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Research Article

On Weak Contractive Cyclic Maps in Generalized Metric Spaces and Some Related Results on Best Proximity Points and Fixed Points

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This paper discusses the properties of convergence of sequences to limit cycles defined by best proximity points of adjacent subsets for two kinds of weak contractive cyclic maps defined by composite maps built with decreasing functions with either the so-called r-weaker Meir-Keeler or (r, r_0)-stronger Meir-Keeler functions in generalized metric spaces. Particular results about existence and uniqueness of fixed points are obtained for the case when the sets of the cyclic disposal have a nonempty intersection. Illustrative examples are discussed.

1. Introduction

The background literature on best proximity points and associated convergence properties in cyclic contractions and proximal contractions in the framework of fixed point theory is abundant. See, for instance, [1–21] and references therein. The literature includes related studies on cyclic contractions and cyclic weak contractions and proximal contractions [1–14, 18–21] and proximal weak contractions [15–17]. See also [22–25] for related results. On the other hand, fixed point theory has a wide amount of applications, for instance, in the study of stability of dynamic systems and differential and difference equations. See, for instance, [21, 22, 26]. In this context, the relevance of cyclic contractions and cyclic nonexpansive mappings is also of interest when strips of the solutions of dynamic systems or difference equations have to lie in different time intervals or due to control actions or external events in distinct defined sets.

The study of contractions in metric and quasi-metric spaces and in generalized metric and quasi-metric spaces has been focused on in a number of papers. See, for instance, [1–4] and references therein. A group of the obtained results are based on the existing background literature on Meir-Keeler contractive-type results. See, for instance, [5, 6]. In particular, the existence of periodic fixed point theorems of weak contractions in the setting of generalized quasi-metric spaces has been studied in [2], while the existence of fixed points for weak contraction mappings in complete generalized metric spaces has been investigated in [3]. The paper has a section of preliminaries where the concepts of Meir-Keeler functions, weaker Meir-Keeler functions, and stronger Meir-Keeler functions are generalized "ad hoc" to be used to define weak generalized contractive mappings involving subsets of a generalized metric space which do not intersect in general. In this context, appropriate nondecreasing functions $\varphi : [0, \infty) \to [0, \infty)$, generalized weaker Meir-Keeler functions $\phi : [D, \infty) \to [D, \infty)$, and stronger Meir-Keeler functions $\psi : [0, \infty) \to [0, 1)$ are used to define the generalized $(\phi - \varphi)$- and $(\phi - \psi)$-weak p-cyclic contraction mappings defined and studied in this paper. Section 3 gives and proves a set of main results on $(\phi - \varphi)$-weak p-cyclic contraction mappings and on generalized $(\phi - \psi)$-weak p-cyclic contraction mappings. Such results are related to boundedness and to convergence properties of generalized distances of sequences of points built through generalized $(\phi - \varphi)$-weak and through generalized $(\phi - \psi)$-weak contractive cyclic maps, either in adjacent subsets or in the same subset, and also on the convergences of sequences either to best proximity points or to fixed points in the case when the subsets of the cyclic disposal intersect. Some illustrative examples adapted to the stated and proved results are also discussed.

2. Preliminaries

Let $\mathbf{Z}$ and $\mathbf{R}$ be the sets of integer numbers and real numbers, respectively, and define their subsets: $\mathbf{Z}_{0+} = \mathbf{Z}_+ \cup \{0\}$, $\mathbf{Z}_+ = \{z \in \mathbf{Z} : z > 0\}$, $\overline{p} = \{1, 2, \ldots, p\}$, $\mathbf{R}_{0+} = \mathbf{R}_+ \cup \{0\}$, and $\mathbf{R}_+ = \{z \in \mathbf{R} : z > 0\}$.

Definition 1. For some given $r \in \mathbf{R}_{0+}$, a mapping $\phi : [r, \infty) \to [r, \infty)$ is said to be a r-weaker Meir-Keeler function if, for each real number $\eta(> r) \in \mathbf{R}_+$, there exists a real number $\delta = \delta(\eta) \in \mathbf{R}_+$ such that $\phi^{n_0}(t) < \eta$ for some $n_0 = n_0(\eta) \in \mathbf{Z}_+$, $\forall t \in [\eta, \eta + \delta)$.

Definition 1 generalizes the two existing definitions below.

Definition 2 (see [1, 2]). A mapping $\phi : [0, \infty) \to [0, \infty)$ which is a 0-weaker Meir-Keeler function is said to be a weaker Meir-Keeler function.

Definition 3 (see [1]). A mapping $\phi : [0, \infty) \to [0, \infty)$ which is a weaker Meir-Keeler function for $n_0 = 1$ is said to be a Meir-Keeler function.

Definition 4. For some given $r, r_0(< 1) \in \mathbf{R}_{0+}$, a mapping $\psi : [r, \infty) \to [r_0, 1)$ is said to be a (r, r_0)-stronger Meir-Keeler function if, for each real number $\eta \in \mathbf{R}_+$, there exist real numbers $\delta = \delta(\eta) \in \mathbf{R}_+$ and $\gamma = \gamma(\eta) \in [0, 1)$ such that $\psi(t) < \gamma$, $\forall t \in [\eta, \eta + \delta)$.

Definition 4 generalizes the existing definition below.

Definition 5 (see [1, 2]). A mapping $\varphi : [0, \infty) \to [0, 1)$ which is a 0-stronger Meir-Keeler function is said to be a stronger Meir-Keeler function.

Through the paper, we will use the mappings ϕ, φ, and ψ which belong to the sets of functions defined below.

Definition 6. For some given $r \in \mathbf{R}_{0+}$, the class Φ_r is the set of r-weaker Meir-Keeler functions $\phi : [r, \infty) \to [r, \infty)$ which satisfy the following:

(ϕ_1) $\phi(t) > r$ for $t > r$ and $\phi(r) = r$.

(ϕ_2) $\{\phi^n(t)\}$ is decreasing for all $t \in [r, \infty)$.

(ϕ_3) For $\{t_n\} \subset [r, \infty)$, one has

- (ϕ_{3_1}) $\limsup_{n\to\infty} \phi(t_n) < \theta$ if $\lim_{n\to\infty} t_n = \theta$ for any given real number $\theta > r$, and
- (ϕ_{3_2}) there exists $\lim_{n\to\infty} \phi(t_n) = r$ if $\lim_{n\to\infty} t_n = r$.

Definition 7. The class Γ_r is the set of nondecreasing functions $\varphi : [0, \infty) \to [0, \infty)$ which satisfy the following:

(φ_1) $\varphi(t) > r$ for $t > r$ and $\varphi(t) = t$ for $t \in [0, r]$.

(φ_2) φ is subadditive; that is, for every $\alpha_1, \alpha_2 \in [r, \infty)$, $\varphi(\alpha_1 + \alpha_2) \le \varphi(\alpha_1) + \varphi(\alpha_2)$.

(φ_3) For all $\{t_n\} \subset [r, \infty)$, $\lim_{n\to\infty} t_n = r$ if and only if $\lim_{n\to\infty} \phi(t_n) = r$.

Definition 8. The class $\psi \in \Psi_r(r_0)$ is the set of (r, r_0)-stronger Meir-Keeler functions $\psi : [r, \infty) \to [r_0, 1)$ for some real constant $r_0 \in (0, 1)$ which satisfy the following:

(ψ_1) $\psi(t) > r_0$ for $t > r$ and $\psi(r) = r_0$.

Definition 9 (see [2, 3]). Let X be a nonempty set. A generalized metric (g.m.) is a mapping $d : X \times X \to \mathbf{R}$ which satisfies

(1) $d(x, y) \ge 0$, $\forall x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$, $\forall x, y \in X$;

(3) $d(x, y) \le d(y, \omega) + d(\omega, z) + d(z, y)$, $\forall x, y \in X$, $\forall \omega, z(\ne \omega) \in X - \{x, y\}$.

Definition 10 (see [2, 3]). Let X be a nonempty set and let $d : X \times X \to \mathbf{R}_{0+}$ be a g.m. on X. Then, (X, d) is said to be a generalized metric space (g.m.s.).

Some basic considerations and properties on a g.m.s. (X, d) are now quoted from [2] to be then invoked in the body of this paper. Let (X, d) be a g.m.s. Then, $\{x_n\} \subset X$ is said to be g.m.s. convergent to $x \in X$ if, for each given $\varepsilon \in \mathbf{R}_+$, $\exists n_0 = n_0(\varepsilon) \in \mathbf{Z}_{0+}$ such that $d(x_n, x) < \varepsilon$, $\forall n(> n_0) \in \mathbf{Z}_+$, and this is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If, for each given $\varepsilon \in \mathbf{R}_+$, $\exists n_0 = n_0(\varepsilon) \in \mathbf{Z}_{0+}$ such that $d(x_n, x_{n+m}) < \varepsilon$, $\forall n(> n_0) \in \mathbf{Z}_+$, $\forall m \in \mathbf{Z}_{0+}$, then $\{x_n\}$ is called a g.m.s. Cauchy sequence in X. If every g.m.s. Cauchy sequence in X is g.m.s. convergent in X, then (X, d) is called a complete g.m.s.. It has been pointed out in [2] that a g.m.s. Cauchy sequence is not necessarily a Cauchy sequence and that a g.m.s. convergent sequence is not necessarily either Cauchy or a convergent sequence.

Example 11 (see [2]). Consider the set $X = \{jt : j \in \overline{5}\}$ for some given $t \in \mathbf{R}_+$ and define $d : X \times X \to \mathbf{R}_{0+}$ as follows for some given $\gamma \in \mathbf{R}_+$:

$$\begin{aligned} &d(x, x) = 0, \quad \forall x \in X \\ &d(x, y) = d(y, x), \quad \forall x, y \in X \\ &d(t, 2t) = 3\gamma; \\ &d(it, 3t) = \gamma, \quad \forall i \in \overline{2} \\ &d(it, 4t) = 2\gamma, \quad \forall i \in \overline{3}; \\ &d(it, 5t) = \left(\frac{3}{2}\right)\gamma; \quad \forall i \in \overline{4}. \end{aligned} \tag{1}$$

Then, $d : X \times X \to \mathbf{R}_{0+}$ is a g.m. and then (X, d) is a g.m.s.. However, $d : X \times X \to \mathbf{R}_{0+}$ is not a metric, and then (X, d) is not a g.m.s., since $d(t, 2t) = 3\gamma > d(t, 3t) + d(3t, 2t) = 2\gamma$.

3. Best Proximity Point and Fixed Point Theorems

We now get some results on $(\phi - \varphi)$-weak contraction mappings and some results on related best proximity points. Given a nonempty abstract set X with $p(\geq 2)$ nonempty subsets X_i, $\forall i \in \overline{p}$, we say that a self-mapping $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ is a p-cyclic self-mapping if $T(X_i) \subseteq X_{i+1}$, $\forall i \in \overline{p}$, with the notation convention that $X_{i+np} \equiv X_i$, $\forall i \in \overline{p}$, $\forall n \in \mathbf{Z}_{0+}$.

In fact, if we extend the above definition to the case $p = 1$, we find trivially that $T : X \mid X_1 \to X_1$ can be considered as an 1-cyclic mapping if $\varnothing \neq T(X_1) \subseteq X_1 \subseteq X$.

Definition 12. Let (X, d) be a g.m.s., let X_i be nonempty subsets of X, having a common distance in-between adjacent subsets $d(X_i, X_{i+1}) = D$, $\forall i \in \overline{p}$, and let $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ be a p-cyclic self-mapping satisfying

$$\begin{aligned} &\varphi(d(Tx, Ty)) \leq \phi(\varphi(d(x, y)));\\ &\qquad \forall (x, y) \in X_i \times X_{i+1},\ \forall i \in \overline{p} \end{aligned} \tag{2}$$

for some $\varphi \in \Gamma_D$ and some $\phi \in \Phi_D$. Then, T is said to be a generalized $(\phi - \varphi)$-weak p-cyclic contraction mapping.

Theorem 13. *Let (X, d) be a g.m.s. and let $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ be a generalized $(\phi - \varphi)$-weak p-cyclic contraction mapping for some $\phi \in \Phi_D$ and some $\varphi \in \Gamma_D$. Then, the following properties hold:*

(i)

$$\begin{aligned} &\lim_{n\to\infty} \phi^{np+j}\left(\varphi\left(d\left(x_0, x_{mp+1}\right)\right)\right) = D;\\ &\qquad \forall m \in \mathbf{Z}_{0+},\ \forall j \in \overline{p-1} \cup \{0\},\\ &\lim_{m\to\infty}\lim_{n\to\infty} \varphi\left(d\left(x_{mp+j}, x_{(m+n)p+j+1}\right)\right)\\ &\quad = \lim_{n\to\infty} \varphi\left(d\left(x_{np+j}, x_{(\ell+n)p+j+1}\right)\right) = D;\\ &\qquad \forall \ell \in \mathbf{Z}_{0+},\ \forall j \in \overline{p-1} \cup \{0\},\\ &\lim_{n,m\to\infty} d\left(x_{mp+j}, x_{(m+n)p+j+1}\right)\\ &\quad = \lim_{n\to\infty} d\left(x_{np+j}, x_{(\ell+n)p+j+1}\right) = D;\\ &\qquad \forall \ell \in \mathbf{Z}_{0+},\ \forall j \in \overline{p-1} \cup \{0\}, \end{aligned} \tag{3}$$

for any sequence $\{x_n\}$ constructed from $x_{n+1} = Tx_n$, $\forall n \in \mathbf{Z}_{0+}$ for some given initial point $x_0 \in \bigcup_{i\in\overline{p}} X_i$.

(ii) Any sequence $\{Tx_n\}$ built from any given initial point $x_0 \in \bigcup_{i\in\overline{p}} X_i$ is bounded.

(iii) Assume, in addition, that (X, d) is a complete g.m.s. and that $z_i \in X_i$ has a best proximity point from X_i to X_{i+1} (i.e., $d(z_i, X_{i+1}) = D$) for some given $i \in \overline{p}$ and that X_{i+1} is approximatively compact with respect to X_i. Then,

$$\begin{aligned} &\{T^{np}x_0\} \longrightarrow z_i,\\ &\{T^{np}x_1\} \longrightarrow z_i,\\ &\{T^{np+1}x_0\} \longrightarrow Tz_i,\\ &\{T^{np+}x_1\} \longrightarrow Tz_i,\\ &\lim_{n\to\infty} \varphi\left(d\left(T^{np}x_0, T^{(m+n)p}x_1\right)\right)\\ &\quad = \lim_{n\to\infty} d\left(T^{np}x_0, T^{(m+n)p}x_1\right) = D,\\ &\lim_{n,\ell\to\infty} \varphi\left(d\left(T^{np}x_0, T^{(n+\ell)p}x_1\right)\right)\\ &\quad = \lim_{n,\ell\to\infty} d\left(T^{np}x_0, T^{(n+\ell)p}x_1\right) = D,\\ &\lim_{n\to\infty} \varphi\left(d\left(T^{np}x_0, T^{(m+n)p+1}x_1\right)\right)\\ &\quad = \lim_{n\to\infty} d\left(T^{np}x_0, T^{(m+n)p+1}x_1\right) = D,\\ &\lim_{n,m\to\infty} \varphi\left(d\left(T^{np}x_0, T^{mp+1}x_1\right)\right)\\ &\quad = \lim_{n,\ell\to\infty} d\left(T^{np}x_0, T^{mp+1}x_1\right) = D,\\ &\lim_{n,m\to\infty} \varphi\left(d\left(T^{np}x_0, T^{mp}x_1\right)\right) = \lim_{n,m\to\infty} d\left(T^{np}x_0, T^{mp}x_1\right)\\ &\quad = 0, \end{aligned} \tag{4}$$

$\forall x_0, x_1 \in X_j$, $\forall j \in \overline{p}$, $\forall m \in \mathbf{Z}_{0+}$, and for any given $\varepsilon \in \mathbf{R}_+$, and there is $N_1 = N_1(\varepsilon) \in \mathbf{Z}_+$ such that, for any positive integers $m > n > N_1$, one has $d(T^{np}x_0, T^{mp+1}x_1) < D + \varepsilon$, $d(T^{np}x_0, T^{mp}x_1) < \varepsilon$, $\varphi(d(T^{np}x_0, T^{mp+1}x_1)) < D + \varepsilon$, and $\varphi(d(T^{np}x_0, T^{mp}x_1)) < \varepsilon$.

Also, all the best proximity points $z_{i+1} = Tz_i = T^j z_{i+1-j}$ from X_i to X_{i+1} are each them unique in X_{i+1} if one of them z_j, for some $j \in \overline{p}$, is unique in X_j and X_i is closed, $\forall i \in \overline{p}$. Also, each best proximity point is also a fixed point of the respective composite mapping $T^p : \bigcup_{j\in\overline{p}} X_j \mid X_i \to \bigcup_{j\in\overline{p}} X_j$, $\forall i \in \overline{p}$, and then p-periodic fixed points of $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$.

Proof. Since $x_0 \in \bigcup_{i\in\overline{p}} X_i$, we can consider equivalently x_0 to be an arbitrary point of X_i for some given arbitrary $i \in \overline{p}$ and we can define the sequence $\{x_n\}$ inductively by $x_{n+1} = Tx_n$,

$\forall n \in \mathbf{Z}_{0+}$. Since $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ is a $(\phi-\varphi)$-weak-cyclic p contraction mapping, one gets from (2) by induction for each $n, m \in \mathbf{Z}_{0+}$ that

$$\begin{aligned}
&\varphi\left(d\left(x_{np+j}, x_{(n+m)p+j+1}\right)\right)\\
&\quad=\varphi\left(d\left(Tx_{np+j-1}, Tx_{(n+m)p+j}\right)\right)\\
&\quad\le\phi\left(\varphi\left(d\left(x_{np+j-1}, x_{(n+m)p+j}\right)\right)\right)\\
&\quad\le\phi\left(\phi\left(\varphi\left(d\left(x_{np+j-2}, x_{(n+m)p+j-1}\right)\right)\right)\right)\\
&\quad=\phi^2\left(\varphi\left(d\left(x_{np+j-2}, x_{(n+m)p+j-1}\right)\right)\right)\cdots\\
&\quad\le\phi^{np+j}\left(\varphi\left(d\left(x_0, x_{mp+1}\right)\right)\right);\\
&\qquad\qquad\forall m, n \in \mathbf{Z}_{0+};\ \forall j \in \overline{p-1}\cup\{0\},
\end{aligned}\tag{5}$$

where $x_{np}, x_{(n+m)p} \in X_i$, $x_{mp+1} \in X_{i+1}$, $x_{np+j}, x_{(n+m)p+j} \in X_{i+j}$, and $x_{np-j} \in X_{i-j+p}$, $\forall n, m \in \mathbf{Z}_{0+}$, $\forall j \in \overline{p-1}\cup\{0\}$. Since $\phi \in \Phi_D$, one has from property (ϕ_2) that $\{\phi^n(t)\}$ is decreasing for all $t \in [D,\infty)$ so that $\{\phi^{np+j}(\varphi(d(x_0, x_{mp+1})))\}$ is decreasing and converges to some limit $\eta_{0j} \in \mathbf{R}_{0+}$, $\forall j \in \overline{p-1}\cup\{0\}$. It is now proved that all the limits $\eta_{0j} = D$, $\forall j \in \overline{p-1}\cup\{0\}$. Since $\phi \in \Phi_D$, it is also a D-weaker Meir-Keeler function $\phi : [D,\infty) \to [D,\infty)$ so that, for each real number $\eta(> D) \in \mathbf{R}_+$, there exists a real number $\delta_j = \delta_j(\eta) \in \mathbf{R}_+$ such that $\phi^{np+j}(t) < \eta$, $\forall n(\ge n_{0j}) \in \mathbf{Z}_+$ for some $n_{0j} = n_{0j}(\eta) \in \mathbf{Z}_+$, for any given $j \in \overline{p-1}\cup\{0\}$, $\forall t \in [\eta, \eta+\delta_j)$. Thus, $\eta = D+\varepsilon$ for any given arbitrary $\eta(> D) \in \mathbf{R}_+$ such that $\varepsilon = \eta - D(\in \mathbf{R}_+)$ is also arbitrary; one has for each given $k \in \overline{p-1}\cup\{0\}$ that if $x_k = Tx_{k-1} = T^k x_0$, then

$$\begin{aligned}
&D \le \phi^{np+j-k}\left(\varphi\left(d\left(x_k, x_{mp+k+1}\right)\right)\right) < D+\varepsilon;\\
&\qquad\qquad\forall j, k \in \overline{p-1}\cup\{0\}
\end{aligned}\tag{6}$$

since $\varphi : [0,\infty) \to [0,\infty)$ is nondecreasing, $(d(x_k, x_{mp+k+1})) \ge D$ and $\varphi(t) \ge D$ for $t \ge D$ with equality standing if and only if $t = D$. Thus, there exist the p identical limits $\lim_{n\to\infty}\phi^{np+j}(\varphi(d(x_{np+k}, x_{(n+m)p+k+1}))) = D$, $\forall j, k \in \overline{p-1}\cup\{0\}$. Then, one gets from (5) and the constraint (ϕ_{3_2}) of the class Φ_D that

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty}\varphi\left(d\left(x_{mp+j}, x_{(m+n)p+j+1}\right)\right)\\
&\quad=\lim_{n\to\infty}\varphi\left(d\left(x_{np+j}, x_{(\ell+n)p+j+1}\right)\right) = D;\\
&\qquad\qquad\forall \ell \in \mathbf{Z}_{0+},\ \forall j \in \overline{p-1}\cup\{0\}.
\end{aligned}\tag{7}$$

This also implies from the constraint (φ_3) of the class Γ_D that $\lim_{n,m\to\infty} d(x_{mp+j}, x_{(m+n)p+j+1}) = \lim_{n\to\infty} d(x_{np+j}, x_{(\ell+n)p+j+1}) = D$, $\forall \ell \in \mathbf{Z}_{0+}$, $\forall j \in \overline{p-1}\cup\{0\}$.

Property (i) has been proved.

To prove Property (ii), we use contradiction arguments by assuming that some sequence $\{T^{pn}x_0\}$, generated by $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ from any $x_0 \in \bigcup_{i\in\overline{p}} A_i$, is unbounded. Thus, for any given real constant $A \in \mathbf{R}_+$, there is $N = N(A) \in \mathbf{Z}_{0+}$ such that, for any $n(\in \mathbf{Z}_{0+}) \ge N$, $d(x_0, T^{pn}x_0) > A$. There is also some $B_0 = B_0(N, A)(> A) \in \mathbf{R}_+$ such that, for any $B(> B_0) \in \mathbf{R}_+$, there is $N_1 = N_1(A)(\ge N) \in \mathbf{Z}_{0+}$ such that $B \ge d(x_0, T^{pn}x_0) > A$ for all $n \in [N, N_1] \cap \mathbf{Z}_{0+}$. This is trivial since, by defining $\alpha = d(x_0, T^{pN}x_0) - A$, we can choose any real constant $B \ge B_0 = A + \alpha$ such that, for some nonempty interval of positive integers $[N, N_1]$, $B \ge d(x_0, T^{pn}x_0) > A$ since the inequality holds by construction for the case $N = N_1$. So if $\{T^{pn}x_0\}$ is unbounded, then there is some subsequence $\{T^{pn_k}x_0\}$ which diverges so that there are some strictly increasing sequence $\{n_k\} \subseteq \mathbf{Z}_{0+}$ and some strictly increasing sequence $\{A_{n_k}\} \subseteq \mathbf{R}_{0+}$ such that $A_{n_{k+1}} \ge d(x_0, T^{pn_k}x_0) > A_{n_k}$ and $d(x_0, T^{pn_{k+1}}x_0) > A_{n_{k+1}}$. Then, there is a strictly increasing sequence of natural numbers $\{n_k\}$ and a strictly increasing sequence of positive real numbers $\{A_{n_k}\}$ such that

$$\begin{aligned}
&d\left(x_0, T^{pn_{k+1}}x_0\right) > A_{n_{k+1}} > A_{n_k},\\
&\qquad A_{n_{k+1}} \ge d\left(x_0, T^{pn_k}x_0\right) > A_{n_k};\\
&\qquad\qquad\forall k \in \mathbf{Z}_{0+}
\end{aligned}\tag{8}$$

and then one gets

$$\begin{aligned}
&\frac{d\left(x_0, T^{pn_{k+1}}x_0\right)}{A_{n_k}} > \frac{d\left(x_0, T^{pn_{k+1}}x_0\right)}{d\left(x_0, T^{pn_k}x_0\right)} > \frac{A_{n_{k+1}}}{d\left(x_0, T^{pn_k}x_0\right)}\\
&\qquad\ge 1;\quad \forall k \in \mathbf{Z}_{0+}.
\end{aligned}\tag{9}$$

On the other hand, one has from the rectangular inequality of the g.m.s. (X, d)

$$\begin{aligned}
&\left|d\left(x_0, T^{pn_{k+1}}x_0\right) - d\left(x_0, T^{pn_k}x_0\right)\right|\\
&\quad\le d\left(T^{pn_{k+2}}x_0, T^{pn_{k+1}}x_0\right) + d\left(T^{pn_{k+2}}x_0, T^{pn_{k+1}}x_0\right),
\end{aligned}\tag{10}$$

$\forall k \in \mathbf{Z}_{0+}$ which leads to $\lim_{k\to\infty}|d(x_0, T^{pn_{k+1}}x_0) - d(x_0, T^{pn_k}x_0)| = 0$ from Property (i). Thus, either $\{d(x_0, T^{pn_k}x_0)\} \to 0$ or there is some positive real sequence $\{\lambda_{n_k}\} \to 1$ such that $d(x_0, T^{pn_{k+1}}x_0) = \lambda_{n_k} d(x_0, T^{pn_k}x_0)$. If $\{d(x_0, T^{pn_k}x_0)\} \to 0$, then $\{T^{pn_k}x_0\}$ is a bounded subsequence of $\{T^{pn}x_0\}$, a contradiction to its claimed unboundedness. Otherwise, if $\{d(x_0, T^{pn_k}x_0)\}$ does not converge to zero while $\{\lambda_{n_k}\} \to 1$, with $\lambda_{n_k} = d(x_0, T^{pn_{k+1}}x_0)/d(x_0, T^{pn_k}x_0)$, then this contradicts (10). As a result, $\{T^{pn}x_0\}$ for any $x_0 \in \bigcup_{i\in\overline{p}} X_i$ and Property (ii) has been proved.

It remains to prove Property (iii). Since X_{i+1} is approximatively compact with respect to X_i if $d(x, y_n) \to d(x, X_{i+1})$ as $n \to \infty$ for some $x \in X_i$ and $\{y_n\} \subset X_{i+1}$, then $\{y_n\}$ has a convergent subsequence $\{y_{n_k}\} \subseteq \{y_n\}$ [6]. Since $z_i \in X_i$ is a best proximity point from X_i to X_{i+1}, $d(z_i, Tz_i) = d(z_i, X_{i+1}) = D$. If $z_i \in X_i$ is the unique best proximity point from X_i to X_{i+1}, then it is a p-periodic point of $T : \bigcup_{j\in\overline{p}} X_j \to \bigcup_{j\in\overline{p}} X_j$ and a fixed point of $T^p : \bigcup_{j\in\overline{p}} \mid X_i \to \bigcup_{j\in\overline{p}} X_j$. This follows by reformulating (5) with initial points $T^p z_i \in X_i$ and $T^{p+1} z_i \in X_{i+1}$ with $z_i \in X_i$ being the unique best proximity point from X_i to

X_{i+1}. One concludes from the properties of the classes Φ_D and Γ_D that $d(T^{np}z_i, T^{np+1}z_i) \to d(T^p z_i, X_{i+1}) = D$ and $d(T^{np}z_i, T^{np+1}z_i) \to d(z_i, X_{i+1}) = D$. Since X_i has a unique best proximity point to X_{i+1}, one concludes that $T^p z_i = z_i$. Next, we prove that $d(T^p z_i, T^{p+1}z_i) = d(z_i, X_{i+1}) = D$. Assume that this property is not true. Thus, there is some $\varepsilon = \varepsilon(n) \in \mathbf{R}_+$ for $n \in \mathbf{Z}_{0+}$ with $\varepsilon_0 = \liminf_{n\to\infty}\varepsilon(n) > 0$ such that one gets from (5) and the constraints (ϕ_2) and (ϕ_{3_2}) for the class Φ_D and the constraint (φ_3) for the class Γ_D that

$$\begin{aligned}\varphi\left(d\left(T^{np}z_i, T^{np+1}z_i\right)\right) &\le \phi^{(n-1)p}\left(\varphi\left(d\left(T^p z_i, T^{p+1}z_i\right)\right)\right) = D + \varepsilon(n) \\ &\le \phi^{np}\left(\varphi\left(d\left(z_i, Tz_i\right)\right)\right)\end{aligned} \tag{11}$$

which leads by taking limits as $n \to \infty$ to the contradiction $D + \varepsilon_0 \le D$. Then, $d(T^{np}z_i, T^{np+1}z_i) = d(T^p z_i, T^{p+1}z_i) = D$, $\forall n \in \mathbf{Z}_{0+}$. We now use again (5) with $x_0 = z_i = T^p z_i = T^{np}z_i \in X_i$, $\forall n \in \mathbf{Z}_{0+}$ and an arbitrary $x_1 \in X_{i+1}$ to yield

$$\begin{aligned}\varphi\left(d\left(z_i, T^{np+1}x_1\right)\right) &\le \phi\left(\varphi\left(d\left(z_i, T^{np}x_1\right)\right)\right) \\ &= \phi^p\left(\varphi\left(d\left(z_i, T^{(n-1)p}x_1\right)\right)\right)\cdots \\ &\le \phi^{np}\left(\varphi\left(d\left(z_i, x_1\right)\right)\right); \\ &\qquad \forall m, n \in \mathbf{Z}_{0+}\end{aligned} \tag{12}$$

and one concludes that $\phi^{np}(\varphi(d(z_i, x_1))) \to D$, $\limsup_{n\to\infty}\varphi(d(z_i, T^{np+1}x_1)) \le D$, and $D \le \varphi(d(z_i, T^{np+1}x_1)) \le \limsup_{n\to\infty}\varphi(d(z_i, T^{np+1}x_1)) \le D$ since $d(z_i, T^{np+1}x_1) \ge D$, $\forall n \in \mathbf{Z}_{0+}$, and then $\varphi(d(z_i, T^{np+1}x_1)) \to D$ as $n \to \infty$ and $d(z_i, T^{np+1}x_1) \to d(z_i, X_{i+1}) = D$ as $n \to \infty$ since $\phi \in \Phi_D$ and $\varphi \in \Gamma_D$. Since X_{i+1} is approximatively compact with respect to X_i, there is a convergent subsequence $\{T^{n_k p+1}x_1\}(\to z_{i+1}) \subseteq \{T^{np+1}x_1\} \subset X_{i+1}$ for the arbitrary given $i \in \overline{p}$.

It is now proved that $\{T^{np+1}x_1\} \to z_{i+1}$ with $z_{i+1} \in X_{i+1}$ for any $x_1 \in X_i$. Assume that this is not the case. Then, there exists a sequence $\{m_k\}$ with $m_k(\in \mathbf{Z}_{0+}) > n_k \ge n_{0k}$ and some $n_{0k} \in \mathbf{Z}_{0+}$such that the subsequence $\{T^{m_k p+1}x_1\} \subseteq \{T^{np+1}x_1\} \subset X_{i+1}$ does not converge to z_{i+1}. Since T is single-valued, if $\{T^{m_k p+1}x_1\}$ does not converge to z_{i+1}, then it does not converge. It cannot have either a convergent subsequence $\{T^{m_{k_j}p+1}x_1\}(\to \hat{z}_{i+1} \ne z_{i+1}) \subseteq \{T^{m_k p+1}x_1\} \subseteq \{T^{np+1}x_1\} \subset X_{i+1}$ and since then z_i would have two distinct images in X_{i+1} which is impossible.

Thus, one gets from (12) that

$$\begin{aligned}D + \varepsilon &\le \varphi\left(d\left(z_i, T^{m_k p+1}x_1\right)\right) \\ &\le \phi^{n_k p}\left(\varphi\left(d\left(z_i, T^{(m_k-n_k)p+1}x_1\right)\right)\right); \\ &\qquad \forall m, n \in \mathbf{Z}_{0+}\end{aligned} \tag{13}$$

for the given $i \in \overline{p}$ and some subsequences $\{m_k\}$, $\{n_k\}$ in $\mathbf{Z}_{0+}$ with $m_k > n_k > N_0 = N_0(\varepsilon)$; since $\phi \in \Phi_D$ is a D-weaker Meir-Keeler function which satisfies (ϕ_2) and (ϕ_{3_2}), $\varphi \in \Gamma_D$ is defined from $[0, \infty)$ to $[0, \infty)$, nondecreasing, and satisfies (φ_1) and (φ_3). Thus, one gets from (13) the following contradiction:

$$D + \varepsilon \le \lim_{k\to\infty}\phi^{n_k p}\left(\varphi\left(d\left(z_i, T^{(m_k-n_k)p}x_1\right)\right)\right) = D \tag{14}$$

so that $\{T^{m_k p+1}x_1\} \to z_{i+1} = Tz_i$, irrespective of the initial point $x_1 \in X_i$, and z_{i+1} is unique if z_i is unique. Since the same contradiction arguments can be used for any claimed nonconvergent subsequence of $\{T^{np+1}x_1\}$, one concludes that any such a sequence as well as the whole sequence converges $\{T^{np+1}x_1\} \to z_{i+1}$ for the given $i \in \overline{p}$. The sequence is a g.m.s. Cauchy sequence since it is convergent and (X, d) is a complete g.m.s.. Since X_{i+1} is closed, then $z_{i+1} \in X_{i+1}$ for the given $i \in \overline{p}$ and it is unique if z_i is unique. Thus, the set of best proximity points $\{z_i \in X_i : i \in \overline{p}\}$ is unique if any of them is unique.

It is now proved that $\lim_{n,m\to\infty}\varphi(d(T^{np}x_0, T^{mp}x_1)) = D$, $\forall x_0 \in X_i$, $\forall x_1 \in X_{i+1}$, and $\forall i \in \overline{p}$. Proceed by contradiction by assuming that there is $\varepsilon \in \mathbf{R}_+$ and some subsequences of nonnegative integers $\{m_k\}$, $\{n_k\}$, and $\{\ell_k\}$ such that one gets for any two distinct initial points $x_0, x_1 \in X_i$ from the subadditivity property (φ_2) of the class Γ_D and the rectangular inequality of the generalized metric

$$\begin{aligned}D + \varepsilon &\le \varphi\left(d\left(T^{n_k p}x_0, T^{(m_k+n_k)p+1}x_1\right)\right) \\ &\le \varphi\left(d\left(T^{n_k p}x_0, T^{(m_k+n_k+\ell_k)p+1}x_0\right)\right) \\ &\quad + \varphi\left(d\left(T^{(m_k+n_k+\ell_k)p+1}x_0, T^{(m_k+n_k+\ell_k+1)p+1}x_1\right)\right) \\ &\quad + \varphi\left(d\left(T^{(m_k+n_k+\ell_k+1)p+1}x_1, T^{(m_k+n_k)p+1}x_1\right)\right).\end{aligned} \tag{15}$$

Note that $\varphi(d(T^{n_k p}x_0, T^{(m_k+n_k+\ell_k)p+1}x_0)) \to D$ as $k \to \infty$ from Property (i) and also

$$\begin{gathered}\left\{T^{(m_k+n_k)p+1}x_1\right\} \longrightarrow Tz_i, \\ \left\{T^{(m_k+n_k+\ell_k)p+1}x_0\right\} \longrightarrow Tz_i, \\ \left\{T^{(m_k+n_k+\ell_k+1)p+1}x_1\right\} \longrightarrow Tz_i \\ \text{as } k \longrightarrow \infty.\end{gathered} \tag{16}$$

If the convergence of these subsequences fails, then the conclusion got from (12) fails and then Property (i) is not true. For instance, if $\{T^{(m_k+n_k)p+1}x_1\} \to Tz_i$ is false, then $\{d(z_i, T^{(n+m)p+1}x_1)\} \to D$ as $n \to \infty$ fails. Then,

$$\begin{gathered}\varphi\left(d\left(T^{n_k p}x_0, T^{(m_k+n_k+\ell_k)p+1}x_0\right)\right) \longrightarrow D, \\ d\left(T^{n_k p}x_0, T^{(m_k+n_k+\ell_k)p+1}x_0\right) \longrightarrow D \\ \text{as } k \longrightarrow \infty,\end{gathered} \tag{17}$$

$$\begin{gathered}\varphi\left(d\left(T^{(m_k+n_k+\ell_k)p+1}x_0, T^{(m_k+n_k+\ell_k+1)p+1}x_1\right)\right) \longrightarrow 0, \\ d\left(T^{(m_k+n_k+\ell_k)p+1}x_0, T^{(m_k+n_k+\ell_k+1)p+1}x_1\right) \longrightarrow 0\end{gathered}$$

as $k \to \infty$ since $z_{i+1} = Tz_i$ is the unique best proximity point in X_{i+1} (since z_i is the unique best proximity point in X_i) and it has been already proved that any sequence in each X_i has to converge to its best proximity point if unique. Also

$$\begin{gathered}\varphi\left(d\left(T^{(m_k+n_k+\ell_k+1)p+1}x_1, T^{(m_k+n_k)p+1}x_1\right)\right) \longrightarrow 0,\\ d\left(T^{(m_k+n_k+\ell_k+1)p+1}x_1, T^{(m_k+n_k)p+1}x_1\right) \quad \text{as } k \longrightarrow \infty.\end{gathered} \tag{18}$$

Thus, the subsequent contradiction is got if $\varphi(d(T^{n_k p}x_0, T^{(m_k+n_k)p+1}x_1)) \to D$ as $k \to \infty$ is not true for any $x_0, x_1 (\neq x_0) \in X_i$:

$$\begin{aligned} D + \varepsilon &\le \liminf_{k\to\infty} \varphi\left(d\left(T^{n_k p}x_0, T^{(m_k+n_k)p+1}x_1\right)\right) \\ &\le \limsup_{k\to\infty} \varphi\left(d\left(T^{n_k p}x_0, T^{(m_k+n_k)p+1}x_0\right)\right) \\ &\quad + \lim_{k\to\infty} \varphi\left(d\left(T^{(m_k+n_k+\ell_k)p+1}x_0, T^{(m_k+n_k+\ell_k+1)p+1}x_1\right)\right) \\ &\quad + \lim_{k\to\infty} \varphi\left(d\left(T^{(m_k+n_k+\ell_k+1)p+1}x_1, T^{(m_k+n_k)p+1}x_1\right)\right) \\ &= \limsup_{k\to\infty} \varphi\left(d\left(T^{n_k p}x_0, T^{(m_k+n_k)p+1}x_1\right)\right) = D. \end{aligned} \tag{19}$$

Then, one gets

$$\begin{aligned} &\lim_{k\to\infty} \varphi\left(d\left(T^{n_k p}x_0, T^{(m_k+n_k)p+1}x_1\right)\right) \\ &\quad = \lim_{k\to\infty} d\left(T^{n_k p}x_0, T^{(m_k+n_k)p+1}x_1\right) = D \\ &\qquad\qquad\qquad\qquad \text{as } k \longrightarrow \infty \\ &\lim_{k\to\infty} \varphi\left(d\left(T^{n_k p}x_0, T^{(n_k+n)p+1}x_1\right)\right) \\ &\quad = \lim_{k\to\infty} d\left(T^{n_k p}x_0, T^{(n_k+n)p+1}x_1\right) = D, \\ &\qquad\qquad\qquad\qquad \forall m \in \mathbf{Z}_+ \text{ as } k \longrightarrow \infty \end{aligned} \tag{20}$$

and the convergence properties of all subsequences distances and points also hold for the whole sequence so that

$$\begin{aligned} &\left\{T^{np}x_0\right\} \longrightarrow z_i \left\{T^{(n+m)p+1}x_1\right\} \longrightarrow Tz_i, \\ &\left\{T^{(n+m+\ell)p+1}x_0\right\} \longrightarrow Tz_i, \\ &\left\{T^{(m+n+\ell+1)p+1}x_1\right\} \longrightarrow Tz_i; \\ &\qquad\qquad\qquad \forall m, \ell \in \mathbf{Z}_{0+} \text{ as } n \longrightarrow \infty, \\ &\lim_{n,m\to\infty} \varphi\left(d\left(T^{np}x_0, T^{(m+n)p+1}x_1\right)\right) \\ &\quad = \lim_{n,m\to\infty} d\left(T^{np}x_0, T^{(m+n)p+1}x_1\right) = D, \\ &\lim_{n\to\infty} \varphi\left(d\left(T^{np}x_0, T^{(m+n)p+1}x_1\right)\right) \\ &\quad = \lim_{n\to\infty} d\left(T^{np}x_0, T^{(m+n)p+1}x_1\right) = D, \quad \forall m \in \mathbf{Z}_+, \end{aligned} \tag{21}$$

$\forall x_0, x_1 \in X_i$ for $i \in \overline{p}$ such that X_i possesses a unique best proximity point $z_i \in X_i$. The two above second limit identities follow since the sequences $\{m_k\} \subset \mathbf{Z}_{0+}$ and $\{\ell_k\} \subset \mathbf{Z}_{0+}$ may be replaced by any positive integers without altering the got final conclusion. Since $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ is a $(\phi - \varphi)$-weak-cyclic p contraction single-valued mapping, then $z_{i+j} = T^j z_i \in X_{i+j}$ for $1 \ge i+j \le p$ are unique best proximity points, so equivalently $z_i \in X_i$, $\forall i \in \overline{p}$, are all unique so that the above limit properties can be extended for any sequences with given initial points $x_0, x_1 \in X_j$, $\forall j \in \overline{p}$. Since $\{m_k\} \subset \mathbf{Z}_{0+}$, $\{n_k\} \subset \mathbf{Z}_{0+}$, and $\{\ell_k\} \subset \mathbf{Z}_{0+}$ are strictly increasing sequences which can be chosen independently of each other, the above conclusion is equivalently enounced as follows: for each given $\varepsilon \in \mathbf{R}_+$, there is some $N_1 = N_1(\varepsilon) \in \mathbf{Z}_{0+}$ such that if $n_k > m_k > N_1$, then $d(T^{n_k p}x_0, T^{m_k p+1}x_1) < D + \varepsilon$.

It is now proved that $d(z_{i+1}, Tz_{i+1}) = D$. Define $x_{np+1} = Tx_{np}$, $\forall n \in \mathbf{Z}_{0+}$ with $\{x_{pn}\} \subset X_{i+1}$ having a convergent subsequence $\{x_{pn_k}\} \to z_{i+1}$ as it has been proved above and pick up any arbitrary initial point $x_0 \in X_i$ such that $T^2 x_0 \in X_{i+2}$ for the given arbitrary $i \in \overline{p}$. Then, one gets by using the rectangular inequality of the g.m.s. (X, d) that

$$\begin{aligned} D &\le d\left(Tz_{i+1}, x_{pn_k}\right) \\ &\le d\left(Tz_{i+1}, x_{pn_k+1}\right) + d\left(x_{pn_{k+\ell}+1}, x_{pn_k+1}\right) \\ &\quad + d\left(x_{pn_{k+\ell}+1}, x_{pn_k}\right) \\ &\le d\left(z_{i+1}, x_{pn_k}\right) + d\left(x_{pn_{k+\ell}+1}, x_{pn_k+1}\right) \\ &\quad + d\left(x_{pn_{k+\ell}+1}, x_{pn_k}\right); \quad \forall \ell \in \mathbf{Z}_{0+}. \end{aligned} \tag{22}$$

Since $\lim_{k\to\infty} d(x_{pn_{k+\ell}+1}, x_{pn_k+1}) = \lim_{k\to\infty} d(x_{pn_{k+\ell}}, x_{pn_k}) = 0$, it follows directly that

$$D = d\left(Tz_{i+1}, z_i\right) = \lim_{k\to\infty} d\left(Tz_{i+1}, x_{pn_k}\right) \tag{23}$$

and Tz_{i+1} is unique since $z_i \in X_i$ is unique and $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ is single-valued with $Tz_{i+1} \in X_{i+1}$ since X_{i+1} is closed. As a result, since $i \in \overline{p}$ is arbitrary, all the best proximity points in-between adjacent subsets are unique if any of them is unique and satisfy $z_{i+j} = T^j z_i$ for any $i, j \in \overline{p}$. It also turns out that $T^p z_i = z_i$; then it is a fixed point of the composite mapping $T^p : \bigcup_{j\in\overline{p}} X_j \mid X_i \to \bigcup_{j\in\overline{p}} X_j$, $\forall i \in \overline{p}$, and then a p-periodic fixed point of $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$. Property (iii) has been proved. □

Note that the classes Φ_D and $\Psi_D(r_0)$ (Definitions 6 and 8) can be redefined as sets of functions from $[0, \infty)$ to $[0, \infty)$ which are identically zero in $[0, D)$. This generalization is irrelevant in practice since the domains of the functions of the classes Φ_D and $\Psi_D(r_0)$ used to define the weak contractive mappings involve distances of points in-between adjacent subsets of the cyclic disposal, so distances are not smaller than D. Theorem 13 and also the relevant subsequent results of the paper can be also got with those extended definitions since $d(x,y) \geq D, \forall (x,y) \in X_i \times X_{i+1}, \forall i \in \overline{p}$, implies that

$$\begin{gathered} \varphi(d(x,y)) \geq D, \\ \phi(\varphi(d(x,y))) \geq D; \\ \forall (x,y) \in X_i \times X_{i+1}, \ \forall i \in \overline{p}, \\ \min_{(x,y)\in X_i\times X_{i+1}} d(x,y) = D. \end{gathered} \tag{24}$$

In the case that the adjacent subsets X_i are nonempty, closed, and intersecting, we can obtain the subsequent result on the existence of a unique fixed point allocated in their intersection.

Theorem 14. *Let (X,d) be a complete g.m.s. and let $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ be a generalized $(\phi - \varphi)$-weak p-cyclic contraction mapping such that that the sets X_i, $\forall i \in \overline{p}$, are closed and intersect for some $\phi \in \Phi_0$ and $\varphi \in \Gamma_0$. Then, there is a unique fixed point $z \in \bigcap_{i\in\overline{p}} X_i$.*

Proof. From Theorem 13(i) for $D = 0$, it follows for any $x_0 \in X_i$ and any $i \in \overline{p}$ that

$$\begin{aligned} &\lim_{n,m\to\infty} d\left(x_{mp+j}, x_{(m+n)p+j+1}\right) \\ &\quad = \lim_{n\to\infty} d\left(x_{np+j}, x_{(\ell+n)p+j+1}\right) = 0; \\ &\qquad \forall \ell \in \mathbf{Z}_{0+}, \ \forall j \in \overline{p-1} \cup \{0\} \end{aligned} \tag{25}$$

for any given $x_0 \in X_i$ and any given $i \in \overline{p}$. Thus, for any given $\varepsilon \in \mathbf{R}_+$, there is $n_0 = n_0(\varepsilon) \in \mathbf{Z}_{0+}$ such that $d(x_{np+k}, x_{mp+k}) < \varepsilon \forall n (\geq n_0) \in \mathbf{Z}_{0+}$, $\forall m (\geq n+1) \in \mathbf{Z}_{0+}$, and $\forall k \in \mathbf{Z}_{0+}$ and $\{x_n\}$ is g.m.s. Cauchy sequence. Assume on the contrary that there are subsequences $\{x_{pn_j+k}\} \subseteq \{x_{pn}\}$ and $\{x_{pm_j+k}\} \subseteq \{x_{pm}\}$ such that $d(x_{pn_j+k}, x_{pm_j+k}) \geq \varepsilon$. But this contradicts (25) so that $\{x_{pn}\}$ is g.m.s. Cauchy sequence for any given initial point $x_0 \in \bigcup_{i\in\overline{p}} X_i$. Since (X,d) is g.m.s. complete, any $\{x_{pn+k}\}$, $\forall k \in \mathbf{Z}_{0+}$, which is g.m.s. Cauchy sequence is convergent to some $z_{i+k} \in X_{i+k}$ if $x_0 \in X_i$ for any given $i \in \overline{p}$, with $k \leq p-i$. It suffices to notice that $X_{i+k} = X_{ni+k} = X_{ni+k-mp}$, $\forall i \in \overline{p}$, $\forall k \in \overline{p-1} \cup \{0\}$, and $\forall n,m \in \mathbf{Z}_+$ with m satisfying the constraint $(ni+k-p)/p \leq m \leq (ni+k-1)/p$ (equivalently, $p \geq ni+k-mp \geq 1$). Since X_i is closed, $\forall i \in \overline{p} \bigcap_{i\in\overline{p}} X_i \neq \varnothing$, $\{x_{pn+k}\} (\subseteq \bigcap_{i\in\overline{p}} X_i) \to z_{i+k}$, $\forall i \in \overline{p}$, $\forall k \in \overline{p-1} \cup \{0\}$, and it follows that $z_j \in \bigcap_{i\in\overline{p}} X_i$, $\forall j \in \overline{p}$. Also z_j is a fixed point of $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$, $\forall j \in \overline{p}$. Otherwise, we would get the contradiction $0 \leftarrow \{d(x_{pn+j}, Tx_{pn+j})\} \to d(z_{i+j}, Tz_{i+j}) > 0$. Since $z_i = T^{k+1} z_i = T(T^k z_i)$, $\forall i \in \overline{p}$, $\forall k \in \mathbf{Z}_{0+}$, and $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ is single-valued, then $z_j = z \in \bigcap_{i\in\overline{p}} X_i$, $\forall j \in \overline{p}$, for the given $x_0 \in X_i$ and $i \in \overline{p}$.

It remains to prove that the fixed point z to which the sequences converge is independent on the initial point $x_0 \in \bigcup_{i\in\overline{p}} X_i$ so that it is unique. Assume that $z = Tz, \widehat{z} = T\widehat{z} (\neq z) \in \bigcap_{i\in\overline{p}} X_i$. We consider them to be initial conditions of two distinct sequences in some pair of adjacent subsets X_i and X_{i+1} for some $i \in \overline{p}$ so as to use the generalized $(\phi - \varphi)$-weak contractive condition (2), and one gets

$$\begin{aligned} &\varphi\left(d\left(Tz_{np+j}, T\widehat{z}_{(n+m)p+j+1}\right)\right) \\ &\quad \leq \phi^{np+j}\left(\varphi\left(d\left(x_0, x_{mp+1}\right)\right)\right); \\ &\qquad \forall m,n \in \mathbf{Z}_{0+}; \ \forall j \in \overline{p-1} \cup \{0\} \end{aligned} \tag{26}$$

which leads to

$$\begin{aligned} &\lim_{n\to\infty} \phi^{np+j}\left(\varphi\left(d\left(Tz_{np+j}, T\widehat{z}_{(n+m)p+j+1}\right)\right)\right) \\ &\quad = \lim_{n\to\infty} \varphi\left(d\left(Tz_{np+j}, T\widehat{z}_{(n+m)p+j+1}\right)\right) = d(z, \widehat{z}) \\ &\quad = \lim_{n\to\infty} \left(d\left(Tz_{np+j}, T\widehat{z}_{(n+m)p+j+1}\right)\right) = 0; \\ &\qquad \forall j,m \in \overline{p-1} \cup \{0\} \end{aligned} \tag{27}$$

which contradicts $z \neq \widehat{z}$. □

Note that Theorem 14 does not require the sets X_i, $\forall i \in \overline{p}$, to be convex sets so as to prove the uniqueness of the fixed point. Simply, the weak contractive condition (2) is used by considering that two claimed distinct fixed points which are in the nonempty intersection of X_i, $\forall i \in \overline{p}$, belong in any case to two adjacent subsets X_i, while the weak contractive condition guarantees that they are identical. We can extend easily Definition 12 and Theorem 13 to the case when the distances in-between adjacent subsets are distinct as follows. The existence and uniqueness of fixed points for generalized $(\phi - \varphi)$-weak (noncyclic) contraction mappings of the type (2) on a g.m.s. (X,d) have been proved in [3].

Sufficient conditions for the uniqueness of uniqueness of the best proximity points in each of the subsets $X_i \subseteq X$, $\forall i \in \overline{p}$, are discussed in the subsequent result which is related to the consideration of a complex structure on the metric space.

Theorem 15. *Let (X,d) be a complete g.m.s. admitting midpoints which has p nonempty closed subsets $X_i \subseteq X$, with $d(X_j, X_{j+1}) = D$, $\forall j \in \overline{p}$, where X is an abstract set, with one of the subsets X_i, for some $i \in \overline{p}$, being strictly convex. Assume that there is some strictly increasing function $\delta : (0,1] \to [0,2]$ such that X has the following uniform convexity property for all $x, y \in X_i$, $p \in X$, $R \in \mathbf{R}_+$, and $r \in [0, 2R]$:*

$$\begin{gathered} \left([\max(d(x,p), d(y,p)) \leq R] \wedge [d(x,y) \geq r]\right) \Longrightarrow \\ \left(d\left(\frac{x+y}{2}, p\right) \leq \left(1 - \delta\left(\frac{r}{R}\right)\right) R\right). \end{gathered} \tag{28}$$

Let $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ be a generalized $(\phi - \varphi)$-weak p-cyclic contraction mapping for some $\phi \in \Phi_0$ and some $\varphi \in \Gamma_0$. Then, $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ has a unique best proximity point $z_j \in X_j$, $\forall j \in \overline{p}$, which is also a p-periodic point of $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ and a fixed point of the composite mapping $T^p : \bigcup_{j\in\overline{p}} X_j \mid X_k \to \bigcup_{j\in\overline{p}} X_j$, $\forall k \in \overline{p}$.

If $\bigcap_{i\in\overline{p}} X_i \neq \varnothing$, then $z_k = z \in \bigcap_{i\in\overline{p}} X_i$, $\forall k \in \overline{p}$, is the unique fixed point of $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$.

Proof. Assume that X_i for some $i \in \overline{p}$ is a strictly convex set. Since $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ is a generalized $(\phi-\varphi)$-weak p-cyclic contraction mapping, it follows from Theorem 13 that $d(T^{pn+j}x, T^{pn+j+1}x) \to D$ as $n \to \infty$, $\forall j \in \overline{p}$, for any given $x \in \bigcup_{j\in\overline{p}} X_j$. Note that there is some integer $j_i = j(i, x_0) \in \overline{p-1} \cup \{0\}$ such that $x = x(x_0) = T^{j_i}x_0 \in X_i$ for any given $x_0 \in \bigcup_{j\in\overline{p}} X_j$ (i.e., $x = x(x_0)$ belongs to the strictly convex subset $X_i \subset X$) for the given initial point $x_0 \in \bigcup_{j\in\overline{p}} X_j$ and $d(T^{pn}x, T^{pn+1}x) = d(T^{pn+j_i}x_0, T^{pn+j_i+1}x_0) \to D$ with $x \in X_i$, $\{T^{pn}x\} \subset X_i$, $\{T^{pn+1}x\} \subset X_{i+1}$, and some $x = x(x_0)$ for any given $x_0 \in \bigcup_{j\in\overline{p}} X_j$. We now take initial conditions $x \in X_i$ for sequences built $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ with $x = x(x_0)$ for any given $x_0 \in \bigcup_{j\in\overline{p}} X_j$ such that $x = x(x_0) = T^{j_i}x_0 \in X_i$ and $\{T^{pn+j_i}x\} \subset X_i$ for some $j_i \in \overline{p-1} \cup \{0\}$. Then, it follows that

$$d\left(T^{pn+j}x, T^{pn+j+1}x\right) \longrightarrow D \quad \text{as } n \longrightarrow \infty,$$
$$d\left(T^{(p+1)n+j}x, T^{pn+j+1}x\right) \longrightarrow D \quad \text{as } n \longrightarrow \infty \tag{29}$$

with $\{T^{pn+j}x\} \subset X_{i+j}$, $T^jx \in X_{i+j}$, $\{T^{pn+j+1}x\} \subset X_{i+j}$, and $T^{j+1}x \in X_{i+j}$. Then, for every given $\varepsilon \in \mathbf{R}_+$, there exists an integer $N_0 = N_0(\varepsilon, x)$ such that $d(T^{pn+j}x, T^{pn+j+1}x) \le D+\varepsilon$, $\forall n(\ge N_0) \in \mathbf{Z}_{0+}$.

One now firstly proves that $d(T^{(n+m)p+j}x, T^{pn+j}z) \to 0$ as $n \to \infty$, $\forall m \in \mathbf{Z}_{0+}$, $\forall x, z \in X_i$; equivalently, $d(T^{pn+j}x, T^{pm+j}z) \to 0$ as $n, m \to \infty$, so that, for every $\varepsilon \in \mathbf{R}_+$, there is $N_1 = N_1(\varepsilon, x, z)$ such that, for all integer numbers $m > n \ge N_1$, $d(T^{pn+j}x, T^{pm+j}z) \le \varepsilon$, $\forall x, z \in X_i$. Assume that the property $d(T^{pn+j}x, T^{pm+j}z) \to 0$ as $n, m \to \infty$, $\forall x, y \in X_i$, does not hold. Thus, there exists some $\varepsilon_0 \in \mathbf{R}_+$ such that, for each $k \in \mathbf{Z}_{0+}$, there exist integers $m_k > n_k \ge k$ for which $d(T^{pn_k+j}x, T^{pm_k+j}z) \ge \varepsilon_0$. Choose $\gamma \in (0,1)$ such that $\varepsilon_0/\gamma > D$ and choose ε such that $0 < \varepsilon < \min(\varepsilon_0/\gamma - D, \delta(\gamma)D/(1-\delta(\gamma)))$. For such a chosen arbitrary constant $\varepsilon \in \mathbf{R}_+$, there exists $N_0 = N_0(\varepsilon, x) \in \mathbf{Z}_{0+}$ such that for all integer numbers $m_k > n_k \ge N_0$, $d(T^{pn_k+j}x, T^{pm_k+j+1}x) \le D+\varepsilon$ for all $x \in X_i$, and also there exists $N_2 = N_2(\varepsilon, z) \in \mathbf{Z}_{0+}$ such that for $m_k > n_k \ge N_2$ and all $z \in X_i$, $d(T^{pn_k+j}z, T^{pm_k+j+1}z) \le D+\varepsilon$, $\forall j \in \overline{p-1} \cup \{0\}$, from Theorem 13(i). Similar distance constraints also hold from Theorem 13(i) for appropriate subsequences of nonnegative integers $\{m_k\}$ and $\{n_k\}$ (with $m_k > n_k$) exceeding certain finite thresholds, dependent on ε, for each initial conditions $x, z \in X_i$ (X_i being a strictly convex subset of X) and $y \in X_{i+1}$; that is, $d(T^{pn_k+j}x, T^{pm_k+j}y) \le D+\varepsilon$ for all $x \in X_i$, $y \in X_{i+1}$ for $m_k > n_k \ge N_0$ and some $n_0 = n_0(\varepsilon, x, y) \in \mathbf{Z}_{0+}$, $\forall j \in \overline{p-1} \cup \{0\}$, and $d(T^{pn_k+j}z, T^{pm_k+j}y) \le D+\varepsilon$ for all $z \in X_i$, some $n_2 = n_2(\varepsilon, z, y) \in \mathbf{Z}_{0+}$, $\forall j \in \overline{p-1} \cup \{0\}$, and all $m_k > n_k \ge n_2$. Now, note that since the set X_i is strictly convex, $\{x_{m_k}\} \subset X_i$ and $\{z_{m_k}\} \subset X_i$, it follows that

$$\left\{\frac{x_{m_k}+z_{n_k}}{2}\right\} \subset X_i,$$
$$d\left(\frac{x_{m_k}+z_{n_k}}{2}, X_{i+1}\right) \ge d\left(X_i, X_{i+1}\right) \ge D; \tag{30}$$
$$\forall k \in \mathbf{Z}_{0+}.$$

Now, choose $n_1 = \max(n_0, n_2)$ so that one gets from (28) for all integers $m_k > n_k \ge n_1$ that the following contradiction holds:

$$\begin{aligned} d\left(\frac{x_{m_k}+z_{n_k}}{2}, y_{n_k}\right) &\le \left(1-\delta\left(\frac{\varepsilon_0}{(D+\varepsilon)}\right)\right)(D+\varepsilon) \\ &= D+\varepsilon-(D+\varepsilon)\,\delta\left(\frac{\varepsilon_0}{(D+\varepsilon)}\right) \\ &< D. \end{aligned} \tag{31}$$

Since ε is arbitrary, it can be chosen so that $\delta(\varepsilon_0/(D+\varepsilon)) > \delta(\gamma D/(D+\varepsilon)) > \varepsilon/(D+\varepsilon)$ since $\delta(\varepsilon)$ is strictly increasing in $(0,1]$. Then, for each given $\varepsilon \in \mathbf{R}_+$, there is some $n_3 = n_3(\varepsilon, x, z)$ such that, for all integer numbers $m > n \ge n_3$, $d(T^{pn+j}x, T^{pm+j}z) \le \varepsilon$, $\forall x, z \in X_i$, such that the set X_i is strictly convex for some $i \in \overline{p}$, and then

$$\begin{aligned} &\lim_{n,\ell\to\infty} d\left(T^{pn+j}x, T^{p\ell+j}z\right) \\ &\quad = \lim_{n\to\infty} d\left(T^{pn+j}x, T^{p(m+n)+j+k}w\right) = 0, \end{aligned} \tag{32}$$

$\forall x, z \in X_i$, $\forall m \in \mathbf{Z}_{0+}$, $\forall j \in \overline{p-1} \cup \{0\}$, and $w \in X_{i-k}$ for any given $k \in \overline{p-1} \cup \{0\}$ implying that $\{T^{pn+j}x\}$, $\{T^{p(m+n)+j+k}w\}$ are sequences in X_{i+j}. From Theorem 13(i) (see (5)), it follows that $d(T^{pn}x, T^{pn+1}z) \to D$ as $n \to \infty$, $\forall x, z \in X_i$. From Theorem 13(ii), it follows that $\{T^{pn}x\}$, $\{T^{pn}z\}$ are bounded sequences in the strictly convex set X_i. From the above discussion in this proof, it follows that $d(T^{pn}x, T^{pn1}z) \to 0$ as $n \to \infty$. Since d is a generalized metric and (X, d) is a complete g.m.s., it follows that $\{T^{pn}x\}$ and $\{T^{pn}z\}$ are bounded and g.m.s. convergent sequences in X_i to $x_i \in X_i$ and $z_i \in X_i$, respectively, and then g.m.s. Cauchy sequences and $d(x_i, X_{i+1}) = d(z_i, X_{i+1}) = D$. Finally, since X_i is strictly convex, there cannot exit two distinct $x_i, z_i \in X_i$ with the minimum distance D to X_{i+1}, and thus $x_i = z_i$. As a result, there exists a unique best proximity point $z_i \in X_i$ such that X_i is strictly convex and closed. Since $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ is single-valued, then $T^jz_i = z_{i+j}$ is the unique best proximity point in X_{i+j}, $\forall j \in \overline{p}$, which satisfies $d(T^jz_i, T^{j+1}z_i) = d(X_{i+j}, X_{i+j+1}) = D$, $\forall j \in \overline{p}$. Then, $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ has a unique best proximity point $z_k \in X_k$, $\forall k \in \overline{p}$ (since X_k is closed, $\forall k \in \overline{p}$), which is also, by construction, a p-periodic point of $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ and a fixed point of the composite mapping $T^p : \bigcup_{j\in\overline{p}} X_j \mid X_k \to \bigcup_{j\in\overline{p}} X_j$, $\forall k \in \overline{p}$. It turns out that if $\bigcap_{i\in\overline{p}} X_i \neq \varnothing$, then $z_k = z \in \bigcap_{i\in\overline{p}} X_i$, $\forall k \in \overline{p}$ is the unique fixed point of $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$. □

Remark 16. (1) Note that (X, d) in Theorem 15 is a uniformly convex metric space (not specifically a uniformly convex Banach space) and X might be an abstract set (nonnecessarily being a linear space). However, it is obvious that any uniformly convex Banach space is also a uniformly convex metric space. In this case, it is assumed that one of the subsets of the cyclic disposal, for which the generalized $(\phi-\varphi)$-weak p-cyclic contraction mapping is defined, is strictly convex and subject to a uniform convexity-type condition (28) for the elements of such a set with respect to the points of X. That condition is related to the uniformly convex structure of the metric space and, equivalently, to the existence of midpoints in the sense that if (28) fails to hold, then the metric space is not uniformly convex. Such a condition, supported by the existence of midpoints, is close to the one satisfied on uniformly convex Banach spaces (X, d) [27] and used in Lemmas 3.7-3.8 and Theorem 3.10 of [6] to prove the existence and uniqueness of fixed points in a 2-cyclic contractive condition on the union of two nonempty closed and convex subsets of X. See also [23–25]. In particular, X could be endowed with a simpler structure than a linear space as, for instance, a group endowed with a composition law, provided that the subsets of the cyclic disposal are closed while just one of them is strictly convex.

(2) Note also that X could even be just an abstract set with nonempty closed subsets with one of them being strictly convex.

(3) Note that Theorem 15 is applicable to the case that $(X, \|\ \|)$ is a uniformly convex Banach space with a norm-induced metric d so that $(X, \|\ \|) \equiv (X, d)$ with nonempty closed convex subsets $X_i \subset X$, $\forall i \in \overline{p}$. This is a direct consequence of the fact that if $(X, \|\ \|) \equiv (X, d)$ is a uniformly convex Banach space, then it is also complete and the norm triangle inequality can be trivially upper-bounded by quadrangular-type sum of norms. This allows adding distances related to distinct subsequences converging to zero to the usual triangle inequality leading to be able to characterize the complete (X, d) also as being a complete g.m.s. [28].

(4) Finally, note that the assumption of the existence of midpoints [27] can also be focused on under the framework of convex structures perhaps at the expenses of a more involved presentation. It is said that a mapping $W(X, X, [0,1]) \to X$ is a convex structure on X if, for each $x, y, z \in X$, $d(z, W(x, y, \lambda)) \le \lambda d(x, z) + (1-\lambda) d(y, z)$, $\forall \lambda \in [0,1]$. See, for instance, [29–33]. Note that $x = W(x, x, \lambda)$ for any $\lambda \in [0,1]$ and $d(y, W(x, x, \lambda)) = d(x, y)$ for any $x, y \in X$. If $z = x$ and $\lambda = 1/2$, then $z_1 = W(x, y, 1/2)$ satisfying $(1/2)d(x, y) = d(x, z_1) = d(x, W(x, y, 1/2)) \le (1/2)d(x, y)$ is the midpoint of the segment $[x, y]$. The convex structure allows characterizing uniformly convex metric spaces as triples (X, d, W) associated with metric spaces (X, d) [32]. See also [28, 34–36]. For a generalized metric and its associate quadrangular constraint of distances, we can define in the same way a generalized convex structure $W(X, X, X, [0,1] \times [0,1]) \to X$ such that, for $w, x, y, z \in Z$, one has

$$\begin{aligned} & d(w, W(x, y, z, \lambda_1, \lambda_2)) \\ & \quad \le \lambda_1 d(w, x) + \lambda_2 d(w, y) \\ & \qquad + (1 - \lambda_1 - \lambda_2) d(w, z); \quad \forall \lambda_1, \lambda_2 \in [0,1] \end{aligned} \tag{33}$$

and the choice $\lambda_1 = \lambda_2 = 1/3$, $w = x$, and $\omega = W(x, y, z, 1/3, 1/3)$ leads to

$$\begin{aligned} & 3d(x, \omega) \\ & \quad = 3\left[d(x, x) + \left(\frac{1}{3}\right) d(x, y) + \left(\frac{1}{3}\right) d(x, z)\right] \\ & \quad = d(x, y) + d(x, z). \end{aligned} \tag{34}$$

Thus, if y is the midpoint of $[x, z]$, then ω is located at 1/3 from x and 2/3 from y of the length $[x, z]$.

Example 17. Consider the set of real numbers $X = \{jt : \pm j \in \overline{5}\}$ for some given $t \in \mathbf{R}_+$ and consider nonempty closed subsets $X_1 = \{jt : j \in \overline{5} \cup \{0\}\}$ and $X_2 = -X_1 \setminus \{0\}$ of X. For some given real constants $\gamma \in \mathbf{R}_+, \gamma_1 \in (0, (3/5)\gamma]$, and $\gamma_2 \in [\max(\gamma, 3\gamma - 2\gamma_1), 1)$, a g.m. $d : X \times X \to \mathbf{R}$ is defined as follows:

$$\begin{aligned} & d(x, x) = 0, \quad \forall x \in X \\ & d(x, y) = d(y, x), \quad \forall x, y \in X \\ & d(0, jt) = d(-jt, 0) = j\gamma_1, \quad j \in \overline{5} \\ & d(t, 2t) = d(-t, -2t) = 3\gamma \\ & d(jt, 3t) = d(-jt, -3t) = \gamma, \quad j \in \overline{2} \\ & d(jt, 4t) = d(-jt, -4t) = 2\gamma, \quad j \in \overline{3} \\ & d(jt, 5t) = d(-jt, -5t) = \left(\frac{3}{2}\right)\gamma, \quad j \in \overline{4} \\ & d(-it, jt) = d(it, -jt) = \gamma_2; \quad i, j \in \overline{5}. \end{aligned} \tag{35}$$

Note that $d(X_1, X_2) = \gamma_1$ and that $d : X \times X \to \mathbf{R}$ is not a metric since $d(\pm t, \pm 2t) = 3\gamma > d(\pm t, \pm 3t) + d(\pm 3t, \pm 2t) = 2\gamma$, and then (X, d) is not a metric space, while it is a g.m.s. since

$$\begin{aligned} \max_{x, y \in X} d(x, y) = 3\gamma = d(t, 2t) & \\ & \le d(t, x) + d(x, y) + d(y, 2t) \\ & = d(t, 0) + d(0, -t) + d(-t, 2t) \\ & = 2\gamma_1 + \gamma_2, \end{aligned} \tag{36}$$

where $x \in X - \{t, 2t\}$ is such that $d(t, x) = \min(d(t, z) : z \in X - \{t, 2t\})$. For instance, take $x = 0$. $y \in X - \{t, 2t, x\}$, for the above $x \in X - \{t, 2t\}$, is such that $d(x, y) = \min(d(x, z) : z \in X - \{t, 2t, x\})$.

For instance, take $x = -t$. Define a 2-cyclic mapping $T : X_1 \cup X_2 \to X_1 \cup X_2$ as follows:

$$\begin{gathered} T(\pm it) = \mp(i-1)t \quad \text{for } i(\in \overline{5}) \geq 2 \\ T(-t) = 0; \\ T(t) = T(0) = -t \end{gathered} \tag{37}$$

which is a generalized $(\phi - \varphi)$-weak p-cyclic contraction for some $\varphi \in \Gamma_{\gamma_1}$, for instance, it can be defined so as to satisfy the constraints $\varphi(\gamma_1) = \gamma_1$ and $\varphi(x) + \varphi(y) \geq \varphi(z) \geq z > \gamma_1$ for $x, yz = x + y \in (\gamma_1, \gamma_2]$, and some $\phi \in \Phi_{\gamma_1}$ chosen to satisfy $\phi(\gamma_1) = \gamma_1$ and $\phi(x) = \rho\gamma_2$ for $x \in (\gamma_1, \gamma_2]$ for some $\rho(\in \mathbf{R}_+) \geq 1$ so as to be nondecreasing and subadditive. Then, T is a generalized $(\phi - \varphi)$-weak 2-cyclic contraction which converges to a limit cycle $\{-1, 0\}$ with the two best proximity points $0 \in X_1$ and $(-1) \in X_2$. Theorem 13 leads to the above conclusion.

Note that a $(\phi - \varphi)$-weak 1-cyclic contraction on X_1 is defined by the sequence $\{t, 2t, 4t, 5t, t, 0\}$, with $\varphi \in \Gamma_{\gamma_1}$ and $\phi \in \Phi_{\gamma_1}$ defined under the same above constraints, leading to the distances in-between adjacent subsets given by the sequence $\{3\gamma, 2\gamma, (3/2)\gamma, (3/2)\gamma, \gamma_1\}$.

Example 18. Reconsider Example 17 with the same set X and the same metric with $X_1 = \{jt : j \in \overline{5} \cup \{0\}\}$ and $X_2 = -X_1$. Then, $X_1 \cap X_2 = \{0\}$ and $d(X_1, X_2) = 0$. The map $T : X_1 \cup X_2 \to X_1 \cup X_2$ is defined by $T(\pm it) = \mp it$ for $i(\in \overline{5}) \geq 2$, $T(t) = T(-t) = 0$, and $T(0) = 0$. Then, T is a generalized 2-cyclic contraction which converges to its fixed point $0 \in X_1 \cap X_2$. Theorem 14 leads to that conclusion.

Example 19. How to combine Examples 17 and 18 with the constraint (28) is now checked, assumed as hypothesis in Theorem 15. Such a constraint is close to the property of uniformly convexity of Banach spaces 8 (see also Remark 16). Now, for all real constant $\varepsilon \in (0, 2]$ such that $d(x, y) \geq \varepsilon$, there is some real constant $\delta \in (0, 1)$ such that $d(x, -y) \leq 2(1 - \delta)$. Several cases can occur:

(a) $2 \geq j\gamma_1 = d(0, x) = d(0, -x) \leq 2(1 - \delta(j))$ for $x \in X - \{0\}$, $j \in \overline{5}$.

This is solved with strictly increasing $\delta(j) = \delta_j$ for $j \in \overline{5}$ if $0 < \delta_j = j\sigma \leq 1 - j\gamma_1/2$; $j \in \overline{5}$, that is, if $\sigma \in (0, (2 - 5\gamma_1)/10)$, $\gamma_1 \in (0, 2/5)$ and $\delta(j) = j\sigma$, $j \in \overline{5}$.

(b) $[2 \geq d(x, y) = \lambda\gamma \Rightarrow d(x, -y) = \gamma_2 \leq 2(1 - \delta(\lambda\gamma))]$,

for $x, y \in X - \{0\}$ and for $x, y \in X_2$ with $\delta(\lambda\gamma) = \lambda\gamma \leq 1 - \gamma_2/2$ for $\lambda = 1, 2, 3, 3/2$ which leads to $\gamma_2 \leq 2(1 - \lambda\gamma)$.

(c) $[2 \geq d(x, -y) = \gamma_2 \Rightarrow d(x, y) = \lambda\gamma \leq 2(1 - \delta(\gamma_2))]$,

for $x, y \in X_1 - \{0\}$ (and, equivalently, for $x, y \in X_2$) with $\delta(\lambda\gamma) = \lambda\gamma \leq 1 - \gamma_2/2$, with the strictly increasing $\delta(\gamma_2) = \rho\gamma_2 \leq 1 - \lambda\gamma/2$, for $\lambda = 1, 2, 3, 3/2$ and some real constant $\rho > 0$.

The various above parametrical constraints are compatible with those in Example 17, and the constraint (28) of Theorem 15 holds, if

$$\begin{gathered} \sigma \in \left(1, \frac{2 - 5\gamma_1}{10}\right), \\ \gamma_1 \in \left(0, \max\left(\frac{2}{5}, \left(\frac{3}{2}\right)\gamma\right)\right), \\ \rho > 0, \\ \gamma_2 \in \left[\max\left(5\sigma, \gamma, 3\gamma - 2\gamma_1\right), \right. \\ \left. \min\left(2 - 6\gamma, \frac{1}{\rho}\left(1 - \frac{3}{2}\right)\gamma\right)\right). \end{gathered} \tag{38}$$

However, due to the discrete nature of the set X, the assumption of Theorem 15 on strict convexity of sets towards the proof of existence of unique best proximity points is not fulfilled by this example.

Definition 20. Let (X, d) be a g.m.s., let X_i be a set of nonempty $p(\geq 2)$ subsets of X distances in-between adjacent subsets $d(X_i, X_{i+1}) = D_i$, $\forall i \in \overline{p}$, and let $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ be a p-cyclic self-mapping satisfying

$$\begin{gathered} \varphi_{i+1}(d(Tx, Ty)) \leq \phi_i(\varphi_i(d(x, y))); \\ \forall (x, y) \in X_i \times X_{i+1}, \ \forall i \in \overline{p} \end{gathered} \tag{39}$$

for some $\varphi_i \in \Gamma_{D_i}$ and some $\phi_i \in \Phi_{D_i}$ and each $i \in \overline{p}$, where the classes Γ_{D_i} and Φ_{D_i} are defined with functions $\varphi_i : [0, \infty) \to [0, \infty)$ and $\phi_i : [D_i, \infty) \to [D_i, \infty)$, $\forall i \in \overline{p}$.

Then, T is said to be a generalized $(\phi - \varphi)$-weak p-cyclic contraction mapping.

Theorem 21. *Let (X, d) be a g.m.s. and let $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ be a generalized $(\phi - \varphi)$-weak p-cyclic contraction mapping. Then, the following properties hold:*

(i)

$$\begin{gathered} \lim_{n\to\infty} \phi^n(i, i-p)\left(\varphi_{i-p-1}\left(d\left(x_0, x_{mp+1}\right)\right)\right) = D_i \\ \lim_{m\to\infty}\lim_{n\to\infty} \varphi_{i-p-1}\left(d\left(x_{mp}, x_{(m+n)p+1}\right)\right) \\ = \lim_{n\to\infty} \varphi_{i-p-1}\left(d\left(x_{np+j}, x_{(\ell+n)p+j+1}\right)\right) = D_i, \end{gathered} \tag{40}$$

$\forall \ell, m \in \mathbf{Z}_{0+}$, $\forall j \in \overline{p-1} \cup \{0\}$, $\forall i \in \overline{p}$, and for any given $x_0 \in X_i$ and any given arbitrary $i \in \overline{p}$, where $\phi(i, i-p) = \phi_{i-p} \cdot \ldots \cdot \phi_{i.-1} \cdot \phi_i$ is a composite mapping of the p functions $\phi_i \in \Phi_{D_i}$ for $i \in \overline{p}$.

(ii) Assume that (X, d) is a complete g.m.s. and that $z_i \in X_i$ has a best proximity point from X_i to X_{i+1} (i.e., $d(z_i, X_{i+1}) = D_i$) for some given $i \in \overline{p}$ and that X_{i+1} is approximatively compact with respect to X_i. Then, there is a best proximity point $z_{i+1} = Tz_i$ from X_{i+1} to X_i which is unique and in X_{i+1} if z_i is unique and X_{i+1} is closed.

Also, if all the subsets $X_i \subseteq X$, $\forall i \in \overline{p}$, are closed, then there are p best proximity points $z_{i+j} = T^j z_i \in X_{i+j}$, $\forall i, j \in \overline{p}$, *which are unique if anyone of them is unique. Each* $z_i \in X_i$ *is also a fixed point of the composite mapping* $T^p : \bigcup_{j\in\overline{p}} X_j \mid X_i \to \bigcup_{j\in\overline{p}} X_j$, $\forall i \in \overline{p}$, *and then a* p*-periodic fixed point of* $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$.

Proof. The proof is close to that of Theorem 13. Let x_0 be an arbitrary point of X_i for some given arbitrary $i \in \overline{p}$ and define the sequence $\{x_n\}$ inductively by $x_{n+1} = Tx_n$, $\forall n \in \mathbf{Z}_{0+}$. Since $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ is a $(\phi - \varphi)$-weak p-cyclic contraction mapping, one gets from (39) by induction for each $n, m \in \mathbf{Z}_{0+}$ that

$$\begin{aligned}
&\varphi_{i+1}\left(d\left(x_{np}, x_{(n+m)p+1}\right)\right)\\
&\quad = \varphi_{i+1}\left(d\left(Tx_{np-1}, Tx_{(n+m)p}\right)\right)\\
&\quad \le \phi_i\left(\varphi_i\left(d\left(x_{np-1}, x_{(n+m)p+}\right)\right)\right)\\
&\quad \le \phi_{i-1}\left(\phi_i\left(\varphi_{i-1}\left(d\left(x_{np-2}, x_{(n+m)p-1}\right)\right)\right)\right)\\
&\quad = \left(\phi_{i-1}\phi_i\right)\left(\varphi_{i-1}\left(d\left(x_{np-2}, x_{(n+m)p-1}\right)\right)\right)\cdots\\
&\quad \le \phi\left(i, i-p\right)\left(\varphi_{i-p}\left(d\left(x_{(n-1)p}, x_{(n+m-1)p-1}\right)\right)\right)\cdots\\
&\quad \le \phi^n\left(i, i-p\right)\left(\varphi_{i-p}\left(d\left(x_0, x_{mp+1}\right)\right)\right);\\
&\quad = \phi^n\left(i, i-p\right)\left(\varphi_i\left(d\left(x_0, x_{mp+1}\right)\right)\right); \quad \forall m, n \in \mathbf{Z}_{0+}
\end{aligned} \tag{41}$$

since $\phi_i \equiv \phi_{i-p}$ and $\varphi_i \equiv \varphi_{i-p}$, $\forall i \in \overline{p}$, where $\phi(i, i-p) = \phi_{i-p}, \ldots, \phi_{i.-1} \cdot \phi_i$, $\forall i \in \overline{p}$. Since $\phi_i \in \Phi_{D_i}$, $\forall i \in \overline{p}$, one has from property (ϕ_2) and the composite mapping structure that $\{\phi^n(i, i-p)(t)\}$ is decreasing for all $t \in [D_i, \infty)$ so that $\{\phi^{np}(\varphi(d(x_0, x_{mp+1})))\}$ is decreasing and converges to some limit $\eta_{0i} \in \mathbf{R}_{0+}$. It is now proved that all the limits $\eta_{0j} = D$, $\forall j \in \overline{p-1} \cup \{0\}$. Since $\phi \in \Phi_D$, it is also a D-weaker Meir-Keeler function $\phi : [D, \infty) \to [D, \infty)$ so that, for each real number $\eta(> D) \in \mathbf{R}_+$, there exist a real number $\delta_j = \delta_j(\eta) \in \mathbf{R}_+$ such that $\phi^{np+j}(t) < \eta$, $\forall n(\ge n_{0j}) \in \mathbf{Z}_+$ for some $n_{0j} = n_{0j}(\eta) \in \mathbf{Z}_+$, for any given $j \in \overline{p-1} \cup \{0\}$, $\forall t \in [\eta, \eta + \delta_j)$. Thus, $\eta = D + \varepsilon$ for any given arbitrary $\eta(> D) \in \mathbf{R}_+$ such that $\varepsilon = \eta - D(\in \mathbf{R}_+)$ is also arbitrary; one has for each given $k \in \overline{p-1} \cup \{0\}$ that if $x_k = Tx_{k-1} = T^k x_0$, then

$$\begin{aligned}
&\left(\left(\phi_{i-1}, \ldots, \phi_{i-j}\right)\phi^n\left(i, i-p\right)\right)\\
&\quad \cdot \left(\varphi_{i-j}\left(d\left(x_k, x_{mp+k+1}\right)\right)\right) < D_{i-j} + \varepsilon,
\end{aligned} \tag{42}$$

$\forall m \in \mathbf{Z}_{0+}$, $\forall j, k \in \overline{p-1} \cup \{0\}$, and $\forall n(\ge n_{0j}) \in \mathbf{Z}_+$, and then there exist the p identical limits $\lim_{n\to\infty}((\phi_{i-1}, \ldots, \phi_{i-j})\phi^n(i, i-p))(\varphi_{i-j}(d(x_k, x_{mp+k+1}))) = D_{i-j}$, $\forall m \in \mathbf{Z}_{0+}$, $\forall j, k \in \overline{p-1} \cup \{0\}$. Then, one gets from (42) that

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty}\varphi_{i+j}\left(d\left(x_{mp+j}, x_{(m+n)p+j+1}\right)\right)\\
&\quad = \lim_{n\to\infty}\varphi_{i+j}\left(d\left(x_{\ell p+j}, x_{(\ell+n)p+j+1}\right)\right) = D_{i+j};\\
&\qquad \forall \ell \in \mathbf{Z}_{0+}, \ \forall j \in \overline{p-1} \cup \{0\}.
\end{aligned} \tag{43}$$

□

The notion of $(\psi - \varphi)$-weak p-cyclic contraction mapping is given below.

Definition 22. Let (X, d) be a g.m.s., let X_i be nonempty subsets of X, having a common distance in-between adjacent subsets $d(X_i, X_{i+1}) = D$, $\forall i \in \overline{p}$, and let $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ be a p-cyclic self-mapping satisfying

$$\begin{aligned}
&\varphi\left(d\left(Tx, Ty\right)\right) \le \psi\left(\varphi\left(d\left(x, y\right)\right)\right)\varphi\left(d\left(x, y\right)\right);\\
&\qquad \forall\left(x, y\right) \in X_i \times X_{i+1}, \ \forall i \in \overline{p}
\end{aligned} \tag{44}$$

for some $\varphi \in \Gamma_D$ and some $\psi \in \Psi_D(r_0)$. Then, T is said to be a generalized $(\psi - \varphi)$-weak p-cyclic contraction mapping.

Theorem 23. *Let* (X, d) *be a g.m.s. and let* $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ *be a generalized* $(\psi - \varphi)$*-weak* p*-cyclic contraction mapping for some function* $\psi \in \Psi_D(r_0)$*, some* $r_0 \in (0, 1)$*, and some* $\varphi \in \Gamma_D$*. Then, the following properties hold:*

(i)

$$\begin{aligned}
&\lim_{n\to\infty}\phi^{np+j}\left(\varphi\left(d\left(x_0, x_{mp+1}\right)\right)\right) = D;\\
&\qquad \forall m \in \mathbf{Z}_{0+}, \ \forall j \in \overline{p-1} \cup \{0\},\\
&\lim_{m\to\infty}\lim_{n\to\infty}\varphi\left(d\left(x_{mp+j}, x_{(m+n)p+j+1}\right)\right)\\
&\quad = \lim_{n\to\infty}\varphi\left(d\left(x_{np+j}, x_{(\ell+n)p+j+1}\right)\right) = D;\\
&\qquad \forall \ell \in \mathbf{Z}_{0+}, \ \forall j \in \overline{p-1} \cup \{0\},\\
&\lim_{n,m\to\infty} d\left(x_{mp+j}, x_{(m+n)p+j+1}\right)\\
&\quad = \lim_{n\to\infty} d\left(x_{np+j}, x_{(\ell+n)p+j+1}\right) = D;\\
&\qquad \forall \ell \in \mathbf{Z}_{0+}, \ \forall j \in \overline{p-1} \cup \{0\}
\end{aligned} \tag{45}$$

for any sequence $\{x_n\}$ *constructed from* $x_{n+1} = Tx_n$, $\forall n \in \mathbf{Z}_{0+}$ *for any given initial point* $x_0 \in \bigcup_{i\in\overline{p}} X_i$.

(ii) Any sequence $\{Tx_n\}$ *built from any given initial point* $x_0 \in \bigcup_{i\in\overline{p}} X_i$ *is bounded.*

(iii) Assume, in addition, that (X, d) *is a complete g.m.s. and that* $z_i \in X_i$ *has a best proximity point from* X_i *to* X_{i+1} *(i.e.,* $d(z_i, X_{i+1}) = D$*) for some given* $i \in \overline{p}$ *and that* X_{i+1} *is approximatively compact with respect to* X_i*. Then, there is a best proximity point* $z_{i+1} = Tz_i$ *from* X_{i+1} *to* X_i *which is unique in* X_{i+1} *if* z_i *is unique in* X_i *which is closed for the given* $i \in \overline{p}$.

Also, if all the subsets X_i *of* X; $\forall i \in \overline{p}$ *are closed, then there are* p *best proximity points* $z_{i+j} = T^j z_i \in X_{i+j}$, $\forall i, j \in \overline{p}$, *which are unique if one of them is unique, with each of them being also a fixed point of the respective composite mapping* $T^p : \bigcup_{j\in\overline{p}} X_j \mid X_i \to \bigcup_{j\in\overline{p}} X_j$, $\forall i \in \overline{p}$, *and then* p*-periodic fixed points of* $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$.

Proof. Since $x_0 \in \bigcup_{i\in\overline{p}} X_i$, we can consider equivalently x_0 to be an arbitrary point of X_i for some given arbitrary $i \in \overline{p}$ and we can define the sequence $\{x_n\}$ inductively by $x_{n+1} = Tx_n$,

$\forall n \in \mathbf{Z}_{0+}$. Since $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ is a $(\psi - \varphi)$-weak-cyclic p contraction mapping, one gets from (44) that

$$\begin{aligned}
&\varphi\left(d\left(x_{np+j}, x_{(n+m)p+j+1}\right)\right) \\
&\quad = \varphi\left(d\left(Tx_{np+j-1}, Tx_{(n+m)p+j}\right)\right) \\
&\quad \le \psi\left(\varphi\left(d\left(x_{np+j-1}, x_{(n+m)p+j}\right)\right)\right) \\
&\qquad \cdot \varphi\left(d\left(x_{np+j-1}, x_{(n+m)p+j}\right)\right) \\
&\quad < \varphi\left(d\left(x_{np+j-2}, x_{(n+m)p+j-1}\right)\right),
\end{aligned} \tag{46}$$

$\forall n, m \in \mathbf{Z}_{0+}$, $\forall j \in \overline{p-1} \cup \{0\}$, provided that $d(x_{np+j-2}, x_{(n+m)p+j-1}) > D$, and then the sequences $\{\varphi(d(x_{np+j-2}, x_{(n+m)p+j-1}))\}$, $\forall j \in \overline{p-1} \cup \{0\}$, bounded from below are strictly decreasing, so convergent to some $\eta(\in \mathbf{R}_+) \ge D$, since $\varphi \in \Gamma_D$, $1 > \psi(t) > r_0$ for $t > D$ and $\psi(D) = r_0 < 1$ since $\psi \in \Psi_D(r_0)$ and the property (ψ_1) of the class $\Psi_D(r_0)$. Note that $x_{np}, x_{(n+m)p} \in X_i$, $x_{mp+1} \in X_{i+1}$, $x_{np+j}, x_{(n+m)p+j} \in X_{i+j}$, and $x_{np-j} \in X_{i-j+p}$, $\forall n, \forall m \in \mathbf{Z}_{0+}$. It is be proved from (46), since $\varphi(D) = D$, that $\eta = D$ so that

$$\begin{aligned}
&\lim_{n\to\infty} \varphi\left(d\left(x_{np+j}, x_{(n+m)p+j+1}\right)\right) \\
&\quad = \lim_{n\to\infty} d\left(x_{np+j}, x_{(n+m)p+j+1}\right) = D; \\
&\qquad \forall j \in \overline{p-1} \cup \{0\}.
\end{aligned} \tag{47}$$

Assume that this is not the case, so that $\eta > D$, and proceed by contradiction to conclude that $\eta = D$. Suppose that $\lim_{n\to\infty}\varphi(d(x_{np+j}, x_{(n+m)p+j+1})) = \eta > D$ for some $j \in \overline{p-1} \cup \{0\}$. Thus, for each $\delta \in \mathbf{R}_+$, there is $n_\delta = n_\delta(\eta) \in \mathbf{Z}_{0+}$ and $\gamma_\eta = \gamma_\eta(\eta) \in [0, 1)$ such that for all $n(\in \mathbf{Z}_{0+}) \ge n_\delta$

$$\begin{aligned}
&\eta \le \varphi\left(d\left(x_{np+j}, x_{(n+m)p+j+1}\right)\right) < \eta + \delta; \\
&\psi\left(\varphi\left(d\left(x_{np+j}, x_{(n+m)p+j+1}\right)\right)\right) < \gamma_\eta.
\end{aligned} \tag{48}$$

$\forall j \in \overline{p-1} \cup \{0\}$, $\forall n \in \mathbf{Z}_{0+}$ since $\{\varphi(d(x_{np+j}, x_{(n+m)p+j+1}))\} \to \eta$ and since $\psi \in \Psi_D(r_0)$ is a stronger Meir-Keeler function. Then, one gets from (46) since $d(x_{np+j}, x_{(n+m)p+j+1}) \ge D$ and $\varphi(d(x_{np+j}, x_{(n+m)p+j+1})) \ge \varphi(D) \ge D$, $\forall j \in \overline{p-1} \cup \{0\}$, $\forall n, m \in \mathbf{Z}_{0+}$, that

$$\begin{aligned}
D < \eta &= \lim_{n\to\infty} \varphi\left(d\left(x_{np+j}, x_{(n+m)p+j+1}\right)\right) = \varphi\left(d\left(Tx_{np+j-1}, Tx_{(n+m)p+j}\right)\right) \\
&\le \max\left[D, \limsup_{n\to\infty}\left(\psi\left(\varphi\left(d\left(x_{np+j-1}, x_{(n+m)p+j}\right)\right)\right) \cdot \varphi\left(d\left(x_{np+j-1}, x_{(n+m)p+j}\right)\right)\right)\right] \\
&\le \max\left[D, \limsup_{n\to\infty}\left(\gamma_{np+j}\varphi\left(d\left(x_{np+j-1}, x_{(n+m)p+j}\right)\right)\right)\right] \cdots \le \max\left[D, \limsup_{n\to\infty}\left(\gamma_\eta^{(n-n_\delta)p+j}\varphi\left(d\left(x_{n_\delta p}, x_{(n_\delta+m)p+1}\right)\right)\right)\right] = D; \\
&\qquad \forall j \in \overline{p-1} \cup \{0\},\ \forall m \in \mathbf{Z}_{0+}
\end{aligned} \tag{49}$$

since $\gamma_\eta < 1$, a contradiction. Then, for any $m \in \mathbf{Z}_{0+}$,

$$\begin{aligned}
&\lim_{n\to\infty} \varphi\left(d\left(x_{np+j}, x_{(n+m)p+j+1}\right)\right) \\
&\quad = \lim_{n\to\infty} d\left(x_{np+j}, x_{(n+m)p+j+1}\right) = D; \\
&\qquad \forall j \in \overline{p-1} \cup \{0\}.
\end{aligned} \tag{50}$$

Property (i) has been proved. Property (ii) is proved in the same ways as its counterpart in Theorem 13 since it is based on Theorem 13(i) which still holds for this theorem and from the rectangular property of distances of the g.m.s. (X, d). Finally, Property (iii) follows from the given assumptions, similar to those of Theorem 13(ii) since (50) has been got from (46) which is a similar property to that used in the proof of Theorem 13(iii). □

Based on (50) and (46) obtained for the proof of Theorem 23, it is direct to prove in a similar way as it has been done for Theorem 14, Theorem 15 (see also Remark 16), and Theorem 21, the following results.

Theorem 24. *Let (X, d) be a complete g.m.s. and let $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ be a generalized $(\psi - \varphi)$-weak p-cyclic contraction mapping such that that the sets X_i; $\forall i \in \overline{p}$ are closed and intersect for some $\psi \in \Psi_0(0)$ and $\varphi \in \Gamma_0$. Then, there is a unique fixed point $z \in \bigcap_{i\in\overline{p}} X_i$.*

Theorem 25. *Let (X, d) be a complete g.m.s. admitting midpoints which has p nonempty closed subsets $X_i \subseteq X$, with $d(X_j, X_{j+1}) = D$, $\forall j \in \overline{p}$, where X is an abstract set, with one of the subsets X_i, for some $i \in \overline{p}$, being strictly convex. Assume that there is some strictly increasing function $\delta : (0, 1] \to [0, 2]$ such that X has the property (28) for all $x, y \in X_i$, $p \in X$, and $R \in \mathbf{R}_+, r \in [0, 2R]$. Let $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ be a generalized $(\psi - \varphi)$-weak p-cyclic contraction mapping for some $\psi \in \Psi_D(r_0)$, some $r_0 \in (0, 1)$, and some $\varphi \in \Gamma_D$. Then, $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ has a unique best proximity point $z_j \in X_j$, $\forall j \in \overline{p}$, which is also a p-periodic point of $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ and a fixed point of the composite mapping $T^p : \bigcup_{j\in\overline{p}} X_j \mid X_k \to \bigcup_{j\in\overline{p}} X_j$, $\forall k \in \overline{p}$.*

If $\bigcap_{i\in\overline{p}} X_i \neq \varnothing$, *then* $z_k = z \in \bigcap_{i\in\overline{p}} X_i$, $\forall k \in \overline{p}$, *is the unique fixed point of* $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$.

Theorem 26. *Let* (X, d) *be a g.m.s. and let* $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$ *be a generalized* $(\psi - \varphi)$*-weak* p*-cyclic contraction mapping. Then, the following properties hold:*

(i)

$$\lim_{n\to\infty} \psi^n (i, i-p) \left(\varphi_{i-p-1} \left(d\left(x_0, x_{mp+1}\right)\right)\right) = D_i$$
$$\lim_{m\to\infty} \lim_{n\to\infty} \varphi_{i-p-1} \left(d\left(x_{mp}, x_{(m+n)p+1}\right)\right) = \lim_{n\to\infty} \varphi_{i-p-1} \left(d\left(x_{np+j}, x_{(\ell+n)p+j+1}\right)\right) = D_i, \tag{51}$$

$\forall \ell, m \in \mathbf{Z}_{0+}$, $\forall j \in \overline{p-1} \cup \{0\}$, $\forall i \in \overline{p}$, *and for any given* $x_0 \in X_i$ *and any given arbitrary* $i \in \overline{p}$ *for some set of functions* $\psi_i \in \Psi_{D_i}(r_{0i})$, $\forall i \in \overline{p}$, *some real constants* $r_{0i} \in (0, 1)$, $\forall i \in \overline{p}$, *where* $\psi(i, i-p) = \psi_{i-p} \cdot \dots \cdot \psi_{i-1} \cdot \psi_i$ *is a composite mapping of* p *such functions.*

(ii) Assume that (X, d) *is a complete g.m.s. and that* $z_i \in X_i$ *has a best proximity point from* X_i *to* X_{i+1} *(i.e.,* $d(z_i, X_{i+1}) = D_i$*) for some given* $i \in \overline{p}$ *and that* X_{i+1} *is approximatively compact with respect to* X_i*. Then, there is a best proximity point* $z_{i+1} = Tz_i$ *from* X_{i+1} *to* X_i *which is unique and in* X_{i+1} *if* z_i *is unique and* X_{i+1} *is closed.*

Also, if all the subsets $X_i \subseteq X$, $\forall i \in \overline{p}$, *are closed, then there are* p *best proximity points* $z_{i+j} = T^j z_i \in X_{i+j}$, $\forall i, j \in \overline{p}$, *which are unique if anyone of them is unique. Each* $z_i \in X_i$ *is also a fixed point of the composite mapping* $T^p : \bigcup_{j\in\overline{p}} X_j \mid X_i \to \bigcup_{j\in\overline{p}} X_j$, $\forall i \in \overline{p}$, *and then a* p*-periodic fixed point of* $T : \bigcup_{i\in\overline{p}} X_i \to \bigcup_{i\in\overline{p}} X_i$.

Example 27. Examples 17–19 can be reformulated for generalized $(\psi - \varphi)$-weak p-cyclic contraction mappings with the replacement $\phi \in \Phi_D \to \psi \in \Psi_D(r_0)$ with some $r_0 \in (\gamma_2, 1)$ since $\gamma_2 \in (0, 1)$ and $D = d(X_1, X_2) < 1$.

Competing Interests

The author declares that he has no competing interests.

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