collapsing star. And this huge concentration will generically manifest as a shell with double-layer properties.

# 12

# Conclusions

## Rotating stars

We have completed a study of the implicit assumptions and/or arguments in the construction of Hartle's model. The starting point was the use of explicit global coordinates in which the perturbations are assumed to be at least $C^1$. This leads also to the argument that the function that drives the first order perturbation depends only on the radial coordinate and the second order admits a finite expansion in Legendre polynomials. Our results can be enumerated as follows:

1. We have studied the use of global coordinates in which the metric is at least $C^1$: In the original work [57], this assumption substituted the matching procedure based on geometrical methods, not available at the time. We have used the perturbed matching theory to second order [79], separating the problem into the interior/exterior spacetimes and matching them in two stages: in a first step we have matched perturbatively to second order two stationary and axisymmetric spacetimes in purely geometrical terms. In a second stage we have included the explicit assumptions and the physics of the model. At this point we also assume the angular structure of the perturbations argued in [57]. The description of the whole perturbed configuration up to second order including interior, exterior and matching is collected in Theorem 5.

2. We have concluded that the assumption of continuity of the metric functions, apart from being inaccurate, leads to wrong results when $E(a) \neq 0$. In practical terms, apart from putting Hartle's model on firm grounds, we have found that the second order function $\tilde{m}_0$ presents a discontinuity in the matching hypersurface $\Sigma_0$, determined by $r = a$, $a$ satisfying $P(a) = 0$ with $P$ the background pressure, when the energy density of the background configuration presents a jump there, i.e. $E(a) \neq 0$,

where $E$ is the background energy density. Since $\tilde{m}_0$ encodes the information about the change in mass due to rotation, the computation of the mass has to be amended whenever the equation of state allows $E(a) \neq 0$.

3. We have performed a deep comparison between this consistent framework and Hartle's results and methods. In particular we discuss how and when the matching determines the deformation of the star.

4. We have verified that the amended change in mass has a correct Newtonian limit. To this aim, we have checked that it agrees with the change in mass calculated following the recipe in [23], where the amending term appears implicitly. For completeness, we formulated the perturbed Newtonian matching conditions for the problem of a fluid ball rotating in equilibrium.

5. We have studied the structure of the angular behaviour of the perturbations. We have proven how the field equations plus regularity conditions at the origin/infinity and the boundary conditions provided by the matching procedure, yield the angular structure of the perturbations argued in [57].

Chapter 8 concludes with the formulation of Theorem 7, that tells us how to construct the global stationary axisymmetric rotating model up to second order by taking just the explicit assumptions in Hartle's classical model [57], i.e. an interior and exterior stationary, axially and equatorially symmetric spacetimes, with a perfect fluid with a barotropic equation of state that rotates rigidly with no convective motions as the interior matter content, and an asymptotically flat vacuum exterior region.

In the following, we detail some of the work that would complete the results presented in this thesis regarding the perturbational approach to rotating stars.

1. The explicit assumption of equatorial symmetry is still used to get rid of the $l = 1$ sector of the perturbations to second order. This part must be studied separately and has not been covered in this thesis.

2. The purely geometrical perturbed matching can be used to generalize Hartle's model to other contexts such as other theories of gravity for which Hartle's model has been generalised already in the literature, to find corresponding corrections to the mass (see the Introduction).

3. A multilayer interior is needed in order to construct a realistic models. A direct generalization of the results in this thesis provides the theoretical tools to let the energy density jump in the transition from one layer to another. We started to

develop a numerical code to check for the change in mass in stars with a core showing a discontinuity in the energy density with the enveloping layer, for instance a core governed by a linear equation of state. Some work regarding multilayer interiors in a perturbative setting has already been done in [45].

4. The mass of rotating stars is central to the study of stability. It would be interesting to see how the amended change in mass contributes to the stability limits.

## On quadratic gravity and double layers

We have applied distribution theory to study the junction conditions in theories of quadratic gravity. The two main results arising from this work are

1. We have found the junction conditions and generalized Israel equations for sources localized in a hypersurface $\Sigma$.

2. The junction conditions imply the existence of double layers in the matching hypersurface, in general.

3. In the abscence of double layers, the generalized Israel equations are identical to the Israel equations derived in GR. Note however that the junction conditions differ, in general, from those in GR.

In the view of these conclusions, the two open lines of work follow

1. We have formulated the matching conditions for quadratic theories of gravity, but due to the intrincate form of their field equations, we have not constructed any explicit model. In order to understand the role of the double layers, finding a physically reasonable explicit model where they show up would be of great interest.

2. The Gauss-Bonnet theory escapes our analysis of quadratic theories of gravity and in fact, it should escape any other work up to date. As seen from the naive study of the $\delta^2$ cancellations, GB seems to get rid of these type of terms without any necessity of restricting the jump in the second fundamental form. A study based on structures more general than standard distributions is necessary to properly formulate the junction conditions for Gauss Bonnet theories, and prove that the outcomes of the cancellation of the $\delta^2$ terms argued in the literature are indeed correct.

# Appendices

A

# Useful formulas

In this appendix we include a collection of formulas that are useful in order to carry out the calculations of Chapter 11. We divide it in four sections. The first one, A.1, is devoted to introduce the intrinsic connection and curvature tensors of $\Sigma$ (this is timelike everywhere). The ambient curvature at points of the embdedded $\Sigma$ and the intrinsic curvature of $\Sigma$ are related in terms of the well known Gauss Codazzi equations. In Section A.2, we address the formulas needed to compute the jump discontinuities of tensors with well defined limits at points of the embdedded $\Sigma$. We work out the case of jumps of product of tensors and jumps of derivatives of tensors. In Section A.3 we give a general formula for the derivative of tensor distributions proportional to the Dirac delta. We end the appendix showing in A.4 that the Ricci identity holds for tensor distributions associated to tensor fields continuous at $\Sigma$. Furthermore, we also discuss the Ricci identity for the $\delta^{\Sigma}$.

## A.1 Concerning $\Sigma$ and its objects

Consider a hypersurface $(\Sigma, h_{ab})$ embedded in a n+1-dimensional spacetime $(\mathcal{V}, g_{\alpha\beta})$. We will later use this construction for the + and $-$ sides. Using the dual bases $\{n^\mu, e^\mu_a\}$ and $\{n_\mu, \omega^a_\mu\}$ introduced in Chapter 2, we have

$$e^\rho_a \nabla_\rho e^\alpha_b = -K_{ab} n^\alpha + \overline{\Gamma}^c_{ab} e^\alpha_c, \quad \text{(A.1)}$$

$$e^\rho_a \nabla_\rho \omega^b_\alpha = -K^b_a n_\alpha - \overline{\Gamma}^b_{ac} \omega^c_\alpha, \quad \text{(A.2)}$$

$$e^\rho_a \nabla_\rho n_\alpha = K_{ab} \omega^b_\alpha \quad \text{(A.3)}$$

where $K_{ab}$ is the second fundamental form introduced in (2.57) and

$$\overline{\Gamma}^c_{ab} := \omega^c_\alpha e^\rho_a \nabla_\rho e^\alpha_b$$

represent the Christoffel symbols of the Levi-Civita connection associated to the first fundamental form $h_{ab}$ of $\Sigma$. In general —unless the jump of the second fundamental form vanishes— there will be two versions, one + and one − of all these equations except for the last one, the connection, which is uniquely defined given that the first fundamental form agrees on both sides (2.2) or (2.4).

The covariant derivative defined by $\overline{\Gamma}$ is denoted by $\overline{\nabla}$. The relationship between $\nabla$ and $\overline{\nabla}$ on $\Sigma$ is ruled by the following formula (given here for a $(1,1)$-tensor field $S^{\beta}_{\alpha}$, but generalizable in the obvious way to arbitrary ranks [82])

$$\omega^a_\beta e^\alpha_b e^\rho_c \nabla_\rho S^\beta_\alpha = \overline{\nabla}_c \overline{S}^a_b + (e^\beta_b S^\rho_\beta n_\rho) K^a_c + (\omega^a_\alpha S^\alpha_\rho n^\rho) K_{cb} \tag{A.4}$$

where, for any tensor field $S$, we denote by $\overline{S}$ its projection to $\Sigma$:

$$\overline{S}^a_b := \omega^a_\alpha e^\beta_b S^\alpha_\beta. \tag{A.5}$$

The equivalent "space-time" version of (A.4) is

$$h^\gamma_\beta h^\alpha_\delta h^\rho_\sigma \nabla_\rho S^\beta_\alpha = \overline{\nabla}_\sigma \overline{S}^\gamma_\delta + (h^\beta_\delta S^\rho_\beta n_\rho) K^\gamma_\sigma + (h^\gamma_\alpha S^\alpha_\rho n^\rho) K_{\sigma\delta}, \tag{A.6}$$

where $\overline{S}^\gamma_\delta$ is the spacetime version of $\overline{S}^a_b$, i.e. $\overline{S}^\gamma_\delta := \omega^b_\delta e^\gamma_a \overline{S}^a_b = h^\gamma_\alpha h^\beta_\delta S^\alpha_\beta$.

Denoting by $\overline{R}^d_{abc}$ the Riemann tensor of $(\Sigma, h_{ab})$, the classical Gauss equation reads (2.66)

$$\omega^d_\alpha R^\alpha_{\beta\gamma\delta} e^\beta_a e^\gamma_b e^\delta_c = \overline{R}^d_{abc} - K_{ac} K^d_b + K_{ab} K^d_c, \tag{A.7}$$

whose contractions are

$$e^\alpha_a e^\gamma_c R_{\alpha\gamma} - n^\alpha n^\gamma R_{\alpha\beta\gamma\delta} e^\beta_a e^\delta_c = \overline{R}_{ac} - K^d_d K_{ac} + K_{ab} K^b_c, \tag{A.8}$$

$$R - 2n^\alpha n^\beta R_{\alpha\beta} = \overline{R} - (K^d_d)^2 + K_{ab} K^{ab} \tag{A.9}$$

where $\overline{R}_{ac}$ and $\overline{R}$ denote the Ricci tensor and scalar curvature of $(\Sigma, h_{ab})$.

Similarly, the classical Codazzi equation reads (2.67)

$$n_\mu R^\mu_{\alpha\beta\gamma} e^\alpha_a e^\beta_b e^\gamma_c = \overline{\nabla}_c K_{ba} - \overline{\nabla}_b K_{ca} \tag{A.10}$$

with contraction

$$n^\alpha R_{\alpha\gamma} e^\gamma_b = \overline{\nabla}_a K^a_b - \overline{\nabla}_b K^d_d. \tag{A.11}$$

As mentioned before, generically there will be two versions of each of the previous equations, one for each ± side of the embdedded $\Sigma$ if this is a matching hypersurface. Thus, for instance (and using space-time notation), (A.9) and (A.11) must have the two versions:

$$R^\pm - 2R^\pm_{\mu\nu} n^\mu n^\nu = \overline{R} - (K^{\pm\rho}{}_\rho)^2 + K^\pm_{\mu\nu} K^{\pm\mu\nu}, \tag{A.12}$$

$$n^\mu R^\pm_{\mu\rho} h^\rho{}_\nu = \overline{\nabla}^\mu K^\pm_{\mu\nu} - \overline{\nabla}_\nu K^{\pm\rho}{}_\rho. \tag{A.13}$$

On the other hand, equation (A.4) at points of the *matching* hypersurface $(\Sigma, h_{ab})$ of the already glued spacetime $\mathcal{V} = \mathcal{V}^+ \cup \mathcal{V}^-$ reads

$$h^\gamma_\beta h^\alpha_\delta h^\rho_\sigma \nabla_\rho S^\beta_\alpha|_\Sigma = \overline{\nabla}_\sigma \overline{S}^\gamma_\delta + (h^\beta_\delta S^\rho_\beta n_\rho) K^{\Sigma\gamma}{}_\sigma + (h^\gamma_\alpha S^\alpha_\rho n^\rho) K^\Sigma_{\sigma\delta}, \tag{A.14}$$

where here we use $h\nabla|_\Sigma$ just to make explicit that $h\nabla$ is being restricted to points at the matching hypersurface $\Sigma$ using the connection given by (2.30), whose restriction to $\Sigma$ is (2.31). Note that whenever both second fundamental forms coincide $[K_{\alpha\beta}] = 0$ on a matching hypersurface $\Sigma$, equation (A.14) reads just as (A.6).

## A.2 Discontinuities

In the computations we need the discontinuities of objects, such as functions and tensor fields, across $\Sigma$. This also implies that we need to control such discontinuities for the *derivatives* of those objects, and for their products. Here we provide the general rules.

Let $A$ and $B$ be any two functions possibly discontinuous across $\Sigma$. Then

$$[AB] = A^+B^+|_\Sigma - A^-B^-|_\Sigma = A^+B^+|_\Sigma - A^+B^-|_\Sigma + A^+B^-|_\Sigma - A^-B^-|_\Sigma = A^+|_\Sigma[B] + [A]B^-|_\Sigma$$

and an equivalent expression interchanging $A \leftrightarrow B$. Adding these two expressions and using (2.23) we get

$$[AB] = A^\Sigma[B] + [A]B^\Sigma. \tag{A.15}$$

Concerning derivatives, let us start with any function $f$ that may be discontinuous across $\Sigma$. If we compute the tangent derivatives on both sides of $\Sigma$ we obtain

$$e^\mu_a [\partial_\mu f] = [e^\mu_a \partial_\mu f] = e^\mu_a \partial_\mu f^+|_\Sigma - e^\mu_a \partial_\mu f^-|_\Sigma = \partial_a f^+|_\Sigma - \partial_a f^+|_\Sigma = \partial_a [f] = e^\mu_a \partial_\mu [f]$$

and thus, by orthogonal decomposition,

$$[\partial_\nu f] = F n_\nu + \omega^a_\nu e^\mu_a \partial_\mu [f] = F n_\nu + h^\mu_\nu \partial_\mu [f] \tag{A.16}$$

where $F$ is a function defined only on $\Sigma$ that measures the discontinuity of the normal derivatives of $f$ across $\Sigma$:

$$F := n^\nu [\partial_\nu f] \, .$$

Consider now the case of a one-form $t_\mu$, again possibly discontinuous across $\Sigma$. A direct computation using (A.16) and (A.15) produces

$$e^\mu_a [\nabla_\mu t_\alpha] = e^\mu_a \left[\partial_\mu t_\alpha - t_\rho \Gamma^\rho_{\mu\alpha}\right] = e^\mu_a \left(\partial_\mu [t_\alpha] - [t_\rho] \Gamma^{\Sigma\rho}_{\mu\alpha}\right) - t^\Sigma_\rho \left[\Gamma^\rho_{\mu\alpha}\right] e^\mu_a = e^\mu_a \nabla_\mu [t_\alpha] - t^\Sigma_\rho \left[\Gamma^\rho_{\mu\alpha}\right] e^\mu_a.$$

Let us remark that the derivative $\nabla_{\vec{e}}$ is restricted to points on $\Sigma$, so that the connection (2.31) must be used. Therefore

$$[\nabla_\mu t_\alpha] = n_\mu T_\alpha + h^\nu_\mu \nabla_\nu [t_\alpha] - t^\Sigma_\rho \left[\Gamma^\rho_{\nu\alpha}\right] h^\nu_\mu \tag{A.17}$$

where $T_\alpha$ is a one-form defined only on $\Sigma$ giving the discontinuity of the normal derivatives of $t_\alpha$ across $\Sigma$,

$$T_\alpha := n^\mu \left[\nabla_\mu t_\alpha\right],$$

and the tangential derivative $h\nabla$ is restricted to $\Sigma$, although it is not explicitly indicated not to overwhelm the expressions. The righthand side of (A.17) can be further computed. First, due to (A.14)

$$\begin{aligned} h^\nu_\mu \nabla_\nu [t_\alpha] &= \overline{\nabla}_\mu \overline{[t_\alpha]} + n^\rho [t_\rho] K^\Sigma_{\mu\alpha} + n_\alpha n^\rho h^\nu_\mu \nabla_\nu [t_\rho] \\ &= \overline{\nabla}_\mu \overline{[t_\alpha]} + n^\rho [t_\rho] K^\Sigma_{\mu\alpha} + n_\alpha \overline{\nabla}_\mu \left([t_\rho] n^\rho\right) - n_\alpha [t^\rho] K^\Sigma_{\rho\mu} \end{aligned}$$

while, for the last summand in (A.17) we use (2.38) and (2.59)

$$-t^\Sigma_\rho \left[\Gamma^\rho_{\nu\alpha}\right] h^\nu_\mu = t^\Sigma_\rho n^\rho [K_{\mu\alpha}] - n_\alpha t^\Sigma_\rho [K^\rho_\mu].$$

Introducing both results into (A.17) we get

$$\begin{aligned} [\nabla_\mu t_\alpha] &= n_\mu T_\alpha + \overline{\nabla}_\mu \overline{[t_\alpha]} + n^\rho [t_\rho] K^\Sigma_{\mu\alpha} + n_\alpha \left(\overline{\nabla}_\mu \left([t_\rho] n^\rho\right) - [t^\rho] K^\Sigma_{\rho\mu}\right) + t^\Sigma_\rho n^\rho [K_{\mu\alpha}] - n_\alpha t^\Sigma_\rho [K^\rho_\mu] \\ &= n_\mu T_\alpha + \overline{\nabla}_\mu \overline{[t_\alpha]} + n^\rho [t_\rho K_{\mu\alpha}] + n_\alpha \left(\overline{\nabla}_\mu \left([t_\rho] n^\rho\right) - [t_\rho K^\rho_\mu]\right). \end{aligned} \tag{A.18}$$

Observe that when there is no jump of the second fundamental form, $[K_{\alpha\beta}] = 0$ ($\Leftrightarrow [\Gamma^\rho_{\nu\alpha}] = 0$), equations (A.17) and (A.18) read, respectively,

$$[\nabla_\mu t_\alpha] = n_\mu T_\alpha + h^\nu_\mu \nabla_\nu [t_\alpha], \tag{A.19}$$

$$[\nabla_\mu t_\alpha] = n_\mu T_\alpha + \overline{\nabla}_\mu \overline{[t_\alpha]} + n^\rho [t_\rho] K_{\mu\alpha} + n_\alpha \left(\overline{\nabla}_\mu \left([t_\rho] n^\rho\right) - [t^\rho] K_{\rho\mu}\right). \tag{A.20}$$

These formulas can be generalized to arbitrary $(p,q)$-tensor fields $T^p_q$ in an obvious way. In that case, the term replacing $T_\alpha$ is simply a tensor field of the same type (and with the same symmetry and trace properties) as $T^p_q$, defined only on $\Sigma$ and measuring the discontinuities of the normal derivatives of $T^p_q$.

## A.3 Derivatives of tensor distributions proportional to $\delta^\Sigma$

Let us consider tensor distributions of type

$$t_{\alpha_1 \dots \alpha_p} \delta^\Sigma$$

where $t_{\alpha_1\dots\alpha_p}$ is any tensor field defined *at least* on $\Sigma$, but not necessarily off $\Sigma$ (for instance $h_{\mu\nu}$ or $n_\mu$ are not defined outside $\Sigma$). We want to compute the covariant derivative of such tensor distributions. Then we have

$$\begin{aligned}\left\langle \nabla_\lambda \left(t_{\alpha_1\dots\alpha_p}\delta^\Sigma\right), Y^{\lambda\alpha_1\dots\alpha_p}\right\rangle &= -\left\langle t_{\alpha_1\dots\alpha_p}\delta^\Sigma, \nabla_\lambda Y^{\lambda\alpha_1\dots\alpha_p}\right\rangle = -\left\langle \delta^\Sigma, t_{\alpha_1\dots\alpha_p}\nabla_\lambda Y^{\lambda\alpha_1\dots\alpha_p}\right\rangle \\ &= -\int_\Sigma t_{\alpha_1\dots\alpha_p}\nabla_\lambda Y^{\lambda\alpha_1\dots\alpha_p} dv = -\int_\Sigma t_{\alpha_1\dots\alpha_p}(n_\lambda n^\rho + h^\rho_\lambda)\nabla_\rho Y^{\lambda\alpha_1\dots\alpha_p} dv.\end{aligned}$$

The first summand here is

$$-\left\langle t_{\alpha_1\dots\alpha_p} n_\lambda n^\rho \delta^\Sigma, \nabla_\rho Y^{\lambda\alpha_1\dots\alpha_p}\right\rangle = \left\langle \nabla_\rho \left(t_{\alpha_1\dots\alpha_p} n_\lambda n^\rho \delta^\Sigma\right), Y^{\lambda\alpha_1\dots\alpha_p}\right\rangle$$

while the second one has derivatives tangent to $\Sigma$ and thus

$$-\int_\Sigma t_{\alpha_1\dots\alpha_p} h^\rho_\lambda \nabla_\rho Y^{\lambda\alpha_1\dots\alpha_p} dv = -\int_\Sigma h^\rho_\lambda \nabla_\rho (t_{\alpha_1\dots\alpha_p} Y^{\lambda\alpha_1\dots\alpha_p}) dv + \int_\Sigma Y^{\lambda\alpha_1\dots\alpha_p} h^\rho_\lambda \nabla_\rho t_{\alpha_1\dots\alpha_p} dv$$

and using (A.6) for the first integral here

$$\begin{aligned} &= -\int_\Sigma \overline{\nabla}_\lambda (\overline{t_{\alpha_1\dots\alpha_p} Y^{\lambda\alpha_1\dots\alpha_p}}) dv - \int_\Sigma K^{\Sigma\rho}{}_\rho\, n_\lambda t_{\alpha_1\dots\alpha_p} Y^{\lambda\alpha_1\dots\alpha_p} dv + \int_\Sigma Y^{\lambda\alpha_1\dots\alpha_p} h^\rho_\lambda \nabla_\rho t_{\alpha_1\dots\alpha_p} dv \\ &= \left\langle \left(h^\rho_\lambda \nabla_\rho t_{\alpha_1\dots\alpha_p} - K^{\Sigma\rho}{}_\rho\, n_\lambda t_{\alpha_1\dots\alpha_p}\right)\delta^\Sigma, Y^{\lambda\alpha_1\dots\alpha_p}\right\rangle\end{aligned}$$

where we have used that, as $Y^{\lambda\alpha_1\dots\alpha_p}$ has compact support, the first total divergence term integrates to zero. Summing up, we have the following basic formula

$$\nabla_\lambda \left(t_{\alpha_1\dots\alpha_p}\delta^\Sigma\right) = \nabla_\rho \left(t_{\alpha_1\dots\alpha_p} n_\lambda n^\rho \delta^\Sigma\right) + \left(h^\rho_\lambda \nabla_\rho t_{\alpha_1\dots\alpha_p} - K^{\Sigma\rho}{}_\rho\, n_\lambda t_{\alpha_1\dots\alpha_p}\right)\delta^\Sigma. \tag{A.21}$$

In particular, for the second derivative of $\underline{\theta}$ one gets

$$\nabla_\nu \nabla_\mu \underline{\theta} = \nabla_\nu (n_\mu \delta^\Sigma) = \nabla_\rho (n_\mu n_\nu n^\rho \delta^\Sigma) + \left(K_{\mu\nu} - K^{\Sigma\rho}{}_\rho n_\mu n_\nu\right)\delta^\Sigma. \tag{A.22}$$

Let us do a remark here. Formula (2.25), or (2.27), is precisely the formula one would derive by using a naif calculation starting from (2.24), applying Leibniz rule and using (2.20). However, such approach *cannot be used* when the tensor distribution to be differentiated involves non-tensorial distributions, such as $\delta^\Sigma$. For instance, the computation of the second covariant derivative of $\underline{\theta}$ starting from (2.20) with such approach provides

$$\nabla_\nu \nabla_\mu \underline{\theta} \not= \nabla_\nu n_\mu\, \delta^\Sigma + n_\mu \nabla_\nu \delta^\Sigma.$$

Neither term on the righthand side is well defined due the the fact that $n_\mu$ exists only on $\Sigma$ and therefore its derivatives non-tangent to $\Sigma$ are not defined at all. Nevertheless, $\nabla_\nu \nabla_\mu \underline{\theta}$ is certainly well defined as a distribution, and one can see from the formula (A.22), obtained by following strictly the rules of tensor-distribution derivation and multiplication.

## A.4 Ricci and Bianchi identities

The Bianchi identity holds in the distributional sense, for a proof consult [82]:

$$\nabla_\rho \underline{R}_{\alpha\beta\nu\mu} + \nabla_\nu \underline{R}_{\alpha\beta\mu\rho} + \nabla_\mu \underline{R}_{\alpha\beta\rho\nu} = 0. \tag{A.23}$$

Concerning the Ricci identity, let us consider a one-form which may have a discontinuity across $\Sigma$. It can be written as 1-covariant tensor and as a one-form distribution as

$$t_\alpha = t^+_\alpha \theta + t^-_\alpha (1-\theta); \qquad \underline{t}_\alpha = t^+_\alpha \underline{\theta} + t^-_\alpha (\underline{1} - \underline{\theta})$$

To compute the derivatives, we need to take $\underline{t}_\alpha$ as a distribution. Then, from (2.27) we first have

$$\nabla_\mu \underline{t}_\alpha = \nabla_\mu t^+_\alpha \underline{\theta} + \nabla_\mu t^-_\alpha (\underline{1} - \underline{\theta}) + [t_\alpha] n_\mu \delta^\Sigma$$

and applying (2.27) to the first part not proportional to $\delta^\Sigma$ we derive

$$\nabla_\lambda \nabla_\mu \underline{t}_\alpha = \nabla_\lambda \nabla_\mu t^+_\alpha \underline{\theta} + \nabla_\lambda \nabla_\mu t^-_\alpha (\underline{1} - \underline{\theta}) + [\nabla_\mu t_\alpha] n_\lambda \delta^\Sigma + \nabla_\lambda \left( [t_\alpha] n_\mu \delta^\Sigma \right). \tag{A.24}$$

Formula (A.21) gives the last term here

$$\nabla_\lambda \left( [t_\alpha] n_\mu \delta^\Sigma \right) = \nabla_\rho \left( [t_\alpha] n_\mu n_\lambda n^\rho \delta^\Sigma \right) + \left( n_\mu h^\rho_\lambda \nabla_\rho [t_\alpha] + [t_\alpha] K_{\lambda\mu} - K^{\Sigma\rho}{}_\rho \, n_\lambda [t_\alpha] n_\mu \right) \delta^\Sigma.$$

Introducing (A.17) into (A.24) and using this last result we arrive at

$$\begin{aligned}(\nabla_\lambda \nabla_\mu - \nabla_\mu \nabla_\lambda) \underline{t}_\alpha \;\; = \;\; & (\nabla_\lambda \nabla_\mu - \nabla_\mu \nabla_\lambda) t^+_\alpha \, \underline{\theta} + (\nabla_\lambda \nabla_\mu - \nabla_\mu \nabla_\lambda) t^-_\alpha (\underline{1} - \underline{\theta}) \\ & - t^\Sigma_\rho \left( n_\lambda [\Gamma^\rho_{\mu\alpha}] - n_\mu [\Gamma^\rho_{\lambda\alpha}] \right) \delta^\Sigma\end{aligned}$$

and using here the Bianchi identity on both $\pm$ regions and expression (2.33) we finally get

$$(\nabla_\lambda \nabla_\mu - \nabla_\mu \nabla_\lambda) \underline{t}_\alpha = -t_\rho \underline{R}^\rho_{\alpha\lambda\mu}. \tag{A.25}$$

Of course, this can be extended to tensor fields of any $(p,q)$ type which may have discontinuities across $\Sigma$.

What about the Ricci identity for tensor distributions not associated to tensor fields? The answer now is much more involved, and it must be treated case by case, because taking covariant derivatives presents several problems. As an illustrative example, let us analyze the case of the second covariant derivative of $\delta^\Sigma$. For the first derivative we have from (A.21)

$$\nabla_\mu \delta^\Sigma = \nabla_\rho \left( n_\mu n^\rho \delta^\Sigma \right) - K^{\Sigma\rho}_\rho n_\mu \delta^\Sigma \tag{A.26}$$

so that defining a one-form distribution $\Delta_\mu$ with support on $\Sigma$ as follows

$$\langle \Delta_\mu, Y^\mu \rangle := -\int_\Sigma n_\mu n^\rho \nabla_\rho Y^\mu dv; \qquad \Delta_\mu = \nabla_\rho \left( n_\mu n^\rho \delta^\Sigma \right) \tag{A.27}$$

we can also write

$$\nabla_\mu \delta^\Sigma = \Delta_\mu - K^{\Sigma\rho}_{\rho} n_\mu \delta^\Sigma.$$

Note, however, that $\Delta_\mu$, and therefore $\nabla_\mu \delta^\Sigma$ too, is only well defined when acting on test vector fields whose *covariant* derivative is locally integrable on $\Sigma$. Thus, the second covariant derivative of $\delta^\Sigma$ is not defined in the general case with a discontinuous connection $\Gamma^\alpha_{\mu\nu}$. To see this, observe that to define $\nabla_\lambda \nabla_\mu \delta^\Sigma$ we need to define $\nabla_\lambda \Delta_\mu$, but this should be according to Definition 5 in Chapter 2

$$\left\langle \nabla_\lambda \Delta_\mu, Y^{\lambda\mu} \right\rangle = -\left\langle \Delta_\mu, \nabla_\lambda Y^{\lambda\mu} \right\rangle \tag{A.28}$$

and this is ill-defined because $\nabla_\lambda Y^{\lambda\mu}$ does *not* have a locally integrable covariant derivative in the sense of functions: actually, its covariant derivative can only be defined in the sense of distributions.

Nevertheless, if the connection is continuous, that is, $[\Gamma^\alpha_{\mu\nu}] = 0$, then (A.28) makes perfect sense because the covariant derivative $\nabla_\rho \nabla_\lambda Y^{\lambda\mu}$ is a locally integrable tensor field. Thus, in this case we can write

$$\left\langle \nabla_\lambda \Delta_\mu, Y^{\lambda\mu} \right\rangle = \int_\Sigma n_\mu n^\rho \nabla_\rho \nabla_\lambda Y^{\lambda\mu} dv \tag{A.29}$$

and we can prove the Ricci identity for distributions such as $\delta^\Sigma$. To that end, a straightforward if somewhat lengthy calculation, using the Ricci identity under the integral and the rest of techniques hitherto explained, leads to the following explicit expression:

$$\begin{aligned} \nabla_\lambda \nabla_\mu \delta^\Sigma \;=\;& \nabla_\rho \nabla_\sigma (n_\mu n_\lambda n^\rho n^\sigma \delta^\Sigma) + \nabla_\rho \{(K_{\lambda\mu} - K^\sigma_\sigma n_\lambda n_\mu) n^\rho \delta^\Sigma\} + \\ & \delta^\Sigma \{K^\sigma_\sigma (K^\rho_\rho n_\lambda n_\mu - K_{\lambda\mu}) + n^\rho n^\sigma R^\Sigma_{\rho\mu\lambda\sigma} + K^\rho_\lambda K_{\mu\rho} + n_\lambda n_\mu K^{\rho\sigma} K_{\rho\sigma}\} + \\ & \delta^\Sigma n_\mu \{\overline{\nabla}_\rho K^\rho_\lambda - \overline{\nabla}_\lambda K^\rho_\rho - n^\rho R^\Sigma_{\rho\lambda}\} \end{aligned}$$

where all the summands are obviously symmetric in $(\lambda\mu)$ except for those in the last line which, by virtue of the contracted Codazzi relation (A.13), become simply $n^\rho n^\sigma R^\Sigma_{\rho\sigma} n_\lambda n_\mu \delta^\Sigma$, so that finally one arrives at the desired result

$$\nabla_\lambda \nabla_\mu \delta^\Sigma - \nabla_\mu \nabla_\lambda \delta^\Sigma = 0.$$

B

# Potential problems with Gaussian coordinates

In the literature on junction conditions [44] or in general when dealing with braneworlds, it is customary to simplify the difficulties of dealing with tensor distributions by using Gaussian coordinates based on the matching hypersurface and a classical Dirac delta "function". This leads to some subtleties very often ignored and, in fact, to unsolvable problems if one is to describe gravitational double layers. In this Appendix we clarify this situation and provide a useful translation between the rigorous and the simplified methods. Choose local Gaussian coordinates $\{y, u^a\}$ based on the matching hypersurface $\Sigma$ given by

$$\Sigma : \quad \{y = 0\}$$

so that the metric reads locally around $\Sigma$ as

$$ds^2 = dy^2 + g_{ab}(y, u^c)dx^a dx^b.$$

We can identify the local coordinates of $\Sigma$ as $\xi^a = u^a$, or in other words, the parametric representation of $\Sigma$ and the tangent vector fields $\vec{e}_a$ are simply

$$\{y = 0, u^a = \xi^a\}, \qquad \vec{e}_a = \left.\frac{\partial}{\partial u^a}\right|_{y=0}.$$

The unit normal is in this case

$$\boldsymbol{n} = dy|_{y=0}$$

and the first fundamental form (2.4) becomes simply

$$h_{ab} = g_{ab}(0, u^c).$$

In what follows, $h$ denotes the determinant of $h_{ab}$. The two regions matched are represented by $y > 0$ and by $y < 0$. A trivial calculation proves that the second fundamental forms inherited from both sides are

$$K^{\pm}_{ab} = \lim_{y\to 0^{\pm}} \partial_y g_{ab} \qquad \Longrightarrow \qquad [K_{ab}] = [\partial_y g_{ab}]\,|_{y=0}.$$

In these coordinates, the $\Sigma$-step function (2.17) can be easily identified with the standard Heaviside step function $\theta(y)$. Thus, its covariant derivative is easily computed

$$\nabla\theta(y) = \delta(y)dy$$

where $\delta(y)$ is the Dirac delta "function". This can be immediately put in correspondence with (2.20) in such a way that, in this coordinate system

$$\delta^{\Sigma} \leftrightarrow \delta(y)\,.$$

Now, if we multiply $\delta(y)$ by any function then

$$F\delta(y) = F|_{y=0}\delta(y) \leftrightarrow F\delta^{\Sigma} = F|_{\Sigma}\delta^{\Sigma}.$$

Observe, however, that a first subtlety arises: when we apply $\delta(y)$ to any test function $Y(x^{\mu})$, we do not simply get $Y|_{y=0}$, but we also need to integrate on $\Sigma$, that is

$$\langle\delta(y), Y\rangle = \int_{y=0} Y(y=0, u^c)\sqrt{-h}\, du^1...du^n.$$

This corresponds to (2.19), after the identification $d\sigma = \sqrt{-h}du^1...du^n$.

The discontinuity of the connection (2.38) together with (2.59) can be expressed by giving the non-zero jumps of the Christoffel symbols

$$[\Gamma^y_{ab}] = -[K_{ab}], \qquad [\Gamma^a_{by}] = [K^a{}_b]$$

and similarly (2.61), (2.62) and (2.64) read (only the non-zero components are shown)

$$H_{yayb} = -[K_{ab}], \qquad H_{yy} = -[K^c_c],\;\; H_{ab} = -[K_{ab}], \qquad \mathcal{G}_{ab} = -[K_{ab}] + [K^c_c]h_{ab}$$

so that, for instance, the Einstein tensor tangent components acquire a term proportional to $\delta(y)$ given by $\mathcal{G}_{ab}\delta(y)$.

If one needs to compute covariant derivatives of the curvature tensors, or the Einstein tensor, as distributions, one must deal with terms such as, say, $\nabla_{\mu}(\mathcal{G}_{ab}\delta(y))$. Eventually one would face the computation of $\nabla_{\mu}\delta(y)$. One might naively write

$$\nabla\delta(y) \not= \delta'(y)dy$$

where $\delta'(y)$ is "the derivative" of the Dirac delta. This is clearly ill-defined, because one does not know how such a $\delta'(y)$ should act on test functions (as minus the integral on $\Sigma$ of the $y$-derivative of the test function?). But worse, even if one could find a proper definition of such a $\delta'(y)$, still the formula would miss the second essential term appearing in (A.26) which is proportional to $\delta^{\Sigma}$ and *depends on the extrinsic properties of the matching hypersurface* via the trace of its second fundamental form.

In order to show how to proceed if one insists in using Gaussian coordinates, the computation of $\nabla\delta(y)$ must go as follows (here $g$ stands for the determinant of $g_{\alpha\beta}$)

$$\begin{aligned}
&\left\langle \nabla\delta(y), \vec{Y} \right\rangle = -\left\langle \delta(y), \nabla_\mu Y^\mu \right\rangle = -\int_{y=0} \nabla_\mu Y^\mu \sqrt{-h} d^n u \\
&= -\int_{y=0} \frac{1}{\sqrt{-g}} \partial_\mu(\sqrt{-g} Y^\mu) \sqrt{-h} d^n u = -\int_{y=0} \frac{1}{\sqrt{-h}} \partial_\mu(\sqrt{-h} Y^\mu) \sqrt{-h} d^n u \\
&= -\int_{y=0} \left( \partial_y Y^y + \partial_a Y^a + Y^\mu \frac{1}{\sqrt{-h}} \partial_\mu \sqrt{-h} \right) \sqrt{-h} d^n u \\
&= -\int_{y=0} \left( \partial_y Y^y + \frac{1}{\sqrt{-h}} \partial_a(\sqrt{-h} Y^a) + Y^y \frac{1}{\sqrt{-h}} \partial_y \sqrt{-h} \right) \sqrt{-h} d^n u \\
&= -\int_{y=0} \left( \partial_y Y^y + Y^y K^{\Sigma a}{}_{a} \right) \sqrt{-h} d^n u.
\end{aligned}$$

In the last step we have used Gauss' theorem. This formula corresponds to (A.26). Observe the fact that the extrinsic curvature $K_{ab}$ is not necessarily equal as computed from either side of $y = 0$ and therefore it is not univocally defined. Hence, a definite prescription of what is its value on $\Sigma$, that is $K^{\Sigma}_{ab}$, must be provided.

The above subtleties and difficulties when using Gaussian coordinates are probably the reasons why double layers were not found in quadratic $F(R)$ or other quadratic theories until they were derived in [98, 99, 100].

# C

# Compatibility integrals

We devote this Appendix to include the treatment of the interior problem using the perturbative framework constructed in [77] for the compatibility of interior problems with the existence of asymptotically flat vacuum exteriors. As we show next, this is an alternative way of showing the discontinuity of the function $m_0$.

The compatibility of the interior problems is formulated as a set of integrals on the interior surface $\Sigma_0^+$ in terms of the metric (and perturbation tensor) functions written in Weyl coordinates (and in the Weyl gauge). We first include a brief review of the procedure in [77].

## C.1 The framework: formulation of the compatibility conditions

### Perturbative approach

Consider the exterior ($E$) family of vacuum spacetimes in Weyl coordinates $\{t, \varphi, \rho, z\}$, which are adapted to the stationary and axial Killing vector fields $\vec{\xi} = \partial_t$ and $\vec{\eta} = \partial_\varphi$. [1] Choose the points of the diffeomorphic spacetimes to be identified using the Weyl gauge (i.e. for equal values of the Weyl coordinates). The family of tensors $g_\varepsilon$ on $\mathcal{V}$ thus reads

$$g_\varepsilon^E \;=\; -e^{2U_\varepsilon}(dt + A_\varepsilon d\varphi)^2 + e^{-2U_\varepsilon}(e^{2k_\varepsilon}(d\rho^2 + dz^2) + \rho^2 d\varphi^2), \qquad \text{(C.1)}$$

where $U_\varepsilon, A_\varepsilon$ and $k_\varepsilon$ are functions smooth in $\rho$, $z$ and $\varepsilon$.

[1] We use initially $E/I$ instead of $+/-$ to refer to the exterior and interior in this Appendix to prevent any possible confussion with previous notation and the use of other sets of coordinates and gauges (Weyl or otherwise).

The functions in (C.1) (except $k_\varepsilon$) are defined intrinsically in terms of the corresponding timelike and axial Killing vectors as follows

$$e^{2U_\varepsilon} = -(\vec{\xi}, \vec{\xi})_{g_\varepsilon}\,, \quad A_\varepsilon = \frac{(\vec{\xi}, \vec{\eta})_{g_\varepsilon}}{(\vec{\xi}, \vec{\xi})_{g_\varepsilon}}. \tag{C.2}$$

Furthermore, $\rho \geq 0$ in (C.1) a scalar, given by

$$\rho^2 = -(\vec{\xi}, \vec{\xi})_{g_\varepsilon}(\vec{\eta}, \vec{\eta})_{g_\varepsilon} + (\vec{\xi}, \vec{\eta})^2_{g_\varepsilon}. \tag{C.3}$$

The vacuum field equations imply the existence (globally once the exterior is simply connected) of the so called twist potential that satisfies

$$d\Omega_\varepsilon = *_\varepsilon(\xi \wedge d\xi) \tag{C.4}$$

where $*_\varepsilon$ is the Hodge dual with respect to $g_\varepsilon$, which related to $A$ by

$$dA_\varepsilon = \rho e^{-4U_\varepsilon} \star d\Omega_\varepsilon, \tag{C.5}$$

where now $\star$ denotes the Hodge dual on the $\{\rho, z\}$ 2-plane defined by $dz = -\star d\rho$ ($\star^2 = -1$). In terms of the potentials $U$ and $\Omega$ the vacuum field equations reduce to the Ernst equations [103]

$$\triangle_\gamma U_\varepsilon + \frac{1}{2}e^{-4U_\varepsilon}(\mathrm{d}\Omega_\varepsilon, \mathrm{d}\Omega_\varepsilon)_\gamma = 0\,, \quad \triangle_\gamma \Omega_\varepsilon - 4(\mathrm{d}\Omega_\varepsilon, \mathrm{d}U_\varepsilon)_\gamma = 0, \tag{C.6}$$

for the metric $\gamma = d\rho^2 + dz^2 + \rho^2 d\varphi^2$, and $k_\varepsilon$ is found by quadratures

$$k_{\varepsilon,\rho} = \rho[U_{\varepsilon,\rho}^{\,2} - U_{\varepsilon,z}^{\,2}]\,, \qquad k_{\varepsilon,z} = 2\rho U_{\varepsilon,z}\, U_{\varepsilon,\rho}\,.$$

The boundary, together with the boundary conditions, that supplement the Ernst equations are put together as follows. Following the construction detailed in Chapter 3, consider now the family of stationary and hypersurfaces $\Sigma^E_\varepsilon$ projected onto $(\mathcal{V}^E, g^E)$. Using the Weyl gauge we specify $\Sigma^E_\varepsilon : \{t = \tau, \rho = \rho_\varepsilon(\mu), z = z_\varepsilon(\mu), \varphi = \phi\}$, where $\{\tau, \mu, \phi\}$ are the chosen coordinates on $\Sigma_0$ [77].

Next, take another family of surfaces $S_\varepsilon$ in the Euclidean space in cylindrical coordinates $(E^3, \gamma)$, where $\gamma = d\rho^2 + dz^2 + \rho^2 d\varphi^2$. For each $\varepsilon$, $S_\varepsilon$ is axially symmetric and compact and it is given by $S_\varepsilon = \{\rho = \rho_\varepsilon(\mu), z = z_\varepsilon(\mu), \varphi = \phi\}$, with $\rho_\varepsilon(\mu)$ and $z_\varepsilon(\mu)$ being the functions that determine $\Sigma^E_\varepsilon$. The only two points $\{\mu_N, \mu_S\}$, the north and south poles of the configuration respectively, where the surface intersects the axis of symmetry are $\rho_\varepsilon(\mu_N) = \rho_\varepsilon(\mu_S) = 0$. At these points, define $z_N := z_\varepsilon(\mu_N)$ and $z_S := z_\varepsilon(\mu_S)$.

The Ernst equations for each $\varepsilon$ live in the domain $D_\varepsilon$, with $D_\varepsilon \subset E^3$ being the exterior region of $S_\varepsilon$ endowed with the flat metric $\gamma$. The boundary data on $S_\varepsilon$ comes from the matching on $\Sigma_\varepsilon^E$ with some given interior, that provides values of the functions and their normal derivatives. The boundary conditions are completed with the asymptotic values $U_\varepsilon \to 1$ and $\Omega_\varepsilon \to 0$. We are thus dealing with an elliptic system complemented with Cauchy boundary data. This is an overdetermined problem and we should not expect solutions to exist for arbitrary data. That expresses the fact that given an arbitrary stationary and axially symmetric interior metric (even if it is perfect fluid, say), there will in general be no stationary and axially symmetric vacuum exterior solution matching with it and also asymptotically flat. The problem we face, then, is the existence for the exterior problem. After finding the perturbed matching and field equations at first and second order, the (necessary and sufficient) conditions on the boundary data for the existence of solutions for the exterior problem at first and second order are obtained. Those conditions on the boundary data will become conditions on the quantities for the interior problem.

**Perturbed Ernst equations**

The perturbations of the potentials at each order are written as

$$\begin{aligned} U_\varepsilon(\rho,z) &= U(\rho,z) + \varepsilon U^{(1)}(\rho,z) + \frac{\varepsilon^2}{2} U^{(2)}(\rho,z) + O(\varepsilon^3) \\ \Omega_\varepsilon(\rho,z) &= \Omega(\rho,z) + \varepsilon \Omega^{(1)}(\rho,z) + \frac{\varepsilon^2}{2} \Omega^{(2)}(\rho,z) + O(\varepsilon^3), \end{aligned}$$

and equivalently for $A_\varepsilon(\rho,z)$, where here and in the following we will be using the notation introduced in Chapter 3, which, note, differs to that used in [77].

The exterior static background metric in Weyl coordinates reads

$$g^E = -e^{2U} dt^2 + e^{-2U} \left[ e^{2k} \left( d\rho^2 + dz^2 \right) + \rho^2 d\varphi^2 \right], \tag{C.7}$$

where the function $U(\rho,z)$ satisfies (C.6) for $\varepsilon = 0$, i.e. the Laplace equation $\Delta_\gamma U = 0$. Recall we try to avoid the use of 0 subindexes to refer to background quantities. The domain corresponds to $D_0 := D_{\varepsilon=0}$, whose boundary is thus given by $S_0 = \{\rho = \rho_0(\mu), z = z_0(\mu), \varphi = \phi\}$ .

Let us stress the fact that the background configuration does not have to be spherically symmetric necessarily. Although we are interested in a spherically symmetric background, we briefly describe in the following the general framework for completeness.

Following the definition given for the perturbation tensors (up to second order), in the

Weyl gauge take they the form

$$
\begin{aligned}
K_1^E &= -2\left[e^{2U}(U^{(1)}dt^2 + A^{(1)}dtd\varphi)\right. \\
&\quad \left. + e^{-2U}\left(U^{(1)}\rho^2 d\varphi^2 - e^{2k}\left(k^{(1)} - U^{(1)}\right)\left(d\rho^2 + dz^2\right)\right)\right], \qquad \text{(C.8)}\\
K_2^E &= -2e^{2U}\left(U^{(2)} + 2U^{(1)^2}\right)dt^2 - 2e^{2U}\left(A^{(2)} + 4A^{(1)}U^{(1)}\right)dtd\varphi \\
&\quad -2\left[e^{2U}A^{(1)^2} + e^{-2U}\rho^2\left(U^{(2)} - 2U^{(1)^2}\right)\right]d\varphi^2 \\
&\quad +2e^{-2U}e^{2k}\left[k^{(2)} - U^{(2)} + 2\left(k^{(1)} - U^{(1)}\right)^2\right]\left(d\rho^2 + \mathrm{d}z^2\right). \qquad \text{(C.9)}
\end{aligned}
$$

The Ernst equations at each order are then the derivatives of (C.6) with respect to $\varepsilon$ at $\varepsilon = 0$ defined on the background space $(D_0, \gamma)$. The equations for $\{U^{(1)}, \Omega^{(1)}\}$ read

$$
\begin{aligned}
&\triangle_\gamma U^{(1)} = 0, \\
&\triangle_\gamma \Omega^{(1)} - 4\left(\mathrm{d}\Omega^{(1)}, \mathrm{d}U\right)_\gamma = 0,
\end{aligned} \qquad \text{(C.10)}
$$

while for $\{U^{(2)}, \Omega^{(2)}\}$ are

$$
\begin{aligned}
&\triangle_\gamma U^{(2)} + e^{-4U}\left(\mathrm{d}\Omega^{(1)}, \mathrm{d}\Omega^{(1)}\right)_\gamma = 0, \\
&\triangle_\gamma \Omega^{(2)} - 4\left(\mathrm{d}\Omega^{(2)}, \mathrm{d}U\right)_\gamma - 8\left(\mathrm{d}\Omega^{(1)}, \mathrm{d}U^{(1)}\right)_\gamma = 0.
\end{aligned} \qquad \text{(C.11)}
$$

It must be stressed that the information about the deformation of $\Sigma_0^E$, will be finally encoded in terms of quantities defined precisely on $S_0$, i.e. $\rho^{(1)}(\mu)$, $\rho^{(2)}(\mu)$, $z^{(1)}(\mu)$, $z^{(2)}(\mu)$ (see below), just as in the general theory of perturbed matchings the matching problem is encoded in the background matching hypersurface, see Chapter 3.

The equations for the twist potential contain extra terms out of the Laplacian operator that can be absorbed using an alternative Laplacian operator in terms of the conformally flat metric $\tilde{\gamma} = e^{-8U}\gamma$:

$$
\begin{aligned}
\triangle_{\tilde{\gamma}}\Omega^{(1)} &= 0, \\
\triangle_{\tilde{\gamma}}\Omega^{(2)} &= 8\left(\mathrm{d}\Omega^{(1)}, \mathrm{d}U^{(1)}\right)_{\tilde{\gamma}}.
\end{aligned}
$$

Therefore all equations for $U^{(1)}$, $U^{(2)}$, $\Omega^{(1)}$, $\Omega^{(2)}$ can be collectively written as

$$
\triangle_{\hat{\gamma}} u = \mathcal{J}, \qquad \text{(C.12)}
$$

where $u = u(\rho, z)$ stands for $U$, $U^{(1)}$, etc..., and $\mathcal{J} = \mathcal{J}(\rho, z)$ represents the inhomogeneous terms in the second order perturbation equations. The metric $\hat{\gamma}$ corresponds to either $\gamma$, for the $U$-equations, or $\tilde{\gamma}$, for the $\Omega$-equations.

### Boundary data

The boundary data for each of the perturbed Ernst equations is developed to second order. Asymptotic flatness imposes $\lim_{\rho^2+z^2\to\infty} U^{(1)} = \lim_{\rho^2+z^2\to\infty} \Omega^{(1)} = \lim_{\rho^2+z^2\to\infty} U^{(2)} = \lim_{\rho^2+z^2\to\infty} \Omega^{(2)} = 0$. On the other hand, given the interior at each order, the matching conditions also provide the values of the functions and their normal derivatives evaluated on the background matching hypersurface $\Sigma_0^E$, and thus on $S_0$ as follows. First we have to characterize the relevant functions for the perturbative matching from the stationary and axisymmetric interior. Let be such an interior $(\mathcal{V}^I, g^I)$ given, matched already to $(\mathcal{V}^E, g^E)$ across static and axially symmetric hypersurfaces $\Sigma_0^I = \Sigma_0^E$. Take a stationary and the axial Killing vectors in $(\mathcal{V}^I, g^I)$, $\vec{\xi}^I$ and $\vec{\eta}^I$ respectively, together with a family of tensors $g_\varepsilon^I$ invariant under such isometries, and compute the functions $V_\varepsilon$, $W_\varepsilon$ and $\alpha_\varepsilon > 0$ by

$$V_\varepsilon = \frac{1}{2}\log\left[-(\vec{\xi^I},\vec{\xi^I})_{g_\varepsilon^I}\right], \quad \mathrm{d}W_\varepsilon = *_\varepsilon(\xi^I\wedge \mathrm{d}\xi^I), \quad \alpha_\varepsilon^2 = -(\vec{\xi^I},\vec{\xi^I})_{g_\varepsilon^I}(\vec{\eta^I},\vec{\eta^I})_{g_\varepsilon^I} + (\vec{\xi},\vec{\eta})^2_{g_\varepsilon^I}. \tag{C.13}$$

These three functions carry all the necessary (and sufficient) information for the present problem regarding the interior geometry [109]. Let also a family $\Sigma_\varepsilon^I$ be given on $(\mathcal{V}^I, g^I)$, and thus their (flow of) normals $\vec{n}_\varepsilon^I$, and assume that they match with their corresponding $\Sigma_\varepsilon^E$, through common coordinates $\{\tau,\mu,\phi\}$ in $\Sigma_0$. Assume finally that $\vec{n}_\varepsilon^I$ point to the interior of $(\mathcal{V}_\varepsilon^I, g_\varepsilon^I)$ and have the same norm as $\partial_\mu$ in $\Sigma_0$, i.e. $(\partial_\mu,\partial_\mu)_{h_\varepsilon}(=(\vec{e},\vec{e})_{g_\varepsilon^E}$ for $\vec{e}=\dot{\rho}_\varepsilon\partial_\rho + \dot{z}_\varepsilon\partial_\rho)$. Then, the functions $\rho_\varepsilon(\mu)$ and $z_\varepsilon(\mu)$ that determine $\Sigma_\varepsilon^E$ are given by [77]

$$\dot{z}_\varepsilon(\mu) = \dot{z}_0 + \varepsilon\dot{z}^{(1)} + \frac{1}{2}\varepsilon^2\dot{z}^{(2)} + O(\varepsilon^3) \;\; = \;\; -\vec{n}_\varepsilon^I(\alpha_\varepsilon)|_{\Sigma_\varepsilon^I} \tag{C.14}$$

$$\rho_\varepsilon(\mu) = \rho_0 + \varepsilon\rho^{(1)} + \frac{1}{2}\varepsilon^2\rho^{(2)} + O(\varepsilon^3) \;\; = \;\; \alpha_\varepsilon|_{\Sigma_\varepsilon^I}. \tag{C.15}$$

It is convenient to define the following functions in order to achieve compact expressions for the boundary data.

$$P_1 = \frac{\dot{\rho}_0\rho^{(1)} + \dot{z}_0 z^{(1)}}{\dot{\rho}_0^2+\dot{z}_0^2}, \qquad Q_1 = \frac{\dot{\rho}_0 z^{(1)} - \dot{z}_0\rho^{(1)}}{\dot{\rho}_0^2+\dot{z}_0^2}, \qquad P_2 = \frac{\dot{\rho}_0\rho^{(2)} + \dot{z}_0 z^{(2)}}{\dot{\rho}_0^2+\dot{z}_0^2},$$
$$Q_2 = \frac{\dot{\rho}_0 z^{(2)} - \dot{z}_0\rho^{(2)}}{\dot{\rho}_0^2+\dot{z}_0^2}, \qquad X_0 = \frac{\ddot{\rho}_0\dot{z}_0 - \ddot{z}_0\dot{\rho}_0}{\dot{\rho}_0^2+\dot{z}_0^2}, \qquad X_1 = \frac{\ddot{\rho}_0\dot{\rho}_0 + \ddot{z}_0\dot{z}_0}{\dot{\rho}_0^2+\dot{z}_0^2}, \tag{C.16}$$

where we are using the dot $\dot{}$ to denote differentiation with respect to $\mu$.

Also, consider the following functions constructed by the inner products of the axial and stationary Killing vectors together with their normal and tangent derivatives to $\Sigma^I_\varepsilon$ [2]

$$
\begin{aligned}
V_\varepsilon = V + \varepsilon V^{(1)} + \frac{1}{2}\varepsilon^2 V^{(2)} + O(\varepsilon^3) &= \frac{1}{2}\log\left(-(\vec{\xi}^I,\vec{\xi}^I)_{g^I_\varepsilon}\right)\Big|_{\Sigma^I_\varepsilon}, \\
\vec{n}V_\varepsilon = \vec{n}V + \varepsilon\vec{n}V^{(1)} + \frac{1}{2}\varepsilon^2\vec{n}V^{(2)} + O(\varepsilon^3) &= \frac{\vec{n}^I_\varepsilon(\vec{\xi}^I,\vec{\xi}^I)_{g^I_\varepsilon}}{2(\vec{\xi}^I,\vec{\xi}^I)_{g^I_\varepsilon}}\Bigg|_{\Sigma^I_\varepsilon}, \qquad \text{(C.17)}\\
\dot{W}_\varepsilon = \varepsilon\dot{W^{(1)}} + \frac{1}{2}\varepsilon^2\dot{W^{(2)}} + O(\varepsilon^3) &= -\frac{(\vec{\xi}^I,\vec{\xi}^I)^2_{g^I_\varepsilon}}{\alpha_\varepsilon}\vec{n}^I_\varepsilon\left(\frac{(\vec{\xi}^I,\vec{\eta}^I)_{g^I_\varepsilon}}{(\vec{\xi}^I,\vec{\xi}^I)_{g^I_\varepsilon}}\right)\Bigg|_{\Sigma^I_\varepsilon}, \\
\vec{n}W_\varepsilon = \varepsilon\vec{n}W^{(1)} + \frac{1}{2}\varepsilon^2\vec{n}W^{(2)} + O(\varepsilon^3) &= \frac{(\vec{\xi}^I,\vec{\xi}^I)^2_{g^I_\varepsilon}}{\alpha_\varepsilon}\frac{d}{d\mu}\left(\frac{(\vec{\xi}^I,\vec{\eta}^I)_{g^I_\varepsilon}}{(\vec{\xi}^I,\vec{\xi}^I)_{g^I_\varepsilon}}\right)\Bigg|_{\Sigma^I_\varepsilon}.
\end{aligned}
$$

All the objects introduced in this section allow us to construct the functions $V(\mu)$, $V^{(1)}(\mu)$, $V^{(2)}(\mu)$, $\vec{n}V(\mu)$, $\vec{n}V^{(1)}(\mu)$, $\vec{n}V^{(2)}(\mu)$, $W^{(1)}(\mu)$, $W^{(2)}(\mu)$, $\vec{n}W^{(1)}(\mu)$, $\vec{n}W^{(2)}(\mu)$ on $\Sigma_0$, explicitly once $g^I_\varepsilon$ and $\Sigma^I_\varepsilon$ are given. Note that since the interior background is assumed to be static, there are no $W(\mu)$ nor $\vec{n}W(\mu)$ terms. In order to avoid confusion with the notation, let us stress that $\vec{n}V$, $\vec{n}V^{(1)}$,... do not denote normal derivatives of $V$, $V^{(1)}$,... but functions constructed following (C.17).

Let us also stress the fact here that we have shown the obtaining of the above functions in terms of a given family $\Sigma^I_\varepsilon$, and therefore, a *flow* of normals $\vec{n}^I_\varepsilon$. That may not be the most convenient way, and one can, in fact, consider only the background $\Sigma^I_0$ and the deformation vectors $\vec{Z}_{1/2}$, through the "unknowns" $P_1, Q_1, P_2$ and $Q_2$ –from which the $\varepsilon$-derivatives at $\varepsilon = 0$ of $\vec{n}^I_\varepsilon$ can be obtained, see [77]–, and construct the above quantities in terms of this information. Since the "flow" version will suit our needs, which is to compare to the original perturbative approach in [57], we do not discuss this further.

Proposition IV.1 in [77] then states that $g^I_\varepsilon$ and $g^E_\varepsilon \equiv g^E + \varepsilon K^E_1 + \frac{1}{2}\varepsilon^2 K^E_2$ match perturbatively to second order on $\Sigma^{I/E}_0$ if and only if the following conditions are satisfied

$$
\begin{aligned}
U|_{\Sigma^E_0} &= V, \qquad \vec{n}(U)|_{\Sigma^E_0} = \vec{n}V, \qquad U^{(1)}|_{\Sigma^E_0} = V^{(1)} - P_1\frac{dV}{d\mu} - Q_1\vec{n}V, \\
\vec{n}(U^{(1)})|_{\Sigma^E_0} &= \vec{n}V^{(1)} + \frac{d}{d\mu}\left(Q_1\frac{dV}{d\mu}\right) - \frac{d\left(P_1\vec{n}V\right)}{d\mu} + Q_1\left(\frac{\dot{\rho}_0}{\rho_0}\frac{dV}{d\mu} - \frac{\dot{z}_0}{\rho_0}\vec{n}V\right), \\
U^{(2)}|_{\Sigma^E_0} &= V^{(2)} - 2P_1\frac{dV^{(1)}}{d\mu} - 2Q_1\vec{n}V^{(1)} + \frac{d}{d\mu}\left(\left(P_1^2 - Q_1^2\right)\frac{dV}{d\mu}\right) + \frac{d\left(2P_1Q_1\vec{n}V\right)}{d\mu} \\
&\quad + \left(-P_2 + P_1^2X_1 + 2P_1Q_1X_0 - Q_1^2X_1 - Q_1^2\frac{\dot{\rho}_0}{\rho_0}\right)\frac{dV}{d\mu} \\
&\quad + \left(-Q_2 - P_1^2X_0 + 2P_1Q_1X_1 + Q_1^2X_0 + Q_1^2\frac{\dot{z}_0}{\rho_0}\right)\vec{n}V,
\end{aligned}
$$

[2]There is a typo in the sign of the last expression of equations (17) in [77]. The corrected expression is given here.

$$\begin{aligned}
\vec{n}(U^{(2)})|_{\Sigma_0^E} &= \vec{n}V^{(2)} + 2\frac{d}{d\mu}\left(Q_1\frac{dV^{(1)}}{d\mu}\right) - 2\frac{d\left(P_1\vec{n}V^{(1)}\right)}{d\mu} + 2Q_1\left(\frac{\dot{\rho}_0}{\rho_0}\frac{dV^{(1)}}{d\mu} - \frac{\dot{z}_0}{\rho_0}\vec{n}V^{(1)}\right)\\
&-\frac{d^2}{d\mu^2}\left(2P_1Q_1\frac{dV}{d\mu}\right) + \frac{d^2}{d\mu^2}\left(\left(P_1^2 - Q_1^2\right)\vec{n}V\right)\\
&+\frac{d}{d\mu}\left\{\left[Q_2 + \left(P_1^2 - Q_1^2\right)X_0 - 2P_1Q_1\frac{\dot{\rho}_0}{\rho_0} - 2P_1Q_1X_1 - Q_1^2\frac{\dot{z}_0}{\rho_0}\right]\frac{dV}{d\mu}\right\}\\
&+\frac{d}{d\mu}\left\{\left[-P_2 + \left(P_1^2 - Q_1^2\right)X_1 + 2P_1Q_1\frac{\dot{z}_0}{\rho_0} + 2P_1Q_1X_0 - Q_1^2\frac{\dot{\rho}_0}{\rho_0}\right]\vec{n}V\right\}\\
&+\left(Q_2 + (P_1^2 - Q_1^2)X_0 - 2P_1Q_1\frac{\dot{\rho}_0}{\rho_0}\right)\left(\frac{\dot{\rho}_0}{\rho_0}\frac{dV}{d\mu} - \frac{\dot{z}_0}{\rho_0}\vec{n}V\right)\\
&+\left(2P_1Q_1X_0 - Q_1^2\frac{\dot{\rho}_0}{\rho_0}\right)\left(\frac{\dot{\rho}_0}{\rho_0}\vec{n}V + \frac{\dot{z}_0}{\rho_0}\frac{dV}{d\mu}\right),
\end{aligned}$$

$$\Omega^{(1)}|_{\Sigma_0^E} = W^{(1)}, \qquad \vec{n}(\Omega^{(1)})|_{\Sigma_0^E} = \vec{n}W^{(1)},$$

$$\Omega^{(2)}|_{\Sigma_0^E} = W^{(2)} - 2P_1\frac{dW^{(1)}}{d\mu} - 2Q_1\vec{n}W^{(1)},$$

$$\begin{aligned}
\vec{n}\left(\Omega^{(2)}\right)|_{\Sigma_0^E} &= \vec{n}W^{(2)} + 2\frac{d}{d\mu}\left(Q_1\frac{dW^{(1)}}{d\mu}\right) - 2\frac{d\left(P_1\vec{n}W^{(1)}\right)}{d\mu}\\
&+2Q_1\left[\left(\frac{\dot{\rho}_0}{\rho_0} - 4\frac{dV}{d\mu}\right)\frac{dW^{(1)}}{d\mu} - \left(\frac{\dot{z}_0}{\rho_0} + 4\vec{n}V\right)\vec{n}W^{(1)}\right]
\end{aligned}$$

In conclusion, this section has been devoted to find the boundary data for the Ernst problems at each order on the surface $S_0$. For this, given an interior region, and given also a prescribed deformation of the hypersurface, the steps to follow are

1. Take an explicit interior model with given background hypersurface $\Sigma_0^I$ and with the three different products of the axial and stationary Killing vectors, together with the perturbations of the hypersurface, together with their flow of normal vectors, compute $V$, $\vec{n}V$, $P_1$, $Q_1$,... (were not the deformation prescribed the functions $P_1$, $Q_1$, $P_2$, $Q_2$ would be left as unknowns).

2. With this quantities at hand, use Proposition IV.1 to build the boundary data for the exterior problem.

### Compatibility conditions

As mentioned, the above boundary conditions overdetermine the elliptic problems, and therefore compatibility conditions for the boundary data arise in order to ensure existence. These compatibility conditions were found in [77] for general static and axially symmetric backgrounds, and, in particular and more explicitly, for spherically symmetric backgrounds. For our purposes here we will only need to focus on this last particular case. Nevertheless, for completeness and to keep a more compact presentation, we review the conditions in the general case in this subsection.

In order to present the compatibility conditions, we need to define first some auxiliary objects. The first ingredients needed are two auxiliary regular $\hat{\gamma}$-harmonic functions on $D_0$, one for each case $\hat{\gamma} = \gamma$ ($U$-problems) and $\hat{\gamma} = \tilde{\gamma}$ ($\Omega$-problems). They are defined in [77] as functions on $D_0$ that solve the Laplace equation $\triangle_{\hat{\gamma}}\psi = 0$ on $D_0$, admit a $C^1$ extension to $\partial D_0$ and decay at infinity in such a way that $\psi\sqrt{\rho^2+z^2}$ is a bounded function on $D_0$. The relevant family of $\gamma$-harmonic functions on $D_0$ is

$$\psi_y(\rho, z) \equiv \frac{1}{\sqrt{\rho^2 + (z-y)^2}}, \qquad y \in (z_S, z_N). \tag{C.18}$$

The corresponding family for $\tilde{\gamma}$ is more involved. It requires first finding a function solution of the PDE

$$\mathrm{d}Z_y = (z-y)\psi_y(\rho, z)\,\mathrm{d}U + \rho\psi_y(\rho, z) \star \mathrm{d}U, \tag{C.19}$$

with boundary condition $\lim_{\rho^2+z^2\to\infty} Z_y = 0$. $Z_y$ can be explicitly integrated when the background is spherically symmetric (see below). The appropriate *regular $\tilde{\gamma}$-harmonic family of functions* is [77]

$$\Psi_y(\rho, z) = \frac{e^{2U-2Z_y}}{\sqrt{\rho^2 + (z-y)^2}}, \qquad y \in (z_S, z_N). \tag{C.20}$$

The remaining auxiliary objects are vector fields, denoted by $T_1$ and $T_2$, related again to $\gamma$ and $\tilde{\gamma}$ respectively, in order to absorb the inhomogeneous terms in (C.12) into surface integrals by using Gauss' identity. These vector fields are formulated, in turn, in terms of three functions $S_1$, $S_2$ and $Z_y^{(1)}$ that vanish at infinity and are solutions of the PDE's

$$\begin{aligned} \mathrm{d}S_1 &= e^{-2U+2Z_y}\left[-\left(1+(z-y)\psi_y\right)\mathrm{d}\Omega^{(1)} - \rho\psi_y \star \mathrm{d}\Omega^{(1)}\right], \\ \mathrm{d}S_2 &= e^{-2U-2Z_y}\left[\left(1-(z-y)\psi_y\right)\mathrm{d}\Omega^{(1)} - \rho\psi_y \star \mathrm{d}\Omega^{(1)}\right], \end{aligned} \tag{C.21}$$

$$\mathrm{d}Z_y^{(1)} = (z-y)\psi_y\,\mathrm{d}U^{(1)} + \rho\psi_y \star \mathrm{d}U^{(1)}. \tag{C.22}$$

In terms of those functions, the vectors $T_1$ and $T_2$ take the form

$$\boldsymbol{T_1} \equiv \frac{1}{2\rho} S_1 \star \mathrm{d}S_2, \tag{C.23}$$

$$\boldsymbol{T_2} \equiv -\frac{4}{\rho} S_2 \star \mathrm{d}\left(Z_y^{(1)} + U^{(1)}\right). \tag{C.24}$$

The existence and uniqueness of the solutions to the previous PDE's is proven in [77].

Given all the definitions introduced in this section, we can quote Theorem V.2 in [77]:

**Theorem 8** *Let $f_0$, $f_1$ be continuous axially symmetric functions on $\Sigma_0$. Then,*

*(i) the Cauchy boundary value problem*

$$\triangle_\gamma U^{(1)} = 0, \qquad U^{(1)}|_{\Sigma_0} = f_0, \qquad \vec{n}\left(U^{(1)}\right)|_{\Sigma_0} = f_1,$$

*admits a regular solution on $D_0$ if and only if*

$$\int_{\mu_S}^{\mu_N} \left[\psi_y\, f_1 - f_0\, \vec{n}(\psi_y)\right] \rho_0|_{\Sigma_0}\, d\mu = 0, \qquad \forall y \in (z_S,\, z_N),$$

*(ii) the Cauchy boundary value problem*

$$\triangle_\gamma \Omega^{(1)} - 4\left(d\Omega^{(1)}, dU\right)_\gamma = 0, \qquad \Omega^{(1)}|_{\Sigma_0} = f_0, \qquad \vec{n}\left(\Omega^{(1)}\right)|_{\Sigma_0} = f_1,$$

*admits a regular solution on $D_0$ if and only if*

$$\int_{\mu_S}^{\mu_N} \left[\Psi_y\, f_1 - f_0\, \vec{n}(\Psi_y)\right] \rho_0 e^{-4U}\Big|_{\Sigma_0} d\mu = 0, \qquad \forall y \in (z_S,\, z_N),$$

*(iii) the Cauchy boundary value problem*

$$\triangle_\gamma U^{(2)} + e^{-4U}\left(d\Omega^{(1)}, d\Omega^{(1)}\right)_\gamma = 0, \qquad U^{(2)}|_{\Sigma_0} = f_0, \qquad \vec{n}\left(U^{(2)}\right)|_{\Sigma_0} = f_1,$$

*admits a regular solution on $D_0$ if and only if*

$$\int_{\mu_S}^{\mu_N} \left[\psi_y\, f_1 - f_0\, \vec{n}(\psi_y) - \boldsymbol{T_1}\left(\vec{n}\right)\right] \rho_0|_{\Sigma_0}\, d\mu = 0, \qquad \forall y \in (z_S,\, z_N), \qquad \text{(C.25)}$$

*and (iv) the Cauchy boundary value problem*

$$\triangle_\gamma \Omega^{(2)} - 8\left(d\Omega^{(1)}, dU^{(1)}\right)_\gamma - 4\left(d\Omega^{(2)}, dU\right)_\gamma = 0, \qquad \Omega^{(2)}|_{\Sigma_0} = f_0, \qquad \vec{n}\left(\Omega^{(2)}\right)|_{\Sigma_0} = f_1,$$

*admits a regular solution on $D_0$ if and only if*

$$\int_{\mu_S}^{\mu_N} \left[\left(\Psi_y\, f_1 - f_0\, \vec{n}(\Psi_y)\right) e^{-4U} - \boldsymbol{T_2}\left(\vec{n}\right)\right] \rho_0\big|_{\Sigma_0}\, d\mu = 0, \qquad \forall y \in (z_S,\, z_N),$$

*where $\psi_y$, $\Psi_y$, $\boldsymbol{T_1}$ and $\boldsymbol{T_2}$ are given in (C.18), (C.20), (C.23) and (C.24) respectively.*

Let us remark that since the above integrals depend on the parameter $y \in (z_S, z_N)$, each one gives an infinite set of conditions.

**Spherically symmetric background**

The background exterior corresponds to the Schwarzschild exterior geometry. In the usual spherical coordinates this reads

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right),$$

These coordinates are related to the Weyl coordinates by

$$\rho = r\sin\theta\sqrt{1-\frac{2m}{r}}, \qquad z = (r-m)\cos\theta. \tag{C.26}$$

From the definitions of the potentials we get

$$U = \frac{1}{2}\log\left(1-\frac{2m}{r}\right), \qquad \Omega = 0. \tag{C.27}$$

Several of the Ernst equations of the perturbed exterior problem can be explicitly solved. In these cases, one can use the explicit solutions and their corresponding tangent/normal derivatives to find restrictions on the boundary data, instead of working with the compatibility conditions in integral form.

In particular, $\Omega^{(1)}$ can be solved using an expansion in Legendre polynomials $\Omega^{(1)} = \sum_{l=0}^{\infty} w_l^{(1)}(x)P_l(\cos\theta)$, where

$$x = \frac{r}{M} - 1$$

is a convenient redefinition of the radial coordinate. The Ernst equation (C.10) is transformed to the Jacobi equation of type $\{-2, 2\}$.

$$\left(1-x^2\right) w_{l,xx}^{(1)} - 2\,(x-2)\, w_{l,x}^{(1)} + l\,(l+1)\, w_l^{(1)} = 0. \tag{C.28}$$

The solutions are the Jacobi polynomials $P_l^{(-2,2)}(x)$ plus another family of functions regular at infinity, which in terms of the associated Legendre functions of the second kind reads

$$\mathcal{F}_l(x) \;=\; \frac{1}{(l+1)(l+2)}\frac{(1-x)}{(1+x)}Q_l^2(x). \tag{C.29}$$

The full solution for the perturbed twist to first order is thus,

$$\Omega^{(1)} = \sum_{l=0}^{\infty} d_l P_l(\cos\theta)\mathcal{F}_l(x). \tag{C.30}$$

Given (C.27), the function $Z_y$ (C.19) is found to be [77]

$$e^{-Z_y} = \frac{yx - m\cos\theta - \sqrt{m^2x^2 + y^2 - 2mxy\cos\theta - m^2\sin^2\theta}}{(y-m)\sqrt{x^2-1}}, \tag{C.31}$$

so that $\psi_y$ together with $\Psi_y$ can be finally written as [77]

$$\psi_y = \frac{1}{\sqrt{m^2x^2 + y^2 - 2mxy\cos\theta - m^2\sin^2\theta}},$$

$$\Psi_y = \frac{\left(yx - m\cos\theta - \sqrt{m^2x^2 + y^2 - 2mxy\cos\theta - m^2\sin^2\theta}\right)^2}{(y-m)^2\,(x+1)^2\sqrt{m^2x^2 + y^2 - 2mxy\cos\theta - m^2\sin^2\theta}}.$$

This case of a spherically symmetric background is particularly useful when studying, for example, perturbations of fluid balls.

## C.2 Application to Hartle's model

In this section we apply the framework introduced in the previous section to the original Hartle's model. We consider Hartle's interior configuration describing a rigidly rotating perfect fluid without convective motions and with a barotropic EOS. This spacetime with boundary is considered as a candidate to be matched to an Ernst vacuum considered in the previous section. To elucidate under what circumstances these spacetimes can be matched (or not), the boundary data (C.1) for the Ernst problem is constructed following proposition IV.1 in [77] and compatibility conditions are evaluated in order to find restrictions on the interior configuration.

We take the family $g^I_\varepsilon$ to be (5.1) initially in the $k$-gauge. The background metric reads (5.4) and the perturbation tensors are (5.2), (5.3). The projected boundaries $\Sigma^I_\varepsilon$ are taken to have the parametric form

$$\Sigma^I_\varepsilon : \{t = \tau, \varphi = \phi, r = a + \varepsilon^2 \xi^H(a, \vartheta), \theta = \pi - \mu \equiv \vartheta\},$$

so that $\Sigma^I_0$ is the sphere at $r = a$. The function $\xi(a, \vartheta)$ therefore describes the deformation in this gauge setting (in the $k$-gauge). For convenience, let us go now to the surface gauge, using $\vec{S}_2 = \xi(r, \theta)\partial_r$ for some function $\xi(r, \theta)$ that extends $\xi(a, \theta)$. In this new gauge the second order perturbation tensor $K_2$ reads as (5.6) with, recall $k_0 = 0$, $C = 0$ and $Y = \xi(r, \theta)$, and $\Sigma^I_\varepsilon$ just read

$$\Sigma^I_\varepsilon : \{t = \tau, \varphi = \phi, r = a, \theta = \pi - \mu \equiv \vartheta\}.$$

In the following, we will use $\vartheta$ as the polar angular coordinate of $\Sigma^I_\varepsilon$. Thence, in all the equations of the previous section C.1 , the dot derivative "$\dot{}$" is equivalent to $-\partial_\vartheta$. Finally, to use the previous results (and those in [77]), it is convenient to perform the change

$$r = M(x + 1),$$

and use $a = M(x_0+1)$. The background matching hypersurface therefore fixes

$$\rho_0(\vartheta) = \sqrt{x_0^2-1}M\sin\vartheta\,, \qquad z_0(\vartheta) = x_0 M\cos\vartheta\,, \qquad \vec{n} = -\sqrt{x_0^2-1}\partial_x. \qquad \text{(C.32)}$$

The explicit assumptions of the original model (equatorial symmetry and invariance under the change $\{t\to -t, \varphi\to -\varphi\}$ so that the perutbation is driven by a rotation) imply that at first order $U^{(1)}=0$, and that $\Omega^{(2)}=0$ at second order.

Since we are interested only at obtaining the discontinuity of $m_0$ we skip the analysis of the first order. We simply take for granted that the function $w$ depends only on $x$, and that the exterior vacuum solution is given already by (6.26). The only contribution to second order order thus comes from the gravitational potential, and therefore the boundary data needed to be calculated is $U^{(2)}|_{\Sigma_0}$, and $\vec{n}(U^{(2)})|_{\Sigma_0}$ and only the compatibility integral (C.25) has to be addressed. Moreover, the fact that $\omega$ is a function of $x$ alone simplifies both the boundary data for the first order required to calculate $T_1$ as well as the second order boundary data itself. The hydrostatic equations evaluated at the boundary will be used in the last expressions in order to write the functions of the background in terms of $M$, $E$ and $P$.

In the chosen gauge the one-form normals to $\Sigma^I_\varepsilon$ take the simple form $\vec{n}_\varepsilon = N_\varepsilon\partial_x$, where $N_\varepsilon$ is determined by the normalisation. Now it is not difficult to compute the $\varepsilon$-derivatives at $\varepsilon=0$ of $\vec{n}_\varepsilon$, and find that the first order normal vanishes and at second order it reads

$$\begin{aligned}\vec{n}^{(2)} &= \left\{\sqrt{x_0^2-1}\left(m+\frac{\xi'}{M}\right)+\frac{4\pi E M^2(x_0+1)^3-x_0}{\sqrt{x_0^2-1}}\frac{\xi}{M}\right\}\partial_x\\ &\quad -\frac{1}{\sqrt{x_0^2-1}}\frac{d}{d\vartheta}\xi(x_0,\vartheta)\partial_\theta,\end{aligned}$$

where we have omitted the argument $(x_0,\vartheta)$ on all the second order functions above not to overload the expressions. We will stick to this notation through the rest of the section. With this we can evaluate the expressions (C.14) and (C.15) to first and second order, to obtain $\rho^{(1)}=0$, $z^{(1)}=0$,

$$\rho^{(2)} = \left\{M\sqrt{x_0^2-1}(h+k+\frac{x_0}{\sqrt{x_0^2-1}}\xi\right\}\sin\vartheta \qquad \text{(C.33)}$$

$$\partial_\vartheta z^{(2)} = -\left(x_0(h+2k-m)+(x_0^2-1)(h'+k')-\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\frac{\xi}{M}\cos\vartheta\right)\right. \qquad \text{(C.34)}$$

$$\left.-\frac{(x_0-1)^3(3x_0-2)M^2\sin^2\vartheta}{2(x_0-1)}\omega^2(x_0)\right)M\sin\vartheta. \qquad \text{(C.35)}$$

The functions $P_2$ and $Q_2$ are obtained from the expressions listed above using (C.16). The set of quantities neeed up to second order for the boundary data are finally evaluated from (C.17) and read

$$\partial_\vartheta W = \partial_\vartheta W^{(2)} = 0\ , \qquad \partial_\vartheta W^{(1)} = (x_0+1)M\sin\vartheta\omega(x_0), \tag{C.36}$$

$$\vec{n}W = \vec{n}W^{(2)} = 0\ , \qquad \vec{n}W^{(1)} = -2\sqrt{x_0^2-1}M\cos\vartheta\omega(x_0), \tag{C.37}$$

$$V = \frac{1}{2}\log\left(\frac{x_0-1}{x_0+1}\right)\ , \qquad V^{(1)} = 0 \tag{C.38}$$

$$V^{(2)} = h + \frac{1}{x_0^2-1}\frac{\xi}{M} - \frac{M^2(x_0+1)^3\sin^2\vartheta}{x_0-1}\omega^2(x_0) \tag{C.39}$$

$$\vec{n}V = -\frac{1}{\sqrt{x_0^2-1}}\ , \qquad \vec{n}V^{(1)} = 0 \tag{C.40}$$

$$\begin{aligned}\vec{n}V^{(2)} &= \frac{1}{x_0^2-1}(m-k) - \sqrt{x_0^2-1}h' + \frac{x_0 - 4\pi EM^2(x_0-1)(x_0+1)^3}{(x_0^2-1)^{3/2}}\frac{\xi}{M}\\ &- \frac{M^2(x_0+1)^3(2x_0-1)\sin^2\vartheta}{(x_0-1)\sqrt{x_0^2-1}}\omega^2(x_0)\end{aligned} \tag{C.41}$$

To evaluate the second order compatibility integral (C.25) we also need to compute the vector $T_1$ (C.23). Nevertheless, we need only the contraction of $T_1$ with the normal vector evaluated at the boundary, which reads

$$\boldsymbol{T_1}(\vec{n}) = -\frac{1}{2}S_1\dot{S}_2, \tag{C.42}$$

where we have used the fact that $(\star\mathrm{d}f)(\vec{n}) = -(\mathrm{d}f)(\vec{e})$, which on $\Sigma_0$ reads $-\partial_\mu f(\rho(\mu), z(\mu))$, for any $f(\rho, z)$. Moreover, since we will only need to evaluate this expression on the hypersurface there is no need to solve the whole PDE for $S_1$ in (C.21). Instead, we project (C.21) onto $\Sigma_0$ (applying it to $\vec{e}$) to obtain $\dot{S}_1$ and integrate the ODE along $\Sigma_0$. After a straigforward calculation we get

$$S_1|_{\Sigma_0} = M\frac{(x_0+1)^2}{x_0-1}\omega\left[(2x_0-1)\cos\mu - \frac{x_0}{M\psi} + \frac{y}{2M}(x_0^2+1)\right]e^{2Z_y} - \frac{J}{M^3}(y-2M), \tag{C.43}$$

where, recall, $Z_y$ is given by (C.31). The expression for $\dot{S}_2$ is obtained directly appyling $\mathrm{d}S_2$ in (C.21) to $\vec{e}$. The explicit expression of (C.42) is found to be

$$\begin{aligned}T_1(\vec{n}|_{\Sigma_0}) &= -\frac{1}{2}S_1\dot{S}_2\\ &= M\sqrt{\frac{(x_0+1)^3}{x_0-1}}\left\{M\frac{(x_0+1)^2}{(x_0-1)}\omega\left[(2x_0-1)\cos\mu - \frac{x_0}{M\psi}\right.\right.\\ &+ \left.\left.\frac{y}{2M}(x_0^2+1)\right] - \frac{J}{M^3}(y-2M)e^{-2Z_y}\right\}\\ &\quad\left[-\frac{1}{2}\sqrt{\frac{x_0+1}{x_0-1}}\tan\mu\,(1-\cos\gamma_y) - \sin\gamma_y\right]\omega\cos\mu\end{aligned} \tag{C.44}$$

We have now every ingredient to evaluate the compatibility condition in the integral form in order to find relations between $h$, $m$, $k$ and $\xi$ and their normal derivatives at $\Sigma_0$. Now, we assume the angular structure given in (8.44), recalling that $k_0 = 0$, and remove the radial derivatives of the functions by using the field equations for the interior, explicitly 6.38 for $m_0'$, 6.40 for $h_0'$, 6.58 to relate $m_2$ to $h_2$ and 6.56 with 6.57 for $h_2'$ and $k_2'$. The integrand is a function of $\mu$ linear in the constants $m_0(a)$, $h_0(a)$, $\xi_0(a)$, $h_2(a)$, $k_2(a)$, and quadratic in the first order term $\omega(a)$ (we go back from $x_0$ to $a$ to recover the initial radial $r$ for convenience). In short, the structure of the equation after the integration of (C.25) from the south to the north pole of $S_0$ is the following

$$\mathcal{I}_{l=0} + (M^2 - y^2)\left\{\mathcal{I}_{l=2} + \mathcal{I}_{(l=1)^2}\right\} = 0, \qquad \forall y \in (-a+M, a-M) \tag{C.45}$$

where $\mathcal{I}_{l=0}$ and $\mathcal{I}_{l=2}$ involve only second order terms of the $l = 0$ and $l = 2$ sectors respectively, and $\mathcal{I}_{(l=1)^2}$ contains squared first order terms.

The equation $\mathcal{I}_{l=0} = 0$ just gives

$$h_0^I(a) + m_0^I(a) + 4\pi \frac{a^2}{a-2M} E(a)\xi_0^H(a) = 0. \tag{C.46}$$

This is, of course, just a relation we have produced for the functions in the interior configuration, we have included a superscript $^I$ for clarity. This relation must be satisfied for the existence of the vacuum (asymptotically flat) exterior, but tells us nothing about the continuity of these functions. However, we know the exterior solution in the same class of coordinates (5.3) at second order for the $l = 0$ sector (see (6.42), (6.43)), which evaluated at $r = a$ can expressed here as

$$\begin{aligned}(a-2M)m_0^E(a) &= \delta M - \frac{J^2}{a^3},\\ h_0^E(a) &= -\frac{\delta M}{a-2M} + \frac{J^2}{a^3(a-2M)}.\end{aligned}$$

This obviously implies the following relation

$$h_0^E(a) + m_0^E(a) = 0.$$

Clearly, if $h_0^I(a) = h_0^E(a)$ is to be satisfied, then

$$m_0^I(a) + 4\pi \frac{a^2}{a-2M} E(a)\xi_0^H(a) = m_0^E(a)$$

must hold. It is now a matter of checking that these $h_0$ and $m_0$'s (in fact $re^{-\lambda}m_0$) correspond to those used in Hartle's framework (4.1) (see Section 7.5), and therefore the "continuity" of $h_0^H$ is incompatible with the "continuity" of $m_0^H$ (recall that $\lambda^I(a) =$

$\lambda^E(a)$). In terms of $m_0^{H(I/E)}(a) = (a-2M)m_0^{I/E}(a)$, the jump, using (4.31) in the last equality

$$m_0^{H(I)}(a) - m_0^{H(E)}(a) = -4\pi a^2 E(a)\xi_0^H(a) = -4\pi\frac{a^3}{M}(a-2M)E(a)p_0^{H*}(a) \qquad \text{(C.47)}$$

which agrees with the previously found jump (7.52).

For completeness we include the results for the $l = 2$ sector. The part of (C.45) proportional to $(M^2 - y^2)$ can be written in terms of $h_2$, $k_2$ and $\omega$. For convenience, the result is given with the coefficient of $k_2$ normalized to 1, and in terms of the associated Legendre functions of the second kind and, and for simplicity, of the coordinate $x_0$. It reads

$$\begin{aligned} k_2^I(x_0) &- \left(\frac{2}{\sqrt{x_0^2-1}}\frac{Q_2^1(x_0)}{Q_2^2} - 1\right) h_2^I(x_0) \\ &+ \frac{M^2(x_0+1)^2}{4}\left(\frac{2(x_0+2)}{\sqrt{x_0^2-1}}\frac{Q_2^1(x_0)}{Q_2^2(x_0)} - 1\right)\omega^2(x_0) = 0. \end{aligned} \qquad \text{(C.48)}$$

It is direct to check now that $h_2^E$ and $k_2^E$ yield the very same relation if we combine the explicit solutions (6.61) and (6.62) to get rid of the constant $A$ therein. Hence, (C.48) is compatible with the continuity of the functions $h_2$ and $k_2$.

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