STABILITY AND ASSIGNMENT OF SPECTRUM IN SYSTEMS WITH DISCRETE TIME LAGS

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The asymptotic stability with a prescribed degree of time delayed systems subject to multiple bounded discrete delays has received important attention in the last years. It is basically proved that the α-stability locally in the delays (i.e., all the eigenvalues have prefixed strictly negative real parts located in $\mathrm{Re}\,s \leq -\alpha < 0$) may be tested for a set of admissible delays including possible zero delays either through a set of Lyapunov's matrix inequalities or, equivalently, by checking that an identical number of matrices related to the delayed dynamics are all stability matrices. The result may be easily extended to check the ε-asymptotic stability independent of the delays, that is, for all the delays having any values, the eigenvalues are stable and located in $\mathrm{Re}\,s \leq \varepsilon \to 0^-$. The above referred number of stable matrices to be tested is 2^r for a set of distinct r point delays and includes all possible cases of alternate signs for summations for all the matrices of delayed dynamics. The manuscript is completed with a study for prescribed closed-loop spectrum assignment (or "pole placement") under output feedback.

1. Stability results

Consider the time-invariant time-delay system

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{r} A_k x(t - h_k), \tag{1.1}$$

where $x \in \mathbb{R}^{\mathbf{n}}$ is the state vector, $h_k \geq 0$ $(k = 1,2,\ldots,r)$ are r point constant delays. The initial conditions of (1.1) are given by any absolutely continuous function $\varphi : [-h,0] \to \mathbb{R}^n$, with possibly finite discontinuities on a subset of zero measure of $[-h,0]$, where $h = \max_{1\leq k\leq r}(h_k)$. The system (1.1) is said to be α-symptotically stable locally in the delays (α-ASLD) for all $h_k \in [0,\overline{h}_k]$ for some $\alpha \in \mathbb{R}_+$, $\overline{h}_k > 0$ $(k = 1,2,\ldots,r)$ (i.e., all the roots of

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$\mathrm{Det}(sI - A_0 - \sum_{k=1}^{r} A_k e^{-h_{ks}}) = 0$ lie in $\mathrm{Re}\, s \le -\alpha < 0$), see [1–5]. The following result was proved in [5].

Result 1.1. The system (1.1) is α-ASLD if there is a real n-matrix $P = P^T > 0$ such that the following Lyapunov's matrix inequality holds

$$PA_0 + A_0^T P + \left[\sum_{k=1}^{r} \pm e^{\overline{h}_{k\alpha}} \left(PA_k + A_k^T P \right) \right]_m \le -2\alpha P \tag{1.2}$$

for $m = 1,2,\dots,2^r \rho \in [-|\rho_0|,0)$, where $[\pm]_m$ denotes all possible 2^r cases of alternating sign. The system (1.1) is asymptotically stable independent of the delays if

$$PA_0 + A_0^T P + \left[\sum_{k=1}^{r} \pm \left(PA_k + A_k^T P \right) \right]_m < 0. \tag{1.3}$$

Result 1.1 was proved based on the subsequent technical fact also proved in [5].

Fact 1.2. For any set of symmetric constant n-matrices $\{T_k;\ k = 0,1,\dots,r\}$, the inequality

$$T_0 \sum_{k=1}^{r} \eta_k T_k \le -2\alpha P \tag{1.4}$$

holds for some $\alpha \in \mathbb{R}_+$, a real n-matrix $P = P^T > 0$, and all real $\eta_k \in [-\eta_{kM}, \eta_{kM}]$ ($k = 1,2,\dots,r$) if and only if it holds at the 2^r vertices of the hyper-rectangle:

$$H := \left\{ \eta = \left(\eta_1, \eta_2, \dots, \eta_r \right)^T \in \mathbb{R}^r \mid \eta_k \in [-\eta_{kM}, \eta_{kM}];\ k = 1,2,\dots,r \right\}. \tag{1.5}$$

The following technical lemma will be then used to prove the main results.

Lemma 1.3. *A set of r real matrices A_i ($i = 1,2,\dots,r$) consists of stability matrices with stability abscissas $(-\alpha_i) < 0$ if and only if the set of r Lyapunov's matrix inequalities*

$$A_i^T P + PA_i \le -2\alpha P; \quad i = 1,2,\dots,r, \tag{1.6}$$

hold for any real constant $\alpha \in (0, \mathrm{Min}_{1 \le i \le r}(\alpha_i)]$ provided that $\mathrm{Min}(\alpha_i)_{1 \le i \le r}$ is sufficiently large.

Proof. Assume that all the A_i are stability matrices with stability abscissas $(-\alpha_i) < 0$ then $(A_i + \alpha_i I)$ are stable matrices and

$$\left(A_i^T + \alpha_i I \right) P + P\left(A_i + \alpha_i I \right) < 0 \tag{1.7}$$

for $i = 1,2,\dots,r$, all real n-matrix $P = P^T > 0$ then

$$A_i^T P + PA_i \le -2\alpha_i P \le -2\alpha P < 0 \tag{1.8}$$

for $i = 1,2,\dots,r$ and all $\alpha > 0$ as specified. To prove the converse, consider three cases for (1.7) to fail and then proceed by contradiction.

Case 1. Assume $(A_i^T+\alpha_i I)P+P(A_i+\alpha_i I)>0$ for at least one $i\in\{1,2,\dots,r\}$. Thus, $(A_i+\alpha_i I)$ is unstable from Lyapunov's instability theorem and one of its eigenvalues has positive real part. Thus, the stability abscissa of A_i exceeds $(-\alpha_i)$ which leads to a contradiction.
Case 2. Assume $(A_i^T+\alpha_i I)P+P(A_i+\alpha_i I)=0$ for at least one $i\in\{1,2,\dots,r\}$. Consider the linear and time-invariant system $\dot{x}(t)=(A_i+\alpha_i I)x(t)$ for any bounded $x(0)=x_0\in\mathbb{R}^{\mathbf{n}}$ with a Lyapunov-Razumikhin function candidate $V(x)=x^T(t)Px(t)$, some real matrix $P=P^T>0$. It turns out that

$$\dot{V}(x)\equiv 0\Longrightarrow V(x)=V(0)<\infty\Longrightarrow \|x(t)\|_2^2\geq\lambda_{\min}(P^{-1})V(0)>0 \tag{1.9}$$

for any $x_0\neq 0$. Thus, it turns out that $x(t)$ cannot tend to zero as $t\to\infty$ if $x_0\neq 0$ and then $(A_i+\alpha_i I)$ is not a stability matrix and thus the stability abscissa of A_i is less than $(-\alpha_i)$ what again leads to a contradiction.
Case 3. Assume that $(A_i^T+\alpha_i I)P+P(A_i+\alpha_i I)$ is indefinite. Decompose $A_k=A_i+\Delta A_{ki}$ for some $1\leq k\leq r$ and all $1\leq i\neq k\leq r$. Thus for any positive definite symmetric square n-matrix Q, there exists a positive definite matrix P such that

$$\begin{aligned}&\left(A_k^TP+\alpha_k I\right)P+P(A_k+\alpha_k I)\\&\quad=\left(A_i^TP+\alpha_i I\right)P+P(A_i+\alpha_i I)+\left(\Delta A_{ki}^TP+P\Delta A_{ki}\right)+2(\alpha_k-\alpha_i)P\\&\quad=-Q+\left(\Delta A_{ki}^TP+P\Delta A_{ki}\right)+2(\alpha_k-\alpha_i)P\end{aligned} \tag{1.10}$$

with $P=\int_0^\infty e^{(A_i^T+\alpha_i I)\tau}Qe^{(A_i^T+\alpha_i I)\tau}d\tau$ satisfying $Q=-(A_i^TP+\alpha_i I)P-P(A_i+\alpha_i I)$.

Note that $\lambda_{\max}(P)\leq K^2/2(\rho_i+\alpha_i)$ for some real constant $K\geq 1$ with $(-\rho_i)<0$ being the stability abscissa of A_i. Thus,

$$\left(A_k^TP+\alpha_k I\right)P+P(A_k+\alpha_k I)<0\quad \text{if } \lambda_{\min}(Q)>2(\|\Delta A_{ki}\|_2+(\alpha_k-\alpha_i))\lambda_{\max}(P) \tag{1.11}$$

what is guaranteed if

$$1\geq\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}\geq\frac{K^2}{\rho_i+\alpha_i}\left[\,|\alpha_i-\alpha_k|+\underset{1\leq i\leq r}{\operatorname{Max}}(\|\Delta A_{ki}\|_2)\right] \tag{1.12}$$

which always holds for sufficiently large ρ_i (i.e., for sufficiently stable A_i) for given $\|\Delta A_{ki}\|_2$, $i=1,2,\dots,r$, and, thus, for sufficiently large $\operatorname{Min}(\alpha_i)_{1\leq i\leq r}$. □

The main result of this section is now stated.

Theorem 1.4. *The subsequent items hold as follows.*

(i) *The system (1.1) is α'-SLD for all $h_k\in[0,\overline{h}_k]$ if the 2^r-matrices $\mathbf{A_m}=A_0+[\sum_{k=1}^r\pm e^{\overline{h}_{k\alpha}}A_k]_m+\alpha I$ are all stability matrices for $m=1,2,\dots,2^r$ and some real $\alpha>0$.*

(ii) *Assume that $\mathbf{A_{m\rho}}=A_0+[\sum_{k=1}^r\pm\rho_kA_k]_m+\alpha I$ are all stability matrices for $\rho^T=(\rho_1,\rho_2,\dots,\rho_r)$ with $\rho_k\geq 1$, $k=1,2,\dots,r$, and some real $\alpha>0$. Then, the system (1.1) is α-ASLD for all delays $h_k\in[0,\overline{h}_k]$ with $\overline{h}_k=(1/\alpha)\ln\rho_k$ for all $k=1,2,\dots,r$.*

(iii) *Assume that*

$$\mathbf{A}_{\mathbf{m}\rho} = A_0 + \left[\sum_{k=1}^{r} \pm \rho_k A_k \right]_m + \alpha I \tag{1.13}$$

are all stability matrices for any real constants $\rho_k > 0$, $k = 1,2,\ldots,r$, and some real $\alpha > 0$. Thus, all the systems of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{r} \pm \frac{\rho_k}{\beta_k} A_k x(t-h_k) \tag{1.14}$$

are α-SLD for any prefixed set of real scalars $\beta_k > 1$ $(k = 1,2,\ldots,r)$ and all delays $h_k \in [0,(1/\alpha)\ln\beta_k]$. If $\beta_k = \rho_k > 1$ for all $k = 1,2,\ldots,r$, then (1.1) is α-ASLD.

(iv) *If $\mathbf{A}'_{\mathbf{m}} = A_0 + [\sum_{k=1}^{r} \pm\rho_k A_k]_m$ are all stability matrices with $\rho_k = 1$ for all $m = 1,2,\ldots,2^r$, $k = 1,2,\ldots,r$, then all the delay systems (1.14) are asymptotically stable independent of the delays for any set $\beta_k > 1$ $(k = 1,2,\ldots,r)$. If $\beta_k = \rho_k = 1$ $(k = 1,2,\ldots,r)$, then (1.1) is asymptotically stable independent of delays.*

Proof. (i) Consider 2^r Lyapunov's matrix equations

$$\mathbf{A}_{\mathbf{m}}^{T} P_m + P_m \mathbf{A}_{\mathbf{m}} = -Q_m = -Q_m^T < 0 \tag{1.15}$$

for $m = 1,2,\ldots,2^r$. Since $\mathbf{A}_{\mathbf{m}}$ are stability matrices, then the unique solutions P_m to the Lyapunov's equation are $P_m = P_m^T = \int_0^\infty e^{\mathbf{A}_{\mathbf{m}}^{\mathbf{T}}\tau} Q_m e^{\mathbf{A}_{\mathbf{m}}\tau}\, d\tau$, $m = 1,2,\ldots,2^r$. On the other hand, $(-Q_m) \leq -2\alpha P_m$ (or, equivalently, $Q_m \geq 2\alpha P_m$) for all $m = 1,2,\ldots,2^r$ if

$$0 < \alpha \leq \frac{1}{2} \frac{\mathrm{Min}_{1\leq i\leq 2^r}\left[\lambda_{\min}(Q_m)\right]}{\mathrm{Max}_{1\leq i\leq 2^r}\left[\lambda_{\max}(P_m)\right]}. \tag{1.16}$$

Note also that for any symmetric positive definite matrices P_m and all $P \geq P_m$,

$$\begin{aligned} \mathbf{A}_{\mathbf{m}}^{\mathbf{T}} P + P\mathbf{A}_{\mathbf{m}} &= \mathbf{A}_{\mathbf{m}}^{\mathbf{T}} P_m + P_m \mathbf{A}_{\mathbf{m}} + \mathbf{A}_{\mathbf{m}}^{\mathbf{T}} \Delta P_m + \Delta P_m \mathbf{A}_{\mathbf{m}} \\ &\leq -Q_m^T + \left(\mathbf{A}_{\mathbf{m}}^{\mathbf{T}} \Delta P_m + \Delta P_m \mathbf{A}_{\mathbf{m}}\right) < 0 \end{aligned} \tag{1.17}$$

with $\Delta P_m = P - P_m$; $m = 1,2,\ldots,2^r$ satisfying

$$\left(\mathbf{A}_{\mathbf{m}}^{\mathbf{T}} \Delta P_m + \Delta P_m \mathbf{A}_{\mathbf{m}}\right) = -Q_m < 0 \tag{1.18}$$

since $\mathbf{A}_{\mathbf{m}}$ is a stability matrix (see the proof of Lemma 1.3). Thus, for any $P \geq P_m$ $(m = 1,2,\ldots,2^r)$,

$$\mathbf{A}_{\mathbf{m}}^{\mathbf{T}} P + P\mathbf{A}_{\mathbf{m}} \leq -2\alpha P \tag{1.19}$$

holds so that P is nonunique and thus the system (1.1) is α-SLD from Result 1.1 and all the set of delays h_k $(k = 1,2,\ldots,r)$ satisfy $\rho_k = e^{\overline{h}_{k\alpha}} \geq e^{h_{k\alpha}} \geq 1$, since $\overline{h}_k \geq 0$, $k = 1,2,\ldots,2^r$, equivalently is guaranteed if

$$0 \le \tau_k \le \tau_{kM} \le 2\frac{\text{Max}_{1\le m\le 2^r}\left[\lambda_{\max}(P_m)\right]}{\text{Min}_{1\le m\le 2^r}\left[\lambda_{\max}(Q_m)\right]}\ln\rho_k \tag{1.20}$$

and the proof of (i) has been completed.

(ii) It follows directly from (i) with $\rho_k = \overline{h}_k\alpha$ for $k = 1,2,\dots,r$ and $\alpha > 0$.

(iii) Consider the nonunique factorizations $\rho_k = \gamma_k\beta_k$, for any sequences $\{\gamma_k;\ k = 1,2,\dots,r\}$, $\{\beta_k;\ k = 1,2,\dots,r\}$, being only subject to the constraints $\beta_k > 1$ for all $k = 1,2,\dots,r$. Thus, it follows from (ii) for $\rho_k = \gamma_k\beta_k \equiv \overline{h}_k\alpha$ $(k = 1,2,\dots,r)$ that if the matrices

$$\mathbf{A}_{\mathbf{m}\beta} = A_0 + \left[\sum_{k=1}^{r} \pm\beta_k\left(\pm\frac{\rho_k}{\beta_k}A_k\right)\right]_m + \alpha I \tag{1.21}$$

are all stability matrices for $m = 1,2,\dots,2^r$, then all the systems (1.14) are α-ASLD.

(iv) It follows from (iii) since the asymptotic stability of the systems (1.14) for all possible values of the delays from zero to infinity, with $\rho_k = \gamma_k\beta_k \equiv \overline{h}_k\alpha = 1$, $\alpha \to 0^+$ and $\overline{h}_k = o(\alpha^{-1}) \to \infty$ for all $k = 1,2,\dots,r$, is guaranteed by testing the 2^r given n-matrices for all $m = 1,2,\dots,2^r$. □

Remarks 1.5. (1) Note that the property of α-asymptotic stability locally in the delays of the system (1.1), which specifies admissibility domains $\lfloor 0,\overline{h}_k\rfloor$ for the delays for $k = 1,2,\dots,r$, may be tested by checking if 2^r matrices

$$\mathbf{A}_{\mathbf{m}} = A_0 + \left[\sum_{k=1}^{r} \pm e^{\overline{h}_k\alpha}A_k\right]_m + \alpha I \tag{1.22}$$

are all stability matrices for $m = 1,2,\dots,2^r$. This property is equivalent to all the matrices

$$\mathbf{A}'_{\mathbf{m}} = A_0 + \left[\sum_{k=1}^{r} \pm e^{\overline{h}_k\alpha}A_k\right]_m \tag{1.23}$$

to be stability matrices with stability abscissas of at least $(-\alpha) < 0$.

(2) Note that $\mathbf{A}_{\mathbf{m}}$ being a stability matrix implies that

$$\widehat{\mathbf{A}}_{\mathbf{m}0} = A_0 + \left[\sum_{k=1}^{r} \pm e^{\overline{h}_k\alpha}A_k\right]_m + \alpha_0 I \tag{1.24}$$

are all stable for $\alpha_0 \in (0,\alpha]$ and $\overline{h}'_k = \overline{h}_k\alpha/\alpha_0$ for $k = 1,2,\dots,r$. Thus, the system (1.1) is also α_0-ASLD for all delays $h_k \in [0,\overline{h}_k\alpha/\alpha_0]$ or $k = 1,2,\dots,r$. As a result, if the system (1.1) is α-ASLD for $h_k \in [0,\overline{h}_k]$, then it is also α_0-ASLD for all $\alpha_0 \in (0,\alpha]$ and delays $h_k \in [0,\overline{h}_k\alpha/\alpha_0]$ for $k = 1,2,\dots,r$.

(3) Note that all $\mathbf{A}_{\mathbf{m}}$ being stability matrices for any $\alpha \ge 0$ implies that A_0 and $(\sum_{k=0}^{r} A_k)$ are both stability matrices. In other words, the delayed dynamics-free auxiliary system $\dot{z}(t) = A_0 z(t)$ and the delay-free system $\dot{z}(t) = (\sum_{k=0}^{r} A_k)z(t)$ are both globally exponentially stable. Both conditions are known to be necessary for stability independent of

the delays (see [3, 5]) and they are obtained in this context as a direct consequence of Theorem 1.4.

2. Output-feedback stabilization with prescribed pole placement

Now system (1.1) is considered as forced and with a measurable output

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{r} A_i x(t-ih) + bu(t),$$
$$y(t) = c^T x(t) + du(t), \tag{2.1}$$

where $h \geq 0$ is now the base delay and $h_i = ih$ $(i = \overline{1,r})$. The change of notation and specification of delays related to a base one h is made by description simplicity reasons. The transfer function of (2.1) is defined in a standard way by using the Laplace transforms of the output and input as $P(s) = [Y(s)/U(s)]_{\varphi\equiv 0}$ leading to

$$P(s) = \frac{B(s)}{A(s)} = c^T \left(sI - \sum_{i=0}^{r} A_i e^{-ihs} \right)^{-1} b + d, \tag{2.2}$$

where $A(s)$ and $B(s)$ are quasipolynomials defined by

$$A(s) = \det\left(sI - \sum_{i=0}^{r} A_i e^{-ihs} \right)$$
$$= \sum_{i=0}^{q} A_i(s) e^{-ihs} = \sum_{i=0}^{n} A_i^*\left(e^{-hs}\right) s^i = \sum_{i=0}^{q}\sum_{k=0}^{n} a_{ik} s^k e^{-ihs} \tag{2.3a}$$

$$B(s) = c^T \operatorname{Adj}\left(sI - \sum_{i=0}^{r} A_i e^{-ihs} \right) b + dA(s)$$
$$= \sum_{i=0}^{q'} B_i(s) e^{-ihs} = \sum_{i=0}^{m} B_i^*\left(e^{-hs}\right) s^i = \sum_{i=0}^{q'}\sum_{k=0}^{m} b_{ik} s^k e^{-ihs} \tag{2.3b}$$

with q and q' being integers satisfying $q' \leq q \leq rn$. For exposition simplicity, it is assumed without loss of generality that $q' = q$. Otherwise, (2.3b) still applies by zeroing the necessary polynomials $B_{(\cdot)}$:

$$B_i(s) = \sum_{i=0}^{m_i} b_{ik} s^k; \qquad A_i(s) = \sum_{i=0}^{n_i} a_{ik} s^k \tag{2.4}$$

are polynomials of respective degrees m_i and n_i $(i = \overline{0,q})$ with $m_i \leq m_0 = m \leq n$ and $n_i \leq n_0 = n$ for $i = \overline{0,q}$ with $m = n$ if and only if $d \neq 0$ in (2.1), that is, the plant is not strictly proper plant and $m \leq n-1$, otherwise. Note that $n = n_0 \geq \operatorname{Max}(m, \operatorname{Max}_{1\leq i\leq q}(n_i, m_i))$ since the transfer function (2.2)-(2.3) obtained from (2.1) is realizable. Alternative polynomials $B_i^*(e^{-hs})$ and $A_i^*(e^{-hs})$ are defined in the same way leading to an equivalent description of (2.1). The following result is the main one of this section.

Theorem 2.1 (Spectrum assignment and closed-loop stability). *Assume that the transfer function (2.2)-(2.3) has no pole-zero cancellation and that the property is not lost under zero delayed dynamics. Thus, the following items hold.*

(i) *There exist infinitely many polynomial pairs* $(R_i(s), S_i(s))$ *which satisfy the* v *nested Diophantine equations of polynomials:*

$$A_0(s)R_i(s) + B_0(s)S_i(s) = \widehat{A}_{mi}(s) - \sum_{l=1}^{i}\left(A_l(s)R_{i-l}(s) + B_l(s)S_{i-l}(s)\right) \quad \text{for } i = \overline{0, v-1} \tag{2.5}$$

for any integer $v \geq 1$. *Furthermore, if* $n_{m0} \geq 2n-1$, *then there is at least a solution* $(R_i(s), S_i(s))$, $i = \overline{0, v-1}$, *which satisfies the following degree constraints:*

$$\begin{gathered} n'_0 = n_{m0} - n, \quad m'_i(s) = n-1 \quad \text{for } i = \overline{0, v-1}, \\ \operatorname{Max}\left(n'_i, m-1\right) = \operatorname{Max}\left(n_{mi}, \operatorname*{Max}_{1 \leq k \leq i}\left(n_k + n'_{i-k}\right)\right) - n. \end{gathered} \tag{2.6}$$

(ii) *If Assumptions* (1)-(2) *hold and* $n_{m0} \geq 2n$, *then it is possible to build infinitely many proper rational functions of the form*

$$Q(s) = \frac{\sum_{l=0}^{v-1}\left[S_l(s) - \Lambda_0(s)A_0(s)\right]e^{-lhs}}{\sum_{l=k}^{v-1}\left[R_l(s) + \Lambda_0(s)B_0(s)\right]e^{-lhs}} \tag{2.7}$$

with existing polynomial solution pairs $(R_i(s) - \Lambda_0(s)A_0(s), S_i(s) + \Lambda_0(s)B_0(s))$ *verifying (2.5) provided that* $(R_i(s), S_i(s))$ *are also solutions to (2.5) where* $\Lambda_0(s) = \lambda_0$ *is any real scalar (i.e., any polynomial of zero degree) if* $n > m$ *and* $\Lambda_0(s)$ *is any arbitrary polynomial of arbitrary degree otherwise. If* $n_{m0} = 2n-1$, *then (2.5) is realizable for* $\Lambda_0(s) = 0$ *if* $n > m$ *and with arbitrary* $\Lambda_0(s)$ *if* $n = m$.

(iii) *Assume that the controller transfer function* $K_v(s) = S(s)/R(s)$ *takes the subsequent specific form*

$$\frac{\sum_{l=0}^{v-1}\left[S_l(s) - \Lambda_0(s)A_0(s)\right]e^{-lhs}}{\sum_{l=k}^{v-1}\left[R_l(s) + \Lambda_0(s)B_0(s)\right]e^{-lhs} + R_v(s)}, \tag{2.8}$$

where $(R_i(s), S_i(s))$ *are pairs of polynomials being any solutions to (2.5),* $i = \overline{0, v-1}$, $\Lambda_0(s)$ *is chosen according to item* (ii), $S_v(s)$ *is an arbitrary polynomial of degree not exceeding* $(n-1)$, *and*

$$\begin{aligned} R_v(s) &= \frac{N_v(s)}{D_v(s)} \\ &= \frac{1}{A(s)}e^{-(l-v)hs}\sum_{l=v}^{v+q}\left[\widehat{A}_{ml}(s) - \left(\sum_{i=\operatorname{Max}(v\cdot l-v+1)}^{\operatorname{Min}(l,q)} A_i(s)R_{l-i}(s) + \sum_{i=\operatorname{Max}(\nu\cdot l-v)}^{\operatorname{Min}(l,q)} B_i(s)S_{l-i}(s)\right)\right]. \end{aligned} \tag{2.9}$$

Then, the closed-loop spectrum satisfies

$$A(s)R(s)+B(s)S(s)=\widehat{A}_m^*(s)=\sum_{i=0}^{v-1}\widehat{A}_{mi}(s)e^{-ihs} \tag{2.10}$$

with the closed-loop being stable with poles in $\widehat{A}_m^*(s)=0$ *and a closed-loop stable cancellation of the plant poles provided that* $\widehat{A}_m^*(s)=\sum_{l=0}^{v-1}A_{ml}(s)e^{-lhs}$ *is a strictly Hurwitzian quasipolynomial satisfying* $n_{m0}\geq 2n-1$.

(iv) *If the suited spectrum satisfies* $n_{m0}\geq 2n-1$ *and the controller is simplified to have a transfer function* $K_v^*(s)=Q(s)$ *(i.e.,* $R_v(s)$ *and* $S_v(s)$ *are zeroed), then the closed-loop spectrum is set to the zeros of*

$$\{\widehat{A}_m^*(s)\}+\left\{\sum_{\ell=v}^{v+q}\sum_{i=\mathrm{Max}(0,l-v)}^{\mathrm{Min}(l,q)}\left[A_i(s)R_{l-i}(s)+B_i(s)S_{l-i}(s)\right]e^{-lhs}\right\} \tag{2.11}$$

without cancellations of the plant poles.

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