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Groups acting on regular rooted trees: congruence subgroup problem and portrait growth

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Contents

Introduction

Groups acting on regular rooted trees have been widely studied since the 1980's, as Grigorchuk discovered the importance of these groups. The first Grigorchuk was introduced by Rostislav Grigorchuk in [26]. This group, known as the first Grigorchuk group, was constructed as a simple counterexample to the General Burnside Problem. The General Burnside Problem asks whether a finitely generated periodic group must be finite. The answer was already known to be negative by an example provided in 1964 by Golod and Shafarevich.

Even if the group was constructed for this purpose, it turned out that this group in particular has very interesting and exotic properties: it has intermediate word growth, it is amenable but not elementary amenable, residually finite and just-infinite, it is not finitely presentable... In fact, it was the first known example of a group of intermediate word growth (see [21]). The problem is known as the Milnor problem. There were already well known examples of groups of polynomial and exponential growth, and Milnor asked in 1968 [32] whether there was something in between, and the first example giving a positive answer to the Milnor problem was the first Grigorchuk group.

After the first Grigorchuk group, many generalisations of it and many different examples of groups acting on regular rooted trees arised. For instance, in 1983, Gupta and Sidki defined a family of p-groups for each odd prime p, which are also a counterexample to the Generalised Burnside Problem, [28]. In another direction, Bartholdi and Sunic defined a family generalising the first Grigorchuk group, all of them being also of intermediate growth (see [6]).

Throughout this thesis, we will present different problems related to groups acting on regular rooted trees and solve them for different examples.

In order to do this, Chapter 1 is devoted to giving basic definitions and properties about groups acting on regular rooted trees. More precisely, we present different ways of describing automorphisms of a regular rooted tree, we define what self-similar groups are, what branch groups are and give tools to prove if a given group is self-similar or branch, for instance.

Later on, the problems that will be discussed in the rest of the chapters are presented. More concretely, we describe the congruence subgroup problem, regarding completions of groups with respect to different topologies; and also the portrait growth, which asks, roughly speaking, how grows the size of some pictures describing the automorphisms of the group.

The last section of this chapter collects the definitions of the groups appearing throughout the thesis. Not only the definitions but also the facts that are easy to prove or already well known about these groups are also presented in this section.

Chapter 2 deals with some basic notions that have given rise to some confusion in the literature. Groups acting on regular rooted trees are called self-similar when, somehow, the action that one can see in any vertex of the tree can already be seen at the root. Then, the question is if at every vertex we recover the whole action of the group or we just recover it partially. This notion, formally defined, splits in three cases: fractal, strongly fractal and super strongly fractal, each of them stronger than the previous one. In the literature sometimes these notions were claimed to be equivalent. In the second chapter, we show by giving explicit examples that these definitions are not equivalent.

Theorem. *There exist groups acting on regular rooted trees which are fractal but not strongly fractal, and there are also strongly fractal groups which are not super strongly fractal.*

This work has led to the paper [37].

The rest of the chapters are devoted to solving the problems described in Chapter 1, each of them for some of the examples presented also in that preliminary chapter.

Chapter 3 is devoted to solving the congruence subgroup problem for the family of multi-GGS-groups. The GGS-groups, named after Grigorchuk, Gupta and Sidki, are a family of groups generalising the Gupta-Sidki family and the so-called second Grigorchuk group. For each odd prime p, a vector in $\mathbb{F}_p^{p-1}$ defines a GGS-group. Indeed, every GGS-group is two generated, and only the definition of one of the generators depends on that vector. Then a multi-GGS-group is a generalisation of the GGS-groups, by adding more generators defined by different defining vectors.

The congruence subgroup problem deals with some special normal subgroups of finite index (which are called congruence subgroups) and asks whether there are many more subgroups of finite index, or if these special ones are all of them. The problem was originally introduced in the context of algebraic groups. More concretely, the problem was firstly studied for the groups $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 2$. In this context the congruence subgroups are the subgroups containing $\ker \pi_m$ where $\pi_m : \mathrm{SL}_n(\mathbb{Z}) \longrightarrow \mathrm{SL}_n(\mathbb{Z}/m\mathbb{Z})$ for some $m \in \mathbb{N}$ (this is the reason for the name 'congruence'). Thus, if every subgroup of finite index is a congruence subgroup we say that the group has the congruence subgroup property. It was shown by Fricke and Klein in the 19th century that for $n = 2$ the group $\mathrm{SL}_n(\mathbb{Z})$ does not have the congruence subgroup property. However, for $n \geq 3$ it does have the congruence subgroup property, and it was shown by Bass, Lazard and Serre (see [7]); and independently by Mennicke (see [30]), in 1964.

In the context of groups acting on regular rooted trees, we say that a subgroup is a congruence subgroup if it contains some level stabilizer. Thus, as for $\mathrm{SL}_n(\mathbb{Z})$, we say that a group has the congruence subgroup property if every subgroup of finite index is a congruence subgroup. The main result of these chapter are the following.

Theorem. *All the multi-GGS-groups apart from $\mathcal{G}$ have the congruence subgroup property and are just infinite.*

Where $\mathcal{G}$ denotes the GGS-group defined by the constant vector, which has a very different behaviour.

Theorem. *The GGS-group $\mathcal{G}$ with constant defining vector has an infinite congruence kernel.*

The first result provides us a way to answer a question made by Barnea about the existence of groups which are finitely generated, non-torsion (or even torsion-free), residually finite and such that their profinite completion is a pro-p group, for p a prime. We show that some of the GGS-groups are such examples providing the answer for both cases. These results have given rise to the papers [14] and [17].

In Chapter 4, we generalise the congruence subgroup problem. In fact, the congruence subgroup problem can be seen from a topological point of view. Given a group, a family of normal subgroups of finite index (under some conditions) forms a system of neighbourhoods of the identity for a topology in the group. Then, the congruence subgroup problem mentioned before, asks whether the topology given by the congruence subgroups and the one given by all finite index subgroups coincide or not. Or, from the point of view of topological completions, if the natural epimorphism from one completion to the other one is an isomorphism or not.

As mentioned above, there are examples of groups not having the congruence subgroup property, for instance, the GGS-group defined by a constant vector. The reason why this happens is because it virtually maps onto $\mathbb{Z}$, and since the congruence completion is a pro-p group, this prevents the group from having the congruence subgroup property. Then, a natural question is whether the appropriate topology to compare with the congruence topology is the pro-p topology in this case, instead of the profinite one. That is, the topology defined by all normal subgroups of p-power index. We prove that, in fact, this is the case for the GGS-group defined by a constant vector. We also prove that the same happens for the Basilica group.

Theorem. *For the GGS-group $\mathcal{G}$ defined by the constant vector and for the Basilica group, the pro-p completion (for p odd and $p = 2$, respectively) coincides with the congruence completion.*

Indeed, we define a more general problem, which we call the $\mathcal{C}$-congruence subgroup problem, where $\mathcal{C}$ denotes a variety of finite groups. We give a sufficient condition for a weakly regular branch group to have the $\mathcal{C}$-congruence subgroup property ($\mathcal{C}$-CSP for short), and using this condition we prove the two examples mentioned before. This condition is the main result of this chapter.

Theorem. *Let $G \leq \operatorname{Aut} T$ be a weakly regular branch group over a subgroup R. Suppose that there exists $H \trianglelefteq G$ such that $R \geq H \geq R' \geq L$ where*

$L := \psi^{-1}(H \times \overset{d}{\ldots} \times H)$. If G has the $\mathcal{C}$-CSP modulo H and H has the $\mathcal{C}$-CSP modulo L, then G has the $\mathcal{C}$-CSP.

These results are collected in [16].

In the last chapter we work with a completely different problem. The problem is about counting the number of elements that have a portrait of size smaller than a given $n \in \mathbb{N}$.

In a group acting on a regular rooted tree, each element of the group can be fully described by decorating a tree. It is enough to decorate each vertex with a permutation, and this describes the action of the element. However, these are infinite trees, and thus, in practise they would not be useful in order to describe elements. In the case of self-similar groups, since the elements act on the subtrees hanging from each vertex as elements of the group again, we can stop decorating the tree by labelling some vertices with the elements in the group. This kind of picture also describes the full action of the element in the tree.

The problem, in the second type of decoration, is how to decide when to stop, and this is why contracting groups are important in this setting. A group is said to be contracting if there exists a finite set of elements in the group, such that for any element in the group, when one starts decorating the tree as described before, at some level all the elements in the portrait belong to this finite set. Such a minimal set is called the nucleus, and thus, one starts decorating the tree with permutations until one finds an element that belongs to the nucleus.

Then one can look at the depth of the portrait of each element, and ask how many are of each depth. Whenever the group is finitely generated, this number will be finite, and thus it makes sense to ask about the growth of this function.

Grigorchuk, in a paper about solved and unsolved problems around the first Grigorchuk group (see Problem 3.5 [18]) asked about growth functions for the portraits of the first Grigorchuk group. In this chapter we answer this question.

Theorem. *There exist positive constants α, β, and γ such that the portrait growth sequence $\{a_n\}_{n=0}^{\infty}$ of the first Grigorchuk group Γ satisfies the inequalities*

$$\alpha e^{\gamma^{2^n}} \leq a_n \leq \beta e^{\gamma^{2^n}},$$

for all $n \geq 0$. Moreover, $\gamma \approx 0.71$.

Actually, we give a way of finding recursive equations for the portrait growth of any contracting regular branch group. We compute also the portrait growth for the GGS-groups defined by a non-symmetric vector and for the Apollonian group.

Theorem. *Let G be a GGS-group defined by a non-symmetric vector $\mathbf{e} \in \mathbb{F}_p^{p-1}$. The portrait growth sequence $\{a_n\}_{n=0}^{\infty}$ of G is given by*

$$a_0 = 1 + 2(p-1)$$
$$a_n = p(x_1 + (p-1)y_1)^{p^{n-1}},$$

where x_1 and y_1 are the number of solutions in $\mathbb{F}_p^p$ of

$$(n_0, \ldots, n_{p-1})C(\mathbf{e}, 0) \odot (n_1, n_2, \ldots, n_{p-1}, n_0) = (0, \ldots, 0),$$

with $n_0 + \cdots + n_{p-1} = 0$ and $n_0 + \cdots + n_{p-1} = 1$, respectively; where $\odot$ denotes the product by coordinates.

Theorem. *The portrait growth sequence $\{a_n\}_{n=0}^{\infty}$ of the Apollonian group is* *given by:*

$$a_n = 3^{\frac{3^n - 1}{2}} 7^{3^n}.$$

These results led to the paper [35].

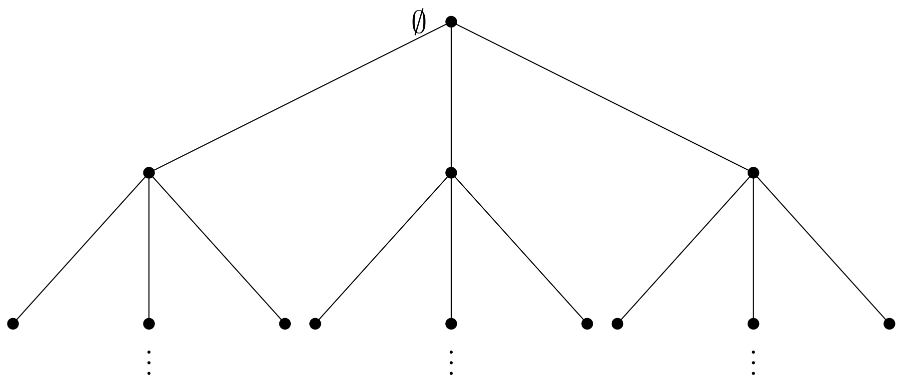
Chapter 1

Preliminaries

1.1 Automorphisms of a regular rooted tree

1.1.1 Definition and general facts

A regular rooted tree is constructed as follows. Given an alphabet X of d letters, consider as the set of vertices X^*, the set of all finite words over the alphabet X. Two vertices $u, v \in X^*$ are joined if $u = vx$ or $v = ux$ for some $x \in X$. The empty word, denoted by $\emptyset$, is called the root of the tree. We will denote by T such a tree, and we will say that T is a d-adic tree when $|X| = d$. For example, this is how the 3-adic tree looks like.



An automorphism of a regular rooted tree is a bijection between vertices that preserves incidence. We denote by $\operatorname{Aut} T$ the set of all automorphisms of the regular rooted tree T, which has a group structure under composition. We write fg for $g \circ f$. Observe that since the automorphisms must preserve incidence, the root of the tree is always fixed. This follows from the fact that it is the unique vertex that has d adjacent vertices. This also implies that any path starting at the root is moved to another path starting at the root, that a vertex of a fixed length is moved to a vertex of the same length, and that if an automorphism fixes some vertex, it must also fix all the vertices belonging to the path going from the root to the fixed vertex.

We denote by L_n the n-th level of the tree, or in other words, the set of vertices representing words of length n over X. Considering only the vertices of length smaller than or equal to a given $n \in \mathbb{N}$, we obtain a finite subtree T_n of T. Then the group of automorphisms $\operatorname{Aut} T_n$ of this finite tree is a quotient of the whole group $\operatorname{Aut} T$. More precisely, we have for each $n \in \mathbb{N}$ the natural projection $\pi_n : \operatorname{Aut} T \longrightarrow \operatorname{Aut} T_n$ whose kernel is the set of all automorphisms fixing the first nth levels. In other words, the stabilizer of the nth level, denoted by $\operatorname{st}(n)$. More precisely,

$$\operatorname{st}(n) = \{g \in \operatorname{Aut} T \mid \ \forall u \in L_n, g(u) = u\}.$$

Thus $\operatorname{Aut} T_n \cong \operatorname{Aut} T/\operatorname{st}(n)$ for each $n \in \mathbb{N}$. Obviously, for each $m \geq n$ we also have the projetions $\pi_{m,n} : \operatorname{Aut} T_m \longrightarrow \operatorname{Aut} T_n$, and they form an inverse system. Thus, considering each $\operatorname{Aut} T_n$ with the discrete topology, the whole group $\operatorname{Aut} T$ is the inverse limite of them,

$$\operatorname{Aut} T = \varprojlim_{n\in\mathbb{N}} \operatorname{Aut} T_n \cong \varprojlim_{n\in N} \operatorname{Aut} T/\operatorname{st}(n).$$

As a consequence, we get that $\operatorname{Aut} T$ is a profinite group.

Observe that each level stabilizer is the intersection of the stabilizers of all vertices in that level. Thus denoting by $\operatorname{st}(u) = \{g \in \operatorname{Aut} T \mid g(u) = u\}$ we can write $\operatorname{st}(n) = \cap_{u\in L_n} \operatorname{st}(u)$.

A way to describe an automorphism g of T is by assigning to each vertex of the tree a permutation of the set X. Then the permutation of S_X assigned to the vertex u describes how g permutes the vertices hanging from $g(u)$. This is called the **label** of g at the vertex $u \in T$ and we denote it by $g_{(u)}$. Thus, the label of g at u is formally defined by

$$g(ux) = g(u)g_{(u)}(x) \text{ for each } x \in X.$$

The tree decorated by the labels of an automorphism at each vertex is the **portrait** of the automorphism.

On the other hand, observe that if we denote by T_u the tree hanging from a vertex $u \in T$, we have that T_u is isomorphic to the whole tree T. Then $\operatorname{Aut} T \cong \operatorname{Aut} T_u$ for each vertex u, and we can describe $g \in \operatorname{Aut} T$ by saying how g acts in the subtree hanging from $g(u)$ for each $u \in T$. This is called the **section** of g at the vertex u, denoted by g_u, and it is formally defined by

$$\forall v \in T, g(uv) = g(u)g_u(v).$$

From the equality in the definition, we get, for instance

$$\begin{aligned} fg(uv) &= fg(u)(fg)_u(v) \\ &= g(f(uv)) = g(f(u)f_u(v)) = g(f(u))g_{f(u)}(f_u(v)) = fg(u)f_u g_{f(u)}(v). \end{aligned}$$

In this way we get that $(fg)_u = f_u g_{f(u)}$. Arguing in a similar way, we get the following useful collection of formulas:

$$\begin{aligned} (fg)_u &= f_u g_{f(u)}, \\ (f^{-1})_u &= (f_{f^{-1}(u)})^{-1}, \\ f_{uv} &= (f_u)_v, \end{aligned} \tag{1.1}$$

and,

$$(f^g)_u = (g_{g^{-1}(u)})^{-1} f_{g^{-1}(u)} g_{g^{-1}f(u)}. \tag{1.2}$$

The formulas above are written for sections, but observe that they are also satisfied for labels.

Taking into account the description given by sections, it turns out that for each $n \in \mathbb{N}$ we can define the following homomorphism

$$\psi_n : \mathrm{st}(n) \longrightarrow \mathrm{Aut}\, T \times \overset{d^n}{\cdots} \times \mathrm{Aut}\, T$$

which sends $g \in \mathrm{st}(n)$ to the d^n-tuple of its sections $(g_{u_1}, \cdots, g_{u_{d^n}})$, with $u_i \in L_n$. Notice that in this case the sections are nothing but the restrictions. For simplicity, we write ψ for ψ_1. Observe also that ψ_0 is nothing but the identity map on $\mathrm{Aut}\, T$. Note that in fact these maps are isomorphisms, which means that $\mathrm{st}(n) \cong \mathrm{Aut}\, T \times \overset{d^n}{\cdots} \times \mathrm{Aut}\, T$ for every $n \in \mathbb{N}$. In the same way, for the stabilizer $\mathrm{st}(u)$ of the vertex u, we have a homomorphism denoted by ψ_u which sends $g \in \mathrm{st}(u)$ to $g_u \in \mathrm{Aut}\, T$.

Lemma 1.1.1. *Let $G \leq \mathrm{Aut}\, T$, then we have*

(i) $\psi(\mathrm{st}_G(n)) = (\mathrm{st}_G(n-1) \times \overset{d}{\ldots} \times \mathrm{st}_G(n-1)) \cap \psi(\mathrm{st}_G(1))$ *for* $n \geq 2$,

(ii) $\mathrm{st}_G(n) = \psi^{-1}(\mathrm{st}_G(n-1) \times \overset{d}{\ldots} \times \mathrm{st}_G(n-1))$ *for* $n \geq 2$,

(iii) $\mathrm{st}_G(n+m) = \psi_n^{-1}(\mathrm{st}_G(m) \times \overset{d^n}{\ldots} \times \mathrm{st}_G(m))$ *for* $m, n \in \mathbb{N}$.

Proof. For (i) there is nothing to prove. In order to see (ii) it suffices to apply ψ^{-1} to (i). By injectivity of ψ and since $\mathrm{st}_G(1) \geq \mathrm{st}_G(n)$ for any $n \in \mathbb{N}$ we obtain the result. Finally, (iii) follows by applying induction to (ii). □

Sometimes it is useful to think of $\mathrm{Aut}\, T$ as a semidirect product.

Proposition 1.1.2. *Let T be the d-adic tree and let us consider the following subgroup for each $n \in \mathbb{N}$:*

$$H_n = \{h \in \mathrm{Aut}\, T \mid h_u = 1 \ \forall u \in L_n\}.$$

Then we have

$$\mathrm{Aut}\, T = H_n \ltimes \mathrm{st}(n).$$

Proof. It is clear that $\mathrm{st}(n) \cap H_n = \{1\}$ and $\mathrm{st}(n) \trianglelefteq \mathrm{Aut}\, T$, since it is the kernel of the epimorphism π_n for each $n \in \mathbb{N}$. Let us see that $\mathrm{Aut}\, T = \langle \mathrm{st}(n), H_n \rangle$.

Let $f \in \mathrm{Aut}\, T$ and define h by taking $h_{(u)} = f_{(u)}$ for all $u \in L_k$, $k < n$, and $h_{(u)} = 1$ otherwise; and g by $g_{(u)} = f_{(u)}$ for $u \in L_k$ for all $k \geq n$ and $g_{(u)} = 1$ otherwise. Clearly $h \in H_n$ and $g \in \mathrm{st}(n)$. Now we want to see that $f = gh$, or which is equivalent, that they have the same portrait, i.e. that for every $u \in L_n$ with $n \in \mathbb{N}$

$$f_{(u)} = (gh)_{(u)} = g_{(u)} h_{(g(u))}.$$

If $k < n$ then for $u \in L_k$, $h_{(u)} = f_{(u)}$ and $g_{(u)} = 1$. Also we have $g(u) = u$, since $g \in \mathrm{st}(n)$. Then, $g_{(u)}h_{(g(u))} = h_{(u)} = f_{(u)}$.

And if $k \geq n$ then $h_{(u)} = 1$ and $g_{(u)} = f_{(u)}$ for each $u \in L_k$. So, since $g(u) \in L_k$, $g_{(u)}h_{(g(u))} = g_{(u)} = f_{(u)}$. □

This gives another way of describing any automorphism combining both labels and sections. Indeed if $X = \{x_1, \ldots, x_d\}$ for any $g \in \mathrm{Aut}\, T$ we can write

$$g = (g_{x_1}, \ldots, g_{x_d})\alpha, \tag{1.3}$$

where g_{x_i} is the section of g at the vertex x_i for every $i = 1, \ldots, d$ and $\alpha = g_{(\emptyset)}$ is the label of g at the root.

Observe that for $f \in \mathrm{st}(n)$ and $g = hg' \in \mathrm{Aut}\, T$, with $h \in H_n$ and $g' \in \mathrm{st}(n)$, by (1.2) we have

$$(f^g)_u = (f_{g^{-1}(u)})^{g_{g^{-1}(u)}} = (f_{h^{-1}(u)})^{g_{h^{-1}(u)}} = (f_{h^{-1}(u)})^{g'_u} \text{ for all } u \in L_n. \tag{1.4}$$

An automorphism is called **rooted automorphism**, if the only non-trivial label of the automorphism is at the root. According to the above notation this is equivalent to $g \in H_1$.

The semidirect group structure together with the fact that $\mathrm{st}(n) \cong \mathrm{Aut}\, T \times \overset{d^n}{\cdots} \times \mathrm{Aut}\, T$ for every $n \in \mathbb{N}$, means that $\mathrm{Aut}\, T$ may also be seen as the iterated permutational wreath product

$$\mathrm{Aut}\, T \cong (\ldots(S_X \wr (S_X \wr S_X))\ldots).$$

The expresion of the right hand side is the invers limit $\varprojlim_{n\in\mathbb{N}} W_n$, where $W_1 = S_X$ and $W_n = S_X \wr W_{n-1}$ for each $n \geq 2$, and where the connecting maps are the natural projections $\pi_{n,n-1} : W_n \longrightarrow W_{n-1}$ whose kernel is the base group $(S_x \times \overset{d}{\ldots} \times S_X)$ for each $n \in \mathbb{N}$. Indeed $\operatorname{Aut} T_n \cong W_n$ for each $n \in \mathbb{N}$.

1.1.2 Self-similarity and branching in subgroups of $\operatorname{Aut} T$

Let us now consider a subgroup G of the whole group $\operatorname{Aut} T$. One can define the restrictions of the previous homomorphisms ψ_n and ψ_u to $\mathrm{st}_G(n) = \mathrm{st}(n) \cap G$ and $\mathrm{st}_G(u) = \mathrm{st}(u) \cap G$ respectively for each $n \in \mathbb{N}$ and $u \in T$.

Definition 1.1.3. *Let $G \leq \operatorname{Aut} T$. We say that G is self-similar if $g_u \in G$ for every $g \in G$ and $u \in T$.*

In particular, for a self-similar group G the images under ψ_u and ψ_n belong to G and $G \times \overset{d^n}{\ldots} \times G$ respectively.

Lemma 1.1.4. *A group $G = \langle S \rangle \leq \operatorname{Aut} T$ is self-similar if and only if $s_x \in G$ for each $s \in S$ and $x \in X$.*

Proof. We proof the "if part" by induction on the length of the vertices. The base case follows from writing each element as a product of elements in S, the hypothesis and from using formulas (1.1). Let now $u \in T$ be a vertex of length n with $n > 1$, and let as assume that the statement is true for every $m < n$. Let us write $u = vx$ where $v \in L_{n-1}$ and $x \in X$. For each $g \in G$ we know by the inductive assumption that $g_v \in G$, and then since we already know it for the case of vertices of length one, we get $g_u = (g_v)_x \in G$. □

First of all, it is worth mentioning that even if in the case of $\operatorname{Aut} T$ these homomorphisms ψ_u and ψ_n are always surjective homomorphisms, this will not be the case in general. We will discuss these notions later on in detail in Chapter 2, but let us just introduce some definitions related to this question.

Definition 1.1.5. *Let $G \leq \operatorname{Aut} T$ be a self-similar group.*

(i) We say that G is fractal *if $\psi_u(\mathrm{st}_G(u)) = G$ for each vertex $u \in T$.*

(ii) We say that G is strongly fractal *if $\psi_x(\mathrm{st}_G(1)) = G$ for each $x \in X$.*

(iii) We say that G is super strongly fractal *if $\psi_u(\mathrm{st}_G(n)) = G$ for each $u \in L_n$ and each $n \in \mathbb{N}$.*

Moreover, observe that even if in the case of $\operatorname{Aut} T$ we have $\psi_n(\mathrm{st}(n)) = \operatorname{Aut} T \times \overset{d^n}{\cdots} \times \operatorname{Aut} T$, for a general self-similar group G, the maps ψ_n need not be surjective onto $G \times \overset{d^n}{\cdots} \times G$. In fact, the image of $\mathrm{st}_G(n)$ under ψ_n may not be a natural direct product inside $\operatorname{Aut} T$, where with natural direct product we mean that there is some $H_i \leq \operatorname{Aut} T$ for each $i \in \{1, \ldots, d^n\}$ such that $\psi_n(\mathrm{st}_G(n)) = H_1 \times \cdots \times H_{d^n}$.

For a self-similar group G, we define the nth rigid stabilizer $\mathrm{rst}_G(n)$, to be the largest subgroup of $\mathrm{st}_G(n)$ such that $\psi_n(\mathrm{rst}_G(n))$ is a natural direct product in $G \times \overset{d^n}{\cdots} \times G$. Defining the rigid stabilizer of a vertex $\mathrm{rst}_G(u)$ as the elements $g \in G$ that have trivial labels outside the subtree T_u hanging from u, that is, $\mathrm{rst}_G(u) = \{g \in G \mid g_{(v)} = 1, v \notin T_u\}$, we get

$$\mathrm{rst}_G(n) = \langle \mathrm{rst}_G(u) \mid u \in L_n \rangle = \prod_{u \in L_n} \mathrm{rst}_G(u).$$

Here we point out some easy but useful facts about the images of rigid stabilizers under ψ. We omit the proof because it only consists in checking both inclusions.

Lemma 1.1.6. *Let $G \leq \operatorname{Aut} T$, then we have*

(i) $\psi_n(\mathrm{rst}_G(n)) = \prod_{u \in L_n} \psi_u(\mathrm{rst}_G(u))$,

(ii) $\psi(\mathrm{rst}_G(n)) = (\mathrm{rst}_G(n-1) \times \cdots \times \mathrm{rst}_G(n-1)) \cap \psi(\mathrm{rst}_G(1))$ *for $n \geq 2$.*

Definition 1.1.7. *A group $G \leq \operatorname{Aut} T$ is said to be level transitive or we say that it acts spherically transitively, if it is transitive on each level L_n.*

Now we can define what a (regular) branch group is. This notion is very helpful in order to be able to work with groups acting on regular rooted trees as we will see in many cases throughout this thesis.

Definition 1.1.8. *Let $G \leq \operatorname{Aut} T$ be self-similar and level transitive. Then*

(i) G is weakly branch if $\operatorname{rst}_G(n)$ is non-trivial for each $n \in \mathbb{N}$,

(ii) G is branch if $\operatorname{rst}_G(n)$ is of finite index in G for each $n \in \mathbb{N}$,

(iii) G is weakly regular branch if there is some non-trivial normal subgroup $K \leq \operatorname{st}_G(1)$ such that $\psi(K) \geq K \times \ldots \times K$,

(iv) G is regular branch if there is a finite index subgroup normal subgroup $K \leq \operatorname{st}_G(1)$ such that $\psi(K) \geq K \times \ldots \times K$.

Observe that if G is weakly regular branch over a subgroup K we automatically get that $\psi_n(\operatorname{rst}_G(n)) \geq K \times \overset{d^n}{\ldots} \times K$. This follows by induction since for $n = 1$ we already have that $\psi(\operatorname{rst}_G(1)) \geq K \times \overset{d}{\ldots} \times K$. Then assuming that the result is true for $n-1$, with $n \geq 1$, since G is weakly branch we get $\psi_n(\operatorname{rst}_G(n-1)) \geq K \times \overset{d^n}{\ldots} \times K$. Given that $\operatorname{rst}_G(n)$ is the largest subgroup containing such a direct subgroup we conclude that $\psi_n(\operatorname{rst}_G(n)) \geq K \times \overset{d^2}{\ldots} \times K$, and so on.

Proposition 1.1.9. *If $G \leq \operatorname{Aut} T$ is (weakly) regular branch then it is (weakly) branch.*

Proof. Since we have $K \times \overset{d^n}{\cdots} \times K \leq \psi_n(\operatorname{rst}_G(n))$ with $K \neq 1$ the "weakly" case follows. For the other case, observe that,

$$\begin{aligned} |G : \mathrm{rst}_G(n)| &= |G : \mathrm{st}_G(n)|\,|\mathrm{st}_G(n) : \mathrm{rst}_G(n)| \\ &= |G : \mathrm{st}_G(n)|\,|\psi_n(\mathrm{st}_G(n)) : \psi_n(\mathrm{rst}_G(n))| \\ &\leq |G : \mathrm{st}_G(n)|\,|G \times \ldots \times G : K \times \ldots \times K| \\ &= |G : \mathrm{st}_G(n)|\,|G : K|^{d^n}. \end{aligned}$$

This index is finite provided that $|G : K| < \infty$. □

The following proposition from [15] is very useful in order to prove that a group is (weakly) regular branch. Even if in the paper the result is written just for the GGS-groups, one easily checks that the statement is true whenever the group is strongly fractal and level transitive.

Proposition 1.1.10. *[15, Proposition 2.18] Let G be a self-similar strongly fractal group acting transitively on each level, and L and N two normal subgroups of G. If $L = \langle X \rangle^G$ for some set X and $(x, 1, \ldots, 1) \in \psi(\mathrm{st}_N(1))$ for all $x \in X$, then*

$$L \times \cdots \times L \leq \psi(\mathrm{st}_N(1)).$$

Recall that we use the notation $\langle X \rangle^G$ for the normal closure in the group G of the subgroup generated by X.

Since we will be interested in analysing the normal subgroups of a given group acting on a regular rooted tree, we state here a very useful lemma that follows from the proof of Theorem 4 in [22].

Lemma 1.1.11. *Let $G \leq \operatorname{Aut} T$ be a group acting level transitively. Then for every non-trivial normal subgroup N there is some $n \in \mathbb{N}$ such that $N \geq \mathrm{rst}_G(n)'$.*

Proof. Let g be any non-trivial element of N. Since g is non-trivial there is some $n \in \mathbb{N}$ for which $g \in \mathrm{st}_G(n-1) \smallsetminus \mathrm{st}_G(n)$. Then we get that $g_u \notin \mathrm{st}_G(1)$ for some $u \in L_{n-1}$, so that there is some $x \in X$ such that $g_u(x) = y \neq x$. Now for every $\xi \in \psi_{ux}(\mathrm{rst}_G(ux))$ we know that there exists some $f \in \mathrm{st}_G(n-1)$ such that $\psi_{n-1}(f) = (1, \ldots, 1, f_u, 1, \ldots, 1)$ and $\psi(f_u) = (1, \ldots, 1, \xi, 1, \ldots, 1)$, where ξ is at position x. Thus, we obtain that

$$\psi_{n-1}([g, f]) = (1, \ldots, 1, [g_u, f_u], 1, \ldots, 1),$$

and by formulas (1.4) we get

$$\begin{aligned}
[g_u, f_u]_x &= (f_{uy}^{-1})^{g_{uy}} f_{ux} = \xi, \\
[g_u, f_u]_y &= (f_{ux}^{-1})^{g_{ux}} f_{uy} = (\xi^{-1})^{g_{ux}}, \\
[g_u, f_u]_z &= (f_{ug_u^{-1}(z)})^{g_{ug_u^{-1}(z)}} f_u z = 1 \text{ for } z \neq x, y.
\end{aligned}$$

That is, we get

$$\psi_n([g, f]) = (1, \ldots, 1, (\xi^{-1})^{g_{ux}}, 1, \ldots, 1, \xi, 1, \ldots, 1),$$

with $(\xi^{-1})^{g_{ux}}$ at position uy and ξ at position ux. Now for any $\eta \in \psi_{ux}(\mathrm{rst}_G(ux))$ there is some $h \in \mathrm{rst}_G(n)$ such that $\psi_n(h) = (1, \ldots, 1, \eta, 1, \ldots, 1)$ and thus we obtain that

$$\psi_n([g, f, h]) = (1, \ldots, 1, [\xi, \eta], 1, \ldots, 1).$$

From here we deduce that $\psi_n(N \cap \mathrm{st}_G(n)) \geq \{1\} \times \cdots \times \{1\} \times \psi_{ux}(\mathrm{rst}_G(ux)') \times$

$\{1\} \times \cdots \times \{1\}$, and by transitivity we obtain

$$\psi_n(N \cap \mathrm{st}_G(n)) \geq \prod_{v \in L_n} \psi_v(\mathrm{rst}_G(v)') = \psi_n(\mathrm{rst}_G(n)'),$$

and the result follows because ψ_n is injective. □

Observe that once we have this result, we can obtain much information about the group if we have enough information about $\mathrm{rst}_G(n)'$ for each $n \in \mathbb{N}$.

Definition 1.1.12. *A group G is said to be just-infinite if every non-trivial normal subgroup has finite index in G.*

Corollary 1.1.13. *Let $G \leq \mathrm{Aut}\, T$ be a regular branch over a subgroup K,*

(i) *if $K' \geq \mathrm{st}_G(m)$ for some $m \in \mathbb{N}$, then every non-trivial normal subgroup N contains some level stabilizer,*

(ii) *if K' is of finite index in G, then G is just-infinite.*

Proof. By Lemma 1.1.11 we know that for every non-trivial normal subgroup N, there is some $n \in \mathbb{N}$ such that $N \geq \mathrm{rst}_G(n)'$. Then by the observation before Proposition 1.1.9, since $\psi_n(\mathrm{rst}_G(n)) \geq K \times \overset{d^n}{\ldots} \times K$, we get that $\psi_n(N \cap \mathrm{st}_G(n)) \geq K' \times \overset{d^n}{\ldots} \times K' \geq \mathrm{st}_G(m) \times \overset{d^n}{\ldots} \times \mathrm{st}_G(m)$.

By (iii) in Lemma 1.1.1 we obtain that

$$N \cap \mathrm{st}_G(n) \geq \psi_n^{-1}(\mathrm{st}_G(m) \times \overset{d^n}{\ldots} \times \mathrm{st}_G(m)) = \mathrm{st}_G(n+m).$$

Assertion (ii) follows because if K' has finite index in G then N also has finite index in G by a similar argument as in the proof of Proposition 1.1.9. □

Observe that if we are able to prove (i) of the corollary above, we automatically will have that the group is just-infinite.

1.2 Several problems related to groups of automorphisms

1.2.1 Completions with respect to different topologies

A topological group is a group with a topology, with the condition that the multiplication and the inversion maps must be continuous with respect to this topology. In general, given a group G and anon-empty family $\mathcal{N}$ of finite index subgroups such that

$$\forall N_1, N_2 \in \mathcal{N} \quad \exists N \in \mathcal{N} \text{ such that } N \leq N_1 \cap N_2,$$

we can always define a topology on G by considering $\mathcal{N}$ as a neighbourhood system of the identity. For instance, the topology defined by considering as $\mathcal{N}$ all the subgroups of finite index, is called the profinite topology on G.

Then, one can ask whether two topologies on the same group are the same or not. This question was firstly formulated for the group $\mathrm{SL}_n(\mathbb{Z})$ with the congruence and the profinite topologies, giving name to the famous Congruence Subgroup Problem. More precisely, the congruence topology is given by considering as the fundamental system of neighbourhoods of the identity the kernels of the following surjective homomorphisms:

$$\pi_k : \mathrm{SL}_n(\mathbb{Z}) \longrightarrow \mathrm{SL}_n(\mathbb{Z}/k\mathbb{Z}),$$

with $k \in \mathbb{N}$. It is clear that these subgroups are of finite index, and then the congruence topology is weaker than the profinite topology. Then, the question is whether they are equal or not. In the 19th century Fricke and Klein showed that the answer is negative for $n = 2$ and later in 1964, Bass, Lazard and Serre, and independently Mennicke proved that they are equal for $n \geq 3$.

One can also formulate the question in terms of completions with respect to a given topology. If a group G has two topologies defined by $\mathcal{N}_1$ and $\mathcal{N}_2$ such that $\mathcal{N}_1 \subseteq \mathcal{N}_2$, and we denote by $\widehat{G}^1$ and $\widehat{G}^2$ the completions of G with respect to these two topologies, we always have a surjective homomorphism

$$\alpha : \widehat{G}^2 \longrightarrow \widehat{G}^1.$$

Then the congruence subgroup problem can be reformulated as finding out what is the kernel of this homomorphism. Obviously, the case of these two topologies being equal corresponds to the case of the kernel being trivial.

In the context of subgroups of $\operatorname{Aut} T$, a natural topology is given by considering the level stabilizers as a system of neighbourhoods of the identity. Observe that this system is given exactly by the kernels of the following surjective homomorphisms:

$$\pi_n : G \longrightarrow G_n, \text{ for } n \in \mathbb{N}$$

where G_n denotes the group of automorphisms induced by the action of G on the finite tree T_n consisting of the first n levels. Thus, by analogy to the case of $\mathrm{SL}_n(\mathbb{Z})$, the level stabilizers are called principal congruence subgroups, and every subgroup containing one of them is called a congruence subgroup. It is clear that the level stabilizers are of finite index in G, and then this topology is weaker than the profinite topology. Thus, we say that G has the congruence subgroup property if these two topologies coincide, or which is equivalent, if every subgroup of finite index is a congruence subgroup. Then, in order to see that a group G has the congruence subgroup property, it will suffice to check whether every $N \leq G$ (or equivalently $N \trianglelefteq G$) of finite index contains some level stabilizer.

In several cases, one can also consider other different topologies. For example, when the group is branch, the rigid stabilizers also give a topology which is weaker than the profinite one and stronger than the congruence one. Another possibility is to consider the pro-p topology for a certain prime p, that is, the topology given by the normal subgroups having index a power of p. This topology is always weaker than the profinite topology, but it is not necessarily comparable to the other two topologies.

In Chapter 3 we will discuss the congruence subgroup problem for the family of the multi-GGS-groups for the case of the profinite topology. Later in Chapter 4 we will define the problem in a more general setting and see some examples where even if the profinite topology does not coincide with the congruence topology, the pro-p topology does coincide.

1.2.2 Portrait growth in contracting self-similar groups

The notion of the word growth of a finitely generated group was first introduced by A.S. Schwarz in [36] and independently by J. Milnor in [31] and [32].

For any two functions $f, g : \mathbb{N} \longrightarrow \mathbb{N}$ we write $f \preceq g$ if there is some $C \in \mathbb{N}$ such that $f(n) \leq g(Cn)$ for every $n \in \mathbb{N}$. Thus, we say that these two functions are equivalent, and we write $f \sim g$, if there are some constants C_1, C_2 such that $f(n) \leq g(C_1 n)$ and $g(n) \leq f(C_2 n)$ for all $n \in \mathbb{N}$. For instance, here are some examples of growth types:

- functions of polynomial growth, $f \sim n^d$ for some $d \in \mathbb{N}$,

- of exponential growth, $f \sim e^n$,
- of intermediate growth, if $f \precsim\!\!\!\!/\; e^n$ and $n^d \precsim\!\!\!\!/\; f$ for every $d \in \mathbb{N}$.

Given a group G generated by a symmetric finite set S, where symmetric means that $S = S^{-1}$, one can define the length of an element $g \in G$: we denote by $\partial(g)$ the length of the shortest word in S^* representing g, where S^* denotes the free monoid over the alphabet S. If we want to emphasize the generating set S we will write $\partial_S(g)$. Then we define the word growth function as $\gamma_G(n) = |B(n)|$ where $B(n) = \{g \in G \mid \partial(g) \leq n\}$. Then, the word growth of the group is the growth type of the equivalence class of γ_G. This can be shown not to be dependent on the generating set.

The most common example of a group having exponential growth is the free group and the ones having polynomial growth are exactly the ones that are virtually nilpotent. This last result was proved by Gromov in [27].

The question posed by John Milnor in 1969 about the existence of groups of superpolynomial and subexponential word growth was known as the Milnor Problem, and it was open until 1984. The first example of a group of intermediate growth was the first Grigorchuk group, and this was proved by Grigorchuk himself in [21].

In this direction, apart from the word growth, when a group is self-similar and contracting one can also measure the size of the portraits.

Definition 1.2.1. *Given a self-similar group $G \leq \operatorname{Aut} T$, we say that G is contracting if there exists a finite set $\mathcal{F} \subseteq G$, such that for every $g \in G$ there is some $n \in \mathbb{N}$ in a way that $g_u \in \mathcal{F}$ for every $u \in L_m$ with $m \geq n$.*

Observe that for any two finite sets $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfying the condition of the above definition, $\mathcal{F}_1 \cap \mathcal{F}_2$ also satisfies the condition. Thus, we can consider the intersection of all such sets, which will be the smallest one satisfying the condition, and we call it the **nucleus** of G, denoted by $\mathcal{N}(G)$.

Note. Observe that this $\mathcal{N}$ here has nothing to do with the one defined in the previous subsection.

In particular, if there exist some constants $\lambda < 1$ and $C \geq 0$ such that for every $g \in G$ written as $g = (g_1, \ldots, g_d)\alpha$ we have

$$\partial_S(g_i) < \lambda \partial_S(g) + C, \tag{1.5}$$

for every $i = 1, \ldots, d$, then the group will be contracting. Here with ∂_S we denote the word length for some finite generating set $S = S^{-1}$.

The reason why (1.5) implies that the group is contracting is because if $\partial_S(g) > \frac{C}{1-\lambda}$, then by (1.5) we get that g_i is strictly shorter than g for each $i = 1, \ldots, d$. Then the elements that possibly have no shortening in their sections are the ones that belong to the finite set

$$\mathcal{F} = \left\{ g \in G \mid \partial_S(g) \leq \frac{C}{1-\lambda} \right\}.$$

Thus, if we start with any $g \in G$, if $g \in \mathcal{F}$ we will be done. Else each g_i will be shorter than g for $i = 1, \ldots, d$. If all of them belong to $\mathcal{F}$ we will finish, else we repeat the process. The sections of the ones already belonging to $\mathcal{F}$ will be again in $\mathcal{F}$ by definition. For those that do not belong to $\mathcal{F}$ the sections on the next level will have length reduction, and thus, after finitely many steps we will get that $g_u \in \mathcal{F}$ for every $u \in L_n$ for some $n \in \mathbb{N}$, as desired.

That way, for any element g in a self-similar contracting group G, we can decorate the tree according to the action of g. First of all, we decompose the element as in (1.3):

$$g = (g_1, \dots, g_d)\alpha.$$

Then we start by decorating the root with α and we repeat the process for each g_i for $i = 1, \dots, d$, unless the element g_i belongs to the nucleus. In that case, we decorate this vertex with the element in the nucleus and we stop there.

In this way, we get a finite decorated tree for each element which is called the **nucleus portrait**. Although it is not the same as the usual portrait (which is an infinite decorated tree) when there is no confusion, for simplicity, we will just say portrait when working with nucleus portraits. A concrete example of such a portrait will be given in Section 5.4.

Once we have a finite tree associated to each element in the group, it makes sense to measure the depth of the portrait of each element. That is, for each $g \in G$ we say that the depth of g, denoted by $d(g)$, is the length of the longest path starting at the root in the portrait of g. Then one can count the number of elements of depth less than or equal to n for each $n \in \mathbb{N}$ and ask about its growth. We will refer to it as the portrait growth of a group.

1.3 Some important groups of automorphisms

1.3.1 The first Grigorchuk group

The first group acting on a regular rooted tree, and one of the most important and studied groups of this type, was introduced by R. Grigorchuk in the 80's [26]. This group is a counterexample to the general Burnside problem, since it is three generated, periodic and infinite. The group is defined as follows.

Definition 1.3.1. *Let T be the 2-adic tree. The first Grigorchuk group $\Gamma \leq \operatorname{Aut} T$ is the group generated by the automorphisms $\{a, b, c, d\}$ where a is the rooted automorphism corresponding to $(1\,2)$, and $b, c, d \in \mathrm{st}_\Gamma(1)$ are defined recursively as follows*

$$\psi(b) = (a, c),$$
$$\psi(c) = (a, d),$$
$$\psi(d) = (1, b).$$

It is already known that this group has the congruence subgroup property, see Theorem 10 in [22], that is, any subgroup of finite index contains some level stabilizer. So, as mentioned before, regarding the questions mentioned in the previous section, there is an unsolved one, which is the question about the portrait growth. In fact, Grigorchuk himself in a paper about problems that are solved and unsolved around this group, mentions this problem (see [18, Problem 3.5]).

In order to answer this question, we need some well known properties of Γ. We collect here some significant properties about this group that will be used later in the thesis. All the proofs can be found in [11].

First of all, observe that Γ is obviously self-similar by Lemma 1.1.4. Then we have the following lemma.

Lemma 1.3.2. *[11, Proposition 30, Exercise 81] Let $K = \langle [a, b] \rangle^\Gamma$. Then we have,*

(i) the subgroup K is generated by $\{t = (ab)^2, v = (bada)^2, w = (abad)^2\}$,

(ii) $|\Gamma : K| = 16$,

(iii) $\psi(K) \geq K \times K$,

(iv) The preimage $\psi^{-1}(K \times K)$ *is generated by* $\{v, v^{t^{-1}}, v^t, w, w^{t^{-1}}, w^t\}$*, and it has index* 4 *in* K*.*

Let us consider $S = S^{-1} = \{a, b, c, d\}$ as a generating set and let us denote by $\partial_S(g)$ the word length for g with respect to S. Then we have the following lemma.

Lemma 1.3.3. *[11, Lemma 46] For any* $g \in \Gamma$ *we have*

$$\partial_S(g_i) \leq \frac{1 + \partial_S(g)}{2},$$

for $i = 1, 2$*.*

Observe that from this lemma we deduce that Γ is contracting by the argument discussed after Definition 1.2.1, and that the nucleus is $\mathcal{N}(\Gamma) = \{1, a, b, c, d\}$.

1.3.2 The GGS-groups and generalizations

The GGS-groups, named after Grigorchuk, Gupta and Sidki, are a family of groups generalizing the so called Gupta-Sidki groups and the Gupta-Fabrikowski group. The Gupta-Sidki groups are a family of groups, each of them acting on the p-adic tree for an odd prime p, generated by the rooted automorphism a which has at the root the label $(1\ 2\ \ldots\ p)$ and the automorphism $b \in \mathrm{st}_G(1)$ defined by $\psi(b) = (a, a^{-1}, 1, \ldots, 1, b)$. The Gupta-Fabrikowski group is the group acting on the 3-adic tree generated by the same a but changing b by $\psi(b) = (a, 1, b)$. The GGS-groups are a family generalizing these groups in a very natural way.

Throughout this section let p be an odd prime and T the p-adic tree.

Definition 1.3.4. *Given a non-zero vector* $\mathbf{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$ *the GGS-group* G *defined by* $\mathbf{e}$ *is the group generated by the rooted automorphism* a *corresponding to the permutation* $(1\,2\,\ldots\,p)$ *and* $b \in \mathrm{st}_G(1)$ *defined as follows:* $\psi(b) = (a^{e_1}, \ldots, a^{e_{p-1}}, b)$.

These groups are known to be torsion groups if and only if $\sum_{i=1}^{p-1} e_i = 0$ in $\mathbb{F}_p$ (see [38]). Thus some of them are non-torsion, and as we shall see in Chapter 3, some are virtually torsion-free. For this family of groups the answer to the three problems posed in Section 1.2 was unknown. We solve the question about the congruence subgroup problem and partially the portrait growth problem. It is worth to mention that the word problem is just solved for the Gupta-Fabrikowski group, which has been shown to have intermediate growth. The question remains open for the rest of them.

We collect here some results from [15] that will be used several times.

Proposition 1.3.5. *[15, Theorem 3.2.1 and Corollary 3.2.5] Let* G *be a GGS-group. Then*

(i) $\mathrm{st}_G(1) = \langle b \rangle^G = \langle b, b^a, \ldots, b^{a^{p-1}} \rangle$;

(ii) $\mathrm{st}_G(2) \leq G' \leq \mathrm{st}_G(1)$;

(iii) $|G : G'| = p^2$ *and* $|G : \gamma_3(G)| = p^3$;

(iv) $\mathrm{st}_G(2) \leq \gamma_3(G)$.

Sometimes in order to simplify notation we will write $b_i = b^{a^i}$ for $i \in \mathbb{Z}$. Notice that it suffices to consider $i = 0, \ldots, p-1$ because $b_i = b_j$ if $i = j$ (mod p). We say that $\mathbf{e}$ is symmetric if $e_i = e_{p-i}$ for $i = 1, \ldots, \frac{p-1}{2}$.

Proposition 1.3.6. *[15, Lemmas 3.3.1 and 3.3.3] Let* G *be a GGS-group* *with non-constant defining vector. Then*

$$\psi(\gamma_3(\mathrm{st}_G(1))) = \gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G).$$

If the defining vector is also non-symmetric, then

$$\psi(\mathrm{st}_G(1)') = G' \times \overset{p}{\cdots} \times G'.$$

Let us consider as a generating set for any GGS-group the set $S = S^{-1} = \{a, a^2, \ldots, a^{p-1}, b, b^2, \ldots, b^{p-1}\}$ and denote the word length by ∂_S.

Lemma 1.3.7. *Let G be a GGS-group, then for every $g \in G$ we have*

$$\partial_S(g_i) \leq \frac{1 + \partial_S(g)}{2},$$

for every $i = 1, \ldots, p$.

Proof. Let $g \in G$ and let $w = a^{i_1} b^{j_1} \ldots a^{i_k} b^{j_k} a^{k+1}$ the shortest word over S representing g, where the only exponents that are allowed to be zero are i_1, i_{k+1}. Then the length of g is $2k-1, 2k$ or $2k+1$. When we look at the sections on the first level, the powers of a do not contribute anything. The only ones contributing some length are the powers of b. Thus $\partial_S(g_i) \leq k \leq \frac{\partial_S(g)+1}{2}$, for $i = 1, \ldots, p$. □

By the same discussion as after Lemma 1.3.3 we deduce that every GGS-group is contracting with $\mathcal{N}(G) = \{1, a, a^2, \ldots, a^{p-1}, b, b^2, \ldots, b^{p-1}\}$.

It is worth mentioning that since two proportional vectors define the same group, in particular we may assume that that the group defined by a constant vector is the one defined by the vector $\mathbf{e} = (1, \ldots, 1)$. Since the GGS-group with constant defining vector plays a very different role in this family of groups, let us denote it by $\mathcal{G}$ from now on.

The GGS-group defined by a constant vector has very different behaviour from the rest. Many of the ingredients for our proofs later come from the analysis of this group developed in [15, Section 4]. Following that paper, we define $y_0 = ba^{-1}$ and $y_i = y_0^{a^i}$ for every integer i and note that $y_i^b = y_i^{aa^{-1}b} = y_{i+1}^{y_1}$. An easy computation shows that $y_{p-1}y_{p-2}\dots y_1y_0 = 1$.

We state the following two lemmas from [15], which will be used in Chapter 3.

Lemma 1.3.8. *[15, Lemma 4.2] If $K = \langle y_0\rangle^{\mathcal{G}}$, then:*

(i) $|\mathcal{G} : K| = p$, *and as a consequence,* $\mathrm{st}_{\mathcal{G}}(n) \leq K$ *for every* $n \geq 2$.

(ii) $K = \langle y_0, \dots, y_{p-1}\rangle$.

(iii) $K' \times \overset{p}{\cdots} \times K' \leq \psi(K') \leq \psi(\mathcal{G}') \leq K \times \overset{p}{\cdots} \times K$. *In particular, $\mathcal{G}$ is a weakly regular branch group over K'.*

Lemma 1.3.9. *[15, Lemmas 4.3 and 4.4] For every element $g \in K$ we have $gg^ag^{a^2}\dots g^{a^{p-1}} \in K'$. Moreover, if $h \in K'$ with $\psi(h) = (h_1, \dots, h_p)$ then $h_p \dots h_1 \in K'$.*

In [1] a generalization of the GGS-groups is given, by adding more generators of the type of the previous b, and they are called multi-GGS-groups.

Definition 1.3.10. *Given a family of linearly independent vectors $\mathbf{e}_1, \dots, \mathbf{e}_r \in \mathbb{F}_p^{p-1}$ the multi-GGS-group G defined by $\mathbf{e}_1, \dots, \mathbf{e}_r$ is the group generated by the rooted automorphism a corresponding to the permutation $(1\,2\,\dots\,p)$ and $b_i \in \mathrm{st}_G(1)$ defined as follows: $\psi(b_i) = (a^{e_{i,1}}, \dots, a^{e_{i,p-1}}, b_i)$, where $\mathbf{e}_i = (e_{i,1}, \dots, e_{i,p-1})$, for $i = 1, \dots, r$.*

The following lemma is a collection of results from [1].

Lemma 1.3.11. *Let $G = \langle a, b_1, \dots, b_r \rangle$ be a multi-GGS-group with defining vectors $\mathbf{e}_1, \dots, \mathbf{e}_r \in \mathbb{F}_p^{p-1}$.*

(i) There is a conjugate $\tilde{G}$ of G in $\operatorname{Aut} T$ defined by $\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}_r}$ such that $\tilde{e}_{i,1} = 1$ for all $\tilde{\mathbf{e}}_i$ with $i = 1, \dots, r$.

(ii) If G is not $\mathcal{G}$ then

$$\psi(\gamma_3(\mathrm{st}_G(1))) = \gamma_3(G) \times \overset{p}{\ldots} \times \gamma_3(G).$$

(iii) $G/G' \cong C_p^{r+1}$.

We remark that the multi-GGS-groups are contained in the Sylow pro-p subgroup of $\operatorname{Aut} T$ consisting of all automorphisms for which the permutation induced at every vertex of T is a power of σ, with $\sigma = (1 \dots p)$. This in particular implies that $\mathrm{st}_G(1)/\mathrm{st}_G(2)$ is abelian, since it is contained in $C_p \times \overset{p}{\ldots} \times C_p$.

Observe also that we can replace each b_i defined by $\mathbf{e}_i$ by other element defined by $\alpha\mathbf{e}_i + \beta\mathbf{e}_j$ for any $\alpha, \beta \in \mathbb{F}_p$ with $\alpha \neq 0$ and any $j = 1, \dots, r$ with $j \neq i$, and we will obtain the same multi-GGS-group. In other words, instead of considering the defining vectors, each vector subspace of $\mathbb{F}_p^{p-1}$ defines authomatically a multi-GGS-group. This means that if we consider the matrix

$$\begin{pmatrix} e_{1,1} & e_{1,2} & \cdots & e_{1,p-1} \\ e_{2,1} & & \cdots & e_{2,p-1} \\ \vdots & & \ddots & \vdots \\ e_{r,1} & e_{r,2} & \cdots & e_{r,p-1} \end{pmatrix},$$

the row echelon form of the above matrix will define the same multi-GGS-group.

1.3.3 The Hanoi Towers group and the Apollonian group

The Hanoi Towers group was introduced by Grigorchuk and Sunic in [19]. These groups are called in this way because they illustrate the well known problem of the Hanoi Towers on three pegs. In the same paper one can find a more general definition of the group simulating the play on d pegs.

Definition 1.3.12. *Let $d \geq 3$ and T the d-adic tree, where $X = \{x_1, \dots, x_d\}$. For $1 \leq i < j \leq d$, we define the element a_{ij} which has the permutation $(x_i\, x_j)$ at the root and for each vertex x on the first level:*

$$(a_{ij})_x = \begin{cases} 1 & \text{if } x = i, j \\ a_{ij} & \text{else.} \end{cases}$$

The Hanoi Towers group is $H = \langle a_{ij} \mid 1 \leq i < j \leq d \rangle$.

For instance the Hanoi Towers group on three pegs can be described using (1.3) as the group H generated by a, b, c where

$$a = (1, 1, a)(1\,2),$$
$$b = (1, b, 1)(1\,3),$$
$$c = (c, 1, 1)(2\,3).$$

The generators represent the movements we are allowed to do in order to solve the game. The game consists on the following: having n disks on 3 pegs, in the initial situation all disks are in one peg, ordered by their size, the biggest on the bottom and the smallest at the top. The aim of the game is to end up with the same configuration but in a different peg. The condition is that one is never allowed to put a disk on top of a smaller one. Then a sequence of n letters on $\{1, 2, 3\}$ encodes a configuration by saying where each disk is. For instance for $n = 4$, the sequence $(1, 2, 1, 3)$ encodes that the biggest disk is on peg 3, the next biggest one on peg 1, the third biggest one on peg 2 and the smallest one on peg 1. Since the disks must be ordered by their size, there is no confusion and the sequence encodes all the information. Then the element a starts reading the sequence and it does nothing while it reads 3 and at the first time it sees a 1 or a 2 it flips it to 2 or 1, and then becomes the identity. This means that a does the only movement we are allowed to do between pegs 1 and 2. Similar for b and c. Obviously, a, b and c are automorphisms of order 2.

The Hanoi Towers group on three pegs has been shown to be regular branch over its commutator H' which is of index 8 in H (see [19]). In the same paper it is also proved that $\psi^{-1}(H' \times H' \times H')$ has index 12 in H'. For the rest of the family it is known that at least they are weakly branch.

The Apollonian group is a subgroup of the Hanoi Towers group on three pegs, and it was introduced in [24] by Grigorchuk, Nekrashevych and Sunic. Some of the following facts are claimed (without proof) in the same paper.

In order to define the Apollonian group as a self-similar group, it is convenient to work with an isomorphic version of the Hanoi Towers group.

Lemma 1.3.13. *The group H is conjugate in $\operatorname{Aut} T$ to the group generated by*

$$a' = (1, 1, b')(1\,2),$$
$$b' = (1, a', 1)(1\,3),$$
$$c' = (c', 1, 1)(2\,3).$$

Proof. Consider the element $g = (h, h, h)(2\,3)$ with $h = (g, g, g)$. Then

$$a^g = (1, a^h, 1)(1\,3),$$
$$b^g = (1, 1, b^h)(1\,2),$$
$$c^g = (c^h, 1, 1)(2\,3).$$

On the other hand, we have

$$a^h = (1, 1, a^g)(1\,2),$$
$$b^h = (1, b^g, 1)(1\,3),$$
$$c^h = (c^g, 1, 1)(2\,3).$$

Observe that a^g and b^h have the same recursive definition, and the same for b^g and a^h; and c^g and c^h. Thus renaming $a^g = b^h$ by b', $b^g = a^h$ by a' and $c^g = c^h$ by c' we obtain the desired result. □

From now on, when working with the Apollonian group, we will always consider the group H to be generated by three automorphisms a, b and c as a', b' and c' in the previous lemma, that is:

$$a = (1, 1, b)(1\,2),$$
$$b = (1, a, 1)(1\,3),$$
$$c = (c, 1, 1)(2\,3).$$

Observe that, being conjugate to the original generators of the Hanoi Towers group, these automorphisms are all of order 2.

Definition 1.3.14. *The Apollonian group A acting on the ternary tree is the group generated by $x = cab, y = abc, z = bca$.*

We collect here some important facts about this group.

Theorem 1.3.15. *Let $A = \langle x, y, z\rangle$ be the Apollonian group. Then*

(i) *A is a normal subgroup of index 4 in H,*

(ii) *A contains the commutator H',*

(iii) *A is regular branch over H',*

(iv) *the subgroup H' is of index 2 in A and corresponds to the words of even length over $S = \{x, y, z, x^{-1}, y^{-1}, z^{-1}\}$.*

Proof. In order to prove (i) it suffices to check that

$$\begin{aligned}(cab)^c &= abc,\\ (cab)^b &= bca,\\ (cab)^a &= acaba = (acb)(bac)(cba),\end{aligned}$$

and similar for the rest of generators of A. It is also easy to check that $H/A = \langle \overline{a}, \overline{b}\rangle \cong C_2 \times C_2$, since $abab = abccab$ and $\overline{c} = \overline{ab}$.

Now, (ii) is clear since H/A is abelian, and since by Theorem 5.1 in [19] we know that H is regular branch over H', so is A. In the same theorem they proved that H/H' is of order 8, which automatically implies that A/H' is or order 2.

Finally, let us denote by E the subgroup of elements of even length over the alphabet $\{x, y, z, x^{-1}, y^{-1}, z^{-1}\}$. We claim that $E = H'$. It is clear that E is normal in A and that the index of E in A is 2. Let us see that the commutators $[a, b] = abab, [a, c] = acac, [b, c] = bcbc$ belong to E and we will be done. This follows from

$$abab = abccab,$$
$$acac = (bca)^{-1}(cab)^{-1},$$
$$bcbc = bcaabc.$$

□

Notice that the group A as defined is not a self-similar group because

$$x = cab = (ca, b, 1)(1\,2),$$
$$y = abc = (a, 1, bc)(1\,3),$$
$$z = bca = (1, ab, c)(2\,3).$$

However, as before, by conjugating with an element in $\operatorname{Aut} T$ we can get a self-similar group which is isomorphic to this one.

Lemma 1.3.16. *The Apollonian group A is isomorphic to the group generated by the following three automorphisms*

$$x' = (1, y', 1)(1\,2),$$
$$y' = (x', 1, 1)(1\,3),$$
$$z' = (1, 1, z')(2\,3).$$

Proof. Let us consider $h = (ch, ah, bh)$. Then we get

$$x^h = (1, y^h, 1)(1\,2),$$
$$y^h = (x^h, 1, 1)(1\,3),$$
$$z^h = (1, 1, z^h)(2\,3),$$

and renaming x^h by x', y^h by y' and z^h by z' we obtain the desired result. □

From now on we will consider A to be generated by x, y, z as x', y', z' in the previous lemma, that is:

$$x = (1, y, 1)(1\,2),$$
$$y = (x, 1, 1)(1\,3),$$
$$z = (1, 1, z)(2\,3).$$

Lemma 1.3.17. *Let A be the Apollonian group. Considering as generating set $S = \{x, y, z, x^{-1}, y^{-1}, z^{-1}\}$, the group A is contracting with $\mathcal{N}(A) = \{1, x, y, z, x^{-1}, y^{-1}, z^{-1}\}$.*

Proof. Let us consider $g \in A$ and let w be the shortest word over S representing g. We will prove by induction on the length of g that there is some $k \in \mathbb{N}$ such that $g_u \in \mathcal{N}(A)$ for every $u \in L_m$ for $m \geq k$.

For the elements of length 1 it is clear from the definition. Let us prove also the case $\partial_S(g) = 2$.

Direct calculation shows

$$
\begin{aligned}
xy &= (1, yx, 1)(1\,2\,3) & xz &= (1, y, z)(1\,3\,2)\\
xy^{-1} &= (1, y, x^{-1})(1\,2\,3) & xz^{-1} &= (z^{-1}, y, 1)(1\,3\,2)\\
yx &= (x, y, 1)(1\,3\,2) & yz &= (xz, 1, 1)(1\,2\,3)\\
yx^{-1} &= (x, 1, y^{-1})(1\,3\,2) & yz^{-1} &= (x, z^{-1}, 1)(1\,2\,3)\\
zx &= (1, 1, zy)(1\,2\,3) & zy &= (x, 1, z)(1\,3\,2)\\
zx^{-1} &= (y^{-1}, 1, z)(1\,2\,3) & zy^{-1} &= (1, x^{-1}, z)(1\,3\,2).
\end{aligned}
$$

This shows that for $k = 2$ any section of an element g of length 2 is of length 1, and then for every vertex u in L_m with $m \geq 2$ we get $g_u \in \mathcal{N}(A)$.

Let us now suppose that the statement is true for any elements of length shorter than n and consider g such that w, the shortest word over S representing g, has length n. Then $w = w's$ with $s \in S$. By the formulas for sections (see 1.1) for every $u \in T$ we have $w_u = w'_u s_{w'(u)}$. Since w' is of length $n-1$ we know that there is some $k \in \mathbb{N}$ such that $w'_u \in \mathcal{N}(A)$, and since $s \in S$ we know that $s_{w'(u)} \in \mathcal{N}(A)$. Thus w_u is a word of length at most 2, and since for those we know that k is at most 2, we obtain that for any $v \in L_m$ with $m \geq k+2$ we will have $g_v \in \mathcal{N}(A)$ as desired. □

1.3.4 The Basilica group

This group was defined by R. Grigorchuk and A. Zuk in [25]. The name came later on, because its Schreier graphs have the shape of the basilica of San Marcos, in Venice. The Schreier graphs of groups acting on rooted trees, even if we will not enter in detail with them, roughly speaking are the orbital graphs on each level. That is, considering a particular vertex u on level $n \in \mathbb{N}$, for each generator s of the group, we match with a labelled edge u and $s(u)$, and we keep doing this for $s(u)$ and so on. This can be seen to converge to a graph when n goes to infinity and that is the Schreier graph of the group.

In the same paper they prove that this group is torsion-free and weakly branch. We collect here the definition and some of these results.

Definition 1.3.18. *Let T be the binary tree. The Basilica group G is generated by two automorphisms a and b defined recursively as follows:*

$$\begin{aligned} a &= (1, b) \\ b &= (1, a)\varepsilon, \end{aligned}$$

where ε denotes the swap at the root.

Lemma 1.3.19. *Let G be the Basilica group. Then,*

(i) *G acts transitively on all levels of T,*

(ii) *G is fractal; that is, $\psi_u(\mathrm{st}_G(u)) = G$ for any $u \in T$,*

(iii) *$\psi(G') \geq G' \times G'$, so G is weakly branch over G',*

(iv) *$\psi(G'') = \gamma_3(G) \times \gamma_3(G)$,*

(v) *$G' = \psi^{-1}(G' \times G') \rtimes \langle [a, b] \rangle$,*

(vi) *$\gamma_3(G) = \psi^{-1}(\gamma_3(G) \times \gamma_3(G)) \rtimes \langle [a, b, b] \rangle$,*

(vii) *$G/G' = \langle aG' \rangle \times \langle bG' \rangle \cong \mathbb{Z} \times \mathbb{Z}$,*

(viii) *G is torsion-free.*

As we will see later on in Chapter 3, the fact that G maps onto two copies of $\mathbb{Z}$ and the fact that when $p = 2$ the whole group $\operatorname{Aut} T$ is a Sylow pro-2 subgroup, prevent G from having the congruence subgroup property. That is, the topology given by level stabilizers cannot be the same as the profinite topology. In Chapter 4 we will see that the congruence topology coincides with the pro-2 topology.

Regarding the word growth, in the same paper, Grigorchuk and Zuk proved that G has exponential word growth. The reason for that is that the monoid generated by a and b is free.

Chapter 2

On the concept of fractality for groups of automorphisms of a regular rooted tree

2.1 Introduction

Given a group $G \leq \operatorname{Aut} T$ the fractality properties of this group may be an interesting tool in order to prove results using induction on the length of a vertex or the length of a word with respect to a given generating set. There are several terms that have been used in order to refer to this concept: self-replicating, recurrent, fractal... There is also some confusion in the literature about the definition and equivalences between concepts that are related to this one, as we shall specify later on. The aim of this chapter is to clarify all these notions.

If G is the whole group $\operatorname{Aut} T$, then the homomorphisms ψ_u and ψ_n are surjective onto $\operatorname{Aut} T$ and $\operatorname{Aut} T \times \overset{d^n}{\cdots} \times \operatorname{Aut} T$, respectively. On the other hand, if G is self-similar then the images of ψ_u and ψ_n are contained in G and $G \times \overset{d^n}{\ldots} \times G$, and we will consider these sets to be the codomains of those maps. It is natural to ask whether ψ_u and ψ_n are also onto in this case. For many interesting groups, ψ_u is known to be onto, i.e. $\psi_u(\mathrm{st}_G(u)) = G$ for each $u \in T$, and the group G is then called **fractal**, recurrent or self-replicating (see [8, 23]). However, in general it is too strong to ask ψ_n to be surjective, and we content ourselves with the image of ψ_n being a subdirect product of $G \times \overset{d^n}{\ldots} \times G$, namely that $\psi_u(\mathrm{st}_G(n)) = G$ for each $u \in L_n$.

In the case of the vertex stabilizers, once the condition is satisfied for the vertices on the first level, the property is inherited by the rest of the vertices. In some papers, the condition of the surjectivity of ψ_u for the whole level stabilizer is only required for $n = 1$; however, as we shall see, it is not always inherited by the rest of the levels. Thus it is necessary to make a distinction between these two concepts. Following terminology from previous papers, G is said to be **strongly fractal** or strongly self-replicating if $\psi_u(\mathrm{st}_G(1)) = G$ for all $u \in L_1$. We say that G is **super strongly fractal** if $\psi_u(\mathrm{st}_G(n)) = G$ for each $n \in \mathbb{N}$ and $u \in L_n$.

Obviously, every super strongly fractal group is also strongly fractal, and every strongly fractal group is fractal, but there is some confusion in the literature about the converse. In several papers, fractal groups are claimed to be the same as strongly fractal groups, or else fractal groups are simply introduced by using the definition of strongly fractal groups (see [3, 8, 9, 10, 12]). In some other papers, a distinction is made between these two concepts (see [4, 23]), but no examples can be found in the literature where a certain fractal group is shown not to be strongly fractal. On the other hand, strongly fractal and super strongly fractal groups have not been clearly distinguished either, see for example the paragraph after Definition 3.6 in [23]. This would mean that being strongly fractal and super strongly fractal are equivalent, but as mentioned before, this is not the case. In fact, it is said that the first Grigorchuk group is an example of this fact. It is true that the first Grigorchuk group is super strongly fractal, but it is not a direct consequence of being strongly fractal, as we shall see at the end of this chapter.

Our aim in this chapter is to fill this gap, and thus we show the following result.

Theorem. *There exist groups acting on regular rooted trees which are fractal but not strongly fractal, and there are also strongly fractal groups which are not super strongly fractal.*

On the one hand, for every $d \geq 3$, we give explicit examples of groups that are fractal but not strongly fractal. More specifically, we show that a certain subgroup of the Hanoi Towers group is of this type. We remark that the restriction to $d \geq 3$ is necessary for these examples to exist, since one can easily show that for $d = 2$ a fractal group is always strongly fractal. In proving that those groups are not strongly fractal, we have obtained a couple of results that allow us to estimate the image of a level stabilizer under ψ_u, which may have some interest of their own. On the other hand, we also give examples of groups which are strongly fractal but not super strongly fractal, and examples of super strongly fractal groups.

2.2 Preliminaries

Let X be a set with d elements and T the d-adic tree. We briefly remember the notions mentioned in the introduction that will play the main role in this chapter. Given a group $G \leq \operatorname{Aut} T$ for each $n \in \mathbb{N}$ we have the homomorphism $\psi_n : \operatorname{st}_G(n) \longrightarrow \operatorname{Aut} T \times \cdots \times \operatorname{Aut} T$ and for each vertex u the homomorphism $\psi_u : \mathrm{st}_G(u) \longrightarrow \mathrm{Aut}\, T$. In the case of G being self-similar these two families of homomorphisms take images in $G \times \cdots \times G$ and G respectively. As mentioned, even if in the case of the whole group of automorphisms $\mathrm{Aut}\, T$ the homomorphisms ψ_n and ψ_u are surjective, in general different situations arise. According to this we have the definitions given in Definition 1.1.5.

Notice that the definition of being super strongly fractal does not imply that ψ_n is surjective from $\mathrm{st}_G(n)$ to $G \times \overset{d^n}{\cdots} \times G$, but only that $\psi_n(\mathrm{st}_G(n))$ is a subdirect product in $G \times \overset{d^n}{\cdots} \times G$. The same remark applies to strongly fractal groups with $n = 1$.

There is a special case in which the first two definitions are equivalent.

Lemma 2.2.1. *Let $G \leq \mathrm{Aut}\, T$ and consider a d-cycle $\sigma \in S_X$. If for each $g \in G$ we have $g_{(\emptyset)} = \sigma^k$ for some $k \in \mathbb{N}$ and G is fractal, then G is strongly fractal.*

Proof. Let $g \in \mathrm{st}_G(x)$ for $x \in X$. Then $\sigma^k(x) = x$ which only happens if $k \equiv 0 \pmod d$. This implies that $g \in \mathrm{st}_G(1)$, so $\mathrm{st}_G(x) = \mathrm{st}_G(1)$. □

Observe that for $d = 2$ the label at the root must be 1 or $(1\,2)$, so according to the previous lemma, in this case being fractal and being strongly fractal are equivalent.

This can be generalised, to obtain another important corollary that follows from the previous lemma in the case $d = p$ where p is a prime. If we consider T to be the p-adic tree, $\mathrm{Aut}\, T$ is a profinite group which has a standard Sylow pro-p subgroup consisting of all automorphisms which have powers of a fixed p-cycle as a label in every vertex. Then, the previous lemma shows that for every subgroup of the Sylow pro-p subgroup being fractal and strongly fractal are equivalent. For example, this happens for the

GGS-groups.

One of our goals is to give examples of subgroups of $\operatorname{Aut} T$ for $d \geq 3$ which are fractal but are not strongly fractal. All the examples that we present will be level transitive groups. Moreover, under the condition of being level transitive it is easier to check if a group is fractal or not.

The following lemma shows that to be fractal and level transitive it is enough if the condition of being fractal is satisfied for some vertex in the first level and the condition of being level transitive holds in the first level.

Lemma 2.2.2. *Let $G \leq \operatorname{Aut} T$. We have*

(i) *if $\psi_x(\mathrm{st}_G(x)) = G$ for every $x \in X$ then G is fractal,*

(ii) *if G is transitive on the first level and $\psi_x(\mathrm{st}_G(x)) = G$ for some $x \in X$, then G is fractal and level transitive.*

Proof. First of all, let us prove (i). We deal by induction on the length of the vertices. Assumption already gives the case $n = 1$, so let us consider an arbitrary vertex $v \in L_{n+1}$ and write $v = xu$ with $u \in L_n$ and $x \in X$. By inductive assumption we know that for any $g \in G$ there is some $h \in \mathrm{st}_G(u)$ such that $\psi_u(h) = g$. On the other hand, there is some $f \in \mathrm{st}_G(x)$ with $\psi_x(f) = h$. Then, $f \in \mathrm{st}_G(v)$ and

$$\psi_v(f) = \psi_u(\psi_x(f)) = \psi_u(h) = g.$$

In order to see (ii), first of all let us see that $\psi_y(\mathrm{st}_G(y)) = G$ for each $y \in X$, and thus by (i) we will already prove that G is fractal. Since G is transitive on the first level for each $y \in X$ there is some $g \in G$ such that $g(x) = y$. Then $\mathrm{st}_G(x)^g = \mathrm{st}_G(y)$. The result follows because using formula (1.2) we get

$$\psi_y(\mathrm{st}_G(x)^g) = \psi_x(\mathrm{st}_G(x))^{g_x} = G^{g_x} = G.$$

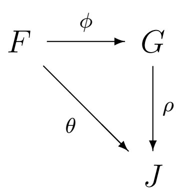
It only remains to prove that G is level transitive. We also prove that by induction on the length of the vertices. Again, the base case is just the hypothesis. Consider $u = vx$ and $u' = v'x'$ with $v, v' \in L_n$ and $x, x' \in X$. By inductive assumption there is some $g \in G$ such that $g(v) = v'$. On the other hand, there is also some $f \in G$ such that $f(g_v(x)) = x'$. Since G is fractal, there is $h \in \mathrm{st}_G(v')$ with $h_{v'} = f$. Thus

$$gh(vx) = h(g(vx)) = h(g(v)g_v(x)) = h(v')h_{v'}(g_v(x)) = v'x'.$$

$\square$

Since we will want to prove that a group is not strongly fractal, we are interested in identifying the first level stabilizer. We present a tool that we have developed in order to do this in the following lemma. Let us denote by ρ the homomorphism from G to S_d that sends each $g \in G$ to the label of g at the root, $g_{(\emptyset)}$.

Lemma 2.2.3. *Let $G \leq \mathrm{Aut}\, T$ and put $J = \rho(G)$. Suppose that we have a presentation $J = \langle Y \mid R \rangle$ and let $\theta : F \longrightarrow J$ be the epimorphism corresponding to this presentation, where F is the free group generated by Y. If there exists a surjective homomorphism $\phi : F \longrightarrow G$ making the following diagram commutative,*



(2.1)

then,

$$\mathrm{st}_G(1) = \langle \phi(R) \rangle^G.$$

Proof. We know that $\ker \theta = \langle R \rangle^F$. On the other hand, since ϕ is surjective, every $g \in G$ can be written as $g = \phi(x)$ for some $x \in F$, and then $g \in \ker \rho$

if and only if $x \in \ker(\rho \circ \phi)$. Consequently,

$$\begin{aligned} \mathrm{st}_G(1) &= \ker\rho = \phi(\ker(\rho \circ \phi)) \\ &= \phi(\ker\theta) = \phi(\langle R\rangle^F) \\ &= \langle \phi(R)\rangle^G. \end{aligned}$$

□

Notice that the actual condition we are asking about ϕ is to be surjective, because by the universal property of free groups we are always able to construct some ϕ making the diagram (2.1) commutative. In other words, the point is whether for each $y \in Y$ we can choose an element $g_y \in \rho^{-1}(\theta(y))$, in such a way that $\{g_y \mid y \in Y\}$ generates the whole group G or not.

Now, in the following lemma we present another new result, which will help us to prove that the image of a level stabilizer under ψ_u is strictly contained in G.

Lemma 2.2.4. *Let $G \leq \mathrm{Aut}\, T$ be a self-similar group. If $K = \langle S\rangle^G \subseteq \mathrm{st}_G(n)$ for some $n \in \mathbb{N}$ and $\psi_u(S) \subseteq N$ for each $u \in L_n$, where $N \trianglelefteq G$, then $\psi_u(K) \subseteq N$ for each $u \in L_n$.*

Proof. Consider $k \in K$ and let us write $k = (s_1^{\epsilon_1})^{g_1} \dots (s_r^{\epsilon_r})^{g_r}$ where $\epsilon_i \in \{-1, 1\}$, $s_i \in S$ and $g_i \in G$ for each $i = 1, \dots, r$. Let $u \in L_n$. Since $K \leq \mathrm{st}_G(n)$ we know that $k \in \mathrm{st}_G(u)$ and we have

$$\psi_u(k) = \psi_u(s_1^{g_1})^{\epsilon_1} \dots \psi_u(s_r^{g_r})^{\epsilon_r}.$$

Thus it is enough to see that $\psi_u(s^g) \in N$ for each $s \in S$, $g \in G$. Since $G \leq \mathrm{Aut}\, T$ and $\mathrm{Aut}\, T = H_n \ltimes \mathrm{st}(n)$ by Proposition 1.1.2, we write each

$g = ht$ where $h \in H_n$ and $t \in \mathrm{st}(n)$. Now by (1.4) we have

$$\psi_u(s^g) = (s_{h^{-1}(u)})^{g_{h^{-1}(u)}}$$

for each $u \in L_n$, and since $\psi_{h^{-1}(u)}(S) \subseteq N$ and N is normal in G, it is enough to check that $g_{h^{-1}(u)}$ belongs to G. This follows from the fact that G is self-similar and we are done. □

Now, let us introduce a stronger version of the previous lemma that will help us to check whether a strongly fractal group is super strongly fractal or only strongly fractal.

Lemma 2.2.5. *Let G be level transitive and super strongly fractal. If $K = \langle S \rangle^G \subseteq \mathrm{st}_G(n)$ for some $n \in \mathbb{N}$, then $\psi_u(K) = \langle \psi_v(S) \mid v \in L_n \rangle^G$ for any $u \in L_n$.*

Proof. Let us denote $N = \langle \psi_v(S) \mid v \in L_n \rangle^G$. Since $\psi_u(S) \subseteq N$ for every $u \in L_n$, which is a normal subgroup, the inclusion $\psi_u(K) \subseteq N$ follows from the previous lemma.

Now, let $g = (\psi_{u_1}(s_1)^{\epsilon_1})^{g_1} \dots (\psi_{u_r}(s_r)^{\epsilon_r})^{g_r} \in N$. Since G is level transitive for every $u_i \in \{u_1, \dots, u_r\}$, there is some $f_i \in G$ such that $f_i(u_i) = u$. Then, by (1.4)

$$\psi_u(s_i^{f_i})^{((f_i)_{u_i})^{-1}} = \psi_{u_i}(s_i).$$

Then we can write $g = (\psi_u(s_1^{f_1})^{\epsilon_1})^{g_1'} \dots (\psi_u(s_r^{f_r})^{\epsilon_r})^{g_r'}$, where $g_i' = ((f_i)_{u_i})^{-1} g_i \in G$. From the fact that G is super strongly fractal, we know that there are some $h_i \in \mathrm{st}_G(n)$ such that $\psi_u(h_i) = g_i'$ for $i = 1, \dots, r$. We conclude because

$$\begin{aligned} g &= (\psi_u(s_1^{f_1})^{\epsilon_1})^{g_1'} \dots (\psi_u(s_r^{f_r})^{\epsilon_r})^{g_r'} \\ &= (\psi_u(s_1^{f_1})^{\epsilon_1})^{\psi_u(h_1)} \dots (\psi_u(s_r^{f_r})^{\epsilon_r})^{\psi_u(h_r)} \\ &= \psi_u((s_1^{\epsilon_1})^{f_1 h_1} \dots (s_r^{\epsilon_r})^{f_r h_r}) \in \psi_u(K). \end{aligned}$$

□

Observe that the result and the proof work in the same way if we replace every n by $n = 1$. Thus we have the same result for strongly fractal groups.

Corollary 2.2.6. *Let G be a strongly fractal group which acts transitively on the first level. If $K = \langle S \rangle^G$ and $K \subseteq \mathrm{st}_G(1)$ then $\psi_x(K) = \langle \psi_y(S) \mid y \in X \rangle^G$ for any $x \in X$.*

Finally let us introduce another lemma that will help us to prove that a group is super strongly fractal. This lemma tells us that, in some cases, it suffices to check whether in each level stabilizer there are elements whose sections at vertices on this level generate the whole group.

Lemma 2.2.7. *Let $G \leq \mathrm{Aut}\, T$ be a self-similar group such that there is a rooted automorphism $a \in G$, with $a_{(\emptyset)}$ a d-cycle. If for each $n \in \mathbb{N}$ we have $\langle \psi_u(\mathrm{st}_G(n)) \mid u \in L_n \rangle = G$, then G is super strongly fractal.*

Proof. The proof works by induction on the length of the vertices. Let $x \in X$ and $g \in G$. We know that there are some $y_1, \dots, y_r \in X$ such that $g = \psi_{y_1}(g_1)^{\epsilon_1} \dots \psi_{y_r}(g_r)^{\epsilon_r}$, where $g_i \in \mathrm{st}_G(1)$ and $\epsilon_i \in \{1, -1\}$. Then for each $i = 1, \dots, r$ we have $a^{j_i}(y_i) = x$ for some $j_i \in \{0, \dots, d-1\}$. Then considering $g_i^{a^{j_i}}$ we get an element on the first level stabilizer such that $(g_i^{a^{j_i}})_x = (g_i)_{y_i}$. Then the element $h = (g_1^{a^{j_1}})^{\epsilon_1} \dots (g_r^{a^{j_r}})^{\epsilon_r} \in \mathrm{st}_G(1)$ satisfies $h_x = g$, so $\psi_x(\mathrm{st}_G(1)) = G$.

Now let us suppose that we know the result for length $n-1$ and let us see it for n. Let $v = x_1 \dots x_n$ and $g \in G$. By assumption we know that $g = \psi_{w_1}(g_1)^{\epsilon_1} \dots \psi_{w_r}(g_r)^{\epsilon_r}$ where $w_i \in L_n$, $g_i \in \mathrm{st}_G(n)$ and $\epsilon_i \in \{1, -1\}$ for each $i = 1, \dots, r$. It suffices to show that for $i = 1, \dots, r$ there is some $h_i \in \mathrm{st}_G(n)$ such that $(h_i)_v = (g_i)_{w_i}$, because then $h = h_1^{\epsilon_1} \dots h_r^{\epsilon_r} \in \mathrm{st}_G(n)$ and $h_v = g$, as desired.

Let w be an arbitrary vertex in L_n. Then $w = y_1 \dots y_n$ with $y_i \in X$. For each $k = 1, \dots, n$ there is some $j_k = 0, \dots, d-1$ such that $a^{j_k}(y_k) = x_k$. By inductive assumption $a \in \psi_u(\mathrm{st}_G(k))$ for every $u \in L_k$, with $k = 1, \dots, n-1$. Thus, for each $k = 1, \dots, n-1$ there is some $f_k \in \mathrm{st}_G(k)$ such that $(f_k)_{y_1 \dots y_k} = a^{j_{k+1}}$. Then if we consider the element $f = a^{j_1} f_1 \dots f_{n-1}$, which belongs to H_n, we obtain that

$$\begin{aligned} f(w) &= (a^{j_1} f_1 \dots f_{n-1})(y_1 \dots y_n) \\ &= f_{n-1}(f_{n-2} \dots (f_1(a^{j_1}(y_1 \dots y_n))) \dots) \\ &= f_{n-1}(f_{n-2} \dots (f_1(x_1 y_2 \dots y_n))) \dots) \\ &= f_{n-1}(f_{n-2} \dots (f_2(x_1 x_2 y_3 \dots y_n))) \dots) \\ &\vdots \\ &= x_1 \dots x_n = v. \end{aligned}$$

Thus, in particular for each $i = 1, \dots, r$ there is some $t_i \in H_n$ such that $t_i(w_i) = v$. Then $h_i = g_i^{t_i} \in \mathrm{st}_G(n)$ and by (1.4)

$$(h_i)_v = (g_i)_{t_i^{-1}(v)} = (g_i)_{w_i}.$$

□

Remark 2.2.8. *In particular, in the conditions of the previous lemma, it is* *enough for a group G to be super strongly fractal to have one vertex $u_n \in L_n$ such that $\psi_{u_n}(\mathrm{st}_G(n)) = G$ for each $n \in \mathbb{N}$.*

2.3 Fractal groups which are not strongly fractal

2.3.1 A subgroup of the Hanoi Towers group

In this section we present an example for each $d \geq 3$ which is fractal but not strongly fractal. Even more, that example is a group which is level transitive. We denote by x_i for $i = 1, \ldots, d$ the elements of X, or what it is the same, the vertices of the first level.

The example that we consider is a subgroup of the Hanoi Towers group, which is defined as follows for each $d \geq 3$, as mentioned in Chapter 1.

For $1 \leq i < j \leq d$, we define the element a_{ij} which has the permutation $(x_i\, x_j)$ at the root and for each vertex on the first level:

$$(a_{ij})_{x_k} = \begin{cases} 1 & \text{if } k = i, j \\ a_{ij} & \text{else.} \end{cases}$$

The Hanoi Towers group is $H = \langle a_{ij} \mid 1 \leq i < j \leq d \rangle$. Although H is strongly fractal (see [19, page 13]), we are going to show that it has a subgroup which is fractal but not strongly fractal.

We consider the subgroup $G = \langle a_{i,i+1} \mid i = 1, \ldots, d-1 \rangle \leq H$. To simplify the notation, we write $b_i = a_{i,i+1}$.

As a consequence of Lemma 1.1.4 it is clear that G is self-similar, because $(b_j)_{x_i} \in G$ for each $j = 1, \ldots, d-1$ and $i = 1, \ldots, d$.

Let us see that G is fractal. Observe that since the element $b_{d-1}b_{d-2}\dots b_1$ has the label $(x_1x_2...x_d)$ at the root, G is transitive on the first level, so by (ii) in Lemma 2.2.2 it is enough to show that $\psi_{x_1}(\mathrm{st}_G(x_1)) = G$.

It suffices to check that each $b_i \in \psi_{x_1}(\mathrm{st}_G(x_1))$. Since $b_i \in \mathrm{st}_G(x_1)$ for $i \neq 1$ and in this case $\psi_{x_1}(b_i) = b_i$, it only remains to check that $b_1 \in \psi_{x_1}(\mathrm{st}_G(x_1))$. To show this, consider the element $b_1^{b_2b_1}$. First of all observe that $(b_1^{b_2b_1})_{(\emptyset)} = (x_1x_2)^{(x_1x_2x_3)} = (x_2x_3)$, so $b_1^{b_2b_1}$ belongs to $\mathrm{st}_G(x_1)$.

On the other hand, using (1.2) we have

$$\begin{aligned}
(b_1^{b_2b_1})_{x_1} &= ((b_2b_1)_{(b_2b_1)^{-1}(x_1)})^{-1}(b_1)_{(b_2b_1)^{-1}(x_1)}(b_2b_1)_{(b_2b_1)^{-1}b_1(x_1)} \\
&= ((b_2b_1)_{x_3})^{-1}(b_1)_{x_3}(b_2b_1)_{x_3} \\
&= ((b_2)_{x_3}(b_1)_{x_2})^{-1}b_1(b_2)_{x_3}(b_1)_{x_2} \\
&= b_1.
\end{aligned}$$

We obtain that $\psi_{x_1}(b_1^{b_2b_1}) = b_1$. Thus, we conclude that $\psi_{x_1}(\mathrm{st}_G(x_1)) = G$ as desired.

Let us now calculate $\mathrm{st}_G(1)$. We have $\rho(G) = \langle \rho(b_i) \mid i = 1,\dots,d-1\rangle = S_d$. We know that a presentation of the group S_d can be obtained by considering as generators $\{\tau_i = (i\,i+1)\}_{i=1,\dots,d-1}$ and the following relations:

$$\begin{aligned}
\tau_i^2 = 1, &\quad i = 1,\dots,d-1, \\
\tau_i\tau_j = \tau_j\tau_i, &\quad |i-j| > 1, \\
(\tau_i\tau_{i+1})^3 = 1, &\quad i = 1,\dots,d-2.
\end{aligned}$$

In order to apply Lemma 2.2.3, let F be the free group generated by $\{\tau_1,\dots,\tau_{d-1}\}$ and $\theta : F \longrightarrow S_d$ the epimorphism corresponding to the presentation above. Thus $\ker\theta = \langle \tau_i^2, [\tau_i,\tau_j], (\tau_i\tau_{i+1})^3 \mid i,j = 1,\dots,d-1, |i-j| >$

$1\rangle^F$. For each $i = 1, \ldots, d-1$ we have $b_i \in \rho^{-1}(\theta(\tau_i))$ and the b_i generate the whole group G. We can define $\phi : F \longrightarrow G$ by sending τ_i to b_i for each $i = 1, \ldots, d-1$. Then ϕ is a surjective homomorphism that makes the diagram commutative. Now, applying the lemma, if

$$S = \{b_i^2, (b_i b_{i+1})^3, [b_i, b_j] \mid i, j = 1, \ldots, d-1, |i-j| > 1\},$$

then we obtain that

$$\mathrm{st}_G(1) = \langle S \rangle^G.$$

Let us see, to conclude, that $\psi_{x_k}(\mathrm{st}_G(1)) \neq G$ for some $k = 1, \ldots, d$. In fact we will see that this happens for any $k \in \{1, \ldots, d\}$.

One can check that

$$(b_i^2)_{x_k} = \begin{cases} b_i^2 & \text{if } k \neq i, i+1, \\ 1 & \text{if } k = i, i+1, \end{cases}$$

which indeed, shows that b_i^2 is the identity element.

Let us now calculate what happens for $((b_i b_{i+1})^3)_{x_k}$. First of all, observe that $(b_i b_{i+1})_{x_k} = (b_i)_{x_k} (b_{i+1})_{b_i(x_k)}$, for $k = 1, \ldots, d$, and since $(b_i)_\emptyset = (x_i\, x_{i+1})$ if $k \neq i, i+1, i+2$ we get $(b_i b_{i+1})_{x_k} = b_i b_{i+1}$, if $k = i$ then $(b_i b_{i+1})_{x_k} = 1$, if $k = i+1$ then $(b_i b_{i+1})_{x_k} = b_{i+1}$ and if $k = i+2$ then $(b_i b_{i+1})_{x_k} = b_i$. Finally we have $(b_i b_{i+1})_\emptyset = (x_i\, x_{i+1}\, x_{i+2})$. Thus since

$$(b_i b_{i+1})^3_{x_k} = (b_i b_{i+1})_{x_k} (b_i b_{i+1})_{(b_i b_{i+1})(x_k)} (b_i b_{i+1})_{(b_i b_{i+1})^2(x_k)}$$

we obtain that

$$((b_ib_{i+1})^3)_{x_k} = \begin{cases} (b_ib_{i+1})^3 & \text{if } k \neq i, i+1, i+2, \\ b_ib_{i+1} & \text{if } k = i, \\ b_{i+1}b_i & \text{if } k = i+1, \\ b_ib_{i+1} & \text{if } k = i+2, \end{cases}$$

and, for $|i-j| > 1$,

$$([b_i, b_j])_{x_k} = \begin{cases} [b_i, b_j] & \text{if } k \neq i, i+1, j, j+1, \\ 1 & \text{else.} \end{cases}$$

To see the importance of the condition $|i-j| > 1$ in the last case, let us calculate for example $[b_i, b_j]_{x_i}$.

$$\begin{aligned} [b_i, b_j]_{x_i} &= (b_i^{-1}b_i^{b_j})_{x_i} \\ &= (b_i^{-1})_{x_i}(b_i^{b_j})_{x_{i+1}} \\ &= ((b_j)_{b_j^{-1}(x_{i+1})})^{-1}(b_i)_{b_j^{-1}(x_{i+1})}(b_j)_{b_j^{-1}b_i(x_{i+1})} \\ &= ((b_j)_{x_{i+1}})^{-1}(b_i)_{x_{i+1}}(b_j)_{x_i} \\ &= b_j^{-1}b_j = 1. \end{aligned}$$

Here it is important that b_j does not move x_i and x_{i+1}, which happens because $|i-j| > 1$. On the other hand, observe that b_i^2 and $[b_i, b_j]$ when $|i-j| > 1$ are the identity automorphism, because they belong to the first level stabilizer and the sections at the first level are just themselves or the identity.

Let $\sigma : S_d \longrightarrow \{1, -1\}$ be the homomorphism sending each permutation to its signature. One can check by calculations as in the previous example, that for any $s \in S$ and $k = 1, \dots, d$ we have $\sigma(\psi_{x_k}(s)_{(\emptyset)}) = 1$ because $\psi_{x_k}(s)$ is always a product of an even number of b_i. Then, if we consider $N = \langle \psi_{x_k}(S) \mid k = 1, \dots, d \rangle^G$ we still have that $\sigma(n_{(\emptyset)}) = 1$ for any $n \in N$.

Now, we have $\mathrm{st}_G(1) = \langle S \rangle^G$ and $\psi_{x_k}(S) \subseteq N$ where N is normal in G, so by Lemma 2.2.4 we conclude that $\psi_{x_k}(\mathrm{st}_G(1)) \subseteq N$. But N cannot be the whole group G because each $n \in N$ has an even permutation at the root and consequently $b_i \notin N$ for each $i = 1, \dots, d-1$. In other words, $\rho(N) \subseteq A_d$ while $\rho(G) = S_d$, so $N \neq G$.

2.3.2 Another example

In this section, we present a different example which is easier, bur based on the same idea.

Let us consider the group $G = \langle a_1, \dots, a_{d-1} \rangle$ where each a_i is defined in the following way:

$$(a_i)_{(\emptyset)} = (x_i \, x_{i+1}),$$

$$(a_1)_{x_j} = \begin{cases} a_2 & \text{for } j = 1, \\ a_{j-1} & \text{for } j = 2, \dots, d-1, \end{cases}$$

and for $i = 2, \dots, d-1$,

$$(a_i)_{x_j} = \begin{cases} a_i & \text{for } j = 1, \\ a_{j-1} & \text{for } j = 2, \dots, d-1. \end{cases}$$

Let us see that G is fractal. Observe that since the element $a_{d-1}a_{d-2}\dots a_1$ has the label $(x_1 \, x_2 \, \dots \, x_d)$ at the root, G is transitive on the first level, so by (ii) Lemma 2.2.2 it is enough to show that $\psi_{x_1}(\mathrm{st}_G(x_1)) = G$.

Let us see that $a_i \in \psi_{x_1}(\mathrm{st}_G(x_1))$ for each $i = 1, \dots, d-1$. Since

$a_2, \dots, a_{d-1} \in \mathrm{st}_G(x_1)$ we already have $a_2, a_3, \dots, a_{d-1} \in \psi_{x_1}(\mathrm{st}_G(x_1))$. We only need to check that $a_1 \in \psi_{x_1}(\mathrm{st}_G(x_1))$.

Observe that $a_1^2 \in \mathrm{st}_G(1) \subseteq \mathrm{st}_G(x_1)$ and $(a_1^2)_{x_1} = a_2 a_1$. Since we already have $a_2 \in \psi_{x_1}(\mathrm{st}_G(x_1))$ we obtain that $a_1 \in \psi_{x_1}(\mathrm{st}_G(x_1))$. Thus, we conclude that $\psi_{x_1}(\mathrm{st}_G(x_1)) = G$ and then, by (ii) Lemma 2.2.2, G is fractal and level transitive.

Let us now calculate $\mathrm{st}_G(1)$. We have $\rho(G) = \langle \rho(a_i) \mid i = 1, \dots, d-1 \rangle = S_d$. As in the previous example, we use the presentation of S_d over the generators $\{\tau_i = (i\, i+1)\}_{i=1,\dots,d-1}$. Since $\rho(a_i) = \tau_i$ and the a_i-s generate the whole group G we can apply Lemma 2.2.3. We conclude that

$$\mathrm{st}_G(1) = \langle a_i^2, (a_i a_{i+1})^3, [a_i, a_j] \mid i, j = 1, \dots, d-1, |i-j| > 1 \rangle^G.$$

Let us see to conclude that $\psi_{x_k}(\mathrm{st}_G(1)) \neq G$ for some $k = 1, \dots, d$. In fact we will see that it happens for any $k \in \{1, \dots, d\}$. Let

$$S = \{a_i^2, (a_i a_{i+1})^3, [a_i, a_j] \mid i, j = 1, \dots, d-1, |i-j| > 1\}.$$

We use again $\sigma : S_d \longrightarrow \{1, -1\}$, the homomorphism sending each permutation to its signature, observe that for any $s \in S$ we have $\sigma(\psi_{x_k}(s)_{(\emptyset)}) = 1$ because $\psi_{x_k}(s)$ is always a product of an even number of a_i-s, and each $\sigma((a_i)_{(\emptyset)}) = -1$. Then, if we consider $N = \langle \psi_{x_k}(S) \rangle^G$ we still have that $\sigma(n_{(\emptyset)}) = 1$ for any $n \in N$.

The result follows by the same argument as in the last paragraph of the previous example.

2.4 Strongly fractal groups which are not super strongly fractal

First of all, let us see that every GGS-group is strongly fractal.

Lemma 2.4.1. *Let G be a GGS-group. Then G is strongly fractal.*

Proof. Let us see that G is fractal. Since G is in the Sylow pro-p subgroup of $\operatorname{Aut} T$ corresponging to the cycle $(1 \dots p)$, this is enough to show that G is strongly fractal because of the discussion after Lemma 2.2.1. Since $\langle a \rangle$ acts transitively on the first level, according to (ii) in Lemma 2.2.2 it suffices to show that $\psi_x(\mathrm{st}_G(x)) = G$ for some x in the first level. Observe that conjugating b by powers of a permutes the sections of b at the first level. In other words,

$$\psi(b_i) = (a^{e_{p-i+1}}, \dots, a^{e_{p-1}}, b, a^{e_1}, \dots, a^{e_{p-i}}),$$

where b_i denotes b^{a^i} for $i \in \mathbb{Z}$ as mentioned in Chapter 1. Then, since $\mathbf{e}$ is non-zero, there is some $i = 1, \dots, p-1$ such that $e_{p-i+1} \neq 0$ and since $b_1, b_i \in \mathrm{st}_G(x_1)$ we obtain that $\psi_{x_1}(\mathrm{st}_G(x_1)) \geq \langle b, a^{e_{p-i+1}} \rangle = G$. We conclude that G is strongly fractal. □

Let us consider the GGS-group $\mathcal{G}$ with constant defining vector.

Proposition 2.4.2. *The group $\mathcal{G}$ is strongly fractal but not super strongly fractal.*

Proof. By the previous lemma it is enough to show that $\mathcal{G}$ is not super strongly fractal. In [15, Theorem 2.4] it is shown that $|\mathcal{G} : \mathrm{st}_{\mathcal{G}}(2)| = p^{t+1}$ where t is the rank of the circulant matrix which has as first row the vector

$(\mathbf{e}, 0) = (1, \ldots, 1, 0)$. In this case the rank is p. It is also proved in [15, Theorem 2.14] that $|\mathcal{G} : \mathrm{st}_{\mathcal{G}}(1)'| = p^{p+1}$.

Now since $\mathrm{st}_{\mathcal{G}}(1)/\mathrm{st}_{\mathcal{G}}(2)$ is abelian, as mentioned after Lemma 1.3.11, we know that $\mathrm{st}_{\mathcal{G}}(1)' \subseteq \mathrm{st}_{\mathcal{G}}(2)$, so we conclude that $\mathrm{st}_{\mathcal{G}}(2) = \mathrm{st}_{\mathcal{G}}(1)'$. Since $\mathrm{st}_{\mathcal{G}}(1) = \langle b_0, \ldots, b_{p-1} \rangle$, we have $\mathrm{st}_{\mathcal{G}}(1)' = \langle [b_i, b_j] \mid i, j = 1, \ldots, p \rangle^{\mathcal{G}}$. Observe that

$$\psi([b_i, b_j]) = (1, \ldots, 1, \underset{j}{[a, b]}, 1, \ldots 1, \underset{i}{[b, a]}, 1 \ldots, 1).$$

By Corollary 2.2.6 we conclude that

$$\psi_{x_1}(\mathrm{st}_{\mathcal{G}}(2)) = \psi_{x_1}(\mathrm{st}_{\mathcal{G}}(1)') = \langle [a, b], [b, a] \rangle^{\mathcal{G}} = \mathcal{G}'.$$

Now again, $\psi([a, b]) = \psi(b_1^{-1} b) = (b^{-1} a, 1, \ldots, 1, a^{-1} b)$. By the same argument as before, we have

$$\psi_{x_1}(\mathcal{G}') = \psi_{x_1}(\langle [a, b] \rangle^{\mathcal{G}}) = \langle b^{-1} a \rangle^{\mathcal{G}}.$$

But then, for the vertex $u = x_1 x_1 \in L_2$ we have that $\psi_u(\mathrm{st}_{\mathcal{G}}(2)) = \langle b^{-1} a \rangle^{\mathcal{G}}$. It is not hard to see that $\mathcal{G}/\mathcal{G}' \cong C_p \times C_p$ (see [15, Theorem 2.1]). Since the image of $\langle ba^{-1} \rangle^{\mathcal{G}}$ in $\mathcal{G}/\mathcal{G}'$ is cyclic, we have $\langle ba^{-1} \rangle^{\mathcal{G}} \neq \mathcal{G}$, and $\mathcal{G}$ is not super strongly fractal. □

Observe that the same proof is true for $p = 2$, that is, if we consider the group acting on the binary tree generated by the rooted automorphism a according to $(1\,2)$ and by $b \in \mathrm{st}(1)$ such that $\psi(b) = (a, b)$, which indeed, is isomorphic to the infinite dihedral group. This happens because the facts proved in [15] that are used are also satisfied in the case $p = 2$ and the proof above also holds in this case. This in particular shows, that even if the concepts of being fractal and strongly fractal coincide for the binary tree, the situation is different for the concepts of being strongly and super strongly fractal.

2.5 Groups which are super strongly fractal

Let us now show that the rest of the family of the GGS-groups is super strongly fractal. Before seeing this, let us prove a result which holds more generally for all the multi-GGS groups different from $\mathcal{G}$.

Lemma 2.5.1. *Let $G = \langle a, b_1, \ldots, b_r \rangle$ be a multi-GGS-group different from $\mathcal{G}$. Then*

$$\psi_x(G') = G \text{ for all } x \in X.$$

Proof. Suppose without loss of generality that b_1 is defined by a non-constant vector, so there exists $i \in \{1, \ldots, p-2\}$ such that $e_{1,i} \neq e_{1,i+1}$. Then $\psi([b_1, a])$ has $a^{e_{1,i}} a^{-e_{1,i+1}} \neq 1$ in the $(i+1)$st coordinate and therefore $\psi([b_1, a]^{a^{1-i}})$ has a non-trivial power of a in the first coordinate. So there exists an element $h \in G'$ such that $\psi(h)$ has a in the first coordinate. For any $j \in \{1, \ldots, r\}$ the element $\psi([b_j, a])$ has $a^{-e_{j,1}} b_j$ in the first coordinate and so the first coordinate of $\psi(h^{e_{j,1}}[b_j, a])$ is b_j. Thus $\psi_{x_1}(G') = G$. Conjugating by powers of a, we obtain the result for the rest of the vertices. □

Proposition 2.5.2. *Let G be a GGS-group different from $\mathcal{G}$. Then G is super strongly fractal.*

Proof. By [15, Lemma 3.3] we know that for $n \geq 3$ we have $\psi(\mathrm{st}_G(n)) = \mathrm{st}_G(n-1) \times \overset{p}{\ldots} \times \mathrm{st}_G(n-1)$. This implies that if $n \geq 3$ then $\psi_x(\mathrm{st}_G(n)) = \mathrm{st}_G(n-1)$ for every $x \in X$, and then for every $v \in L_{n-2}$ we have $\psi_v(\mathrm{st}_G(n)) = \mathrm{st}_G(2)$. Since we already know that $\psi_x(\mathrm{st}_G(1)) = G$ for every $x \in X$, if we show that $\psi_x(\mathrm{st}_G(2)) = \mathrm{st}_G(1)$ for each $x \in X$ we will be done, because then for every $u = vxy \in L_n$ we have

$$\begin{aligned}\psi_{vxy}(\mathrm{st}_G(n)) &= \psi_y(\psi_x(\psi_v(\mathrm{st}_G(n)))) \\ &= \psi_y(\psi_x(\mathrm{st}_G(2))) \\ &= \psi_y(\mathrm{st}_G(1)) \\ &= G.\end{aligned}$$

As mentioned after Lemma 1.3.11, since $\mathrm{st}_G(1)/\,\mathrm{st}_G(2)$ is abelian for every multi-GGS-group, we know that $\mathrm{st}_G(1)' \leq \mathrm{st}_G(2)$. Now as in the proof of Proposition 2.4.2, we obtain that $\psi_x(\mathrm{st}_G(1)') = G'$. Finally, by Lemma 2.5.1 we know that for any vertex $xy \in L_2$ we have

$$G = \psi_y(G') = \psi_y(\psi_x(\mathrm{st}_G(1)')) \leq \psi_{xy}(\mathrm{st}_G(2)).$$

Since the other inclusion is always satisfied the result follows. □

Even more, we are also able to show that all the multi-GGS-groups with $r \geq 2$ are also super strongly fractal. In order to prove this, we need an auxiliary result that will also be helpful in Chapter 3. As mentioned in Lemma 1.3.11, every multi-GGS-group apart from $\mathcal{G}$ is known to be regular branch over $\gamma_3(G)$. We improve this result by showing that all of them are regular branch over G' for $r \geq 2$.

Lemma 2.5.3. *If the multi-GGS-group G is generated by $r \geq 2$ directed generators and the rooted automorphism a, then $\psi(\mathrm{st}_G(1)')) = G' \times \overset{p}{\ldots} \times G'$. In particular, G is regular branch over its commutator subgroup G'.*

Proof. Since $\psi(\mathrm{st}_G(1)) \leq G \times \overset{p}{\ldots} \times G$, we need only show the '$\geq$' inclusion in the statement.

Suppose that b_1 has non-symmetric defining vector. Then $H = \langle a, b_1 \rangle \leq G$ is a GGS-group defined by a non-symmetric vector, and thus we already know that $\psi(\mathrm{st}_H(1)') = H' \times \cdots \times H'$. Thus we get that

$$([a, b_1], 1, \ldots, 1) \in \psi(\mathrm{st}_H(1)') \leq \psi(\mathrm{st}_G(1)').$$

Now by Lemma 1.3.11 we can assume that $e_{1,1} = 1$, and thus for any other directed generator b_i we have,

$$\psi([b_1, b_i^a]) = ([a, b_i], 1 \ldots, 1, [b_1, a^{e_{i,p-1}}]).$$

Therefore, since $H' \times \cdots \times H' \in \psi(\mathrm{st}_G(1)')$, we obtain that $([a, b_i], 1 \ldots, 1) \in \psi(\mathrm{st}_G(1)')$. Thus $(x, 1, \ldots, 1) \in \psi(\mathrm{st}_G(1)')$ for each normal generator x of G' and by Proposition 1.1.10 we obtain that $G' \times \overset{p}{\ldots} \times G' \leq \psi(\mathrm{st}_G(1)')$.

Now suppose that all b_i-s are defined by symmetric vectors ($e_{i,j} = e_{i,p-j}$ for every $i, j \in \{1, \ldots, p-1\}$). Again, by (i) in Lemma 1.3.11 we may assume that $e_{i,1} = 1$ for $i = 1, \ldots, r$. As pointed out after the same lemma, since we can consider the row echelon form, we may assume that $\mathbf{e}_i = (0, *, \ldots, *, 0)$ for $i = 2, \ldots, r$. We thus obtain that

$$\psi([b_1, b_i^a]) = ([a, b_i], 1, \ldots, 1),$$

for $i = 2, \ldots, r$.

Now let $e_{2,j}$ be the first non-trivial entry in $\mathbf{e}_2$ and $\alpha \in \mathbb{F}_p$ such that $e_{2,j}\alpha = 1$. As mentioned after Lemma 1.3.11, we can also replace $\mathbf{e}_1$ by

$\mathbf{e}_1 - k\mathbf{e}_2$ where $e_{1,j} + ke_{2,j} = 0$ so that $e_{1,j} = 0$.

$$\begin{aligned}\psi([b_1^{a^j}, (b_2)^\alpha]) &= (1, \ldots, 1, [b_1, a], 1, \ldots, 1, [a^{e_{1,p-j}}, b_2^\alpha]) \\ &= (1, \ldots, 1, [b_1, a], 1, \ldots, 1, 1),\end{aligned}$$

where the last equality follows because $e_{1,p-j} = e_{1,j} = 0$. Repeating the same argument as in the previous case we obtain the result. □

Now we can prove that all the multi-GGS-groups with $r \geq 2$ are super strongly fractal.

Proposition 2.5.4. *Let $G = \langle a, b_1, \ldots, b_r \rangle$ be a multi-GGS group with $r \geq 2$. Then G is super strongly fractal.*

Proof. Since G is strongly fractal, we already know that $\psi_x(\mathrm{st}_G(1)) = G$ for every $x \in L_1$. Now let us see that $\psi_u(\mathrm{st}_G(n)) = G$ for $u \in L_n$ with $n \geq 2$.

Since G is regular branch over G' we can consider for each $n \in \mathbb{N}$ the subgroup $K_n = \psi_n^{-1}(G' \times \overset{p^n}{\ldots} \times G')$. Then we have

$$\psi_n(K_n) = G' \times \overset{p^n}{\ldots} \times G' \subseteq (\mathrm{st}_G(1) \times \overset{p^n}{\ldots} \times \mathrm{st}_G(1)) \cap \psi_n(\mathrm{st}_G(n)) = \psi_n(\mathrm{st}_G(n+1)).$$

Thus $K_n \subseteq \mathrm{st}_G(n+1)$, and for each $u \in L_n$, one has $G' = \psi_u(K_n) \subseteq \psi_u(\mathrm{st}_G(n+1))$. Finally for each $x \in X$ we obtain

$$G = \psi_x(G') \subseteq \psi_{ux}(\mathrm{st}_G(n+1)),$$

so that $\psi_v(\mathrm{st}_G(n+1)) = G$ for each $v \in L_{n+1}$. □

Finally, as mentioned in the introduction of this chapter, in [23, page 85] it is said that first Grigorchuk group Γ is super strongly fractal. Let us see that, indeed, it is, even if it is not a direct consequence of being strongly fractal. The proof of this fact is similar to the previous example.

Proposition 2.5.5. *The group* Γ *is super strongly fractal.*

Proof. In [3, Theorem 4.3] it is shown that $\psi(\mathrm{st}_\Gamma(n)) = \mathrm{st}_\Gamma(n-1)\times \mathrm{st}_\Gamma(n-1)$ for $n \geq 4$. Since a, the rooted automorphism corresponding to the permutation $(1\,2)$ is in Γ, by Lemma 2.2.7 it suffices to show that $\langle \psi_{u_n}(\mathrm{st}_\Gamma(n)) \mid u_n \in L_n\rangle = \Gamma$ when $n = 1, 2, 3$.

Observe that since $bc = d$, it is enough to check that a and two of the three generators $\{b, c, d\}$ are in the image of $\mathrm{st}_\Gamma(n)$ for $n = 1, 2, 3$.

For $n = 1$ it follows from the definition of the elements b, c, d. Let us see the cases $n = 2$ and $n = 3$.

It is easy to calculate and check that $d, (ac)^4 \in \mathrm{st}_\Gamma(2)$ and that

$$\begin{aligned} \psi_{x_2x_1}(d) &= a, \\ \psi_{x_2x_2}(d) &= c, \\ \psi_{x_2x_2}((ac)^4) &= b. \end{aligned}$$

To conclude, the element $g = (ab)^4(adabac)^2$ belongs to $\mathrm{st}_\Gamma(3)$ and

$$\begin{aligned} \psi_{x_1x_2x_1}(g) &= d, \\ \psi_{x_2x_2x_1}(g) &= a, \\ \psi_{x_2x_2x_2}(g) &= c. \end{aligned}$$

This proves that Γ is super strongly fractal. $\square$

Chapter 3

On the congruence subgroup property for the multi-GGS-groups

3.1 Introduction

As mentioned in the introduction of the thesis, the congruence subgroup property for subgroups of $\mathrm{Aut}\,T$ is defined by analogy with the same property for linear algebraic groups [7]. More precisely, a subgroup G of $\mathrm{Aut}\,T$ satisfies the congruence subgroup property if each of its finite index subgroups contains some level stabilizer $\mathrm{st}_G(n) = G \cap \mathrm{st}(n)$. Taking the subgroups $\{\mathrm{st}_G(n) \mid n \in \mathbb{N}\}$ as a neighbourhood basis for the identity gives a topology on G, the congruence topology. The completion $\overline{G}$ of G with respect to this topology, which is called the *congruence completion* of G, is a profinite group which is isomorphic to the closure of G in $\mathrm{Aut}\,T$. On the other hand, G also embeds in its profinite completion $\widehat{G}$, and $\widehat{G}$ maps onto $\overline{G}$. Now G satisfying the congruence subgroup property is tantamount to the map $\widehat{G} \to \overline{G}$ being an isomorphism. The congruence subgroup problem asks whether this is the case and, if not, whether it is possible to determine the kernel of this map, which is called the *congruence kernel* of G.

In [22, Examples 10.1 and 10.2], Grigorchuk showed that the GGS-group corresponding to $\mathbf{e} = (1, 0, \ldots, 0)$ is just infinite and satisfies the congruence subgroup property for $p \geq 5$, and that the same holds for all the GGS-groups with $e_{p-3} = e_{p-2} = e_{p-1} = 0$, provided that $p \geq 7$. Vovkivsky proved that all torsion GGS-groups are just infinite [38, Theorem 4], and then Pervova showed that torsion GGS-groups satisfy the congruence subgroup property [33]. Observe that, according to [38, Theorem 1], a GGS-group with defining vector $\mathbf{e}$ is torsion if and only if $e_1 + \cdots + e_{p-1} = 0$. As a consequence, many vectors of $\mathbb{F}_p^{p-1}$ define non-torsion GGS-groups. Our first main result is the generalization of Pervova's theorem on the congruence subgroup property to all the GGS-groups other than $\mathcal{G}$.

Theorem 3.1.1. *All the multi-GGS-groups apart from $\mathcal{G}$ have the congruence subgroup property and are just infinite.*

We prove this result first for the GGS-groups defined by a non-constant vector. Our proof is based on a general criterion of Bartholdi and Grigorchuk for a regular branch group to have the congruence subgroup property which, in particular, also yields that the groups in Theorem 3.1.1 are just infinite. Also, it does not rely on the results of Pervova for torsion GGS-groups.

In [2], Barnea asked about the existence of infinite finitely generated residually finite non-torsion groups whose profinite completion is a pro-p group, and also whether such groups may even be torsion-free. Observe that Theorem 3.1.1 shows that the profinite completion of a GGS-group with non-constant defining vector is the same as its congruence completion and, in particular, a pro-p group. We will prove that some of these GGS-groups are virtually torsion-free, and then passing to a torsion-free subgroup will allow us to answer Barnea's questions in the positive.

The GGS-group with constant defining vector has a completely different behaviour.

Theorem 3.1.2. *The GGS-group $\mathcal{G}$ with constant defining vector has an infinite congruence kernel.*

We do not yet have a concrete description of this infinite kernel. Previous work on the congruence subgroup problem for groups acting on rooted trees was done by Bartholdi, Siegenthaler and Zalesskii [5], where they developed tools to determine the congruence kernel of branch groups. However, these tools are not available to us, as the GGS-group with constant defining vector is not a branch group (although it is weakly branch). We also prove this fact, which had been mentioned for the case $p = 3$ in [3, Proposition 7.3].

Theorem 3.1.3. *The GGS-group $\mathcal{G}$ with constant defining vector is not a branch group.*

Finally, we prove that in the family of multi-GGS-groups, which generalises the family of the GGS-groups, the answer is still the same. That is, all the multi-GGS-groups except the GGS-group defined by the constant vector have the congruence subgroup property. This concludes the proof of Theorem 3.1.1.

3.2 Congruence Subgroup Problem for $\operatorname{Aut} T$

Before we prove all the results mentioned in the introduction, in order to motivate the chapter, let us see what happens with the congruence subgroup problem for the whole group $\operatorname{Aut} T$. Whereas the properties of being fractal, strongly fractal and super strongly fractal are trivially satisfied in the whole group $\operatorname{Aut} T$, this is not the case any more when we ask about the congruence subgroup property, as we will see.

Proposition 3.2.1. *Let $G = H \ltimes N$ be any semidirect product. Then the abelianization of G is*

$$G^{ab} \cong H^{ab} \times \frac{N^{ab}}{[H, N^{ab}]}.$$

Proof. First of all observe that $[H, N^{ab}]$ makes sense, since the action of H in N induces an action on N^{ab} defined by $\overline{n}^h = \overline{n^h}$, where $\overline{n}$ denotes nN'.

Now we have $[G, G] = [H, H][N, G]$ and it is clear that $[N, G]$ is normal in G' and $[N, G] \cap [H, H] = \{1\}$. Thus $G' = H' \ltimes [N, G]$. Finally one obtains that $G^{ab} \cong H^{ab} \times \frac{N}{[N,G]}$. Since $N' \leq [N, G]$ there is a projection $\pi : N^{ab} \longrightarrow N/[N, G]$. If we show that the kernel is equal to $[N^{ab}, H]$ we obtain the result. It is clear that $\pi(\overline{n}^{-1}\overline{n^h}) = n^{-1}n^h = 1 \pmod{[N, G]}$. On the other hand $\overline{[n, g]} = \overline{[n, hm]} = \overline{[n, h][n, m]^h} = 1 \pmod{[H, N^{ab}]}$ and the result follows. □

Corollary 3.2.2. *Let $G = C \wr_X D$ where X is a finite set and D acts on X. Then*

$$G^{ab} \cong D^{ab} \times (C^{ab} \times \overset{|X/D|}{\cdots} \times C^{ab}),$$

where X/D denotes the D-orbits of X.

Proof. Since $C \wr_X D \cong D \ltimes (C \times \overset{|X|}{\cdots} \times C)$ by the previous proposition we get

$$G^{ab} \cong D^{ab} \times \frac{(C^{ab} \times \overset{|X|}{\cdots} \times C^{ab})}{[D, (C^{ab} \times \cdots \times C^{ab})]}.$$

Thus it only remains to show that

$$\frac{(C^{ab}\times \overset{|X|}{\cdots} \times C^{ab})}{[D,(C^{ab}\times\cdots\times C^{ab})]} \cong (C^{ab}\times \overset{|X/D|}{\cdots} \times C^{ab}).$$

Let $X_1,\dots,X_k$ denote the different D-orbits of X. Now we have a surjective homomorphism $\alpha : C^{ab}\times \overset{|X|}{\cdots} \times C^{ab} \longrightarrow C^{ab}\times \overset{|X/D|}{\cdots} \times C^{ab}$ which sends each $(c_1,\dots,c_{|X|})$ to $(\prod_{c_i\in X_1} c_i,\dots,\prod_{c_i\in X_k} c_i)$. Thus if we see that $\ker\alpha = [D,(C^{ab}\times \overset{|X|}{\cdots} \times C^{ab})]$ we will be done. First of all for any generator we have

$$\begin{aligned}\alpha([(c_1,\dots,c_{|X|}),d]) &= \alpha((c_1^{-1},\dots,c_{|X|}^{-1})(c_{d^{-1}(1)},\dots,c_{d^{-1}(|X|)}))\\ &= (\prod_{c_i\in X_1} c_i^{-1},\dots,\prod_{c_i\in X_k} c_i^{-1})(\prod_{c_i\in X_1} c_i,\dots,\prod_{c_i\in X_k} c_i)\\ &= (1,\dots,1).\end{aligned}$$

On the other hand, it is easy to check that any element of the kernel can be written as an element in $[D,(C^{ab}\times \overset{|X|}{\cdots} \times C^{ab})]$ and hence the result follows.

□

We already know that $\operatorname{Aut} T_n \cong (\dots((S_d \wr S_d)\wr S_d)\dots)\wr S_d$. Then, by the previous corollary we obtain that $(S_d\wr S_d)^{ab}\cong C_2\times C_2$. Applying this iteratively we get that $(\operatorname{Aut} T_n)^{ab} = \frac{\operatorname{Aut} T}{(\operatorname{Aut} T)' \operatorname{st}(n)} \cong C_2\times \overset{n}{\cdots}\times C_2$.

On the one hand, observe that the number of open subgroups in $\operatorname{Aut} T = \varprojlim_{n\in\mathbb{N}} \operatorname{Aut} T/\operatorname{st}(n)$ is countable, because the open subgroups are the ones that contain some level stabilizer. Since each quotient $\operatorname{Aut} T/\operatorname{st}(n)$ is finite, there is a finite number of such subgroups for each $n\in\mathbb{N}$, thus a countable number altogether.

On the other hand, if we consider the two inverse systems

$$\{\operatorname{Aut} T/\operatorname{st}(n), \pi_{n,m}, \mathbb{N}\} \text{ and } \{\operatorname{Aut} T/(\operatorname{Aut} T)' \operatorname{st}(n), \pi'_{n,m}, \mathbb{N}\},$$

with $\pi_{n,m}, \pi'_{n,m}$ being the natural projections for $n \geq m$, together with the projections $\pi_n : \operatorname{Aut} T/\operatorname{st}(n) \longrightarrow \operatorname{Aut} T/(\operatorname{Aut} T)' \operatorname{st}(n)$ for each $n \in \mathbb{N}$, we get a continuous homomorphism

$$\tilde{\pi} : \operatorname{Aut} T \longrightarrow \varprojlim_{n \in \mathbb{N}} \operatorname{Aut} T/(\operatorname{Aut} T)' \operatorname{st}(n) \cong \varprojlim_{n \in \mathbb{N}} (C_2 \times \overset{n}{\cdots} \times C_2) \cong C_2^{\mathbb{N}},$$

which is onto because each π_n is so. Thus, $\operatorname{Aut} T$ has at least as many subgroups of finite index as $C_2^{\mathbb{N}}$. The group $C_2^{\mathbb{N}}$ can be seen as a vector space over $\mathbb{F}_2$. Thus, it has a basis, and the basis must be uncountable, because otherwise $C_2^{\mathbb{N}}$ would be countable. Removing one element of the basis we get a subgroup of finite index of $C_2^{\mathbb{N}}$ and thus, it has uncountably many subgroups of finite index. This shows that with respect to the profinite topology $\operatorname{Aut} T$ has an uncountable number of open subgroups, and thus it cannot be the same topology as the one given by the level stabilizers. This shows that $\operatorname{Aut} T$ does not have the congruence subgroup property.

Since for any infinitely generated pro-p group there are non-open subgroups of finite index, the same argument applies to any pro-p subgroup of $\operatorname{Aut} T$.

3.3 The GGS-groups with non-constant defining vector

In this section we prove Theorem 3.1.1, i.e. that the GGS-groups with a non-constant defining vector have the congruence subgroup property.

The results mentioned in Proposition 1.3.6 in the first chapter show that all the GGS-groups with non-constant defining vector are regular branch over $\gamma_3(G)$, and even over G' when the defining vector is not symmetric. As a consequence, they are branch groups.

Our proof that the GGS-groups with a non-constant defining vector have the congruence subgroup property relies on (i) of Corollary 1.1.13. Indeed, by the corollary, in order to show that G has the congruence subgroup property it suffices to show that G'' and $\gamma_3(G)'$ contain some level stabilizer, depending on whether the defining vector is symmetric or not.

In the rest of this section we will show that, if G is a GGS-group with non-symmetric defining vector, then G'' contains some level stabilizer of G, and that the same property holds for non-constant symmetric defining vector, with $\gamma_3(G)'$ in the place of G''. This will complete the proof of Theorem 3.1.1.

Lemma 3.3.1. *If G is a GGS-group with non-constant defining vector, then $\psi(G')$ is a subdirect product of $G \times \overset{p}{\cdots} \times G$.*

We omit the proof of this lemma because it is very similar to the proof of Lemma 2.5.1.

Lemma 3.3.2. *If G is a GGS-group with non-constant symmetric defining vector, then $\psi(\gamma_3(G))$ is a subdirect product of $G \times \overset{p}{\cdots} \times G$.*

Proof. First of all, observe that if $p = 3$ and the defining vector of G is symmetric, then it must be constant. Hence $p \geq 5$. We have

$$\psi([b,a,a]) = (b^{-1}a^{e_1}b^{-1}a^{e_{p-1}}, a^{e_2-2e_1}b, a^{e_1-2e_2+e_3}, \ldots$$
$$\ldots, a^{e_{p-3}-2e_{p-2}+e_{p-1}}, a^{-e_{p-1}}ba^{e_{p-2}-e_{p-1}}).$$

Since $\mathbf{e}$ is non-constant and symmetric, there exists $i \in \{1, \ldots, (p-3)/2\}$ such that $e_i \neq e_{i+1}$. Let us choose i as large as possible subject to that condition. This choice, together with $e_{(p-1)/2} = e_{(p+1)/2}$, yields that $e_{i+1} = e_{i+2}$. Consequently $e_i - 2e_{i+1} + e_{i+2} = e_i - e_{i+1} \neq 0$, and the coordinate of $\psi([b,a,a])$ in position $i+2$ is a generator of $\langle a \rangle$. Since we also have $a^{e_2-2e_1}b$ in the second position of $\psi([b,a,a])$, the result follows as in the proof of Lemma 2.5.1. □

We can now prove part of Theorem 3.1.1.

Theorem 3.3.3. *Let G be a GGS-group with non-constant defining vector. Then G has the congruence subgroup property and is just infinite.*

Proof. By Corollary 1.1.13, it suffices to show that G'' or $\gamma_3(G)'$ contain some level stabilizer, according as the defining vector $\mathbf{e}$ is non-symmetric or non-constant symmetric.

Assume first that $\mathbf{e}$ is non-symmetric. We have $\gamma_3(G) = \langle [g,a], [g,b] \mid g \in G' \rangle^G$. By Proposition 1.3.6, for each $g \in G'$ there exists $h \in \mathrm{st}_G(1)'$ such that $\psi(h) = (g, 1, \ldots, 1)$. On the other hand, by Lemma 3.3.1, there exist $x, y \in G'$ such that $\psi(x) = (a, *, \ldots, *)$ and $\psi(y) = (b, *, \ldots, *)$, where each $*$ denotes an undetermined element of G. Then $\psi([h,x]) = ([g,a], 1, \ldots, 1)$ and $\psi([h,y]) = ([g,b], 1, \ldots, 1)$ belong to $\psi(G'')$, and consequently, by Proposition 1.1.10, $\psi(G'') \geq \gamma_3(G) \times \cdots \times \gamma_3(G)$. Since $\mathrm{st}_G(2) \leq \gamma_3(G)$ by (iv) of

Proposition 1.3.5, we conclude that

$$\psi(G'') \geq \mathrm{st}_G(2) \times \cdots \times \mathrm{st}_G(2) = \psi(\mathrm{st}_G(3)),$$

and $G'' \geq \mathrm{st}_G(3)$, as desired.

Now we assume that $\mathbf{e}$ is non-constant symmetric. Arguing as above, by combining Proposition 1.3.6 and Lemma 3.3.2, we get that $\psi(\gamma_3(G)') \geq \gamma_4(G) \times \cdots \times \gamma_4(G)$. If we show that $\mathrm{st}_G(3) \leq \gamma_4(G)$ then $\mathrm{st}_G(4) \leq \gamma_3(G)'$, and we are done. By (iii) of Proposition 1.3.5, we have $|\,\mathrm{st}_G(1) : \gamma_3(G)| = p^2$. Hence $\mathrm{st}_G(1)' \leq \gamma_3(G)$ and $\gamma_3(\mathrm{st}_G(1)) \leq \gamma_4(G)$. Then

$$\begin{aligned}\psi(\gamma_4(G)) \geq \psi(\gamma_3(\mathrm{st}_G(1))) &= \gamma_3(G) \times \cdots \times \gamma_3(G)\\ &\geq \mathrm{st}_G(2) \times \cdots \times \mathrm{st}_G(2) = \psi(\mathrm{st}_G(3)),\end{aligned}$$

by using Proposition 1.3.6. Thus $\mathrm{st}_G(3) \leq \gamma_4(G)$, which completes the proof.

□

3.4 Barnea's questions on profinite groups

In [2], Barnea posed the following two questions:

(i) Is there an infinite finitely generated residually finite non-torsion group such that its profinite completion is pro-p?

(ii) Is there an infinite finitely generated residually finite torsion-free group such that its profinite completion is pro-p?

According to Theorem 3.1.1, the profinite completion of a GGS-group G with non-constant defining vector is the same as its congruence completion. Since G lies in a Sylow pro-p subgroup of $\operatorname{Aut} T$, the index $|G : \mathrm{st}_G(n)|$ is a power of p for all $n \geq 1$. Thus the profinite completion of G is a pro-p group. By considering non-constant vectors $\mathbf{e}$ with coordinates satisfying $e_1 + \cdots + e_{p-1} \neq 0$, we get groups that answer in the positive Barnea's first question. Note that given a group G having the congruence subgroup property, for any subgroup H of finite index in G, we know that every $N \leq H$ of finite index is a subgroup of finite index in G and thus $N \geq \mathrm{st}_G(n) \geq \mathrm{st}_H(n)$ for some $n \in \mathbb{N}$. That is, the congruence subgroup property is hereditary for finite index subgroups. Thus, in order to answer the second question, we consider the GGS-group with defining vector $\mathbf{e} = (1, \ldots, 1, 0)$ and show that it is virtually torsion-free. In the case $p = 3$, this GGS-group is known as the Fabrykowski-Gupta group, and it was shown to be virtually torsion-free in [3, Theorem 6.4].

To start with, we identify which finite index subgroup should be shown to be torsion-free, using the following criterion.

Proposition 3.4.1. *Let G be a regular branch group over a subgroup K and suppose that G has the congruence subgroup property. If $\mathcal{P}$ is a property of groups which is hereditary for subgroups then G virtually has $\mathcal{P}$ if and only if K has $\mathcal{P}$.*

Proof. Since K has finite index in G, the 'if' direction is clear. To show the 'only if' part, suppose that G virtually has $\mathcal{P}$ and has the congruence subgroup property. Thus there exists some n such that $\mathrm{st}_G(n)$ has $\mathcal{P}$ and therefore $\mathrm{rst}_G(n)$ has $\mathcal{P}$. Since G is regular branch over K, we have $\psi_n(\mathrm{rst}_G(n)) \geq K \times \cdots \times K$ and therefore K must have $\mathcal{P}$. □

As a consequence, a natural strategy in order to answer Barnea's second question in the affirmative is to consider a GGS-group G with non-symmetric defining vector and examine whether G' is torsion-free. We will show that this is the case for the group with defining vector $\mathbf{e} = (1, \ldots, 1, 0)$ for every odd prime p, although the proof is valid for other vectors too, as explained at the end of the section.

We need the following two lemmas.

Lemma 3.4.2. *Let G be a GGS-group and let $h \in \mathrm{st}_G(1)$. Then the following conditions are equivalent:*

(i) $h \in G'$.

(ii) *If $\psi(h) = (h_1, \ldots, h_p)$, then $h_1 \ldots h_p \in G'$.*

(iii) $\psi((ha)^p) \in G' \times \cdots \times G'$.

Proof. Let $\Phi : \mathrm{st}_G(1) \longrightarrow G/G'$ be the homomorphism given by $\Phi(h) = h_1 \ldots h_p G'$, where $\psi(h) = (h_1, \ldots, h_p)$. Clearly, we have $\Phi(h^a) = \Phi(h)$ for all $h \in \mathrm{st}_G(1)$, and then $\Phi(b_i) = \Phi(b)$ for all $i \in \mathbb{Z}$. If we write $h \in \mathrm{st}_G(1)$ in the form $h = b_{i_1}^{r_1} \ldots b_{i_k}^{r_k}$, with $r_1, \ldots, r_k \in \mathbb{Z}$, it follows that $\Phi(h) = \Phi(b)^{r_1 + \cdots + r_k}$. Since G/G' is elementary abelian and $\Phi(b)$ is non-trivial, we have $h_1 \ldots h_p \in G'$ if and only if $r_1 + \cdots + r_k = 0$ in $\mathbb{F}_p$. Now, by Theorem 2.11 in [15], the latter condition is equivalent to $h \in G'$. This proves that (i) and (ii) are equivalent.

Now we prove the equivalence between (ii) and (iii). Since

$$(ha)^p = hh^{a^{p-1}}h^{a^{p-2}} \ldots h^a,$$

the ith component of $\psi((ha)^p)$ is $h_i h_{i+1} \ldots h_{i+p-1}$, where the indices are to be reduced modulo p to the interval $[1, p]$, and the result follows. □

Lemma 3.4.3. *Let G be a GGS-group and let $g \in G$ be such that $g^p = 1$. Then $g \in \langle a \rangle G' \cup \langle b \rangle G' \cup G'$.*

Proof. Suppose for a contradiction that $g = fb^r a^s$, with $f \in G'$ and $r, s \not\equiv 0 \pmod p$. By considering a suitable power of g, we may assume that $s = 1$. Since $\psi(g^p) = (1, \ldots, 1)$, it follows from the previous lemma that $fb^r \in G'$, which is a contradiction. □

Theorem 3.4.4. *Let G be the GGS-group defined by the vector $\mathbf{e} = (1, \ldots, 1, 0)$. Then G' is torsion-free.*

Proof. The GGS-group G lies in a Sylow pro-p subgroup of $\operatorname{Aut} T$, and consequently a torsion element must be of p-power order. Thus it suffices to show that G' has no elements of order p.

Let us consider an arbitrary element $g \in G'$. Assume first that $g \in G' \setminus \mathrm{st}_G(1)'$. By Theorems 2.11 and 2.14 in [15], the set

$$\{b_1^{i_1} \ldots b_p^{i_p} \mid i_1 + \cdots + i_p \equiv 0 \pmod p\}$$

is a transversal of $\mathrm{st}_G(1)'$ in G'. Thus we can write $g = b_1^{i_1} \ldots b_p^{i_p} h$ with $h \in \mathrm{st}_G(1)'$, $(i_1, \ldots, i_p) \in \mathbb{F}_p^p \setminus \{(0, \ldots, 0)\}$ and $i_1 + \cdots + i_p = 0$. By replacing g with a suitable conjugate, we may assume that $i_1 \neq 0$. We have

$$\psi(g) = \psi(b_1^{i_1} \ldots b_p^{i_p})\psi(h) = (a^{m_1} b_{k_1}^{i_1} f_1, \ldots, a^{m_p} b_{k_p}^{i_p} f_p), \tag{3.1}$$

for some $k_j \in \mathbb{Z}$ and $f_j \in G'$, and with

$$m_j = \Big(\sum_{r=1}^{p} i_r\Big) - i_j - i_{j+1} = -(i_j + i_{j+1}) \tag{3.2}$$

for every $j \in \{1, \ldots, p\}$ (where we put $i_{p+1} = i_1$). We claim that m_j and i_j are both non-zero for some j. To this end, let j be as large as possible subject to the condition $m_1 = \cdots = m_{j-1} = 0$. Then by (3.2) we have

$i_j = (-1)^{j-1} i_1 \neq 0$, and so if $j \leq p$ we are done. Otherwise, if $m_1, \ldots, m_p$ are all 0, we get the contradiction $2i_1 = 0$. This proves the claim.

It follows from Lemma 3.4.3 that the jth component of $\psi(g)$ is not of order p, and therefore neither is g.

Now we assume that $g \in \mathrm{st}_G(1)'$. Thus we can consider the largest integer $n \geq 0$ for which

$$\psi_n(g) \in \mathrm{st}_G(1)' \times \overset{p^n}{\cdots} \times \mathrm{st}_G(1)'.$$

Then $g \in \mathrm{st}_G(n+1)$ and, since $\psi(\mathrm{st}_G(1)') = G' \times \overset{p}{\cdots} \times G'$ by Proposition 1.3.6, the vector $\psi_{n+1}(g)$ has a component in $G' \smallsetminus \mathrm{st}_G(1)'$. By the previous paragraph, g is not of order p also in this case. □

Notice that in the above proof, equation (3.2) is equivalent to

$$(m_1, \ldots, m_p) = (i_1, \ldots, i_p)C,$$

where C is the circulant matrix

$$C = \begin{pmatrix} 0 & e_1 & \cdots & e_{p-1} \\ e_{p-1} & 0 & \cdots & e_{p-2} \\ \vdots & \ddots & \ddots & \vdots \\ e_1 & e_2 & \cdots & 0 \end{pmatrix}$$

whose ith row corresponds to the powers of a in b_i with $1 \leq i \leq p$ for a GGS-group with defining vector $\mathbf{e}$. Thus the proof is valid not just for the vector $(1, \ldots, 1, 0)$ but for any non-symmetric vector $\mathbf{e}$ such that the following condition holds: for every non-zero $(i_1, \ldots, i_p) \in \mathbb{F}_p^p$ with $\sum_{r=1}^p i_r = 0$, there exists $j \in \{1, \ldots, p\}$ such that $m_j i_j \neq 0$. Indeed, the paragraph below equation (3.2) is the proof that the vector $(1, \ldots, 1, 0)$ satisfies this condition. A slight modification of the proof shows that, more generally, the vector $\mathbf{e} = (1, \dots, 1, \lambda)$ also satisfies the required condition, provided that $\lambda \in \mathbb{F}_p \smallsetminus \{1, 2\}$. This gives many more examples of virtually torsion-free GGS-groups with non-symmetric defining vector.

3.5 The GGS-groups with constant defining vector

In this section we prove that the GGS-group $\mathcal{G}$ with constant defining vector is not a branch group and does not have the congruence subgroup property.

Recall the definition and facts about the subgroup K and of the elements y_i for $i = 0, \dots, p-1$, mentioned in Lemma 1.3.8 and the comments just before this lemma, respectively. We start by determining the structure of the quotient $\mathcal{G}/K'$. We need the following lemma.

Lemma 3.5.1. *The elements $y_0, \dots, y_{p-1}$ have infinite order.*

Proof. It suffices to prove the claim for y_0. If the order of y_0 is finite, then it must be a power of p, say p^n, since $\mathcal{G}$ is contained in a Sylow pro-p subgroup of $\operatorname{Aut} T$. Now,

$$y_0^p = (ba^{-1})^p = bb^a \dots b^{a^{p-1}} \in \mathrm{st}_{\mathcal{G}}(1),$$

and

$$\psi(y_0^p) = (aba^{p-2}, a^2ba^{p-3}, \dots, ba^{p-1}) = (y_{p-1}, y_{p-2}, \dots, y_0).$$

Thus the last coordinate of $\psi(y_0^{p^n})$ is $y_0^{p^{n-1}}$, which must be 1. This is a contradiction. $\square$

Proposition 3.5.2. *The quotient group $\mathcal{G}/K'$ is isomorphic to the semidirect product*

$$P = \langle d \rangle \ltimes \langle c_0, \dots, c_{p-2} \rangle \cong C_p \ltimes (C_\infty \times \overset{p-1}{\cdots} \times C_\infty),$$

where $c_i^d = c_{i+1}$ *for* $i = 0, \ldots, p-3$ *and* $c_{p-2}^d = (c_0 \ldots c_{p-2})^{-1}$*, and the isomorphism maps* K/K' *to the kernel of the semidirect product. In particular,* K/K' *is torsion-free.*

Proof. Taking into account that $y_i^a = y_{i+1}$ for all i and that $y_{p-1} \ldots y_1 y_0 = 1$, the assignments $c_i \mapsto y_i K'$ and $d \mapsto aK'$ define a surjective homomorphism α from P to $\mathcal{G}/K'$, by von Dyck's Theorem. Thus we only need to show that $\ker \alpha = 1$. By way of contradiction, assume that the kernel of α contains an element $w \neq 1$.

Put $C = \langle c_0, \ldots, c_{p-2} \rangle$, which is a free abelian group of rank $p-1$. If $w \in P \smallsetminus C$ then $P = \langle w \rangle C$ and $\alpha(P) = \alpha(C) = K/K'$, which is a contradiction. Thus $w \in C$. If m is the order of the torsion subgroup of $C/\langle w \rangle$ then $C/\langle w \rangle \cong C_\infty \times \overset{p-2}{\cdots} \times C_\infty \times F$ with $|F| = m$. Since $\alpha(\langle w \rangle C^{p^m}) = (K/K')^{p^m}$, it follows that $|K/K' : (K/K')^{p^m}| \leq |C : \langle w \rangle C^{p^m}| \leq p^{m(p-2)} m$. Now, by [15, Theorem 4.6], the quotient $\mathcal{G}/K' \operatorname{st}_{\mathcal{G}}(n)$ is a p-group of maximal class of order p^{n+1} for every $n \geq 1$. Let us choose $n = m(p-1)$. Then the group $K/K' \operatorname{st}_{\mathcal{G}}(n)$ is homocyclic of rank $p-1$ and exponent p^m (see [13, Theorem 4.9] or [29, Corollary 3.3.4]). Thus $|K/K' \operatorname{st}_{\mathcal{G}}(n) : (K/K' \operatorname{st}_{\mathcal{G}}(n))^{p^m}| \geq p^n$. But then $p^n = p^{m(p-1)} \leq |K/K' \operatorname{st}_{\mathcal{G}}(n) : (K/K' \operatorname{st}_{\mathcal{G}}(n))^{p^m}| \leq |K/K' : (K/K')^{p^m}| \leq p^{m(p-2)} m$, which gives a contradiction because $p^m > m$. Thus $\ker \alpha = 1$, as desired. □

We can now prove Theorem 3.1.2.

Proof of Theorem 3.1.2. Let $\widehat{\mathcal{G}}$ and $\overline{\mathcal{G}}$ be the profinite and congruence completions of $\mathcal{G}$, respectively, and let C be the congruence kernel of $\mathcal{G}$, i.e. the kernel of the natural homomorphism from $\widehat{\mathcal{G}}$ onto $\overline{\mathcal{G}}$.

Consider a prime q other than p. By Proposition 3.5.2, the factor group $\mathcal{G}/K'$ is a semidirect product with kernel K/K' isomorphic to $C_\infty \times \overset{p-1}{\cdots} \times C_\infty$

and complement isomorphic to C_p. For every $n \in \mathbb{N}$, let K_n be the normal subgroup of $\mathcal{G}$ defined by the condition $K_n/K' = (K/K')^{q^n}$. Then $|\mathcal{G} : K_n| = pq^{n(p-1)}$.

A basic result in profinite group theory (see [34, Proposition 3.2.2]) states that there is a one-to-one correspondence Φ between the subgroups of $\mathcal{G}$ which are open in the profinite topology of $\mathcal{G}$ and the open subgroups of $\widehat{\mathcal{G}}$. The map Φ takes an open subgroup $H \leq \mathcal{G}$ to the closure of H in $\widehat{\mathcal{G}}$ (having identified $\mathcal{G}$ with its image in $\widehat{\mathcal{G}}$). Moreover, Φ preserves the indices between subgroups. Thus, if $U_n = \Phi(K_n)$ then

$$pq^{n(p-1)} = |\mathcal{G} : K_n| = |\widehat{\mathcal{G}} : U_n| = |\widehat{\mathcal{G}} : U_nC|\,|U_nC : U_n|. \tag{3.3}$$

Now, $\widehat{\mathcal{G}}/U_nC$ is a finite quotient of

$$\widehat{\mathcal{G}}/C \cong \overline{\mathcal{G}} \cong \varprojlim_{n\in\mathbb{N}} \mathcal{G}/\operatorname{st}_{\mathcal{G}}(n),$$

which is a pro-p group. Consequently $|\widehat{\mathcal{G}} : U_nC|$ is a power of p, and then by (3.3),

$$q^{n(p-1)} \mid |U_nC : U_n| = |C : U_n \cap C|$$

for all $n \in \mathbb{N}$. We conclude that C is infinite, as desired. □

Our next purpose is to prove Theorem 3.1.3, i.e. that $\mathcal{G}$ is not a branch group. This means that the techniques developed so far (in [5]) for the calculation of the congruence kernel of a subgroup of $\operatorname{Aut} T$ are not available in this case. We need the following easy lemma.

Lemma 3.5.3. *Let G be a subgroup of* $\operatorname{Aut} T$*, and assume that* $|G : \operatorname{rst}_G(n)|$ *is finite for some n. If H is a finite index subgroup of G, then* $|H : \operatorname{rst}_H(n)|$ *is also finite.*

Proof. Let m be the index of H in G. Then

$$\begin{aligned} |\operatorname{rst}_G(n) : \operatorname{rst}_H(n)| &= \Big| \prod_{u \in L_n} \operatorname{rst}_G(u) : \prod_{u \in L_n} \operatorname{rst}_H(u) \Big| \\ &= \prod_{u \in L_n} |\operatorname{rst}_G(u) : \operatorname{rst}_G(u) \cap H| \leq m^{|L_n|} \end{aligned}$$

is finite, and the result follows. □

Proof of Theorem 3.1.3. Let $L = \psi^{-1}(K' \times \cdots \times K')$. By Lemma 1.3.8, we have $L \subseteq \operatorname{rst}_{\mathcal{G}'}(1)$. We claim that equality holds. To that purpose, we consider an element $g \in \operatorname{rst}_{\mathcal{G}'}(x)$, with $x \in X$, and we prove that $g \in L$. By definition of rigid stabilizer of a vertex, all coordinates of $\psi(g)$ are equal to 1, except possibly the one corresponding to position x, say, h. Observe that $h \in K$, since $\psi(\mathcal{G}') \subseteq K \times \cdots \times K$ by Lemma 1.3.8. If

$$g^* = g g^a \dots g^{a^{p-1}},$$

then $g^* \in K'$ by Lemma 1.3.9. Now $\psi(g^*) = (h, \dots, h)$ and, by applying the second part of Lemma 1.3.9, we get $h^p \in K'$. Since $h \in K$ and K/K' is torsion-free by Proposition 3.5.2, it follows that $h \in K'$. Thus $\psi(g) \in K' \times \cdots \times K'$, and $g \in L$, as desired.

Now assume by way of contradiction that $\mathcal{G}$ is a branch group. Then $|\mathcal{G} : \operatorname{rst}_{\mathcal{G}}(1)|$ is finite, and by Lemma 3.5.3 and the previous paragraph, $|\mathcal{G}' : L|$ is also finite. Now observe that $L \leq K'$ by Lemma 1.3.8. Therefore the factor group $\mathcal{G}/K'$ is finite, which is a contradiction, according to Proposition 3.5.2. □

3.6 The multi-GGS-groups have the congruence subgroup property

We will now prove that the result of Theorem 3.1.1 extends to the multi-GGS-groups, that is, that all of them apart from the GGS-group defined by the constant vector have the congruence subgroup property.

In Lemma 2.5.3 we have already proved that every multi-GGS-group different from $\mathcal{G}$ is regular branch over G'. Thus, as before, by Corollary 1.1.13, it suffices to show that G'' contains some level stabilizer. This will be shown in Corollary 3.6.4.

Before proving this, we need some auxiliary results.

Lemma 3.6.1. *Let G be any multi-GGS-group. Then $\mathrm{st}_G(1)' \leq \gamma_3(G)$.*

Proof. Since $\mathrm{st}_G(1)$ is normally generated by $b_1, \ldots, b_r$ (hence, generated by the conjugates of $b_1, \ldots, b_r$ by powers of a), we have that $\mathrm{st}_G(1)'$ is normally generated by commutators of the form $[b_i^{a^m}, b_j^{a^n}]$ with $i, j \in \{1, \ldots, r\}$ and $m, n \in \mathbb{F}_p$. Now notice that $[b_i^{a^m}, b_j^{a^n}] = [b_i[b_i, a^m], b_j[b_j, a^n]]$ which is congruent modulo $\gamma_3(G)$ to $[b_i, b_j] = 1$. Thus all normal generators of $\mathrm{st}_G(1)'$ are contained in $\gamma_3(G) \trianglelefteq G$, which proves our claim. □

Lemma 3.6.2. *Let $G = \langle a, b_1, \ldots, b_r \rangle$ be a multi-GGS-group with $r \geq 2$. Then*

$$\psi_2(G'') \geq G' \times \overset{p^2}{\cdots} \times G'.$$

Proof. By Lemma 2.5.1 we know that there exist $x, y_i \in G'$ such that $\psi(x) = (a, *, \ldots, *)$ and $\psi(y_i) = (b_i, *, \ldots, *)$ for each $i \in \{1, \ldots, r\}$ (where $*$ denotes unknown, unimportant elements). On the other hand, by Lemma 2.5.3, for each $h \in G'$ there is some $g \in G'$ such that $\psi(g) = (h, 1, \ldots, 1)$. Thus

$\psi([x,g]) = ([a,h],1,\dots,1)$ and $\psi([y_i,g]) = ([b_i,h],1,\dots,1)$ for $i = 1,\dots,r$. Now, $[x,g],[y_i,g] \in G''$ implies that

$$\psi(G'') \geq \gamma_3(G) \times \overset{p}{\dots} \times \gamma_3(G).$$

Finally, Lemma 3.6.1 and another application of Lemma 2.5.3 yield the result.

□

Let us first establish some notation. Set

$$G_n = \frac{G}{\mathrm{st}_G(n)}, \qquad \overline{G}_n = \frac{G_n}{G_n'}, \qquad \widehat{G}_n = \frac{G_n}{\mathrm{st}_{G_n}(1)'}.$$

Observe that in the same way in which $\psi : \mathrm{st}_G(1) \to G \times \overset{p}{\dots} \times G$ holds, we also have $\psi_{(n)} : \mathrm{st}_{G_n}(1) \to G_{n-1} \times \overset{p}{\dots} \times G_{n-1}$. Denoting by π_n the projection from G to G_n, the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{st}_G(1) & \xrightarrow{\quad\psi\quad} & G \times \overset{p}{\dots} \times G \\
\big\downarrow{\scriptstyle \pi_n} & & \big\downarrow{\scriptstyle \pi_{n-1} \times \overset{p}{\dots} \times \pi_{n-1}} \\
\mathrm{st}_{G_n}(1) & \xrightarrow{\psi_{(n)}} & G_{n-1} \times \overset{p}{\dots} \times G_{n-1}
\end{array}$$

Moreover, since $\psi : \mathrm{st}_G(1)' \longrightarrow G' \times \overset{p}{\dots} \times G'$ is an isomorphism, the map

$$\widehat{\psi}_{(n)} : \frac{\mathrm{st}_{G_n}(1)}{\mathrm{st}_{G_n}(1)'} \longrightarrow \overline{G}_{n-1} \times \overset{p}{\dots} \times \overline{G}_{n-1}$$

is well defined.

Proposition 3.6.3. *Let $G = \langle a, b_1, \dots, b_r \rangle$ be a multi-GGS-group. Then $G' \geq \mathrm{st}_G(r+1)$.*

Proof. We will prove by induction on $n \in \mathbb{N}$ that $d(\overline{G}_n) \geq n$ for every $n = 2,\dots,r+1$. This implies in particular that $\overline{G}_{r+1}$ is generated by $r+1$

elements, and then $|\overline{G}_{r+1}| = |G : G'|$, which implies that $G' = G' \operatorname{st}_G(r+1)$ and the result follows.

Observe that $d(G_n) = d(\overline{G}_n) = d(\widehat{G}_n)$, because $G_n^p \leq G_n'$ and then $\Phi(G_n) = G_n'$. Since G_n' and $\operatorname{st}_{G_n}(1)'$ are contained in $\Phi(G_n)$, the minimum number of generators does not change.

The case $n = 2$ is obvious because if G_2 is generated by one element, since $G_2 = \langle \overline{a} \rangle \ltimes \operatorname{st}_G(1)/\operatorname{st}_G(2)$ we necessarily have $G_2 = \langle \overline{a} \rangle$ and then $\operatorname{st}_G(1) = \operatorname{st}_G(2)$, which is a contradiction. Let us suppose the statement holds true for $n \leq r$, that is $d(\overline{G}_n) \geq n$. Since $\overline{G}_n$ is elementary abelian, and it is generated by the projections of the generators of G, we can choose a basis and we may assume that $\overline{G}_n = \langle \overline{a}, \overline{b}_1, \dots, \overline{b}_{n-1}, \dots \rangle$ where the first n generators are linearly independent in $\overline{G}_n$. We want to prove the case $n+1$. Suppose for a contradiction that $\overline{G}_{n+1}$ can be generated by n elements. Then since $\overline{G}_n$ is a quotient of $\overline{G}_{n+1}$ and it is generated by $\langle \overline{a}, \overline{b}_1, \dots, \overline{b}_{n-1}, \dots \rangle$ the first n being linearly independent, we get that $\overline{G}_{n+1} = \langle \overline{a}, \overline{b}_1, \dots, \overline{b}_{n-1} \rangle$. Now since $\Phi(G_{n+1}) \leq G_{n+1}'$, we also get that $\widehat{G}_{n+1} = \langle \widehat{a}, \widehat{b}_1, \dots, \widehat{b}_{n-1} \rangle$. Then

$$\widehat{b}_n = \widehat{b}_1^{i_{1,0}} (\widehat{b}_1^{\widehat{a}})^{i_{1,1}} \dots (\widehat{b}_1^{\widehat{a}^{p-1}})^{i_{1,p-1}} \dots \widehat{b}_{n-1}^{i_{n-1,0}} (\widehat{b}_{n-1}^{\widehat{a}})^{i_{n-1,1}} \dots (\widehat{b}_{n-1}^{\widehat{a}^{p-1}})^{i_{n-1,p-1}} \tag{3.4}$$

with $i_{j,k} \in \mathbb{F}_p$. But then the images under $\widehat{\psi}_{(n)}$ of the element on the left hand side and right hand side must be equal in $\overline{G}_n$. On the one hand we have

$$\widehat{\psi}_{(n)}(\widehat{b}_n) = (\overline{a}^{e_{n,1}}, \dots, \overline{a}^{e_{n,p-1}}, \overline{b}_n).$$

On the other hand, the image under $\widehat{\psi}_{(n)}$ of the right hand side in (3.4) will have $\overline{a}^{\alpha_k} \overline{b}_1^{i_{1,k}} \dots \overline{b}_{n-1}^{i_{n-1,k}}$ at position k, for some α_k. Since $\overline{b}_1, \dots, \overline{b}_{n-1}$ are linearly independent, if both sides must be equal we are forced to have $i_{j,k} = 0$ for $k \neq 0$. Now looking at the powers of a in each entry, this means that

$\mathbf{e}_n = i_{1,0}\mathbf{e}_1 + \cdots + i_{n-1,0}\mathbf{e}_{n-1}$, which is impossible, because the defining vectors are linearly independent. Thus, $d(\overline{G}_{n+1}) \geq n+1$ and the theorem follows by induction. □

Corollary 3.6.4. *Let G be a multi-GGS-group different from $\mathcal{G}$. Then G has the congruence subgroup property and it is just infinite.*

As remarked previously, it suffices to show, by Corollary 1.1.13 and Lemma 2.5.3, that G'' contains some level stabilizer. Lemma 3.6.2 and Proposition 3.6.3 yield that

$$\psi_2(G'') \geq \mathrm{st}_G(r+1) \times \overset{p^2}{\cdots} \times \mathrm{st}_G(r+1) \geq \psi_2(\mathrm{st}_G(r+3)).$$

Thus $G'' \geq \mathrm{st}_G(r+3)$.

Theorem 3.3.3 and Corollary 3.6.4 together give rise to Theorem 3.1.1.

Chapter 4

$\mathcal{C}$-congruence subgroup property for weakly branch groups

4.1 Introduction

As mentioned in the introduction of the thesis the topology given by the level stabilizers may be compared to other topologies. The classical congruence subgroup property is the special case when we compare the congruence topology with the profinite topology. However, as we have seen in the previous section, they do not always coincide.

The aim of this chapter is to introduce a more general notion of the congruence subgroup property, which will be called $\mathcal{C}$-congruence subgroup property, or $\mathcal{C}$-CSP for short, where $\mathcal{C}$ denotes a variety of finite groups (see Section 4.2 for the definition of variety). Given such a variety, one can define a topology on G by considering as a neighbourhood system of the identity the normal subgroups of G such that the quotients by them belongs to $\mathcal{C}$. This topology is called the pro-$\mathcal{C}$ topology on G. Then the $\mathcal{C}$-congruence subgroup problem asks whether the topology given by the level stabilizers coincides with the pro-$\mathcal{C}$ topology given by a variety $\mathcal{C}$ of finite groups or not.

In the previous chapter, we have already seen a particular case, the group $\mathcal{G}$, where the topology given by the level stabilizers does not coincide with the profinite one. The reason of the group not having the congruence subgroup property was having finite quotients which were not p-groups, while every quotient by an stabilizer was a p-group. Thus in this case, it makes sense to ask whether the appropriate topology to compare the congruence topology with is the pro-p topology, which is given by the variety of all finite p-groups.

Thus, in this chapter, we give a sufficient condition for a weakly branch group to have the $\mathcal{C}$-CSP. After this, we prove that the group $\mathcal{G}$ has the $\mathcal{C}$-CSP for the variety of finite p-groups with $p \geq 3$ and that the same happens for the Basilica group, with $p = 2$.

It is worth mentioning that there are examples of branch groups for which every $G/\operatorname{st}_G(n)$ belongs to the variety of p-groups and G does not possess the p-CSP. For instance, a similar proof of the fact of the whole group $\operatorname{Aut} T$ not having the congruence subgroup property given in Section 3.2 works in the same way for the Sylow pro-p subgroup of $\operatorname{Aut} T$ in which all the GGS-groups lie. Another example comes from the EGS-groups defined in [33] by E. Pervova. These groups are regular branch over the commutator subgroup (which is of p-power index), but G' does not contain any level stabilizers.

On the other hand, we should point out that although the concrete cases discussed here are all p-CSP cases, the sufficient condition is given in the general setting. Therefore, it may be helpful to compare the topology of a weakly regular branch group given by the level stabilizers with topologies given by other kind of varieties of finite groups, such as those of nilpotent groups or solvable groups.

4.2 Varieties of finite groups and natural direct products

Before we start proving our main result, let us introduce some basic definitions and useful lemmas related to varieties that will help us later on. We follow the definitions given in [34].

Definition 4.2.1. *Let $\mathcal{C}$ be a non-empty class of finite groups. We say that $\mathcal{C}$ is a* variety of finite groups *if the following properties are satisfied:*

($\mathcal{C}_1$) *it is closed under taking subgroups, that is, if $G \in \mathcal{C}$ and $H \leq G$ then $H \in \mathcal{C}$,*

($\mathcal{C}_2$) *it is closed under quotients, that is, if $G \in \mathcal{C}$ and $N \trianglelefteq G$ then $G/N \in \mathcal{C}$,*

($\mathcal{C}_3$) *it is closed under taking finite direct products, that is, if $G_1, \ldots, G_k \in \mathcal{C}$ then $\prod_{i=1}^{k} G_i \in \mathcal{C}$.*

Our aim is to endow a group G with a topology given by a collection of normal subgroups of finite index. Observe that for any family $\mathcal{C}$ of finite groups, one can always consider the set $\mathcal{N}_{\mathcal{C}} = \{N \trianglelefteq G \mid G/N \in \mathcal{C}\}$. In order to simplify notation, we will denote by $N \trianglelefteq_{\mathcal{C}} G$ the fact that $N \trianglelefteq G$ and $G/N \in \mathcal{C}$.

Lemma 4.2.2. *Let G be a group and $\mathcal{C}$ a variety of finite groups. Then*

(i) *if $N_1, N_2 \trianglelefteq_{\mathcal{C}} G$ then $N_1 \cap N_2 \trianglelefteq_{\mathcal{C}} G$,*

(ii) *if $N \trianglelefteq_{\mathcal{C}} G$ and $N \leq K \trianglelefteq G$ then $K \trianglelefteq_{\mathcal{C}} G$,*

(iii) *if* $N \trianglelefteq_{\mathcal{C}} G$ *and* $N \leq K \leq G$ *then* $N \cap K \trianglelefteq_{\mathcal{C}} K$*,*

(iv) *if* $\alpha : G_1 \longrightarrow G_2$ *is a homomorphism and* $N_1 \trianglelefteq_{\mathcal{C}} G_1$ *then* $\alpha(N_1) \trianglelefteq_{\mathcal{C}} \alpha(G_1)$.

Proof. We start by proving (i). Let $M = N_1 \cap N_2$. Observe that $G/N_1 \times G/N_2 \in \mathcal{C}$ because of $(\mathcal{C}_3)$. On the other hand the map sending the element $gM \in G/M$ to $(gN_1, gN_2) \in G/N_1 \times G/N_2$ is an injective homomorphism and thus G/M is isomorphic to a subgroup of $G/N_1 \times G/N_2$, which belongs to $\mathcal{C}$ by $(\mathcal{C}_1)$.

Observe that (ii) follows from $(\mathcal{C}_2)$ because $G/K \cong \frac{G/N}{K/N}$.

In order to prove (iii) observe that since KN/N is a subgroup of G/N it belongs to $\mathcal{C}$. And by the second isomorphism theorem we get that $K/(N \cap K) \in \mathcal{C}$.

Finally, let us prove (iv). First of all, notice that $\alpha(N_1)$ is normal in $\alpha(G_1)$. Let us consider the natural projection π from $\alpha(G_1)$ to the quotient $\alpha(G_1)/\alpha(N_1)$. Then the kernel of the composition $\pi \circ \alpha$ is $\alpha^{-1}(\alpha(N_1))$ which is normal in G_1 and contains N_1. Then by (ii) $\alpha^{-1}(\alpha(N_1)) \trianglelefteq_{\mathcal{C}} G_1$, and since $\pi \circ \alpha$ is surjective, we get that $G_1/\alpha^{-1}(\alpha(N_1)) \cong \alpha(G_1)/\alpha(N_1)$ and the result follows. □

As mentioned in Section 1.2, for a family $\mathcal{N}$ of normal subgroups in G to define a neighbourhood system of the identity for some topology in G, it suffices to satisfy the following condition:

for every $N_1, N_2 \in \mathcal{N}$ there is some $M \in \mathcal{N}$ such that $N_1 \cap N_2 \geq M$.

Observe that (i) in the above lemma shows that a variety of finite groups defines a topology in a group considering $\mathcal{N}_{\mathcal{C}}$ as a neighbourhood system of the identity.

Finally, we introduce a key lemma relating varieties of finite groups and natural direct products. Recall that given a group $H = H_1 \times \cdots \times H_m$, we say that a subgroup of H is a natural direct product if it is of the form $J_1 \times \cdots \times J_m$ with $J_i \leq H_i$ for $i = 1, \ldots, m$.

Lemma 4.2.3. *Let $G = G_1 \times \cdots \times G_m$ and $H \leq G$. Then*

(i) *The largest natural direct product contained in H is*

$$J = (H \cap G_1) \times \cdots \times (H \cap G_m),$$

where we identify G_i with $\{1\} \times \cdots \times \{1\} \times G_i \times \{1\} \times \cdots \times \{1\}$ for $i = 1, \ldots, m$.

(ii) *If $H \trianglelefteq G$ then $J \trianglelefteq G$ and if moreover $|G : H| < \infty$ then $|G : J| < \infty$.*

(iii) *If $H \trianglelefteq_{\mathcal{C}} G$, for some variety of finite groups $\mathcal{C}$, then $J \trianglelefteq_{\mathcal{C}} G$.*

Proof. For (i) observe that it is clear that J is a natural direct product contained in H. Moreover, if there were a larger one, then $J' = J_1 \times \cdots \times J_m$ with some J_i strictly containing $H \cap G_i$, this automatically would imply that J' is not contained in H, which is a contradiction.

Since the elements of G are m-tuples and H is normal in G it is clear that J is also normal in G. Now if $|G : H| < \infty$, in order to see that J is of finite index in G it suffices to check that each $H \cap G_i$ is of finite index in G_i for $i = 1, \ldots, m$. By the second isomorphism theorem, we have

$$\frac{G_i}{G_i \cap H} \cong \frac{G_i H}{H},$$

and since H is of finite index in G we obtain the desired result.

Finally, (iii) also follows by the same argument. Provided that $\mathcal{C}$ is closed under direct products it suffices to prove that $H \cap G_i \trianglelefteq_{\mathcal{C}} G_i$. Again, by the second isomorphism theorem we get that

$$\frac{G_i}{G_i \cap H} \cong \frac{G_i H}{H},$$

and since $H \trianglelefteq_{\mathcal{C}} G$ and $\mathcal{C}$ is closed under subgroups, $H \trianglelefteq_{\mathcal{C}} G_i H$ and the result follows.

□

4.3 A sufficient condition for a weakly regular branch group to have the $\mathcal{C}$-CSP

Let $G \leq \operatorname{Aut} T$ and let $\mathcal{C}$ be a variety of finite groups.

Standing Assumption. *Throughout these section we assume that $G/\operatorname{st}_G(n) \in \mathcal{C}$ for every $n \in \mathbb{N}$.*

Definition 4.3.1. *A group $G \leq \operatorname{Aut} T$ has the $\mathcal{C}$-congruence subgroup property (abbreviated to $\mathcal{C}$-CSP) if every normal subgroup N in G such that $G/N \in \mathcal{C}$ contains some level stabilizer in G.*

Also we say that G has the $\mathcal{C}$-CSP modulo $M \trianglelefteq G$ if every normal subgroup N such that $G/N \in \mathcal{C}$ and containing M also contains some level stabilizer in G.

For instance, in the case of the GGS-groups, and in particular for the one defined by the constant vector denoted by $\mathcal{G}$, we know that $\mathcal{G}/\operatorname{st}_{\mathcal{G}}(n)$ belongs to the varieties of finite p-groups, finite nilpotent groups and finite solvable groups. Since we will focus on the case of the variety of finite p-groups, we introduce the specific definitions for this particular case.

Let $\mathcal{A}$ be a Sylow pro-p subgroup of $\operatorname{Aut} T$, where T is the p-adic tree.

Definition 4.3.2. *A group $G \leq \mathcal{A}$ has the p-congruence subgroup property (abbreviated to p-CSP) if every normal subgroup of index a power of p in G contains some level stabilizer in G.*

Also we say that G has the p-CSP modulo $M \trianglelefteq G$ if every normal subgroup of p-power index in G and containing M also contains some level stabilizer in G.

We start with a simple but very useful proposition that will be used repeatedly.

Lemma 4.3.3. *Let $G \leq \operatorname{Aut} T$ and $N \trianglelefteq M \trianglelefteq G$. If G has the $\mathcal{C}$-CSP modulo M and M has the $\mathcal{C}$-CSP modulo N then G has the $\mathcal{C}$-CSP modulo N.*

Proof. First of all, observe that it makes sense to consider the $\mathcal{C}$-CSP for M, because by (iii) in Lemma 4.2.2 we have $\operatorname{st}_G(n) \cap M = \operatorname{st}_M(n) \trianglelefteq_{\mathcal{C}} M$ for every $n \in \mathbb{N}$.

Now let $H \trianglelefteq_{\mathcal{C}} G$ be such that $H \geq N$. We have to prove that $H \geq \operatorname{st}_G(n)$ for some $n \in \mathbb{N}$. Since M has the $\mathcal{C}$-CSP modulo N and since again by (iii) in Lemma 4.2.2 $H \cap M \trianglelefteq_{\mathcal{C}} M$, there is some $m \in \mathbb{N}$ such that $\operatorname{st}_M(m) \leq H \cap M$. Now since $H, \operatorname{st}_G(m) \trianglelefteq_{\mathcal{C}} G$, by (i) in Lemma 4.2.2 we know that $\operatorname{st}_G(m) \cap H \trianglelefteq_{\mathcal{C}} G$, and by (ii) in the same lemma $(\operatorname{st}_G(m) \cap H)M \trianglelefteq_{\mathcal{C}} G$. Thus there is some $l \in \mathbb{N}$ such that $\operatorname{st}_G(l) \leq (\operatorname{st}_G(m) \cap H)M$. Taking $n := \max\{m, l\}$, we have

$$\begin{aligned} \operatorname{st}_G(n) &= \operatorname{st}_G(l) \cap \operatorname{st}_G(m) \leq (\operatorname{st}_G(m) \cap H)M \cap \operatorname{st}_G(m) \\ &= (\operatorname{st}_G(m) \cap H)(M \cap \operatorname{st}_G(m)) \\ &\leq (\operatorname{st}_G(m) \cap H)(H \cap M) \leq H, \end{aligned}$$

where the second equality follows by the modular law. □

Recall that if a group $G \leq \operatorname{Aut} T$ is weakly regular branch over a normal subgroup R then $\psi(R) \geq R \times \overset{d}{\ldots} \times R$ with $R \neq 1$. In particular, $\psi(\mathrm{rst}_G(1)) \geq R \times \overset{d}{\ldots} \times R$ and, by induction, $\psi_n(\mathrm{rst}_G(n)) \geq R \times \overset{d^n}{\ldots} \times R$ for all $n \in \mathbb{N}$, as mentioned after Definition 1.1.8.

Theorem 4.3.4. *Let $G \leq \operatorname{Aut} T$ be a weakly regular branch group over a subgroup R. Suppose that there exists $H \trianglelefteq G$ such that $R \geq H \geq R' \geq L$ where $L := \psi^{-1}(H \times \overset{d}{\ldots} \times H)$. If G has the $\mathcal{C}$-CSP modulo H and H has the $\mathcal{C}$-CSP modulo L, then G has the $\mathcal{C}$-CSP.*

Proof. Put $L_0 := H$, $L_1 := L = \psi^{-1}(H \times \overset{d}{\ldots} \times H) \leq R'$ and

$$L_n := \psi_n^{-1}(H \times \overset{d^n}{\ldots} \times H) \leq \psi_{n-1}^{-1}(R' \times \overset{d^{n-1}}{\cdots} \times R') \leq \mathrm{rst}_G(n-1)'$$

for $n \in \mathbb{N}, n \geq 2$.

We will show by induction on n that G has the $\mathcal{C}$-CSP modulo L_n for each $n \in \mathbb{N}$. Then, as G is weakly regular branch, it is in particular transitive on all levels of T, so by Lemma 1.1.11, for each non-trivial $N \trianglelefteq G$ there exists $n \in \mathbb{N}$ such that $N \geq \mathrm{rst}_G(n)' \geq L_{n+1}$, whence the result follows.

There is nothing to show for the base case as we have assumed that G has the $\mathcal{C}$-CSP modulo H. It will suffice to show that L_n has the $\mathcal{C}$-CSP modulo L_{n+1} for all $n \in \mathbb{N}$ and then inductively apply Lemma 4.3.3.

Fix $n \in \mathbb{N}$ and let $L_{n+1} \leq N \trianglelefteq_{\mathcal{C}} L_n$. Then by (iv) in Lemma 4.2.2,

$$L \times \overset{d^n}{\ldots} \times L \leq \psi_n(N) \trianglelefteq_{\mathcal{C}} H \times \overset{d^n}{\ldots} \times H.$$

By Lemma 4.2.3 we know that $J = J_1 \times \cdots \times \ldots J_{d^n}$ where $J_i = \psi_n(N) \cap H$ for $i = 1, \ldots, d^n$ is such that $L \times \cdots \times L \leq J \leq \psi_n(N) \leq H \times \cdots \times H$ with $J \trianglelefteq_{\mathcal{C}} H \times \cdots \times H$. Since each $H/J_i \cong \{1\} \times \cdots \times \{1\} \times H/J_i \times \{1\} \times \cdots \times \{1\} \leq$

$H/J_1 \times \cdots \times H/J_{d^n}. \in \mathcal{C}$ and $\mathcal{C}$ is closed under subgroups, we get that $H/J_i \in \mathcal{C}$ for $i = 1, \ldots, d^n$. Thus $L \leq J_i \trianglelefteq_{\mathcal{C}} H$ and since H has the $\mathcal{C}$-CSP modulo L, there is some $m_i \in \mathbb{N}$ such that $\mathrm{st}_H(m_i) \leq J_i$. Taking the maximum, m, of the m_i, we obtain

$$\mathrm{st}_H(m) \times \overset{d^n}{\cdots} \times \mathrm{st}_H(m) \leq \psi_n(N).$$

Thus

$$\mathrm{st}_{L_n}(m+n) = \psi_n^{-1}(\mathrm{st}_H(m) \times \overset{d^n}{\cdots} \times \mathrm{st}_H(m)) \leq N$$

as required. □

4.4 Example 1: the GGS-groups with constant defining vector.

Let p be an odd prime and let $\mathcal{G} \leq \mathrm{Aut}\, T$ be the GGS-group defined by the constant vector, and let again $K = \langle ba^{-1} \rangle^{\mathcal{G}}$. We have shown in the previous chapter that $\mathcal{G}$ does not have the CSP, because it virtually maps onto $\mathbb{Z}$ and therefore has many finite quotients that are not p-groups (this is also the general strategy used in the context of arithmetic groups). However, as we next see, it does have the p-CSP.

This automatically gives us the answers for the cases of varieties of solvable and nilpotent groups. Observe that for the family $\mathcal{C}_s$ of finite solvable groups, $\mathcal{G}$ will not have the $\mathcal{C}_s$-CSP. The reason is similar to the case of the usual CSP. Since $\mathcal{G}/K' \cong C_p \ltimes (C_\infty \times \overset{p-1}{\cdots} \times C_\infty)$ by Proposition 3.5.2, the group has quotients that are solvable but not of p-power index, and since all level stabilizers are of p-power index the topologies cannot coincide.

On the other hand, if we consider $\mathcal{C}_n$ the family of finite nilpotent groups, the answer will be positive, that is, $\mathcal{G}$ has the $\mathcal{C}_n$-CSP. Suppose $N \trianglelefteq G$ is such that G/N is nilpotent. Then $N \geq \gamma_i(\mathcal{G})$ for some $i \in \mathbb{N}$. Since $\mathcal{G}/\mathcal{G}'$ is of exponent p, by (iii) in Lemma 1.3.5, each quotient $\gamma_i(\mathcal{G})/\gamma_{i+1}(\mathcal{G})$ is also of exponent p, and since they are finitely generated, they will be finite. Thus, each $\gamma_i(\mathcal{G})$ is of finite index in $\mathcal{G}$ and moreover of index a power of p. Once we prove that $\mathcal{G}$ has the p-CSP we will be proving, in particular, that each $\gamma_i(\mathcal{G})$ contains some level stabilizer, and thus $\mathcal{G}$ has the $\mathcal{C}_n$-CSP.

We know from [15] that $\mathcal{G}$ is weakly regular branch over K'.

Proposition 4.4.1. *For each $n \in \mathbb{N}$, the nth rigid stabilizer satisfies*

$$\psi_n(\mathrm{rst}_{\mathcal{G}}(n)) = K' \times \overset{p^n}{\ldots} \times K'.$$

Proof. Observe that by the discussion after Proposition 1.1.9 we already know that $\psi_n(\mathrm{rst}_G(n)) \geq K' \times \overset{p^n}{\ldots} \times K'$. Now if we prove the statement for $n = 1$, since $\psi(\mathrm{rst}_G(2)) \leq \mathrm{rst}_G(1) \times \cdots \times \mathrm{rst}_G(1)$ we get $\psi_2(\mathrm{rst}_G(2)) \leq K' \times \overset{p^2}{\ldots} \times K'$ and inductively the same for the rest of the levels. Thus, it suffices to show the claim for $n = 1$.

By the proof of Theorem 3.1.3, we have $\psi(\mathrm{rst}_{\mathcal{G}'}(1)) = K' \times \overset{p}{\ldots} \times K'$. Thus we only need to prove that $K \geq \mathrm{rst}_{\mathcal{G}}(1)$, since then $\mathrm{rst}_{\mathcal{G}}(1) = \mathrm{rst}_K(1) = \mathrm{rst}_{\mathcal{G}'}(1)$, where the latter equality holds because $\mathcal{G}' = \mathrm{st}(1) \cap K$. We will in fact show the stronger statement $\mathrm{st}_{\mathcal{G}}(1)' \geq \mathrm{rst}_{\mathcal{G}}(x)$ for some $x \in L_1$, (and therefore for all $x \in L_1$, as $\mathrm{st}_{\mathcal{G}}(1)$ is normal in G, which acts transitively on L_1) from which the claim follows as $K \geq \mathcal{G}' \geq \mathrm{st}_{\mathcal{G}}(1)'$. Suppose that there is some g such that $\psi(g) = (h, 1, \ldots, 1) \in \mathrm{rst}_{\mathcal{G}}(x) \setminus \mathrm{st}_{\mathcal{G}}(1)'$. Then we can write $g = b^{i_0} b_1^{i_1} \ldots b_{p-1}^{i_{p-1}} t$ where $t \in \mathrm{st}_{\mathcal{G}}(1)'$ and $i_j \in \mathbb{F}_p$ for $j = 0, \ldots, p-1$. Now

$$\psi(g) = (a^* b^{i_1} a^* t_1, \ldots, a^* b^{i_0} a^* t_n) = (h, 1, \ldots, 1),$$

where $t_i \in G'$ for $i = 1, \dots, p$ and the $*$ denote unimportant exponents. Then, necessarily, $i_j = 0$ for $j \neq 1$, and consequently

$$\psi(g) = (b^{i_1} t_1, a^{i_1} t_2, \dots, a^{i_1} t_{p-1}) = (h, 1, \dots, 1),$$

implies that also $i_1 = 0$. Thus $g \in \mathrm{st}_{\mathcal{G}}(1)'$, as required. □

Proposition 4.4.2 (See Proposition 3.5.2 and [29] Example 7.4.14, Section 8.2)**.** *The quotient $\mathcal{G}/K'$ is isomorphic to the integral uniserial space group $C_p \ltimes \mathbb{Z}[\theta]$ where θ is a primitive pth root of unity and the generator of C_p acts by multiplication by θ. In particular, each normal subgroup of p-power index in $\mathcal{G}/K'$ is precisely $\gamma_i(\mathcal{G})K'/K'$ for some $i \in \mathbb{N}$.*

Corollary 4.4.3. *The group $\mathcal{G}$ has the p-CSP modulo K'.*

Proof. In [15, Theorem 4.6] it is proved that $G/K' \, \mathrm{st}_G(n)$ is of maximal class and order p^{n+1} for every $n \in \mathbb{N}$. Then we get that $\mathrm{st}_G(n)K' = \gamma_n(G)K'$ for every $n \in \mathbb{N}$ and by Proposition 4.4.2 the result follows. □

Lemma 4.4.4. *The subgroup K has the p-CSP modulo K'.*

Proof. Let us consider $N \trianglelefteq_p K$ such that $N \geq K'$. We want to see that $N \geq \mathrm{st}_K(m)$ for some $m \in \mathbb{N}$. Since K is of finite index in $\mathcal{G}$ the normal core of N in $\mathcal{G}$ will be the intersection of a finite number of conjugates of N, say $N_{\mathcal{G}} = N_1 \cap N_2 \cap \dots \cap N_k$, where $N_i = N^{g_i}$ for some $g_i \in \mathcal{G}$, $i = 1, \dots, k$. Now

$$|K : N_{\mathcal{G}}| = |K : N_1||N_1 : N_1 \cap N_2| \dots |N_1 \cap \dots \cap N_{k-1} : N_1 \cap \dots N_k|.$$

But each $N_1 \cap \dots \cap N_{j-1}/N_1 \cap \dots \cap N_j \cong (N_1 \cap \dots \cap N_{j-1})N_j/N_j$. Since each N_i has the same index as N in K, all these indices are powers of p, and thus $N_{\mathcal{G}} \trianglelefteq_p \mathcal{G}$. Moreover since K' is normal in $\mathcal{G}$ we also have $K' \leq N_{\mathcal{G}}$. Then by the previous corollary $N_{\mathcal{G}} \geq \mathrm{st}_{\mathcal{G}}(m)$ for some m and since $N_{\mathcal{G}} \leq K$ we have $N \geq N_{\mathcal{G}} \geq \mathrm{st}_K(m)$. □

Define $K_1 := K', K_2 := \psi^{-1}(K' \times \overset{p}{\ldots} \times K') = \mathrm{rst}_{\mathcal{G}}(1)$.

Let us consider the following maps:

$$\begin{array}{rcl} S : \mathrm{st}_{\mathcal{G}}(1) & \to & \mathcal{G}/K_1 \times \overset{p-2}{\ldots} \times \mathcal{G}/K_1 \\ g = (g_1, \ldots, g_p) & \mapsto & (g_1 K_1, \ldots, g_{p-2} K_1) \end{array},$$

and for $n \geq 3$,

$$\begin{array}{rcl} \pi_n : K/K_1 \times \overset{p-2}{\ldots} \times K/K_1 & \to & K/\,\mathrm{st}_{\mathcal{G}}(n)K_1 \times \overset{p-2}{\ldots} \times K/\,\mathrm{st}_{\mathcal{G}}(n)K_1 \\ (g_1 K_1, \ldots, g_{p-2} K_1) & \mapsto & (g_1\,\mathrm{st}_{\mathcal{G}}(n)K_1, \ldots, g_{p-2}\,\mathrm{st}_{\mathcal{G}}(n)K_1) \end{array},$$

Observe that $\ker \pi_n = \mathrm{st}_{\mathcal{G}}(n)K_1/K_1 \times \overset{p-2}{\ldots} \times \mathrm{st}_{\mathcal{G}}(n)K_1/K_1$. Then we have the following properties.

Lemma 4.4.5. *[15, Theorem 4.5] With the above notation, we have*

(i) *the map S restricted to K_1 has kernel K_2 and image $K/K_1 \times \overset{p-2}{\ldots} \times K/K_1$,*

(ii) *the kernel of the composition $\pi_n \circ S$ is $(\mathrm{st}_{\mathcal{G}}(n+1) \cap K_1)K_2$.*

Let us write $S_n = \pi_n \circ S$.

Proposition 4.4.6. *The group K_1 has the p-CSP modulo K_2.*

Proof. Let N be a subgroup such that $K_2 \leq N \trianglelefteq_p K_1$. Then by (iv) of Lemma 4.2.2 and by (i) of Lemma 4.4.5

$$S(N) \trianglelefteq_p K/K_1 \times \overset{p-2}{\ldots} \times K/K_1.$$

For $i = 1, \ldots, p-2$, the intersection of $S(N)$ with the ith direct factor $(K/K_1)_i$ in $S(K_1)$ is of p-power index in K, by (iii) of Lemma 4.2.2. By

Lemma 4.4.4, it contains $(\mathrm{st}_{\mathcal{G}}(n_i)K_1/K_1)_i$ for some $n_i \in \mathbb{N}$. Taking $n = \max\{n_i \mid i = 1, \ldots, p-2\}$ yields $S(N) \geq \mathrm{st}_{\mathcal{G}}(n)K_1/K_1 \times \overset{p-2}{\cdots} \times \mathrm{st}_{\mathcal{G}}(n)K_1/K_1$. In other words, $S(N) \geq \ker \pi_n$, and thus $N \geq S^{-1}(\ker \pi_n) = \ker S_n = (\mathrm{st}_{\mathcal{G}}(n+1) \cap K_1)K_2$. □

We must now separate the proof into two cases: $p = 3$ and $p \geq 5$. This happens because we would like to apply Theorem 4.3.4 to $\mathcal{G}$ with $H = R = K_1$. The only remaining hypothesis to check is that $K_1' \geq K_2$. However, this only holds when $p \geq 5$, which is implicit in the proof of [15, Lemma 4.2 (iii)]. In fact, by (ii) in Lemma 4.4.5, $K_1/K_2 \cong K/K_1 \times \overset{p-2}{\cdots} \times K/K_1$ and hence it is abelian, so that $K_1' = K_2$. In particular, this and Proposition 4.4.1 imply that $\mathrm{rst}_{\mathcal{G}}(n)' = \mathrm{rst}_{\mathcal{G}}(n+1)$ for each $n \geq 1$.

Corollary 4.4.7. *For every prime $p \geq 5$, the GGS-group $\mathcal{G} \leq \mathrm{Aut}\, T$ with constant vector has the p-CSP, but not the CSP.*

Let us now prove the remaining case, so that from now on $p = 3$. The following result can be found in [3, Proposition 7.2].

Lemma 4.4.8. *Let $\mathcal{G}$ and K be as before, then we have*

$$\psi(\mathcal{G}'') = K' \times K' \times K'.$$

Proof. One inclusion is clear because $\psi(\mathcal{G}') \leq K \times K \times K$ by (iii) in Lemma 1.3.8. For the other one, observe that we have $\psi([b, a]) = (y_1, 1, y_1^{-1})$ and $\psi([b^{-1}, a]^a) = (y_0, y_0^{-1}, 1)$. Thus $\psi([[b, a], [b^{-1}, a]^a]) = ([y_0, y_1], 1, 1)$ and, since $K' = \langle [y_0, y_1] \rangle^{\mathcal{G}}$, the result follows by Proposition 1.1.10. □

In order to apply Theorem 4.3.4 with $R = K_1 = K'$ and $H = K_2 = \mathcal{G}''$ we must check that $K'' \geq \psi^{-1}(\mathcal{G}'' \times \mathcal{G}'' \times \mathcal{G}'')$.

Proposition 4.4.9. *Let $\mathcal{G}$ and K as before. We have*

$$\mathcal{G}'' \leq \gamma_3(K).$$

Proof. Since $\mathcal{G}' = \langle [a,b] \rangle^{\mathcal{G}}$, we have

$$\mathcal{G}'' = \langle [[a,b],[a,b]^g] \mid g \in \mathcal{G} \rangle^{\mathcal{G}}.$$

Observe that $\gamma_3(K)$ is a normal subgroup in $\mathcal{G}$; hence it suffices to prove that $[[a,b],[a,b]^g] \in \gamma_3(K)$ for every $g \in \mathcal{G}$. We already know that $\mathcal{G}/K \cong C_3$ and we can take as coset representatives $\{1, a, a^2\}$. Then let us write $g = ka^i$ with $i \in \mathbb{F}_3$ and $k \in K$. If $i = 0$ there is nothing to prove, because

$$\begin{aligned}
[[a,b],[a,b]^g] &= [[a,b],[a,b][a,b,g]] \\
&= [[a,b],[a,b,g]],
\end{aligned}$$

and since $\mathcal{G}' \leq K$, it is clear that the element belongs to $\gamma_3(K)$.

Let us suppose that $g = ka^i$ with $i = 1, 2$ and $k \in K$. Now we have

$$\begin{aligned}
[[a,b],[a,b,ka^i]] &= [[a,b],[a,b,a^i][a,b,k]^{a^i}] \\
&= [[a,b],[a,b,k]^{a^i}][[a,b],[a,b,a^i]]^{[a,b,k]^{a^i}}.
\end{aligned}$$

It is clear that the first factor is in $\gamma_3(K)$. On the other hand we have

$$\begin{aligned}
\psi([a,b]) &= (b^{-1}a, 1, a^{-1}b), \\
\psi([a,b,a]) &= ((a^{-1}b)^2, b^{-1}a, b^{-1}a), \\
\psi([a,b,a^2] &= (a^{-1}b, a^{-1}b, (b^{-1}a)^2),
\end{aligned}$$

and this shows that the second factor is trivial for $i = 1, 2$.

□

Proposition 4.4.10. *Let* $\mathcal{G}$ *and* K *as before. We have* $\psi(K'') \geq \mathcal{G}'' \times \mathcal{G}'' \times \mathcal{G}''$.

Proof. By Proposition 4.4.9, it suffices to prove the following containment

$$\psi(K'') \geq \gamma_3(K) \times \gamma_3(K) \times \gamma_3(K).$$

Since $\mathcal{G}$ is weakly regular branch over K', we know that for every $k_1 \in K'$ there is some $g_1 \in K'$ such that $\psi(g_1) = (k_1, 1, 1)$. On the other hand, since $\psi([y_0, y_1]) = (y_2, y_0, y_1)$ we get that K' is subdirect in $K \times K \times K$. Thus, for every $k_2 \in K$ there is some $g_2 \in K'$ such that $\psi(g_2) = (k_2, *, *)$. Finally, we obtain

$$\psi([g_1, g_2]) = ([k_1, k_2], 1, 1),$$

and by Proposition 1.1.10, the result follows. □

Now we will apply Theorem 4.3.4 with $R = K' = K_1$ and $H = \mathcal{G}'' = K_2$. By Proposition 4.4.6 we only need to prove that $K_2 = \mathcal{G}''$ has the p-CSP modulo $\psi^{-1}(\mathcal{G}'' \times \mathcal{G}'' \times \mathcal{G}'')$. Now by Lemma 4.4.8 we get that $\mathcal{G}''/\psi^{-1}(\mathcal{G}'' \times \mathcal{G}'' \times \mathcal{G}'') \cong K_1/K_2 \times K_1/K_2 \times K_1/K_2$, and then using again Proposition 4.4.6 the result follows.

4.5 Example 2: the Basilica group

Let us consider the Basilica group, defined in the first chapter of the thesis, so that $G = \langle a, b\rangle$ with

$$a = (1, b),$$
$$b = (1, a)\epsilon,$$

where ϵ denotes the permutation at the root $(1\,2)$.

Observe that since $G/G' \cong \mathbb{Z} \times \mathbb{Z}$ (see Lemma 1.3.19) the same proof as for the constant vector GGS-group shows that the Basilica group does not have the congruence subgroup property. However, as we will see, it has the 2-CSP.

Lemma 4.5.1. *Let $A := \langle a\rangle^G$ and $B := \langle b\rangle^G$. Then $G' = A \cap B$.*

Proof. Since $G' = \langle [a, b]\rangle^G$ and $[a, b] \in A \cap B$, it is clear that $G' \leq A \cap B$. Moreover, $A \cap B \subseteq \langle aG'\rangle \cap \langle bG'\rangle = G'$, where the last equality comes from (vii) in Lemma 1.3.19. □

Lemma 4.5.2. *With A and B as above we have*

(i) $\mathrm{rst}_G(1) = A$ *with* $\psi(A) = B \times B$,

(ii) $\psi_{n-1}(\mathrm{rst}_G(n)) = G' \times \overset{2^{n-1}}{\cdots} \times G'$.

Proof. The first item is Lemma 3 of [25].

For the rest of the levels, we already know by Lemma 1.1.6 that

$$\psi(\mathrm{rst}_G(n)) = (\mathrm{rst}_G(n-1) \times \mathrm{rst}_G(n-1)) \cap \psi(\mathrm{rst}_G(1)).$$

Then $\psi(\mathrm{rst}_G(2)) = (A \times A) \cap (B \times B) = G' \times G'$, and the rest follows because $\psi(G') \geq G' \times G'$. □

In view of the fact that $\psi(\gamma_3(G)) \geq \psi(G'') = \gamma_3(G) \times \gamma_3(G)$ (recall Lemma 1.3.19), we will take $R = G'$ and $H = \gamma_3(G)$ to apply Theorem 4.3.4. Note that we even have $L_n \leq \mathrm{rst}_G(n)'$ for all $n \in \mathbb{N}$, in the notation of that theorem. It only remains to show that G and $\gamma_3(G)$ have the 2-CSP modulo $\gamma_3(G)$ and $\psi^{-1}(\gamma_3(G) \times \gamma_3(G))$, respectively. The rest of this section is devoted to proving this.

Proposition 4.5.3. *The quotient $G'/\gamma_3(G)$ is infinite cyclic.*

Proof. To prove the statement, it suffices to show that $[a, b]$, whose image generates $G'/\gamma_3(G)$, has infinite order modulo $\gamma_3(G)$. Suppose for a contradiction that it has finite order $n \in \mathbb{N}$. Then, we know that $\psi([a, b]^n) = ((b^n)^a, b^{-n}) \in \psi(\gamma_3(G))$. Now by (vi) in Lemma 1.3.19, we know that $\psi(\gamma_3(G)) = \langle \psi([a, b, b]) \rangle \ltimes (\gamma_3(G) \times \gamma_3(G))$, and since $\psi([a, b, b]) = ((b^{-2})^a, bb^a)$ we get that

$$\psi([a, b]^n) = ((b^n)^a, b^{-n}) = (h_1, h_2)((b^{-2t})^a, (bb^a)^t),$$

with $h_i \in \gamma_3(G)$ for $i = 1, 2$ and $t \in \mathbb{Z}$. In particular, $(b^n)^a \equiv (b^{-2t})^a \pmod{G'}$ implies that $n = -2t$. But then $b^{-2t}(bb^a)^t$ is in $\gamma_3(G)$ and modulo $\gamma_3(G)$ we get the following equalities

$$b^{-2t}(bb^a)^t = b^{-2t}(b^2[b, a])^t = b^{-2t}b^{2t}[b, a]^t = [a, b]^{-t} = 1,$$

implying that $[a, b]^{-t} \in \gamma_3(G)$ and contradicting the minimality of n. □

As a consequence, we get a description for $G/\gamma_3(G)$.

Definition 4.5.4. *The integral Heinseberg group is the group*

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

It is well known that $H = \langle x, y, z \mid [x,z] = [y,z] = 1, [x,y] = z \rangle \cong \langle x \rangle \ltimes \langle y, z \rangle \cong \mathbb{Z} \ltimes (\mathbb{Z} \times \mathbb{Z})$, where

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proposition 4.5.5. *The quotient $G/\gamma_3(G)$ is isomorphic to the integral Heisenberg group.*

Proof. We already know that $G/B = \langle aB \rangle \cong \mathbb{Z}$. Then, by the projective property of free groups, we know that $G/\gamma_3(G)$ splits over $B/\gamma_3(G)$. Let us see that $B/\gamma_3(G) \cong \mathbb{Z} \times \mathbb{Z}$. We already know that $B/G' = \langle bG' \rangle \cong \frac{B/\gamma_3(G)}{G'/\gamma_3(G)} \cong \mathbb{Z}$ is cyclic. On the other hand since $G/\gamma_3(G)$ is of nilpotency class 2 we know that $G'/\gamma_3(G)$ is in the center and by Proposition 4.5.3 $G'/\gamma_3(G) = \langle [a,b]G' \rangle \cong \mathbb{Z}$. Thus, we conclude that $B/\gamma_3(G)$ is abelian and isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Hence

$$\begin{aligned} G/\gamma_3(G) &= \langle a\gamma_3(G) \rangle \ltimes B/\gamma_3(G), \\ &= \langle a\gamma_3(G) \rangle \ltimes (\langle b\gamma_3(G) \rangle \times \langle [a,b]\gamma_3(G) \rangle), \\ &\cong \mathbb{Z} \ltimes (\mathbb{Z} \times \mathbb{Z}). \end{aligned}$$

Since it is clear that $G/\gamma_3(G)$ satisfies the relations of the presentation above, the groups are isomorphic. □

Lemma 4.5.6. *If* $g \in G'$ *is such that* $\psi(g) = (g_1, g_2)$ *then* $g_2 g_1 \in G'$. *Similarly if* $g \in G' \operatorname{st}_G(n)$ *then* $g_2 g_1 \in G' \operatorname{st}_G(n-1)$.

Proof. Let us define $\varphi : G \times G \longrightarrow G/G'$ by sending each (g_1, g_2) to $g_2 g_1 G'$. Then we want to prove that $G' \leq \ker(\varphi \circ \psi)$. It is clear that $\psi^{-1}(G' \times G')$ is contained in the kernel, and since $G' \geq \psi^{-1}(G' \times G')$ it suffices to check that the image is in the kernel for the generators of G' modulo $\psi^{-1}(G' \times G')$, that is, for $[a, b]$. The result follows because $\psi([a, b]) = (b^a, b^{-1})$. □

Proposition 4.5.7. *The group* G *has the* 2*-CSP modulo* $\gamma_3(G)$.

Proof. It suffices to prove that G, A, and G' have the 2-CSP modulo A, G' and $\gamma_3(G)$, respectively, and apply Lemma 4.3.3 twice. Since $G/A \cong A/G' \cong G'/\gamma_3(G) \cong \mathbb{Z}$, it is enough to show that $|G : A \operatorname{st}_G(n)|$, $|A : G' \operatorname{st}_A(n)|$ and $|G' : \gamma_3(G) \operatorname{st}_{G'}(n)|$ tend to infinity with n. Indeed, since in $\mathbb{Z}$ the subgroups of index a power of 2 are totally ordered, this will imply that any normal subgroup N of index a power of 2 in, for instance, $A \leq N \leq G$ will satisfy that $N \geq \operatorname{st}_G(n)A$ for some $n \in \mathbb{N}$ as desired.

We first prove by induction that $b^{2^n} \notin A \operatorname{st}_G(2n+1)$. The base step, $b \notin A \operatorname{st}_G(1) = \operatorname{st}_G(1)$, is clear. Now assume that $b^{2^{n-1}} \notin A \operatorname{st}_G(2n-1)$ and suppose for a contradiction that $b^{2^n} \in A \operatorname{st}_G(2n+1) = \langle a \rangle G' \operatorname{st}_G(2n+1)$. By Lemma 1.3.19 (v), we can write $A \operatorname{st}_G(2n+1) = \langle a \rangle \langle [a, b] \rangle \psi^{-1}(G' \times G') \operatorname{st}_G(2n+1)$. So there are $i, j \in \mathbb{Z}$ such that

$$\psi([a, b]^j a^i b^{2^n}) = ((b^a)^j a^{2^{n-1}}, b^{i-j} a^{2^{n-1}}) \in G' \operatorname{st}_G(2n) \times G' \operatorname{st}_G(2n).$$

Consider $b^{i-j} a^{2^{n-1}} \in G' \operatorname{st}_G(2n)$. Lemma 4.5.6 implies that $a^{i-j} b^{2^{n-1}} \in G' \operatorname{st}_G(2n-1) \leq A \operatorname{st}_G(2n-1)$. This implies that $b^{2^{n-1}} \in A \operatorname{st}_G(2n-1)$, which is a contradiction. The claim follows by induction.

This easily implies that $a^{2^n} \notin G' \operatorname{st}_A(2n+2)$ for each $n \in \mathbb{N}$. For, $a^{2^n} = (1, b^{2^n})$ and, since $b^{2^n} \notin G' \operatorname{st}_A(2n+1)$, Lemma 4.5.6 yields that a^{2^n} cannot be in $G' \operatorname{st}_A(2n+2)$.

Finally, let us prove that $|G' : \gamma_3(G) \operatorname{st}_{G'}(n)|$ tends to infinity. Suppose that it does not. Then there is some $n \in \mathbb{N}$ and $m_n \in \mathbb{N}$ such that $[a,b]^{2^n} \in \gamma_3(G) \operatorname{st}_{G'}(m)$ for every $m \geq m_n$. Let n be the smallest natural number with this property. Then, we get that

$$\psi([a,b]^{2^n}) = ((b^{2^n})^a, b^{-2^n}),$$

belongs to $\psi(\langle [a,b,b] \rangle)(\gamma_3(G) \operatorname{st}_G(m-1) \times \gamma_3(G) \operatorname{st}_G(m-1))$, for every $m \geq m_n$. Now since $\psi([a,b,b]) = ((b^{-2})^a, bb^a)$, we get that for some $k \in \mathbb{Z}$

$$\begin{aligned}\psi([a,b]^{2^n}) &= ((b^{2^n})^a (b^{-2k})^a, b^{-2^n}(bb^a)^k) \\ &= ((b^{2^n-2k})^a, b^{-2^n+2k}[b,a]^k),\end{aligned}$$

belongs to $(\gamma_3(G) \operatorname{st}_G(m-1) \times \gamma_3(G) \operatorname{st}_G(m-1))$for every $m \geq m_n$.

Since $b^{2^n} \notin G' \operatorname{st}_G(2n+1)$, we know that the order of b in $G/G' \operatorname{st}_G(2n+1)$ is at least 2^{n+1}. Now if $b^{2^n-2k} \in \gamma_3(G) \operatorname{st}_G(m-1) \leq G' \operatorname{st}_G(m-1)$ for every $m \geq m_n$ we necessarily must have $2^{n+1} | (2^n - 2k)$. But then, if ν_2 denotes the 2-adic valuation, we get that $n+1 = \nu_2(2^{n+1}) \leq \nu_2(2^n - 2k)$. Now if $\nu_2(2^n) \neq \nu_2(-2k)$, we get that $\nu_2(2^n - 2k) = \min\{\nu_2(2^n), \nu_2(2k)\} \leq n$ which is a contradiction. Thus we necessarily have $\nu_2(-2k) = n$ which means that $k = 2^{n-1}\alpha$ with α odd.

Now since $G/\gamma_3(G) \operatorname{st}_G(m-1)$ is a 2-group, looking at the second component we get $[b,a]^{2^{n-1}} \in \gamma_3(G) \operatorname{st}_{G'}(m-1)$ for $m \geq m_n$ contradicting the minimality of n. □

Proposition 4.5.8. *The group $\gamma_3(G)$ has the 2-CSP modulo $\psi^{-1}(\gamma_3(G) \times$*

$\gamma_3(G))$.

Proof. This is proved like the previous result. Since $\gamma_3(G) = \psi^{-1}(\gamma_3(G) \times \gamma_3(G)) \rtimes \langle [a,b,b] \rangle$ by Lemma 1.3.19, we need only show that

$$|\gamma_3(G) : \psi^{-1}(\gamma_3(G) \times \gamma_3(G)) \, \mathrm{st}_{\gamma_3(G)}(n)|$$

tends to infinity with n as in the previous proposition. Writing $d := [a,b,b]$, as before, we have $\psi(d) = (a^{-1}b^{-2}a, ba^{-1}ba)$.

Suppose that $d^{2^n} \in \psi^{-1}(\gamma_3(G) \times \gamma_3(G)) \, \mathrm{st}_G(2n+4)$. Then $\psi(d^{2^n}) = ((b^{-2^{n+1}})^a, (bb^a)^{2^n}) \in \gamma_3(G) \, \mathrm{st}_G(2n+3) \times \gamma_3(G) \, \mathrm{st}_G(2n+3)$ implies that $b^{2^{n+1}} \in G' \, \mathrm{st}_G(2n+3)$ which is a contradiction by the proof of the previous proposition. □

Chapter 5

Portrait growth in contracting regular branch groups

5.1 Introduction

As mentioned in the first chapter of the thesis, groups acting on regular rooted trees have had an important role regarding the word growth problem as the first examples of intermediate word growth was of this nature. Similarly to the word growth, in the case of self-similar contracting groups one can also ask about the portrait growth.

In this chapter we will work on the portrait growth. Indeed, we give a constructive way of finding recursive equations for the sequence of the portrait growth for any regular branch contracting group. Then we calculate these equations explicitly for some of them and conclude that all have doubly exponential portrait growth. We conjecture that this will happen whenever the group is regular branch and contracting, but we do not have a general proof yet.

In particular, we show this for the first Grigorchuk group, answering a question posed by Grigorchuk in Problem 3.5 of [18].

Theorem 5.1.1. *There exist positive constants α, β, and γ such that the portrait growth sequence $\{a_n\}_{n=0}^{\infty}$ of the first Grigorchuk group Γ satisfies the inequalities*

$$\alpha e^{\gamma^{2^n}} \leq a_n \leq \beta e^{\gamma^{2^n}},$$

for all $n \geq 0$. Moreover, $\gamma \approx 0.71$.

We also study the portrait growth for the non-symmetric GGS-groups and for the Apollonian group, which have a closed formula for the portrait growth sequence.

Theorem 5.1.2. *Let G be a GGS-group defined by a non-symmetric vector $\mathbf{e} \in \mathbb{F}_p^{p-1}$. The portrait growth sequence $\{a_n\}_{n=0}^{\infty}$ of G is given by*

$$a_0 = 1 + 2(p-1)$$
$$a_n = p(x_1 + (p-1)y_1)^{p^{n-1}},$$

where x_1 and y_1 are the number of solutions in $\mathbb{F}_p^p$ of

$$(n_0, \ldots, n_{p-1})C(\mathbf{e}, 0) \odot (n_1, n_2, \ldots, n_{p-1}, n_0) = (0, \ldots, 0),$$

with $n_0 + \cdots + n_{p-1} = 0$ and $n_0 + \cdots + n_{p-1} = 1$, respectively; where $\odot$ denotes the product by coordinates.

The definition of $C(\mathbf{e}, 0)$ is given at the begining of Section 5.5. For instance, for the Gupta-Sidki 3-group, which corresponds to the GGS-group defined by $e = (1, -1)$ with $p = 3$, one can check that the sequence is exactly $a_n = 3 \cdot 9^{3^{n-1}}$.

Finally we also present the portrait growth of the Apollonian group.

Theorem 5.1.3. *The portrait growth sequence* $\{a_n\}_{n=0}^{\infty}$ *of the Apollonian group is given by:*

$$a_n = 3^{\frac{3^n-1}{2}} 7^{3^n}.$$

5.2 Portrait growth sequence on a regular branch contracting group

As mentioned in Subsection 1.2.2, given a self-similar contracting group one can describe any element by a finite tree. Then, the longest path in this finite tree starting at the root is called the depth of the element.

For a contracting group G let us denote by $d(g)$ the depth of any $g \in G$. One can consider for each $n \in \mathbb{N}$ the set

$$\{g \in G \mid d(g) \leq n\},$$

which is finite, and ask about the growth function of G with respect to this depth. This is what is called the portrait growth of a group.

We now focus on regular branch groups, since their structure gives us a way to describe the portrait growth function in a recursive way.

Let $G \leq \operatorname{Aut} T$, where T is the d-adic tree, be a contracting regular branch group over a subgroup K. We consider a transversal $S = \{s_1, \dots, s_k\}$ of K in G and denote by $p_n(s_i) = |\{g \in s_i K \mid d(g) \leq n\}|$ and by $p_n = |\{g \in G \mid d(g) \leq n\}|$, the sizes of the sets consisting of the elements of depth less than or equal to n, in each coset and the whole group G, respectively. Then we have $p_n = \sum_{i=1}^{k} p_n(s_i)$.

Now, if we consider a transversal for $\psi^{-1}(K \times \cdots \times K)$ in K, denoted by $R = \{r_1, \ldots, r_l\}$, we know that for each $i = 1, \ldots, k$ and $j = 1, \ldots, l$ we have

$$s_i r_j = (g_1, \ldots, g_d)\alpha \equiv (s_{ij1}, \ldots, s_{ijd})\alpha \pmod{K \times \cdots \times K},$$

with $s_{ijm} \in S$ for $m = 1, \ldots, d$, where α denotes the permutation at the root according to (1.3). Thus $p_n(s_i) = \sum_{j=1}^{l} p_{n-1}(s_{ij1}) \ldots p_{n-1}(s_{ijd})$, and

$$p_n = \sum_{i=1}^{k} \sum_{j=1}^{l} p_{n-1}(s_{ij1}) \ldots p_{n-1}(s_{ijd}).$$

From a theoretical point of view, this can be applied to any contracting regular branch group in order to obtain a recursive formula for the sequence p_n.

The following lemma is a small observation that will be helpful later on.

Lemma 5.2.1. *Let G be a regular branch contracting group which is branch over a subgroup K, and let S be a transversal of K in G. If a is a rooted automorphism of G then for any $s \in S$ we have*

$$p_n(s) = p_n(as) = p_n(sa) = p_n(s^a) \text{ for } n \geq 1.$$

Proof. Observe that for any $g \in sK$ and $u \in L_n$ for $n \geq 1$ we have

$$(ag)_u = a_u g_{a(u)} = g_{a(u)},$$
$$(ga)_u = g_u a_{g(u)} = g_u,$$
$$(g^a)_u = g_{(a^{-1}(u)}.$$

Thus there is a bijection between elements of depth n in sK, asK, saK and s^aK, which implies the statement. □

Observe that in the previous lemma it is important that a is rooted, not for $p_n(s^a)$, but for $p_n(as)$ and $p_n(sa)$ since it contributes to the sections trivially.

5.3 Growth functions and sequences of doubly exponential growth

First of all let us introduce the definition of doubly exponential growth.

Definition 5.3.1. *Given a sequence of positive real numbers $\{a_n\}_{n\in\mathbb{N}}$ we say that it grows doubly exponentially if there exist some positive constants α, β and some $\gamma, d > 1$ such that*

$$\alpha e^{\gamma d^n} \leq a_n \leq \beta e^{\gamma d^n},$$

for every $n \in \mathbb{N}$.

In order to determine that the sequences obtained for our groups below are doubly exponential we need an auxiliary result for doubly exponential sequences.

Lemma 5.3.2. *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers and d a constant with $d > 1$. The following are equivalent.*

(i) *There exist positive constants A and B such that, for all $n \geq 0$,*

$$Aa_n^d \leq a_{n+1} \leq Ba_n^d.$$

(ii) *There exist positive constants α, β, and γ such that, for all $n \geq 0$,*

$$\alpha e^{\gamma d^n} \leq a_n \leq \beta e^{\gamma d^n}.$$

Moreover, if (i) *holds, the sequence* $\left\{\frac{\ln a_n}{d^n}\right\}_{n=0}^{\infty}$ *is convergent, the series* $\sum_{n=0}^{\infty}\frac{1}{d^{n+1}}\ln\frac{a_{n+1}}{a_n^d}$ *is convergent,*

$$\gamma = \lim_{n\to\infty}\frac{\ln a_n}{n^d} = \ln a_0 + \sum_{n=0}^{\infty}\frac{1}{d^{n+1}}\ln\frac{a_{n+1}}{a_n^d},$$

and α *and* β *can be chosen to be* e^{-M} *and* e^M*, respectively, where*

$$M = \frac{1}{d-1}\max\{|\ln A|, |\ln B|\}.$$

Proof. (ii) implies (i). We have, for all n,

$$\frac{\alpha}{\beta^d}a_n^d \leq \frac{\alpha}{\beta^d}\left(\beta e^{\gamma d^n}\right)^d = \alpha e^{\gamma d^{n+1}} \leq a_{n+1} \leq \beta e^{\gamma d^{n+1}} = \frac{\beta}{\alpha^d}\left(\alpha e^{\gamma d^n}\right)^d \leq \frac{\beta}{\alpha^d}a_n^d,$$

so it suffices to consider $A = \frac{\alpha}{\beta^d}$ and $B = \frac{\beta}{\alpha^d}$. (i) implies (ii). Let $b_n = \ln a_n$, for all n. Since $a_{n+1} = a_n^d\frac{a_{n+1}}{a_n^d}$ we have, for all n,

$$b_{n+1} = db_n + \ln\frac{a_{n+1}}{a_n^d},$$

and therefore

$$b_n = d^n\left(b_0 + \sum_{i=0}^{n-1}\frac{1}{d^{i+1}}\ln\frac{a_{i+1}}{a_i^d}\right).$$

For all i, we have $\ln A \leq \ln\frac{a_{i+1}}{a_i^d} \leq \ln B$, and therefore

$$\left|\ln\frac{a_{i+1}}{a_i^d}\right| \leq \max\{|\ln A|, |\ln B|\} = (d-1)M,$$

which implies that, for every n, the series $\sum_{i=n}^{\infty}\frac{1}{d^{i+1}}\ln\frac{a_{i+1}}{a_i^d}$ is absolutely convergent (by comparison to $\sum_{i=n}^{\infty}\frac{1}{d^{i+1}}(d-1)M = \frac{M}{d^n}$).

Let $\gamma = b_0 + \sum_{i=0}^{\infty}\frac{1}{d^{i+1}}\ln\frac{a_{i+1}}{a_i^d}$ and $r_n = \sum_{i=n}^{\infty}\frac{1}{d^{i+1}}\ln\frac{a_{i+1}}{a_i^d}$, for all n. Then, for all n, $b_n = d^n(\gamma - r_n)$ and

$$a_n = e^{\gamma d^n} e^{-d^n r_n}.$$

Since $|r_n| \leq \frac{M}{d^n}$, we obtain that, for all n,

$$e^{-M} e^{\gamma d^n} \leq a_n \leq e^M e^{\gamma d^n}.$$

The last inequalities imply that the sequence $\left\{\frac{\ln a_n}{d^n}\right\}_{n=0}^{\infty}$ converges to γ. □

Notice that the lemma also provides a way of calculating γ once a given sequence satisfies (i).

Finally let us point out an observation that will be useful later on.

Lemma 5.3.3. *Let G be a contracting regular branch group acting on the d-adic tree and $\{a_n\}_{n\geq 0}$ its portrait growth sequence. Then we have*

$$a_{n+1} \leq |G : \mathrm{st}_G(1)| a_n^d, \text{ for } n \geq 0.$$

Proof. The proof of this fact is just a combinatorial observation. Since any element of depth $n+1$ must have sections at the first level of depth n, the posibilities are at most a_n^d. On the other hand, since at the root an element may have $|G : \mathrm{st}_G(1)|$ different labels, we get the inequality. □

5.4 Portrait growth in the first Grigorchuk group

Denote by Γ the first Grigorchuk group and let us recall the definition given in Chapter 1.

Definition 5.4.1. *Let T be the binary tree. The first Grigorchuk group Γ is the group generated by the rooted automorphism a, and by $b, c, d \in \mathrm{st}_\Gamma(1)$, where b, c and d are defined recursively as follows:*

$$\psi(b) = (a, c),$$
$$\psi(c) = (a, d),$$
$$\psi(d) = (1, b).$$

By Lemma 1.3.3 we already know that Γ is contracting with nucleus $\mathcal{N}(\Gamma) = \{a, b, c, d, 1\}$.

For instance, this would be the portrait of the element *bacadb* in Γ.

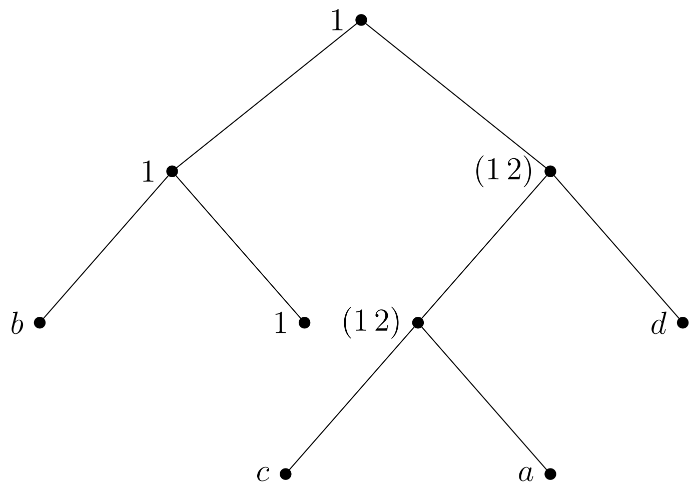


Figure 5.1: Portrait of the element *bacadb*

On the other hand, remember that by Lemma 1.3.2 we know that the group Γ is regular branch over the subgroup $K = \langle [a, b] \rangle^\Gamma$ and that $|\Gamma : K| = 16$.

Theorem 5.4.2. *The portrait growth sequence $\{a_n\}_{n=0}^\infty$ of the first Grig-*

orchuk group Γ *is given by*

$$a_0 = 5$$
$$a_n = 2x_n + 4y_n + 2z_n + 2X_n + 4Y_n + 2Z_n, \text{ for } n \geq 1,$$

where x_n, y_n, z_n, X_n, Y_n, *and* Z_n, *for* $n \geq 1$, *satisfy the system of recursive relations*

$$\begin{aligned}
x_{n+1} &= x_n^2 + 2y_n^2 + z_n^2, \\
y_{n+1} &= x_nY_n + Y_nz_n + X_ny_n + y_nZ_n, \\
z_{n+1} &= X_n^2 + 2Y_n^2 + Z_n^2, \\
X_{n+1} &= 2x_ny_n + 2y_nz_n, \\
Y_{n+1} &= x_nX_n + 2y_nY_n + z_nZ_n, \\
Z_{n+1} &= 2X_nY_n + 2Y_nZ_n,
\end{aligned}$$

with initial conditions

$$\begin{aligned}
&x_1 = y_1 = z_1 = Y_1 = 1, \\
&X_1 = 2, \\
&Z_1 = 0.
\end{aligned}$$

Proof. As mentioned above, it is known that Γ is a regular branch group over $K = \langle [a,b] \rangle^\Gamma$ and $|\Gamma : K| = 16$. We set $B = \langle b \rangle^\Gamma$. In Chapter VIII, Proposition 25 of [11] it is shown that $\Gamma/B = \langle \overline{a}, \overline{d} \rangle$, which is isomorphic to the dihedral group of 8 elements. Then a transversal of B in Γ is given by $\{1, d, ada, dada, a, ad, da, dad\}$. On the other hand, in Proposition 30 (ii) of the same book and chapter it is shown that B/K has order 2 and hence, a transversal of K in B is given by $\{1, b\}$. In this way, we get that a transversal for K in Γ is given by

$$T = \{\ 1, d, ada, dada,\ \ a, ad, da, dad,\ \ b, c, aca, cada,\ \ ba, ac, ca, cad\ \}.$$

Denote by $p_n(t)$ the number of portraits of depth no greater than n in Γ representing elements in the coset tK.

On the other hand, from Lemma 1.3.2 we know that $K/\psi^{-1}(K \times K)$ is generated by the image of $abab$ and that it is of order 4. Thus we get that a transversal of $\psi^{-1}(K \times K)$ in K is given by

$$S = \{1, abab, (abab)^2, baba\}.$$

By Lemma 5.2.1, observe that we have $t \in T \cap \mathrm{st}_G(1)$ we will have

$$p_{n+1}(t) = p_{n+1}(at) = p_{n+1}(t^a) = p_{n+1}(ta), \text{ for } n \geq 0, t \in T. \tag{5.1}$$

Thus we only need to calculate the equations for the representatives in $\{1, c, dada, b, d, cada\}$. We have

$$\begin{aligned}
\psi(1) &= (1,1) & \psi(abab) &= (ca, ac)\\
\psi((abab)^2) &= (dada, dada) & \psi(baba) &= (ac, ca)\\
\psi(c) &= (a,d) & \psi(cabab) &= (aca, cad)\\
\psi(c(abab)^2) &= (dad, ada) & \psi(cbaba) &= (c, ba)\\
\psi(dada) &= (b,b) & \psi(dadaabab) &= (da, ad)\\
\psi(dada(abab)^2) &= (cada, cada) & \psi(dadababa) &= (ad, da)\\
\psi(b) &= (a,c) & \psi(babab) &= (aca, dad)\\
\psi(b(abab)^2) &= (dad, aca) & \psi(bbaba) &= (c, a)\\
\psi(d) &= (1,b) & \psi(dabab) &= (ca, ad)\\
\psi(d(abab)^2) &= (dada, cada) & \psi(dbaba) &= (ac, da)\\
\psi(cada) &= (ba, d) & \psi(cadaabab) &= (ada, cad)\\
\psi(cada(abab)^2) &= (cad, ada) & \psi(cadababa) &= (d, ba),
\end{aligned}$$

where the sections are already written modulo K as representatives in T.

Thus we obtain, for $n \geq 0$,

$$\begin{aligned}
p_{n+1}(1) &= p_n(1)^2 + 2p_n(ac)p_n(ca) + p_n(dada)^2,\\
p_{n+1}(c) &= p_n(a)p_n(d) + p_n(dad)p_n(ada) + p_n(c)p_n(ba) + p_n(aca)p_n(cad),\\
p_{n+1}(dada) &= p_n(b)^2 + 2p_n(ad)p_n(da) + p_n(cada)^2\\
p_{n+1}(b) &= 2p_n(a)p_n(c) + 2p_n(dad)p_n(aca), \qquad (5.2)\\
p_{n+1}(d) &= p_n(1)p_n(b) + p_n(ac)p_n(da) + p_n(ca)p_n(ad) + p_n(dada)p_n(cada),\\
p_{n+1}(cada) &= 2p_n(d)p_n(ba) + 2p_n(ada)p_n(cad),
\end{aligned}$$

with initial conditions

$$p_0(1) = p_0(a) = p_0(b) = p_0(c) = p_0(d) = 1,$$
$$p_0(t) = 0, \text{ for } t \in T \setminus \{1, a, b, c, d\}.$$

Direct calculations, based on (5.2), give

$$p_1(b) = 2$$
$$p_1(cada) = 0$$
$$p_1(t) = 1, \text{ for } t \in \{1, c, d, dada\}.$$

Now if we denote, for $n \geq 1$,

$$\begin{aligned} x_n &= p_n(1) = p_n(a), \\ y_n &= p_n(c) = p_n(ac) = p_n(aca) = p_n(ca), \\ z_n &= p_n(dada) = p_n(dad), \\ X_n &= p_n(b) = p_n(ba), \\ Y_n &= p_n(d) = p_n(ad) = p_n(ada) = p_n(da), \\ Z_n &= p_n(cada) = p_n(cad) \end{aligned}$$

we obtain, for $n \geq 1$,

$$a_n = 2x_n + 4y_n + 2z_n + 2X_n + 4Y_n + 2Z_n,$$

where x_n, y_n, z_n, X_n, Y_n, and Z_n satisfy the recursive relations and initial conditions as claimed, which follows from (5.2). □

Now we can proof the main theorem.

Proof of Theorem 5.1.1. By Lemma 5.3.2, it suffices to show that there exist some positive constants A, B such that for each $n \in \mathbb{N}$ we have

$$Aa_n^2 \leq a_{n+1} \leq Ba_n^2.$$

By Lemma 5.3.3 considering $B = 2$ the inequality on the right hand side is satisfied.

For the other one, we want a constant A such that

$$a_{n+1} - Aa_n^2 \geq 0.$$

Using the expressions obtained for a_n for $n \geq 1$ this can be written in terms of $x_n, y_n, z_n, X_n, Y_n, Z_n$ and choosing $A = \frac{1}{4}$ we obtain that

$$a_{n+1} - Aa_n^2 = (x_n - z_n + X_n - Z_n)^2 \geq 0.$$

Finally, the approximation of γ may be calculated using the observation after Proposition 5.3.2. □

5.5 Portrait growth in non-symmetric GGS-groups

Let us now consider the GGS-groups defined by a non-symmetric defining vector.

Recall that by Lemma 1.3.7 we know that for

$$S = S^{-1} = \{a, a^2, \dots, a^{p-1}, b, \dots, b^{p-1}\},$$

G is contracting and that $\mathcal{N}(G) = S \cup \{1\}$.

We denote by

$$C(\mathbf{e}, 0) = \begin{pmatrix} e_1 & e_2 & \dots & e_{p-1} & 0 \\ 0 & e_1 & \dots & e_{p-2} & e_{p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_2 & \dots & e_{p-1} & 0 & e_1 \end{pmatrix}$$

the circulant matrix of the vector $(e_1, \dots, e_{p-1}, 0)$.

Proof of Theorem 5.1.2. By Lemma 1.3.6 we know that a GGS-group defined by a non-symmetric vector is regular branch over G'. Moreover, we know that G' of index p^2 in G, and a transversal of G' in G is given by

$$S = \{a^i b^j \mid i, j = 0, \dots, p-1\}.$$

For each pair $(i, j) \in \{0, \dots, p-1\}^2$ denote by $p_n(i, j)$ the number of portraits of depth no greater than n in G representing elements of the coset $a^i b^j G'$.

We have

$$a^i b^j \equiv a^i b^{n_0} (b^a)^{n_1} (b^{a^2})^{n_2} \dots (b^{a^{p-1}})^{n_{p-1}} \pmod{G'},$$

whenever $j = n_0 + \dots + n_{p-1}$ in $\mathbb{F}_p$. And then,

$$a^i b^j \equiv a^i (a^{i_1} b^{n_1}, \dots, a^{i_{p-1}} b^{n_{p-1}}, a^{i_0} b^{n_0}) \pmod{G' \times \dots \times G'},$$

where $(i_1, \ldots, i_{p-1}, i_0) = (n_0, \ldots, n_{p-1})C(\mathbf{e}, 0)$. We obtain that

$$p_{n+1}(i,j) = \sum_{n_0+\cdots+n_{p-1}=j} \prod_{r=1}^{p} p_n(i_r, n_r).$$

First of all, observe that the decomposition of $p_{n+1}(i,j)$ does not depend on i, so we can write

$$p_{n+1}(j) = \sum_{n_0+\cdots+n_{p-1}=j} \prod_{r=1}^{p} p_n(i_r, n_r), \tag{5.3}$$

and then for $n \geq 1$ we have $a_n = p\sum_{j=0}^{p-1} p_n(j)$, where we multiply by p because we have to sum for each $i = 0, \ldots, p-1$.

On the other hand, the initial conditions are the following

$$\begin{aligned} &p_0(0,0) = p_0(i,0) = p_0(0,j) = 1 \text{ for } i,j \in \{1,\ldots,p-1\}, \\ &p_0(i,j) = 0 \text{ otherwise.} \end{aligned}$$

Then, the previous formula gives that $p_1(0)$ is the number of solutions in $\mathbb{F}_p^p$ of

$$(n_0, \ldots, n_{p-1})C(\mathbf{e}, 0) \odot (n_1, n_2, \ldots, n_0) = (i_1 n_1, \ldots, i_0 n_0) = (0, \ldots, 0),$$

with $n_0 + \cdots + n_{p-1} = 0$, and that $p_1(j)$ is the number of solutions of the same equation but with $n_0 + \cdots + n_{p-1} = j$.

Now, let us prove by induction that $p_n(1) = p_n(j)$ for $n \geq 1$ and $j \neq 0$. Observe that for $n = 1$ if $(n_0, \ldots, n_{p-1})$ is a solution for $(n_0, \ldots, n_{p-1})C(\mathbf{e}, 0) = (0, \ldots, 0)$ with $n_0 + \cdots + n_{p-1} = 1$, then $(jn_0, \ldots, jn_{p-1})$ it is also a solution with $n_0 + \cdots + n_{p-1} = j$. And the other way around, if we start with a solution such that the sum is equal to j, multiplying by the inverse of j

in $\mathbb{F}_p$ we get a solution that sums up 1. Thus, there is a bijection between the soltutions and hence $p_1(1) = p_1(j)$ for $j \neq 0$. Let us now assume that $p_n(1) = p_n(j)$ for $n \geq 1$ and let us prove it for $n+1$. By (5.3) and since for $n \geq 1$ we know that $p_n(i,j) = p_n(j)$. Hence, we have

$$p_{n+1}(j) = \sum_{n_0+\cdots+n_{p-1}=j} \prod_{r=1}^{p} p_n(n_r).$$

Now, as before, since each tuple $(n_0, \ldots, n_{p-1})$ with $n_0 + \cdots + n_{p-1} = j$ can be obtained as a tuple summing up 1 and multiplied by j, we have

$$p_{n+1}(j) = \sum_{n_0+\cdots+n_{p-1}=1} \prod_{r=1}^{p} p_n(jn_r).$$

Finally, by inductive assumption, since $j \neq 0$, we have $p_n(jn_r) = p_n(n_r)$ for each $r = 0, \ldots, p-1$, and hence

$$p_{n+1}(j) = \sum_{n_0+\cdots+n_{p-1}=1} \prod_{r=1}^{p} p_n(n_r) = p_{n+1}(1).$$

Let us denote $x_n = p_n(0)$ and $y_n = p_n(1)$, so that $a_n = p(x_n + (p-1)y_n)$ for $n \geq 1$.

Observe that by (5.3) we have

$$x_{n+1} = \sum_{n_0+\cdots+n_{p-1}=0} \prod_{n_i=0} x_n \prod_{n_i \neq 0} y_n,$$
$$y_{n+1} = \sum_{n_0+\cdots+n_{p-1}=1} \prod_{n_i=0} x_n \prod_{n_i \neq 0} y_n$$

and then we get that

$$x_{n+1} = \sum_{l=0}^{p} x_n^{p-l} y_n^l \binom{p}{l} z_l$$
$$y_{n+1} = \sum_{l=0}^{p} x_n^{p-l} y_n^l \binom{p}{l} z'_l,$$

where z_l is the number of non-zero solutions of $n_1 + \cdots + n_l = 0$ and z'_l the number of non-zero solutions of $n_1 + \cdots + n_l = 1$, where by non-zero solution we mean that none of the n_i-s is zero for $i = 1, \ldots, l$.

For z_l and z'_l one has the relations

$$z_{l+1} = (p-1)z'_l$$
$$z'_{l+1} = z_l + (p-2)z'_l,$$

with initial conditions $z_1 = 0$ and $z'_1 = 1$. Solving this system we obtain that

$$z_l = \frac{1}{p}((p-1)^l - (-1)^{l-1}(p-1)),$$
$$z'_l = \frac{1}{p}((p-1)^l - (-1)^l),$$

which gives us

$$x_{n+1} = \frac{1}{p}(x_n + (p-1)y_n)^p + \frac{p-1}{p}(x_n - y_n)^p,$$
$$y_{n+1} = \frac{1}{p}(x_n + (p-1)y_n)^p - \frac{1}{p}(x_n - y_n)^p.$$

Finally, we get that

$$x_{n+1} + (p-1)y_{n+1} = (x_n + (p-1)y_n)^p,$$

and we conclude that

$$a_n = p(x_1 + (p-1)y_1)^{p^{n-1}}.$$

□

5.6 Portrait growth in the Apollonian group

As mentioned in the first chapter, the Apollonian group is a subgroup of the Hanoi Towers group and it was introduced by Grigorchuk, Nekrashevych and Sunic in [24]. Let us recall the definition.

Definition 5.6.1. *The Apollonian group A acting on the ternary tree is the group generated by the following three automorphisms*

$$x = (1, y, 1)(1\,2),$$
$$y = (x, 1, 1)(1\,3),$$
$$z = (1, 1, z)(2\,3).$$

As proved in Lemma 1.3.17, considering the generating set $S = \{x, y, z\}$, the group is contracting and $\mathcal{N}(A) = \{1, x, y, z, x^{-1}, y^{-1}, z^{-1}\}$.

Proof of Theorem 5.1.3. As mentioned in the first chapter when we introduced the Apollonian group, in [20] it is shown that the Hanoi Towers group is regular branch over its commutator, which is of index 8 in the group, and also that the index of the three copies of the commutator subgroup on the commutator subgroup is 12.

In Theorem 1.3.15 we have already seen that A has index 4 in the Hanoi Towers group. We also know that it is regular branch over E the subgroup

$(1,1,1)$	$=1$	$\equiv 1,$	$(1,y,1)(1\,2)$	$=x$	$\equiv x,$
$(x,y,1)(1\,3\,2)$	$=yx$	$\equiv 1,$	$(y,yx,1)(2\,3)$	$=xyx$	$\equiv x,$
$(x,yx,y)(1\,2\,3)$	$=(yx)^2$	$\equiv 1,$	$(yx,yx,y)(1\,3)$	$=x(yx)^2$	$\equiv x,$
$(y,y,1)$	$=x^2$	$\equiv 1,$	$(y,y^2,1)(1\,2)$	$=x^3$	$\equiv x,$
$(x,1,x)$	$=y^2$	$\equiv 1,$	$(1,yx,x)(1\,2)$	$=xy^2$	$\equiv x,$
$(1,z,z)$	$=z^2$	$\equiv 1,$	$(z,y,z)(1\,2)$	$=xz^2$	$\equiv x,$
$(yx,y^2,1)(1\,3\,2)$	$=x^2yx$	$\equiv 1,$	$(y^2,y^2x,1)(2\,3)$	$=x^3yx$	$\equiv x,$
$(x^2,y,x)(1\,3\,2)$	$=y^3x$	$\equiv 1,$	$(y,yx^2,x)(2,3)$	$=xy^3x$	$\equiv x,$
$(x,zy,z)(1\,3\,2)$	$=z^2yx$	$\equiv 1,$	$(zy,yx,z)(2\,3)$	$=xz^2yx$	$\equiv x,$
$(yx,y^2x,y)(1\,2\,3)$	$=x^2(yx)^2$	$\equiv 1,$	$(y^2x,y^2x,y)(1\,3)$	$=x^3(yx)^2$	$\equiv x,$
$(x^2,yx,xy)(1\,2\,3)$	$=y^2(yx)^2$	$\equiv 1,$	$(yx,yx^2,xy)(1\,3)$	$=xy^2(yx)^2$	$\equiv x,$
$(x,zyx,zy)(1\,2\,3)$	$=z^2(yx)^2$	$\equiv 1,$	$(zyx,yx,zy)(1\,3)$	$=xz^2(yx)^2$	$\equiv x,$

Table 5.1: The cosets of $E \times E \times E$ decomposing the cosets of E

of A of index 2, which is the image of the commutator of the Hanoi Towers group under an isomorphism given by a certain conjugation in $\operatorname{Aut} T$. Since E consist of the elements that are represented by words of even length over the alphabet $\{x,y,z\}$, a transversal for E in A is given by $T = \{1,x\}$. Let us denote by X_n and Y_n the number of portraits of depth at most n of the cosets $1E$ and xE respectively.

We still have that the index of $\psi^{-1}(E\times E\times E)$ in E is 12, and a transversal is given by

$$T' = \{1, yx, (yx)^2, x^2, y^2, z^2, x^2yx, y^3x, z^2yx, x^2(yx)^2, y^2(yx)^2, z^2(yx)^2\}.$$

Then Table 5.1 shows the equations for the representatives $\{1,x\}$ giving that

$$X_{n+1} = 3X_n^3 + 9X_nY_n^2,$$
$$Y_{n+1} = 3Y_n^3 + 9X_n^2Y_n.$$

And then we obtain that

$$a_{n+1} = X_{n+1} + Y_{n+1} = 3(X_n + Y_n)^3 = 3a_n^3.$$

Taking into account that $a_0 = 7$ one can check by induction that the result follows.

□

Laburpena euskeraz

Tesi honek zuhaitz errotu erregularren automorfismoen taldeen inguruko problema batzuk ebaztea du helburu. Talde hauek, Grigorchuk-en lehen taldea konkretuki, 80ko hamarkadan izan ziren lehenengoz definituak (ikus [26]), eta geroztik luze eta zabal ikertu da beraien inguruan. Talde hauek interesgarri izatearen arrazoi nagusia dauzkaten ezaugarri bitxiak direla esan genezake. Esate baterako, Grigorchuk-en lehen taldea Burnsideren problema orokorraren kontra adibide bat da, hau da, talde finituki sortua, periodikoa eta infinitua. Ordurako ezagunak ziren jadanik beste adibide konplexuago batzuk, baina talde hau definitzearen helburua hasiera batean Burnsideren problema orokorrarentzat adibide sinple bat ematea izan zen. Gerora, ordea, beste propietate berezi asko dauzkala frogatu ahal izan da. Horien artean garrantzitsuena, Milnor-en problemari ([32]) erantzuna eman ziona. Jakina zen ordurako bazirela taldeak hitzen hazkunde polinomiala eta exponentziala zeukatenak, baina erdibideko hazkundea zeukan talderik ba ote zen galdetu zuen Milnorrek. Erantzuna baiezkoa zen, eta Grigorchuk-en lehen taldea izan zen lehen adibidea ([21]). Geroztik hainbat problema ikertu dira talde hauekin erlazionatuta, eta hainbat orokorpen ezberdin eta talde berri definitu dira alor honetan.

Zuhaitz errotu erregularrak, honela eraikitzen dira: Izan bedi X multzo finitu bat, d elementu dauzkana. Orduan X^*, hau da, multzoa alfabeto bezala kontsideratuz osa daitezkeen hitz finituen multzoak osatzen du zuhaitzaren erpinen multzoa. Bi erpin $u, v \in X^*$ ertz batez lotuta egongo dira baldin eta $u = vx$ edo $v = ux$ bada $x \in X$ baterako. Honela eraikitako zuhaitza T bidez adieraziko dugu, eta d-adikoa dela esango dugu X-k d elementu baldin badauzka.

Izan bedi T zuhaitz d-adikoa. Zuhaitzaren automorfismo bat, erpinen arteko bijekzio bat da, zeinek ertzen loturak errespetatzen dituen. Automorfismo guztien multzoa $\operatorname{Aut} T$ bezala adieraziko dugu, eta konposaketarekiko talde bat osatzen du.

Tesiaren 1. kapituluan zuhaitzen automorfismoen inguruko definizio garrantzitsuenak ematen dira. Adibidez, $G \leq \operatorname{Aut} T$ *self-similar* edo *(weakly) branch* izatea zer den.

Ondoren, talde hauekin erlazionatutako problema ezberdinak azaltzen dira. Batetik, *congruence subgroup problem* bezala ezagutzen dena. Problema honek talde infinituen indize finituko azpitaldeak hobeto ezagutzea du helburu. Hasiera batean talde aljebraikoentzako planteatu zen, hain zuzen ere $\mathrm{SL}_n(\mathbb{Z})$ taldeentzako. Problemak, nolabait esateko, galdetzen du ea indizie finituko azpitalde guztiak ezagutzeko nahikoa den azpitalde finituen familia konkretu bat ezagutzea. Edo beste era batera esanda, ea indize finituko azpitalde guztiek eta familia konkretu horrek topologia bera definitzen duten taldean. Lehen aipatutako $\mathrm{SL}_n(\mathbb{Z})$ taldeen kasuan, familia hori $\{\ker(\pi_m : \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/m\mathbb{Z}))\}_{m \in \mathbb{N}}$ da, eta hortik datorkio 'congruence' izena. Beraz, bi topologiak berdinak diren kasuan, taldeak *congruence subgroup property* duela esaten da.

Zuhaitzen automorfismoen taldeentzat problemaren analogoa planteatzerako orduan, familia berezi bezala maila bakoitzeko estabilizatzaileak hartzen ditugu kontuan; hau da, $\mathrm{st}_G(n)$ da n luzerako erpinak finko uzten dituzten automorfismoek osatzen duten azpitaldea. Beraz, galdera litzateke ea $G \leq \operatorname{Aut} T$-ren indize finituko azpitaldeek eta estabilizatzaileek topologia bera definitzen duten G-n.

Planteatzen den bigarren problemak talde bateko elementuen deskribapenarekin du zerikusia. Zuhaitz d-adiko baten automorfismo bakoitza erpin bakoitzari permutazio bat esleituta deskriba daiteke, non permutazio horrek adierazten duen nola mugitzen dituen automorfismoak erpin horretatik zintzilik dauden d erpinak. Gainera, taldea *self-similar* den kasuetan, dekorazio hori puntu batean amai daiteke, permutazio baten ordez talde bereko automorfismo bat jarriz. Horrek adieraziko luke erpin horretatik behera elementuak egiten duen ekintza automorfismo horrek zuhaitz osoan egiten duen ekintzarekin deskriba daitekeela. Kontua da era honetan ez dagoela garbi noiz amaitu behar dugun dekorazioa eta noiz jarraitu. Horregatik, taldea *contracting* deritzona izatea garrantzizkoa da. Izan ere, kasu horretan elementu multzo finitu bat existitzen da, nukleo deritzona, non edozein elementuren dekorazioan puntu batetik aurrera nukleo horretako elementuetan erortzen garen. Orduan elementu bakoitza gisa horretan dekoratuta, hau da, nukleoko elementu batekin topo egiterakoan gelditu ezkero, elementuaren sakonerari (*depth*) buruz hitz egin dezakegu. Elementu baten sakonera litzateke elementuaren dekorazioan errotik hasita dagoen bide luzeenaren luzera. Behin elementu bakoitzari sakonera bat esleituta, galdera naturala da zein den sakoneraren hazkundea, *portrait growth* bezala ezagutzen dena. Galdera hau Grigorchuk-ek egin zuen [18] artikuluan Grigorchuk-en lehen taldeari buruz.

Lehen kapituluarekin bukatzeko tesian zehar agertuko diren talde ezberdinen definizioa ematen da: Grigorchuk-en lehen taldea, GGS-taldeen familia, Hanoi-ren dorreen taldea eta Apollonian taldea eta Basilica taldea. Talde bakoitzaren definizioaz gain ezagunak diren zenbait propietate garrantzitsu aipatzen dira, baita gerora beharrezkoak izango diren batzuk enuntziatu eta frogatu ere.

Bigarren kapituluan literaturan nahasmena sortu duen kontzeptu bat argitzen dugu. Zuhaitzen automorfismoen talde bat *fractal* dela esaten da baldin eta erpin bakoitzean talde osoaren ekintza berreskura badaiteke, nolabait esateko. Zenbait artikulutan esaten zen hori eta lehen mailako estabilizatzailearen erpin bakoitzeko proiekzioa supraiektiboa izatea baliokideak zirela. Egia da bigarrenak lehena inplikatzen duena, baina alderantzizkoa ez da egia. Beste zenbait artikulutan bereizketa egina zegoen eta gogorragoa den baldintza honi *strongly fractal* esaten zitzaion. Edozein kasutan, inon ez zen adibiderik ematen fractal izan eta strongly fractal ez zenarena. Guk bi adibide eraikitzen ditugu. Bestalde, talde bat fractal izan dadin, nahikoa da lehen mailako erpinetan fractal izateko baldintza batetzen badu. Artikulu batean esaten zen strongly fractal-ekin ere gauza bera gertatzen zela. Adibideak emanez ikusten dugu ez dela horrela, eta beraz hirugarren honi, hau da, lehen mailan bakarrik ez, maila denetan strongly fractal izateko baldintza betetzeari *super strongly fractal* izena eman diogu. Adibideak emanez erakusten dugu bi propietate hauek ere ez direla baliokideak. Emaitza hauek [37] artikuluan publikatuak izan dira.

Hirugarren kapituluan aurrerago aipatutako *congruence subgroup problem* aztertzen dugu GGS-taldeen familiarentzat. Talde hauek zuhaitz p-adikoaren automorfismoen taldeak dira p zenbaki lehen bakoitia izanik. Bi elementuk sortzen dituzte eta elementuetako bat bektore baten arabera definitzen da. Honela, bektore bakoitzak talde bat definitzen du. Aldez aurretik jakina zen ([33]), talde hauek periodikoak diren kasuan badaukatela congruence subgroup property. Hau da, indize finituko azpitaldeek eta estabilizatzaileek topologia bera definitzen dutela taldean. Kontua da, talde hauek periodikoak direla baldin eta soilik baldin definizio bektorearen osagaien batura zero bada $\mathbb{F}_p$-n (ikus [38]). Guk kasu guztietarako ematen dugu erantzuna. Hasteko, frogatzen dugu G taldea bektore ez-konstante batek definituriko GGS-taldea bada, orduan congruence subgroup property daukala.

Emaitza honi esker, Barnea-k egindako galdera bat erantzuteko gai izan gara (ikus [2]). Izan ere, galdetzen zuen ea existitzen ziren finituki sortuak, erresidualki finituak, ez periodikoak ziren talde infinituak zeintzuen konplezio profinitua pro-p taldea zen. GGS-taldeetako asko ez direnez periodikoak eta $\operatorname{Aut} T$-ren Sylow-en pro-p talde batean bizi direnez, Barnea-ren galderarako adibideak direla frogatzen du aurreko emaitzak. Gainera, Barnea-k bigarren galdera bat egiten du, ez periodiko beharrean tortsio-askeak (*torsion-free*) izateko eskatuz. Talde hauetako batzuk birtualki tortsio-askeak direla frogatzen dugu, eta beraz bigarren galderari ere erantzuna ematen diogu.

Bektore konstante bidez definituriko GGS-taldearen kasua ($\mathcal{G}$ bidez adierazi duguna tesi osoan zehar) erabat ezberdina da. Izan ere konplezio profinitutik estabilizatzaileekiko konpleziora dagoen epimorfismo naturalak isomorfismo izan behar luke congruence subgroup property izateko. Bestela esanda, epimorfismo horren nukleoa, *congruence kernel* deiturikoa, tribiala izan behar da. Guk frogatzen dugu $\mathcal{G}$-ren kasuan nukleo hau infinitua dela, eta beraz, ez daukala congruence subgroup property.

Hau gertatzearen arrazoia da, $\mathcal{G}$ taldeak indize finituko azpitalde bat duela zein $\mathbb{Z}$-ra proiektatzen den. Beraz, $\mathcal{G}$-n existitzen dira indizea p-ren berretura ez duten indize finituko azpitaldeak. Nola estabilizatzaile denen indizea p-ren berretura den, horrek zuzenean garamatza bi topologiak ezin daitezkeela berdinak izan ondorioztatzera. Beraz, galdera naturala da, indize finituko azpitalde guztiak hartu ordez, indizea p-ren berretura duten azpitalde normalak kontsideratuz gero, ea orduan bat datozen estabilizatzaileen topologia eta azken hau. Hori da hain zuzen ere laugarren kapituluaren motibazioa. Aurretik aipatutako emaitzak, [14] artikuluan bilduta daude.

Bukatzeko, kapitulu berean, GGS-taldeen orokorpena diren multi-GGS taldeentzako ere orokortzen ditugu aurreko bi emaitzak. Talde hauek, sortzaile berriak gehituz eraikitzen dira. Hala, sortzaile bakoitza bektore ezberdin batek definitzen du. Kasu honetan beraz, emaitza da $\mathcal{G}$ ez den edozein multi-GGS taldek baduela congruence subgroup property. Hemen aurki daiteke aipatutako emaitza: [17].

Laugarren kapituluan congruence subgroup problem-aren orokorpen bat planteatzen dugu. Aldez aurretik esan gisan, kasu batzuetan beste topologia batzuk egokiagoak izan daitezke estabilizatzaileenarekin konparatzerako garaian.

Izan bedi $\mathcal{C}$ talde finituen pseudo-barietate bat. Hau da, talde finituen multzo bat, itxia dena azpitaldeekiko, zatidurekiko eta biderketa kartesiar finituekiko. Orduan, oro har G talde infinitu bat izanik, $\mathcal{N}_\mathcal{C} = \{N \trianglelefteq G \mid G/N \in G\}$-k topologia bat definitzen du G-n, pro-$\mathcal{C}$ topologia deritzona. Baldin eta G zuhaitz errotu erregular baten automorfismoen taldea bada, eta $G/\operatorname{st}_G(n) \in \mathcal{C}$ betetzen bada $n \in \mathbb{N}$ guztietarako, orduan estabilizatzaileen topologia konparagarria da pro-$\mathcal{C}$ topologiarekin. Bi topologiak bat datozenean esango dugu G-k $\mathcal{C}$-congruence subgroup property daukala ($\mathcal{C}$-CSP laburtuta).

Kapitulu berean, problema definitzeaz gain, weakly regular branch diren zuhaitzen automorfismoen taldeentzako baldintza nahikoa den bat ematen dugu $\mathcal{C}$-CSP izan dezaten.

Baldintza hori baliatuz frogatzen dugu $\mathcal{G}$-k baduela p-CSP, eta baita Basilica taldeak 2-CSP duela ere bai. Nahiz eta gu p-talde finituen barietateekin baino ez garen aritzen, aipagarria da baldintzak orokorrean balio duela, eta beraz, interesgarria litzateke adibideak topatzea zeintzuentzat talde nilpotente finituen edo ebazgarri finituen pseudo-barietateentzako betetzen den propietatea, adibidez. Emaitza hauek [16] artikuluan daude jasota.

Bukatzeko, bostgarren eta azken kapituluan lehenago aipatutako portrait growth-aren problemaz arduratzen gara. Hasteko, regular branch diren taldeentzako bide bat ematen dugu ekuazio errekurtsibo batzuk eskuratu ahal izateko. Horrela, n sakonera duten elementuen kopurua kalkulatzeko gaitasuna izango dugu $n-1$ sakonera dutenen kopurua ezagututa.

Ondoren Grigorchuk-en lehen taldea, bektore ez-simetrikoek definitutako GGS-taldeak eta Apollonian taldearentzat kalkulatu egiten ditugu aipatutako ekuazioak. Hiru kasuetan, teknika ezberdinak erabiliz, gai gara frogatzeko hazkundea exponentzial bikoitza dela. Gure konjetura da gauza bera beteko dela edozein regular branch eta contracting den taldetan, baina momentuz ez gara frogatzeko gai izan. Emaitza hauek [35] artikuluan topa daitezke.

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