Universidad del País Vasco / Euskal Herriko Unibertsitatea

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Boltzmann-en banaketa: aplikazio ekonomikoak / La distribución de Boltzmann: aplicaciones económicas

The Boltzmann distribution: Economic Applications

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Uztaila 2017/Julio 2017/July 2017

The Boltzmann distribution: economic applications

Abstract

This work is focused on two applications to economy of the Boltzmann probability distribution of statistical mechanics. The first one describes the distribution of money while the second one the allocation of CO_2 emission permits. The first application has been the instructive model in our case as it involves a very simplified idealization of the economy and it has helped us to submerge into the world of Econophysics. We have digged much deeper into the second application from which we have suggested a new hypothetical application for TAC allocation in fisheries. One way or another, all cases studied consist of partitioning of a limited resource among multiple agents.

Keywords: Econophysics, Boltzmann distribution, money distribution, allocation of permits, partitioning, entropy maximization.

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1 Introduction

Econophysics is a new interdisciplinary research field where physicists apply statistical physics methods to problems in economy. By means of mathematical methods from statistical physics, econophysics studies statistical properties of complex economic systems consisting of a large number of interacting economic agents. It applies to economic and financial data, increasingly available with the development of computers and the Internet. These applications are based on agent-based modeling[1] and simulations, where concepts such as scaling and universality are used.

Indeed, economic and financial systems suffer from the same limitation as systems in astrophysics, atmospherics physics and geophysics in the sense that principles of the theory of probabilities are needed to describe these systems. This is so because properties from microscopic equations of motion cannot be derived. Further, the empirical analysis performed on financial or economic data is not equivalent to the usual experimental investigation carried out in physical sciences. Thus, it is impossible to perform large-scale experiments in economics and finance that could verify a given theory.

Historically, as far as we know, back in 1936, Ettore Majorana stated for the first time the analogy between statistical laws in physics and in social sciences, and this is the point where interest in financial and economic systems originates. By that time, this was considered an unconventional perspective, but with time both physics and economics became more formal in their research, and the relation between social sciences and statistical physics was left behind.

It was not until 1974 that the statistical physicist Elliott Montroll coauthored the book Introduction to Quantitative Aspects of Social Phenomena (Montroll and Badger (1974)). Another early attempt to bring together the leading theoretical physicists and economists at the Santa Fe Institute was not entirely successful (Anderson, Arrow, and Pines (1988)). Physicist Stanley was the first to use the term "econophysics" arguing that "behavior of large numbers of humans (as measured, e.g., by economic indices) might conform to analogs of the scaling laws that have proved useful in describing systems composed of large numbers of inanimate objects" (Mantegna and Stanley (1999)). Although the actual status of econophysics within physics and economics fields is still diffuse, it can be said that for the late 1990s, the attempts to apply statistical physics to social phenomena consolidated into a

[1] Agent-based model (ABM) is a type of computational model for simulating actions and interactions of autonomous agents (both individual or collective entities) with a view to assessing their effects on the system as a whole.

solid movement of econophysics and sociophysics.[2] Econophysics focuses on the economic behavior of humans, where more quantitative data is available, while sociophysics studies social issues. The boundary between these two topics is not that sharp, and both fields are frequently mixed (Chakrabarti, Chakraborti, and Chatterjee (2006)).[3]

On the other hand, the number of economists following this tendency is growing, but still most of the econophysics papers have been published in physics journals: (i) The journal Physica A: Statistical Mechanics and its Applications has emerged as the leader in econophysics publications and has even attracted submissions from some well known economists. (ii) Several well-known economists, Blume (1993) and Foley (1994) among them, applied statistical physics to economic problems. Gradually, reputable economics journals as Games and Economic Behavior and Journal of Economic Theory are also starting to publish econophysics papers. (iii) There is a Physics and Astronomy Classification Scheme PACS number for econophysics, and Physical Review E has published many papers on this subject. (iv) Econophysics sessions are included in the annual meetings of physical societies and statistical physics conferences (see http://www.unifr.ch/econophysics/.)

When modern econophysics began to emerge in the middle of 1990s, analysis of financial markets attracted most of the attention. The financial branch of econophysics applies concepts from probability theory to financial time series in order to gain new understanding of how financial markets behave. Financial markets are exceptionally well defined complex systems. They can be constantly followed in time scales of seconds while every economic transaction is virtually recorded and there is a huge amount of this data available for research. Financial markets are open systems that present many of the properties that characterize complex systems in which many subunits interact nonlinearly in the presence of feedback.

Another direction within economics is the application of statistical mechanics to social and economic inequality. Entropy maximization concept brings up the Boltzmann distribution of energy, which is the fundamental equilibrium law in statistical mechanics based on the conservation of energy and it used in a number of fields. This fact suggests the possible generalization that any conserved quantity in a big statistical system follows such probability distribution in equilibrium. Following this idea Drăgulescu and Yakovenko (2000) study the probability distributions of money, wealth, and

[2] Sociophysics is the study of social and political behavior using tools and concepts from the physics of disordered matter.

[3] The New Palgrave Dictionary of Economics (Rosser, 2008) includes an entry on econophysics.

income in a closed society.[4] More recently the Boltzmann distribution has also been applied to allocation problems (see Park *et al.* (2012).)

This work is focused on two applications to economy of the Boltzmann probability distribution of statistical mechanics, where the two share the objective of **partitioning a limited resource among multiple agents**. The first application describes the distribution of money based on the pairwise money transfer models presented in Drăgulescu and Yakovenko (2000). Due to the fact that it involves a very simplified idealization of the economy, this application has been a much more instructive exercise in order to submerge into the world of Econophysics. We have studied much more into depth the second application, carried out by Park et *al.* (2012), that describes the allocation of CO_2 emission permits through the Boltzmann distribution. After developing different exercises for a certain empirical data, we have outlined a hypothetical model for TAC allocation in fisheries in the European Union, application suggested by Park et *al.* (2012).

The work is organized as follows: In Section 2 some basic notions about the Boltzmann probability distribution are introduced. Section 3 contains models of the application of the distribution of money. Section 4 details in first place the application of trading emission permits to pollute. Different exercises are carried out in order to analyze the properties of the distribution. The last part of this section briefly describes an application to the allocation of fishing quotas in the European Union. Section 5 brings out the results and conclusions from the work. The Appendix in the Supplement, formally develops statistical mechanics required to understand these applications regarding the Boltzmann distribution of energy.

2 The Boltzmann distribution

In this section we are going to informally describe the Boltzmann distribution of energy and the concepts used throughout the work. For their formal derivation,see the Appendix in the Supplement.

Let us consider an isolated system with a great number of particles that can occupy one of the energy levels E_i. At any time t, the particles are distributed among the different levels so that n_0 particles have energy E_0, n_1 particles have energy E_1 and so on. The total number of particles is

[4] See Banerjee and Yakovenko (2010) and Yakovenko and Barkley (2009). See also http://physics.umd.edu/~yakovenk/econophysics/

$N = n_0 + n_1 + ...$ and given that the system is isolated, the total energy remains constant so that $E = n_0E_0 + n_1E_1 + n_2E_2 + ...$ Due to interactions and collisions between the particles, numbers $n_0, n_1, n_2...$ are continuously changing. We can suppose that for each macroscopic state of the system, there is a distribution of the particles among the different levels that is more probable that any other. Once this distribution is reached, the system is said to be in *equilibrium.* So numbers $n_0, n_1, n_2...$ can then fluctuate around the equilibrium state without making any macroscopic change. Let us determine the way in which the particles of an isolated system distribute among the permitted energy levels.

In classical statistics, the particles of a system are identical but indistinguishable. That means, that at macroscopic scale we cannot distinguish a particle from the other, but we can imagine that at microscopic scale we have some sort of procedure to identify the individual particles. It is important to differentiate the macrostate and the microstate of the system. Macroscopic properties, such as pressure and temperature, are determined by the macrostate of the system. The corresponding microstates to such macrostate are experimentally indistinguishable between them. The number of microstates that correspond to the given macrostate is represented by Ω and is:

$$\Omega = \frac{N!}{n_0!n_1!n_2!.....} \qquad (1)$$

Postulate 1; Equiprobability a priori: A closed system[5] in equilibrium has the same probability to be in any of it's accessible[6] microstates.

In other words, probability to find the system in state r is $P_r = \frac{1}{\Omega}$. Thus, the macrostate that has the largest number of microstates is the most probable macrostate of the system. Distribution $\{n_0, n_1, n_2, n_3...\}$ for which Ω has it's greatest value, is the *most probable distribution* of the particles of the system among its energy levels (equilibrium state of the system).

In a system in thermal equilibrium, interactions between particles (collisions) change the system towards a macrostate that has practically the same number of microstates than the initial one. In order to find the most probable distribution of an isolated system, the number of microstates Ω has to be calculated first so that the total energy remain constant. Maximum value of Ω is obtained when the distribution of particles n_i is a decreasing function

[5] Closed system has energy E, number of particles N and volume V all constant.

[6] Accessible microstates meaning to have compatible properties with the properties specified from the system: energy within the system's energy interval, equal number of particles to represent the number of particles of both the microstate and the system.

of the energy levels E_i:

$$n_i = C \cdot \exp(-\beta E_i) \qquad (2)$$

Positive quantity β determines the speed at which n_i decreases when energy E_i increases. Constant C is determined so the total number of particles that occupy accessible energy levels is N. Function (2) is called the **Boltzmann distribution of energy** and provides the most probable distribution in which particles in a physical system in equilibrium occupy the i-th states with energy E_i.

Based on the Boltzmann distribution, the probability P_i that a system can be found in a certain state i with energy E_i is inversely proportional to the exponential function of the energy:

$$P(\varepsilon_i) = C' \cdot \exp(-\frac{\varepsilon_i}{k_B T}) \qquad (3)$$

where C' is a normalization constant, T is the temperature of the system and k_B is the Boltzmann constant.[7] Expression (3) shows that the states with lower energy will have higher probability of being occupied than the states with higher energy. The system in this case may range from an atom, to a gas in a tank, and this is the reason why the Boltzmann distribution is very useful and can be applied in such a variety of cases.

In addition, it is known from thermodynamics that an isolated system tends to its equilibrium state by maximization of its entropy S. Entropy S is the measure of disorder in the system. Such disorder in physical statistics can be translated as the measure of number of accessible microstates (measure of unpredictability). The process of entropy maximization in statistical physics translates as an isolated system maximizing the number of accessible microstates Ω (or $\ln \Omega$) given by the following relation:

$$S = k_B \ln \Omega \qquad (4)$$

Expression (4) is the bridge between thermodynamics (macrostate) on the left and microscopic world on the right. Note that when $\Omega = 1$ ($S = 0$) there is a single microstate accessible to the system and the microscopic state of the system is completely predictable. According to this, we might introduce the *maximum entropy principle* that states that the probability distribution that best represents the current state of knowledge is the one with the largest entropy.

[7] This could be generalized to the continuous case: $P(\varepsilon)d\varepsilon = C' \cdot \exp(-\frac{\varepsilon}{k_B T})d\varepsilon$, which is the probability of the system to be in a state with energy between ε and $\varepsilon + d\varepsilon$.

Given the simplicity of the derivation of (2) *i.e.* the statistical character of the system and the conservation of energy, Drăgulescu and Yakovenko (2000) wondered if any conserved quantity in a big statistical system in equilibrium follows the Boltzmann probability distribution. In the same way that statistical mechanics studies collections of atoms, economy considers a huge amount of participating agents. For that reason, economy may be perceived as big statistical system and the Boltzmann distribution can be naturally applied.

3 The Boltzmann distribution of money

In this section we discuss some applications in economics of the Boltzmann probability distribution carried out by Drăgulescu and Yakovenko (2000).

Given that energy is the conserved quantity in statistical mechanics, *money* is now considered to be the conserved quantity in a closed economy with total amount of money M and many economic agents N. Therefore, if in statistical mechanics energy follows the Boltzmann distribution that gives probability distribution of energy $P(\varepsilon) = C \cdot \exp(-\frac{\varepsilon}{T})$, we wonder whether or not money follows such distribution

$$P(m) = D \cdot \exp(-\frac{m}{T_m}) \qquad (5)$$

where m is the money, D a normalization constant and T_m is the "money temperature" which are found from normalization conditions

$$\int_0^\infty P(m)dm = 1 \text{ and } \int_0^\infty m \cdot P(m)dm = \frac{M}{N} \qquad (6)$$

from which $D = \frac{1}{T}$ and $T_m = \frac{M}{N}$ are calculated.[8] From the preceding relation, "money temperature" T_m represents the average amount of money per economic agent.

This equilibrium distribution $P(m)$ is derived in the same way as the equilibrium distribution of energy $P(\varepsilon)$ in statistical mechanics. One way is the derivation followed in physics by bringing two subsystems into thermal contact (see Supplement) while the energy of the composite system remains constant, but just taking into account that money is conserved and additive: $m = m_1 + m_2$. $P(m)$ can also be derived within the canonical ensemble

[8] In this case, the Boltzmann constant k_B is set to unity so the temperature is measured in energy units and defines the average energy per particle $T \sim < \varepsilon >$ up to a numerical coefficient of the order of 1.

(see Supplement) by means of maximization of entropy. This leads to the definition of the entropy of money distribution

$$S = \int_0^\infty dm P(m) \cdot \ln P(m) \qquad (7)$$

which when maximized with respect to the total number of economic agents in a bin (as in state i in physics) with the constrains of the total money M and the total number of agents N, generates the Boltzmann distribution for $P(m)$.

It is important to remark the difference between money distribution and wealth distribution. Money is just a fraction of the total wealth while wealth includes non conserved material products that can be manufactured, destroyed and consumed (foods for example can be eaten or can go rot and disappear). Measuring units also bring this lack of conservation up. As opposed to goods, money is always measured in a the same unit. Thus only money flow is going under study here.

3.1 Pairwise money transfer without debt

Now we are going to model a very simplified economic situation for the study of the probability distribution of money $P(m)$ among economic agents. Pairwise money transfer models represent a primitive market. Even in modern economies dominated by big enterprises, these models are attractive because of their simplicity as they result very instructive. We consider that the economy is in equilibrium and therefore we are searching for a *stationary probability distribution of money* $P(m)$. First, idealizations must be done in order to guarantee system's stability and statistical equilibrium in the model:

The process we are going to consider is the economic transaction between agents where agent i pays the amount of money Δm to the agent j. The process requires the following assumptions:

A.1: The number of participating economic agents (individuals or corporations) in the system is fixed and very large $N \gg 1$. Money can only be transferred between these economic agents.

A.2: Money manufacture is not permitted. An increase in material production does not produce an automatic increase in money fund. Only a central bank has the monopoly of changing the money balance. Money transaction is given in one country with only one currency, so the total amount of money in the system is conserved. Then the economy under study is a closed system. This is the same as physics starts from the study of ideal closed systems as in thermal equilibrium and then generalizes by getting rid of restrictive conditions.

A.3: Money is on exchange of economic activity and it is used as means of exchange goods or services. We do not consider the transaction of goods or services since they are not perdurable.

A.4: Debt (considered as negative money from individual economic agents' point of view) is not allowed as it may disturb money conservation. In analogy with physics where kinetic energy of atoms is $\varepsilon_i \geq 0$, agent's money $m_i \geq 0$ for all i. Agent i cannot buy anything from other agents if has no money $m_i = 0$, but can receive money from other agents.

A.5: In order money transaction to happen, we assume each agent has enough money to pay $m_i \geq \Delta m$.

The agents' money balances develop as:

$$m_i \longrightarrow m_i' = m_i - \Delta m \text{ and } m_j \longrightarrow m_j' = m_j + \Delta m \qquad (8)$$

This equation in economy corresponds in physics to the transfer of energy from one molecule to another in molecular collisions in a gas.. Hence the money is conserved during the transaction:

$$m_i + m_j = m_i' + m_j' \qquad (9)$$

3.1.1 Computer simulations

In order to justify these conjectures, Drăgulescu and Yakovenko (2000) carried out some computer simulations of money transfers between agents with the following characteristics:

All agents start with the same amount of money. For example \$1000, as represented by the double vertical line in Figure 1.

Pair of agents i and j are selected randomly and transaction begins: agent i (randomly chosen to be the "loser") transfers amount $\Delta m \geq 0$ to agent j (randomly chosen to be the "winner") as represented in (8). Recall that if loser i does not have enough money to pay ($m_i < \Delta m$), then money transaction does not happen, and another pair of agents must me picked. Transaction process is repeated many times.

Several different exchange rules for Δm were considered in simulations by Drăgulescu and Yakovenko (2000): (a) small fixed amount $\Delta m = 1$, (b) random fraction v ($0 \leq v \leq 1$) of the average money in the system $\Delta m = v\frac{M}{N}$and (c) random fraction v ($0 \leq v \leq 1$) of the average money of the pair $\Delta m = \frac{v(m_i+m_j)}{2}$.

The final stationary distribution was found to be the same for all three trades.

Time evolution of the distribution of money is illustrated in computer animations by Chen and Yakovenko (2007) and by Wright (2007):
http://demonstrations.wolfram.com/StatisticalMechanicsOfMoney/

(a) Transferring fixed constant quantity $\Delta m = \$1$ represents the economical case where all agents sell their products for the same price (\$1)[9]. Figure 1 shows six different snapshots from computer animation 1.

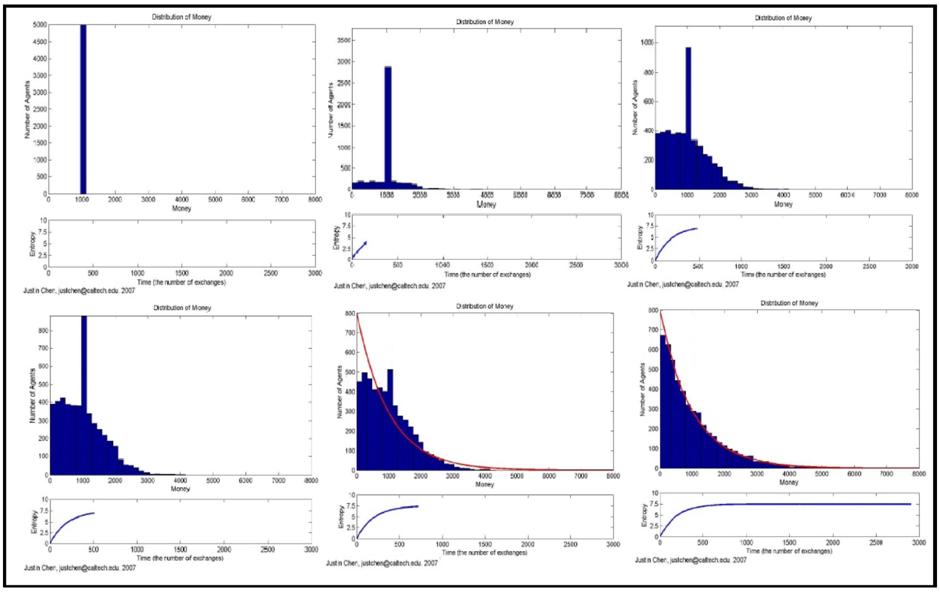

Figure 1: Time evolution of distribution of money and of entropy represented by six different moments in computer animation 1. For each instant, the figure on the top is the distribution of money (histogram). The figure underneath is the entropy as a function of time. Solid red curve fits the Boltzmann distribution.

While agents exchange money, the initial delta-function distribution $P(m) = \delta(m - \frac{M}{N})$ first broadens symmetrically to a Gaussian curve as shown in Figure 1. Due to the imposed boundary condition of no debt $m \geq 0$, probability distribution starts to concentrate around $m = 0$ state. As a result, $P(m)$ becomes biased (asymmetric) and in the course of time, money distribution reaches the expected stationary exponential shape as shown in Figure 2:

[9] Agents paying the same prices Δm for the same products, independent of their money balances m, seems very appropriate for the modern anonymous economy, especially for purchases over the Internet.

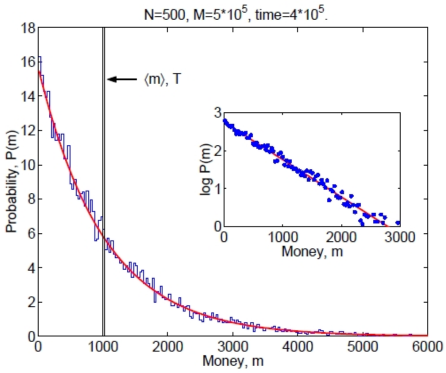

Figure 2: Histogram and points: stationary probability distribution $P(m)$. Solid red curve fits the Boltzmann distribution $P(m)$. Vertical lines represent the initial distribution of money \$1000.

(b) Transferring $\Delta m = v(\frac{M}{N})$ a random fraction of the average amount of money per agent, where v is a uniformly distributed random number between 0 and 1, represents a wide variety of prices for different products in real economy. Agents can buy and consume varied quantities among a variety of products (simple or complex, cheap or expensive...). In Figure 3, six snapshots from computer animation 2 are shown in order to see time evolution. Again, with time, the same distribution is reached.

(c) Transferring $\Delta m = \frac{v(m_i+m_j)}{2}$ a random fraction of the average amount of money of the two agents, where v is a uniformly distributed random number between 0 and 1, also reaches the same stationary probability distribution in (5) with time.

As shown in the bottom figure of Figures 1 and 3, computer animations also represent the growth of the entropy of money distribution (7): S starts from initial value 0 where all agents have the same money. When transactions starts S increases with time until saturation at the maximal value when reached statistical equilibrium. In the Figure 4 we can compare time evolution of entropy for the presented exchange models (a) and (b) in the text.

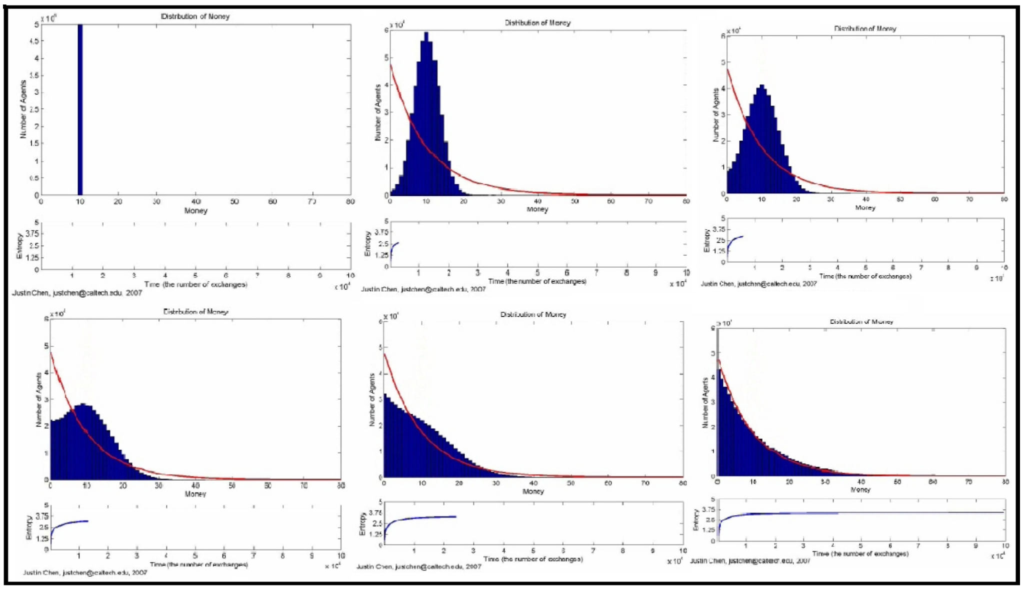

Figure 3: Time evolution of distribution of money and of entropy represented by six different moments in computer animation 2. For each instant, the figure on the top is the distribution of money (histogram). The figure underneath is the entropy as a function of time. Solid red curve fits the Boltzmann distribution.

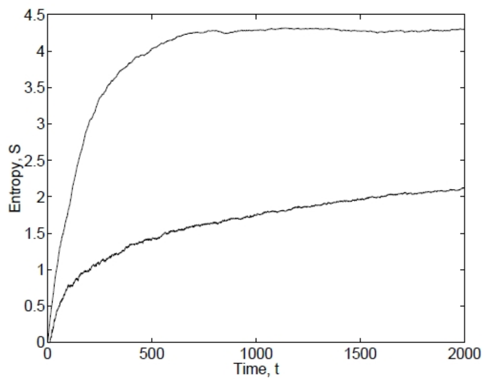

Figure 4: Time evolution of entropy. Top curve shows entropy time-evolution computed for the model of exchange (b) of the text while the bottom curve for model (a). The time scale for the bottom curve is 500 times greater than indicated so ends at the time 10^6. Thus, exchange in model (a) is much slower than in model (b), but systems in both models eventually reach the Boltzmann state of maximal entropy.

Let us point out the following observations:

1. Boundary condition at $m = 0$ represents the ground state of energy in statistical mechanics. Without this boundary condition, the probability distribution of money would not reach a stationary state.

2. Money temperature T_m is equal to the average amount of the available money per agent $\frac{M}{N}$, thus it depends on the boundary conditions.

At this point, despite the differences in rules for transfer Δm , the final probability distribution of money (5) for pairwise money transfer seems to be universal in these settings but as we shall see, this is not the case.

3.2 Models with debt

In cases up until now, debt was not permitted as in theory disturbed money conservation. We proceed to consider two cases of pairwise money transfer Δm between agents i and j where now debt is allowed $m_i < 0$ and check whether probability distribution of money $P(m)$ still tends to the exponential function (5).

Money conservation is the main principle not to skip. Following assumptions are required for these processes:

A.1: The bank is a big reservoir of money apart from the system of ordinary economic agents.

A.2: Transactions may happen when agents have no money to pay Δm by borrowing money from the bank. Then agent's money balance (positive money M) increases while earns a debt (negative money D). Total money balance of the agent remains the same. Thus, borrowing money from the bank still satisfies the generalized conservation law of the total money represented by money balance: $M_b = M - D$ where M_b is the monetary base (money in the central bank and not available to the public).

A.3: Interests rates on borrowed money are not considered.

A.4: When an agent with a negative balance receives money, the agent uses this money to repay the debt until balance becomes positive.

Observe that $m = 0$ is not the ground state any more in neither of the following models.

3.2.1 Unlimited debt without any restrictions on agents' debt

As in (8), transaction now sets a new negative balance of agent i :

$$m_i' = m_i - \Delta m < 0 \qquad (10)$$

First simulations were carried out without any restrictions on debt and the results were that the probability distribution of money $P(m)$ never stabilizes. As opposed to cases without debt ($m \geq 0$), now $P(m)$ distribution with time keeps spreading symmetrically to a Gaussian without limits towards $m = +\infty$ and $m = -\infty$ and the system never reaches a stationary state. The fact of debt being unlimited, or not having any boundary condition (like $m = 0$ before) is clearly the reason why $P(m)$ never reaches the exponential form.

This model of unlimited debt in a system represents financial crisis where some agents go into debt with negative balances $m < 0$ and without producing anything in exchange while other agents become richer with positive balances $m > 0$.

3.2.2 Limited agents' debt

In order to prevent the growth of debt with no limits and so to ensure the stability of the system, new boundary conditions must be imposed. Drăgulescu and Yakovenko (2000) made computer simulations of a simple model as in the previous section with boundary condition for all agents i, $m_i \geq -m_d$ where m_d is the debt limit of each agent. Results are shown in Figure 5. As expected, the stationary money distribution $P(m)$ recovers the exponential shape, but now with the new boundary condition at $m = -m_d$ instead of at $m = 0$.

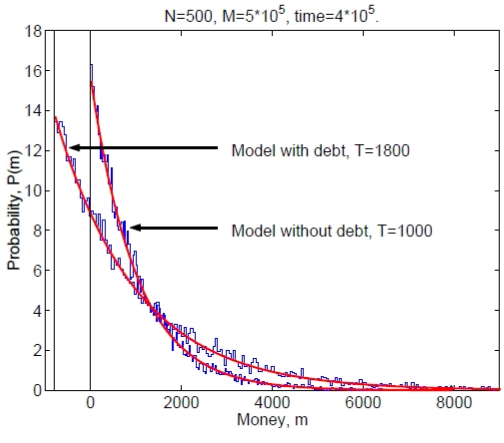

Figure 5: Computer simulation for two different cases of limited debt: curve for $m_d = 800$ and curve for $m_d = 0$.

By allowing agents to go into limited m_d debt, the amount of money available to each agent has increased m_d. As a consequence, money temperature

($T_m = m_d + \frac{M_b}{N}$) increases correspondingly. The higher the temperature, the broader money distribution. This means that debt increases inequality between agents.

We have presented through distinct idealized models how the Boltzmann probability distribution does not necessarily hold for any conservative model in economic settings.

Other exchange models have been presented in the econophysics literature, and different distributions have been found. Drăgulescu and Yakovenko (2000) carried out further studies in the universality of the money distribution. They brought up symmetry conditions of the models showing that the Boltzmann distribution (5) can be derived from the Boltzmann kinetic equation. It turns out that the Boltzmann distribution is the stationary solution of the Boltzmann kinetic equation. Nevertheless, in the absence of detailed knowledge of real microscopic dynamics of economic exchanges, the Boltzmann distribution seems to be an appropriate starting point.

4 The Boltzmann distribution applied to allocation of permits

In this section we present the application of the Boltzmann distribution to the allocation of CO_2 emission permits based on the paper by Park *et al.* (2012). The paper contains an empirical application of the model for years 2007-2008. After describing the theoretical model, we do not go into their results but we do update the empirical application for years 2010-2011. In addition we develop two new exercises with the aim of a deeper study of the properties of the Boltzmann distribution. Finally, following the work by Park *et al.* (2012), we outline a possible application of the Boltzmann distribution to fisheries.

It is known that CO_2 emissions to the atmosphere cause irreversible damage in our planet. International reaction is needed in order to confront such problem and world CO_2 emissions must be reduced to slow global warming down. The *Kyoto Protocol* is an international environmental treaty whose aim is to stabilize atmospheric greenhouse gas concentrations to prevent dangerous interference with climate. It was embraced on December 1997 in Kyoto (Japan) and entered into force on February in 2005. Without impositions, it requires to set limits for individual countries on greenhouse gas emissions. There are 197 countries that meet annually in conferences to deal with climate change and discuss how to achieve the treaty's aims. The Protocol requires the developed countries to reduce current emissions that are historically guilty for the actual greenhouse gases levels in the atmosphere.

One of the proposals to control pollution is a market-based system known as *emission permits trading*. This system allows countries to decide how to accomplish policy targets, and works as follows: a central authority allocates or sells a limited number of permits to pollute. Polluters are required to hold permits in equal amount to their emissions. The ones that want to increase their emissions must buy permits to the ones that wish to sell them. In general, these permits are sold in the international market at the prevailing market price, so there is permit transfer between countries. Indeed the flow and value of what is traded depends on its initial allocation.

The *initial allocation of permits* is a disputed task as it has to be *fair* for the countries that take part in the system. The total amount of permits may be distributed in different ways among participants. But the key question is who has the right to clean away air. If the answer is that industry has the right to pollute, permits are allocated for free based on past pollution. But if no one has the right to pollute without compensating society, then polluters have to pay for it. There are two known proposals: (i) Grandfathering allocates permits for free based on historical emissions. (ii) Auctioning of the permits consists of selling permits to polluters that bid highest.

Both systems have limitations: Grandfathering rewards those firms that have polluted excessively in the past. Auctioning may damage to countries with incipient industry, which would hardly get any permit.

Park et al. (2012) argue that Boltzmann probability distribution provides a simple and natural rule for allocating emission permits among countries by describing the *most probable distribution* of emissions permits.

4.1 The theoretical model of allocation of CO_2 emission permits

First, we present the replacement of the concepts of the Boltzmann distribution from statistical mechanics (3) into the emission permit allocation frame.

4.1.1 Variable definition

-The *system* in statistical mechanics is reinterpreted as CO_2 *emissions trading system.*

-*Particles* in statistical mechanics correspond to *emission permits units.*

-The $i-th$ *state* in statistical mechanics is the $j-th$ *individual* in $i-th$ *country.*

-*Energy* ε_i *of the* $i-th$ *substate* in statistical mechanics corresponds to *energy per capita* E_i *of the country* i.[10] E_i of country i is negatively proportional to the actual CO_2 emissions per capita of country i.

An essential *assumption* is required: all individuals in a country make the same contribution to the total CO_2 emissions of such country i. Therefore, the probability that a unit emission permit is allocated to a country i should be proportional to its total population C_i.

4.1.2 The distribution

In accordance to the variable replacement the Boltzmann distribution for permit allocation is defined as the probability P_i that a unit permit is allocated to a country i, which for now, we might only say is inversely proportional to the exponential function of the allocation energy per capita E_i:

$$P_i \ \alpha \ C_i \exp(-\beta E_i), \ \beta > 0 \qquad (11)$$

where positive constant β corresponds to the positive temperatures of a system in physics.

The authors present a diagram that illustrates the idea of permit allocation using the Boltzmann distribution that we reproduce in the following figure:

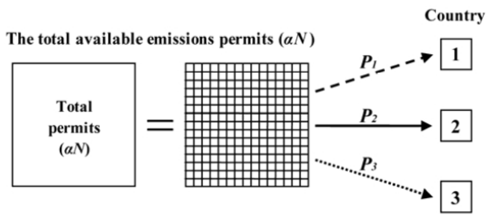

Figure 6: Permit allocation using the Boltzmann distribution. The total available emissions permits αN (first box) is divided into N (number of unit emissions permits) pieces of unit permit α (same box, different distribution). Then the emissions permits are allocated to country i ($i = 1, 2, 3$), based on probability P_i.

To develop the model the following ingredients are also required:

- There are n participating countries: each country i ($i = 1, 2, 3, \ldots, n$) has population C_i and energy per capita E_i.

[10] E_i is now an intensive variable (independent to the size of the system) for the country i.

- αN is the total available emission permits allocated to the countries where α is the unit permit and N is the total number of permits. (Note that by making unit α sufficiently small, the number of unit permits N can be made large enough for the Boltzmann statistics to be applied).

- N_i^j represents the number of unit permits that are allocated to the $j-$th individual in a country i which depends on its energy per capita E_i.

- During the allocation process, emission permits cannot be taken away and cannot be added. The total number of available unit permits N is conserved. So we add the number of unit permits that are allocated to the $j-$th individual in country i N_j^i, for all population in the country C_i and for all n countries:

$$\sum_{i=1}^{n}\sum_{j=1}^{C_i} N_i^j = N \qquad (12)$$

- Let E be the total global allocation energy which must be conserved. E has to be regulated based on the agreement among the participating countries and it controls the general distribution of the emissions permits over countries. It is calculated by the product of the allocation energy per capita of a country i E_i and the number of unit permits allocated to the $j-$th individual in a country i N_i^j all added for all population of a country i for all n participants countries:

$$\sum_{i=1}^{n}\sum_{j=1}^{C_i} (N_i^j E_i) = E \qquad (13)$$

As in statistical mechanics, the Boltzmann distribution for permit allocation is valid under constraints (12) and (13). Further, since the Boltzmann distribution is based on entropy maximization, the greatest number of unit permit that individual $j-$th in country i can receive is defined as:

$$N_i^j = A\exp(-\beta E_i) \qquad (14)$$

with constant A. Note that N_i^j is only a function of energy per capita of E_i. Constant A can be calculated by means of restriction in (12):

$$\sum_{i=1}^{n}\sum_{j=1}^{C_i} N_i^j = \sum_{i=1}^{n}\sum_{j=1}^{C_i} A\exp(-\beta E_i) = \sum_{i=1}^{n} AC_i\exp(-\beta E_i) = N \qquad (15)$$

Solving for A:

$$A = \frac{1}{\sum_{i=1}^{n} C_i\exp(-\beta E_i)} \qquad (16)$$

Therefore the **probability that a unit permit is allocated to a country** i is the ratio between all the permits allocated to a country and the total number of the available permits:

$$P_i = \frac{\sum_{j=1}^{C_i} N_i^j}{N} = \frac{C_i N_i^j}{N} = \frac{C_i \exp(-\beta E_i)}{\sum_{i=1}^{n} C_i \exp(-\beta E_i)} \qquad (17)$$

Expression (17) verifies expression (11) in which P_i is proportional to the population C_i of country i and inversely proportional to the exponential function of the allocation energy per capita E_i.

Finally, the **amount of emission permits allocated to country** i is the product between the total available emissions permits αN and the probability P_i of getting one:

$$P_i \cdot \alpha N = \alpha N \frac{C_i \exp(-\beta E_i)}{\sum_{i=1}^{n} C_i \exp(-\beta E_i)} \qquad (18)$$

4.1.3 The β value

The β value has to be determined by all the countries in the system. Let us illustrate what the meaning of this parameter is by means of analyzing the limit cases.

1. Suppose that $\beta \to 0$, then the probability that a unit emissions permit is allocated to a country i in (17) only depends on the populations of the participating countries. Hence, for two countries with similar energy per capita E_i the country with larger population is rewarded. This situation can be interpreted as *egalitarian* (which in this setting means "all people have an equal right to pollute or to be protected from pollution") since every individual in each country is assigned with the same number of unit permits.

2. Suppose that $\beta \to \infty$, then probability in (17) takes values different from 0 only for the countries with the lowest allocation energy per capita E_i. These countries will share all the available emission permits. But all other countries with close to null probability will not be satisfied as they obtain few permits.

Recall that the allocation energy per capita E_i of a country i is assumed to be negatively proportional to the CO_2 emissions per capita of a country i (the essential assumption). Hence, from limit cases we conclude that countries with higher CO_2 emissions per capita will prefer larger values of β, while countries with lower CO_2 emissions per capita prefer smaller β values. So a proper β value can be difficult to find as there is no value that satisfies all countries.

Authors suggest a value for β, using the least square y calculation between the allocated permits and demand. When y has its minimum value at a β value, β is considered a reference point.

$$y = \sum_{i=1}^{n} (permits_i - demand_i)^2 \qquad (19)$$

4.2 An updated empirical analysis

Park *et al.* (2012) illustrate the application using data from 2007 and 2008 for Canada, China, Germany, Italy, Japan, Russia, the United Kingdom and the United States ($n = 8$). In order to see the performance of the application, we have carried out analogous calculations and graphical representations for the same countries but for 2010 and 2011 which is the latest available data.[11]

The data required are the CO_2 emissions and the population for the selected countries in years 2010 and 2011 shown in Table 1 along with the calculation of emission per capita in 2011, that is, the quotient between emissions and population. For this example the total amount of allowed permits for 2011 is the 97% of the total emissions of the preceding year: $\alpha N = 18,160,003$ (1000 metric tons) is total available permits that will be allocated among the countries. We have made the assumption that the energy per capita E_i of a country i is the negative value of CO_2 emissions per capita of a country i in 2011.

[11] Sources:
- Emissions:
http://data.un.org/Data.aspx?q=co2+emissions&d=MDG&f=seriesRowID%3a749
- Population:
http://data.un.org/Data.aspx?q=population+datamart%5bWDI%5d&d=WDI&f=Indicator_Code%3aSP.POP.TOTL

Country i	CO2 Emissions (1000 metric tons) in 2010	CO2 Emissions (1000 metric tons) in 2011	Population Ci	Emissions per capita (metric tons) (-Ei)
Canada	496.105	485.436	34.342.780	14,14
China	8.256.969	9.019.518	1.344.130.000	6,71
Germany	750.697	729.458	81.797.673	8,92
Italy	405.361	397.994	59.379.449	6,70
Japan	1.168.919	1.187.657	127.817.277	9,29
Russia	1.742.540	1.808.073	142.960.868	12,65
UK	492.192	448.236	63.258.918	7,09
US	5.408.869	5.305.570	311.718.857	17,02
Total	**18.721.652**	**19.381.942**	**2.165.405.822**	**82,51**

Table 1: The carbon dioxide emissions in 2010 and 2011, population in 2011 and CO_2 per capita in 2011.

As we have said, agreement on the value of β is crucial in the allocation. The following graph shows how this parameter affects the amount of emission permits for each country. We consider a reasonable range of β values $[0, 3]$ and we plug in the data from Table 1 into expression (18) for each country. The result is largely affected by the β value, as shown in Figure 7.

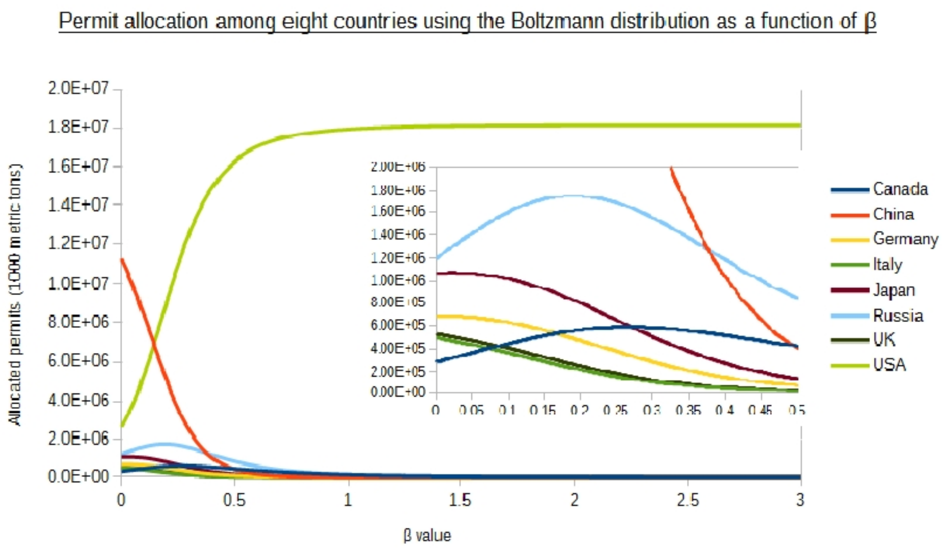

Figure 7: Permit allocation in countries using the Boltzmann distribution as a function of the β value. A zoom-in has been done in order to appreciate the curves in the more denser area, shown on the right side of the image.

From Figure *7* we can see the following situation:

1. In the United States (light green in Figure *7*) the amount of emissions permits allocated are least when β is 0. The amount increases rapidly up to $\beta \sim 0.5$, and beyond that it increases slower until approaches the total amount of available emissions permits. Permits allocated to China (red) are most when β is 0. The amount rapidly decreases up to $\beta \sim 0.5$ and then approaches 0. In Italy and in the UK permits allocated are the largest when the β is 0 and then the amount monotonously decreases up to the $\beta \sim 0.3$. For the rest of the countries (Canada, Germany, Japan and Russia) the amount of permits first increases and then gradually decreases to 0 in the given β value range.

2. The United States meets its demand in 2011 (see Table *1*) when β is 0.0973. Therefore, the United States prefers larger β values than 0.0973. Canada prefers β values between 0.1273 and 0.7636 (between the two values of β at which the allocated permits meet their actual CO_2 emissions in 2011). China prefers β values smaller than 0.0462; Italy prefers β values smaller than 0.0779 and the United Kingdom prefers smaller βs than 0.0701. The three remaining countries (Germany, Japan, Russia) cannot obtain sufficient permits to meet their actual emissions in 2011. The reason for that is that the total allowed emissions permits in 2011 (97% of 2010 emissions: 18,160,003) is less than the total CO_2 emissions in 2011 (19,381,942).

3. Another result that can be seen is the permit allocation dispute between China and the United States. China produces the greatest CO_2 emissions, but has the smallest CO_2 emissions per capita due to its huge population while the United States has the greatest CO_2 emissions per capita (see Table 1). This observation brings up a challenging question in the reduction of the CO_2 emissions. Which country should be most responsible for future reduction: the one that produces the greatest CO_2 emissions overall (China), or the one that produces the greatest CO_2 emissions per capita (the United States)? In permit allocation by means of the Boltzmann distribution, the responsibility can be shifted between both by making changes in the β value, as shown in Figure 7. When β is 0.1399, both countries have equal amount of permits. Below that value, China receives more permits than the United States so the United States would be more responsible for pollution. Above such value the United States receives more permits than China, leaving more responsibility to China. This result shows the flexibility of applic-

ation of the models of the Boltzmann distribution and the importance of determining a proper β value.

Next, as suggested, we derive the least square calculation for β between the allocated permits (results from Figure 7) and demand (actual CO_2 emissions in 2011). The least square value y has its minimum at the β value of 0.0880, at which the difference between the allocated permits and demand becomes the smallest for the eight participating countries. This β value provides a reference point for the permit allocation.

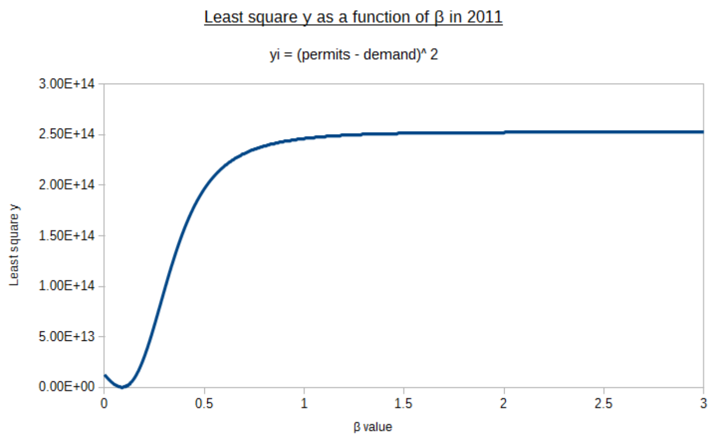

Figure 8: Function $y(\beta)$ where its minimum yields for $\beta = 0.0880$.

Results for β=0.0880

Considering the value for β of 0.0880 Table 2 summarizes the results of emissions permit allocation. The probability P_i values for permit allocation are distributed between 0.02 and 0.48 for the eight countries. The number of emissions permits allocated to the countries ranges from 383,672 to 8,692,547 (1000 metric tons).

Country i	2011 CO2 Emissions (1000 metric tons)	Boltzmann Probability Pi for B = 0.0880	Allocated permits (1000 metric toncs) to country (for B = 0.088)	Difference in 2011= allocated – emitted
Canada	485.436	0,02	427.071	-58.365
China	9.019.518	0,48	8.692.547	-326.972
Germany	729.458	0,04	642.552	-86.906
Italy	397.994	0,02	383.672	-14.323
Japan	1.187.657	0,06	1.037.284	-150.373
Russia	1.808.073	0,09	1.559.328	-248.745
UK	448.236	0,02	423.010	-25.226
US	5.305.570	0,28	4.994.540	-311.029
Total	**19.381.942**	**1,00**	**18.160.003**	**-1.221.939**

Table 2: Emissions permits of the selected eight countries using the Boltzmann distribution for $\beta \ = 0.0880$: CO_2 emissions in 2011 (1000 metric tons), the Boltzmann probability P_i, the allocated permits (1000 metric tons) and the difference (allocated permits – actual CO_2 emissions in 2011).

From results in Table 2, in Figure 9 we might compare the allocated permits using the Boltzmann distribution with the reference β value ($\beta = 0.0880$) and the actual CO_2 emissions in 2011.

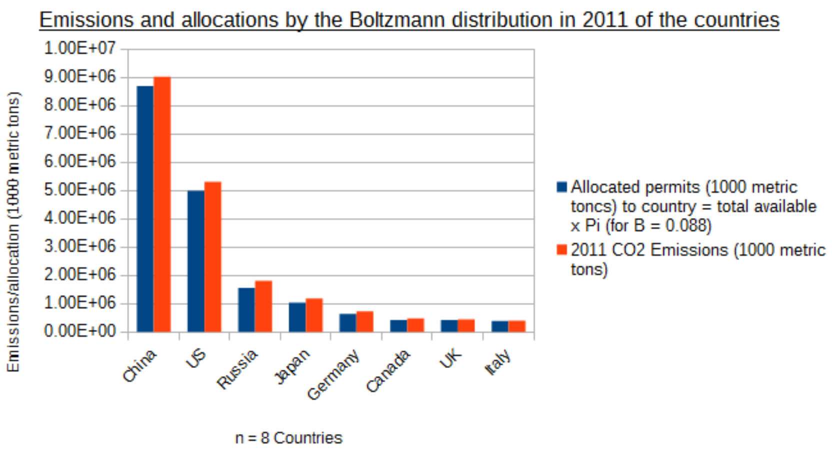

Figure 9: Permit allocation using the Boltzmann distribution with the least square reference β value ($\beta = 0.0880$). It shows the CO_2 emissions in 2011 of the participating countries and the emissions permits allocated.

Comparison between the two numerical exercises

Let us compare the results presented in the Park *et al.* paper which slightly differ from our preceding results.

1. Unlike in Park *et al.* (2012) work, we observe how none of the countries exceed their demands and that they all emitted more than the permits they got. This does not necessarily imply that for this particular example, permits cannot be traded between the selected countries. There may trade between the countries but in this case we cannot distinct at first sight which country will be the buyer and which country will be the seller. This will depend on the market price of the permits. Typically developed countries will be the buyers of the permits. Nevertheless, regarding the fact that this empirical analysis shows permit allocation among this particular eight countries, if more countries are either added or subtracted, then permit sellers and permit buyers could be different.

2. All countries seems to have changed similarly and follow the same behavior. Of course they all meet their demands for different β values due to the difference in their CO_2 emissions in 2007 and 2008. Again, the total amount of available permit is taken to be the 97% of the emissions from 2007 that make 17,084,135 permits. The only country that has changed its behavior from 2008 and 2011 is the United Kingdom where in 2008 was not able to obtain sufficient permits to meet its demand. The competition between China and the United States is still detectable.

3. The least square value of β has diminished. In Park *et al.* (2012) work y was 0.0966, while in our empirical exercise is of 0.0880.

4. The probability P_i values for permit allocation are distributed in a similar way and the number of emissions permits allocated to the countries ranges from 392,870 to 7,079,729 (1000 metric tons).

5. As opposed to the analogue problem for the date in 2010, China and the United States exceeded their demands while the remaining six countries (Canada, Germany, Italy, Japan, Russia, and the United Kingdom), did not. For the data in 2008, after the initial permit allocation, emissions trading can occur between permit sellers (China and the United States) and permit buyers (Canada, Germany, Italy, Japan, Russia, and the United Kingdom). Authors consider that it is also possible that extra emission permits can be considered and supplied through developing technology, tree planting, reducing CO_2 emissions by carbon tax during emissions trading.

4.2.1 Further analysis

With the aim of studying the behavior of the allocation of permits induced by the Boltzmann distribution we modify the empirical analysis, in two different ways: we reduce the number of countries and the targets of emission reduction. China and the United States have been left aside because they are the more polluting countries in the preceding selection. Therefore we execute the same exercise for the remaining six countries ($n = 6$ Canada, Germany, Italy, Japan, Russia and the United Kingdom). Then we perform the analysis for these six countries with a different target to analyze the consistency of the method.

a) A change in the number of countries (n) : First of all, we execute the exercise in the same way differing only in the selected countries. For this reason all of the totals in Table 1 (last row) change as shown in Table 3. We find that the total amount of allowed permits to be allocated among the countries is now $\alpha N = 4,904,140$ (1000 metric tons). Recall that this number comes from taking the 97% of the total of the CO_2 emissions in the year 2010 which is 5,055,814 (1000 metric tons). Next, we specify the results:

1. As known, the β value determines the allocation of permits but the allocation results are different. Russia is the only country that first increases the amount of allocated permits with increasing β and then gradually decreases. Canada the amount of permits allocated reaches a minimum when β is 0 and then increases gradually. The remaining four countries (Italy, Germany, Japan and the United Kingdom) reach their maximum amount when β is 0, and then monotonously decreases up to the $\beta \sim 0.5$.

2. The least square value y has its minimum at the β value of 0.1025, at which the difference between the allocated permits and demand becomes the smallest for the six participating countries. Even though this value of β is considered a reference point, it has considerably changed.

3. Considering the value for β of 0.1025, Table 3 summarizes the results of emission permit allocation for the six countries. As we can appreciate in the last column of the following table, Canada and Italy exceed their demands while the remaining four countries (Germany, Japan, Russia, and the United Kingdom), do not. Thus, after the initial permit allocation, emissions trading can occur between permit sellers (Canada and Italy) and permit buyers (Germany, Japan, Russia, and the United Kingdom).

Country i	2011 CO2 Emissions (1000 metric tons)	Boltzmann Probability Pi for B = 0.1025	Allocated permits (1000 metric toncs) to country (for B = 0.1025)	Difference in 2011= allocated – emitted
Canada	485.436	0,10	493.737	8.301
Germany	729.458	0,14	688.703	-40.755
Italy	397.994	0,08	398.202	208
Japan	1.187.657	0,23	1.117.768	-69.889
Russia	1.808.073	0,36	1.764.209	-43.864
UK	448.236	0,09	441.520	-6.716
Total	**5.056.854**	**1,00**	**4.904.140**	**-152.714**

Table 3: Emissions permits of the new selection of six countries for $\beta = 0.1025$: CO_2 emissions in 2011 (1000 metric tons), the probability P_i, the allocated permits (1000 metric tons) and the difference (allocated permits – actual CO_2 emissions in 2011). The total amount of permits to be allocated is αN = 4,904,140 (1000 metric tons).

From this analysis, we can confirm that the differences between the countries are not that big as in the preceding analysis and that the range of results is not that broad. This is due to getting rid of the most pollutant global powers. As expected, the results on permit allocation depends on which countries have been selected.

b) The robustness of the Boltzmann distribution: One interesting question is how robust is the Boltzmann distribution in the following sense. Once an allocation has been determined, assume that some agents take their awards and leave and the situation is then reassessed. The property, called consistency in rationing problems, requires that the awards derived in the new situation for the remaining agents coincide with the ones they received initially (see Young 1994). Following this property, an interesting question is to consider the same $n = 6$ countries (Canada, Germany, Italy, Japan, Russia and the United Kingdom), but limit the total amount of allowed permits to be allocated among the countries αN in a completely different way. Instead of taking the 97% of the total of the CO_2 emissions of the six countries in the year 2010, we add the allocated permits of those six countries when the allocation was done among eight countries (Canada, Germany, Italy, Japan, Russia, and the United Kingdom and also with China and the United States). Therefore, the total amount of allowed permits to be allocated is $\alpha N = 4,472,916$ (1000 metric tons). Then we wonder whether the allocation induced by the Boltzmann distribution gives the same amount. The following table is based on Table 2 but without China and the United States:

Country i	2011 CO2 Emissions (1000 metric tons)	Allocated permits (1000 metric toncs) to country (for B = 0.088)
Canada	485.436	427.071
Germany	729.458	642.552
Italy	397.994	383.672
Japan	1.187.657	1.037.284
Russia	1.808.073	1.559.328
UK	448.236	423.010
Total	**5.056.854**	**4.472.916**

Table 4: Emissions permits of the six countries using the Boltzmann distribution for $\beta = 0.0880$: CO_2 emissions in 2011 (1000 metric tons) and the total amount of allowed permits to be allocated among the countries is $\alpha N = 4,472,916$ (1000 metric tons). This table is the same as Table 2, but taking out data for China and the United States.

Once the limited amount of permits is set, we develop the same probability distribution among the six countries and here the results:

Country i	CO2 Emissions (1000 metric tons) in 2011	Boltzmann Probability Pi for B = 0.0880	Allocated permits (1000 metric toncs) to country (for B = 0.088)
Canada	485.436	0,10	427.071
Germany	729.458	0,14	642.552
Italy	397.994	0,09	383.672
Japan	1.187.657	0,23	1.037.284
Russia	1.808.073	0,35	1.559.328
UK	448.236	0,09	423.010
Total	**5.056.854**	**1,00**	**4.472.916**

Table 5: Emissions permits of the new selection of six countries using the Boltzmann distribution for $\beta = 0.0880$: CO_2 emissions in 2011 (1000 metric tons), the Boltzmann probability P_i and the allocated permits (1000 metric tons). The total amount of allowed permits to be allocated among the countries is $\alpha N = 4,472,916$ (1000 metric tons).

As we can see, the Boltzmann probability for each country P_i has completely changed, but each country is allocated with the same number of emissions permits that when China and the United States were considered. The fourth column of Table 4 and Table 5 are identical. This result implies that this method of allocation is consistent.

Furthermore, by comparison between exercise (a) and exercise (b) in our work, we might notice how the total amount of allowed permits to be allocated among the countries αN is greater in exercise (a), so exercise (b) has a more restricted number of permits to allocate among the countries. This restriction is applied to all countries, that is, none of the countries is being benefited from this restriction and they all get less permits than in exercise (a). We might extend this concept to politics and say, how when the emissions policy is more restrictive, all the countries will have to reduce their emissions.

4.3 A tentative application: TAC allocation in fisheries

The aim of this subsection is to suggest how the Boltzmann distribution could be applicable to *Total Allowable Catches* (TAC) allocation in fisheries as Park *et al.* (2012) state. Fish species have a limited reproductive capacity. If fishing were not controlled then stocks could cease to be economically viable. We describe how the process of TAC allocation could be done considering the European Union (EU) policy in fisheries.

1. All EU countries are under the *Common Fisheries Policy* whose aim is to ensure high long-term fishing profits for all stocks. The TAC allocation policy was developed in 1983. Each TAC is the fishing limit (expressed in tonnes) set for most of the commercial fish stock. Countries are responsible for ensuring that their fishing quotas have not been exceeded, while the *European Commission* establishes the TAC for each fish stock. In general, TAC are annually set for the majority of the stocks and each fish stock is based on scientific advice on the stock status. When a country has fished all the available quota of a species, fishery has to be shut down. Both the TAC and the quotas are settled annually.

2. TAC allocation is established on the basis of catches in the past.

3. Once the allocation is done, countries can exchange quotas.

As in the preceding application, an agreement on the initial allocation of the TAC which satisfies the demands of all countries is complicated. Hence we outline how this could be done by means of the Boltzmann distribution. To proceed we suggest the replacement of the concepts used in physics into the concepts used in TAC allocation:

i The concept of a physical system is replaced by the TAC system which consists in all participating countries.

ii The concept of the physical particle is replaced by that of the unit quota.

iii The concept of the physical substates is replaced by that of fishing efforts of the participating countries.

iv The energy ε_i of a physical substate i, is replace is by the catch per unit of effort of country i.

We admit that the selection of these variables are arguable and that more work is needed to find an accurate measure of fishing effort, but let us justify the selection of these variables:

According to *Food and Agriculture Organization of the United Nations*, FAO, *fishing effort* is a measure of the "amount of fishing" considering persons and tools used in the fishing activity; for example, the number of hours or days spent fishing, numbers of hooks used (in long-line fishing), kilometers of nets used, etc. Fishing effort can be expressed uniformly as the number of boat-days per species. *Catch* refers to the fishes taken together and is usually computed within the logical context of a limited geographical area for given reference period and for a specific boat/gear category. *Catch Per Unit Effort* is defined as the quotient between the total fishing catch divided by the total amount of effort invested in the catch.[12]

To proceed we must assume that fishing efforts in a country i, get the same catches across all the fishing areas. This implies that we are assuming the same kind of vessels and techniques etc. in the fishing. Therefore, the probability that a unit quota is allocated to a country i should be proportional to its total effort.

Several tries for finding data have been done, but we have found difficulties. For instance, the *saithe* is a fish whose TAC was reduced between the years 2014 and 2015. This species is fished in geographical areas of several sea-territories not all belonging to EU. In addition, we were not successful in finding the fishing effort of each of these countries. These two facts, the geographical division and the lack of information on the effort, make this model hard to fit the reality in the designed direction. Indeed, more work has to be done in order to analyze this allocation problem.

5 Conclusions

First of all, let us summarize the principal concepts in physics in which our work is based on: The Boltzmann distribution of energy in (2) provides the

[12] http://www.fao.org/docrep/004/y2790e/y2790e03.htm

most probable distribution in which particles in a physical system in equilibrium occupy the i-th states with energy E_i. Based on the Boltzmann distribution, expression (3) provides the probability P_i to find a system in a certain state i with energy ϵ_i, which is inversely proportional to the exponential function of the energy. Thus, the states with lower energy will have higher probability of being occupied than the states with higher energy. These referred systems, in physics, may range from an atom, to a gas in a tank, and this is the reason why the Boltzmann distribution is very useful and can be applied in such a variety of cases. In addition, in thermodynamics entropy S is the measure of disorder in the system, *i.e.* the measure of unpredictability. In the same way in statistical physics an isolated system tends to its equilibrium state by maximization of its entropy S, we might say that an isolated system maximizes the number of accessible microstates Ω given by relation (4) that connects the macroscopic and the microscopic worlds. Finally, the maximum entropy principle states that the probability distribution that best represents the current state of knowledge is the one with the largest entropy.

Our starting point in this work was the layout in Drăgulescu and Yakovenko (2000) of speculating whether any conserved quantity in a big statistical system in equilibrium, such as economy, follows the Boltzmann distribution. In the same way energy is conserved in statistical mechanics, money is conserved in a closed economy with total amount of money M and many economic agents N. In accordance, the probability distribution of money is given by (5). It is important to remark that the authors do not claim that the real economy is in equilibrium. The same way most of the physical world around us is not in equilibrium either, the concept of statistical equilibrium is just a useful reference point for studying non-equilibrium phenomena. However, the bigger the ensemble, the more relevant the concept of statistical equilibrium is.

The first economic models considered for the study of the probability distribution of money $P(m)$ among economic agents were pairwise money transfer models without debt representing a primitive market. In these models, agents exchange money Δm under different assumptions and restrictive conditions. Then, development of money balances were studied and represented in computer simulations. In our work, we have presented some of these idealized models and we have showed how the Boltzmann probability distribution does not necessarily hold for any conservative model in economic settings. When debt is permitted, money conservation still prevails, but the factor that makes the difference is of course the boundary condition. *i.e.* if debt is limited or not. The probability distribution of money $P(m)$ never stabilizes as the system never reaches a stationary state. When agents' debt is limited, results show how $P(m)$ recovers the exponential shape, but just shifted up the new limit compared to the limit of no debt.

Once we were familiar with the simplified models presented in Drăgulescu and Yakovenko (2000), we focus on studying the problem of allocation of CO_2 emission permits among polluting countries introduced by Park *et al.* (2012). For this problem the Boltzmann distribution provides the most probable allocation of permits among countries. The concept of "most probable" from physics is interpreted as "fair" in permit allocation. Grandfathering rewards those countries that have polluted more in the past, while auctioning favors rich countries. However, the Boltzmann distribution for allocation of permits does not choose who has the right to pollute and distributes the permits in an impartial way. The probability distribution depends on population and emissions per capita of each country and all individuals in a country are considered to pollute the same. What this distribution does, is to proportionally allocate permits to a country depending on the limit set in order to reduce the CO_2 pollution.

The β value has to be determined by the countries, so to find a proper value might be a difficult task. The limit cases show that when β tends to 0 (egalitarian situation), the probability of getting emission permits depends only on the population of the countries. While if β tends to ∞, probability is greater for the countries with the lowest energy per capita. Results show how all countries seek β values between 0 and 1: countries with the higher CO_2 emissions per capita prefer larger values, while countries with lower emissions per capita prefer smaller values.

From the updated empirical analysis, the result that stands out is the dispute between China (that produces the greatest CO_2 emissions caused by the huge population) and the United States (that has the greatest CO_2 emissions per capita). Which of these two global powers should be more responsible for future reduction? The flexibility of this allocation of permits can make them share this responsibility by shifting the value of β.

By comparing authors' results for 2008 and our results for 2011, for a small group of countries, we observe the following: The CO_2 emissions in the year 2010 are less than in 2007, fact that makes the number of total available permits to allocate among the countries smaller. Consequently the reference value of β calculated by least square is smaller while the probability values that determine the permit allocation follow a similar pattern.

Indeed, the results depend on the number of countries selected. By eliminating the two global powers the number of available allocation permits is diminished and all the countries get results that change in the same direction.

Further, for the small group of countries we have also considered the case in which the CO_2 emissions is exactly the sum of what they got initially. In this case the application of Boltzmann distribution to this new situation gives the same initial allocation. As we have said in text, this shows the robustness of the Boltzmann distribution as an allocation procedure. This exercise invites us to a deeper study of the mathematical properties of the Boltzmann distribution that could be carried out in a future.

Summing up, Park *et al.* (2012) suggest that the Boltzmann distribution can be applied to other allocation problems such as environmental air and water pollution control, water and food supply management, and fisheries management. Following this suggestion we have outlined how the Boltzmann distribution could be applied to *Total Allowable Catches* (TAC) allocation in fisheries. As we have commented in text, our selection of variables are arguable and more work is needed in order to take this model into practice by means of empirical data. This could also be a very interesting path to follow in future studies.

To conclude, we might say that the Boltzmann probability distribution of statistical mechanics applied to economy in cases of partitioning a limited resource among multiple agents can be applied in a variety of systems that surround us. Overall, econophysics research field is still emerging and many situations are still to be discovered.

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