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Characterization of (asymptotically) Kerr–de Sitter-like spacetimes at null infinity[*]

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Abstract

We investigate solutions $(\mathcal{M}, g)$ to Einstein's vacuum field equations with positive cosmological constant Λ which admit a smooth past null infinity $\mathscr{I}^-$ à la Penrose and a Killing vector field whose associated Mars–Simon tensor (MST) vanishes. The main purpose of this work is to provide a characterization of these spacetimes in terms of their Cauchy data on $\mathscr{I}^-$. Along the way, we also study spacetimes for which the MST does not vanish. In that case there is an ambiguity in its definition which is captured by a scalar function Q. We analyze properties of the MST for different choices of Q. In doing so, we are led to a definition of 'asymptotically Kerr–de Sitter-like spacetimes', which we also characterize in terms of their asymptotic data on $\mathscr{I}^-$.

Keywords: characterization, Kerr–de Sitter-like, cosmological constant, null infinity, Mars–Simon tensor

1. Introduction

This is the first in a series of at least two papers [20] in which we (resp. some of us) analyze the asymptotic structure, and a certain initial value problem, for vacuum solutions of

Einstein's equations

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{1.1}$$

on a 4-dimensional spacetime $(\mathcal{M}, g)$, where g is smooth and Λ is a ('cosmological') constant. We focus on the case $\Lambda > 0$ but compare occasionally with $\Lambda = 0$. Space–time indices are Greek, while coordinates in $1 + 3$ splits are denoted by $\{x^\alpha\} = \{t, x^i\}$ (rather than $\{x^0, x^i\}$), with corresponding tensorial indices. Our conventions for the signature, the curvature tensor $R_{\mu\nu\sigma}{}^\kappa$, the Weyl tensor $C_{\mu\nu\sigma}{}^\kappa$, the Ricci tensor $R_{\mu\sigma}$ and the scalar curvature R follow e.g. [33]. The Levi-Civita connection of g is denoted by ∇.

The setting of our work is an asymptotic structure *à la Penrose* [9, 25]. By that we mean that an appropriate conformal rescaling of $(\mathcal{M}, g)$

$$g \mapsto \widetilde{g} = \Theta^2 g, \quad \mathcal{M} \overset{\phi}{\hookrightarrow} \widetilde{\mathcal{M}}, \quad \Theta|_{\phi(\mathcal{M})} > 0, \tag{1.2}$$

leads to an *unphysical spacetime* $(\widetilde{\mathcal{M}}, \widetilde{g})$ which admits a representation of null infinity

$$\mathscr{I} = \{\Theta = 0, \ \mathrm{d}\Theta \neq 0\} \bigcap \partial\phi(\mathcal{M})$$

through which the unphysical metric $\widetilde{g}$ and the conformal factor Θ can be smoothly extended. $\mathscr{I}$ is a smooth hypersurface which consists of two (not necessarily connected) subsets: future and past null infinity, distinguished by the absence of endpoints of past or future causal curves contained in $(\mathcal{M}, g)$, respectively. In this paper we will normally denote by $\mathscr{I}^-$ and $\mathscr{I}^+$ chosen connected components of past and future null infinity, respectively. Clearly, all initial value results in this paper starting from $\mathscr{I}^-$ have obvious 'final value counterparts' obtained via replacing $\mathscr{I}^-$ by $\mathscr{I}^+$, 'future' by 'past', etc. We will implicitly identify $\mathcal{M}$ with its image $\phi(\mathcal{M}) \subset \widetilde{\mathcal{M}}$, so that we can write $\widetilde{g} = \Theta^2 g$. Indices of physical and unphysical fields will be raised and lowered with g and $\widetilde{g}$, respectively.

In this setting, Friedrich [9, 10] has shown that, in terms of suitable variables, the field equations (1.1) become a regular, symmetric hyperbolic system on $(\widetilde{\mathcal{M}}, \widetilde{g})$. We recall these 'metric conformal field equations' (MCFE) in section 2.1. An important member of the MCFE is the rescaled Weyl tensor

$$\widetilde{d}_{\alpha\beta\gamma}{}^\delta := \Theta^{-1} C_{\alpha\beta\gamma}{}^\delta. \tag{1.3}$$

and key properties of $C_{\alpha\beta\gamma}{}^\delta$ and $\widetilde{d}_{\alpha\beta\gamma}{}^\delta$ are the following:

  I. $C_{\alpha\beta\gamma}{}^\delta$ vanishes on $\mathscr{I}$, whence $\widetilde{d}_{\alpha\beta\gamma}{}^\delta$ extends regularly to $\mathscr{I}$.
 II. $C_{\alpha\beta\gamma}{}^\delta$ satisfies a regular, linear, homogeneous symmetric hyperbolic system on $(\mathcal{M}, g)$.
III. $\widetilde{d}_{\alpha\beta\gamma}{}^\delta$ satisfies a regular, linear, homogeneous symmetric hyperbolic system on $(\widetilde{\mathcal{M}}, \widetilde{g})$.

These properties, together with stability of solutions of symmetric hyperbolic systems, are the key ingredients in uniqueness and stability results of asymptotically simple spacetimes $(\widetilde{\mathcal{M}}, \widetilde{g})$ as defined in definition 9.1. of [11]; the latter definition includes the requirements that $(\widetilde{\mathcal{M}}, \widetilde{g})$ has a compact Cauchy hypersurface and every maximally extended null geodesic has a past endpoint on $\mathscr{I}^-$ and a future endpoint on $\mathscr{I}^+$. We give here first a uniqueness result for de Sitter spacetime and then a sketchy version of the stability result (theorem 9.8 of [11]) which applies in particular to de Sitter.

Theorem 1.1

> ***Uniqueness of de Sitter.*** *Let smooth data for the MCFE be given on a $\mathscr{I}^-$ which is topologically $\mathbb{S}^3$ and such that $\widetilde{d}_{\alpha\beta\gamma}{}^{\delta}|_{\mathscr{I}^-}$ vanishes identically. Then the evolving spacetime $(\widetilde{\mathscr{M}}, \widetilde{g})$ is isometric to de Sitter.*
>
> ***Stability of aymptotically simple solutions.*** *Given an asymptotically simple spacetime $(\widetilde{\mathscr{M}}, \widetilde{g})$, then any data for the MCFE on $\mathscr{I}^-$ which are close to the data for $(\widetilde{\mathscr{M}}, \widetilde{g})$ (in terms of suitable Sobolev norms) evolve to an asymptotically simple spacetime.*

A motivation for the present work is to generalize these uniqueness and stability results to more general solutions of (1.1). Using again properties I.-III. above, it is straightforward to generalize the above results on asymptotically simple solutions to corresponding 'semiglobal' results for any concrete family of solutions, where 'semiglobal' means the domain of dependence of $\mathscr{I}^-$. On the other hand, and needless to say, any fully *global* results for solutions which are not asymptotically simple but contain horizons and singularities involve 'cosmic censorship' issues and will be very complicated. The main targets of the present work are Kottler (Schwarzschild–de Sitter) and Kerr–de Sitter (KdS) spacetimes for which the topology of each connected component of $\mathscr{I}$ is $\mathbb{R} \times \mathbb{S}^2$. Our main achievement is a *semi-global* uniqueness result, namely theorem 1.3, for a class of solutions which includes Kerr–de Sitter. What makes our result highly non-trivial is its particular formulation which we *expect to be* useful for the fully global problem, for reasons given below.

In this uniqueness result, and from now onwards, we assume that $(\mathscr{M}, g)$ admits a non-trivial Killing vector field (KVF) X,

$$(\mathscr{L}_X g)_{\mu\nu} \equiv 2\nabla_{(\mu}X_{\nu)} = 0. \tag{1.4}$$

Since X^μ is a KVF, $F_{\mu\nu} := \nabla_\mu X_\nu$ is a two-form: $F_{(\mu\nu)} = 0$.

The main purpose of this assumption is to achieve a simplification and to permit the use of a special technique. However, as an aside we note that the existence of the isometry *might* change the character of the stability problem substantially. To see this on a heuristic basis, consider data for the MCFE on $\mathscr{I}^-$ which are at the same time Killing initial data, and which are *close to* Kerr–de Sitter in a suitable sense. Now consider the time-evolution of such data, and assume that the spacetime can be extended beyond its ('cosmological') Cauchy horizon (as it is the case for Schwarzschild–de Sitter and Kerr–de Sitter). In this extension, the isometry should become timelike, and now another conjecture, namely uniqueness of stationary black-holes, should lead to Kerr–de Sitter in the region between the event and the cosmological horizon. Extending backwards to the domain of dependence of $\mathscr{I}^-$ suggests that the 'near Kerr–de Sitter' data will actually be Kerr–de Sitter in the above setting. Accordingly, the existence of the isometry, together with reasonable global assumptions, can turn a stability into a uniqueness problem. This 'effect' is of course familiar from uniqueness results for stationary, asymptotically flat solutions.

While obtaining global results sketched above is far beyond our present scope, it motivates our local analysis, in particular the use of the so-called Mars–Simon tensor (MST) [17, 18, 31] in theorem 1.3. This tensor is defined as follows:

$$\mathcal{S}_{\mu\nu\sigma\rho} := \mathcal{C}_{\mu\nu\sigma\rho} + Q\, \mathcal{U}_{\mu\nu\sigma\rho}, \tag{1.5}$$

in terms of the quantities

$$\mathcal{C}_{\mu\nu\sigma\rho} := C_{\mu\nu\sigma\rho} + iC^\star_{\mu\nu\sigma\rho}, \tag{1.6}$$

$$\mathcal{U}_{\mu\nu\sigma\rho} := -\mathcal{F}_{\mu\nu}\mathcal{F}_{\sigma\rho} + \frac{1}{3}\mathcal{F}^2\mathcal{I}_{\mu\nu\sigma\rho}, \tag{1.7}$$

$$\mathcal{I}_{\mu\nu\sigma\rho} := \frac{1}{4}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma} + i\eta_{\mu\nu\sigma\rho}), \tag{1.8}$$

$$\mathcal{F}_{\mu\nu} := F_{\mu\nu} + iF^{\star}_{\mu\nu}, \tag{1.9}$$

$$\mathcal{F}^2 := \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}. \tag{1.10}$$

In these expressions $\eta_{\mu\nu\sigma\rho}$ is the volume form of g, $\star$ the corresponding Hodge dual and Q is a function. $\mathcal{F}_{\mu\nu}$ and $\mathcal{C}_{\alpha\beta\gamma\delta}$ are self-dual, i.e. they satisfy $\mathcal{F}^{\star}_{\mu\nu} = -i\mathcal{F}_{\mu\nu}$ and $\mathcal{C}^{\star}_{\alpha\beta\gamma\delta} = -i\mathcal{C}_{\alpha\beta\gamma\delta}$. The symmetric double two-form $\mathcal{I}_{\mu\nu\sigma\rho}$ plays a natural role as a metric in the space of self-dual two-forms, in the sense that $\mathcal{I}_{\mu\nu\sigma\rho}\mathcal{W}^{\sigma\rho} = \mathcal{W}_{\mu\nu}$ for any self-dual two-form $\mathcal{W}_{\mu\nu}$. In connection with this definition and its applications, there now arise naturally two *a priori* independent problems:

1. Classify the solutions of (1.1) for which there *exists* a Q such that the MST (1.5) vanishes.
2. Prescribe the function Q such that properties I-III above (or a subset thereof) hold for the MST.

Problem 1 has been settled in [17–19] for case $\Lambda = 0$, while the extension to $\Lambda \neq 0$ was accomplished in [21]. The classes of solutions characterized in this way include Kerr and Kerr–de Sitter, respectively, and these solutions can in fact be singled out by supplementing the condition $\mathcal{S}_{\mu\nu\sigma\rho} = 0$ with suitable 'covariant' conditions.

As to problem 2 for $\Lambda = 0$, one sets

$$Q = 6\sigma^{-1} \tag{1.11}$$

in terms of the 'Ernst potential' σ, defined up to an additive complex constant (called 'σ-constant' henceforth) by

$$\partial_{\beta}\sigma = 2X^{\alpha}\mathcal{F}_{\alpha\beta}. \tag{1.12}$$

The corresponding MST then in fact satisfies a linear, homogeneous, symmetric hyperbolic system, irrespective of how the σ-constant has been chosen [16]. In the asymptotically flat setting the MST vanishes at infinity (which holds again for any choice of the σ-constant provided that the ADM mass is non-zero). The σ-constant is fixed uniquely in a natural way by requiring the Ernst potential to vanish at infinity. We remark that this symmetric hyperbolic system, or rather the wave equation which can be derived from it, has been used in uniqueness proofs for stationary, asymptotically flat black holes [1, 16].

In analogy with (1.3) we now define

$$\widetilde{\mathcal{T}}_{\alpha\beta\gamma}{}^{\delta} := \Theta^{-1}\mathcal{S}_{\alpha\beta\gamma}{}^{\delta}. \tag{1.13}$$

For $\Lambda > 0$, key properties of these tensors can be summarized as follows (I-III is shown in the present work while IV is a reformulation of a result of [21]; II and IV in fact hold for any sign of the cosmological constant):

I. There exists a function Q_0 such that the corresponding MST $\mathcal{S}^{(0)}_{\alpha\beta\gamma\delta}$ vanishes on $\mathscr{I}$, whence $\widetilde{\mathcal{T}}^{(0)}_{\alpha\beta\gamma\delta}$ extends regularly to $\mathscr{I}$.

II. There exists a function $Q^{(ev)}$ such that the corresponding MST $\mathcal{S}^{(ev)}_{\alpha\beta\gamma\delta}$ satisfies a linear, homogeneous symmetric hyperbolic system on $(\mathscr{M}, g)$.

III. $\mathcal{T}^{(ev)}_{\alpha\beta\gamma\delta}$ satisfies a linear, homogeneous symmetric hyperbolic system on $(\widetilde{\mathcal{M}}, \widetilde{g})$ which is of 'Fuchsian type' at $\mathscr{I}$.

IV. When $\mathcal{S}_{\alpha\beta\gamma\delta}$ is required to vanish identically for some Q, then $Q = Q_0 = Q^{(ev)}$.

Conditions I-III stated above for the MST should be compared with the corresponding conditions stated earlier for the Weyl tensor. Unfortunately, or maybe for a deeper reason, there appears to be no universal definition of Q which satisfies I–III simultaneously.

We proceed with explaining these findings in some detail, and with describing their arrangement in the following sections. The function Q_0 is introduced in (2.59), and property I is shown in proposition 2.3. Next, theorem 2.8 gives necessary and sufficient conditions on the data in order for $\widetilde{\mathcal{T}}^{(0)}_{\alpha\beta\gamma\delta}$ to vanish on $\mathscr{I}$. These conditions agree with conditions (i) and (ii) in theorem 1.3 quoted below.

On the other hand, in (4.19)–(4.21) we define a *class of functions* $Q^{(ev)}$ for which we show in section 4.4 that the corresponding MST $\mathcal{S}^{(ev)}_{\alpha\beta\gamma\delta}$ satisfies a linear, homogeneous symmetric hyperbolic system, which gives property II (and from which one readily derives a system of wave equations). For the rescaled tensor $\widetilde{\mathcal{T}}^{(ev)}_{\alpha\beta\gamma\delta}$ we then obtain equations of the same form on $\widetilde{\mathcal{M}}$ (cf lemmas 4.11 and 4.13). The appropriate definition of $Q^{(ev)}$ involves a 'σ-(integration)-constant' (called 'a' in (4.11)), and in analogy with the case $\Lambda = 0$ mentioned before there is again a natural way (namely (4.31)) of fixing the constant from the asymptotic conditions. However, in contrast to the case $\Lambda = 0$, the resulting $\mathcal{S}^{(ev)}_{\alpha\beta\gamma\delta}$ does not vanish automatically on $\mathscr{I}^-$, whence $\mathcal{T}^{(ev)}_{\alpha\beta\gamma\delta}$ is not necessarily regular there. In definition 4.5 we call solutions for which $\mathcal{T}^{(ev)}_{\alpha\beta\gamma\delta}$ (with the optimal σ-constant) can be regularized on $\mathscr{I}^-$ (and agrees with $\widetilde{\mathcal{T}}^{(0)}{}_{\mu\nu\sigma}{}^{\rho}$ on $\mathscr{I}^-$) 'asymptotically Kerr–de Sitter like'. This class can be characterized in terms of the data as follows (this is a shortened version of theorem 4.3):

Theorem 1.2.　*Consider a $\Lambda > 0$-vacuum spacetime which admits a smooth $\mathscr{I}^-$ and a KVF X. Denote by Y the CKVF induced, in the conformally rescaled spacetime, by X on $\mathscr{I}^-$.*

The condition

$$\widetilde{\mathcal{T}}^{(ev)}{}_{\mu\nu\sigma}{}^{\rho}|_{\mathscr{I}^-} = \widetilde{\mathcal{T}}^{(0)}{}_{\mu\nu\sigma}{}^{\rho}|_{\mathscr{I}^-}$$

holds if and only if Y^j is a common eigenvector of $\widehat{C}_{ij}$ and D_{ij}, where $\widehat{C}_{ij}$ is the Cotton–York tensor (2.72) and $D_{ij} = d_{titj}|_{\mathscr{I}^-}$.

This now suggests considering a Cauchy problem for the MCFE on $\mathscr{I}^-$, starting from asymptotically Kerr–de Sitter-like data. However, in contrast to the evolution equation for the rescaled Weyl tensor $\widetilde{d}_{\alpha\beta\gamma\delta}$, the coefficients in the evolution equation for $\widetilde{\mathcal{T}}^{(ev)}_{\alpha\beta\gamma\delta}$ are not regular at $\mathscr{I}$ and even not necessarily so off some neighborhood of $\mathscr{I}$. (Non-regularities may occur already in the evolution equations for $\mathcal{S}^{(ev)}_{\alpha\beta\gamma\delta}$). Regarding $\mathscr{I}$, we are now dealing with a linear homogeneous *Fuchsian* symmetric hyperbolic system (lemma 4.11). Adapting results available in the literature, we prove in lemma 4.14 a local uniqueness theorem for regular solutions of a class of Fuchsian systems which includes the present one. We then apply this result to 'trivial' data satisfying $\widetilde{\mathcal{T}}^{(ev)}_{\alpha\beta\gamma\delta}|_{\mathscr{I}^-} = 0$, which we call 'Kerr–de Sitter-like'. Our preliminary uniqueness result, lemma 4.15, now yields local-in-time uniqueness of these solutions, and implies that $\mathcal{S}^{(ev)}_{\alpha\beta\gamma\delta}$ vanishes near $\mathscr{I}^-$. However, this conclusion does not immediately extend to the whole domain of dependence of $\mathscr{I}^-$ since the evolution equation is manifestly regular only in some neighborhood of (and excluding) $\mathscr{I}^-$. Nevertheless, the

required result does follow from the classification results of [21], so $\mathcal{S}^{(\mathrm{ev})}_{\alpha\beta\gamma\delta} \equiv 0$ indeed holds on the domain of dependence of $\mathscr{I}^-$.

Altogether this yields the following classification result for Kerr–de Sitter like spacetimes in terms of data on $\mathscr{I}^-$, which may be considered as counterpart of the first part of 1.1 above:

Theorem 1.3. *Let* (Σ, h) *be a Riemannian 3-manifold which admits a CKVF* Y *with* $|Y|^2 > 0$, *complemented by a TT tensor* D_{ij} *to asymptotic Cauchy data. Then there exists a maximal globally hyperbolic* $\Lambda > 0$*-vacuum spacetime* $(\mathscr{M}, g)$ *which admits a KVF* X^i *with* $X^i|_{\mathscr{I}^-} = Y^i$ *and such that the associated MST vanishes, and* Σ *represents past null infinity* $\mathscr{I}^-$ *with* $g_{ij}|_{\mathscr{I}^-} = h_{ij}$ *and* $d_{titj}|_{\mathscr{I}^-} = D_{ij}$ *if and only if*

  i. $\widehat{C}_{ij} = \sqrt{\frac{\Lambda}{3}} C_{\mathrm{mag}} |Y|^{-5} \left(Y_i Y_j - \frac{1}{3} |Y|^2 h_{ij} \right)$ *for some constant* C_{mag}, *and*

  ii. $D_{ij} = C_{\mathrm{el}} |Y|^{-5} \left(Y_i Y_j - \frac{1}{3} |Y|^2 h_{ij} \right)$ *for some constant* C_{el}.

Spacetimes with vanishing MST have very different properties depending on the values taken by the free constants in the family. In particular, the maximal domain of dependence of $\mathscr{I}$ may or may not be extendible across a Killing horizon, and these different behaviours occur even within the class of spacetimes with vanishing C_{mag}. In this latter case, $\mathscr{I}$ is locally conformally flat and the data consist simply in a choice of a conformal Killing vector Y in (a domain of) $\mathbb{S}^3$ and a choice of a constant C_{el}. An interesting (and probably difficult) question is whether it is possible to identify directly at $\mathscr{I}$ the behaviour of its domain of dependence in the large. In particular, it would be interesting to see if the properties of Y at its zeroes can be related to the existence of a Killing horizon across which the domain of dependence of $\mathscr{I}$ can be extended.

We remark that we do not obtain a counterpart to the stability result (part 2 of theorem 1.1). Recall also the remark after (1.4) in connection with the significance of the stability problem in the presence of isometries.

In the final section 5 we analyze the relations between the vanishing of the rescaled MST (1.13) (or the corresponding condition on the data) and the existence of other conformal Killing vector fields on $\mathscr{I}$, and we discuss the extension of the latter to Killing vector fields on $\mathscr{M}$. This result, given in proposition 5.9, will be relevant for the classification of spacetimes with vanishing $\mathcal{S}^{(\mathrm{ev})}_{\alpha\beta\gamma\delta}$ and conformally flat $\mathscr{I}$ presented in the subsequent paper mentioned above already [20].

2. The Mars–Simon tensor (MST) at null infinity

2.1. The conformally rescaled spacetime

In this section we collect key equations which are gauge-independent, and which hold irrespectively of the sign (or vanishing) of Λ.

In the asymptotic setting described in the introduction the pair $(\widetilde{g}, \Theta)$ satisfies the *metric conformal field equations (MCFE)* on $\widetilde{\mathscr{M}}$ [13] (we use tildes for all geometric objects associated to $\widetilde{g}$),

$$\widetilde{\nabla}_\rho \widetilde{d}_{\mu\nu\sigma}{}^\rho = 0, \tag{2.1}$$

$$\widetilde{\nabla}_\mu \widetilde{L}_{\nu\sigma} - \widetilde{\nabla}_\nu \widetilde{L}_{\mu\sigma} = \widetilde{\nabla}_\rho \Theta \, \widetilde{d}_{\nu\mu\sigma}{}^\rho, \tag{2.2}$$

$$\widetilde{\nabla}_\mu \widetilde{\nabla}_\nu \Theta = -\Theta \widetilde{L}_{\mu\nu} + \tilde{s}\, \tilde{g}_{\mu\nu}, \tag{2.3}$$

$$\widetilde{\nabla}_\mu \tilde{s} = -\widetilde{L}_{\mu\nu} \widetilde{\nabla}^\nu \Theta, \tag{2.4}$$

$$2\Theta\tilde{s} - \widetilde{\nabla}_\mu \Theta \widetilde{\nabla}^\mu \Theta = \Lambda/3, \tag{2.5}$$

$$\widetilde{R}_{\mu\nu\sigma}{}^\kappa[\tilde{g}] = \Theta \tilde{d}_{\mu\nu\sigma}{}^\kappa + 2(\tilde{g}_{\sigma[\mu} \widetilde{L}_{\nu]}{}^\kappa - \delta_{[\mu}{}^\kappa \widetilde{L}_{\nu]\sigma}), \tag{2.6}$$

where the Riemann tensor $\widetilde{R}_{\mu\nu\sigma}{}^\kappa[\tilde{g}]$ is to be regarded as a differential operator on $\tilde{g}$, while $\widetilde{L}_{\mu\nu} := \frac{1}{2}\widetilde{R}_{\mu\nu} - \frac{1}{12}\widetilde{R}\tilde{g}_{\mu\nu}$, $\tilde{d}_{\mu\nu\sigma}{}^\rho := \Theta^{-1}\widetilde{C}_{\mu\nu\sigma}{}^\rho$ are, respectively, the Schouten and rescaled Weyl tensor of $\tilde{g}$, and

$$\tilde{s} := \frac{1}{4}\Box_{\tilde{g}}\Theta + \frac{1}{24}\widetilde{R}\Theta. \tag{2.7}$$

Let us now express the MST in terms of unphysical fields on $(\widetilde{\mathcal{M}}, \tilde{g})$. We first of all note that the push-forward $\widetilde{X}^\mu$ of the KVF X^μ, which we identify with X^μ, satisfies the *unphysical Killing equations* [23]

$$\widetilde{F}_{(\mu\nu)} = 0 \quad \text{and} \quad \widetilde{F} = 4\widetilde{X}^\mu \widetilde{\nabla}_\mu \log \Theta, \tag{2.8}$$

where

$$\widetilde{F}_{\mu\nu} := (\widetilde{\nabla}_\mu \widetilde{X}_\nu)_{\mathrm{tf}}, \quad \widetilde{F} := \widetilde{\nabla}_\mu \widetilde{X}^\mu. \tag{2.9}$$

and the symbol $(.)_{\mathrm{tf}}$ denotes the trace-free part of the corresponding $(0, 2)$-tensor. $\widetilde{F}_{\mu\nu}$ is hence a two-form and we can define $\widetilde{C}_{\mu\nu\sigma\rho}$, $\widetilde{U}_{\mu\nu\sigma\rho}$, $\widetilde{I}_{\mu\nu\sigma\rho}$, $\widetilde{F}_{\mu\nu}$ and $\widetilde{\mathcal{F}}^2$ using definitions analogous to (1.6)–(1.10) with all geometric objects referred to $(\widetilde{\mathcal{M}}, \tilde{g})$. The following relations are found via a simple computation.

$$\mathcal{C}_{\mu\nu\sigma}{}^\rho = \widetilde{\mathcal{C}}_{\mu\nu\sigma}{}^\rho, \tag{2.10}$$

$$\mathcal{I}_{\mu\nu\sigma}{}^\rho = \Theta^{-2}\widetilde{\mathcal{I}}_{\mu\nu\sigma}{}^\rho, \tag{2.11}$$

$$\begin{aligned}
F_{\mu\nu} &= \nabla_\mu(\Theta^{-2}\widetilde{X}_\nu) \\
&= \Theta^{-2}(\widetilde{F}_{\mu\nu} + \frac{1}{4}\tilde{g}_{\mu\nu}\widetilde{F}) + \Theta^{-3}(2\widetilde{X}_{[\mu}\widetilde{\nabla}_{\nu]}\Theta - \tilde{g}_{\mu\nu}\widetilde{X}^\sigma \widetilde{\nabla}_\sigma \Theta) \\
&= \Theta^{-2}(\widetilde{F}_{\mu\nu} + \Theta^{-1}\widetilde{H}_{\mu\nu}),
\end{aligned} \tag{2.12}$$

$$\mathcal{F}_{\mu\nu} = \Theta^{-2}(\widetilde{\mathcal{F}}_{\mu\nu} + \Theta^{-1}\widetilde{\mathcal{H}}_{\mu\nu}), \tag{2.13}$$

$$\mathcal{F}^2 = \widetilde{\mathcal{F}}^2 + 2\Theta^{-1}\widetilde{\mathcal{F}}_{\alpha\beta}\widetilde{\mathcal{H}}^{\alpha\beta} + \Theta^{-2}\widetilde{\mathcal{H}}^2, \tag{2.14}$$

$$\begin{aligned}
\mathcal{U}_{\mu\nu\sigma}{}^\rho &= \Theta^{-2}\widetilde{\mathcal{U}}_{\mu\nu\sigma}{}^\rho - \Theta^{-3}\left(\widetilde{\mathcal{F}}_{\mu\nu}\widetilde{\mathcal{H}}_\sigma{}^\rho + \widetilde{\mathcal{H}}_{\mu\nu}\widetilde{\mathcal{F}}_\sigma{}^\rho - \frac{2}{3}\widetilde{\mathcal{F}}_{\alpha\beta}\widetilde{\mathcal{H}}^{\alpha\beta}\widetilde{\mathcal{I}}_{\mu\nu\sigma}{}^\rho\right) \\
&\quad - \Theta^{-4}\left(\widetilde{\mathcal{H}}_{\mu\nu}\widetilde{\mathcal{H}}_\sigma{}^\rho - \frac{1}{3}\widetilde{\mathcal{H}}^2\widetilde{\mathcal{I}}_{\mu\nu\sigma}{}^\rho\right),
\end{aligned} \tag{2.15}$$

where we have set

$$\widetilde{H}_{\mu\nu} := 2\widetilde{X}_{[\mu}\widetilde{\nabla}_{\nu]}\Theta, \tag{2.16}$$

$$\widetilde{\mathcal{H}}_{\mu\nu} := \widetilde{H}_{\mu\nu} + i\widetilde{H}^{\star}_{\mu\nu}. \tag{2.17}$$

We want to investigate how the MST behaves when approaching the conformal boundary $\mathscr{I}$. Note that the conformal Killing equation implies that $\widetilde{X}^{\mu}$ admits a smooth extension across $\mathscr{I}$ [14], in particular the tensor $\widetilde{\mathcal{U}}_{\mu\nu\sigma}{}^{\rho}$ is a regular object there.

We observe that the following relations are fulfilled. The first two are general identities for self-dual two-forms and the third one is a consequence of $\widetilde{H}$ being a simple two-form,

$$\widetilde{\mathcal{F}}_{\alpha\beta}\widetilde{\mathcal{H}}^{\alpha\beta} = 2\widetilde{F}_{\alpha\beta}\widetilde{H}^{\alpha\beta} + 2i\widetilde{F}^{\alpha\beta}\widetilde{H}^{\star}_{\alpha\beta} = 2\widetilde{\mathcal{F}}_{\alpha\beta}\widetilde{H}^{\alpha\beta}, \tag{2.18}$$

$$\widetilde{\mathcal{F}}^2 = 2\widetilde{F}^2 + 2i\widetilde{F}^{\star}_{\alpha\beta}\widetilde{F}^{\alpha\beta}, \tag{2.19}$$

$$\widetilde{\mathcal{H}}^2 = 2\widetilde{H}^2, \tag{2.20}$$

$$\widetilde{X}^{\mu}\widetilde{\mathcal{H}}_{\mu\nu} = \widetilde{X}^2\widetilde{\nabla}_{\nu}\Theta - \frac{1}{4}\Theta\widetilde{F}\widetilde{X}_{\nu}. \tag{2.21}$$

Here and henceforth we write $\widetilde{T}^2 := \widetilde{T}_{\alpha_1\cdots\alpha_p}\widetilde{T}^{\alpha_1\cdots\alpha_p}$ for any $(0, p)$-*spacetime*-tensor, while we write $|T|^2 := T_{i_1\cdots i_p}T^{i_1\cdots i_p}$ for any $(0, p)$-*space*-tensor on $\mathscr{I}$. Since we never write down explicitly the second component of a vector, the reader will not get confused by this notation.

Moreover, the MCFE and the unphysical Killing equations imply that

$$\widetilde{\mathcal{H}}^2 = -\frac{4}{3}\Lambda\widetilde{X}^2 + 8\Theta\tilde{s}\widetilde{X}^2 - \frac{1}{4}\Theta^2\widetilde{F}^2, \tag{2.22}$$

$$\widetilde{\mathcal{F}}_{\alpha\beta}\widetilde{\mathcal{H}}^{\alpha\beta} = \Theta\widetilde{X}^{\alpha}\widetilde{\nabla}_{\alpha}\widetilde{F} + 4\Theta\widetilde{X}^{\alpha}\widetilde{X}^{\beta}\widetilde{L}_{\alpha\beta} - 4s\widetilde{X}^2 + 2i\widetilde{F}^{\mu\nu}\widetilde{H}^{\star}_{\mu\nu}. \tag{2.23}$$

2.2. Cauchy data at $\mathscr{I}^-$

Let us henceforth assume a positive cosmological constant

$$\Lambda > 0. \tag{2.24}$$

We consider a connected component $\mathscr{I}^-$ of past null infinity. As in [24], to which we refer the reader for further details, we use adapted coordinates $(x^0 = t, x^i)$ with $\mathscr{I}^- = \{t = 0\}$ and impose a *wave map gauge condition* with

$$\widetilde{R} = 0, \quad \tilde{s}|_{\mathscr{I}^-} = 0, \quad \widetilde{g}_{tt}|_{\mathscr{I}^-} = -1, \quad \widetilde{g}_{ti}|_{\mathscr{I}^-} = 0, \quad \widetilde{W}^{\sigma} = 0, \quad \check{g}_{\mu\nu} = \widetilde{g}_{\mu\nu}|_{\mathscr{I}^-}. \tag{2.25}$$

The gauge freedom to prescribe $\widetilde{R}$ and $\tilde{s}|_{\mathscr{I}^-}$ reflects the freedom to choose the conformal factor Θ, which is treated as an unknown in the MCFE. It is well-known that the freedom to choose coordinates near a spacelike hypersurface with induced metric h_{ij} can be employed to prescribe $\widetilde{g}_{tt}|_{\mathscr{I}^-}$ and $\widetilde{g}_{ti}|_{\mathscr{I}^-}$, as long as $(\widetilde{g}_{tt} - h^{ij}\widetilde{g}_{ti}\widetilde{g}_{tj})|_{\mathscr{I}^-} < 0$ is satisfied.

The remaining freedom to choose coordinates is captured by the wave map gauge condition, a generalization of the classical harmonic gauge condition, and requires the vanishing of the so-called wave gauge vector

$$H^{\sigma} := g^{\alpha\beta}(\Gamma^{\sigma}_{\alpha\beta} - \check{\Gamma}^{\sigma}_{\alpha\beta}) - W^{\sigma} = 0, \tag{2.26}$$

where $\check{g}_{\mu\nu}$ denotes some target metric, the $\check{\Gamma}^{\sigma}_{\alpha\beta}$'s are the associated connection coefficients, and the W^{σ}'s are the gauge source functions, which can be arbitrarily prescribed [8]. The target metric is introduced for the wave gauge vector to become a tensor. Here, as in [24], we have chosen $\check{g}_{\mu\nu}$ to be independent of t and to agree with $\widetilde{g}_{\mu\nu}$ on $\mathscr{I}^-$. The gauge has been

chosen in such a way that $\partial_t \widetilde{g}_{\mu\nu}$ vanishes on $\mathscr{I}^-$, in order to make the computations as simple as possible. Given arbitrary coordinates the wave map gauge can be realized by solving wave equations.

Viewing the MCFE as an evolution problem with initial data on $\mathscr{I}^-$, the free data are a (connected) Riemannian 3-manifold (Σ, h_{ij}), which represents $\mathscr{I}^-$ in the emerging space-time,[5] and a TT tensor D_{ij} (i.e.: trace-free and divergence-free) which satisfies the relation

$$D_{ij} = \widetilde{d}_{titj}|_{\mathscr{I}^-} \tag{2.27}$$

once the asymptotic Cauchy problem has been solved.

Theorem 2.1 [9]. *Let (Σ, h_{ij}) be a Riemanian 3-manifold, D_{ij} a symmetric $(0, 2)$-tensor and $\Lambda > 0$. Then, if and only if D_{ij} is a TT tensor, the tuple (Σ, h_{ij}, D_{ij}) defines an (up to isometries) unique maximal globally hyperbolic development (in the unphysical spacetime) of the Λ-vacuum field equations where Σ can be embedded, with embedding ι, such that $\iota(\Sigma)$ represents $\mathscr{I}^-$ with $\iota^*\widetilde{g}_{ij}|_\Sigma = h_{ij}$ and $\iota^*\widetilde{d}_{titj}|_\Sigma = D_{ij}$.*

For simplicity, we will often identify Σ with its image under ι and drop all reference to the embedding.

It is a property of the spacelike Cauchy problem that all transverse derivatives can be computed algebraically from the initial data (here h_{ij} and D_{ij}). In the gauge (2.25) the MCFE (2.1)–(2.6) enforce the following relations on $\mathscr{I}^-$, cf [9, 24],

$$\overline{\widetilde{g}}_{tt} = -1, \quad \overline{\widetilde{g}}_{ti} = 0, \quad \overline{\widetilde{g}}_{ij} = h_{ij}, \quad \overline{\partial_t \widetilde{g}_{\mu\nu}} = 0, \tag{2.28}$$

$$\overline{\Theta} = 0, \quad \overline{\partial_t \Theta} = \sqrt{\frac{\Lambda}{3}}, \quad \overline{\partial_t \partial_t \Theta} = 0, \tag{2.29}$$

$$\overline{\partial_t \partial_t \partial_t \Theta} = -\frac{1}{2}\sqrt{\frac{\Lambda}{3}}\widehat{R}, \quad \overline{\partial_t \partial_t \partial_t \partial_t \Theta} = 0, \tag{2.30}$$

$$\overline{\widetilde{s}} = 0, \quad \overline{\partial_t \widetilde{s}} = \frac{1}{4}\sqrt{\frac{\Lambda}{3}}\,\widehat{R}, \tag{2.31}$$

$$\overline{\widetilde{L}}_{ij} = \widehat{L}_{ij}, \quad \overline{\widetilde{L}}_{ti} = 0, \quad \overline{\widetilde{L}}_{tt} = \frac{1}{4}\widehat{R}, \tag{2.32}$$

$$\overline{\partial_t \widetilde{L}_{ij}} = -\sqrt{\frac{\Lambda}{3}}\,D_{ij}, \quad \overline{\partial_t \widetilde{L}_{ti}} = \frac{1}{4}\partial_i\widehat{R}, \quad \overline{\partial_t \widetilde{L}_{tt}} = 0, \tag{2.33}$$

$$\overline{\widetilde{d}}_{titj} = D_{ij}, \quad \overline{\widetilde{d}}_{tijk} = \sqrt{\frac{3}{\Lambda}}\,\widehat{C}_{ijk}, \tag{2.34}$$

$$\overline{\partial_t \widetilde{d}_{titj}} = \sqrt{\frac{3}{\Lambda}}\,\widehat{B}_{ij}, \quad \overline{\partial_t \widetilde{d}_{tijk}} = 2\widehat{\nabla}_{[j}D_{k]i}. \tag{2.35}$$

$$\overline{\widetilde{\Gamma}}^k_{ij} = \widehat{\Gamma}^k_{ij}, \quad \overline{\widetilde{\Gamma}}^t_{ij} = \overline{\widetilde{\Gamma}}^t_{ti} = \overline{\widetilde{\Gamma}}^t_{tt} = \overline{\widetilde{\Gamma}}^k_{tt} = \overline{\widetilde{\Gamma}}^k_{ti} = 0, \tag{2.36}$$

$$\overline{\widetilde{R}}_{tijk} = 0, \quad \overline{\widetilde{R}}_{titj} = -\widehat{L}_{ij} + \frac{1}{4}h_{ij}\widehat{R}, \tag{2.37}$$

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[5] It is actually merely the conformal class of the Riemannian 3-manifold which matters geometrically. This will be relevant in paper II [20].

$$\overline{\partial_t \widetilde{R}_{tijk}} = \widehat{C}_{ijk} - \frac{1}{2}h_{i[j}\widehat{\nabla}_{k]}\widehat{R}, \quad \overline{\partial_t \widetilde{R}_{titj}} = 2\sqrt{\frac{\Lambda}{3}}D_{ij}. \tag{2.38}$$

An overbar will be used to denote the restriction of spacetime objects to $\mathscr{I}^-$, if not explicitly stated otherwise (in the latter cases it will denote 'complex conjugation'). We use the symbol $\widehat{}$ to denote objects associated to the induced Riemannian metric h_{ij}, in particular $\widehat{C}_{ijk}$, $\widehat{L}_{ij}$ and $\widehat{B}_{ij}$ denote the Cotton, Schouten and Bach tensor, respectively, of h_{ij}. Recall that they are defined by

$$\widehat{C}_{ijk} := \widehat{\nabla}_k \widehat{L}_{ij} - \widehat{\nabla}_j \widehat{L}_{ik}, \qquad \widehat{L}_{ij} := \widehat{R}_{ij} - \frac{1}{4}\widehat{R}h_{ij}, \tag{2.39}$$

$$\widehat{B}_{ij} := -\widehat{\nabla}^k \widehat{C}_{ijk} = \widehat{\nabla}^k \widehat{\nabla}_i \widehat{L}_{jk} - \widehat{\nabla}_k \nabla^k \widehat{L}_{ij}. \tag{2.40}$$

Note that due to (2.36) the actions of $\widetilde{\nabla}_t$ and ∂_t, as well as $\widetilde{\nabla}_i$ and $\widehat{\nabla}_i$, respectively, coincide on $\mathscr{I}^-$, so we can use them interchangeably.

Whenever X^μ is a KVF of the physical spacetime, the vector field

$$Y^i := \widetilde{X}^i|_{\mathscr{I}^-} \tag{2.41}$$

is a conformal Killing vector field (CKVF) of $(\mathscr{I}^-, h_{ij})$, i.e.

$$(\mathscr{L}_Y h)_{ij} \equiv 2\widehat{\nabla}_{(i}Y_{j)} = \frac{2}{3}\widehat{\nabla}_k Y^k h_{ij}, \tag{2.42}$$

which fulfills the *KID equations* [24]

$$\mathscr{L}_Y D_{ij} + \frac{1}{3}D_{ij}\widehat{\nabla}_k Y^k = 0, \tag{2.43}$$

and vice versa:

Theorem 2.2 [**24**].   *Let (Σ, h_{ij}) be a Riemannian 3-manifold, D_{ij} a symmetric $(0, 2)$-tensor on Σ and $\Lambda > 0$. Then, the tuple $(\Sigma, h_{ij}, D_{ij}, Y^i)$ defines an (up to isometries) unique, in the unphysical spacetime maximal globally hyperbolic Λ-vacuum spacetime with a smooth $\mathscr{I}^-$, represented by $\iota(\Sigma)$, with $\iota^* \widetilde{g}_{ij}|_\Sigma = h_{ij}$ and $\iota^* \widetilde{d}_{titj}|_\Sigma = D_{ij}$, which contains a Killing vector field X with $\overline{\widetilde{X}}^i = Y^i$, if and only if D_{ij} is a TT tensor and Y is a conformal Killing vector field on (Σ, h_{ij}) which satisfies the KID equations (2.43).*

Moreover, $\widetilde{X}^\mu$ satisfies

$$\overline{\widetilde{X}}^t = 0, \quad \overline{\widetilde{\nabla}_t \widetilde{X}}^t = \frac{1}{3}\widehat{\nabla}_i Y^i, \quad \overline{\widetilde{\nabla}_t \widetilde{X}}^i = 0. \tag{2.44}$$

From what has been shown in [24] one easily derives the following expressions on $\mathscr{I}$,

$$\overline{\widetilde{F}} = \frac{4}{3}\widehat{\nabla}_i Y^i, \tag{2.45}$$

$$\Delta_h Y_i = -\widehat{L}_{ij}Y^j - \frac{1}{4}\widehat{R}Y_i - \frac{1}{3}\widehat{\nabla}_i \widehat{\nabla}_j Y^j, \tag{2.46}$$

$$\Delta_h \overline{\widetilde{F}} = -Y^i \widehat{\nabla}_i \widehat{R} - \frac{1}{2} \widehat{R} \overline{\widetilde{F}} \tag{2.47}$$

$$\overline{\widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{X}_t} = 0, \tag{2.48}$$

$$\overline{\widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{X}_i} = \widehat{L}_{ij} Y^j - \frac{1}{4} \widehat{R} Y_i + \frac{1}{3} \widehat{\nabla}_i \widehat{\nabla}_j Y^j, \tag{2.49}$$

$$\overline{\widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{X}_t} = -\frac{1}{4} \Delta_h \overline{\widetilde{F}}, \tag{2.50}$$

$$\overline{\widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{X}_k} = -2 \sqrt{\frac{\Lambda}{3}} D_{kl} Y^l, \tag{2.51}$$

$$\overline{\widetilde{\nabla}_t \widetilde{F}} = 0, \tag{2.52}$$

$$\overline{\widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{F}} = \Delta_h \overline{\widetilde{F}}. \tag{2.53}$$

2.3. The function Q

2.3.1. A necessary condition for vanishing MST. Our aim is to characterize initial data on a spacelike $\mathscr{I}^-$ which lead to a vanishing MST. We have not specified the function Q yet. Nonetheless, let us assume for the time being that $\Theta^{-4} Q$ does not tend to zero at $\mathscr{I}$. Then, it follows from (2.10) and (2.15) that a necessary condition for the MST to vanish on $\mathscr{I}$ is

$$\left[\widetilde{\mathcal{H}}_{\mu\nu} \widetilde{\mathcal{H}}_\sigma{}^\rho - \frac{1}{3} \widetilde{\mathcal{H}}^2 \widetilde{\mathcal{I}}_{\mu\sigma}{}^\rho \right] \bigg|_{\mathscr{I}} = 0. \tag{2.54}$$

A straightforward computation on a spacelike $\mathscr{I}$ in the wave map gauge (2.25) shows that this is the case if and only if

$$0 = \left[\widetilde{\mathcal{H}}_{ti} \widetilde{\mathcal{H}}_{tj} - \frac{1}{3} \widetilde{\mathcal{H}}^2 \widetilde{\mathcal{I}}_{titj} \right] \bigg|_{\mathscr{I}} = \frac{\Lambda}{3} (Y_i Y_j)_{\mathrm{tf}} \iff Y^i = 0. \tag{2.55}$$

This already implies [24] that the KVF X^μ is trivial. Hence, $\Theta^{-4} Q$ must necessarily go to zero whenever the MST vanishes on a spacelike $\mathscr{I}$. In the next section we in fact show that

$$Q = O(\Theta^5) \tag{2.56}$$

holds automatically for an appropriate definition of Q.

2.3.2. Definition and asymptotic behavior of the MST. In order to analyze the situation where $\mathcal{S}_{\mu\nu\sigma\rho}$ vanishes, it is natural to define Q in such a way that a certain scalar constructed from $\mathcal{S}_{\mu\nu\sigma\rho}$ vanishes automatically. This tensor has the same algebraic properties as the Weyl tensor, so all its traces are identically zero and cannot be used to define Q. A convenient choice is to require

$$\mathcal{S}_{\mu\nu\sigma\rho} \mathcal{F}^{\mu\nu} \mathcal{F}^{\sigma\rho} = 0, \tag{2.57}$$

or, equivalently,

$$Q\mathcal{F}^4 = \frac{3}{2}\mathcal{F}^{\mu\nu}\mathcal{F}^{\sigma\rho}\mathcal{C}_{\mu\nu\sigma\rho} = 6F^{\mu\nu}F^{\sigma\rho}\mathcal{C}_{\mu\nu\sigma\rho}. \tag{2.58}$$

The function Q necessarily needs to satisfy (2.58) whenever the MST vanishes. Let us restrict attention to the case where $\mathcal{F}^2$ has no zeros. In fact, $\mathcal{F}^2 = -\frac{4}{3}\Lambda\Theta^{-2}\,|Y|^2 + O(\Theta^{-1})$, so, at least sufficiently close to $\mathscr{I}$, it suffices to assume that Y has no zeros on $\mathscr{I}$. Then (2.58) determines Q. From now on this choice of Q will be denoted by Q_0,

$$Q_0 := \frac{3}{2}\mathcal{F}^{-4}\mathcal{F}^{\mu\nu}\mathcal{F}^{\sigma\rho}\mathcal{C}_{\mu\nu\sigma\rho}, \tag{2.59}$$

and the corresponding MST by $\mathcal{S}^{(0)}_{\mu\nu\sigma\rho}$. When we want to emphasize the metric g with respect to which $\mathcal{S}^{(0)}_{\mu\nu\sigma\rho}$ is defined, we will write $\mathcal{S}^{(0)}_{\mu\nu\sigma\rho}[g]$.

As has already been done for the other fields appearing in the definition of the MST, we express Q_0 in terms of the unphysical fields. First of all we set

$$\widetilde{\mathcal{D}}_{\mu\nu\sigma\rho} = \Theta^{-1}\widetilde{\mathcal{C}}_{\mu\nu\sigma\rho}. \tag{2.60}$$

Making use of the various relations (2.10)–(2.21) we find that[6]

$$Q_0 = -6\mathcal{F}^{-4}F^{\mu\nu}F_\rho{}^\sigma\mathcal{C}_{\mu\nu\sigma}{}^\rho \tag{2.61}$$

$$= 6\Theta^4\widetilde{\mathcal{C}}_{\mu\nu\sigma\rho}\frac{(\widetilde{H}^{\mu\nu} + \widetilde{F}^{\mu\nu}\Theta)(\widetilde{H}^{\sigma\rho} + \widetilde{F}^{\sigma\rho}\Theta)}{[\widetilde{\mathcal{H}}^2 + 2\widetilde{\mathcal{F}}_{\alpha\beta}\widetilde{\mathcal{H}}^{\alpha\beta}\Theta + \widetilde{\mathcal{F}}^2\Theta^2]^2} \tag{2.62}$$

$$= 6\Theta^5\widetilde{\mathcal{D}}_{\mu\nu\sigma\rho}\frac{\widetilde{H}^{\mu\nu}\widetilde{H}^{\sigma\rho} + 2\widetilde{H}^{\mu\nu}\widetilde{F}^{\sigma\rho}\Theta + \widetilde{F}^{\sigma\rho}\widetilde{F}^{\mu\nu}\Theta^2}{[\widetilde{\mathcal{H}}^2 + 2\widetilde{\mathcal{F}}_{\alpha\beta}\widetilde{\mathcal{H}}^{\alpha\beta}\Theta + \widetilde{\mathcal{F}}^2\Theta^2]^2} \tag{2.63}$$

$$= \frac{3}{2}\Theta^5\widetilde{H}^{-4}\widetilde{\mathcal{D}}_{\mu\nu\sigma\rho}(\widetilde{H}^{\mu\nu}\widetilde{H}^{\sigma\rho} + 2\widetilde{H}^{\mu\nu}\widetilde{F}^{\sigma\rho}\Theta - 2\widetilde{H}^{-2}\widetilde{H}^{\mu\nu}\widetilde{H}^{\sigma\rho}\widetilde{\mathcal{F}}_{\alpha\beta}\widetilde{\mathcal{H}}^{\alpha\beta}\Theta)$$
$$+ O(\Theta^7). \tag{2.64}$$

Using $\Lambda > 0$ and the relations (2.22) and (2.23), which in particular imply

$$\widetilde{H}^{-2} = -\frac{3}{2}\Lambda^{-1}\widetilde{X}^{-2} + O(\Theta^2) \tag{2.65}$$

(note that $\tilde{s} = O(\Theta)$ due to (2.25)), we find the following expression for Q_0,

$$Q_0 = \frac{27}{8}\Theta^5\Lambda^{-2}\widetilde{X}^{-4}\widetilde{\mathcal{D}}_{\mu\nu\sigma\rho}(\widetilde{H}^{\mu\nu}\widetilde{H}^{\sigma\rho} + 2\widetilde{H}^{\mu\nu}\widetilde{F}^{\sigma\rho}\Theta$$
$$+ 6i\Lambda^{-1}\widetilde{X}^{-2}\widetilde{H}^{\mu\nu}\widetilde{H}^{\sigma\rho}\widetilde{H}^{\alpha\beta}\widetilde{F}^\star_{\alpha\beta}\Theta) + O(\Theta^7). \tag{2.66}$$

We conclude that, in the wave map gauge (2.25),

$$(\Theta^{-5}Q_0)|_{\mathscr{I}} = \frac{27}{8}\Lambda^{-2}\widetilde{X}^{-4}\widetilde{\mathcal{D}}_{\mu\nu\sigma\rho}\widetilde{H}^{\mu\nu}\widetilde{H}^{\sigma\rho} = \frac{9}{2}\Lambda^{-1}\,|Y|^{-4}Y^iY^j\widetilde{\mathcal{D}}_{titj}. \tag{2.67}$$

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[6] Not all orders given here and later in several instances are needed for our calculations. Nevertheless, we have chosen to write them down for the sake of completeness.

2.4. Properties of the MST on $\mathscr{I}$

Proposition 2.3. *Consider a spacetime $(\mathcal{M}, g)$, solution to Einstein's vacuum field equations with $\Lambda > 0$, which admits a smooth conformal extension through $\mathscr{I}$ and which contains a KVF X with $\widetilde{X}^2|_{\mathscr{I}} > 0$. Then the MST $\mathcal{S}^{(0)}_{\mu\nu\sigma}{}^{\rho}[\Theta^{-2}\widetilde{g}_{\alpha\beta}]$ corresponding to X with $Q = Q_0$ defined by (2.59) vanishes on $\mathscr{I}$.*

Proof. The Weyl tensor is known to vanish on $\mathscr{I}$. Since $\mathcal{U}_{\mu\nu\sigma}{}^{\rho} = O(\Theta^{-4})$ by (2.15) and $Q_0 = O(\Theta^5)$ by (2.66), the lemma is proved. $\qquad\qquad\qquad\square$

Corollary 2.4. *The rescaled MST*

$$\widetilde{\mathcal{T}}^{(0)}_{\mu\nu\sigma}{}^{\rho}[\Theta, \widetilde{g}_{\alpha\beta}] := \Theta^{-1}\mathcal{S}^{(0)}_{\mu\nu\sigma}{}^{\rho}[\Theta^{-2}\widetilde{g}_{\alpha\beta}]$$

is regular at $\mathscr{I}$.

2.5. The rescaled MST on $\mathscr{I}$

In this section we determine the behavior of the rescaled MST $\widetilde{\mathcal{T}}^{(0)}_{\mu\nu\sigma}{}^{\rho}$ at $\mathscr{I}$. For the tensor $\mathcal{U}_{\mu\nu\sigma}{}^{\rho}$ we find, using (2.15), (2.22) and (2.23)

$$\Theta^4 \mathcal{U}_{\mu\nu\sigma}{}^{\rho} = -\left(\widetilde{\mathcal{H}}_{\mu\nu}\widetilde{\mathcal{H}}_{\sigma}{}^{\rho} + \frac{4}{9}\Lambda\widetilde{X}^2\widetilde{\mathcal{I}}_{\mu\nu\sigma}{}^{\rho}\right)$$
$$- \Theta\left(\widetilde{\mathcal{F}}_{\mu\nu}\widetilde{\mathcal{H}}_{\sigma}{}^{\rho} + \widetilde{\mathcal{H}}_{\mu\nu}\widetilde{\mathcal{F}}_{\sigma}{}^{\rho} - \frac{4}{3}i\widetilde{H}^{\alpha\beta}\widetilde{F}^{\star}_{\alpha\beta}\widetilde{\mathcal{I}}_{\mu\nu\sigma}{}^{\rho}\right) + O(\Theta^2). \tag{2.68}$$

Now we are ready to evaluate the rescaled MST $\widetilde{\mathcal{T}}^{(0)}_{\mu\nu\sigma\rho} = \widetilde{g}_{\rho\alpha}\widetilde{\mathcal{T}}^{(0)}_{\mu\nu\sigma}{}^{\alpha}$ on $\mathscr{I}$. From (2.67) and (2.68),

$$\widetilde{\mathcal{T}}^{(0)}_{\mu\nu\sigma\rho}|_{\mathscr{I}} = \widetilde{\mathcal{D}}_{\mu\nu\sigma\rho} - \frac{9}{2}\Lambda^{-1}|Y|^{-4}Y^iY^j\widetilde{\mathcal{D}}_{titj}(\widetilde{\mathcal{H}}_{\mu\nu}\widetilde{\mathcal{H}}_{\sigma\rho} - \frac{1}{3}\widetilde{\mathcal{H}}^2\widetilde{\mathcal{I}}_{\mu\nu\sigma\rho}). \tag{2.69}$$

Since the rescaled MST is a self-dual Weyl field, its independent components on $\mathscr{I}$ are $\widetilde{\mathcal{T}}^{(0)}_{titj}|_{\mathscr{I}}$. Employing the various relations collected in section 2.2, it follows that, in the wave map gauge (2.25),

$$\left(\widetilde{\mathcal{H}}_{ti}\widetilde{\mathcal{H}}_{tj} - \frac{1}{3}\widetilde{\mathcal{H}}^2\widetilde{\mathcal{I}}_{titj}\right)\Bigg|_{\mathscr{I}} = \frac{\Lambda}{3}(Y_iY_j)_{\text{tf}}, \tag{2.70}$$

$$\widetilde{\mathcal{D}}_{titj}|_{\mathscr{I}} = D_{ij} - i\sqrt{\frac{3}{\Lambda}}\widehat{C}_{ij}. \tag{2.71}$$

Here $\widehat{C}_{ij}$ denotes the Cotton–York tensor

$$\widehat{C}_{ij} = -\frac{1}{2}\widehat{\eta}_i{}^{kl}\widehat{C}_{jkl} \quad\Longleftrightarrow\quad \widehat{C}_{ijk} = -\widehat{\eta}_{jk}{}^l\widehat{C}_{il}, \tag{2.72}$$

which is a TT tensor, and $\widehat{\eta}_{jkl}$ denotes the canonical volume 3-form relative to h_{ij}.

Note that D_{ij} and $\widehat{C}_{ij}$ correspond to the asymptotic electric and magnetic part, respectively, of the conformal Weyl tensor. We observe that (2.71) immediately implies that $\mathscr{I}$ will be locally conformally flat, i.e. has vanishing Cotton–York tensor, if and only if the magnetic part of the rescaled Weyl tensor $\widetilde{d}_{\mu\nu\sigma\rho}$ vanishes at $\mathscr{I}$, cf [4].

Proposition 2.5. *Consider a spacetime $(\mathcal{M}, g)$, solution to Einstein's vacuum field equations with $\Lambda > 0$, which admits a smooth conformal extension through $\mathscr{I}$ and which contains a KVF X with $\widetilde{X}^2|_{\mathscr{I}} > 0$. Then, the rescaled MST $\widetilde{\mathcal{T}}^{(0)}_{\mu\nu\sigma}{}^{\rho}$ satisfies*

$$\widetilde{\mathcal{T}}^{(0)}_{titj}|_{\mathscr{I}} = D_{ij} - \frac{3}{2}\,|Y|^{-4}Y^kY^lD_{kl}\,(Y_iY_j)_{\mathrm{tf}} - i\,\sqrt{\frac{3}{\Lambda}}\left(\widehat{C}_{ij} - \frac{3}{2}\,|Y|^{-4}Y^kY^l\widehat{C}_{kl}\,(Y_iY_j)_{\mathrm{tf}}\right).$$

(Recall that $\widetilde{\mathcal{T}}^{(0)}_{titj}|_{\mathscr{I}}$ comprises all independent components.)

According to proposition 2.5, the rescaled MST vanishes on $\mathscr{I}$ if and only if

$$D_{ij} - \frac{3}{2}\,|Y|^{-4}Y^kY^lD_{kl}\,(Y_iY_j)_{\mathrm{tf}} = 0, \tag{2.73}$$

$$\widehat{C}_{ij} - \frac{3}{2}\,|Y|^{-4}Y^kY^l\widehat{C}_{kl}\,(Y_iY_j)_{\mathrm{tf}} = 0. \tag{2.74}$$

We solve (2.73) on $\mathscr{I}^-$. (2.74) can be treated in exactly the same manner.

We define

$$d := Y^iY^jD_{ij}. \tag{2.75}$$

Applying $\widehat{\nabla}^j$ to (2.73) and employing the fact that the constraints equations enforce D_{ij} to be a TT tensor, we are led to the equation

$$Y_i\left(Y^j\widehat{\nabla}_j d + \frac{1}{3}d\widehat{\nabla}_j Y^j\right) - \frac{1}{3}\,|Y|^2\widehat{\nabla}_i d - \frac{1}{6}d\widehat{\nabla}_i\,|Y|^2 = 0, \tag{2.76}$$

after using the following two consequences of the conformal Killing equation for Y,

$$Y^j\widehat{\nabla}_j\,|Y|^2 = \frac{2}{3}\,|Y|^2\widehat{\nabla}_l Y^l, \tag{2.77}$$

$$Y^j\widehat{\nabla}_j Y_i = \frac{2}{3}Y_i\widehat{\nabla}_l Y^l - \frac{1}{2}\widehat{\nabla}_i\,|Y|^2. \tag{2.78}$$

Contraction of (2.76) with Y^i gives

$$Y^j\widehat{\nabla}_j d + \frac{1}{3}d\widehat{\nabla}_j Y^j = 0. \tag{2.79}$$

Inserting this into (2.76) yields

$$2\widehat{\nabla}_i d + d\widehat{\nabla}_i \log|Y|^2 = 0. \tag{2.80}$$

The general solution of this equation is, using that $\mathscr{I}^-$ is connected,

$$d = \frac{2}{3}C_{\mathrm{el}}\,|Y|^{-1}, \quad C_{\mathrm{el}} = \mathrm{const}. \tag{2.81}$$

It follows that necessarily

$$D_{ij} = C_{\mathrm{el}}\,|Y|^{-5}(Y_iY_j)_{\mathrm{tf}}, \tag{2.82}$$

which is, indeed, a TT tensor satisfying (2.73):

Lemma 2.6. *Let (Σ, h) be an n-dimensional Riemannian manifold. Let Y be a vector field on Σ with $|Y|^2 \neq 0$, and denote by $\widehat{\nabla}$ the connection associated to h. Then $D_{ij} := |Y|^{-n-2}(Y_iY_j)_{\mathrm{tf}}$ is a TT-tensor if and only*

$$Y^j(\widehat{\nabla}_{(i}Y_{j)})_{\mathrm{tf}} = 0. \tag{2.83}$$

(So in particular if Y is a CKVF.)

Proof. We compute the divergence of D_{ij},

$$\widehat{\nabla}^j D_{ij} = -(n+2)|Y|^{-n-4}Y_i Y^j Y^k (\widehat{\nabla}_{(j}Y_{k)})_{\mathrm{tf}} + 2\,|Y|^{-n-2}Y^j(\widehat{\nabla}_{(i}Y_{j)})_{\mathrm{tf}}, \tag{2.84}$$

and observe that D_{ij} is a TT-tensor if (2.83) holds. Conversely, contraction of (2.84) with Y^i yields

$$Y^i\widehat{\nabla}^j D_{ij} = -n\,|Y|^{-n-2}Y^i Y^j (\widehat{\nabla}_{(i}Y_{j)})_{\mathrm{tf}}, \tag{2.85}$$

which we insert into (2.84),

$$\widehat{\nabla}^j D_{ij} = \frac{n+2}{n}\,|Y|^{-2}Y_i Y^j\widehat{\nabla}^k D_{jk} + 2\,|Y|^{-n-2}Y^j(\widehat{\nabla}_{(i}Y_{j)})_{\mathrm{tf}}. \tag{2.86}$$

It follows that if D_{ij} is a TT-tensor then (2.83) holds, which completes the proof of the lemma. $\qquad\square$

Similarly, one shows that for some constant C_{mag}

$$\widehat{C}_{ij} = \sqrt{\frac{\Lambda}{3}}\,C_{\mathrm{mag}}\,|Y|^{-5}(Y_i Y_j)_{\mathrm{tf}}. \tag{2.87}$$

Remark 2.7. If Y^i is a CKVF, (2.82) defines, away from zeros of Y, a TT-tensor D_{ij} which satisfies the KID equations (2.43). On the other hand, a solution of (2.43) always satisfies (2.79).

Up to this stage we had to assume that $|Y|^2 > 0$ on $\mathscr{I}$. In fact, the above considerations reveal that this follows from the assumption of the existence of a smooth $\mathscr{I}$ whenever the rescaled tensor $\widetilde{\mathcal{T}}^{(0)}_{\mu\nu\sigma}{}^{\rho}$ vanishes there: The CKVF Y is not allowed to vanish in some open region of $\mathscr{I}$, because this would imply that the corresponding KVF would vanish in the domain of dependence of that region. Let us assume that $|Y(p)|^2 = 0$ for some $p \in \mathscr{I}$. Then it follows from (2.82)–(2.87) that, for either $C_{\mathrm{el}} \neq 0$ or $C_{\mathrm{mag}} \neq 0$,

$$\widetilde{d}_{\mu\nu\sigma\rho}\widetilde{d}^{\mu\nu\sigma\rho}|_{\mathscr{I}} = (4\widetilde{d}_{titj}\widetilde{d}^{titj} + 4\widetilde{d}_{tijk}\widetilde{d}^{tijk} + \widetilde{d}_{ijkl}\widetilde{d}^{ijkl})|_{\mathscr{I}} \tag{2.88}$$

$$= 8D_{ij}D^{ij} - \frac{24}{\Lambda}\widehat{C}_{ij}\widehat{C}^{ij} = \frac{16}{3\,|Y|^6}(C_{\mathrm{el}}^2 - C_{\mathrm{mag}}^2) \tag{2.89}$$

or

$$\widetilde{d}^{\star}_{\mu\nu\sigma\rho}\widetilde{d}^{\mu\nu\sigma\rho}|_{\mathscr{I}} = (4\widetilde{\eta}_{ti}{}^{jk}\widetilde{d}_{jktl}\widetilde{d}^{titl} + 2\widetilde{\eta}_{ti}{}^{jk}\widetilde{d}_{jklm}\widetilde{d}^{tilm})|_{\mathscr{I}} \tag{2.90}$$

$$= -16\sqrt{\frac{3}{\Lambda}}\widehat{C}_{ij}D^{ij} = -\frac{32}{3\,|Y|^6}C_{\mathrm{el}}C_{\mathrm{mag}} \tag{2.91}$$

(we used that $\widetilde{d}_{ijkl}|_{\mathscr{I}} = -\widetilde{\eta}_{ij}{}^{tm}\widetilde{\eta}_{kl}{}^{tn}\widetilde{d}_{tmtn}$), diverges at p, so that p actually cannot belong to the (unphysical) manifold. This argument does not apply when $C_{\mathrm{el}} = C_{\mathrm{mag}} = 0$. In this case the metric h is conformally flat and D_{ij} vanishes, so the data at $\mathscr{I}^-$ correspond to data for the de Sitter metric. The maximal de Sitter data is $\mathscr{I}^- = \mathbb{S}^3$ with h the standard round metric. This space has ten linearly independent conformal Killing vectors, which generically vanish at

some points. In this case the points where the conformal Killing vector vanishes do belong to $\mathscr{I}^-$ [7]. This is why we need to exclude de Sitter explicitly in the following theorem.

Theorem 2.8. *Consider a spacetime $(\mathscr{M}, g)$ solution to Einstein's vacuum field equations with $\Lambda > 0$, which admits a smooth conformal extension through $\mathscr{I}$ and which contains a KVF X. Denote by h the Riemannian metric induced by $\widetilde{g} = \Theta^2 g$ on $\mathscr{I}$, and by Y the CKVF induced by X on $\mathscr{I}$. Assume that $(\mathscr{M}, g)$ is not locally isometric to the de Sitter spacetime. Then $|Y|^2 > 0$, and the rescaled MST $\widetilde{\mathcal{T}}^{(0)}_{\mu\nu\sigma}{}^{\rho} = \Theta^{-1} \mathcal{S}^{(0)}_{\mu\nu\sigma}{}^{\rho}$ corresponding to X with $Q = Q_0$ defined by (2.59) vanishes on a connected component $\mathscr{I}^-$ of $\mathscr{I}$ if and only if the following relations hold:*

i. $\widehat{C}_{ij} = \sqrt{\frac{\Lambda}{3}} C_{\mathrm{mag}} |Y|^{-5} \left(Y_i Y_j - \frac{1}{3} |Y|^2 h_{ij} \right)$ for some constant C_{mag}, where $\widehat{C}_{ij}$ is the Cotton–York tensor of the Riemannian 3-manifold $(\mathscr{I}^-, h)$, and

ii. $D_{ij} = \widetilde{d}_{titj}|_{\mathscr{I}^-} = C_{\mathrm{el}} |Y|^{-5} \left(Y_i Y_j - \frac{1}{3} |Y|^2 h_{ij} \right)$ for some constant C_{el}.

3. The functions *c* and *k* and their restrictions to $\mathscr{I}$

3.1. The functions c and k and their constancy

Following [21], we define four real-valued functions b_1, b_2, c and k by the system (we make the assumption $Q\mathcal{F}^2 - 4\Lambda \neq 0$; later on it will become clear that this holds automatically near a regular $\mathscr{I}$)

$$b_2 - ib_1 = -\frac{36Q(\mathcal{F}^2)^{5/2}}{(Q\mathcal{F}^2 - 4\Lambda)^3}, \tag{3.1}$$

$$c = -X^2 - \mathrm{Re}\left(\frac{6\mathcal{F}^2(Q\mathcal{F}^2 + 2\Lambda)}{(Q\mathcal{F}^2 - 4\Lambda)^2} \right), \tag{3.2}$$

$$k = \left| \frac{36\mathcal{F}^2}{(Q\mathcal{F}^2 - 4\Lambda)^2} \right| \nabla_\mu Z \nabla^\mu Z - b_2 Z + cZ^2 + \frac{\Lambda}{3} Z^4, \tag{3.3}$$

where

$$Z = 6 \, \mathrm{Re}\left(\frac{\sqrt{\mathcal{F}^2}}{Q\mathcal{F}^2 - 4\Lambda} \right). \tag{3.4}$$

We note that the expression (3.3) for k as given in [21] has two typos both in the statement of theorem 1 and of theorem 6.

A remark is in order concerning the appearance of square roots of the complex function $\mathcal{F}^2$. In the setting of [21], the function $\mathcal{F}^2$ is shown to be nowhere vanishing so we can prescribe the choice of square root at one point and extend it by continuity to the whole manifold. Since $\mathcal{F}^2$ does not vanish, no branch point of the root is ever met and $\sqrt{\mathcal{F}^2}$ is smooth everywhere. Moreover, the function $\mathcal{F}^2$ has strictly negative real part in a neighborhood of $\mathscr{I}$ (see (3.8) below). We can thus fix the square root $\sqrt{\mathcal{F}^2}$ in this neighborhood by choosing the positive branch near $\mathscr{I}$, namely the branch that takes positive real numbers and gives positive real values. We will use this prescription for any function which is non-zero in a neighborhood of infinity.

---

[7] Note that in the de Sitter case we have $\mathcal{C}_{\alpha\beta\mu\nu} = 0 = Q_0$, so the MST associated to *any* KVF vanishes identically.

The following result is proven in [21, theorems 4 and 6]

Theorem 3.1. *Let $(\mathcal{M}, g)$ be a Λ-vacuum spacetime which admits a KVF X such that the MST vanishes for some function Q. Assume further that the functions $Q\mathcal{F}^2$ and $Q\mathcal{F}^2 - 4\Lambda$ are not identically zero. Then:*

   i. $\mathcal{F}^2$ and $Q\mathcal{F}^2 - 4\Lambda$ are nowhere vanishing,
  ii. Q is given by (2.59), i.e. $Q = Q_0$, and
 iii. b_1, b_2, c and k are constant.

Remark 3.2. If $\Lambda > 0$ and $(\mathcal{M}, g)$ admits a smooth $\mathscr{I}$, has vanishing MST and is not locally isometric to the de Sitter spacetime, it follows from (3.5)–(3.8) below that Q, $\mathcal{F}^2$ and $Q\mathcal{F}^2 - 4\Lambda$ will never be identically zero. In other words, b_1, b_2, c and k are constant whenever the MST vanishes in a spacetime $(\mathcal{M}, g)$ as above.

Combining theorem 3.1 and remark 3.2 it follows that a $\Lambda > 0$-vacuum spacetime admitting a KVF X with vanishing associated MST for some Q and for which $\mathcal{F}^2 = 0$ somewhere cannot admit a smooth $\mathscr{I}$, unless the spacetime is locally isometric to de Sitter. Although *a priori* interesting, this result turns out to be empty since it has been proven in [22] that all spacetimes with vanishing MST and null Killing form $\mathcal{F}$ (somewhere, and hence everywhere) have necessarily $\Lambda \leqslant 0$.

The above functions (3.1)–(3.3), or rather their restrictions to $\mathscr{I}$, turn out to be crucial for the classification of vacuum spacetimes with vanishing MST (and conformally flat $\mathscr{I}$, cf [20]). Our next aim will therefore be to find explicit expressions for them in terms of the data at $\mathscr{I}$ under the assumption that the MST vanishes for some choice of Q. We wish to find expressions at null infinity that make sense (and generally cease to be constant) for any Λ-vacuum spacetime with a smooth conformal compactification and a KVF.

Employing the relations collected in sections 2.2, 2.3 and 2.5 we find that, under the assumption that the MST vanishes,

$$(\Theta^{-5}Q_0) = 3\Lambda^{-1}|Y|^{-5}(C_{\text{el}} - iC_{\text{mag}}) + O(\Theta), \tag{3.5}$$

$$Q_0\mathcal{F}^2 - 4\Lambda = -4\Lambda + O(\Theta^3), \tag{3.6}$$

$$\mathcal{F}^2 = -\frac{4}{3}\Lambda\Theta^{-2}\widetilde{X}^2 + 2\widetilde{F}^2 - \frac{1}{4}\widetilde{F}^2 + 2i\widetilde{F}^{\star}_{\alpha\beta}\widetilde{F}^{\alpha\beta}$$
$$+ 2\widetilde{X}^{\alpha}\widetilde{\nabla}_{\alpha}\widetilde{F} + 8\widetilde{X}^{\alpha}\widetilde{X}^{\beta}\widetilde{L}_{\alpha\beta} + 4i\Theta^{-1}\widetilde{F}^{\mu\nu}\widetilde{H}^{\star}_{\mu\nu} \tag{3.7}$$

$$= -\frac{4}{3}\Lambda\Theta^{-2}\widetilde{X}^2 + 4i\sqrt{\frac{\Lambda}{3}}\,\Theta^{-1}Y_k N^k + |N|^2 - \frac{4}{9}f^2$$
$$+ \frac{8}{3}Y^i\widehat{\nabla}_i f + 8Y^iY^j\widehat{L}_{ij} + O(\Theta). \tag{3.8}$$

Here

$$N^k := \text{curl } Y^k = \widehat{\eta}^{ijk}\widehat{\nabla}_i Y_j \tag{3.9}$$

denotes the curl of Y, and

$$f := \widehat{\nabla}_i Y^i \tag{3.10}$$

its divergence.

　　　Let us determine the trace of (3.1)–(3.3) in the unphysical, conformally rescaled spacetime on $\mathscr{I}$ under the assumption that the MST vanishes for some choice of Q. With (3.5) and (3.8), we observe that, on $\mathscr{I}$, equation (3.1) yields

$$(b_2 - ib_1)|_{\mathscr{I}} = \left(\frac{9}{16}\Lambda^{-3}(\Theta^{-5}Q_0)(\Theta^2\mathcal{F}^2)^{5/2}\right)\bigg|_{\mathscr{I}} = \frac{2}{\Lambda}\sqrt{\frac{3}{\Lambda}}\,(C_{\text{mag}} + iC_{\text{el}}). \tag{3.11}$$

From (3.2) and (3.8) we conclude that

$$\frac{\Lambda}{3}c\bigg|_{\mathscr{I}} = \frac{\Lambda}{3}\left(-\Theta^{-2}\widetilde{X}^2 - \frac{3}{4}\Lambda^{-1}\text{Re}(\mathcal{F}^2)\right)\bigg|_{\mathscr{I}} \tag{3.12}$$

$$= -\frac{1}{4}\,|N|^2 + \frac{1}{9}f^2 - \frac{2}{3}Y^i\widehat{\nabla}_i f - 2Y^iY^j\widehat{L}_{ij}. \tag{3.13}$$

Note that this implies that

$$\mathcal{F}^2 = -\frac{4}{3}\Lambda\Theta^{-2}\widetilde{X}^2 + 4i\Theta^{-1}\widetilde{F}^{\mu\nu}\widetilde{H}^{\star}_{\mu\nu} - \frac{4}{3}\Lambda c$$

$$- 8D_{ij}Y^iY^j\Theta - 4i\sqrt{\frac{3}{\Lambda}}N^k\widetilde{\nabla}_t\widetilde{\nabla}_t\widetilde{X}_k\Theta + O(\Theta^2) \tag{3.14}$$

$$= -\frac{4}{3}\Lambda(\Theta^{-2}\,|\widetilde{X}|^2 + c) + 4i\sqrt{\frac{\Lambda}{3}}\,\Theta^{-1}Y_kN^k + O(\Theta). \tag{3.15}$$

Next, we compute the function Z (here an overbar means 'complex conjugation'),

$$Z = \text{Re}\left(\frac{6\sqrt{\mathcal{F}^2}}{Q_0\mathcal{F}^2 - 4\Lambda}\right)$$

$$= \frac{6\,\text{Re}\,(\sqrt{\mathcal{F}^2}\,(\overline{Q_0\mathcal{F}^2 - 4\Lambda}))}{(Q_0\mathcal{F}^2 - 4\Lambda)(\overline{Q_0\mathcal{F}^2 - 4\Lambda})}$$

$$= \left(\frac{3}{8}\Lambda^{-2} + O(\Theta^3)\right)\text{Re}\,(\sqrt{\mathcal{F}^2}\,(\overline{Q_0\mathcal{F}^2 - 4\Lambda}))$$

$$= -\frac{3}{2}\Lambda^{-1}\Theta^{-1}\,\text{Re}\,(\sqrt{\Theta^2\mathcal{F}^2}) + O(\Theta^2).$$

equation (3.15) yields

$$\text{Re}\,(\sqrt{\Theta^2\mathcal{F}^2}) = |Y|^{-1}Y_kN^k\Theta + O(\Theta^3). \tag{3.16}$$

Thus

$$Z = -\frac{3}{2}\Lambda^{-1}\,|Y|^{-1}Y_kN^k + O(\Theta^2), \tag{3.17}$$

and we deduce from (3.3) that

$$k|_{\mathscr{I}} = \left(\left|-\frac{9}{4\Lambda^2}\Theta^2\mathcal{F}^2(\widetilde{\nabla}_t Z)^2 + \frac{9}{4\Lambda^2}\Theta^2\mathcal{F}^2\widetilde{\nabla}_i Z\widetilde{\nabla}^i Z\right| - b_2 Z + cZ^2 + \frac{1}{3}\Lambda Z^4\right)\bigg|_{\mathscr{I}}$$

$$= \left(\frac{3}{\Lambda}\,|Y|^2\widehat{\nabla}_i Z\widehat{\nabla}^i Z - b_2 Z + cZ^2 + \frac{1}{3}\Lambda Z^4\right)\bigg|_{\mathscr{I}}.$$

From the conformal Killing equation for Y we find that

$$\Lambda |Y| \widehat{\nabla}_i Z|_{\mathscr{I}} = -\frac{1}{2} f \; |Y|^{-2} Y_i Y_k N^k - \frac{3}{4} \; |Y|^{-2} Y_k N^k \widehat{\eta}_{ijl} Y^j N^l + \frac{3}{2} \widehat{\nabla}_i (Y_k N^k),$$

whence, using (3.13),

$$
\begin{aligned}
\Lambda^2 \; |Y|^2 \widehat{\nabla}_i Z \widehat{\nabla}^i Z|_{\mathscr{I}} &= \frac{1}{4} f^2 \; |Y|^{-2} (Y_k N^k)^2 - \frac{3}{4} f \; |Y|^{-2} Y^i \widehat{\nabla}_i (Y_k N^k)^2 \\
&\quad + \frac{9}{16} \; |Y|^{-2} (Y_k N^k)^2 |N|^2 - \frac{9}{16} \; |Y|^{-4} (Y_k N^k)^4 \\
&\quad - \frac{9}{8} \; |Y|^{-2} \widehat{\eta}_{ijl} Y^j N^l \widehat{\nabla}^i (Y_k N^k)^2 + \frac{9}{4} \widehat{\nabla}_i (Y_j N^j) \widehat{\nabla}^i (Y_k N^k) \\
&= \frac{1}{2} f^2 \; |Y|^{-2} (Y_k N^k)^2 - \frac{3}{4} f \; |Y|^{-2} Y^i \widehat{\nabla}_i (Y_k N^k)^2 \\
&\quad - \frac{3}{2} \; |Y|^{-2} (Y_k N^k)^2 Y^i \widehat{\nabla}_i f - \frac{9}{2} \; |Y|^{-2} (Y_k N^k)^2 Y^i Y^j \widehat{L}_{ij} \\
&\quad - \frac{9}{8} \; |Y|^{-2} \widehat{\eta}_{ijl} Y^j N^l \widehat{\nabla}^i (Y_k N^k)^2 + \frac{9}{4} \widehat{\nabla}_i (Y_j N^j) \widehat{\nabla}^i (Y_k N^k) \\
&\quad - \frac{1}{3} \Lambda^3 c Z^2 - \frac{1}{9} \Lambda^4 Z^4,
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\frac{\Lambda}{3} \right)^3 k \bigg|_{\mathscr{I}} &= \frac{1}{18} \; |Y|^{-2} (Y_k N^k)^2 (f^2 - 3 Y^i \widehat{\nabla}_i f - 9 Y^i Y^j \widehat{L}_{ij}) + \frac{1}{2} (\widehat{C}_{ij} Y^i Y^j) Y_k N^k \\
&\quad - \frac{1}{8} \widetilde{\nabla}_i \log |Y|^2 \widetilde{\nabla}^i (Y_k N^k)^2 + \frac{1}{4} \widetilde{\nabla}_i (Y_j N^j) \widetilde{\nabla}^i (Y_k N^k),
\end{aligned}
\tag{3.18}
$$

where we have used $b_2 = 6 \Lambda^{-2} \sqrt{\frac{\Lambda}{3}} \, C_{\text{mag}}$ and have replaced C_{mag} in terms of the Cotton–York tensor $C_{\text{mag}} = \frac{3}{2} \sqrt{\frac{3}{\Lambda}} |Y| \widehat{C}_{ij} Y^i Y^j$. Expression (3.18) provides a simplified formula for k on $\mathscr{I}$ in terms of Y.

　It follows from theorem 3.1 and remark 3.2 that the right-hand sides of (3.13) and (3.18) are constant, whenever the MST vanishes for some function Q.

3.2. The functions $\widehat{c}(Y)$ and $\widehat{k}(Y)$

In the previous section we have introduced the spacetime functions c and k and computed their restrictions on $\mathscr{I}$ in terms of the induced metric h and the CKVF Y, whenever the MST vanishes. Here we regard these restrictions as functions which are intrinsically defined on some Riemannian 3-manifold, whence we are led to the following

Definition 3.3. Let (Σ, h) be a Riemannian 3-manifold which admits a CKVF Y. Then, guided by (3.13) and (3.18), we set

$$\widehat{c}(Y) := -\frac{1}{4} \; |N|^2 + \frac{1}{9} f^2 - \frac{2}{3} Y^i \widehat{\nabla}_i f - 2 Y^i Y^j \widehat{L}_{ij}, \tag{3.19}$$

$$\widehat{k}(Y) := \frac{1}{18}\,|Y|^{-2}(Y_k N^k)^2(f^2 - 3Y^i\widehat{\nabla}_i f - 9Y^i Y^j \widehat{L}_{ij}) + \frac{1}{2}(\widehat{C}_{ij}Y^i Y^j)Y_k N^k$$

$$- \frac{1}{8}\widehat{\nabla}_i \log |Y|^2 \widehat{\nabla}^i (Y_k N^k)^2 + \frac{1}{4}\widehat{\nabla}_i (Y_j N^j)\widehat{\nabla}^i (Y_k N^k). \tag{3.20}$$

The spacetime functions c and k, (3.2) and (3.3), have been introduced in [21] in the setting of a vanishing MST, where they arise naturally as integration constants. Concerning $\widehat{c}(Y)$ and $\widehat{k}(Y)$, in general one cannot expect them to be constant. However, let us assume that the Cotton–York tensor satisfies condition (2.87). Our main result, theorem 1.3, (which is a reformulation of theorem 4.16) now implies the following: Choosing an initial D_{ij} according to (2.82), (Σ, h) can be extended to a $\Lambda > 0$ vacuum spacetime for which Σ represents $\mathscr{I}^-$ and to which Y extends as a KVF such that the associated MST vanishes for some function Q. One then deduces from the results in [21] that c and k, and therefore also $\widehat{c}(Y)$ and $\widehat{k}(Y)$ are constant:

Lemma 3.4. *Let (Σ, h) be a Riemannian 3-manifold which admits a CKVF Y with $|Y|^2 > 0$ and such that $\widehat{C}_{ij} = C\,|Y|^{-5}(Y_i Y_j)_{\mathrm{tf}}$ with C constant. Then the functions $\widehat{c}(Y)$ and $\widehat{k}(Y)$ as given by (3.19) and (3.20) are constant.*

In particular, in the case of $\widehat{k}(Y)$, it is far from obvious that the condition (2.87) implies that this function is constant, but the proof via the extension of (Σ, h) to a vacuum spacetime provides an elegant tool to prove that. As already indicated above, the constants $\widehat{c}$ and $\widehat{k}$ play a decisive role in the classification of $\Lambda > 0$-vacuum spacetimes which admit a conformally flat $\mathscr{I}$ and a KVF w.r.t. which the associated MST vanishes [20].

3.3. Constancy of $\widehat{c}(Y)$

Let us focus attention on the function $\widehat{c}(Y)$. In section 4.2 we will introduce an alternative definition of the function Q which permits the derivation of evolution equations. It turns out that the associated MST will in general not be regular at $\mathscr{I}$, and that the constancy of $\widehat{c}(Y)$ is a necessary condition to ensure regularity. Let us therefore consider the issue under which condition the function $\widehat{c}(Y)$ is constant. The aim of this section is to prove the following lemma.

Lemma 3.5. *Let (Σ, h) be a 3-dimensional oriented Riemannian manifold which admits a CKVF, Y, $\mathscr{L}_Y h = (2/3)f h$ with f and N_i as defined in (3.9) and (3.10). Then the function $\widehat{c}(Y)$ introduced in (3.19) satisfies the following identity*

$$\widehat{\nabla}_l \widehat{c}(Y) = -2\widehat{\eta}^{m}_{\ li}Y^i\widehat{C}_{mj}Y^j = \widehat{C}_{jli}Y^j Y^i.$$

In particular, if, for some smooth function $H : \Sigma \mapsto \mathbb{R}$, the Cotton–York tensor (2.72) satisfies

$$\widehat{C}_{ij} = H(Y_i Y_j)_{\mathrm{tr}} \tag{3.21}$$

and Σ is connected, then the proportionality function necessarily takes the form $H = C\,|Y|^{-5}$ where C is a constant, and $\widehat{c}(Y)$ is constant over the manifold.

Remark 3.6. The lemma implies in particular that $\hat{c}$ is constant if and only if Y^j is an eigenvector of the Cotton–York tensor.

Proof. From the conformal Killing equation $\nabla_{(i} Y_{j)} = \frac{1}{3} f h_{ij}$ it follows

$$\widehat{\nabla}_j \widehat{\nabla}_k f = -3(\mathscr{L}_Y \widehat{L})_{jk}, \tag{3.22}$$

$$\widehat{\nabla}_i \widehat{\nabla}_j Y_l = Y_m \widehat{R}^m_{\ ijl} + \frac{1}{3}(h_{jl} \widehat{\nabla}_i f + h_{il} \widehat{\nabla}_j f - h_{ij} \widehat{\nabla}_l f). \tag{3.23}$$

Evaluating $|N|^2$ as

$$|N|^2 = \hat{\eta}_{ijk} \hat{\eta}^{ilm} \widehat{\nabla}^j Y^k \widehat{\nabla}_l Y_m = (\delta^l_j \delta^m_k - \delta^l_k \delta^m_j) \widehat{\nabla}^j Y^k \widehat{\nabla}_l Y_m$$
$$= \widehat{\nabla}^j Y^k \widehat{\nabla}_j Y_k - \widehat{\nabla}^j Y^k \widehat{\nabla}_k Y_j,$$

we can write $\hat{c}(Y) = -\frac{1}{4} \widehat{\nabla}_i Y_j (\widehat{\nabla}^i Y^j - \widehat{\nabla}^j Y^i) + \frac{1}{9} f^2 - \frac{2}{3} Y^i \widehat{\nabla}_i f - 2 Y^i Y^j \widehat{L}_{ij}$. It is convenient to split $\hat{c}(Y)$ in two terms

$$\hat{c}_1 := -\frac{2}{3} Y^i \widehat{\nabla}_i f - 2 Y^i Y^j \widehat{L}_{ij} + \frac{1}{9} f^2,$$
$$\hat{c}_2 := -\frac{1}{4} \widehat{\nabla}_i Y_j (\widehat{\nabla}^i Y^j - \widehat{\nabla}^j Y^i).$$

so that $\hat{c} = \hat{c}_1 + \hat{c}_2$. We start with $\widehat{\nabla}_l c_1$,

$$\widehat{\nabla}_l \hat{c}_1 = -\frac{2}{3} \widehat{\nabla}_l Y^i \widehat{\nabla}_i f - \frac{2}{3} Y^i \widehat{\nabla}_l \widehat{\nabla}_i f - 4(\widehat{\nabla}_l Y^i) Y^j \widehat{L}_{ij} - 2 Y^i Y^j \widehat{\nabla}_l \widehat{L}_{ij} + \frac{2}{9} f \widehat{\nabla}_l f$$
$$= -\frac{2}{3} \widehat{\nabla}_l Y^i \widehat{\nabla}_i f + 2 Y^i (\mathscr{L}_Y \widehat{L})_{li} - 4(\widehat{\nabla}_l Y^i) Y^j \widehat{L}_{ij} - 2 \eta^m_{\ li} Y^i \widehat{C}_{mj} Y^j$$
$$\quad - 2 Y^j [(\mathscr{L}_Y L)_{lj} - \widehat{L}_{lm} \widehat{\nabla}_j Y^m - \widehat{L}_{mj} \widehat{\nabla}_l Y^m] + \frac{2}{9} f \widehat{\nabla}_l f$$
$$= -\frac{2}{3} \widehat{\nabla}_l Y^i \widehat{\nabla}_i f - 4 Y^j \widehat{L}_{i[j} \widehat{\nabla}_{l]} Y^i - 2 \hat{\eta}^m_{\ li} Y^i \widehat{C}_{mj} Y^j + \frac{2}{9} f \widehat{\nabla}_l f, \tag{3.24}$$

where in the second equality we inserted (3.22) and $Y^i \widehat{\nabla}_l \widehat{L}_{ij} = Y^i \widehat{\nabla}_i \widehat{L}_{lj} + Y^i \hat{\eta}^m_{\ lj} \widehat{C}_{mj} = (\mathscr{L}_Y \widehat{L})_{lj} - \widehat{L}_{lm} \widehat{\nabla}_j Y^m - \widehat{L}_{mj} \widehat{\nabla}_l Y^m + Y^i \hat{\eta}^m_{\ li} \widehat{C}_{mj}$ and in the third one obvious cancellations have been applied. Concerning $\widehat{\nabla}_l \hat{c}_2$ we find, after a simple rearrangement of indices,

$$\widehat{\nabla}_l \hat{c}_2 = -\frac{1}{2} \widehat{\nabla}_l \widehat{\nabla}_i Y_j (\widehat{\nabla}^i Y^j - \widehat{\nabla}^j Y^i) = -Y_m \widehat{R}^m_{\ lij} \widehat{\nabla}^i Y^j - \frac{1}{3} \widehat{\nabla}_i f (\widehat{\nabla}^i Y_l - \widehat{\nabla}_l Y^i),$$

where in the second equality we used (3.23) and the antisymmetry of $(\widehat{\nabla}^i Y^j - \widehat{\nabla}^j Y^i)$. We now use the Riemann tensor decomposition in three dimensions,

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$$\widehat{R}^{m}_{\ lij} = \delta^{m}_{i}\widehat{L}_{lj} - \delta^{m}_{j}\widehat{L}_{li} + \widehat{L}^{\ m}_{i}h_{lj} - \widehat{L}^{\ m}_{j}h_{li},$$

to obtain

$$\widehat{\nabla}_l \hat{c}_2 = -(Y^m \widehat{L}_{mi} + \frac{1}{3}\widehat{\nabla}_i f)(\widehat{\nabla}^i Y_l - \widehat{\nabla}_l Y^i) - Y_i \widehat{L}_{lj}(\widehat{\nabla}^i Y^j - \widehat{\nabla}^j Y^i). \qquad (3.25)$$

Combining (3.24) and (3.25)

$$\begin{aligned}
\widehat{\nabla}_l \hat{c}\,(Y) &= -Y^m \widehat{L}_{mi}(\widehat{\nabla}^i Y_l + \widehat{\nabla}_l Y^i) + Y_i \widehat{L}_{lj}(\widehat{\nabla}^i Y^j + \widehat{\nabla}^j Y^i) - \frac{1}{3}\widehat{\nabla}_i f\,(\widehat{\nabla}^i Y_l + \widehat{\nabla}_l Y^i) \\
&\quad + \frac{2}{9}f\widehat{\nabla}_l f - 2\widehat{\eta}^{\ m}_{li}Y^i \widehat{C}_{mj}Y^j \\
&= -2\widehat{\eta}^{\ m}_{li}Y^i \widehat{C}_{mj}Y^j,
\end{aligned}$$

where in the second equality we used the conformal Killing equation.

Now, whenever (2.87) holds we have $\widehat{C}_{mj}Y^j = \frac{2}{3}H\,|Y|^2 Y_m$ and $\widehat{\nabla}_l \hat{c}\,(Y) = 0$ so that $\hat{c}\,(Y)$ is constant over the (connected) manifold Σ. The fact that H is necessarily of the form $H = C\,|Y|^{-5}$ was already shown in the proof of proposition 2.5. $\qquad \square$

Remark 3.7. A similar lemma holds for three-dimensional manifolds of arbitrary signature. The term $|N|^2$ in $\hat{c}\,(Y)$ needs to be replaced by $\epsilon\,|N|^2$ where ϵ is an appropriate sign depending on the signature.

Another problem of interest is to find necessary and sufficient conditions which ensure the constancy of $\hat{k}\,(Y)$. Since this expression is of higher order in Y than $\hat{c}\,(Y)$, this is expected to be somewhat more involved.

4. Evolution of the MST

4.1. The Ernst potential on $\mathscr{I}$

In this section we make no assumption concerning the MST, so all the results hold generally for any $(\Lambda > 0)$-vacuum spacetime admitting a KVF X and a smooth conformal compactification.

Using the results of section 2.1 the so-called Ernst one-form of X, $\sigma_\mu := 2X^\alpha \mathcal{F}_{\alpha\mu}$, has the following asymptotic expansion

$$\sigma_\mu = 2X^\alpha \mathcal{F}_{\alpha\mu} \qquad (4.1)$$

$$= 2X^\alpha(\Theta^{-3}\widetilde{\mathcal{H}}_{\alpha\mu} + \Theta^{-2}\widetilde{\mathcal{F}}_{\alpha\mu}) \qquad (4.2)$$

$$= 2\Theta^{-3}\widetilde{X}^\alpha\left(\widetilde{H}_{\alpha\mu} + \frac{i}{2}\widetilde{\eta}_{\alpha\mu}^{\ \ \nu\sigma}\widetilde{H}_{\nu\sigma} + \Theta(\widetilde{F}_{\alpha\mu} + \frac{i}{2}\widetilde{\eta}_{\alpha\mu}^{\ \ \nu\sigma}\widetilde{F}_{\nu\sigma})\right) \qquad (4.3)$$

$$= 2\Theta^{-3}\widetilde{X}^\alpha\left(2\widetilde{X}_{[\alpha}\widetilde{\nabla}_{\mu]}\Theta + \Theta((\widetilde{\nabla}_\alpha\widetilde{X}_\mu)_{\mathrm{tf}} + \frac{i}{2}\widetilde{\eta}_{\alpha\mu}^{\ \ \nu\sigma}\widetilde{\nabla}_\nu\widetilde{X}_\sigma)\right). \qquad (4.4)$$

It is known (see e.g. [32]) that this covector field has an ('Ernst-') potential $\sigma_\mu = \partial_\mu \sigma$, at least locally. Taking the following useful relations into account,

$$Y^i \widehat{\eta}_i{}^{jk} \widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{\nabla}_j \widetilde{X}_k |_{\mathscr{I}} = 2\widehat{C}_{ij} Y^i Y^j,$$

$$\widetilde{\nabla}_t \left(-|\widetilde{X}|^2 \Theta^{-2} + \frac{3}{\Lambda} i\widetilde{H}^{\alpha\beta} \widetilde{F}^\star_{\alpha\beta} \Theta^{-1} \right) \bigg|_{\mathscr{I}} = 2\sqrt{\frac{\Lambda}{3}} \, |Y|^2 \Theta^{-3} - iY_i N^i \Theta^{-2}$$

$$- \frac{1}{2}\sqrt{\frac{3}{\Lambda}} \widehat{R} \, |Y|^2 \Theta^{-1} + \frac{2}{3}\sqrt{\frac{3}{\Lambda}} D_{kl} Y^k Y^l$$

$$+ \frac{i}{2}\frac{3}{\Lambda} (2\widehat{C}_{kl} Y^k Y^l + N^k \overline{\widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{X}_k}) + O(\Theta),$$

a somewhat lengthy computation making extensive use of the equations (2.28)–(2.38) and the Killing relations (2.44)–(2.53) reveals that (as before an overbar means 'restriction to $\mathscr{I}$')

$$\sigma_t = 2\Theta^{-2} \widetilde{X}^t \widetilde{\nabla}_t \widetilde{X}_t + \frac{1}{2}\Theta^{-2} \widetilde{X}^t \widetilde{\nabla}_\beta \widetilde{X}^\beta + 2\Theta^{-3} \widetilde{X}^i \widetilde{X}_i \widetilde{\nabla}_t \Theta$$

$$+ 2\Theta^{-2} \widetilde{X}^i \widetilde{\nabla}_i \widetilde{X}_t + i\Theta^{-2} \widetilde{X}^i \widetilde{\eta}_{it}{}^{jk} \widetilde{\nabla}_j \widetilde{X}_k + O(\Theta) \tag{4.5}$$

$$= 2\sqrt{\frac{\Lambda}{3}} \, |Y|^2 \Theta^{-3} - iY_i N^i \Theta^{-2} - \frac{1}{2}\sqrt{\frac{3}{\Lambda}} \widehat{R} \, |Y|^2 \Theta^{-1} + \frac{2}{3}\sqrt{\frac{3}{\Lambda}} D_{kl} Y^k Y^l$$

$$- \frac{i}{2}\frac{3}{\Lambda} (2\widehat{C}_{ij} Y^i Y^j + N^i \overline{\widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{X}_i}) + O(\Theta) \tag{4.6}$$

$$= \widetilde{\nabla}_t \bigg[-\widetilde{X}^2 \Theta^{-2} + \frac{3}{\Lambda} i\widetilde{H}^{\alpha\beta} \widetilde{F}^\star_{\alpha\beta} \Theta^{-1} + q(x^i)$$

$$- i\frac{3}{\Lambda}\sqrt{\frac{3}{\Lambda}} (2\widehat{C}_{kl} Y^k Y^l + N^k \overline{\widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{X}_k})\Theta + O(\Theta^2) \bigg] \tag{4.7}$$

for some function $q(x^i)$. To determine this function it is necessary to compute σ_i up to and including constant order. Using the relation

$$\widehat{\nabla}_i (Y_k N^k) = \frac{1}{3} f N_i + 2\widehat{\eta}_{ij}{}^k Y^j Y^l \widehat{L}_{kl} + \frac{2}{3}\widehat{\eta}_{ij}{}^k Y^j \widehat{\nabla}_k f, \tag{4.8}$$

another lengthy calculation gives via (2.28)–(2.38) and (2.44)–(2.53)

$$\sigma_i = -\widehat{\nabla}_i |Y|^2 \Theta^{-2} + i\sqrt{\frac{3}{\Lambda}} \widehat{\nabla}_i (Y_k N^k) \Theta^{-1} + \frac{1}{\Lambda}\bigg[-Y_i \left(\Delta_h - \frac{\widehat{R}}{3} \right) f$$

$$- Y_i \overline{\widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{X}^t} + 3i\widehat{\eta}_{ij}{}^k Y^j \overline{\widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{X}_k} + 3\overline{\widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{X}^j} \widehat{\nabla}_j Y_i$$

$$- \frac{1}{2} |Y|^2 \widehat{\nabla}_i \widehat{R} + \frac{1}{2} Y_i Y^j \widehat{\nabla}_j \widehat{R} + 3Y^j \overline{\widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{\nabla}_j \widetilde{X}_i} \bigg] + O(\Theta) \tag{4.9}$$

$$
= -\widehat{\nabla}_i |Y|^2 \Theta^{-2} + i\sqrt{\frac{3}{\Lambda}}\,\widehat{\nabla}_i(Y_k N^k)\Theta^{-1}
$$

$$
+ \frac{1}{\Lambda}\Bigg[\underbrace{\frac{2}{3}Y_i\left(\Delta_h f + \frac{1}{2}\widehat{R}f + \frac{3}{4}Y^j\widehat{\nabla}_j\widehat{R}\right)}_{=0\text{ by }(3.22)} - 6i\sqrt{\frac{\Lambda}{3}}\,\widehat{\eta}_{ij}{}^k D_{kl}Y^j Y^l
$$

$$
- 3\overline{\widetilde{\nabla}_t\widetilde{\nabla}_t\widetilde{X}^j}\,\widehat{\nabla}_i Y_j + 2f\overline{\widetilde{\nabla}_t\widetilde{\nabla}_t\widetilde{X}_i} - \frac{1}{2}|Y|^2\widehat{\nabla}_i\widehat{R} - 3Y^j\overline{\widetilde{\nabla}_t\widetilde{\nabla}_t\widetilde{\nabla}_i\widetilde{X}_j}\Bigg] + O(\Theta) \tag{4.10}
$$

$$
= \widetilde{\nabla}_i\left(-\widetilde{X}^2\Theta^{-2} + \frac{3}{\Lambda}i\widetilde{H}^{\alpha\beta}\widetilde{F}^{\star}_{\alpha\beta}\Theta^{-1} - a + O(\Theta)\right) - 2i\sqrt{\frac{3}{\Lambda}}\,\widehat{\eta}_{ij}{}^k D_{kl}Y^j Y^l \tag{4.11}
$$

for some complex constant a (the 'σ-constant' introduced in the Introduction). As one should expect, the last term in (4.11) has, at least locally, a potential, supposing that Y is a CKVF and D_{ij} a TT tensor which together satisfy the KID equations (2.43): Indeed, setting

$$
P_i := -2\sqrt{\frac{3}{\Lambda}}\,\widehat{\eta}_{ij}{}^k D_{kl}Y^j Y^l, \tag{4.12}
$$

we find that

$$
\widehat{\eta}_i{}^{jk}\widehat{\nabla}_j P_k = 2\sqrt{\frac{3}{\Lambda}}\,Y^j\left(\mathscr{L}_Y D_{ij} + \frac{1}{3}fD_{ij}\right) = 0 \tag{4.13}
$$

$$
\implies \quad \widehat{\nabla}_{[i}P_{j]} = 0. \tag{4.14}
$$

On the simply connected components of the initial 3-manifold this implies

$$
P_i = \widehat{\nabla}_i p \quad \text{for some real-valued function} \quad p = p(x^i). \tag{4.15}
$$

(The fact that p is only determined up to some constant is reflected in the σ-constant a introduced above.) Thus,

$$
\sigma_i = \widetilde{\nabla}_i\left(-\widetilde{X}^2\Theta^{-2} + \frac{3}{\Lambda}i\widetilde{H}^{\alpha\beta}\widetilde{F}^{\star}_{\alpha\beta}\Theta^{-1} + ip(x^j) - a + O(\Theta)\right). \tag{4.16}
$$

Altogether, we conclude that $q(x^i) = ip(x^i) - a$ for some not yet specified $a \in \mathbb{C}$, and that

$$
\sigma = -\widetilde{X}^2\Theta^{-2} + \frac{3}{\Lambda}i\widetilde{H}^{\alpha\beta}\widetilde{F}^{\star}_{\alpha\beta}\Theta^{-1} + ip(x^j) - a
$$

$$
- i\frac{3}{\Lambda}\sqrt{\frac{3}{\Lambda}}(2\widehat{C}_{kl}Y^k Y^l + N^k\widetilde{\nabla}_t\widetilde{\nabla}_t\widetilde{X}_k)\Theta + O(\Theta^2) \tag{4.17}
$$

$$
= -|Y|^2 \Theta^{-2} + i\sqrt{\frac{3}{\Lambda}} Y_i N^i \Theta^{-1} + \frac{1}{3\Lambda} f^2 - \frac{3}{\Lambda} Y^i \overline{\widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{X}_i}
$$

$$
+ ip(x^j) - a + \frac{3}{\Lambda}\left(\frac{2}{3} D_{kl} Y^k Y^l - i\sqrt{\frac{3}{\Lambda}} \widehat{C}_{kl} Y^k Y^l \right.
$$

$$
\left. - \frac{i}{4}\sqrt{\frac{3}{\Lambda}} N^k (2\overline{\widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{X}_k} + Y_k \widehat{R}) \right)\Theta + O(\Theta^2). \tag{4.18}
$$

Proposition 4.1. *Consider a $\Lambda > 0$-vacuum spacetime which admits a KVF and a smooth $\mathscr{I}$. Then the Ernst potential σ can be computed explicitly near $\mathscr{I}$, where it admits the expansion (4.18).*

4.2. Alternative definition of the function Q

In order to derive evolution equations it is convenient (cf [16] for the $\Lambda = 0$-case) to define, *for each Ernst potential σ*, a new function $Q = Q_{ev}$ by the following set of equations,

$$
Q_{ev} := \frac{3J}{R} - \frac{\Lambda}{R^2}, \tag{4.19}
$$

$$
R := -\frac{1}{2}i\sqrt{\mathcal{F}^2}, \tag{4.20}
$$

$$
J := \frac{R + \sqrt{R^2 - \Lambda\sigma}}{\sigma}. \tag{4.21}
$$

and all square roots are chosen with the same prescription as explained above, cf page 16. Alternatively, we could have defined $R = +(i/2)\sqrt{\mathcal{F}^2}$. Then the expression for J would have changed accordingly. The choice (4.20) is preferable because then the real part of R approaches minus infinity at $\mathscr{I}$, in agreement with the usual behavior of Boyer-Lindquist type coordinates near infinity in Kerr–de Sitter and related metrics [21]. Note that the definition of J above implies the identity

$$
\sigma J^2 - 2JR + \Lambda = 0, \tag{4.22}
$$

which will be useful later. The MST associated with the choice $Q = Q_{ev}$ will be denoted by $\mathcal{S}^{(ev)}_{\mu\nu\sigma\rho}$.

It follows from (3.14) that

$$
R^2 = -\frac{1}{4}\mathcal{F}^2 \tag{4.23}
$$

$$
= \frac{\Lambda}{3}\Theta^{-2}\widetilde{X}^2 - i\Theta^{-1}\widetilde{H}^{\alpha\beta}\widetilde{F}^{\star}_{\alpha\beta} + \frac{\Lambda}{3}c(x^j)
$$

$$
+ \left(2D_{ij}Y^iY^j + i\sqrt{\frac{3}{\Lambda}} N^k \overline{\widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{X}_k} \right)\Theta + O(\Theta^2) \tag{4.24}
$$

$$= \frac{\Lambda}{3}\Theta^{-2}\,|Y|^2 - i\sqrt{\frac{\Lambda}{3}}\,Y_i N^i \Theta^{-1} - \frac{1}{9}f^2 + Y^i \overline{\widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{X}_i} + \frac{\Lambda}{3}c(x^j)$$

$$+ \left(\frac{4}{3}D_{kl}Y^k Y^l - i\sqrt{\frac{3}{\Lambda}}\,\widehat{C}_{kl}Y^k Y^l + \frac{i}{4}\sqrt{\frac{3}{\Lambda}}\,N^k (2\overline{\widetilde{\nabla}_t \widetilde{\nabla}_t \widetilde{X}_k} + Y_k \widehat{R})\right)\Theta$$

$$+ O(\Theta^2). \tag{4.25}$$

One remark is in order: In this section we do *not* assume that the MST vanishes, so there is no reason why the real function c, which has been defined on $\mathscr{I}$ in (3.13), should be constant. From (4.18) and (4.25) we observe that

$$\Xi := \sigma + \frac{3}{\Lambda}R^2 = c(x^j) + ip(x^j) - a + \frac{6}{\Lambda}\widetilde{\mathcal{D}}_{tktl}|_{\mathscr{I}}Y^k Y^l \Theta + O(\Theta^2), \tag{4.26}$$

whence

$$Q_{\mathrm{ev}} = \frac{3R^2 - \Lambda\sigma + 3R\sqrt{R^2 - \Lambda\sigma}}{\sigma R^2} \tag{4.27}$$

$$= 3\frac{2R^2 - \frac{\Lambda}{3}\Xi - 2R^2\sqrt{1 - \frac{\Lambda}{4}\frac{\Xi}{R^2}}}{R^2\left(\Xi - \frac{3}{\Lambda}R^2\right)} \tag{4.28}$$

$$= \frac{\Lambda^2}{12}\frac{\Xi}{R^4} + O\left(\frac{\Xi^2}{R^6}\right). \tag{4.29}$$

Remark 4.2. The expressions (4.27) and (4.28) for Q_{ev} rely on the choice of $R = -\sqrt{R^2}$. The final expression (4.29) is however independent of this choice, as it must be. This final expression ensures that, in an appropriate setting, Q_{ev} coincides with Q_0, as will be shown in theorem 4.16. It should also be emphasized that this expression does not admit a limit $\Lambda \searrow 0$. This is because, when $\Lambda = 0$, the function $\mathcal{F}^2$ approaches zero at infinity and the definition of square root needs to be worked out differently.

We have

$$Q_{\mathrm{ev}} = \begin{cases} \frac{3}{4}(c(x^j) + ip(x^j) - a)\,|Y|^{-4}\Theta^4 + O(\Theta^5) & \text{if } a \not\equiv c(x^j) + ip(x^j), \\ \frac{9}{2}\Lambda^{-1}\,|Y|^{-4}\widetilde{\mathcal{D}}_{tktl}Y^k Y^l \Theta^5 + O(\Theta^6) & \text{if } a \equiv c(x^j) + ip(x^j). \end{cases} \tag{4.30}$$

The rescaled MST with $Q = Q_{\mathrm{ev}}$ will be regular on $\mathscr{I}$ if and only if $Q_{\mathrm{ev}} = O(\Theta^5)$, i.e. if and only if both functions c and p are constant and the σ-constant a has been chosen such that

$$\mathrm{Re}(a) = c \quad \text{and} \quad \mathrm{Im}(a) = p. \tag{4.31}$$

We remark that with this choice of a the function Q_{ev} is completely determined.

Note that the potential p will be constant if and only if the covector field P_i vanishes. The constancy of c has been analyzed in lemma 3.5. Comparison with (2.67) then leads to the following result, a shortened version of which has been stated as theorem 1.2 in the Introduction:

Theorem 4.3. *Consider a* $\Lambda > 0$-*vacuum spacetime*[8] *which admits a smooth* $\mathscr{I}^-$ *and a KVF* X. *Denote by* Y *the CKVF induced, in the conformally rescaled spacetime, by* X *on* $\mathscr{I}^-$. *If and only if*

 i. $\hat{\eta}_{ij}{}^k \hat{C}_{kl} Y^j Y^l = 0$ *(so that the function* $c = \frac{3}{\Lambda}\hat{c}(Y)$ *is constant on* $\mathscr{I}^-$), *and*
 ii. $\hat{\eta}_{ij}{}^k D_{kl} Y^j Y^l = 0$ *(so that* $P_i = 0$ *whence its potential* p *is constant*),

 there exists a unique σ-*constant* a, *given by* $a = c + ip$, *which leads via* Q_{ev} *to a rescaled MST* $\widetilde{\mathcal{T}}{}^{(\text{ev})}_{\mu\nu\sigma\rho}$ *which is regular at* $\mathscr{I}^-$. *In that case the leading order terms of* $Q_{\text{ev}} = O(\Theta^5)$ *and* $Q_0 = O(\Theta^5)$ *coincide,*

$$\lim_{\Theta \to 0}(\Theta^{-5}Q_{\text{ev}}) = \lim_{\Theta \to 0}(\Theta^{-5}Q_0).$$

In particular,

$$\widetilde{\mathcal{T}}{}^{(\text{ev})}_{\mu\nu\sigma}{}^{\rho}|_{\mathscr{I}^-} = \widetilde{\mathcal{T}}{}^{(0)}_{\mu\nu\sigma}{}^{\rho}|_{\mathscr{I}^-}.$$

Remark 4.4. For initial data of the form (2.82)–(2.87), which are necessary for the MST to vanish for some choice of Q, the conditions (i) and (ii) are satisfied.

It is worth to emphasize the roles of $\hat{C}_{ij}$ and D_{ij} which enter theorem 2.8 as well as theorem 4.3 in a completely symmetric manner.

4.3. (Asymptotically) KdS-like spacetimes

In theorem 4.3 we have obtained necessary and sufficient conditions for the rescaled MST $\widetilde{\mathcal{T}}{}^{(\text{ev})}_{\mu\nu\sigma\rho}$ to be regular at $\mathscr{I}^-$. As we shall see in section 4.4.2, this tensor satisfies a homogeneous symmetric hyperbolic Fuchsian system with data prescribed at $\mathscr{I}^-$. The zero data is such that its propagation stays zero. The resulting spacetime has vanishing MST and hence either a Kerr–de Sitter metric or one of the related metrics classified in [21]. We call such spacetimes *KdS-like*:

Definition 4.5. Let $(\mathcal{M}, g)$ be a $\Lambda > 0$-vacuum spacetime admitting smooth conformal compactification and corresponding null infinity $\mathscr{I}$. $(\mathcal{M}, g)$ is called 'Kerr–de Sitter-like' at a connected component $\mathscr{I}^-$ of $\mathscr{I}$ if it admits a KVF X which induces a CKVF Y on $\mathscr{I}^-$, such that the rescaled MST $\widetilde{\mathcal{T}}{}^{(\text{ev})}_{\mu\nu\sigma\rho}$ vanishes at $\mathscr{I}^-$.

Note that also the Kerr-NUT-de Sitter spacetime belongs to the class of KdS-like spacetimes. In [20] we analyze in detail KdS-like space-times which admit a conformally flat $\mathscr{I}$.

The case where the tensor $\widetilde{\mathcal{T}}{}^{(\text{ev})}_{\mu\nu\sigma\rho}$ is merely assumed to be finite at $\mathscr{I}^-$ obviously includes the zero case (i.e. Kerr–de Sitter and related metrics) and at the same time excludes many other Λ-vacuum spacetimes with a smooth $\mathscr{I}$. It makes sense to call such spacetimes *asymptotically Kerr–de Sitter-like*. We put forward the following definition:

Definition 4.6. Let $(\mathcal{M}, g)$ be a $\Lambda > 0$-vacuum spacetime admitting smooth conformal compactification and corresponding null infinity $\mathscr{I}$. $(\mathcal{M}, g)$ is called 'asymptotically Kerr–de Sitter-like' at a connected component $\mathscr{I}^-$ of $\mathscr{I}$ if it admits a KVF X which induces a CKVF Y

---

[8] Since P_i needs to vanish in this setting, P_i is exact and we need not assume that $\mathscr{I}^-$ is simply connected in order to get a globally defined Ernst potential.

on $\mathscr{I}^-$, which satisfies $|Y|^2 > 0$, such that the conditions (i) and (ii) in theorem 4.3 are satisfied, or, equivalently, such that the rescaled MST $\widetilde{\mathcal{T}}^{(\mathrm{ev})}_{\mu\nu\sigma\rho}$ is regular at $\mathscr{I}^-$.

Remark 4.7. As will be shown later (cf corollary 4.17), KdS-like space-times have a vanishing MST, whence, as shown in section 2.5, the condition $|Y|^2 > 0$ follows automatically. In the asymptotically KdS-like case, though, the conditions (i) and (ii) in theorem 4.3 might be compatible with zeros of Y.

An interesting open problem is to classify 'asymptotically Kerr–de Sitter-like' spacetimes.

4.4. Derivation of evolution equations for the (rescaled) MST

Based on the corresponding derivation for $\Lambda = 0$ in [16], we will show that the MST

$$\mathcal{S}^{(\mathrm{ev})}_{\mu\nu\sigma\rho} = \mathcal{C}_{\mu\nu\sigma\rho} + Q_{\mathrm{ev}}\mathcal{U}_{\mu\nu\sigma\rho}, \tag{4.32}$$

with Q_{ev} as defined in (4.19)–(4.21), satisfies a symmetric hyperbolic system of evolution equations as well as a system of wave equations.

4.4.1. An analog to the Bianchi equation.

First, we derive an analog to the Bianchi equation $\nabla_\rho \mathcal{C}_{\mu\nu\sigma}{}^\rho = 0$ for the MST $\mathcal{S}^{(\mathrm{ev})}_{\mu\nu\sigma}{}^\rho$. For this we set

$$\mathcal{W}_{\alpha\beta} := R^{-1}\mathcal{F}_{\alpha\beta}, \tag{4.33}$$

so that

$$Q_{\mathrm{ev}}\mathcal{U}_{\alpha\beta\mu\nu} = \left(\frac{3J}{R} - \frac{\Lambda}{R^2}\right)\left(-\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu} + \frac{1}{3}\mathcal{F}^2\mathcal{I}_{\alpha\beta\mu\nu}\right) \tag{4.34}$$

$$= (\Lambda - 3JR)\left(\mathcal{W}_{\alpha\beta}\mathcal{W}_{\mu\nu} + \frac{4}{3}\mathcal{I}_{\alpha\beta\mu\nu}\right). \tag{4.35}$$

Differentiation yields

$$\begin{aligned}\nabla_\rho(Q_{\mathrm{ev}}\mathcal{U}_{\alpha\beta\mu\nu}) = &-3\nabla_\rho(JR)\left(\mathcal{W}_{\alpha\beta}\mathcal{W}_{\mu\nu} + \tfrac{4}{3}\mathcal{I}_{\alpha\beta\mu\nu}\right) \\ &+ (\Lambda - 3JR)(\mathcal{W}_{\mu\nu}\nabla_\rho\mathcal{W}_{\alpha\beta} + \mathcal{W}_{\alpha\beta}\nabla_\rho\mathcal{W}_{\mu\nu}).\end{aligned} \tag{4.36}$$

First of all let us calculate the covariant derivative of $\mathcal{W}_{\mu\nu}$. From

$$\nabla_\rho\mathcal{F}_{\mu\nu} = X^\sigma\left(\mathcal{C}_{\mu\nu\sigma\rho} + \frac{4}{3}\Lambda\mathcal{I}_{\mu\nu\sigma\rho}\right) \tag{4.37}$$

$$= X^\sigma\left(\mathcal{S}_{\mu\nu\sigma\rho} - Q_{\mathrm{ev}}\mathcal{U}_{\mu\nu\sigma\rho} + \frac{4}{3}\Lambda\mathcal{I}_{\mu\nu\sigma\rho}\right) \tag{4.38}$$

$$= \frac{3JR - \Lambda}{2R^2}\sigma_\rho\mathcal{F}_{\mu\nu} + 4JRX^\sigma\mathcal{I}_{\mu\nu\sigma\rho} + X^\sigma\mathcal{S}_{\mu\nu\sigma\rho}, \tag{4.39}$$

$$\nabla_\rho\mathcal{F}^2 = 2\mathcal{F}^{\mu\nu}\nabla_\rho\mathcal{F}_{\mu\nu} \tag{4.40}$$

$$= -4(2JR - \Lambda)\sigma_\rho + 2X^\sigma \mathcal{F}^{\mu\nu} \mathcal{S}_{\mu\nu\sigma\rho} \tag{4.41}$$

$$= \frac{1}{3}(2Q_{\mathrm{ev}}\mathcal{F}^2 + 4\Lambda)\sigma_\rho + 2X^\sigma \mathcal{F}^{\mu\nu} \mathcal{S}_{\mu\nu\sigma\rho}, \tag{4.42}$$

$$\nabla_\rho R = -\frac{1}{2}\nabla_\rho \sqrt{-\mathcal{F}^2} = -\frac{1}{8}R^{-1}\nabla_\rho \mathcal{F}^2 \tag{4.43}$$

$$= \frac{1}{6}\left(2Q_{\mathrm{ev}}R - \frac{\Lambda}{R}\right)\sigma_\rho - \frac{1}{4}X^\sigma \mathcal{W}^{\mu\nu}\mathcal{S}_{\mu\nu\sigma\rho} \tag{4.44}$$

$$= \left(J - \frac{\Lambda}{2R}\right)\sigma_\rho - \frac{1}{4}X^\sigma \mathcal{W}^{\mu\nu}\mathcal{S}_{\mu\nu\sigma\rho}, \tag{4.45}$$

we deduce that

$$\nabla_\rho \mathcal{W}_{\mu\nu} = R^{-1}\nabla_\rho \mathcal{F}_{\mu\nu} - R^{-1}\mathcal{W}_{\mu\nu}\nabla_\rho R \tag{4.46}$$

$$= \frac{J}{2R}\sigma_\rho \mathcal{W}_{\mu\nu} + 4JX^\sigma \mathcal{I}_{\mu\nu\sigma\rho} + R^{-1}X^\sigma \mathcal{S}_{\mu\nu\sigma\rho}$$
$$+ \frac{1}{4R}X^\sigma \mathcal{W}_{\mu\nu}\mathcal{W}^{\alpha\beta}\mathcal{S}_{\alpha\beta\sigma\rho}, \tag{4.47}$$

$$\nabla^\nu \mathcal{W}_{\mu\nu} = \frac{J}{2R}\sigma^\nu \mathcal{W}_{\mu\nu} + 3JX_\mu + \frac{1}{4R}X^\sigma \mathcal{W}_\mu{}^\rho \mathcal{W}^{\alpha\beta}\mathcal{S}_{\alpha\beta\sigma\rho} \tag{4.48}$$

$$= 2JX_\mu + \frac{1}{4R}X^\sigma \mathcal{W}_\mu{}^\rho \mathcal{W}^{\alpha\beta}\mathcal{S}_{\alpha\beta\sigma\rho}, \tag{4.49}$$

where we used that $\sigma^\rho \mathcal{F}_{\mu\rho} = \frac{1}{2}\mathcal{F}^2 X_\mu$ [21].

Taking the derivative of (4.22), we obtain

$$\nabla_\rho(JR) = \frac{J^2\sigma}{R - J\sigma}\left(\frac{R}{2\sigma}\sigma_\rho - \nabla_\rho R\right) \tag{4.50}$$

$$= \frac{J}{R - J\sigma}\left(\frac{-3JR^2 + 2R\Lambda + \Lambda\sigma J}{2R}\sigma_\rho + \frac{1}{4}J\sigma X^\sigma \mathcal{W}^{\mu\nu}\mathcal{S}_{\mu\nu\sigma\rho}\right) \tag{4.51}$$

$$= \frac{J}{2R}(3JR - \Lambda)\sigma_\rho + \frac{J^2\sigma}{4(R - J\sigma)}X^\sigma \mathcal{W}^{\mu\nu}\mathcal{S}_{\mu\nu\sigma\rho}. \tag{4.52}$$

Altogether we find

$$\nabla_\rho(Q_{\mathrm{ev}}\mathcal{U}_{\alpha\beta\mu}{}^\rho) = J(3JR - \Lambda)\left(2X_\mu \mathcal{W}_{\alpha\beta} - \frac{2}{R}\sigma^\rho \mathcal{I}_{\alpha\beta\mu\rho} - 4X^\sigma \mathcal{W}_\mu{}^\rho \mathcal{I}_{\alpha\beta\sigma\rho}\right)$$
$$+ (\Lambda - 3JR)R^{-1}X^\sigma \mathcal{W}_\mu{}^\rho\left(\mathcal{S}_{\alpha\beta\sigma\rho} + \frac{1}{2}\mathcal{W}_{\alpha\beta}\mathcal{W}^{\gamma\delta}\mathcal{S}_{\gamma\delta\sigma\rho}\right)$$
$$- \frac{3J^2\sigma}{4(R - J\sigma)}X^\sigma \mathcal{W}^{\gamma\delta}\mathcal{S}_{\gamma\delta\sigma\rho}\left(\mathcal{W}_{\alpha\beta}\mathcal{W}_\mu{}^\rho + \frac{4}{3}\mathcal{I}_{\alpha\beta\mu}{}^\rho\right). \tag{4.53}$$

Using

$$\mathcal{F}_{(\mu}{}^{\sigma}\mathcal{I}_{\nu)\sigma\alpha\beta} = \frac{1}{4}g_{\mu\nu}\mathcal{F}_{\alpha\beta}, \tag{4.54}$$

cf [16, equation (4.37)], we obtain

$$\frac{2}{R}\sigma^{\rho}\mathcal{I}_{\alpha\beta\mu\rho} + 4X^{\sigma}\mathcal{W}_{\mu}{}^{\rho}\mathcal{I}_{\alpha\beta\sigma\rho} = 4X^{\sigma}(\mathcal{W}_{\sigma}{}^{\rho}\mathcal{I}_{\alpha\beta\mu\rho} + \mathcal{W}_{\mu}{}^{\rho}\mathcal{I}_{\alpha\beta\sigma\rho}) \tag{4.55}$$

$$= 2X_{\mu}\mathcal{W}_{\alpha\beta}, \tag{4.56}$$

and thus

$$\nabla_{\rho}\mathcal{S}_{\alpha\beta\mu}^{(\text{ev})\,\rho} = (\Lambda - 3JR)\mathcal{W}_{\mu}{}^{\rho}\left(R^{-1}X^{\sigma}\mathcal{S}_{\alpha\beta\sigma\rho}^{(\text{ev})} + \frac{1}{2}R^{-1}X^{\sigma}\mathcal{W}_{\alpha\beta}\mathcal{W}^{\gamma\delta}\mathcal{S}_{\gamma\delta\sigma\rho}^{(\text{ev})}\right)$$
$$- \frac{3J^2\sigma}{4(R - J\sigma)}X^{\sigma}\mathcal{W}^{\gamma\delta}\mathcal{S}_{\gamma\delta\sigma\rho}^{(\text{ev})}\left(\mathcal{W}_{\alpha\beta}\mathcal{W}_{\mu}{}^{\rho} + \frac{4}{3}\mathcal{I}_{\alpha\beta\mu}{}^{\rho}\right) \tag{4.57}$$

$$= \frac{\Lambda - 3JR}{R}\mathcal{W}_{\mu}{}^{\rho}X^{\sigma}\mathcal{S}_{\alpha\beta\sigma\rho}^{(\text{ev})} - \frac{J^2\sigma}{R - J\sigma}X^{\sigma}\mathcal{I}_{\alpha\beta\mu}{}^{\rho}\mathcal{W}^{\gamma\delta}\mathcal{S}_{\gamma\delta\sigma\rho}^{(\text{ev})}$$
$$- \frac{\Lambda}{4R}\frac{R + 2J\sigma}{R - J\sigma}X^{\sigma}\mathcal{W}_{\alpha\beta}\mathcal{W}^{\gamma\delta}\mathcal{W}_{\mu}{}^{\rho}\mathcal{S}_{\gamma\delta\sigma\rho}^{(\text{ev})}. \tag{4.58}$$

Expressed in terms of $\mathcal{F}_{\mu\nu}$ and Q_{ev} we finally end up with the desired equation for the MST, which may be regarded as an analog to the Bianchi equation,

$$\nabla_{\rho}\mathcal{S}_{\alpha\beta\mu}^{(\text{ev})\,\rho} = \left(4\Lambda\frac{5Q_{\text{ev}}\mathcal{F}^2 + 4\Lambda}{Q_{\text{ev}}\mathcal{F}^2 + 8\Lambda}\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\rho} + \frac{2}{3}\mathcal{F}^2\frac{Q_{\text{ev}}^2\mathcal{F}^4 - 2\Lambda Q_{\text{ev}}\mathcal{F}^2 - 8\Lambda^2}{Q_{\text{ev}}\mathcal{F}^2 + 8\Lambda}\mathcal{I}_{\alpha\beta\mu\rho}\right)$$
$$\times \mathcal{F}^{-4}X^{\sigma}\mathcal{F}^{\gamma\delta}\mathcal{S}_{\gamma\delta\sigma}^{(\text{ev})\,\rho} - Q_{\text{ev}}\mathcal{F}_{\mu\rho}X^{\sigma}\mathcal{S}_{\alpha\beta\sigma}^{(\text{ev})\,\rho} \tag{4.59}$$

$$= -4\Lambda\frac{5Q_{\text{ev}}\mathcal{F}^2 + 4\Lambda}{Q_{\text{ev}}\mathcal{F}^2 + 8\Lambda}\mathcal{U}_{\alpha\beta\mu\rho}\mathcal{F}^{-4}X^{\sigma}\mathcal{F}^{\gamma\delta}\mathcal{S}_{\gamma\delta\sigma}^{(\text{ev})\,\rho}$$
$$+ Q_{\text{ev}}X^{\sigma}\left(\frac{2}{3}\mathcal{I}_{\alpha\beta\mu\rho}\mathcal{F}^{\gamma\delta}\mathcal{S}_{\gamma\delta\sigma}^{(\text{ev})\,\rho} - \mathcal{F}_{\mu\rho}\mathcal{S}_{\alpha\beta\sigma}^{(\text{ev})\,\rho}\right) \tag{4.60}$$

$$=: \mathcal{J}(\mathcal{S}^{(\text{ev})})_{\alpha\beta\mu}. \tag{4.61}$$

Here we have introduced the shorthand $\mathcal{J}(\mathcal{S}^{(\text{ev})})_{\alpha\beta\mu}$ for the righthand side, which is a double (2, 1)-form, linear and homogeneous in $\mathcal{S}_{\alpha\beta\sigma}^{(\text{ev})\,\rho}$, with the following properties

$$\mathcal{J}(\mathcal{S}^{(\text{ev})})_{\alpha\beta\mu} = \mathcal{J}(\mathcal{S}^{(\text{ev})})_{[\alpha\beta]\mu}, \quad \mathcal{J}(\mathcal{S}^{(\text{ev})})_{[\alpha\beta\mu]} = 0, \quad \mathcal{J}(\mathcal{S}^{(\text{ev})})^{\rho}{}_{\beta\rho} = 0. \tag{4.62}$$

It is also self-dual in the first pair of anti-symmetric indices: $\mathcal{J}^{\star}(\mathcal{S}^{(\text{ev})})_{\alpha\beta\mu} = -i\mathcal{J}(\mathcal{S}^{(\text{ev})})_{\alpha\beta\mu}$.

Using the fact that the MST $\mathcal{S}_{\mu\nu\sigma}^{(\text{ev})\,\rho}$ has all the algebraic symmetries of the Weyl tensor, we immediately obtain a Bianchi-like equation from (4.61) for the rescaled MST $\widetilde{\mathcal{T}}_{\mu\nu\sigma}^{(\text{ev})\,\rho}$ in the conformally rescaled 'unphysical' spacetime,

$$\widetilde{\nabla}_{\rho}\widetilde{\mathcal{T}}_{\alpha\beta\mu}^{(\text{ev})\,\rho} = \Theta^{-1}\nabla_{\rho}\mathcal{S}_{\alpha\beta\mu}^{(\text{ev})\,\rho} = \Theta^{-1}\mathcal{J}(\mathcal{S}^{(\text{ev})})_{\alpha\beta\mu} = \mathcal{J}(\widetilde{\mathcal{T}}^{(\text{ev})})_{\alpha\beta\mu}. \tag{4.63}$$

4.4.2. A symmetric hyperbolic system satisfied by the (rescaled) MST. We now want to show that the equations (4.61) and (4.63) contain a system of linear first-order symmetric

hyperbolic equations in their respective spacetimes. Given that (4.61) and (4.63) have exactly the same structure, it is enough to perform the analysis for any one of the two systems, or to a model equivalent system in a given spacetime. Then, a further analysis of the regularity of the system (4.63) near $\mathscr{I}$ is also needed, and this will be done later in this section.

Let $\mathcal{S}_{\alpha\beta\lambda}{}^{\mu}$ represent either $\mathcal{S}^{(\mathrm{ev})\,\mu}_{\alpha\beta\lambda}$ or $\widetilde{\mathcal{T}}^{(\mathrm{ev})\,\mu}_{\alpha\beta\lambda}$, or for that matter, any other self-dual symmetric and traceless double (2, 2)-form satisfying a system of equations such as (4.61) or (4.63):

$$\nabla_{\rho}\mathcal{S}_{\gamma\mu\nu}{}^{\rho} = \mathcal{J}_{\gamma\mu\nu}(\mathcal{S}), \tag{4.64}$$

where $\mathcal{J}_{\gamma\mu\nu}(\mathcal{S})$ is a self-dual double (2, 1)-form, linear and homogeneous in $\mathcal{S}_{\alpha\beta\sigma}{}^{\rho}$, with the properties given in (4.62). Employing the fact that the rescaled MST satisfies all the algebraic symmetries of the rescaled Weyl tensor we find that this system is equivalent to (cf [16, 26])

$$3\nabla_{[\sigma}\mathcal{S}_{\mu\nu]\alpha\beta} = -\frac{1}{2}\eta_{\sigma\mu\nu\kappa}\eta^{\delta\rho\kappa}\nabla_{\gamma}\mathcal{S}_{\delta\rho\alpha\beta} = \eta_{\sigma\mu\nu}{}^{\kappa}\nabla_{\gamma}(\mathcal{S}_{\alpha\beta\kappa}{}^{\gamma})^{\star} = -i\eta_{\sigma\mu\nu}{}^{\kappa}\nabla_{\gamma}\mathcal{S}_{\alpha\beta\kappa}{}^{\gamma} \tag{4.65}$$

that is to say

$$3\nabla_{[\sigma}\mathcal{S}_{\mu\nu]\alpha\beta} = -i\eta_{\mu\nu\sigma}{}^{\rho}\mathcal{J}(\mathcal{S})_{\alpha\beta\rho}. \tag{4.66}$$

Observe that each of (4.64) and (4.66) contains 8 complex (16 real) independent equations for only 5 complex (10 real) unknowns, hence they are overdetermined.

Systems of this type have been analyzed many times in the literature (in order to see if they comprise a symmetric hyperbolic system), especially in connection with the Bianchi identities [5, 6]. Here, to check that the systems (4.64), or (4.66), and therefore (4.61) and (4.63), contain symmetric hyperbolic evolution equations we use the general ideas exposed in [15], which were applied to systems more general than—and including—those of type (4.64) or (4.66) and discussed at length in [30]. The goal is to find a 'hyperbolization', in the sense of [15]. To that end, simplifying the calculations in [30], pick *any* timelike vector v^{μ} and contract (4.64) with

$$-v_{[\alpha}\delta^{\nu}_{\beta]}v^{[\gamma}v_{[\delta}\delta^{\mu}_{\epsilon]} \tag{4.67}$$

and add the result to the contraction of (4.66) with

$$-\frac{1}{2}v^{[\lambda}\delta^{\tau}_{\alpha}\delta^{\nu}_{\beta]}v^{[\gamma}v_{[\delta}\delta^{\mu}_{\epsilon]} \tag{4.68}$$

to arrive at the following system

$$Q^{\tau\,\lambda\nu\gamma\mu}_{\alpha\beta\delta\epsilon}\nabla_{\tau}\mathcal{S}_{\lambda\nu\gamma\mu} = \mathcal{J}_{\alpha\beta\delta\epsilon}(\mathcal{S}) \tag{4.69}$$

with

$$Q^{\tau\,\lambda\nu\gamma\mu}_{\alpha\beta\delta\epsilon} = v^{[\gamma}v_{[\delta}\delta^{\mu}_{\epsilon]}\left(g^{\tau[\lambda}v_{[\alpha}\delta^{\nu}_{\beta]} + \delta^{\tau}_{[\alpha}v^{[\lambda}\delta^{\nu}_{\beta]} - \frac{1}{2}v^{\tau}\delta^{\lambda}_{[\alpha}\delta^{\nu}_{\beta]}\right).$$

By construction, the righthand side of (4.69) is linear in $\mathcal{J}_{\gamma\mu\nu}(\mathcal{S})$ and *a fortiori* linear in $\mathcal{S}_{\gamma\mu\nu\rho}$, so that its explicit expression is unimportant. The system (4.69) is symmetric hyperbolic. To prove it, we have to check two properties of $Q^{\tau\,\lambda\nu\gamma\mu}_{\alpha\beta\delta\epsilon}$: it must be Hermitian in $\lambda\nu\gamma\mu \leftrightarrow \alpha\beta\delta\epsilon$, and there must exist a one-form u_{τ} such that its contraction with $Q^{\tau\,\lambda\nu\gamma\mu}_{\alpha\beta\delta\epsilon}$ is positive definite.

The first condition can be easily checked by first noticing that $Q^{\tau\,\lambda\nu\gamma\mu}_{\alpha\beta\delta\epsilon}$ happens to be real and then contracting it with two arbitrary self-dual trace-free double (2, 2) forms, say $\mathcal{A}_{\lambda\nu\gamma\mu}$ and $\mathcal{B}^{\alpha\beta\delta\epsilon}$. The result is

$$v^\beta v^\lambda v^\mu \left[\mathcal{A}^\tau{}_{\rho\lambda\sigma} \mathcal{B}_\beta{}^\rho{}_\mu{}^\sigma + \mathcal{B}^\tau{}_{\rho\lambda\sigma} \mathcal{A}_\beta{}^\rho{}_\mu{}^\sigma - \frac{1}{2} \delta^\tau_\beta \mathcal{A}_{\alpha\rho\lambda\sigma} \mathcal{B}^{\alpha\rho}{}_\mu{}^\sigma \right],$$

which is manifestly symmetric under the interchange of $\mathcal{A}$ and $\mathcal{B}$. Thus, the matrix of the system (4.69) is Hermitian.

With regard to the second condition, we contract $Q^\tau{}_{\alpha\beta\delta\epsilon}{}^{\lambda\nu\gamma\mu}$ with $\mathcal{A}_{\lambda\nu\gamma\mu}$, $\overline{\mathcal{A}}^{\alpha\beta\delta\epsilon}$, and with u_τ. We stress that in this section an overbar means 'complex conjugation' rather than 'restriction to $\mathscr{I}$'. We get

$$u_\tau v^\beta v^\lambda v^\mu (\mathcal{A}^\tau{}_{\rho\lambda\sigma} \overline{\mathcal{A}}_\beta{}^\rho{}_\mu{}^\sigma + \overline{\mathcal{A}}^\tau{}_{\rho\lambda\sigma} \mathcal{A}_\beta{}^\rho{}_\mu{}^\sigma) = 2 \, u_\tau v^\beta v^\lambda v^\mu (\mathcal{A}^\tau{}_{\rho\lambda\sigma} \overline{\mathcal{A}}_\beta{}^\rho{}_\mu{}^\sigma)$$

and note that the expression in brackets is precisely the Bel–Robinson superenergy tensor $t_{\tau\beta\lambda\mu}$ of the self-dual Weyl-type tensor $\mathcal{A}_{\alpha\beta\lambda\mu}$ [26, 29]. It is known that this tensor satisfies the dominant property [26, 29], that is, $t_{\tau\beta\lambda\mu} v_1^\tau v_2^\beta v_3^\lambda v_4^\mu > 0$ for arbitrary future-pointing timelike vectors v_1^τ, v_2^β, v_3^λ, v_4^μ. Thus, the previous expression is positive for *any* timelike u_τ with the same time orientation as v^τ, as required. Actually, by choosing $u^\tau = v^\tau$ it becomes (twice) the so-called super-energy density of $\mathcal{A}_{\alpha\beta\lambda\mu}$ relative to v^τ:

$$W_v(\mathcal{A}) := v^\tau v^\beta v^\lambda v^\mu \mathcal{A}_{\tau\rho\lambda\sigma} \overline{\mathcal{A}}_\beta{}^\rho{}_\mu{}^\sigma \tag{4.70}$$

which is non-negative, vanishing if and only if so does the full $\mathcal{A}_{\alpha\beta\lambda\mu}$ [29].

In order to see the relation between the found symmetric hyperbolic system (4.69) and the original equations (4.64) or (4.66), we take into account that $Q^\tau{}_{\alpha\beta\delta\epsilon}{}^{\lambda\nu\gamma\mu}$ is non-degenerate as an operator acting on 2-forms in its '$\gamma\mu$, $\delta\epsilon$' part so that (4.69) is actually fully equivalent to

$$3v^\lambda \nabla_{[\lambda} \mathcal{S}_{\tau\nu]\gamma\mu} + 2\nabla_\rho \mathcal{S}_{\gamma\mu[\nu}{}^\rho v_{\tau]} = iv^\lambda \mathcal{J}_{\gamma\mu\sigma}(\mathcal{S})\eta^\sigma{}_{\lambda\tau\nu} + 2\mathcal{J}_{\gamma\mu[\nu}(\mathcal{S})v_{\tau]}.$$

There is some redundancy here due to the equivalence of (4.64) and (4.66). To optimize the expression of this symmetric hyperbolic system we note that, via the identity (4.65), it can be rewritten as

$$\overline{\mathcal{I}}^{\tau\nu}{}_{\lambda\sigma}(\nabla_\rho \mathcal{S}_{\gamma\mu[\nu}{}^\rho - \mathcal{J}_{\gamma\mu[\nu}(\mathcal{S}))v_{\tau]} = 0$$

which is easily seen to be equivalent to

$$(\nabla_\rho \mathcal{S}_{\gamma\mu[\nu}{}^\rho - \mathcal{J}_{\gamma\mu[\nu}(\mathcal{S}))v_{\tau]} = 0 \tag{4.71}$$

with v_β any timelike vector. The linear symmetric hyperbolic set (4.71) constitutes the *evolution* equations of our system. Note that, taking into account trace and symmetry properties, there are precisely 5 complex (10 real) independent equations in (4.71), which is the number of independent unknowns.

The complete system (4.64) is re-obtained by adding the constraints, which can be written for any given spacelike hypersurface Σ with timelike normal n^μ as (cf [30], section 4)

$$n^\nu (\nabla_\rho \mathcal{S}_{\gamma\mu\nu}{}^\rho - \mathcal{J}_{\gamma\mu\nu}(\mathcal{S})) = 0. \tag{4.72}$$

Notice, first of all, that only derivatives *tangent* to Σ appear in (4.72). Observe furthermore that (4.72) contains 3 complex (6 real) independent equations which adds up with (4.71) to the number of equations of the original system (4.64), rendering the former two equations fully equivalent with the latter one. To check this directly, contract (4.71) with v^τ to get

$$(-v^2 \delta^\sigma_\nu + v^\sigma v_\nu)(\nabla_\rho \mathcal{S}_{\gamma\mu\sigma}{}^\rho - \mathcal{J}_{\gamma\mu\sigma}(\mathcal{S})) = 0,$$

where

$$h^\sigma{}_\gamma := \delta^\sigma_\gamma - v^{-2} v_\gamma v^\sigma \tag{4.73}$$

is the projector orthogonal to v^μ, which immediately leads to (4.64) by taking into account (4.72)— e.g., by simply choosing v^μ pointing along n^μ.

As mentioned before, (4.71) contains 5 equations for the 5 complex independent unknowns in $\mathcal{S}_{\gamma\mu\sigma}{}^\rho$. A convenient way of explicitly expressing this fact is by recalling the following identity

$$\mathcal{S}_{\alpha\beta\lambda\mu} = 2[(h_{\alpha[\lambda} - v^{-2}v_\alpha v_{[\lambda})\mathcal{E}_{\mu]\beta} + (h_{\beta[\mu} - v^{-2}v_\beta v_{[\mu})\mathcal{E}_{\lambda]\alpha}$$
$$- iv^{-2}v^\rho\eta_{\rho\lambda\mu\sigma}v_{[\alpha}\mathcal{E}_{\beta]}{}^\sigma - iv^{-2}v^\rho\eta_{\rho\alpha\beta\sigma}v_{[\lambda}\mathcal{E}_{\mu]}{}^\sigma]$$

in terms of the spatial 'electric-magnetic' tensor defined for any timelike v^τ by

$$\mathcal{E}_{\beta\mu} := -v^{-2}v^\alpha v^\lambda \mathcal{S}_{\alpha\beta\lambda\mu}. \tag{4.74}$$

Observe the following properties

$$\mathcal{E}_{\beta\mu} = \mathcal{E}_{\mu\beta}, \quad \mathcal{E}_{\beta\mu}v^\mu = 0, \quad \mathcal{E}^\mu{}_\mu = 0.$$

Thus, $\mathcal{E}_{\beta\mu}$ contains 5 complex independent components and exactly the same information as the full $\mathcal{S}_{\gamma\mu\sigma}{}^\rho$. Note that the density (4.70) is then expressed simply as

$$W_v(\mathcal{S}) = \mathcal{E}_{\mu\nu}\bar{\mathcal{E}}^{\mu\nu}. \tag{4.75}$$

In any orthonormal basis with its timelike 't'-part aligned with v^μ, the five independent components of $\mathcal{E}_{\mu\nu}$ are given simply by

$$\mathcal{E}_{ij} = \mathcal{S}_{titj}, \qquad \mathcal{S}_{tijl} = i\eta^t{}_{kjl}\mathcal{E}^k_i$$

where the second equation follows from the self-duality of $\mathcal{S}$. Using this, the evolution equations (4.71) become simply

$$\nabla_\rho \mathcal{S}_{t(ij)}{}^\rho = \nabla_t\mathcal{E}_{ij} + i\eta^t{}_{lk(j}\nabla^l\mathcal{E}^k_{i)} = \mathcal{J}(\mathcal{S})_{t(ij)} \tag{4.76}$$

while the constraint equations (4.72) (with n^τ pointing along v^τ) read

$$\nabla_\rho\mathcal{S}_{tit}{}^\rho = \nabla_j\mathcal{E}_i{}^j = \mathcal{J}(\mathcal{S})_{tit}. \tag{4.77}$$

We will use these expressions later for the case $\mathcal{S}_{\alpha\beta\gamma\delta} = \widetilde{\mathcal{T}}^{(ev)}_{\alpha\beta\gamma\delta}$, to prove uniqueness of the solutions to (4.63).

All in all, as a generalization of [16, theorem 4.5 and 4.7] we have obtained

Lemma 4.8

i. The MST $\mathcal{S}^{(ev)}_{\mu\nu\sigma\rho}$ satisfies, for any sign of the cosmological constant Λ, a linear, homogeneous symmetric hyperbolic system of evolution equations in $(\mathcal{M}, g)$.

ii. The rescaled MST $\widetilde{\mathcal{T}}^{(ev)}_{\mu\nu\sigma\rho}$ satisfies, for any sign of the cosmological constant Λ, a linear, homogeneous symmetric hyperbolic system of evolution equations in $(\widetilde{\mathcal{M}}, \tilde{g})$.

Remark 4.9. An alternative route to arrive at the same result is by using spinors, see [8, 12]. In this formalism [26] the (rescaled) MST is represented by a fully symmetric spinor Υ_{ABCD}, and equations (4.64) are written in the following form

$$\nabla^A{}_{A'}\Upsilon_{ABCD} = L_{A'BCD}$$

where $L_{A'BCD} = L_{A'(BCD)}$ is the spinor associated to $\mathcal{J}_{\gamma\mu\nu}(\mathcal{S})$. Then this is easily put in symmetric hyperbolic form, writing it as in [12], section 4, for the Bianchi equations.

Remark 4.10. Note that the denominator $Q_{ev}\mathcal{F}^2 + 8\Lambda$ in the equation (4.59) for $\mathcal{S}^{(ev)\,\rho}_{\mu\nu\sigma}$ might have zeros. Furthermore, the Ernst potential may have zeros so that Q_{ev} blows up. An analogous problem arises for $\widetilde{\mathcal{T}}^{(ev)}_{\mu\nu\sigma}{}^\rho$, cf (4.84) below. In fact, it follows from (4.18), (4.78), (4.80) and (4.92) below, that this cannot happen sufficiently close to $\mathscr{I}$ (for $\Lambda > 0$). It is not clear, though, whether the evolution equations remain regular off some neighborhood of $\mathscr{I}$. Moreover, it will be shown in the subsequent section that $\mathcal{J}(\widetilde{\mathcal{T}}^{(ev)})_{\alpha\beta\mu}$ is singular at $\mathscr{I}$ due to the vanishing of the conformal factor Θ there –see (4.86) below–, whence one actually has to deal with a Fuchsian system.

4.4.3. Behavior of the Bianchi-like system for $\widetilde{\mathcal{T}}^{(ev)}_{\mu\nu\sigma}{}^\rho$ near $\mathscr{I}$. Let us analyze the behavior of the system (4.63) near $\mathscr{I}$. Note that we are *not* assuming *a priori* that $\widetilde{\mathcal{T}}^{(ev)}_{\mu\nu\sigma}{}^\rho$ is regular at $\mathscr{I}$. First of all we employ the following expansions which have been derived in section 2.1, and which do not rely on any gauge choice,

$$Q_{ev} = O(\Theta^4), \tag{4.78}$$

$$\mathcal{F}_{\mu\nu} = \widetilde{\mathcal{H}}_{\mu\nu}\Theta^{-3} + O(\Theta^{-2}), \tag{4.79}$$

$$\mathcal{F}^2 = \Theta^{-2}\widetilde{\mathcal{H}}^2 + O(\Theta^{-1}), \tag{4.80}$$

$$g^{\mu\nu} = \Theta^2\widetilde{g}^{\mu\nu}, \tag{4.81}$$

$$\mathcal{I}_{\alpha\beta\mu\nu} = \Theta^{-4}\widetilde{\mathcal{I}}_{\alpha\beta\mu\nu}, \tag{4.82}$$

$$\mathcal{U}_{\alpha\beta\mu}{}^\nu = -\left(\widetilde{\mathcal{H}}_{\alpha\beta}\widetilde{\mathcal{H}}_\mu{}^\nu - \frac{1}{3}\widetilde{\mathcal{H}}^2\widetilde{\mathcal{I}}_{\alpha\beta\mu}{}^\nu\right)\Theta^{-4} + O(\Theta^{-3}). \tag{4.83}$$

This yields

$$\mathcal{J}(\widetilde{\mathcal{T}}^{(ev)})_{\alpha\beta\mu} = -4\Lambda\frac{5Q_{ev}\mathcal{F}^2 + 4\Lambda}{Q_{ev}\mathcal{F}^2 + 8\Lambda}\mathcal{U}_{\alpha\beta\mu}{}^\nu\mathcal{F}^{-4}X^\sigma g_{\rho\nu}g^{\gamma\kappa}g^{\delta\varkappa}\mathcal{F}_{\kappa\varkappa}\widetilde{\mathcal{T}}^{(ev)}_{\gamma\delta\sigma}{}^\rho$$

$$+ Q_{ev}X^\sigma\left(\frac{2}{3}\mathcal{I}_{\alpha\beta\mu\rho}g^{\gamma\kappa}g^{\delta\varkappa}\mathcal{F}_{\kappa\varkappa}\widetilde{\mathcal{T}}^{(ev)}_{\gamma\delta\sigma}{}^\rho - \mathcal{F}_{\mu\rho}\widetilde{\mathcal{T}}^{(ev)}_{\alpha\beta\sigma}{}^\rho\right) \tag{4.84}$$

$$= -2\Lambda\mathcal{U}_{\alpha\beta\mu}{}^\nu\mathcal{F}^{-4}X^\sigma g_{\rho\nu}g^{\gamma\kappa}g^{\delta\varkappa}\mathcal{F}_{\kappa\varkappa}\widetilde{\mathcal{T}}^{(ev)}_{\gamma\delta\sigma}{}^\rho + (O(\Theta)\widetilde{\mathcal{T}}^{(ev)})_{\alpha\beta\mu} \tag{4.85}$$

$$= 2\Lambda\Theta^{-1}\left(\widetilde{\mathcal{H}}_{\alpha\beta}\widetilde{\mathcal{H}}_{\mu\rho} - \frac{1}{3}\widetilde{\mathcal{H}}^2\widetilde{\mathcal{I}}_{\alpha\beta\mu\rho}\right)\widetilde{\mathcal{H}}^{-4}\widetilde{\mathcal{H}}^{\gamma\delta}\widetilde{X}^\sigma\widetilde{\mathcal{T}}^{(ev)}_{\gamma\delta\sigma}{}^\rho + (O(1)\widetilde{\mathcal{T}}^{(ev)})_{\alpha\beta\mu}. \tag{4.86}$$

In adapted coordinates (t, x^i) where $\mathscr{I}^- = \{t = 0\}$, we have

$$X^i = \widetilde{X}^i = Y^i + O(\Theta), \quad X^t = \widetilde{X}^t = O(\Theta). \tag{4.87}$$

Let us further assume a gauge where

$$\widetilde{g}_{tt}|_\mathscr{I} = -1, \quad \widetilde{g}_{ti}|_\mathscr{I} = 0. \tag{4.88}$$

In particular this implies by (2.5) that the conformal factor Θ satisfies

$$\Theta = \sqrt{\frac{\Lambda}{3}} t + O(1). \tag{4.89}$$

Moreover, as for the wave map gauge (2.25), which is compatible with (4.88), we find

$$\widetilde{\mathcal{H}}_{ti} = -\sqrt{\frac{\Lambda}{3}} Y_i + O(\Theta), \tag{4.90}$$

$$\widetilde{\mathcal{H}}_{ij} = i\sqrt{\frac{\Lambda}{3}} \hat{\eta}_{ijk} Y^k + O(\Theta), \tag{4.91}$$

$$\widetilde{\mathcal{H}}^2 = -4\frac{\Lambda}{3} |Y|^2 + O(\Theta). \tag{4.92}$$

Using further that

$$\widetilde{\mathcal{H}}^{\gamma\delta} \widetilde{\mathcal{T}}^{(\mathrm{ev})\,\rho}_{\gamma\delta\sigma} = 2\widetilde{H}^{\gamma\delta} \widetilde{\mathcal{T}}^{(\mathrm{ev})\,\rho}_{\gamma\delta\sigma} = 4\widetilde{X}^\gamma \widetilde{\nabla}^\delta \Theta \widetilde{\mathcal{T}}^{(\mathrm{ev})\,\rho}_{\gamma\delta\sigma} = 4\sqrt{\frac{\Lambda}{3}} Y^i \widetilde{\mathcal{T}}^{(\mathrm{ev})\,\rho}_{ti\sigma} + (O(\Theta)\widetilde{\mathcal{T}}^{(\mathrm{ev})})_\sigma{}^\rho, \tag{4.93}$$

we find that the system (4.63) has the following structure near $\mathscr{I}$,

$$\widetilde{\nabla}_\rho \widetilde{\mathcal{T}}^{(\mathrm{ev})\,\rho}_{\alpha\beta\mu} = \frac{9}{2}\sqrt{\frac{\Lambda}{3}} \Lambda^{-1}\Theta^{-1} |Y|^{-4} (\widetilde{\mathcal{H}}_{\alpha\beta}\widetilde{\mathcal{H}}_{\mu\rho} - \frac{1}{3}\widetilde{\mathcal{H}}^2 \widetilde{\mathcal{I}}_{\alpha\beta\mu\rho}) Y^i Y^k \widetilde{\mathcal{T}}^{(\mathrm{ev})\,\rho}_{tik}$$
$$+ (O(1)\widetilde{\mathcal{T}}^{(\mathrm{ev})})_{\alpha\beta\mu}, \tag{4.94}$$

in adapted coordinates and whenever (4.88) holds.

As explained in the previous section the system (4.63) splits into a symmetric hyperbolic system of evolution equations and a system of constraint equations for $\widetilde{\mathcal{T}}^{(\mathrm{ev})}_{\alpha\beta\mu\nu}$. However, this requires an appropriate gauge choice. A convenient way to realize such a gauge is to impose the condition (4.88) also off the initial surface,

$$\tilde{g}_{tt} = -1, \quad \tilde{g}_{ti} = 0. \tag{4.95}$$

It is well known that these *Gaussian normal coordinates* [33] are obtained by shooting geodesics normally to $\mathscr{I}$; the coordinate t is then chosen to be an affine parameter along these geodesics, while the coordinates $\{x^i\}$ are transported from $\mathscr{I}$ by requiring them to be constant along these geodesics.

Setting $\mathcal{S}_{\alpha\beta\gamma\delta} = \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{\alpha\beta\gamma\delta}$ in (4.76) and (4.77) and using (2.70), which follows from (4.90)–(4.92), we find in the unphysical spacetime (we have $\mathcal{E}_{ij} \equiv \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{titj}$)

$$\widetilde{\nabla}_\rho \widetilde{\mathcal{T}}^{(\mathrm{ev})\,\rho}_{tit} \equiv \widetilde{\nabla}_j \mathcal{E}_i{}^j \tag{4.96}$$

$$= \frac{9}{2}\sqrt{\frac{\Lambda}{3}} \Lambda^{-1}\Theta^{-1} |Y|^{-4} (\widetilde{\mathcal{H}}_{ti}\widetilde{\mathcal{H}}_{tj} - \frac{1}{3}\widetilde{\mathcal{H}}^2 \widetilde{\mathcal{I}}_{titj}) Y^l Y^k \widetilde{\mathcal{T}}^{(\mathrm{ev})\,j}_{tlk}$$
$$+ (O(1)\widetilde{\mathcal{T}}^{(\mathrm{ev})})_{tit} \tag{4.97}$$

$$= \frac{1}{2}\sqrt{\frac{\Lambda}{3}} \Theta^{-1} |Y|^{-2} Y^l Y^k \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{tlki} + (O(1)\widetilde{\mathcal{T}}^{(\mathrm{ev})})_{tit} \tag{4.98}$$

$$= -\frac{1}{2} i \sqrt{\frac{\Lambda}{3}} \, \Theta^{-1} \, |Y|^{-2} \widehat{\eta}_i{}^{jk} Y^l Y_{[j} \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{k]tlt} + (O(1)\widetilde{\mathcal{T}}^{(\mathrm{ev})})_{tit} \tag{4.99}$$

$$\equiv -\frac{1}{2} i \sqrt{\frac{\Lambda}{3}} \, \Theta^{-1} \, |Y|^{-2} \widehat{\eta}_i{}^{jk} Y^l Y_{[j} \mathcal{E}_{k]l} + (O(1)\mathcal{E})_{tit} \tag{4.100}$$

for the constraint equations, and

$$\widetilde{\nabla}_\rho \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{t(ij)}{}^\rho \equiv \widetilde{\nabla}_t \mathcal{E}_{ij} + i\widehat{\eta}_{(j}{}^{lk} \widetilde{\nabla}_l \mathcal{E}_{i)k} \tag{4.101}$$

$$= -\frac{9}{2} \sqrt{\frac{\Lambda}{3}} \, \Lambda^{-1} \Theta^{-1} \, |Y|^{-4} (\widetilde{\mathcal{H}}_{t(i} \widetilde{\mathcal{H}}_{|t|j)} - \frac{1}{3} \widetilde{\mathcal{H}}^2 \widetilde{\mathcal{I}}_{t(i|t|j)}) Y^k Y^l \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{tktl}$$

$$+ \frac{9}{2} \sqrt{\frac{\Lambda}{3}} \, \Lambda^{-1} \Theta^{-1} \, |Y|^{-4} (\widetilde{\mathcal{H}}_{t(i} \widetilde{\mathcal{H}}_{j)k} - \frac{1}{3} \widetilde{\mathcal{H}}^2 \widetilde{\mathcal{I}}_{t(ij)k}) Y^l Y^m \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{tlm}{}^k$$

$$+ (O(1)\widetilde{\mathcal{T}}^{(\mathrm{ev})})_{(tij)} \tag{4.102}$$

$$= \sqrt{\frac{\Lambda}{3}} \, |Y|^{-2} \Theta^{-1} \left(\frac{3}{2} Y_{(i} Y^l \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{|t|j)tl} - 3 \, |Y|^{-2} Y_i Y_j Y^k Y^l \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{tktl} \right.$$

$$\left. + \frac{1}{2} h_{ij} Y^k Y^l \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{tktl} \right) + (O(1)\widetilde{\mathcal{T}}^{(\mathrm{ev})})_{t(ij)} \tag{4.103}$$

$$\equiv \sqrt{\frac{\Lambda}{3}} \, |Y|^{-2} \Theta^{-1} \left(\frac{3}{2} Y_{(i} Y^l \mathcal{E}_{j)l} - 3 \, |Y|^{-2} Y_i Y_j Y^k Y^l \mathcal{E}_{kl} + \frac{1}{2} h_{ij} Y^k Y^l \mathcal{E}_{kl} \right)$$

$$+ (O(1)\mathcal{E})_{t(ij)} \tag{4.104}$$

for the evolution equations.

Note that the equations (4.99) and (4.103) hold regardless of the gauge as long as the asymptotic gauge condition (4.88) is ensured. However, the global gauge condition (4.95) (or an analogous one, cf section 4.4.2) is needed to ensure that this realizes the splitting into constraint and evolution equations.

The divergent terms in both constraint and evolution equations are regular if and only if

$$Y^l Y_{[j} \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{k]tlt}|_{\mathscr{I}} = 0 = Y^l Y_{(j} \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{k)tlt}|_{\mathscr{I}} \quad \Longleftrightarrow \quad Y^j \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{titj}|_{\mathscr{I}} \equiv Y^j \mathcal{E}_{ij}|_{\mathscr{I}} = 0$$

$$\Longleftrightarrow \quad (\widetilde{\mathcal{H}}^{\alpha\beta} \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{\mu\nu\alpha\beta})|_{\mathscr{I}} = 0.$$

For the sake of consistency, we check that these conditions hold if and only if the spacetime is asymptotically KdS-like. Indeed

$$0 = Y^l Y_{[j} \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{k]tlt}|_{\mathscr{I}} = Y^l Y_{[j} D_{k]l} - i\sqrt{\frac{3}{\Lambda}} Y^l Y_{[j} \widehat{C}_{k]l} \tag{4.105}$$

holds if and only if Y^k is an eigenvector of both D_{jk} and $\widehat{C}_{jk}$, or in other words if and only if

$$Y^k D_{jk} = |Y|^{-2} Y^k Y^l D_{kl} Y_j, \quad Y^k \widehat{C}_{jk} = |Y|^{-2} Y^k Y^l \widehat{C}_{kl} Y_j, \tag{4.106}$$

which are precisely the conditions defining asymptotically KdS-like spacetimes in definition 4.6. Analogously, the divergent term in the evolution equations will be regular if and only if

$$0 = Y^l Y_{(j} \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{k) t l t}|_{\mathscr{I}} = Y^l Y_{(j} \mathcal{E}_{k) l}|_{\mathscr{I}}$$

$$= Y^l Y_{(j} D_{k) l} - |Y|^{-2} Y^m Y^n D_{mn} Y_j Y_k$$

$$- i \sqrt{\frac{3}{\Lambda}} (Y^l Y_{(j} \widehat{C}_{k) l} - |Y|^{-2} Y^m Y^n \widehat{C}_{mn} Y_j Y_k), \tag{4.107}$$

which is automatically true in the asymptotically KdS-like setting as follows from (4.106).

In summary, the evolution equations (4.76) for $\mathcal{S}_{\mu\nu\sigma}{}^{\rho} = \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{\mu\nu\sigma}{}^{\rho}$ constitute a symmetric hyperbolic system in the unphysical spacetime with a righthand side of the form

$$\mathcal{J}(\widetilde{\mathcal{T}}^{(\mathrm{ev})})_{\alpha\beta\mu} = \frac{1}{\Theta} \mathcal{N}(\widetilde{\mathcal{T}}^{(\mathrm{ev})})_{\alpha\beta\mu} \tag{4.108}$$

where $\mathcal{N}(\widetilde{\mathcal{T}}^{(\mathrm{ev})})$ denotes a linear map $\mathcal{N}(\widetilde{\mathcal{T}}^{(\mathrm{ev})})_{\alpha\beta\mu} = N_{\alpha\beta\mu}{}^{\rho\nu\sigma}{}_{\kappa} \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{\rho\nu\sigma}{}^{\kappa}$ being $N_{\alpha\beta\mu}{}^{\rho\nu\sigma}{}_{\kappa}$ a smooth tensor field up to and including $\mathscr{I}$, at least in some neighborhood of $\mathscr{I}$, cf remark 4.12. Equations with such divergent terms are called *Fuchsian* in the literature. We state the existence and properties of the evolution equation for $\widetilde{\mathcal{T}}^{(\mathrm{ev})}_{\rho\sigma\mu}{}^{\nu}$ as a lemma.

Lemma 4.11. *The rescaled MST* $\widetilde{\mathcal{T}}^{(\mathrm{ev})}_{\rho\sigma\mu}{}^{\nu}$ *with* $Q = Q_{\mathrm{ev}}$, *satisfies a symmetric hyperbolic, linear, homogeneous Fuchsian system of evolution equations near* $\mathscr{I}$.

Remark 4.12. As already discussed in remark 4.10, it is not clear that the evolution system remains regular outside some neighborhood of $\mathscr{I}$, in its whole domain of dependence.

4.4.4. A wave equation satisfied by the (rescaled) MST. We now recall that (4.66) holds in particular for $\widetilde{\mathcal{T}}^{(\mathrm{ev})}_{\alpha\beta\lambda}{}^{\mu}$, and with tildes on all quantities. Application of $\widetilde{\nabla}^{\sigma}$ yields, together with with (4.63), the linear, homogeneous wave equation

$$\Box_{\widetilde{g}} \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{\alpha\beta\mu\nu} = -2\widetilde{\nabla}^{\sigma} \widetilde{\nabla}_{[\mu} \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{\nu]\sigma\alpha\beta} - i\widetilde{\eta}_{\mu\nu\sigma}{}^{\rho} \widetilde{\nabla}^{\sigma} \mathcal{J}(\widetilde{\mathcal{T}}^{(\mathrm{ev})})_{\alpha\beta\rho} \tag{4.109}$$

$$= -2\widetilde{\nabla}_{[\mu} \mathcal{J}(\widetilde{\mathcal{T}}^{(\mathrm{ev})})_{|\alpha\beta|\nu]} - i\widetilde{\eta}_{\mu\nu}{}^{\sigma\rho} \widetilde{\nabla}_{\sigma} \mathcal{J}(\widetilde{\mathcal{T}}^{(\mathrm{ev})})_{\alpha\beta\rho} - 2\widetilde{R}_{\kappa[\mu} \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{|\alpha\beta|\nu]}{}^{\kappa}$$

$$- 2\widetilde{R}_{\alpha\kappa[\mu}{}^{\sigma} \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{\nu]\sigma\beta}{}^{\kappa} + 2\widetilde{R}_{\beta\kappa[\mu}{}^{\sigma} \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{\nu]\sigma\alpha}{}^{\kappa} + \widetilde{R}_{\mu\nu\sigma}{}^{\kappa} \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{\alpha\beta\kappa}{}^{\sigma}. \tag{4.110}$$

Of course the same reasoning can be applied to $\mathcal{S}^{(\mathrm{ev})}_{\alpha\beta\gamma\delta}$, and we are led to the following

Lemma 4.13

i. *The MST* $\mathcal{S}^{(\mathrm{ev})}_{\mu\nu\sigma\rho}$ *satisfies, for any sign of the cosmological constant* Λ, *a linear, homogeneous system of wave equations in* $(\mathcal{M}, g)$.

ii. *The MST* $\widetilde{\mathcal{T}}^{(\mathrm{ev})}_{\mu\nu\sigma\rho}$ *satisfies, for any sign of the cosmological constant* Λ, *a linear, homogeneous system of wave equations in* $(\widetilde{\mathcal{M}}, \widetilde{g})$.

Some care is needed concerning the regularity of the coefficients in these wave equations, cf remark 4.10.

It follows from (4.86) that (4.110) is a linear, homogeneous wave equation *of Fuchsian type* at $\mathscr{I}$. Indeed, using adapted coordinates (x^0, x^i) and imposing the asymptotic gauge condition (4.88) a more careful calculation which uses (4.87), (4.89), (4.90)–(4.92) and (4.93) shows (note that $\mathcal{E}_{ij} \equiv \widetilde{\mathcal{T}}^{(\mathrm{ev})}_{titj}$ encompasses all independent components of the rescaled MST),

$$\Box_{\tilde{g}} \mathcal{E}_{ij} = 3\sqrt{\frac{\Lambda}{3}} \, |Y|^{-4}\Theta^{-1}\left((Y_i Y_j)_{\mathrm{tf}} Y^k Y^l \widetilde{\nabla}_t \mathcal{E}_{kl} - \frac{1}{2}\, |Y|^2 (Y^k Y_{(i} \widetilde{\nabla}_{|t|}\mathcal{E}_{j)k})_{\mathrm{tf}} \right)$$

$$- \Lambda \, |Y|^{-4}\Theta^{-2}\left((Y_i Y_j)_{\mathrm{tf}} Y^k Y^l \mathcal{E}_{kl} - \frac{1}{2}\, |Y|^2 (Y^k Y_{(i}\mathcal{E}_{j)k})_{\mathrm{tf}} \right)$$

$$+ i\sqrt{\frac{\Lambda}{3}} \, |Y|^{-4}\Theta^{-1}\hat{\eta}_{(i}{}^{lm}\left(\frac{1}{2}\,|Y|^2 Y^k Y_{|l|}\widetilde{\nabla}_{m|}\mathcal{E}_{j)k} - \frac{1}{2}\,|Y|^2 Y^k Y_{|l|}\widetilde{\nabla}_{j)}\mathcal{E}_{mk} \right.$$

$$\left. - 3 Y_{j)} Y_l Y^k Y^n \widetilde{\nabla}_m \mathcal{E}_{kn} + |Y|^2 Y_{j)} Y^k \widetilde{\nabla}_m \mathcal{E}_{kl} \right)$$

$$+ (O(1)\widetilde{\nabla}\mathcal{E})_{ij} + (O(\Theta^{-1})\mathcal{E})_{ij}. \tag{4.111}$$

4.5. Uniqueness for the evolution equation for $\widetilde{\mathcal{T}}^{(\mathrm{ev})}{}_{\rho\sigma\mu}{}^{\nu}$

Existence of solutions of quasilinear symmetric hyperbolic Fuchsian systems with prescribed asymptotics at $\mathscr{I}$ has been analyzed in the literature mainly in the analytic case. For the merely smooth case, there exist results by Claudel and Newman [7], Rendall [28], and more recently Ames *et al* [2, 3]. The results in these papers involve a number of algebraic requirements, as well as global conditions in space. It is an interesting problem to see whether any of these results can be adapted to our setting here, in particular in order to prove a localized existence result in which an appropriate singular behavior of $\widetilde{\mathcal{T}}^{(\mathrm{ev})}{}_{\rho\sigma\mu}{}^{\nu}$ is prescribed on some domain B of $\mathscr{I}^-$ and existence and uniqueness of a corresponding solution is shown in the domain of dependence of B. This would also require studying the impact of the constraint equations and their preservation under evolution.

For the purposes of this section, where we aim to show that the necessary conditions listed in items (i) and (ii) of theorem 2.8 for the vanishing of the rescaled MST tensor $\widetilde{\mathcal{T}}^{(\mathrm{ev})}{}_{\rho\sigma\mu}{}^{\nu}$ in a neighborhood of $\mathscr{I}$ are also sufficient, we merely need a localized uniqueness theorem for the evolution system with trivial initial data.

We state and prove such a result in a more general context by adapting some of the ideas in [2]. Then we show that this result applies to the evolution system satisfied by $\widetilde{\mathcal{T}}^{(\mathrm{ev})}{}_{\rho\sigma\mu}{}^{\nu}$.

4.5.1. A localized uniqueness theorem for symmetric hyperbolic Fuchsian systems.

Let $(\mathscr{M}, \tilde{g})$ be an $(n+1)$-dimensional spacetime and $\mathscr{I}$ a spacelike hypersurface. Choose coordinates in a neighborhood of $\mathscr{I}$ so that $\mathscr{I} = \{t = 0\}$, and the metric is such that $\tilde{g}_{tt} = -1$ and $\tilde{g}_{ti} = 0$, $i = 1, \cdots, n$. Let us consider the first order, homogenous linear symmetric hyperbolic system of PDEs

$$A^t \partial_t u + A^i \partial_i u + \frac{1}{t} N u = 0 \tag{4.112}$$

where $u : \mathscr{M} \mapsto \mathbb{C}^m$ is the unknown, A^t, A^i and N are $m \times m$ matrices which depend smoothly on the spacetime coordinates (t, x^i). We assume that $\mathbb{C}^m$ is, at each spacetime point (t, x), endowed with a positive definite sesquilinear product $\langle u, v \rangle$ such that the endomorphisms $A^\mu(t, x)$, for $\mu = t, i$, are Hermitian with respect to this product

$$\langle u, A^\mu v \rangle = \langle \overline{A}^\mu u, v \rangle.$$

Define $N_0(x) := N(t = 0, x)$. Our main assumption is that $A^t + N_0$ is strictly positive definite with respect to $\langle , \rangle$ at every point $p \in \mathscr{I}$. Since both the inner product and N depend smoothly on t and x, the same holds for a sufficiently small spacetime neighborhood of any point $p \in \mathscr{I}$. The domain of dependence of (4.112) is defined in the usual way (namely, the standard definition in terms of causal curves, with causality at $T_p\mathscr{M}$ being defined as

$k \in T_p\mathcal{M}$ being future timelike (causal) iff $k^t > 0$ and $A^\mu|_p k_\mu$ is negative definite (semidefinite), where k_μ is obtained from k by lowering indices with respect to $\tilde{g}$). We also make the assumption that $k = \partial_t$ is future timelike in this sense.

We want to prove that the PDE (4.112) with trivial initial data on a domain $B \subset \mathscr{I}$ vanishes identically in the domain of dependence of B, denoted by $D(B)$. More precisely

Lemma 4.14. *Let $B \subset \mathscr{I}$ be a domain with compact closure. Let u be a C^1 map $u : \mathcal{M} \mapsto \mathbb{C}^m$ which vanishes at B and solves (4.112). Assume that A^t and $A^t + N_0$ are positive definite, then u vanishes at $D(B)$.*

Proof. The proof is adapted from the basic energy estimate lemma 2.7 in [2]. Since our setup is simpler, we can use a domain of dependence-type argument, instead of a global-in-space argument as in [2, 3]. First note that since u vanishes at $t = 0$, and it is C^1,

$$\partial_t u|_{t=0,x} = \lim_{t \to 0} \frac{u(t, x)}{t} := u_1$$

with $u_1 : \mathscr{I} \mapsto \mathbb{C}^m$ continuous. Taking the limit of (4.112) as $t \to 0$ with $(0, x) \in B$ and using $u(0, x) = 0$ it follows

$$(A^t + N_0)u_1 = 0 \qquad \Longrightarrow \qquad u_1 = 0,$$

because $A^t + N_0$ is positive definite, and hence has trivial kernel. Let us consider the real quantity

$$\mathcal{Z}^\mu := e^{-kt}\left\langle \frac{u}{t}, A^\mu \frac{u}{t} \right\rangle, \qquad k \in \mathbb{R},$$

and consider its (coordinate) divergence. Since the product $\langle,\rangle$ depends on the spacetime point, we denote by $\langle,\rangle_\mu$ the bilinear form (at each spacetime point) defined by

$$\partial_\mu \langle u, v \rangle = \langle \partial_\mu u, v \rangle + \langle u, \partial_\mu v \rangle + \langle u, v \rangle_\mu, \qquad \forall u, v \in \mathbb{R}^m.$$

It follows

$$\begin{aligned}
\partial_\mu \mathcal{Z}^\mu = &- \frac{e^{-kt}}{t^2}\left(k + \frac{2}{t}\right)\langle u, A^t u \rangle + \frac{e^{-kt}}{t^2}(2\langle u, A^\mu \partial_\mu u \rangle + \langle u, (\partial_\mu A^\mu)u \rangle + \langle u, A^\mu u \rangle_\mu) \\
= &- \frac{2}{t^3}e^{-kt}\langle u, (A^t + N)u \rangle \\
&+ e^{-kt}\left(\left\langle \frac{u}{t}, (-kA^t + \partial_\mu A^\mu)\frac{u}{t} \right\rangle + \left\langle \frac{u}{t}, A^\mu \frac{u}{t} \right\rangle_\mu\right),
\end{aligned}$$

(4.113)

where in the first equality we have used that A^μ is Hermitian w.r.t. $\langle,\rangle$ and in the second equality we have used the fact that u satisfies (4.112). Consider now a domain $V \subset \mathcal{M}$ bounded by three smooth hypersurfaces-with-boundary $B \subset \mathscr{I}$, $B_T \subset \{t = T\}$ and Σ, whose union is a compact topological hypersurface. Note that B and B_T are spacelike (i.e. their normal is timelike). We choose V so that Σ is achronal and that its outward normal (defined as the normal one-form which contracted with any outward directed vector is positive) is past causal. Consider the domain $V_\epsilon = V \cap \{t \geqslant \epsilon\}$ for $\epsilon > 0$ small enough. The boundary splits as $\partial V_\epsilon = B_\epsilon \cup \Sigma_\epsilon \cup B_T$, with obvious notation.

We integrate $\partial_\mu \mathcal{Z}^\mu$ on V_ϵ with respect to the spacetime volume form $\eta = F dt dx$ and use the Gauss identity. Denote by n an outward normal to ∂V, then

$$\int_V (\partial_\mu \mathcal{Z}^\mu) F \, \mathrm{d}t \mathrm{d}x = \int_V \partial_\mu (\mathcal{Z}^\mu F) \, \mathrm{d}t \mathrm{d}x - \int_V (\partial_\mu F) \, \mathcal{Z}^\mu \mathrm{d}t \mathrm{d}x$$
$$= \int_{\partial V} \mathcal{Z}^\mu n_\mu \, \mathrm{d}S - \int_V (\partial_\mu F) \, \mathcal{Z}^\mu \mathrm{d}t \mathrm{d}x,$$

where $\mathrm{d}S$ is the induced volume form on ∂V corresponding to the choice of normal n. Note in particular that, as a vector, n^μ points *inwards* both on B_ϵ and on B_T, so they can be taken simply to be $n = \partial_t$ on B_ϵ and $n = -\partial_t$ on B_T, i.e. $\boldsymbol{n} = -\mathrm{d}t$ on B_ϵ and $\boldsymbol{n} = \mathrm{d}t$ on B_T. Inserting (4.113) and splitting the integral at the boundary in three pieces yields

$$\int_{B_T} \mathrm{e}^{-kT} \left\langle \frac{u}{T}, A^t \frac{u}{T} \right\rangle \mathrm{d}S - \int_{B_\epsilon} \mathrm{e}^{-k\epsilon} \left\langle \frac{u}{\epsilon}, A^t \frac{u}{\epsilon} \right\rangle \mathrm{d}S$$
$$= - \int_{\Sigma_\epsilon} \mathrm{e}^{-kt} \left\langle \frac{u}{t}, (A^\mu n_\mu) \frac{u}{t} \right\rangle \mathrm{d}S$$
$$- \int_V \frac{2}{t^3} \mathrm{e}^{-kt} \langle u, (A^t + N) u \rangle \boldsymbol{\eta}$$
$$+ \int_V \mathrm{e}^{-kt} \left[\left\langle \frac{u}{t}, (-kA^t + (\partial_\mu F) A^\mu + \partial_\mu A^\mu) \frac{u}{t} \right\rangle + \left\langle \frac{u}{t}, A^\mu \frac{u}{t} \right\rangle_\mu \right] \boldsymbol{\eta} := I_{V_\epsilon}.$$

The matrix $A^\mu n_\mu$ on Σ_ϵ is positive semidefinite because n is past causal. We now choose T small enough so that $A^t + N$ is positive definite on V and k large enough so that the last term in I_{V_ϵ} is negative (recall that V has compact closure). Thus we have $I_{V_\epsilon} \leqslant 0$ and in fact strictly negative unless $u = 0$. Thus

$$\int_{B_T} \mathrm{e}^{-kT} \left\langle \frac{u}{T}, A^t \frac{u}{T} \right\rangle \mathrm{d}S - \int_{B_\epsilon} \mathrm{e}^{-k\epsilon} \left\langle \frac{u}{\epsilon}, A^t \frac{u}{\epsilon} \right\rangle \mathrm{d}S \leqslant 0.$$

We now take the limit $\epsilon \to 0$ and use the fact that $\frac{u}{\epsilon} \to u_1 = 0$ to conclude

$$\int_{B_T} \mathrm{e}^{-kT} \left\langle \frac{u}{T}, A^t \frac{u}{T} \right\rangle \mathrm{d}S \leqslant 0.$$

Since the product $\langle , \rangle$ is positive definite, it follows $u = 0$ on B_T. As a consequence $I_V = 0$ which implies $u = 0$ on V. It is clear that the domain of dependence $D(B)$ can be exhausted by such V's, so $u = 0$ on $D(B)$ as claimed. □

4.5.2. Application to the Fuchsian system satisfied by $\widetilde{\mathcal{T}}^{(\mathrm{ev})}{}_{\rho\sigma\mu}{}^\nu$. In this section we show that the symmetric hyperbolic evolution system (4.76) for $\mathcal{S}_{\rho\sigma\mu}{}^\nu = \widetilde{\mathcal{T}}^{(\mathrm{ev})}{}_{\rho\sigma\mu}{}^\nu$ satisfies all the conditions of lemma 4.14 in the unphysical space-time and conclude that the unique solution with vanishing data at $\mathscr{I}^-$ is trivial and hence, since the system is linear and homogeneous, we also get uniqueness of all solutions given regular initial data at $\mathscr{I}^-$.

We choose coordinates $\{t, x^i\}$ on a neighborhood of $\mathscr{I}^-$ satisfying $\tilde{g}_{tt} = -1$, $\tilde{g}_{ti} = 0$, and $\mathscr{I}^- = \{t = 0\}$. The induced metrics on the hypersurfaces Σ_t of constant $x^0 = t$ are denoted by h^t, with corresponding volume forms $\hat{\eta}^t$. Rewriting (4.104) by moving all the Christoffel symbol terms to the right-hand side and using (4.89) we have

$$\partial_t \mathcal{E}_{ij} + i\hat{\eta}_{(j}{}^{lk} \partial_l \mathcal{E}_{i)k} = -\frac{1}{t} \frac{1}{|Y|^4} \left(3 Y_i Y_j Y^k - \frac{1}{2} h_{ij} |Y|^2 Y^k - \frac{3}{2} |Y|^2 Y_{(i} \delta^k_{j)} \right) Y^l \mathcal{E}_{lk}$$
$$+ (O(1) \mathcal{E})_{ij}. \tag{4.114}$$

The unkown is the complex symmetric and trace-free tensor $\mathcal{E}_{ij}$ introduced in (4.74). The system (4.114) is of the form (4.112) with $u = \{\mathcal{E}_{ij}\} \in \mathbb{C}^5$, $A^t = \mathrm{Id}_5$ and

$$A^{l\,nk}_{ij} = i\eta^{tl(k}{}_{(j}\delta^{n)}_{i)}.$$

Take the sesquilinear product $\langle,\rangle$ defined by $\langle \mathcal{E}, \widehat{\mathcal{E}} \rangle = \mathcal{E}_{ij}\overline{\widehat{\mathcal{E}}}^{ij}$ (indices lowered and raised with h^t), which is obviously positive definite –its norm leading to the density (4.75). It is straightforward to check that A^μ is Hermitian with respect to this product and A^t is obviously positive definite.

It remains to check that $A^t + N_0 = \mathrm{Id}_5 + N_0$ is positive definite. The endomorphism N_0 is, from (4.114),

$$N_0(\mathcal{E})_{ij} := \frac{1}{|Y|^4}\left(3Y_i Y_j Y^k - \frac{1}{2}h_{ij}\,|Y|^2 Y^k - \frac{3}{2}\,|Y|^2 Y_{(i}\delta^k_{j)}\right)Y^l\mathcal{E}_{lk}$$

so that

$$\langle \mathcal{E}, (\mathrm{Id}_5 + N_0)\mathcal{E}\rangle = \mathcal{E}_{ij}\overline{\mathcal{E}}^{ij} + \frac{3}{|Y|^4}\overline{\mathcal{E}}^{ij}Y_i Y_j \mathcal{E}_{kl}Y^k Y^l - \frac{3}{2}\frac{1}{|Y|^2}Y_i\overline{\mathcal{E}}^{ik}Y^l\mathcal{E}_{lk}.$$

To see if this has a sign we introduce the following objects orthogonal to Y^i

$$c_{ij} := \mathcal{E}_{ij} - \frac{2}{|Y|^2}Y_{(i}c_{j)} - \frac{1}{|Y|^4}Y_i Y_j \mathcal{E}_{kl}Y^k Y^l,$$

$$c_i := Y^k\mathcal{E}_{ki} - \frac{1}{|Y|^2}Y_i\mathcal{E}_{kl}Y^k Y^l$$

and the previous expression can be rewritten as

$$\langle \mathcal{E}, (\mathrm{Id}_5 + N_0)\mathcal{E}\rangle = c_{ij}\overline{c}^{ij} + \frac{1}{2}\frac{1}{|Y|^2}c_i\overline{c}^i + \frac{5}{2}\frac{1}{|Y|^4}\mathcal{E}_{kl}Y^k Y^l\overline{\mathcal{E}}_{ij}Y^i Y^j$$

which is manifestly positive definite. We have thus proven

Lemma 4.15. *Let $(\mathcal{M}, g)$ be a spacetime admitting a smooth conformal compactification. If $(\mathcal{M}, g)$ admits a Killing vector field for which the rescaled MST $\widetilde{T}^{(ev)}_{\mu\nu\sigma}{}^\rho$ vanishes at $\mathscr{I}^-$, then $\widetilde{T}^{(ev)}_{\mu\nu\sigma}{}^\rho$ vanishes in a neighborhood of $\mathscr{I}^-$.*

The characteristics of the symmetric hyperbolic system (4.114) coincide with those of the propagational part of the Bianchi equation, and are computed and discussed in [12, section 4]. It is shown there that they form null *and timelike* hypersurfaces.

4.6. Main result

We end up with the following main result:

Theorem 4.16. *Consider a spacetime $(\mathcal{M}, g)$, or rather its conformally rescaled counterpart, in wave map gauge (2.25),[9] solution to Einstein's vacuum field equations*

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[9] In fact, it suffices if $\widetilde{R}$ and $\widetilde{W}^\sigma$, including certain transverse derivatives thereof, vanish on $\mathscr{I}^-$. Moreover, a corresponding result must also hold for non-vanishing gauge source functions. We leave it to the reader to work this out.

with $\Lambda > 0$, which admits a smooth conformal extension through $\mathscr{I}^-$ and which contains a KVF X. Denote by h the Riemannian metric induced by $\widetilde{g} = \Theta^2 g$ on $\mathscr{I}^-$, and by Y the CKVF induced by X on $\mathscr{I}^-$. Then, there exists a function Q, namely $Q = Q_0$ ($=Q_{\mathrm{ev}}$ for an appropriate choice of the σ-constant), for which the MST $\mathcal{S}_{\mu\nu\sigma}{}^\rho$ corresponding to X vanishes in the domain of dependence of $\mathscr{I}^-$ if and only if the following relations hold:

 i. *$\widehat{C}_{ij} = \sqrt{\frac{\Lambda}{3}} \, C_{\mathrm{mag}} \, |Y|^{-5} \left(Y_i Y_j - \frac{1}{3} |Y|^2 h_{ij} \right)$ for some constant C_{mag}, where $\widehat{C}_{ij}$ is the Cotton–York tensor of the Riemannian 3-manifold $(\mathscr{I}^-, h)$, and*
 ii. *$D_{ij} = \widetilde{d}_{titj}|_{\mathscr{I}^-} = C_{\mathrm{el}} \, |Y|^{-5} \left(Y_i Y_j - \frac{1}{3} |Y|^2 h_{ij} \right)$ for some constant C_{el}, where $\widetilde{d}_{\mu\nu\sigma}{}^\rho$ is the rescaled Weyl tensor of the unphysical spacetime $(\widetilde{\mathscr{M}}, \widetilde{g})$.*

Proof. Theorem 2.8 shows that (i) and (ii) are necessary conditions. Conversely, if (i) and (ii) hold, it follows from theorems 2.8 and 4.3 that there exists a choice of the σ-constant *a* in Q_{ev} for which the rescaled MST $\widetilde{\mathcal{T}}^{(\mathrm{ev})}_{\alpha\beta\mu\nu}$ vanishes on $\mathscr{I}^-$. It then follows from lemma 4.15 that it vanishes in some neighborhood of $\mathscr{I}^-$. However, once we know that the MST vanishes in such neighborhood, it follows from the results in [21] that the metric has to take one of the local forms given there. But for all these metrics the MST vanishes globally. So we can conclude that it vanishes in fact in the whole domain of dependence of $\mathscr{I}^-$.   □

Recall the definition 4.5 of Kerr–de Sitter-like spacetimes. Lemma 4.15 and theorem 4.16 lead to the following characterization of KdS-like space-times:

Corollary 4.17. *Let $(\mathscr{M}, g)$ be a $\Lambda > 0$-vacuum spacetime admitting smooth conformal compactification and corresponding null infinity $\mathscr{I}$ as well as a KVF X. Then the following statements are equivalent:*

 i. *$(\mathscr{M}, g)$ is Kerr–de Sitter-like at a connected component $\mathscr{I}^-$.*
 ii. *There exists a function Q such that the MST associated to X vanishes in the domain of dependence of $\mathscr{I}^-$.*
 iii. *The CKVF Y induced by X on $\mathscr{I}^-$ satisfies the conditions (i) and (ii) of theorem 4.16.*

Reformulated in terms of an asymptotic Cauchy problem theorem 4.16 becomes theorem 1.3 given in the Introduction.

5. A second conformal Killing vector field

5.1. Existence of a second conformal Killing vector field

It follows from [21, theorem 4] that (here overbars mean 'complex conjugate of')

$$\varsigma^\mu = \frac{4}{|Q\mathcal{F}^2 - 4\Lambda|^2} X^\sigma \overline{\mathcal{F}}_\sigma{}^\rho \mathcal{F}_\rho{}^\mu + \mathrm{Re}\left(\frac{\mathcal{F}^2}{(Q\mathcal{F}^2 - 4\Lambda)^2} \right) X^\mu \tag{5.1}$$

is another KVF which commutes with X, supposing that $(\mathscr{M}, g)$ is a Λ-vacuum spacetime for which Q, $\mathcal{F}^2$ and $Q\mathcal{F}^2 - 4\Lambda$ are not identically zero and whose MST vanishes w.r.t. the KVF X (cf [17, 27] for the $\Lambda = 0$-case). Note that ς may be trivial or merely a multiple of X.

However, expression (5.1) can be taken as definition of a vector field ς in any Λ-vacuum spacetime admitting a KVF X and such that $\mathcal{F}^2 \neq 0$ and $Q\mathcal{F}^2 - 4\Lambda \neq 0$. We take $Q = Q_0$ and assume that $(\mathcal{M}, g)$ admits a smooth $\mathscr{I}$, but we do *not* assume that the MST vanishes.

A somewhat lengthy computation reveals that (recall that f and N denote divergence and curl of Y, respectively)

$$\varsigma^t = -\frac{2}{|Q_0\mathcal{F}^2 - 4\Lambda|^2}\Theta^4 h^{ij}\overline{\sigma}_j \mathcal{F}_{it} + \text{Re}\left(\frac{\mathcal{F}^2}{(Q_0\mathcal{F}^2 - 4\Lambda)^2}\right)\widetilde{X}^t \tag{5.2}$$

$$= -\frac{1}{8\Lambda^2}\Theta^4 h^{ij}\overline{\sigma}_j \mathcal{F}_{it} + \frac{1}{16\Lambda^2}\text{Re}(\mathcal{F}^2)\widetilde{X}^t + O(\Theta) \tag{5.3}$$

$$= O(\Theta), \tag{5.4}$$

$$\varsigma^i = \frac{2}{|Q_0\mathcal{F}^2 - 4\Lambda|^2}\Theta^4 h^{ij}(h^{kl}\overline{\sigma}_l \mathcal{F}_{kj} + \overline{\sigma}_t \mathcal{F}_{jt}) + \text{Re}\left(\frac{\mathcal{F}^2}{(Q_0\mathcal{F}^2 - 4\Lambda)^2}\right)\widetilde{X}^i$$
$$= \frac{1}{8\Lambda^2}\Theta^4 h^{ij}(h^{kl}\overline{\sigma}_l \mathcal{F}_{kj} + \overline{\sigma}_t \mathcal{F}_{jt}) + \frac{1}{16\Lambda^2}\text{Re}(\mathcal{F}^2)\widetilde{X}^i + O(\Theta) \tag{5.5}$$

$$= \frac{1}{8\Lambda^2}Y_k N^k N^i + \frac{1}{8\Lambda^2}Y^i\left(-\frac{1}{2}|N|^2 + \frac{2}{9}f^2 - 2\hat{c}\right)$$
$$- \frac{1}{2\Lambda^2}|Y|^2\left(\hat{L}^i_{\ k}Y^k + \frac{1}{3}\widehat{\nabla}^i f\right) + \frac{1}{12\Lambda^2}f\hat{\eta}^{ikl}Y_k N_l + O(\Theta) \tag{5.6}$$

$$= \frac{1}{8\Lambda^2}Y_k N^k N^i + \frac{1}{2\Lambda^2}Y^i\left(Y^k Y^l \hat{L}_{kl} + \frac{1}{3}Y^k \widehat{\nabla}_k f\right)$$
$$- \frac{1}{2\Lambda^2}|Y|^2\left(\hat{L}^i_{\ k}Y^k + \frac{1}{3}\widehat{\nabla}^i f\right) + \frac{1}{12\Lambda^2}f\hat{\eta}^{ikl}Y_k N_l + O(\Theta), \tag{5.7}$$

where we used (3.6), (4.6), (4.8), (4.10) as well as the following relations:

$$\text{Re}(\mathcal{F}^2) = -\frac{4}{3}\Lambda|Y|^2\Theta^{-2} - 4Y^i\overline{\widetilde{\nabla}_t\widetilde{\nabla}_t\widetilde{X}_i} + \frac{4}{9}f^2 - \frac{4}{3}\Lambda c + O(\Theta), \tag{5.8}$$

$$\mathcal{F}_{it} = \sqrt{\frac{\Lambda}{3}}Y_i\Theta^{-3} - \frac{i}{2}N_i\Theta^{-2} - \frac{1}{2}\sqrt{\frac{3}{\Lambda}}\Theta^{-1}\left(\overline{\widetilde{\nabla}_t\widetilde{\nabla}_t\widetilde{X}_i} + \frac{\widehat{R}}{2}Y_i\right) + O(1), \tag{5.9}$$

$$\mathcal{F}_{ij} = i\sqrt{\frac{\Lambda}{3}}\hat{\eta}_{ijk}Y^k\Theta^{-3} + \frac{1}{2}\hat{\eta}_{ijk}N^k\Theta^{-2} + O(\Theta^{-1}), \tag{5.10}$$

$$\widetilde{X}^t = \frac{1}{3}\sqrt{\frac{3}{\Lambda}}f\Theta + O(\Theta^3), \tag{5.11}$$

$$\widetilde{X}^i = Y^i + \frac{1}{2}\frac{3}{\Lambda}\overline{\widetilde{\nabla}_t\widetilde{\nabla}_t\widetilde{X}^i}\Theta^2 + O(\Theta^3). \tag{5.12}$$

We conclude that the vector field ς is always tangential to $\mathscr{I}$, and not just in the setting where the MST vanishes.

Proceeding in the same manner as we did for the functions $\widehat{c}$ and $\widehat{k}$, we regard (5.5) as the definition of a vector field on some Riemannian 3-manifold:

Definition 5.1. Let (Σ, h) be a Riemannian 3-dimensional manifold which admits a CKVF Y. Then we define the vector field

$$\widehat{\varsigma}^i(Y) := \frac{9}{4} Y_k N^k N^i + 9Y^i \left(Y^k Y^l \widehat{L}_{kl} + \frac{1}{3} Y^k \widehat{\nabla}_k f \right) - 9 \, |Y|^2 \left(\widehat{L}^i{}_k Y^k + \frac{1}{3} \widehat{\nabla}^i f \right) + \frac{3}{2} f \widehat{\eta}^{ikl} Y_k N_l. \tag{5.13}$$

As in the case of $\widehat{c}$ and $\widehat{k}$, one can use a spacetime argument where Σ is embedded as 'null infinity' into a $\Lambda > 0$-vacuum spacetime with vanishing MST to prove that $\widehat{\varsigma}(Y)$, which in that case reads $\widehat{\varsigma}^i(Y) = 18\Lambda^2 \varsigma^i|_{\mathscr{I}}$, is a (possibly trivial) CKVF which commutes with Y, supposing that $|Y|^2 > 0$ and (2.87) hold. Irrespective of that, one can raise the question under which conditions $\widehat{\varsigma}(Y)$ is a CKVF and under which conditions it commutes with Y.

For this purpose let us compute the covariant derivative of $\widehat{\varsigma}$. A lengthy calculation which uses (4.8), (3.22), (3.23), the conformal Killing equation for Y as well as the relation

$$\widehat{\nabla}_i N_j = \frac{2}{3} \widehat{\eta}_{ij}{}^k \widehat{\nabla}_k f - \widehat{\eta}_j{}^{kl} \widehat{R}_{klim} Y^m \tag{5.14}$$

gives

$$\begin{aligned}
\widehat{\nabla}_i \widehat{\varsigma}_j &= 3\widehat{\nabla}_i Y_j Y^k \widehat{\nabla}_k f + 3Y_j \widehat{\nabla}_i Y^k \widehat{\nabla}_k f + 3Y^k Y_j \widehat{\nabla}_i \widehat{\nabla}_k f - 6Y^k \widehat{\nabla}_i Y_k \widehat{\nabla}_j f \\
&\quad + 3Y^k \widehat{\nabla}_j Y_k \widehat{\nabla}_i f - 3 \, |Y|^2 \widehat{\nabla}_i \widehat{\nabla}_j f + 3f \widehat{\nabla}_i Y^k \widehat{\nabla}_j Y_k + 3fY^k \widehat{\nabla}_i \widehat{\nabla}_j Y_k \\
&\quad + \frac{9}{4} N_j \widehat{\nabla}_i (Y_k N^k) + \frac{9}{4} Y_k N^k \widehat{\nabla}_i N_j - 2fY_j \widehat{\nabla}_i f - f^2 \widehat{\nabla}_i Y_j \\
&\quad + 9\widehat{\nabla}_i Y_j Y^k Y^l \widehat{L}_{kl} + 18Y_j Y^k \widehat{\nabla}_i Y^l \widehat{L}_{kl} + 9Y_j Y^k Y^l \widehat{\nabla}_i \widehat{L}_{kl} \\
&\quad - 18\widehat{\nabla}_i Y_l Y^k Y^l \widehat{L}_{jk} - 9 \, |Y|^2 \widehat{\nabla}_i Y^k \widehat{L}_{jk} - 9 \, |Y|^2 Y^k \widehat{\nabla}_i \widehat{L}_{jk} \tag{5.15} \\[4pt]
&= \frac{3}{4} f \, |N|^2 h_{ij} - \frac{9}{2} \underbrace{Y_{[(i} N_k \widehat{\nabla}_{l]} f \widehat{\eta}_{j)}{}^{kl}}_{= \frac{1}{3} h_{ij} \widehat{\eta}^{klm} Y_k N_l \widehat{\nabla}_m f} - \frac{27}{2} \underbrace{Y^m Y_{[j} N_k \widehat{L}_{l]m} \widehat{\eta}_i{}^{kl}}_{= \frac{1}{3} h_{ij} Y^m Y_p N_k \widehat{L}_{lm} \widehat{\eta}^{pkl}} \\
&\quad + \frac{3}{2} \widehat{\eta}_{ijl} (N^l Y^k \widehat{\nabla}_k f + Y_k N^k \widehat{\nabla}^l f + 3Y_m N^m Y^k \widehat{L}_k{}^l + 3N^l Y^k Y^m \widehat{L}_{km} \\
&\quad - \frac{1}{3} f^2 N^l) + \frac{3}{2} \widehat{\eta}_{[i}{}^{kl} (N_{j]} Y_k \widehat{\nabla}_l f - Y_{j]} N_k \widehat{\nabla}_l f - 3Y_k N_l \widehat{\nabla}_{j]} f) \\
&\quad - 2fY_{[i} \widehat{\nabla}_{j]} f - 6fY^k Y_{[i} \widehat{L}_{j]k} + 9Y^m N_l Y_k \widehat{L}_{m[i} \widehat{\eta}_{j]}{}^{kl} \\
&\quad + 9Y_j \widehat{\eta}_{ik}{}^l \widehat{C}_{lp} Y^k Y^p - 9 \, |Y|^2 \widehat{\eta}_{ik}{}^l \widehat{C}_{lj} Y^k \\
&\quad + \frac{9}{2} \underbrace{(3Y^m N_{[k} Y_l \widehat{L}_{m]i} \widehat{\eta}_j{}^{kl} - Y_j N_k Y_l \widehat{L}_{mi} \widehat{\eta}^{mkl})}_{= 0}. \tag{5.16}
\end{aligned}$$

Employing again the fact that Y is a CKVF one shows that Y and $\widehat{\varsigma}$ commute,

$$[Y, \widehat{\varsigma}]_i = Y^j \widehat{\nabla}_j \widehat{\varsigma}_i - \widehat{\varsigma}^j \widehat{\nabla}_j Y_i \tag{5.17}$$

$$= Y^j \widehat{\nabla}_j \widehat{\varsigma}_i + \frac{1}{2} \widehat{\eta}_{ijk} \widehat{\varsigma}^j N^k - \frac{1}{3} f \widehat{\varsigma}_i \tag{5.18}$$

$$= \frac{3}{4} f \, |N|^2 Y_i - \frac{9}{4} Y_i \widehat{\eta}^{klm} Y_k N_l \widehat{\nabla}_m f - \frac{9}{2} Y_i Y^m Y_p N_k \widehat{L}_{lm} \widehat{\eta}^{pkl}$$

$$\quad - \frac{1}{2} Y^j \widehat{\eta}_{ijl} \left(-\frac{3}{2} N^l Y^k \widehat{\nabla}_k f - f^2 N^l + \frac{9}{2} Y_k N^k \widehat{\nabla}^l f + 9 Y_m N^m Y^k \widehat{L}_k{}^l \right)$$

$$\quad + \frac{3}{4} \widehat{\eta}_i{}^{kl} |Y|^2 N_k \widehat{\nabla}_l f + 3 f Y^j Y^k Y_i \widehat{L}_{jk} - 3 f \, |Y|^2 Y^k \widehat{L}_{ik}$$

$$\quad + f Y^j Y_i \widehat{\nabla}_j f - f \, |Y|^2 \widehat{\nabla}_i f + \frac{1}{2} \widehat{\eta}_{ijk} \widehat{\varsigma}^j N^k - \frac{1}{3} f \widehat{\varsigma}_i \tag{5.19}$$

$$= \frac{9}{4} (3 \widehat{\eta}_i{}^{jk} Y^l Y_{[j} N_k \widehat{\nabla}_{l]} f - Y_i \widehat{\eta}^{klm} Y_k N_l \widehat{\nabla}_m f)$$

$$\quad + \frac{9}{2} (3 \widehat{\eta}_i{}^{jk} Y^m N_{[j} Y_m \widehat{L}_{k]l} Y^l - Y_i Y^m Y_j N_k \widehat{L}_{lm} \widehat{\eta}^{jkl}) \tag{5.20}$$

$$= 0. \tag{5.21}$$

Lemma 5.2. *Let* (Σ, h) *be a Riemannian 3-manifold which admits a CKVF Y. Let the vector field* $\widehat{\varsigma}(Y)$ *be given by (5.13). Then*

$$[Y, \widehat{\varsigma}] = 0.$$

We further deduce from (5.16) that

$$(\widehat{\nabla}_{(i} \widehat{\varsigma}_{j)})_{\text{tf}} = 9 Y_{(i} \widehat{\eta}_{j)k}{}^l \widehat{C}_{lp} Y^k Y^p - 9 \, |Y|^2 \widehat{\eta}_{(i|k|}{}^l \widehat{C}_{j)l} Y^k, \tag{5.22}$$

i.e. $\widehat{\varsigma}(Y)$ will be a (possibly trivial) CKVF if and only if

$$Y_{(i} \widehat{\eta}_{j)k}{}^l \widehat{C}_{lp} Y^k Y^p = |Y|^2 \widehat{\eta}_{(i|k|}{}^l \widehat{C}_{j)l} Y^k \tag{5.23}$$

$$\Longleftrightarrow \quad (\widehat{C}_{lp} Y_{(i} - Y_p \widehat{C}_{l(i)}) \widehat{\eta}_{j)k}{}^l Y^k Y^p = 0. \tag{5.24}$$

Lemma 5.3. *Let* (Σ, h) *be a Riemannian 3-manifold which admits a CKVF Y. Then the vector field* $\widehat{\varsigma}(Y)$, *defined in (5.13), is a (possibly trivial) CKVF if and only if (5.24) holds.*

Remark 5.4. In particular (5.24) is fulfilled supposing that (2.87) holds as one should expect from the results in [21].

Remark 5.5. Observe that, from (2.72), condition (5.24) can be re-expressed as

$$Y_p Y^k (Y^p \widehat{C}_{(ij)k} - Y_{(i} \widehat{C}^p{}_{j)k}) = 0.$$

5.2. Properties of the KID equations

In this section we study the case where the KID equations on a spacelike $\mathscr{I}^-$ of a $\Lambda > 0$-vacuum spacetime admit at least two solutions, as it is the case for e.g. Kerr–de Sitter, or, more generally, for spacetimes with vanishing MST [21].

Recall the KID equations [24]

$$\mathscr{L}_Y D_{ij} + \frac{1}{3} D_{ij} \widehat{\nabla}_k Y^k = 0. \tag{5.25}$$

Consider two CKVFs Y and ζ on the Riemannian 3-manifold $(\mathscr{I}^-, h)$ which both assumed to solve the KID equations. Then also their commutator, which is obviously a CKVF, provides another (possibly trivial) solution of the KID equations,

$$\mathscr{L}_{[Y,\zeta]} D_{ij} + \frac{1}{3} D_{ij} \widehat{\nabla}_k [Y, \zeta]^k = 0. \tag{5.26}$$

This reflects the well-known fact that KVFs together with the commutator form a Lie algebra.

Let us continue assuming that Y and ζ are two CKVFs on $(\mathscr{I}^-, h)$ which solve the KID equations, and let us further assume that D_{ij} satisfies condition (2.82) (in particular, we assume $|Y|^2 > 0$). Then the KID equations (5.25) for ζ can be written as (set $V := [Y, \zeta]$ and assume $C_{el} \neq 0$)

$$2Y_{(i} V_{j)} + (h_{ij} - 5 |Y|^{-2} Y_i Y_j) Y_k V^k = 0. \tag{5.27}$$

Contraction with Y^j yields

$$0 = |Y|^2 V_i - 3Y_i Y_j V^j. \tag{5.28}$$

Another contraction with Y^i gives

$$Y^k V_k = 0. \tag{5.29}$$

Inserting this into (5.28) we find that (5.27) is equivalent to

$$V = [Y, \zeta] = 0. \tag{5.30}$$

We have proven the following:

Lemma 5.6. *Let $(\mathscr{I}^-, h_{ij})$ be a Riemannian 3-manifold which admits a CKVF Y with $|Y|^2 > 0$. Denote by $(\mathscr{M}, g_{\mu\nu})$ the $\Lambda > 0$-vacuum spacetime constructed from the initial data h_{ij} and $D_{ij} = C_{el} |Y|^{-5}\left(Y_i Y_j - \frac{1}{3} |Y|^2 h_{ij}\right)$, $C_{el} \neq 0$. Then any other vector field on $(\mathscr{I}^-, h_{ij})$ extends to a KVF of $(\mathscr{M}, g_{\mu\nu})$ if and only if it is a CKVF of $(\mathscr{I}^-, h_{ij})$ which commutes with Y.*

Remark 5.7. Note that the unphysical Killing equations imply that a KVF in the physical spacetime can be non-trivial if and only if the induced CKVF on $\mathscr{I}$ is non-trivial (compare [24]).

Remark 5.8. Assume that $\mathscr{I}$ is conformally flat. Then it can be shown that there exists at least one independent CKVF ζ which commutes with Y. It then follows from lemma 5.6 that the emerging spacetime admits at least two KVFs. This provides a simple proof that

$\Lambda > 0$-vacuum spacetimes with vanishing MST and conformally flat $\mathscr{I}$ have at least two KVFs (cf [21, theorem 4]).

Moreover, we have the following

Proposition 5.9. *Let* (Σ, h_{ij}) *be a Riemannian 3-manifold which admits a CKVF* Y *with* $|Y|^2 > 0$. *Assume further that its Cotton–York tensor satisfies* $\widehat{C}_{ij} = C\,|Y|^{-5}\Big(Y_i Y_j - \frac{1}{3}\,|Y|^2 h_{ij}\Big)$, $C = $ const. *Then* (Σ, h_{ij}) *admits a second, independent CKVF* ζ *which commutes with* Y.

Proof. One more time we use a spacetime argument: There exists a $\Lambda > 0$-vacuum spacetime $(\mathscr{M}, g_{\mu\nu})$ with a KVF X such that the associated MST vanishes, such that (Σ, h_{ij}) can be identified with past null infinity, and such that $X^i|_\Sigma = Y^i$. It follows from the classification results in [21] that $(\mathscr{M}, g_{\mu\nu})$ admits a second independent KVF which commutes with X, and which induces a CKVF on $\mathscr{I}^-$ with the asserted properties. $\qquad\square$

Remark 5.10. The second CKVF ζ may or may not be ς as given in definition 5.1. The statement of the proposition is that, even when ς happens to be linearly dependent to Y, there is still another independent CKVF on (Σ, h_{ij}).

Proposition 5.9 might be useful to classify Riemannian 3-manifolds which admit a CKVF which is related to the Cotton–York tensor via (2.74).

Acknowledgments

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