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An analytic superfield formalism for tree superamplitudes in D=10 and D=11

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Abstract: Tree amplitudes of 10D supersymmetric Yang-Mills theory (SYM) and 11D supergravity (SUGRA) are collected in multi-particle counterparts of analytic on-shell superfields. These have essentially the same form as their chiral 4D counterparts describing $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA, but with components dependent on a different set of bosonic variables. These are the D=10 and D=11 spinor helicity variables, the set of which includes the spinor frame variable (Lorentz harmonics) and a scalar density, and generalized homogeneous coordinates of the coset $\frac{SO(D-2)}{SO(D-4)\otimes U(1)}$ (internal harmonics).

We present an especially convenient parametrization of the spinor harmonics (Lorentz covariant gauge fixed with the use of an auxiliary gauge symmetry) and use this to find (a gauge fixed version of) the 3-point tree superamplitudes of 10D SYM and 11D SUGRA which generalize the 4 dimensional anti-MHV superamplitudes.

Keywords: Field Theories in Higher Dimensions, Scattering Amplitudes, Supergravity Models, Superspaces

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Contents

1 Introduction

An impressive recent progress in calculation of multi-loop amplitudes of d=4 supersymmetric Yang-Mills (SYM) and supergravity (SUGRA) theories, especially of their maximally supersymmetric versions $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA [1–5], was reached in its significant part with the use of spinor helicity formalism and of its superfield generalization [6, 7, 9–13]. This latter works with superamplitudes depending on additional fermionic variables and unifying a number of different amplitudes of the bosonic and fermionic fields from the SYM or SUGRA supermultiplet.

The spinor helicity formalism for D=10 SYM was developed by Caron-Huot and O'Connel in [14] and for D=11 supergravity in [15] (more details can be found in [16]). The progress in the latter was reached due to the observation that the 10D spinor helicity variables of [14] can be identified with spinor Lorentz harmonics or spinor moving frame variables used for the description of massless D=10 superparticles in [17–19]. (Similar observation was made and used in D=5 context in [20]). The spinor helicity formalism of [15] uses 11D spinor harmonics of [21–24].

As far as the generalization of D=4 superamplitudes is concerned, in [14] a kind of Clifford superfield representation of the amplitudes of 10D SYM was constructed. However, this later happened to be quite nonminimal and difficult to apply. Then the subsequent papers [25–28] used the D=10 spinor helicity formalism of [14] in the context of type II supergravity where the natural complex structure helped to avoid the use of the above mentioned Clifford superfields.[1] An alternative, constrained superfield formalism was proposed for 11D SUGRA amplitudes in [15]; its 10D SYM cousin will be briefly described here (see also [31] and [16] for details). In it the superamplitudes carry the indices of 'little groups' $\mathrm{SO}(D-2)_i$ of the light-like momenta $k_{a(i)}$ of i-th scattered particles and obey a set of differential equations involving fermionic covariant derivatives $D^+_{q(i)}$. This formalism is quite different from the 4D superamplitude approach; some efforts on development of the necessary technique and on deeper understanding of its structure are still required to be accomplished to make possible its efficient application to physically interesting problems.

In this paper we develop a simpler *analytic superfield formalism* for the description of 11D SUGRA and 10D SYM amplitudes. In it the superamplitudes are multiparticle counterparts of an on-shell analytic superfields, which depend on the fermionic variable in exactly the same manner as the chiral superfields describing $\mathcal{N}=8$ SUGRA and $\mathcal{N}=4$ SYM. However, the component fields in these analytic superfields depend on another set of bosonic variables including some internal harmonic variables (see [32–34]) $w_q^A, \bar{w}_{qA}$ parametrizing the coset $\frac{\mathrm{Spin}(D-2)}{\mathrm{Spin}(D-4)\otimes \mathrm{U}(1)}$. These are used to split the set of $(2\mathcal{N})$ real spinor fermionic coordinates θ^-_q of the natural on-shell superspaces of 11D SUGRA and 10D SYM on the set of $\mathcal{N}$ complex spinor coordinates η^-_A and its complex conjugate $\bar{\eta}^{-A}$. The *analytic* on-shell superfields describing 11D SUGRA and 10D SYM depend on η^-_A but not on $\bar{\eta}^{-A}$ and, in this sense, are similar to the *chiral* on-shell superfields describing $\mathcal{N}=8$ SUGRA and $\mathcal{N}=4$ SYM. However, as in higher dimensional case $\eta^-_A = \theta^-_q \bar{w}_{qA}$ is formed with the use of harmonic variable $\bar{w}_{qA}$, we call these superfields *analytic* rather than chiral.

We show how the analytic superamplitudes are constructed from the basic constrained superamplitudes of 10D SYM and 11D SUGRA and the set of complex $(D-2)$ component null-vectors $U_{I\,i}$ related to the internal frame associated to i-th scattered particle. We describe the properties of analytic superamplitudes and present a convenient parametrization of the spinor harmonics (gauge fixing with respect to a set of auxiliary symmetries acting on spinor frame variables), which allows to establish relations between D=10, 11 superamplitudes and their 4d counterparts. Using such relation we have found a gauge fixed expressions for the on-shell 3-point tree superamplitudes. These can be used as basic elements of the analytic superamplitude formalism based on a generalization of the BCFW recurrent relations [7]. The derivation and application of these latter, as well as the use of analytic superamplitudes to gain new insight for further development of the constrained superamplitude formalism will be the subject of future papers.

The rest of this paper has the following structure.

In the remaining part of the Introduction, after a resume of our notation, we briefly review $D=4$ spinor helicity and on-shell superfield description of $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA. In section 2 we describe the D=10 spinor helicity formalism. In section 3 we review briefly the on-shell superfield description of 10D SYM [21]. Analytic on-shell superfield

[1]An interesting recent analysis of the divergences of higher dimensional maximal SYM theory [29, 30] avoids an explicit use of the 10D spinor helicity formalism but assumes some generic properties of the amplitudes in this formalism.

JHEP05(2018)103

approach is developed in section 4. The spinor helicity formalism, constrained on-shell superfield and analytic on-shell superfield descriptions of D=11 SUGRA are presented in section 5. In section 6 we introduce the analytic D=10 and D=11 superamplitudes and describe their properties and their relation with constrained superamplitudes. A real supermomentum, which is supersymmetric invariant due to the momentum conservation, is introduced there.

A convenient parametrization of the spinor harmonics is described in section 7. Its study indicated the necessity to impose a relation between internal harmonics corresponding to different scattered particles, which then allowed to associate a complex spinor frames to each of them. In section 7 we also present a convenient gauge fixing of the auxiliary gauge symmetries which leads to a simple gauge fixed form of both real and complex spinor harmonics. This has been used to obtain gauge fixed expressions for 3-point analytic superamplitudes of 10D SYM and 11D SUGRA, which can be found in section 8. We conclude in section 9.

Appendix A is devoted to spinor frame re-formulation of 4D spinor helicity formalism, which is useful for comparison of 4D and 10/11D (super)amplitudes. Appendix B shows how to obtain the BCFW-like deformation of the 10/11D spinor helicity and complex fermionic variables from the deformation of real spinor frame and real fermionic variables found in [14, 15].

1.1 Notation

As we will use many different types of indices, for reader convenience we resume the index notation here.

The equations in D=10 and D=11 cases often have similar structure and we use similar notations in these two cases. To describe these in a universal manner and also to stress this similarity, it is convenient to introduce parameters $\mathcal{N}$ and s, which take values $\mathcal{N} = 4, 8$ and $s = 1, 2$ for the case 10D SYM and 11D SUGRA, respectively,

$$\text{10D SYM} : \qquad \mathcal{N} = 4\,, \qquad s = 1\,, \tag{1.1}$$

$$\text{11D SUGRA} : \qquad \mathcal{N} = 8\,, \qquad s = 2\,. \tag{1.2}$$

These characterize the number of supersymmetries and maximal spin of the quanta of the dimensionally reduced theories, $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA. Clearly, $s = \mathcal{N}/4$.

The symbols from the beginning of the Greek alphabet denote $Spin(1, D-1)$ indices (this is to say, indices of the minimal spinor representation of $\mathrm{SO}(1, D-1)$)

$$\alpha, \beta, \gamma, \delta = 1, \ldots, 4\mathcal{N}\,.$$

Notice that, when we consider D=4 SYM and SUGRA, we use the complex Weyl spinor indices $\alpha, \beta = 1, 2$ and $\dot{\alpha}, \dot{\beta} = 1, 2$ so that the above equations do not apply.

The spinor indices of the small group $\mathrm{SO}(D-2)$ (indices of $\mathrm{Spin}(D-2)$) are denoted by

$$q, p = 1, \ldots, 2\mathcal{N} \qquad \text{and} \qquad \dot{q}, \dot{p} = 1, \ldots, 2\mathcal{N}\,.$$

In the case of D=11 the dotted $Spin(9)$ indices are identical to undotted, $\dot{q} = q$, while for D=10 they are transformed by different (although equivalent) 8s and 8c representations of SO(8).

The vector indices of SO$(D-2)$ are denoted by

$$I, J, K, L = 1, \ldots, (D-2)$$

while

$$\check{I}, \check{J}, \check{K}, \check{L} = 1, \ldots, (D-4)$$

are vector indices of 'tiny' group SO$(D-4)$. Spinor indices of SO$(D-4)$ (Spin$(D-4)$ indices) are denoted by

$$A, B, C, D = 1, \ldots, \mathcal{N} .$$

The latter notation also applies to the 4D dimensional reduction of 11D and 10D theories, where A, B, C, D denote the indices of the fundamental representation of SU$(\mathcal{N})$ R-symmetry group.

Finally, $a, b, c, d = 0, 1, \ldots, (D-1)$ are D-vector indices. In D=4 we also use $\mu, \nu, \rho = 0, 1, 2, 3$ to stress the difference from $D = 10$ and $D = 11$.

The symbols $i, j = 1, \ldots, n$ are used to enumerate the scattered particles described by n-point (super)amplitude.

1.2 D=4 spinor helicity formalism

In spinor helicity formalism the scattering amplitudes of n massless particles $\mathcal{A}(1, \ldots, n) := \mathcal{A}(p_{(1)}, \varepsilon_{(1)}; \ldots, p_{(n)}, \varepsilon_{(n)})$ are considered to be homogeneous functions of n pairs of 2-component bosonic Weyl spinors $\lambda^{\alpha}_{(i)} = (\bar{\lambda}^{\dot{\alpha}}_{(i)})^*$ ($\alpha = 1, 2$; $\dot{\alpha} = 1, 2$),

$$\mathcal{A}(1, \ldots, n) := \mathcal{A}(p_{(1)}, \varepsilon_{(1)}; \ldots; p_{(n)}, \varepsilon_{(n)}) = \mathcal{A}(\lambda_{(1)}, \bar{\lambda}_{(1)}; \ldots; \lambda_{(n)}, \bar{\lambda}_{(n)}) . \tag{1.3}$$

The spinor $\lambda^{\alpha}_{(i)}$ carries the information about momentum and polarization of i-th particle. In particular, i-th light-like 4-momentum $p^{\mu}_{(i)}$ is determined in terms of $\lambda^{\alpha}_{(i)} = (\bar{\lambda}^{\dot{\alpha}}_{(i)})^*$ by Cartan-Penrose relation ($\alpha = 1, 2$, $\dot{\alpha} = 1, 2$, $\mu = 0, \ldots, 3$) [41, 42]

$$p_{_{A\dot{A}(i)}} := p_{\mu(i)} \sigma^{\mu}_{\alpha\dot{\alpha}} = 2\lambda_{\alpha(i)} \bar{\lambda}_{\dot{\alpha}(i)} \qquad \Leftrightarrow \qquad p_{\mu(i)} = \lambda_{(i)} \sigma_{\mu} \bar{\lambda}_{(i)} . \tag{1.4}$$

Here $\sigma^{\mu}_{\alpha\dot{\alpha}}$ are relativistic Pauli matrices obeying $\sigma^{\mu}{}_{\alpha\dot{\alpha}} \sigma_{\mu\beta\dot{\beta}} = 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}$ with $\epsilon_{\alpha\beta} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \epsilon_{\dot{\alpha}\dot{\beta}}$. This identity explains equivalence of two forms of the Cartan-Penrose representation (1.4) and also allows to show that $p_{a(i)} p^{a}_{(i)} = 0$.

The n-particle amplitude is restricted by n helicity constraints

$$\hat{h}_{(i)} \mathcal{A}(1, \ldots, n) = h_i \mathcal{A}(1, \ldots, n) , \tag{1.5}$$

where the operator

$$\hat{h}_{(i)} := \frac{1}{2} \left(\lambda^{\alpha}_{(i)} \frac{\partial}{\partial \lambda^{\alpha}_{(i)}} - \bar{\lambda}^{\dot{\alpha}}_{(i)} \frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}}_{(i)}} \right) \tag{1.6}$$

counts the difference between degrees of homogeneity in $\lambda^{\alpha}_{(i)}$ and $\bar{\lambda}^{\dot{\alpha}}_{(i)}$. Its eigenvalue h_i, the helicity of i-th particle, defines the amplitude homogeneity property with respect to the phase transformations of $\lambda^{\alpha}_{(i)}$ and $\bar{\lambda}^{\dot{\alpha}}_{(i)}$,

$$\mathcal{A}(\dots, e^{i\beta_i}\lambda^{\alpha}_{(i)}, e^{-i\beta_i}\bar{\lambda}^{\dot{\alpha}}_{(i)}, \dots) = e^{2ih_i\beta_i}\mathcal{A}(\dots, \lambda^{\alpha}_{(i)}, \bar{\lambda}^{\dot{\alpha}}_{(i)}, \dots)\,. \tag{1.7}$$

It is quantized: the amplitude is a well defined function of complex variable $\lambda^{\alpha}_{(i)}$ if and only if β_i is equivalent to $\beta_i + 2\pi$, and this happens when $2h_i \in \mathbb{Z}$. In the case of gluons $h_i = \pm 1$ and in the case of gravitons $h_i = \pm 2$.

1.3 D=4 superamplitudes and on-shell superfields

A superamplitude of $\mathcal{N} = 4$ SYM or $\mathcal{N} = 8$ supergravity depends, besides n sets of complex bosonic spinors, on n sets of complex fermionic variables $\eta^A_{(i)}$ $((\eta^A_{(i)})^* = \bar{\eta}_{A(i)})$ carrying the index of fundamental representation of the SU($\mathcal{N}$) R-symmetry group $A, B = 1, \dots, \mathcal{N}$,

$$\mathcal{A}(1; \dots; n) = \mathcal{A}(\lambda_{(1)}, \bar{\lambda}_{(1)}, \eta_{(1)}; \dots; \lambda_{(n)}, \bar{\lambda}_{(n)}, \eta_{(n)})\,, \qquad \eta^A_{(i)}\eta^B_{(j)} = -\eta^B_{(j)}\eta^A_{(i)}\,. \tag{1.8}$$

It obeys n super-helicity constraints,

$$\hat{h}_{(i)}\mathcal{A}(\{\lambda_{(i)}, \bar{\lambda}_{(i)}, \eta^A_{(i)}\}) = \frac{\mathcal{N}}{4}\mathcal{A}(\{\lambda_{(i)}, \bar{\lambda}_{(i)}, \eta^A_{(i)}\})\,, \qquad A = 1, \dots, \mathcal{N} \tag{1.9}$$

with

$$2\hat{h}_{(i)} = \lambda^{\alpha}_{(i)}\frac{\partial}{\partial\lambda^{\alpha}_{(i)}} - \bar{\lambda}^{\dot{\alpha}}_{(i)}\frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}}_{(i)}} + \eta^A_i\frac{\partial}{\partial\eta^A_i}\,. \tag{1.10}$$

It is important that the dependence of amplitude on fermionic variables is holomorphic: it depends on η^A_i but is independent of $\bar{\eta}_{A(i)} = (\eta^A_{(i)})^*$. Furthermore, according to (1.10), the degrees of homogeneity in these fermionic variables is related to the helicity h_i characterizing dependence on bosonic spinors. Hence, decomposition of superamplitude on the fermionic variables involves amplitudes of different helicities.

These superamplitudes can be regarded as multiparticle generalizations of the so-called on-shell superfields

$$\Phi(\lambda, \bar{\lambda}, \eta^A) = f^{(+s)} + \eta^A\chi_A + \frac{1}{2}\eta^B\eta^A s_{AB} + \dots + \eta^{\wedge(\mathcal{N}-1)}{}_A\bar{\chi}^A + \eta^{\wedge\mathcal{N}} f^{(-s)}\,, \tag{1.11}$$

$$\eta^{\wedge\mathcal{N}} = \frac{1}{\mathcal{N}!}\eta^{A_1}\dots\eta^{A_\mathcal{N}}\epsilon_{A_1\dots A_\mathcal{N}}\,, \quad \eta^{\wedge(\mathcal{N}-1)}{}_A = \frac{1}{(\mathcal{N}-1)!}\eta^{B_2}\dots\eta^{B_\mathcal{N}}\epsilon_{AB_2\dots B_\mathcal{N}}\,, \tag{1.12}$$

which obey the super-helicity constraint

$$\hat{h}\Phi(\lambda, \bar{\lambda}, \eta) = s\Phi(\lambda, \bar{\lambda}, \eta)\,, \quad s = \frac{\mathcal{N}}{4}\,, \quad 2\hat{h} = +\lambda^{\alpha}\frac{\partial}{\partial\lambda^{\alpha}} - \bar{\lambda}^{\dot{\alpha}}\frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}}} + \eta^A\frac{\partial}{\partial\eta^A}\,, \quad A = 1, \dots, \mathcal{N}. \tag{1.13}$$

The chiral superfields on a real superspace $\Sigma^{(4|2\mathcal{N})} = \{\lambda, \bar{\lambda}, \eta, \bar{\eta}\}$ obeying eq. (1.13) describe the on-shell states of $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA. They can be considered as homogeneous superfields on *chiral on-shell superspace*

$$\Sigma^{(4|\mathcal{N})} = \{\lambda, \bar{\lambda}, \eta\} \tag{1.14}$$

JHEP05(2018)103

satisfying eq. (1.13), which just fixes the charge of superfield with respect to a phase transformations of its arguments.[2]

Such on-shell superfields can be obtained by quantization of $D = 4$ Brink-Schwarz superparticle with $\mathcal{N}$-extended supersymmetry in its Ferber-Shirafuji formulation [45, 46] (see also [47, 48] as well as [49] and [50]). This observation has served us as an important guide: in [16] we show how to obtain the 10D and 11D on-shell superfield formalism from D=10 and D=11 superparticle quantization. Here we will not consider superparticle quantization but describe briefly the resulting constrained on-shell superfields and constrained superamplitude formalism of [15, 16] and use these as a basis to search for the analytic on-shell superfields and analytic superamplitude formalism.

To conclude our brief review, let us present the expressions for basic building blocks of the 4D superamplitude formalism, the 3-point superamplitudes of D=4 $\mathcal{N} = 4$ SYM theory. These are two: the anti-MHV ($\overline{\text{MHV}}$)

$$\mathcal{A}^{\overline{\text{MHV}}}(1,2,3) = \frac{1}{<12><23><31>}\,\delta^4\left(\eta_{A(1)} <23> + \eta_{A(2)} <31> + \eta_{A(3)} <12>\right) \tag{1.15}$$

and the MHV superamplitude

$$\mathcal{A}^{\text{MHV}}(1,2,3) = \frac{1}{[12]\,[23]\,[31]}\,\delta^8\left(\bar{\lambda}_{\dot{\alpha}1}\eta_{A1} + \bar{\lambda}_{\dot{\alpha}2}\eta_{A2} + \bar{\lambda}_{\dot{\alpha}3}\eta_{A3}\right). \tag{1.16}$$

Here we set the SYM coupling constant to unity and use the standard notation for the contraction of 4D Weyl spinors

$$\begin{aligned} <ij> &= <\lambda_i\lambda_j> = \lambda_i^{\alpha}\lambda_{\alpha j} = \epsilon^{\alpha\beta}\lambda_{\beta i}\lambda_{\alpha j}\,, \\ [\,ij\,] &= <ij>^* = [\bar{\lambda}_i\bar{\lambda}_j] = \bar{\lambda}_i^{\dot{\alpha}}\bar{\lambda}_{\dot{\alpha}j} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\lambda}_{\dot{\beta}i}\bar{\lambda}_{\dot{\alpha}j}\,. \end{aligned} \tag{1.17}$$

2 Spinor helicity formalism in D=10

As we have already mentioned in the Introduction, the D=10 spinor helicity formalism [14] can be constructed using the spinor (moving) frame or Lorentz harmonic variables. To describe these it is convenient to start with introducing the vector frame variables or vector harmonics (called light-cone harmonics in [51, 52]).

2.1 Vector harmonics

The property of vector harmonic variables are universal so that, instead of specifying ourselves to D=10 dimensional case, we write the equations of this section for arbitrary number D of spacetime dimensions. This will allow us to refer on these equations when considering spinor helicity formalism for 11D supergravity.

[2] The relative charges of bosonic and fermionic coordinates of this phase transformations can be restored from the relation between supertwistors and standard superspace coordinates [45]. In superamplitude context these relations can be found e.g. in [2].

Let us consider a vector frame

$$u_{ai}^{(b)} = \left(\frac{1}{2} \left(u_{a(i)}^{\#} + u_{ai}^{=} \right), u_{ai}^{I}, \frac{1}{2} \left(u_{ai}^{\#} - u_{ai}^{=} \right) \right) \quad \in \quad \mathrm{SO}(1, D-1)\,. \tag{2.1}$$

It can be associated with D-dimensional light-like momentum $k_{a(i)}$, $k_{a(i)}k^a_{(i)} = 0$, by the condition that one of the light-like vectors of the frame, say $u_{ai}^{=} = u_{ai}^{0} - u_{ai}^{(D-1)}$, is proportional to this $k_{a(i)}$,

$$k^a_{(i)} = \rho^{\#}_{(i)} u^{a=}_{(i)}\,. \tag{2.2}$$

The additional index i will enumerate particles scattered in the process described by an on-shell amplitude. Below in this section, to lighten the equations, we will omit this index when this does not lead to a confusion.

The condition (2.1) implies $u_a^{(c)} \eta^{ab} u_b^{(d)} = \eta^{(c)(d)}$, which can be split into [51, 52]

$$u_a^{=} u^{a=} = 0\,, \tag{2.3}$$

$$u_a^{\#} u^{a\#} = 0\,, \qquad u_a^{=} u^{a\#} = 2\,, \tag{2.4}$$

$$u_a^{I} u^{a=} = 0\,, \qquad u_a^{I} u^{a\#} = 0\,, \qquad u_a^{I} u^{aJ} = -\delta^{IJ}\,, \tag{2.5}$$

and also $u_a^{(c)} \eta_{(c)(d)} u_b^{(d)} = \eta_{ab}$, which can be written in the form of

$$\delta_a{}^b = \frac{1}{2} u_a^{=} u^{b\#} + \frac{1}{2} u_a^{\#} u^{b=} - u_a^{I} u^{bI}\,. \tag{2.6}$$

Notice that the sign indices $^{=}$ and $^{\#}$ of two light-like elements of the vector frame (see (2.3) and (2.4)) indicate their weights under the transformations of SO(1, 1) subgroup of the Lorentz group SO$(1, D-1)$,

$$u_a^{=} \mapsto e^{-2\alpha} u^{a=}\,, \qquad u_a^{\#} \mapsto e^{+2\alpha} u^{a\#}\,, \qquad u_a^{I} \mapsto u^{aI}\,. \tag{2.7}$$

It is convenient to change the basis and to consider the splitting of the vector frame matrix (2.1) on two light-like and $(D-2)$ orthogonal vectors in the form [51]

$$u_a^{(b)} = \left(u_a^{=}, u_a^{\#}, u_a^{I} \right)\,, \qquad u_c^{(a)} u^{c(b)} = \eta^{(a)(b)} = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -\delta^{IJ} \end{pmatrix}\,. \tag{2.8}$$

This is manifestly invariant under the direct product SO$(1,1) \otimes$ SO$(D-2)$ of the above scaling symmetry (2.7) and the rotation group SO$(D-2)$ mixing the spacelike vectors u_a^I,

$$SO(D-2)\ : \qquad u_a^{=} \mapsto u_a^{=}\,, \qquad u_a^{\#} \mapsto u_a^{\#}\,, \qquad u_a^{I} \mapsto u_a^{J} \mathcal{O}^{JI}\,, \qquad \mathcal{O}\mathcal{O}^T = I\,. \tag{2.9}$$

If only one light-like vector $u_a^{=}$ of the frame is relevant, as it will be the case in our discussion below, the transformations mixing $u_a^{\#}$ and u_a^I can be also considered as a symmetry. These are so-called $K_{(D-2)}$ transformations

$$\begin{aligned} K_{(D-2)}\ : \quad & u_a^{=} \mapsto u_a^{=}\,, \\ & u_a^{\#} \mapsto u_a^{\#} + u_a^{I}\, K^{\#I} + \frac{1}{4} u_a^{=} (K^{\#I} K^{\#I})\,, \quad u_a^{I} \mapsto u_a^{I} + \frac{1}{2} u_a^{=} K^{\#I} \end{aligned} \tag{2.10}$$

(identified in [17, 18] as conformal boosts of the conformal group of Euclidean space).

JHEP05(2018)103

To make the associated momentum (2.2) invariant under SO(1, 1) transformations (2.7), we have to require that

$$\rho^{\#} \mapsto \; e^{+2\alpha}\rho^{\#} \, , \tag{2.11}$$

and this explains the index $^{\#}$ of ρ multiplier in (2.2). Of course, we can use (2.11) to set $\rho^{\#} = 1$. However, it happens to be much more convenient to keep SO(1, 1) unfixed and to use it as identification relation (gauge symmetry acting on) vector harmonics (2.1).

The complete expression for light-like momentum (2.2) is invariant under $H_B = [\mathrm{SO}(1,1) \otimes \mathrm{SO}(D-2)] \subset\!\!\!\!\times K_{(D-2)}$ transformations (2.7), (2.9), (2.10). This is the Borel subgroup of $\mathrm{SO}(1, D-1)$ so that $\mathrm{SO}(1, D-1)/H_B$ coset is compact; actually it is isomorphic to the sphere $\mathbb{S}^{(D-2)}$. If we use H transformations as identification relation on the set of vector harmonics, these can be considered as a kind of homogeneous coordinates of such a sphere [17, 18]

$$\left\{ \left(u_a^{=}, u_a^{\#}, u_a^{I} \right) \right\} \; = \frac{\mathrm{SO}(1, D-1)}{[\mathrm{SO}(1,1) \otimes \mathrm{SO}(D-2)] \otimes K_{(D-2)}} = \mathbb{S}^{(D-2)} \, . \tag{2.12}$$

Such a treatment as constrained homogeneous coordinates of the coset makes the vector frame variable similar to the internal coordinate of harmonic superspaces introduced in [32, 33], and stays beyond the name *vector harmonics* or *vector Lorentz harmonics*, which we mainly use for them.

In the context of (2.2), $\mathbb{S}^{(D-2)}$ in (2.12) can be identified with the celestial sphere of a D-dimensional observer. Notice that this is in agreement with the fact that a light-like D-vector defined up to a scale factor can be considered as providing homogeneous coordinates for the $\mathbb{S}^{(D-2)}$ sphere

$$\{u_a^{=}\} = \mathbb{S}^{(D-2)} \, . \tag{2.13}$$

The usefulness of seemingly superficial construction with the complete frame (2.12) becomes clear when we consider spinor frame variables, which provide a kind of square roots of the light-like vectors of the Lorentz frame.

2.2 Spinor frame in D=10

To each vector frame $u_b^{(a)}$ we can associate a spinor frame described by $\mathrm{Spin}(1, D-1)$ valued matrix $V_\alpha^{(\beta)} \in \mathrm{Spin}(1, D-1)$ related to $u_b^{(a)}$ by the condition of the preservation of D-dimensional Dirac matrices

$$V\Gamma_b V^T = u_b^{(a)}\Gamma_{(a)} \, , \qquad V^T\tilde{\Gamma}^{(a)}V = \tilde{\Gamma}^b u_b^{(a)} \quad , \tag{2.14}$$

and also of the charge conjugation matrix if such exists in the minimal spinor representation of D-dimensional Lorentz group,

$$VCV^T = C \, , \qquad \text{if} \quad C \text{ exists for given } D \, . \tag{2.15}$$

In the case of D=10, where the minimal Majorana-Weyl (MW) spinor representation is 16-dimensional, the $\mathrm{SO}(1,1)\times \mathrm{SO}(8)$ invariant splitting of vector frame in (2.1) is reflected by splitting the spinor frame matrix on two rectangular blocks, $v^+_{\alpha\dot{q}}$ and $v^-_{\alpha q}$,

$$V_\alpha^{(\beta)} = \left(v^+_{\alpha\dot{q}},\, v^-_{\alpha q}\right) \in \mathrm{Spin}(1,D-1)\,, \tag{2.16}$$

which are called *spinor frame variables* or *Lorentz harmonic* (*spinor Lorentz harmonic*). Their sign indices $^\pm$ indicate their scaling properties with respect to the $\mathrm{SO}(1,1)$ transformations, and their columns are enumerated by indices of different, c-spinor and s-spinor representations of $\mathrm{SO}(8)$ group,

$$D = 10: \qquad \alpha = 1,\ldots,16\,, \quad \dot{q} = 1,\ldots,8\,, \quad q = 1,\ldots,8\,. \tag{2.17}$$

The set of constraints on 10D Lorentz harmonics are given by eqs. (2.14) in which $\Gamma^a_{\alpha\beta} = \sigma^a_{\alpha\beta} = \sigma^a_{\beta\alpha}$ and $\tilde{\Gamma}^{a\,\alpha\beta} = \tilde{\sigma}^{a\,\alpha\beta} = \tilde{\sigma}^{a\,\beta\alpha}$ are 16×16 generalized Pauli matrices, which obey $\sigma^a\tilde{\sigma}^b + \sigma^b\tilde{\sigma}^a = 2\eta^{ab}\mathbb{I}_{16\times 16}$. We prefer to write this relation in the universal form

$$\Gamma^a_{\alpha\gamma}\tilde{\Gamma}^{b\,\gamma\beta} + \Gamma^b_{\alpha\gamma}\tilde{\Gamma}^{a\,\gamma\beta} = 2\delta_\alpha{}^\beta\,, \tag{2.18}$$

which also describes the properties of symmetric 32×32 11D Dirac matrices introduced below (see section 5.2).

The charge conjugation matrix does not exist in 10D Majorana -Weyl spinor representation so that there is no way to rise or to lower the spinor indices. The elements of the inverse of the spinor frame matrix

$$V_{(\beta)}{}^\alpha = \begin{pmatrix} v_{\dot{q}}^{-\alpha} \\ v_q^{+\alpha} \end{pmatrix} \in \mathrm{Spin}(1,D-1) \tag{2.19}$$

are introduced as additional variables, which obey the constraints

$$V_\alpha^{(\beta)}V_{(\beta)}{}^\gamma := v^+_{\alpha\dot{q}}v_{\dot{q}}^{-\gamma} + v^-_{\alpha q}v_q^{+\gamma} = \delta_\alpha{}^\gamma \tag{2.20}$$

and

$$\begin{aligned} v_{\dot{q}}^{-\alpha}v^+_{\alpha\dot{p}} &= \delta_{\dot{q}\dot{p}}\,, \qquad & v_{\dot{q}}^{-\alpha}v^-_{\alpha q} &= 0\,, \\ v_q^{+\alpha}v^+_{\alpha\dot{p}} &= 0\,, \qquad & v_q^{+\alpha}v^-_{\alpha p} &= \delta_{qp}\,. \end{aligned} \tag{2.21}$$

For brevity, we will call $v_{\dot{q}}^{-\alpha}$ and $v_q^{+\alpha}$ *inverse harmonics.*

The constraints (2.14) can be split on the following set of $\mathrm{SO}(1,1)\otimes \mathrm{SO}(8)$ covariant relations

$$u_a^{=}\Gamma^a_{\alpha\beta} = 2v_{\alpha q}{}^-v_{\beta q}{}^-\,, \qquad u_a^{=}\delta_{qp} = v_q^-\tilde{\Gamma}_a v_p^-\,, \tag{2.22}$$

$$v_{\dot{q}}^+\tilde{\Gamma}_a v_{\dot{p}}^+ = u_a^{\#}\delta_{\dot{q}\dot{p}}\,, \qquad 2v_{\alpha\dot{q}}{}^+v_{\beta\dot{q}}{}^+ = \Gamma^a_{\alpha\beta}u_a^{\#}\,, \tag{2.23}$$

$$v_q^-\tilde{\Gamma}_a v_{\dot{p}}^+ = u_a^I\gamma^I_{q\dot{p}}\,, \qquad 2v_{(\alpha|q}{}^-\gamma^I_{q\dot{q}}v_{|\beta)\dot{q}}{}^+ = \Gamma^a_{\alpha\beta}u_a^I\,, \tag{2.24}$$

where $\gamma^I_{q\dot{p}} = \tilde{\gamma}^I_{\dot{p}q}$ with $I = 1, \ldots, 8$ are SO(8) Clebsh-Gordan coefficients obeying

$$\gamma^I \tilde{\gamma}^J + \gamma^J \tilde{\gamma}^I = \delta^{IJ} I_{8\times 8}\,, \qquad \tilde{\gamma}^I \gamma^J + \tilde{\gamma}^J \gamma^I = \delta^{IJ} I_{8\times 8}\,. \tag{2.25}$$

Although the constraints for the inverse harmonics (2.19)

$$u_a^{=} \tilde{\Gamma}^{a\,\alpha\beta} = 2 v_{\dot{q}}^{-\alpha} v_{\dot{q}}^{-\beta}\,, \qquad u_a^{=} \delta_{\dot{q}\dot{p}} = v_{\dot{q}}^{-} \Gamma_a v_{\dot{p}}^{-}\,, \tag{2.26}$$

$$v_q^{+} \Gamma_a v_p^{+} = u_a^{\#} \delta_{qp}\,, \qquad 2 v_q^{+\alpha} v_q^{+\beta} = \tilde{\Gamma}^{a\alpha\beta} u_a^{\#}\,, \tag{2.27}$$

$$v_{\dot{q}}^{-} \Gamma_a v_p^{+} = -u_a^I \gamma^I_{p\dot{q}}\,, \qquad 2 v_{\dot{q}}^{-(\alpha} \gamma^I_{q\dot{q}} v_q^{+\beta)} = -\tilde{\Gamma}^{a\alpha\beta} u_a^I\,, \tag{2.28}$$

can be obtained from (2.22)–(2.24) and (2.21), it is convenient to keep their form in mind.

The constraints (2.22) allow us to treat harmonic $v_{\alpha q}^{-}$ as a kind of square root of the light-like vector $u_a^{=}$ of the vector frame. Similar to this latter, $v_{\alpha q}^{-}$ can be also treated as a constrained homogeneous coordinates of the coset isomorphic to the celestial sphere

$$\{v_{\alpha q}^{-}\} \in \mathbb{S}^8\,. \tag{2.29}$$

Actually, eq. (2.29) abbreviates the spinorial counterparts of (2.12) and (2.13); the complete form of the first of these is

$$\{(v_{\alpha\dot{q}}^{+}, v_{\alpha q}^{-})\} = \frac{\mathrm{Spin}(1,9)}{[\mathrm{SO}(1,1)\otimes \mathrm{Spin}(8)] \,\subset\!\!\!\!\times K_8} = \mathbb{S}^8\,, \tag{2.30}$$

where K_{D-2} (K_8 in our 10D case) leaves $v_{\alpha q}^{-}$ invariant and acts on the complementary harmonics $v_{\alpha\dot{q}}^{+}$ by

$$K_{D-2}\ : \qquad v_{\alpha\dot{q}}^{+} \mapsto v_{\alpha\dot{q}}^{+} + \frac{1}{2} K^{\# I} v_{\alpha p}^{-} \gamma^I_{p\dot{q}}\,. \tag{2.31}$$

In a model with $[\mathrm{SO}(1,1)\otimes \mathrm{Spin}(D-2)] \,\subset\!\!\!\!\times K_{D-2}$ gauge symmetry $v_{\alpha\dot{q}}^{+}$ does not carry degrees of freedom: any $v_{\alpha\dot{q}}^{+}$ forming $\mathrm{Spin}(1, D-1)$ matrix with given $v_{\alpha q}^{-}$ can be obtained from some reference solution of this condition, $v_{\alpha\dot{q}0}^{+}$, by K_{D-2} transformations (2.31). This justifies the simplified form of (2.29) where only $v_{\alpha p}^{-}$ are presented as the constrained homogeneous coordinates of the sphere.

2.3 D=10 spinor helicity formalism

When the vector frame is attached to a light-like momentum as in (2.2),

$$k_a = \rho^{\#} u_a^{=}\,, \tag{2.32}$$

the constraints (2.22) for the associated spinor frame imply that the following D=10 counterparts of the D=4 Cartan-Penrose relations (1.4) hold:

$$k_a \Gamma^a_{\alpha\beta} = 2\rho^{\#} v_{\alpha q}^{-} v_{\beta q}^{-}\,, \qquad \rho^{\#} v_q^{-} \tilde{\Gamma}_a v_p^{-} = k_a \delta_{qp}. \tag{2.33}$$

In D=10 we should also mention the existence of the similar relations for the inverse harmonics (2.19),

$$k_a \tilde{\Gamma}^{a\,\alpha\beta} = 2\rho^{\#} v_{\dot{q}}^{-\alpha} v_{\dot{q}}^{-\beta}\,, \qquad \rho^{\#} v_{\dot{q}}^{-} \Gamma_a v_{\dot{p}}^{-} = k_a \delta_{\dot{q}\dot{p}}\,. \tag{2.34}$$

JHEP05(2018)103

Contracting the first equations in (2.33) and in (2.34) with $v_{\dot{q}}^{-\beta}$ and $v_{\alpha q}^{-}$, and using (2.21) we easily find that these obey the massless Dirac equations (or, better to say, $D=10$ Weyl equations)

$$k_a \Gamma^a_{\alpha\beta} v_{\dot{q}}^{-\beta} = 0\,, \qquad k_a \tilde{\Gamma}^{a\,\alpha\beta} v_{\beta q}^{-} = 0\,. \tag{2.35}$$

Thus, they can be identified, up to a scaling factor, with D=10 spinor helicity variables of [14]:

$$\lambda_{\alpha q} = \sqrt{\rho^{\#}} v_{\alpha q}^{-}. \tag{2.36}$$

The polarization spinor of the D=10 fermionic fields [14] can be associated with the inverse harmonics $v_{\dot{q}}^{-\alpha}$:

$$\lambda^{\alpha}_{\dot{q}} = \sqrt{\rho^{\#}} v_{\dot{q}}^{-\alpha}\,. \tag{2.37}$$

2.4 D=10 SYM multiplet in the Lorentz harmonic spinor helicity formalism

The polarization vector of the vector field can be identified with spacelike vectors u_a^I of the frame adapted to the light-like momentum of the particle by (2.32) (cf. [14]) so that the on-shell field strength of the D=10 gauge field can expressed by

$$D=10: \quad F_{ab} = k_{[a} u_{b]}{}^{I}\, w^I = \rho^{\#} u^{=}_{[a} u_{b]}{}^{I}\, w^I\,, \quad a=0,1,\ldots,9\,, \quad I=1,\ldots,8 \tag{2.38}$$

in terms of one SO(8) vector w^I. It is easy to check that both Bianchi identities and Maxwell equations in momentum representations are satisfied, $k_{[a}F_{bc]} = 0 = k_a F^{ab}$.

As we have already said, the polarization spinor can be identified with the spinor frame variable $v_{\dot{q}}^{-\alpha}$. Hence, in the linear approximation, the on-shell states of spinor superpartner of the gauge field can be described by

$$\chi^{\alpha} = v_{\dot{q}}^{-\alpha} \psi_{\dot{q}} \tag{2.39}$$

in terms of a fermionic SO(8) c-spinor $\psi_{\dot{q}}$. Indeed, due to (2.35), the field (2.39) solves the free Dirac equation.

When the formalism is applied to external particles of scattering amplitudes, the bosonic w^I and fermionic $\psi_{\dot{q}}$ are considered to be dependent on $\rho^{\#}$ and on spinors harmonics $v_{\alpha q}^{-}$ related to the momentum of the particle through (2.33),

$$w^I = w^I(\rho^{\#}, v_q^{-}) \qquad \text{and} \qquad \psi_q = \psi_q(\rho^{\#}, v_q^{-}) \tag{2.40}$$

When describing the on-shell states of the SYM multiplet, it is suggestive to replace $\rho^{\#}$ by its conjugate coordinate and consider the field on the nine-dimensional space $\mathbb{R} \otimes \mathbb{S}^8$:

$$w^I = w^I(x^{=}, v_q^{-}) \qquad \text{and} \qquad \psi_q = \psi_q(x^{=}, v_q^{-})\,. \tag{2.41}$$

The supersymmetry acts on these 9d fields by

$$\delta_\epsilon \psi_{\dot{q}}(x^{=}, v_q^{-}) = \epsilon^{-q} \gamma^I_{q\dot{q}}\, w^I(x^{=}, v_q^{-})\,, \qquad \delta_\epsilon w^I(x^{=}, v_q^{-}) = 2i\epsilon^{-q} \gamma^I_{q\dot{q}} \partial_{=} \psi_{\dot{q}}(x^{=}, v_q^{-})\,, \tag{2.42}$$

where 8 component fermionic ϵ^{-q} is the contraction of the constant fermionic spinor ϵ^{α} with the spinor frame variable,

$$\epsilon^{-q} = \epsilon^{\alpha} v_{\alpha q}^{-}\,. \tag{2.43}$$

JHEP05(2018)103

3 Constrained on-shell superfield description of 10D SYM

The above described fields of the spinor helicity formalism for 10D SYM can be collected in on-shell superfields, which can be considered as one-particle prototypes of tree superamplitudes. A constrained on-shell superfield formalism for linearized 10D SYM was proposed in [21]. We briefly describe that in this section and, in the next section 4, use it as a starting point to obtain a new analytic superfield description of 10D SYM.

3.1 On-shell superspace for 10D SYM

In [21] the constrained superfields describing 10D SYM are defined on the real *on-shell superspace* with bosonic coordinates $x^{=}$ and $v^{-}_{\alpha q}$, and fermionic coordinates θ^{-}_{q}

$$\begin{aligned} \Sigma^{(9|8)}: \qquad & \{(x^{=}, \theta^{-}_{q}, v^{-}_{\alpha q})\}\,, \qquad \{v^{-}_{\alpha q}\} = \mathbb{S}^{8}\,, \\ & q = 1, \ldots, 8\,, \qquad\qquad \alpha = 1, \ldots, 16\,. \end{aligned} \tag{3.1}$$

The 10D supersymmetry acts on the coordinates of $\Sigma^{(9|8)}$ by

$$\delta_{\epsilon} x^{=} = 2i\theta^{-}_{q}\, \epsilon^{\alpha} v^{-}_{\alpha q}\,, \qquad \delta_{\epsilon}\theta^{-}_{q} = \epsilon^{\alpha} v^{-}_{\alpha q}\,, \qquad \delta_{\epsilon} v^{-}_{\alpha q} = 0\,. \tag{3.2}$$

This specific form indicates that our on-shell superspace $\Sigma^{(9|8)}$ can be regarded as invariant subspace of the D=10 *Lorentz harmonic superspace*, i.e. of the direct product of standard 10D and 11D superspaces and of the internal sector parametrized by Lorentz harmonics $(v^{+}_{\alpha\dot{q}}, v^{-}_{\alpha q}) \in \mathrm{Spin}(1,9)$ considered as homogeneous coordinates of the coset $\frac{\mathrm{Spin}(1,9)}{\mathrm{Spin}(1,1)\otimes \mathrm{Spin}(8)}$.

The generic unconstrained superfield on $\Sigma^{(9|8)}$ (3.1) contains too many component fields so that on-shell superfield describing linearized D=10 SYM should obey some superfield equations. Such equations have been proposed in [21]. To write them in a compact form we will need the fermionic derivatives covariant under (3.2)

$$D^{+}_{q} = \partial^{+}_{q} + 2i\theta^{-}_{q}\partial_{=}\,, \qquad \partial_{=} := \frac{\partial}{\partial x^{=}}\,, \qquad \partial^{+}_{q} := \frac{\partial}{\partial \theta^{-}_{q}}\,, \qquad q = 1, \ldots, 8\,. \tag{3.3}$$

These carry the s-spinor indices of Spin(8) group and obey $d = 1$ $\mathfrak{N} = 8$ extended supersymmetry algebra

$$\{D^{+}_{q}, D^{+}_{p}\} = 4i\delta_{qp}\partial_{=}\,. \tag{3.4}$$

A one particle counterpart of a superamplitude is actually given by Fourier images of the superfield on (3.1) with respect to $x^{=}$. These will depend on the set of coordinates $(\rho^{\#}, \theta^{-}_{q}, v^{-}_{\alpha q})$, where $\rho^{\#}$ is a momentum conjugate to $x^{=}$. The fermionic covariant derivative acting on such Fourier-transformed on-shell superfields reads

$$D^{+}_{q} = \partial^{+}_{q} + 2\rho^{\#}\theta^{-}_{q}\,, \tag{3.5}$$

and obeys

$$\{D^{+}_{q}, D^{+}_{p}\} = 4\rho^{\#}\delta_{qp}\,. \tag{3.6}$$

3.2 On-shell superfields and superfield equations of 10D SYM

The basic superfield equations of D=10 SYM [21]

$$D = 10 \ : \qquad D_q^+ \Psi_{\dot q} = \gamma^I_{q\dot q} V^I \, , \qquad q = 1, \dots, 8 \, , \quad \dot q = 1, \dots, 8 \, , \quad I = 1, \dots, 8 \tag{3.7}$$

are imposed on the fermionic superfield $\Psi_{\dot q} = \Psi_{\dot q}(x^{=}, \theta^-_{\dot q}, v_{\alpha\dot q}{}^-)$ carrying c-spinor index of SO(8). The superfield V^I is defined by eq. (3.7) itself, which also imply that it obeys

$$D_q^+ V^I = 2i\gamma^I_{q\dot q} \partial_{=} \Psi_{\dot q} \, . \tag{3.8}$$

This equation shows that there are no other independent components in the constrained on-shell superfield $\Psi_{\dot q}$.

4 An analytic on-shell superfield description of 10D SYM

In this section we present an *analytic* superfield formalism for the on-shell D=10 SYM, which is alternative to both the Clifford superfield approach of [14] and to the constrained superfield formalism, which we have described above (more details can be found in [16]). We begin by solving the equations of the constrained on-shell superfields of 10D SYM from [21] in terms of one analytic on-shell superfield. In section 6 we generalize this for the case of superamplitudes and describe an analytic superamplitude formalism.

4.1 From constrained to unconstrained on-shell superfield formalism

To arrive at our unconstrained superfield formalism it is convenient to write the superspace equations (3.8) and (3.7) for on-shell superfields describing 10D SYM [21] in the form of

$$D_q^+ W^I = 2i\gamma^I_{q\dot q} \Psi_{\dot q} \, , \tag{4.1}$$
$$D_q^+ \Psi_{\dot q} = \gamma^I_{q\dot q} \, \partial_{=} W^I \, , \qquad q = 1, \dots, 8 \, , \quad \dot q = 1, \dots, 8 \, , \quad I = 1, \dots, 8 \, . \tag{4.2}$$

The superfield V^I in (3.7) and (3.8) is related to W^I by $V^I = \partial_{=} W^I$. After such a redefinition, we can discuss the bosonic superfield W^I as fundamental and state that $\Psi_{\dot q}$ is defined by the γ-trace part of (4.1). The first terms in its decomposition on fermionic coordinates are

$$W^I = w^I + 2i\theta^- \gamma^I \psi + i\theta^- \gamma^{IJ} \theta^- \partial_{=} w^I - \frac{2}{3} \theta^- \gamma^{IJ} \theta^- \, \theta^- \gamma^I \partial_{=} \psi + \dots \, .$$

We are going to show that, after breaking SO(8) symmetry down to its SO(6)=SU(4) subgroup, eq. (4.1) splits into a chirality condition for a single complex superfield ($\Phi = W^7 + iW^8$) and other parts which, together with (4.2), allow to determine $\Psi_{\dot q}$ and all the remaining components of W^I in terms of this single chiral superfield.

4.2 SU(4) invariant solution of the constrained superfield equations

Breaking $SO(8) \mapsto SO(6) \otimes SO(2) \approx SU(4) \otimes U(1)$, we can split the vector representation $\mathbf{8}_v$ of SO(8) on **6+1+1** of SO(6),

$$W^I = (W^{\check I}, W^7, W^8), \qquad \check I = 1, \dots, 6 \, . \tag{4.3}$$

Then introducing

$$\Phi = \frac{W^7 - iW^8}{2}, \qquad \bar{\Phi} = \frac{W^7 + iW^8}{2}, \qquad \Psi_q = \gamma^8_{q\dot{q}} \Psi_{\dot{q}}\,, \tag{4.4}$$

we find that (4.1) implies

$$\begin{aligned} D^+_q \Phi &= \left(\delta_{qp} + i(\gamma^7\tilde{\gamma}^8)_{qp}\right) \Psi_p\,, \\ D^+_q \bar{\Phi} &= -\left(\delta_{qp} - i(\gamma^7\tilde{\gamma}^8)_{qp}\right) \Psi_p. \end{aligned} \tag{4.5}$$

It is important to notice that the matrices

$$\mathcal{P}^{\pm}_{qp} = \frac{1}{2}\left(\delta_{qp} \pm i(\gamma^7\tilde{\gamma}^8)_{qp}\right)\,, \tag{4.6}$$

are orthogonal projectors

$$\mathcal{P}^+\mathcal{P}^+ = \mathcal{P}^+\,, \qquad \mathcal{P}^-\mathcal{P}^- = \mathcal{P}^-\,, \qquad \mathcal{P}^+\mathcal{P}^- = 0 \tag{4.7}$$

$$\mathcal{P}^+ + \mathcal{P}^- = \mathbb{I}\,, \qquad (\mathcal{P}^+)^* = \mathcal{P}^-\,, \tag{4.8}$$

and hance that (4.5) implies

$$\left(\delta_{qp} - i(\gamma^7\tilde{\gamma}^8)_{qp}\right) D^+_p \Phi = 0\,, \qquad \left(\delta_{qp} + i(\gamma^7\tilde{\gamma}^8)_{qp}\right) D^+_p \bar{\Phi} = 0. \tag{4.9}$$

As, according to (4.8), the projectors $\mathcal{P}^+$ and $\mathcal{P}^-$ are complementary and complex conjugate, we can introduce complex 8×4 matrix $w_q{}^A$ and its complex conjugate $\bar{w}_{qA}$ such that

$$\left(\delta_{qp} + i(\gamma^7\tilde{\gamma})^8_{qp}\right) = 2w_q{}^A\bar{w}_{pA}\,, \qquad \left(\delta_{qp} - i(\gamma^7\tilde{\gamma})^8_{qp}\right) = 2\bar{w}_{qA}w_p{}^A. \tag{4.10}$$

In terms of these rectangular blocks eqs. (4.9) can be written as chirality (analyticity) conditions

$$\bar{D}^+_A \Phi = 0\,, \qquad D^{+A}\bar{\Phi} = 0\,, \tag{4.11}$$

with

$$\bar{D}^+_A = \bar{w}_{pA} D^+_q\,, \qquad D^+_A = w_q{}^A D^+_q\,. \tag{4.12}$$

The remaining parts of eqs. (4.5) determine the fermionic superfield $\Psi_{\dot{q}}$,

$$\Psi_{\dot{q}} = w_{\dot{q}}{}^A\bar{\Psi}^{+A} + \bar{w}_{\dot{q}A}\Psi^+_A\,, \qquad \Psi^+_A = -\frac{i}{4}D^+_A\Phi\,, \qquad \bar{\Psi}^{+A} = -\frac{i}{4}\bar{D}^+_A\bar{\Phi}\,. \tag{4.13}$$

Eq. (4.2) allows us to find also the derivatives of the remaining 6 components $W^{\check{I}}$ of the SO(8) vector superfield W^I,

$$\partial_= W^{\check{I}} = \frac{1}{8}(\gamma^{\check{I}}\tilde{\gamma}^8)_{qp} D^+_q D^+_p (\Phi - \bar{\Phi})\,. \tag{4.14}$$

To conclude, we have solved the equations for constrained on shell superfields of 10D SYM [21] in terms of one chiral (analytic) on-shell superfield Φ and its c.c. $\bar{\Phi}$ (4.4).

4.3 The on-shell superfields are analytic rather than chiral

Our solution breaks explicitly the manifest $SO(D-2) = SO(8)$ 'little group' invariance of the constrained superfield formalism down to $SO(D-4) = SO(6)$ (called 'tiny group' in [26]). Actually, one can avoid this explicit $SO(8) \mapsto SO(6) \otimes SO(2) \approx SU(4) \otimes U(1)$ symmetry breaking by using the method of harmonic superspaces [32, 34]. To this end we must write the general solution of the constrained superfield equations in a formally SO(8) invariant form by introducing a 'bridge' coordinates parametrizing $SO(8)/[SU(4) \otimes U(1)]$ coset: the SO(8) valued matrix

$$U_I^{(J)} = \left(U_I{}^{\check{J}}, U_I{}^{(7)}, U_I{}^{(8)} \right) = \left(U_I{}^{\check{J}}, \frac{1}{2}\left(U_I + \bar{U}_I\right), \frac{1}{2i}\left(U_I - \bar{U}_I\right) \right) \in SO(8)\,. \tag{4.15}$$

This is transformed by multiplication on SO(8) matrix from the left and by multiplication by $SO(6) \times SO(2) \subset SO(8)$ matrix from the right. The conditions of orthogonality of the $U_I^{(J)}$ matrix (4.15), $U_I^{(J)} U_I^{(K)} = \delta^{(J)(K)}$, imply that the complex vector U_I is null and has the norm equal to 2,

$$U_I U_I = 0\,, \qquad \bar{U}_I \bar{U}_I = 0\,, \qquad U_I \bar{U}_I = 2\,, \tag{4.16}$$

as well as that it is orthogonal to six mutually orthogonal real vectors $U_I{}^{\check{I}}$

$$U_I U_I{}^{\check{J}} = 0\,, \qquad \bar{U}_I U_I{}^{\check{J}} = 0\,, \qquad U_I{}^{\check{J}} U_I{}^{\check{K}} = \delta^{\check{J}\check{K}}\,. \tag{4.17}$$

Now we can easily define SO(8) covariant counterparts of the projectors in (4.6)

$$\begin{aligned} \mathcal{P}^+_{qp} &= \frac{1}{2}\left(\delta_{qp} + i(\gamma^I \tilde{\gamma}^J)_{qp} U_I^{(7)} U_J^{(8)} \right) = \frac{1}{4}\gamma^I \tilde{\gamma}^J \bar{U}_I U_J\,, \\ \mathcal{P}^-_{qp} &= \frac{1}{2}\left(\delta_{qp} - i(\gamma^I \tilde{\gamma}^J)_{qp} U_I^{(7)} U_J^{(8)} \right) = \frac{1}{4}\gamma^I \tilde{\gamma}^J U_I \bar{U}_J\,. \end{aligned} \tag{4.18}$$

Furthermore, we can define the 8×8 SO(8) valued matrices $w_q^{(p)}$ and $w_{\dot{q}}^{(\dot{p})}$, which are related to (4.17) by

$$\gamma^I_{q\dot{p}} U_I^{(J)} = w_q^{(p)} \gamma^{(J)}_{(p)(\dot{q})} w_{\dot{p}}^{(\dot{q})}\,, \qquad w_{q'}^{(p)} w_{q'}^{(q)} = \delta^{(p)(q)}\,, \qquad w_{\dot{p}'}^{(\dot{q})} w_{\dot{p}'}^{(\dot{p})} = \delta^{(\dot{q})(\dot{p})}\,. \tag{4.19}$$

The elements of these real matrices can be combined in two rectangular 8×4 complex conjugate blocks

$$w_q^A = (\bar{w}_{qA})^*\,, \qquad w_{\dot{q}}^A = (\bar{w}_{\dot{q}A})^*\,, \qquad A = 1, 2, 3, 4\,. \tag{4.20}$$

These obey

$$w_q{}^A \bar{w}_{pA} + \bar{w}_{qA} w_p{}^A = \delta_{qp}\,, \tag{4.21}$$

$$\bar{w}_{qB} w_q{}^A = \delta_B{}^A\,, \qquad w_q{}^A w_q{}^B = 0\,, \qquad \bar{w}_{qA} \bar{w}_{qB} = 0\,, \tag{4.22}$$

and factorize the orthogonal projectors (4.18)

$$\mathcal{P}^+_{qp} = \frac{1}{4}\gamma^I \tilde{\gamma}^J \bar{U}_I U_J = w_q{}^A \bar{w}_{pA}\,, \qquad \mathcal{P}^-_{qp} = \frac{1}{4}\gamma^I \tilde{\gamma}^J U_I \bar{U}_J = \bar{w}_{qA} w_p{}^A \tag{4.23}$$

(cf. (4.10)).

With a suitable choice of representation of 8d Clebsch-Gordan coefficients $\gamma^I_{q\dot q} = \tilde\gamma^I_{\dot q q}$, the first equation in (4.19) can be split into

$$\varPsi^{\check J}_{q\dot p} := \gamma^I_{q\dot p} U^{\check J}_I = i w^A_q \sigma^{\check J}_{AB} w^B_{\dot p} + i\bar w_{qA} \tilde\sigma^{\check J AB} \bar w_{\dot p B}\,, \tag{4.24}$$

$$\varPsi_{q\dot p} := \gamma^I_{q\dot p} U_I = 2\bar w_{qA} w^A_{\dot p}\,, \qquad \bar\varPsi_{q\dot p} := \gamma^I_{q\dot p}\bar U_I = 2 w^A_q \bar w_{\dot p A}\,. \tag{4.25}$$

In (4.24)

$$\sigma^{\check I}_{AB} = -\sigma^{\check I}_{BA} = -(\tilde\sigma^{\check I AB})^* = \frac{1}{2}\epsilon_{ABCD}\tilde\sigma^{\check I CD}\,, \quad \check I = 1,\dots,6\,, \quad A,B,C,D = 1,\dots,4 \tag{4.26}$$

are 6d Clebsch-Gordan coefficients which obey

$$\sigma^{\check I}\tilde\sigma^{\check J} + \sigma^{\check J}\tilde\sigma^{\check I} = 2\delta^{\check I\check J}\delta_A{}^B\,, \quad \sigma^{\check I}_{AB}\tilde\sigma^{\check I CD} = -4\delta_{[A}{}^C\delta_{B]}{}^D\,, \quad \sigma^{\check I}_{AB}\,\sigma^{\check I}_{CD} = -2\epsilon_{ABCD}\,. \tag{4.27}$$

Using (4.25) and (4.22), it is not difficult to check that eqs. (4.23) are satisfied.[3]

The above bridge coordinates or *harmonic variables* [32–34] can be used to define the SO(8) invariant version of complex covariant derivatives (4.12), and of complex linear combinations of 8 bosonic superfields W^I

$$\Phi = W^I U_I\,, \qquad \bar\Phi = W^I \bar U_I\,, \tag{4.28}$$

which are analytic and anti-analytic, (4.11).

The expression for fermionic superfield $\Psi_{\dot q}$ can be written in the form of (4.13), but now with w and $\bar w$ factorizing the covariant projectors (4.23). It is also not difficult to write the covariant counterpart of the expression (4.14) for other 6 projections $W^{\check I} = W^J U^{\check I}_J$ of the 8-vector superfield W^I. However, a more straightforward expression for $W^{\check I} = W^J U^{\check I}_J$ in terms of Φ reads

$$W^{\check I} = -\bar{\mathbb D}^{\check I}\Phi\,, \tag{4.29}$$

where

$$\bar{\mathbb D}^{\check J} = \frac{1}{2}\bar U_I \frac{\partial}{\partial U^{\check J}_I} - U^{\check J}_I \frac{\partial}{\partial U_I} + \frac{i}{2}\sigma^{\check J}_{AB} w^B_q \frac{\partial}{\partial \bar w_{qA}} - \frac{i}{2}\tilde\sigma^{\check J AB}\bar w_{\dot q B}\frac{\partial}{\partial w^A_{\dot q}}\,, \tag{4.30}$$

is one of the covariant harmonic derivatives (first introduced in [32] and [33] for SU(2)/U(1) and SU(3)/[U(1)×U(1)] harmonic variables). In our case the other covariant derivatives are

$$\mathbb D^{\check J} = \frac{1}{2} U_I \frac{\partial}{\partial U^{\check J}_I} - U^{\check J}_I \frac{\partial}{\partial \bar U_I} + \frac{i}{2}\tilde\sigma^{\check J AB}\bar w_{qB}\frac{\partial}{\partial w^A_q} - \frac{i}{2}\sigma^{\check J}_{AB} w^B_{\dot q}\frac{\partial}{\partial \bar w_{\dot q A}}\,, \tag{4.31}$$

[3] In this calculation and below the following identity is useful

$$\gamma^I_{q\dot p} = \bar U_I \bar w_{qA} w^A_{\dot p} + U_I w^A_q \bar w_{\dot p A} + U^{\check J}_I\left(i w^A_q \sigma^{\check J}_{AB} w^B_{\dot p} + i\bar w_{qA}\tilde\sigma^{\check J AB}\bar w_{\dot p B}\right).$$

conjugate to (4.30), and

$$\mathbb{D}^{(0)} = U_I \frac{\partial}{\partial U_I} - \bar{U}_I \frac{\partial}{\partial \bar{U}_I} + \frac{1}{2}\left(\bar{w}_{qA}\frac{\partial}{\partial \bar{w}_{qA}} - w_q^A \frac{\partial}{\partial w_q^A}\right) + \frac{1}{2}\left(w_{\dot{q}}^A \frac{\partial}{\partial w_{\dot{q}A}^A} - \bar{w}_{\dot{q}A}\frac{\partial}{\partial \bar{w}_{\dot{q}A}}\right), \tag{4.32}$$

$$\begin{aligned}\mathbb{D}^{\check{I}\check{J}} = &\frac{1}{2}\left(U_K^{\check{I}}\frac{\partial}{\partial U_K^{\check{J}}} - U_K^{\check{J}}\frac{\partial}{\partial U_K^{\check{I}}}\right) + \frac{i}{2}\sigma^{\check{I}\check{J}}{}_B{}^A\left(w_q^B \frac{\partial}{\partial w_q^A} - \bar{w}_{qA}\frac{\partial}{\partial \bar{w}_{qB}}\right) + \\ &+ \frac{i}{2}\sigma^{\check{I}\check{J}}{}_B{}^A\left(w_{\dot{q}}^B \frac{\partial}{\partial w_{\dot{q}}^A} - \bar{w}_{\dot{q}A}\frac{\partial}{\partial \bar{w}_{\dot{q}B}}\right),\end{aligned} \tag{4.33}$$

providing the differential operator representation of the U(1) and Spin(6) = SU(4) generators on the space of internal harmonics. These covariant derivatives preserve all the constraints on harmonic variables, eqs. (4.21), (4.22), (4.24) and (4.25), and form the $so(8)$ algebra.

One can easily check that, by construction, our analytic superfield (4.28) obeys

$$\mathbb{D}^{\check{J}}\Phi = 0\,, \tag{4.34}$$

$$\mathbb{D}^{\check{I}\check{J}}\Phi = 0\,, \tag{4.35}$$

$$\mathbb{D}^{(0)}\Phi = \Phi\,. \tag{4.36}$$

These equations are consistent with the analyticity conditions (4.11) as

$$[\mathbb{D}^{\check{J}}, \bar{D}_A^+] = 0\,. \tag{4.37}$$

4.4 Analytic superfields and harmonic on-shell superspace

Thus, we have solved the superfield equations for constrained on-shell superfields of $D = 10$ SYM in term of one complex analytic superfield Φ obeying the chirality-type equation (4.11) with complex fermionic derivatives (4.12) defined with the use of $\frac{\mathrm{Spin}(8)}{\mathrm{Spin}(6)\otimes \mathrm{U}(1)} = \frac{\mathrm{SO}(8)}{\mathrm{SU}(4)\otimes \mathrm{U}(1)}$ coset coordinates (4.21), (4.22), which we, following [32–34], call harmonic variables or internal harmonics.

These analytic superfields are actually defined on a 'harmonic on-shell superspace' which can be understood as direct product of the on-shell superspace (3.1) and the $\frac{\mathrm{Spin}(D-2)}{\mathrm{Spin}(D-4)\otimes \mathrm{U}(1)}$ coset

$$\begin{aligned}&\Sigma^{(3(D-3)|2\mathcal{N})} = \{(x^{=}, v_{\alpha q}^{-}; \bar{w}_{qA}, w_q^A; \theta_q^-)\}\,, \\ &\{x^{=}\} = \mathbb{R}^1\,, \quad \{v_{\alpha q}^{-}\} = \mathbb{S}^{D-2}\,, \quad \{(\bar{w}_{qA}, w_q^A)\} = \frac{\mathrm{Spin}(D-2)}{\mathrm{Spin}(D-4)\otimes \mathrm{U}(1)}\,.\end{aligned} \tag{4.38}$$

Here and below in (4.42), to exclude the literal repetition of the same equations, we write them in the form applicable both for $D = 10$ and $D = 11$ cases, for which

$$q = 1, \ldots, 2\mathcal{N}\,, \qquad \alpha = 1, \ldots, 4\mathcal{N}\,, \qquad \mathcal{N} = \begin{cases} 4 & \text{for } D = 10\,, \\ 8 & \text{for } D = 11\,. \end{cases} \tag{4.39}$$

Supersymmetry acts on the coordinates of the harmonic on-shell superspace by (cf. (3.2))

$$\delta_\epsilon x^{=} = 2i\theta^-_q\, \epsilon^\alpha v^-_{\alpha q}\,, \quad \delta_\epsilon \theta^-_q = \epsilon^\alpha v^-_{\alpha q}\,, \quad \delta_\epsilon v^-_{\alpha q} = 0\,, \quad \delta_\epsilon \bar{w}_{qA} = 0 = \delta_\epsilon w_q{}^A\,, \tag{4.40}$$

and leaves invariant the covariant derivatives (3.3)

$$D^+_q = \partial^+_q + 2i\theta^-_q \partial_{=}\,, \qquad D_{=} = \partial_{=}\,, \tag{4.41}$$

as well as $\bar{D}^+_A = \bar{w}_{qA} D^+_q$ used to define analytic superfields Φ by $\bar{D}^+_A \Phi = 0$, (4.11).

To see that the analytic superfields are actually functions on a sub-superspace of (4.38), we have to pass to the analytic coordinate basis.

4.5 Analytical basis and analytic subsuperspace of the harmonic on-shell superspace

The presence of additional harmonic variables allows to change the coordinate basis of the harmonic on-shell superspace $\Sigma^{(3(D-3)|2\mathcal{N})}$ to the following *analytical basis*

$$\begin{aligned} \Sigma^{(3(D-3)|2\mathcal{N})} &= \{(x^{=}_L, v^-_{\alpha q}; \bar{w}_{qA}, w^A_q; \eta^-_A, \bar{\eta}^{-A})\}\,, \\ x^{=}_L &:= x^{=} + 2i\eta^-_A \bar{\eta}^{-A}\,, \qquad \eta^-_A := \theta^-_q \bar{w}_{qA}\,, \qquad \bar{\eta}^{-A} = \theta^-_q w^A_q\,. \end{aligned} \tag{4.42}$$

The supersymmetry acts on the coordinates of this basis by

$$\delta_\epsilon x^{=}_L = 4i\eta^-_A \bar{\epsilon}^{-A}\,, \qquad \delta_\epsilon \eta^-_A = \epsilon^-_A\,, \qquad \delta_\epsilon \bar{\eta}^{-A} = \bar{\epsilon}^{-A}\,, \tag{4.43}$$

where

$$\epsilon^-_A = \epsilon^\alpha v^-_{\alpha q} \bar{w}_{qA}\,, \qquad \bar{\epsilon}^{-A} = \epsilon^\alpha v^-_{\alpha q} w^A_q\,. \tag{4.44}$$

It is generated by the differential operators

$$\bar{Q}^+_A = \bar{\partial}^+_A + 4i\eta^-_A \partial^L_{=}\,, \qquad Q^{+A} = \partial^+_A \tag{4.45}$$

and leaves invariant the covariant derivatives[4]

$$\bar{D}^+_A = \bar{\partial}^+_A \equiv \frac{\partial}{\partial \bar{\eta}^{-A}}\,, \qquad D^{+A} = \partial^+_A + 4i\bar{\eta}^{-A}\partial^L_{=}\,, \qquad D_{=} = \partial^L_{=} \equiv \frac{\partial}{\partial x^{=}_L}\,. \tag{4.46}$$

The harmonic covariant derivatives in the analytical basis have the form

$$D^{\check{J}} = \mathbb{D}^{\check{J}} - \frac{i}{2}\eta^-_A \tilde{\sigma}^{\check{J}AB} \frac{\partial}{\partial \bar{\eta}^{-B}}\,, \tag{4.47}$$

$$\overline{D}^{\check{J}} = \overline{\mathbb{D}}^{\check{J}} - \frac{i}{2}\bar{\eta}^{-A} \sigma^{\check{J}}_{AB} \frac{\partial}{\partial \eta^-_B}\,, \tag{4.48}$$

$$D^{(0)} = \mathbb{D}^{(0)} + \frac{1}{2}\eta^-_A \frac{\partial}{\partial \eta^-_A} - \frac{1}{2}\bar{\eta}^{-A} \frac{\partial}{\partial \bar{\eta}^{-A}}\,, \tag{4.49}$$

$$D^{\check{I}\check{J}} = \mathbb{D}^{\check{I}\check{J}} + \frac{i}{2}\bar{\eta}^{-B} \sigma^{\check{I}\check{J}}{}_B{}^A \frac{\partial}{\partial \bar{\eta}^{-A}} - \frac{i}{2}\sigma^{\check{I}\check{J}}{}_B{}^A \eta^-_A \frac{\partial}{\partial \eta^-_B}\,, \tag{4.50}$$

where $\mathbb{D}^{\check{J}}$, $\overline{\mathbb{D}}^{\check{J}}$, $\mathbb{D}^{(0)}$ and $\mathbb{D}^{\check{I}\check{J}}$ formally coincides with (4.31), (4.30), (4.32) and (4.33).

[4] To be rigorous, one might want to write the L symbol also on the fermionic derivatives in (4.46), $\bar{\partial}^+_A \mapsto \bar{\partial}^+_{A\,L}$, $\partial^{+A} \mapsto \partial^{+A}_L$. We, however, prefer to make the formulae lighter and write this symbol on the bosonic derivative $\partial^L_{=}$ only.

It is not difficult to see that supersymmetry (4.43) leaves invariant *analytical on-shell superspace* $\Sigma_L^{(3(D-3)|\mathcal{N})}$, a sub-superspace of $\Sigma^{(3(D-3)|2\mathcal{N})}$ with coordinates

$$\Sigma_L^{(3(D-3)|\mathcal{N})} = \{(x_L^{=}, v_{\alpha q}^{-}; \bar{w}_{qA}, w_q^A; \eta_A^{-})\}\,. \tag{4.51}$$

The above defined analytic superfields are superfields on this analytic sub-superspace,

$$\Phi = \Phi(x_L^{=}, \eta_A^{-}; v_{\alpha q}^{-}; \bar{w}_{qA}, w_q^A) \qquad \Leftrightarrow \qquad \bar{D}_A^{+}\Phi = 0\,. \tag{4.52}$$

The supersymmetry transformation of the analytical superfields are defined by $\Phi'(x_L^{=\prime}, \eta_A^{-\prime}; \ldots) = \Phi(x_L^{=}, \eta_A^{-}; \ldots)$, or equivalently,

$$\begin{aligned}\Phi'(x_L^{=}, \eta_A^{-}; \ldots) &= e^{-(\bar{\epsilon}^{-A}\bar{Q}_A^{+} + \epsilon_A^{-}Q^{+A})}\Phi(x_L^{=}, \eta_A^{-}; \ldots) = \\ &= e^{-4i\bar{\epsilon}^{-A}\eta_A^{-}\partial_{=}^{L} - \epsilon_A^{-}\partial_L^{+A}}\Phi(x_L^{=}, \eta_A^{-}; \ldots)\,.\end{aligned} \tag{4.53}$$

For our discussion of the amplitudes, it will be useful to consider a Fourier image of an analytic superfield with respect to $x_L^{=}$,

$$\Phi(x_L^{=}, \eta_A^{-}; v_{\alpha q}^{-}; \bar{w}_{qA}, w_q^A) = \int d\rho^{\#} e^{-i\rho^{\#}x_L^{=}}\Phi(\rho^{\#}, \eta_A^{-}; v_{\alpha q}^{-}; \bar{w}_{qA}, w_q^A)\,. \tag{4.54}$$

The supersymmetry acts on this Fourier image as

$$\Phi'(\rho^{\#}, \eta_A^{-}; \ldots) = e^{-4\rho^{\#}\bar{\epsilon}^{-A}\eta_A^{-} - \epsilon_A^{-}\partial^{+A}}\Phi(\rho^{\#}, \eta_A^{-}; \ldots) \tag{4.55}$$

$$= \exp\{-4\rho^{\#}\bar{\epsilon}^{-A}\eta_A^{-}\}\;\Phi(\rho^{\#}, \eta_A^{-} - \epsilon_A^{-}; \ldots)\,. \tag{4.56}$$

The analytic on-shell superfield can be decomposed in series on complex fermionic variable,

$$\Phi(\rho^{\#}, v_{\alpha q}^{-}; w, \bar{w}; \eta_A) = \phi^{(+)} + \eta_A\psi^{+1/2\,A} + \frac{1}{2}\eta_B\eta_A\phi^{AB} + (\eta)^{\wedge 3\,A}\psi_A^{-1/2} + (\eta^{+})^{\wedge 4}\phi^{(-)}\,. \tag{4.57}$$

In this description 8 fermions $\psi_{\dot{q}}$ and 8 bosons w^I of the SO(8) covariant constrained superfield formalism are split into **4+4** and **1+6+1** representations of SO(6) $\approx$ SU(4)

$$\psi^{\alpha} \leftrightarrow (\Psi_{\dot{q}}) = (\psi^{+1/2\,A}, \psi_A^{-1/2})\,, \tag{4.58}$$

$$A_{\mu} \leftrightarrow (w^I) = (\phi^{(+)}, \phi^{AB}, \phi^{(-)})\,. \tag{4.59}$$

The sign and numerical superscripts of the fields describe their charge with respect to U(1) group acting on $\eta_A = \eta_A^{-}$. The origin of the analytic superfield in components of SO(8) vectors suggests that its charge is equal to +1, $\Phi = \Phi^{(+)}$.

This can be expressed by the differential equation

$$\hat{h}^{(10D)}\Phi(\rho^{\#},\, v_{\alpha q}^{-}; w, \bar{w}; \eta_A) = \Phi(\rho^{\#}, v_{\alpha q}^{-}; w, \bar{w}; \eta_A)\,, \tag{4.60}$$

where

$$\hat{h}^{(10D)} := \frac{1}{2}\bar{w}_{qA}\frac{\partial}{\partial\bar{w}_{qA}} - \frac{1}{2}w_q^A\frac{\partial}{\partial w_q^A} + \frac{1}{2}\eta_A\frac{\partial}{\partial\eta_A} \tag{4.61}$$

is the 10D counterpart of the superhelicity operator. It is easy to see that, when acting on an analytic superfield, $\hat{h}^{(10D)}$ coincides with covariant harmonic derivative $D^{(0)}$ (4.49),

$$\hat{h}^{(10D)} = D^{(0)}|_{\text{on analytic superfields}} \, , \tag{4.62}$$

so that eq. (4.60) for analytical Φ coincides with $D^{(0)}\Phi = \Phi$. Indeed, while the central basis form of this latter equation is given by (4.36) with $\mathbb{D}^{(0)}$ from (4.32), in the analytical basis we have to use the covariant derivative (4.49) so that our analytic superfield, being independent of $\bar{\eta}^{-A}$, obeys

$$(D^{(0)} - 1)\Phi(\rho^{\#},\, v_{\alpha q}^{-}; w, \bar{w}; \eta_A) = \left(\mathbb{D}^{(0)} + \frac{1}{2}\left(\eta_A^{-} \frac{\partial}{\partial \eta_A^{-}}\right) - 1 \right) \Phi(\rho^{\#},\, v_{\alpha q}^{-}; w, \bar{w}; \eta_A) = 0\, . \tag{4.63}$$

This is identical to (4.60) with $\hat{h}^{(10D)}$ given in (4.61).

Similarly, the analytic basis counterparts of eqs. (4.34) and (4.35), $D^{\check{J}}\Phi = 0$ and $D^{\check{I}\check{J}}\Phi = 0$, include the derivatives (4.47) and (4.50) and read

$$D^{\check{J}}\Phi(\rho^{\#},\, v_{\alpha q}^{-}; w, \bar{w}; \eta_A) = \mathbb{D}^{\check{J}}\Phi(\rho^{\#},\, v_{\alpha q}^{-}; w, \bar{w}; \eta_A) = 0\, , \tag{4.64}$$

$$\mathbb{D}^{\check{I}\check{J}}\Phi(\rho^{\#},\, v_{\alpha q}^{-}; w, \bar{w}; \eta_A) = \frac{i}{2}\sigma^{\check{I}\check{J}}{}_B{}^A \eta_A^{-} \frac{\partial}{\partial \eta_B^{-}}\, \Phi(\rho^{\#},\, v_{\alpha q}^{-}; w, \bar{w}; \eta_A)\, . \tag{4.65}$$

In both cases the analyticity (4.11) of the superfield Φ,

$$\bar{D}_A^{+}\Phi = \frac{\partial}{\partial \bar{\eta}^{-A}} \Phi(\rho^{\#},\, v_{\alpha q}^{-}; w, \bar{w}; \eta_A) = 0\, , \tag{4.66}$$

has been used.[5]

As we have already noticed, the spectrum of the component fields described by the analytical superfield (4.57) formally coincides with the fields of $\mathcal{N} = 4$ D=4 SYM. However, these fields depend on different set of bosonic variables: on 1+8+12=21

$$\{\rho^{\#}\} = \mathbb{R}^1_+\, , \qquad \{v_{\alpha q}^{-}\} \;= \mathbb{S}^8\, , \qquad \{(w_q{}^A, \bar{w}_{Aq})\} \;=\; \frac{\mathrm{SO}(8)}{\mathrm{SU}(4)\otimes \mathrm{U}(1)} \tag{4.67}$$

instead of 3=4-1 non-pure gauge components of $(\lambda, \bar{\lambda}) = \mathbb{CP}^2$ in 4D case.

[5] A reader with experience in harmonic superspace formulation of $\mathcal{N} = 2$ $D = 4$ supersymmetric matter and gauge theories might notice the similarity of eqs. (4.11) and (4.34) with basic equations of the hypermultiplet superfield q^+, which read $\bar{D}^+_{\dot{\alpha}} q^+ = 0 = D^+_\alpha q^+$ and $D^{++}q^+ = 0$. In central basis of $\mathcal{N} = 2$ harmonic superspace $\bar{D}^+_{\dot{\alpha}} = u^{+i}\bar{D}_{\dot{\alpha}i}$, $D^+_\alpha = u_i^+ D^i_\alpha$ and $D^{++} = u_i^+ \frac{\partial}{\partial u_i^-}$ where $D^i_\alpha = (\bar{D}_{\dot{\alpha}i})^*$ are standard $\mathcal{N} = 2$ fermionic covariant derivatives, $i, j = 1, 2$ and $\epsilon^{ij}u_i^+ u_j^- = 1$. It is well-known [32, 34] that the first of these equations (Grassmann analyticity conditions) are dynamical and the last is purely algebraic in this basis. However, after passing to an analytical basis the role of the equations interchange: the Grassmann analyticity conditions define a subclass of superfields, analytic superfields, while $D^{++}q^+ = 0$ becomes dynamical equation for the analytic superfield. One might wonder whether similar interchange effect occurs in our formalism. The answer is negative as far as our D=10 on-shell superspace description is oriented on collecting inside an analytic superfield the on-shell degrees of freedom of the SYM: we cannot distinguish algebraic and dynamical equations in this framework. Furthermore, as we will stress and discuss below, our internal harmonics are actually pure gauge variables.

The indices $A, B = 1, \ldots, 4$ of the fermionic coordinates and of some of the on-shell component fields of 10D SYM are transformed by SU(4). However, in distinction to the rigid SU(4) R-symmetry group of $\mathcal{N} = 4$ D=4 SYM, in ten dimensional theory SU(4) is a gauge symmetry: it is used as identification relation on the set of harmonic variables $w_q{}^A, \bar{w}_{Aq}$ (4.22) making them generalized homogeneous coordinates of the $\frac{\mathrm{SO}(8)}{\mathrm{SU}(4)\otimes \mathrm{U}(1)}$ coset.

4.6 On $(w, \bar{w})$-dependence of the analytic superfields. Complex spinor harmonics

The meaning of the dependence of analytic superfields (4.52) on the additional set of internal harmonic variables $(w, \bar{w})$ requires some clarification. As we have discussed in section 3, the 10D counterpart of the 4D helicity spinors $(\lambda, \bar{\lambda})$ is provided by the spinor harmonic variables $(v_{\dot{q}}^+, v_q^-)$. However, these are real while to define an analytic or chiral superfield, similar to the ones used in the on-shell superfield description of $\mathcal{N} = 4$ SYM in D=4, we need to have a complex structure. The role of internal harmonics $(w, \bar{w})$ is to introduce such a complex structure without breaking explicitly the Spin(8) gauge symmetry. This 'little group' symmetry, acting on the spinor harmonics $(v_{\dot{q}}^+, v_q^-)$ and used to identify them with the homogeneous coordinates of the celestial sphere $\mathbb{S}^8 = \frac{\mathrm{Spin}(1,9)}{[\mathrm{Spin}(6)\otimes\mathrm{Spin}(2)\otimes\mathrm{SO}(1,1)]\otimes\!\!\!\!\times K_8}$ (2.30), acts also on $(w, \bar{w})$. Moreover, $(w, \bar{w})$ are pure gauge with respect to this Spin(8) symmetry, so that the only invariant content encoded in them is the above mentioned complex structure.

One can formally fix the gauge with respect to SO(8) symmetry by setting $U_I = \delta_I^7 + i\delta_I^8$. The residual symmetry of this gauge, in which $(w, \bar{w})$ are determined by (4.10), is $\mathrm{SU}(4)\otimes \mathrm{U}(1)$, and the Spin(8) symmetry acting on the spinor harmonic reduces to this smaller subgroup. In this language, the analytic superfields depend on the spinor harmonics only, but these parametrize the coset

$$\frac{\mathrm{Spin}(1,9)}{[\mathrm{Spin}(6)\otimes\mathrm{Spin}(2)\otimes\mathrm{SO}(1,1)]\otimes\!\!\!\!\times K_8} = \frac{\mathrm{Spin}(1,9)}{[\mathrm{SU}(4)\otimes\mathrm{U}(1)\otimes\mathrm{SO}(1,1)]\otimes\!\!\!\!\times K_8} \tag{4.68}$$

instead of (2.30).

This is tantamount to saying that the analytic superfields (4.52) depend on the set of *complex spinor harmonics* composed of $(v_{\dot{q}}^+, v_q^-)$ and $(w, \bar{w})$ according to

$$v_{\alpha A}^- := v_{\alpha q}^- \bar{w}_{qA}\,, \quad \bar{v}_\alpha^{-A} := v_{\alpha p}^- w_p{}^A\,, \quad v_{\alpha A}^+ := v_{\alpha\dot{p}}^+ \bar{w}_{\dot{p}A}\,, \quad \bar{v}_\alpha^{+A} := v_{\alpha\dot{p}}^+ w_{\dot{p}}{}^A\,, \tag{4.69}$$

$$v_A^{-\alpha} := v_{\dot{q}}^{-\alpha} \bar{w}_{\dot{q}A}, \quad \bar{v}^{-A\alpha} := v_{\dot{q}}^{-\alpha} w_{\dot{q}}{}^A, \quad v_A^{+\alpha} := v_q^{+\alpha} \bar{w}_{qA}, \quad \bar{v}^{+A\alpha} := v_q^{+\alpha} w_q{}^A. \tag{4.70}$$

After taking into account the constraints and identification relations, one concludes that these parametrize the coset (4.68). Resuming, (4.52) can be equivalently written in the form

$$\Phi = \Phi(x_L^{=}, \eta_A^-; v_{\alpha A}^-, \bar{v}_\alpha^{-A}) \qquad \Leftrightarrow \qquad \bar{D}_A^+ \Phi = 0\,. \tag{4.71}$$

We, however, find more convenient at this stage to think about dependence of analytic superfields on real spinor harmonics, parametrizing the celestial sphere $\mathbb{S}^8$ $(= \frac{\mathrm{Spin}(1,9)}{[\mathrm{Spin}(8)\otimes\mathrm{SO}(1,1)]\otimes\!\!\!\!\times K_8})$, and on the set of internal harmonic variables parametrizing the coset $\frac{\mathrm{Spin}(8)}{\mathrm{SU}(4)\otimes\mathrm{U}(1)}$, in spite of these latter are pure gauge in our case.

4.6.1 Origin of internal harmonics

To clarify the passage from (4.52) to (4.71) and the auxiliary nature of internal harmonics, it is instructive to discuss how the analytic superfield Φ can be obtained in quantization of massless 10D superparticle. The reader not interested in this issue can pass directly to section 5.

More details on quantization in present notation can be found in [16] (see also [23] for 11D case). Here we begin from stating that in a Lorentz-analytic coordinate basis[6] the 10D (and 11D) massless superparticle has no fermionic first class constraints but only 8 (16) second class fermionic constraints d^+_q. They have the form $d^+_q = \pi^+_q + 2i\theta^-_q \rho^\# \approx 0$, where π^+_q is the momentum variable conjugate to fermionic coordinate function $\theta^-_q = \theta^-_q(\tau)$ and $\rho^\# = \rho^\#(\tau)$ is momentum conjugate to the bosonic coordinate function $x^= = x^=(\tau)$ depending on particle proper time variable τ. These d^+_q are the classical mechanics counterparts of the fermionic covariant derivatives in (3.3). Their second class nature is reflected by the Poisson bracket (P.B.) relations

$$\{d^+_q, d^+_p\}_{\text{P.B.}} = -4i\rho^\# \delta_{qp} \,. \tag{4.72}$$

The second class constraints can be resolved by passing to Dirac brackets (D.B.). Then the dynamical system has no fermionic constraints but the fermionic coordinate variable obey

$$\{\theta^-_q, \theta^-_p\}_{\text{D.B.}} = -\frac{i}{4\rho^\#}\delta_{qp} \,. \tag{4.73}$$

After quantization (in the momentum representation with respect to $x^=$), the algebra of fermionic operators $\hat{\theta}^-_q$ is

$$\{\hat{\theta}^-_q, \hat{\theta}^-_p\} = \frac{1}{4\rho^\#}\delta_{qp} \,, \tag{4.74}$$

and we have to find a representation of this Clifford-like algebra on the superparticle state vectors ('wavefunctions').

The appearance on this way of the constrained superfields (5.28) and of the Clifford superfield formalism by Caron-Huot and O'Connel is discussed in [16] (see also concluding section 9 for a brief discussion). To arrive at the analytic superfields formalism, we need to split 8 (16) Clifford-like variables $\hat{\theta}^-_q$ on the set of 4 complex fermionic coordinates and 4 momenta conjugate to these. Such an 'oscillator' (creation and annihilation operator) representation of Clifford algebra is well known, but it requires to introduce a complex structure, which breaks the SO(8) symmetry of the 8-dimensional Clifford algebra down to U(4).

A generic complex structure can be described by (complex linear combinations of the) columns of an SO(8) valued matrix, w^A_q and $\bar{w}_{qA}$ (4.20) obeying (4.22). They can be used to split $\hat{\theta}^-_q$ on the counterparts of creation and annihilation operators,

$$\hat{\eta}^-_A = \hat{\theta}^-_q \bar{w}_{qA} \,, \qquad \hat{\bar{\eta}}^{-A} = \hat{\theta}^-_q w^A_q \,, \qquad \{\hat{\eta}^-_A, \hat{\bar{\eta}}^{-B}\} = \frac{1}{4\rho^\#}\delta_A{}^B \,. \tag{4.75}$$

[6]The ten bosonic and 16 fermionic coordinates of the Lorentz-analytical coordinate basis of Lorentz harmonic 10D superspace are constructed from the standard superspace coordinates x^a, θ^α and Lorentz harmonics as $x^= := x^a u^=_a$, $x^\# := x^a u^\#_a$, $x^I := x^a u^I_a + i\theta^-_q \gamma^I_{q\dot{q}}\theta^+_{\dot{q}}$, $\theta^-_q = \theta^\alpha v^-_{\alpha q}$ and $\theta^+_{\dot{q}} = \theta^\alpha v^+_{\alpha\dot{q}}$ (see [16] and refs. therein).

Then, we can quantize superparticle in η-representation, in which $\hat{\eta}^-_A = \eta^-_A$ and $\hat{\bar{\eta}}^{-B} \propto \frac{\partial}{\partial \eta^- B}$, and the wavefunction depends on η^-_A. Such a wavefunction do not depend on the complex conjugate fermionic coordinate $\bar{\eta}^{-B}$ and, hence, is a counterpart of chiral superfield, an analytical superfield.

Actually in such a way, we arrive at the analytical superfield (4.57), which depend, besides the above mentioned η^-_A and bosonic $\rho^{\#}$ (or its conjugate coordinate $x^=$) also on the Lorentz harmonic variables $v^-_{\alpha q}$, parametrizing the celestial sphere of the 10D observer (2.29) and the above variables $w^A_q, \bar{w}_{qA}$.

Such a description corresponds to introducing $w^A_q(\tau), \bar{w}_{qA}(\tau)$ as additional variables of particle mechanics, so that the splitting of real $\theta^-_q(\tau)$ on $\eta^-_A(\tau)$ and $\bar{\eta}^{-A}(\tau)$, as in (4.75), can be performed already at the level of superparticle action. This is also in correspondence with the general ideology of harmonic superfield approach [32–34] (see [53–55] for quantization of superparticle in the standard $\mathcal{N} \geq 2$ D=4 harmonic superspace). However, in our case there is a peculiarity related to the fact that such internal harmonic coordinate functions in the superparticle action will be pure gauge with respect to SO(8) gauge symmetry. Furthermore, this is the same SO(8) gauge symmetry which was used as an identification relation on the set of Lorentz harmonic variables and allowed to treat them as homogeneous coordinates of the celestial sphere (2.29). Hence, on one hand, we can fix the SO(8) gauge by setting $w^A_q, \bar{w}_{qA}$ to some constant values thus breaking SO(8) gauge symmetry down to SU(4) × U(1). But on the other hand, after that we cannot use SO(8) gauge symmetry as an identification relation on the Lorentz harmonic variables. Then these latter cannot be considered as parametrizing the celestial sphere, but rather are homogeneous coordinates of a bigger coset (4.68). The number of additional (with respect to celestial sphere) dimensions of this coset coincides with the dimension of the coset SO(8)/[SU(4) × U(1)].

Such a gauge fixing leads us to the wavefunction (4.71) seemingly dependent on smaller number of harmonic variables. However, as we have just explained, the number of degrees of freedom in the Lorentz harmonic variables serving as an argument of the wavefunction (4.71) is the same as the sum of the number of degrees of freedom in Lorentz harmonic variables (2.29) and internal harmonics (4.67) the superfield (4.52) depend on.

As we have already said, we prefer the second description, in which SO(8) gauge symmetry is used as an identification relation on the set of Lorentz harmonics, which then parametrize the celestial sphere $\mathbb{S}^8$, and the internal harmonics describe the degrees of freedom of the coset SO(8)/[SU(4) × U(1)].

4.6.2 Comment on harmonic integration

If we were constructing the harmonic superspace actions for field theories in terms of our analytic superfields, then at some stage we would need to define and to use the integration over the internal harmonic variables $w^A_q, \bar{w}_{qA}$. In particular, the Lagrangians of such actions should be defined as an integral over SO(8)/[SU(4)×U(1)] coset. Such a problem, although interesting, goes beyond the scope of this paper where we use only on-shell superfields and their multiparticle generalizations, tree superamplitudes.

When working with the on-shell superfield description of free supermultiplets and tree amplitudes, we can always treat the analytic superfields/superamplitudes as encoding the constrained superfields/superemplitudes and their components, particle amplitudes, which are independent of internal harmonics. The internal harmonic variables enter such encoding relations in a linear manner: see (4.28) and its superamplitude generalization (6.13). However, the further development of the formalism, and particularly its generalization to loop amplitudes, might require to introduce and to use the integration over SO(8)/[SU(4)×U(1)] coset parametrized by internal harmonics.

5 Spinor helicity formalism and on-shell superfield descriptions of the linearized 11D SUGRA

5.1 Spinor helicity formalism in D=11

In this section we develop $D = 11$ spinor helicity formalism [15] on the basis of the spinor moving frame approach to 11D superparticle [23, 24]. This uses the Lorentz harmonics which can be considered as square roots of the vector frame variables, the 11D version of vector harmonics introduced in [51, 52]. The description of this latter basically coincide with that given in section 2.1 for their 10D cousin, but with setting D=11, $a, b, c = 0, 1 \ldots, 10$, and $I, J, K = 1, \ldots, 9$ in the appropriate places.

This is to say the vector frame is described by eqs. (2.1) or (2.8) with $D = 11$, and is "attached" to a light-like 11-momentum k_{ai} by eq. (2.2). The Lorentz harmonic variables forming the vector frame matrix are constrained by (2.3)–(2.6) and defined up to the transformations (2.7), (2.9), (2.10) which allow to identify them as homogeneous coordinates of the coset (2.12) with $D = 11$. This last equation is equivalent 11D version of (2.13) where in the l.h.s. the light-like vector $u_a^=$ is defined modulo its scaling transformations (as resulting from acting on it by SO(1,1) symmetry of the set of vector harmonics (2.12)).

5.2 Spinor frame and spinor helicity formalism in D=11

11D spinor harmonics are defined as rectangular blocks of Spin(1,10)-valued spinor frame matrix

$$V_\alpha^{(\beta)} = \left(v_{\alpha q}^{+}, \, v_{\alpha q}^{-} \right) \in \mathrm{Spin}(1,10)\,. \tag{5.1}$$

This is defined as a kind of square root of the vector frame matrix (2.12) by constraints

$$V\Gamma_b V^T = u_b^{(a)} \Gamma_{(a)}\,, \qquad V^T \tilde{\Gamma}^{(a)} V = \tilde{\Gamma}^b u_b^{(a)}\;, \tag{5.2}$$

$$VCV^T = C\,. \tag{5.3}$$

Here C is the 11D charge conjugation matrix, which is imaginary and antisymmetric $C_{\gamma\beta} = -C_{\beta\gamma} = -(C_{\gamma\beta})^*$. Γ_b and $\tilde{\Gamma}^b$ in (5.2) are real symmetric 32×32 matrices $\Gamma^a_{\alpha\beta} = \Gamma^a_{\beta\alpha} = \Gamma_\alpha^{a\,\gamma} C_{\gamma\beta}$ and $\tilde{\Gamma}^{a\,\alpha\beta} = \tilde{\Gamma}^{a\,\alpha\beta} = C^{\alpha\gamma}\Gamma_\gamma^{a\,\beta}$, obeying (2.18). They are constructed as products of 11D Dirac matrices $\Gamma_\alpha^{a\,\gamma} = -(\Gamma_\alpha^{a\,\gamma})^*$ obeying the Clifford algebra, $\Gamma^a\Gamma^b + \Gamma^b\Gamma^a = 2\eta^{ab}\mathbb{I}_{32\times 32}$, and of the above described charge conjugation matrix.

In (5.1) α, β, γ are indices of 32 dimensional Majorana spinor representation of SO(1,10) and q, p are spinor indices of SO(9),

$$D = 11: \qquad \alpha, \beta, \gamma = 1, \ldots, 32 \quad \text{and} \quad q, p = 1, \ldots, 16\,. \tag{5.4}$$

JHEP05(2018)103

Notice that, in distinction with D=10, both spinor harmonics in 11D spinor frame matrix (5.1) carry Spin(9) indices of the same type. Furthermore, the existence of charge conjugation matrix allows to construct the elements of inverse spinor frame matrix obeying

$$\begin{aligned} v_q^{-\alpha} v_{\alpha p}^{+} &= \delta_{qp}\,, \qquad & v_q^{-\alpha} v_{\alpha q}^{-} &= 0\,, \\ v_q^{+\alpha} v_{\alpha p}^{+} &= 0\,, \qquad & v_q^{+\alpha} v_{\alpha p}^{-} &= \delta_{qp} \end{aligned} \tag{5.5}$$

in terms of the same spinor harmonics:

$$D = 11 \ : \qquad v_q^{+\alpha} = iC^{\alpha\beta} v_{\beta q}^{+}\,, \qquad v_q^{-\alpha} = -iC^{\alpha\beta} v_{\beta q}^{-}\,. \tag{5.6}$$

The constraints (5.2) can be split on the following set of $\mathrm{SO}(1,1)\otimes\mathrm{SO}(9)$ covariant relations

$$u_a^{=}\Gamma^a_{\alpha\beta} = 2v_{\alpha q}{}^{-} v_{\beta q}{}^{-}\,, \qquad\qquad u_a^{=}\delta_{qp} = v_q^{-}\tilde{\Gamma}_a v_p^{-}\,, \tag{5.7}$$

$$v_q^{+}\tilde{\Gamma}_a v_p^{+} = u_a^{\#}\delta_{qp}\,, \qquad\qquad 2v_{\alpha q}{}^{+} v_{\beta q}{}^{+} = \Gamma^a_{\alpha\beta} u_a^{\#}\,, \tag{5.8}$$

$$v_q^{-}\tilde{\Gamma}_a v_p^{+} = u_a^{I}\gamma^I_{qp}\,, \qquad\qquad 2v_{(\alpha|q}{}^{-}\gamma^I_{qq} v_{|\beta)q}{}^{+} = \Gamma^a_{\alpha\beta} u_a^{I}\,, \tag{5.9}$$

where $\gamma^I_{qp} = \gamma^I_{pq}$ are nine dimensional 16×16 Dirac matrices, $I = 1,\ldots,9$. In the Majorana spinor representation of SO(9) the charge conjugation matrix is symmetric and we identify it with δ_{qp}.

The K_{D-2} transformations of Lorentz harmonics in D=11 are given by

$$K_9 \ : \qquad v_{\alpha q}^{+} \mapsto v_{\alpha q}^{+} + \frac{1}{2}K^{\#I} v_{\alpha p}^{-}\gamma^I_{pq}\,, \qquad v_{\alpha q}^{-} \mapsto v_{\alpha q}^{-}\,. \tag{5.10}$$

When $[\mathrm{SO}(1,1)\otimes\mathrm{Spin}(9)] \subset\!\!\!\!\times K_9$ can be used as an identification relation, i.e. in the model which possesses gauge symmetry under these transformations, the spinor harmonics can be considered as the coordinates of the coset of the Lorentz group isomorphic to $\mathbb{S}^9$ sphere,

$$\{v_{\alpha \dot q}^{+}, v_{\alpha q}^{-}\} \ \in \ \frac{\mathrm{Spin}(1,10)}{[\mathrm{SO}(1,1)\otimes\mathrm{Spin}(9)] \subset\!\!\!\!\times K_9} = \mathbb{S}^9\,. \tag{5.11}$$

The fact that the vector frame is adapted to the light-like 11-momentum k_a by eq. (2.2) imply

$$k_a\Gamma^a_{\alpha\beta} = 2\rho^{\#} v_{\alpha q}^{-} v_{\beta q}^{-}\,, \qquad \rho^{\#} v_q^{-}\tilde{\Gamma}_a v_p^{-} = k_a\delta_{qp}\,, \tag{5.12}$$

which can be equivalently written as

$$k_a\tilde{\Gamma}^{a\,\alpha\beta} = 2\rho^{\#} v_q^{-\alpha} v_q^{-\beta}\,, \qquad \rho^{\#} v_q^{-}\Gamma_a v_p^{-} = k_a\delta_{qp}\,. \tag{5.13}$$

These relations imply that the spinor harmonics $v_{\alpha q}^{-}$ obey the Dirac equation

$$k_a\tilde{\Gamma}^{a\,\alpha\beta} v_{\beta q}^{-} = 0 \qquad \Leftrightarrow \qquad k_a\Gamma^a_{\alpha\beta} v_q^{-\beta} = 0 \tag{5.14}$$

and hence define the helicity spinor

$$\lambda_{\alpha q} = \sqrt{\rho^{\#}}\, v_{\alpha q}^{-}\,. \tag{5.15}$$

The polarization spinor in D=11 can be obtained from helicity spinor with the use of charge conjugation matrix,

$$D = 11 \ : \qquad \lambda_q^{\alpha} = \sqrt{\rho^{\#}}\, v_q^{-\alpha} = iC^{\alpha\beta}\lambda_{\beta q}\,. \tag{5.16}$$

5.3 Linearized D=11 SUGRA in the Lorentz harmonic spinor helicity formalism

The linearized on-shell field strength of 3-form gauge field of 11D SUGRA (called 'formon' in [56]) can be expressed by

$$F_{abcd} = k_{[a}u_b{}^I u_c{}^J u_{d]}{}^K A_{IJK} \tag{5.17}$$

in terms of light-like momentum (2.32), spacelike vectors $u_b{}^I$ of the frame adapted to the momentum by (2.32), and an antisymmetric SO(9) tensor $A_{IJK} = A_{[IJK]}$ (in **84** of SO(9)). The linearized on-shell expression for the Riemann tensor reads

$$R_{ab}{}^{cd} = k_{[a}u_{b]}{}^I k^{[c}u^{d]J} h_{IJ}\,, \tag{5.18}$$

where the second rank SO(9) tensor h_{IJ} is symmetric and traceless (in **44** of SO(9))

$$h_{IJ} = h_{JI}\,, \qquad h_{II} = 0\,. \tag{5.19}$$

We express these properties by writing $h_{IJ} = h_{((IJ))}$. Finally, the gravitino field strength solving the Rarita-Schwinger equation is expressed in terms of γ-traceless SO(9) vector-spinor Ψ_{Iq} (**128** of SO(9)) by

$$\mathcal{T}_{ab}{}^{\alpha} = k_{[a}u^I_{b]}v_q^{-\alpha}\Psi_{Iq} = \rho^{\#}u^{=}_{[a}u^I_{b]}v_q^{-\alpha}\Psi_{Iq}\,, \qquad \gamma^I_{qp}\Psi_{Ip} = 0\,. \tag{5.20}$$

The set of on-shell fields h_{IJ}, A_{IJK}, Ψ_{Ip} can be used to describe the supergravity multiplet in light-cone gauge [57]. In our spinor helicity/spinor frame description, which can be deduced from the on-shell superfield formalism of [21], these fields depend on the density $\rho^{\#}$ and spinor harmonics $v^-_{\alpha q}$ (homogeneous coordinates of $\mathbb{S}^9$ (2.29)) related to the momentum by (2.33),

$$A_{IJK} = A_{[IJK]}(\rho^{\#}, v_q^-)\,, \qquad h_{IJ} = h_{((IJ))}(\rho^{\#}, v_q^-)\,, \qquad \Psi_{Iq} = \Psi_{Iq}(\rho^{\#}, v_q^-)\,. \tag{5.21}$$

In the next section we will use the (superfield generalization) of the Fourier images of the above fields defined on $\mathbb{R}\otimes\mathbb{S}^9$ space,

$$A_{IJK} = A_{[IJK]}(x^{=}, v_q^-)\,, \qquad h_{IJ} = h_{((IJ))}(x^{=}, v_q^-)\,, \qquad \Psi_{Iq} = \Psi_{Iq}(x^{=}, v_q^-)\,. \tag{5.22}$$

5.4 Constrained on-shell superfield description of 11D SUGRA

A constrained on-shell superfield formalism for linearized 11D SUGRA was proposed in [21] and was generalized for the case of superamplitudes in [15] (see [16] for details).

The constrained on-shell superfields are functions on the real on-shell superspace

$$\begin{aligned} \Sigma^{(10|16)}: \qquad & \{(x^{=}, \theta^-_q, v^-_{\alpha q})\}\,, \qquad \{v^-_{\alpha q}\} = \mathbb{S}^9\,, \\ & q = 1,\ldots,16\,, \qquad\qquad \alpha = 1,\ldots,32\,, \end{aligned} \tag{5.23}$$

where the 11D supersymmetry acts as follows (cf. (3.2))

$$\delta_\epsilon x^{=} = 2i\theta^-_q\,\epsilon^\alpha v^-_{\alpha q}\,, \qquad \delta_\epsilon\theta^-_q = \epsilon^\alpha v^-_{\alpha q}\,, \qquad \delta_\epsilon v^-_{\alpha q} = 0\,. \tag{5.24}$$

The fermionic covariant derivatives have the form

$$D_q^+ = \partial_q^+ + 2i\theta_q^- \partial_= \,, \qquad \partial_= := \frac{\partial}{\partial x^=} \,, \qquad \partial_q^+ := \frac{\partial}{\partial \theta_q^-} \,, \qquad q = 1, \ldots, 16 \,. \tag{5.25}$$

They obey $d = 1$ $\mathfrak{N} = 16$ supersymmetry algebra (cf. (3.4))

$$\{D_q^+, D_p^+\} = 4i\delta_{qp}\partial_= \,. \tag{5.26}$$

The linearized 11D supergravity was described in [21] by a bosonic antisymmetric tensor superfield

$$A^{IJK} = A^{[IJK]}(x^=, \theta_q^-, v_{\alpha q}^-) \tag{5.27}$$

which obeys the superfield equation

$$D_q^+ A^{IJK} = 3i\gamma_{qp}^{[IJ}\Psi_p^{K]} \,, \qquad \gamma_{qp}^I \Psi_p^I = 0 \,, \qquad q, p = 1, \ldots, 16 \,, \quad I = 1, \ldots, 8, 9 \,. \tag{5.28}$$

The consistency of eq. (5.28) requires that gamma-traceless fermionic Ψ_{Iq} obeys

$$D_q^+ \Psi_p^I = \frac{1}{18}\left(\gamma_{qp}^{IJKL} + 6\delta^{I[J}\gamma_{qp}^{KL]}\right)\partial_= A^{JKL} + 2\partial_= H_{IJ}\gamma_{qp}^J \tag{5.29}$$

with symmetric traceless tensor superfield H_{IJ} satisfying

$$D_q^+ H_{IJ} = i\gamma_{qp}^{(I}\Psi_p^{J)} \,, \qquad H_{IJ} = H_{JI} \,, \qquad H_{II} = 0 \,. \tag{5.30}$$

Actually any of the three equations, (5.28), (5.29) or (5.30), can be chosen as the fundamental; then other two will be reproduced as its consistency conditions. All the on-shell degrees of freedom of the 11D SUGRA can be extracted from any of the three constrained superfields H_{IJ}, A_{IJK} or Ψ_{Iq}.

5.5 Analytic on-shell superfields of 11D SUGRA

Similar to the case of 10D SYM, the 11D SUGRA can be also described by one complex superfield in $\mathcal{N} = 8$ extended analytic superspace. This is to say, the system of superfield equations (5.28), (5.29), (5.30) can be solved in terms of one analytic (chiral-like) superfield carrying charge 2 under the U(1) subgroup of SO(9) $\subset$ Spin(1, 10) acting naturally on the $\frac{\mathrm{Spin}(9)}{\mathrm{Spin}(7)\otimes \mathrm{U}(1)}$ coset.

5.5.1 $\frac{\mathbf{SO(9)}}{\mathbf{SO(7)\times SO(2)}}$ harmonic variables

Following the line described in section 4.3 for the case of D=10 SYM, let us introduce internal vector harmonics providing a set of constrained homogeneous coordinates for the $\frac{\mathrm{SO}(9)}{\mathrm{SO}(7)\times \mathrm{SO}(2)}$ coset (cf. (4.15))

$$U_I^{(J)} = \left(U_I{}^{\check{J}}, U_I{}^{(8)}, U_I{}^{(9)}\right) = \left(U_I{}^{\check{J}}, \frac{1}{2}\left(U_I + \bar{U}_I\right), \frac{1}{2i}\left(U_I - \bar{U}_I\right)\right) \in \mathrm{SO}(9) \,. \tag{5.31}$$

The condition (5.31) is equivalent to the set of relations which are described by (4.16) and (4.17), but now with $I, J = 1, \ldots, 9$ and $\check{I}, \check{J} = 1, \ldots, 7$. eqs. (4.16), in their turn, imply that the symmetric 16×16 matrices

$$\Psi_{qp} := U_I \gamma^I_{qp} \,, \qquad \bar{\Psi}_{qp} := \bar{U}_I \gamma^I_{qp} \tag{5.32}$$

are nilpotent

$$\Psi\Psi = 0 \,, \qquad \bar{\Psi}\bar{\Psi} = 0 \,, \tag{5.33}$$

and their anticommutator is proportional to unity matrix

$$\Psi\bar{\Psi} + \bar{\Psi}\Psi = 4 \,. \tag{5.34}$$

Hence, $\Psi\bar{\Psi}/4$ and its complex conjugate $\bar{\Psi}\Psi/4$ are orthogonal projectors and thus can be factorized

$$\mathcal{P}^+_{qp} = \frac{1}{4}\bar{\Psi}\Psi = w_q{}^A \bar{w}_{pA} \,, \qquad \mathcal{P}^-_{qp} = \frac{1}{4}\Psi\bar{\Psi} = \bar{w}_{pA} w_q{}^A \tag{5.35}$$

in terms of complex 16×8 matrices $w_q{}^A = (\bar{w}_{pA})^*$, which obey eqs. (4.21) and (4.22).

Let us introduce the Spin(9) valued matrix $w_q^{(p)}$ providing a bridge between spinor representations of SO(9) and SO(7), and also a kind of 'square root' of $U_I^{(J)}$ of (5.31).

$$w_q^{(p)} \in \text{Spin}(9) \,, \qquad U_I^{(J)} \gamma^I_{qp} = w_q^{(q')} \gamma^{(J)}_{(q')(p')} w_p^{(p')} \,. \tag{5.36}$$

Then we can calculate the projectors (5.35) in its term and find

$$2w_q{}^A \bar{w}_{pA} = w_q^{(q')} (I + i\gamma^8\gamma^9)_{(q')(p')} w_p^{(p')} \,, \quad 2\bar{w}_{qA} w_p{}^A = w_q^{(q')} (I - i\gamma^8\gamma^9)_{(q')(p')} w_p^{(p')} \,. \tag{5.37}$$

These equations make manifest that complex 16× 8 matrices $w_q{}^A$ and $\bar{w}_{qA}$ are combinations of the columns of a real 16×16 Spin(9) valued matrix $w_q^{(p)}$. Thus, the space parametrized by $w_q{}^A$ and $\bar{w}_{pA}$ is Spin(9) group manifold. Now, if we assume the Spin(7)⊗Spin(2) gauge symmetry and use it as an identification relation in this space, we can treat $w_q{}^A$ and $\bar{w}_{pA}$ as homogeneous coordinates of the coset $\frac{\text{Spin}(9)}{\text{Spin}(7)\otimes\text{Spin}(2)}$,

$$\{w_q{}^A, \bar{w}_{qA}\} = \frac{\text{Spin}(9)}{\text{Spin}(7)\otimes \text{U}(1)} \,, \tag{5.38}$$

and call them $\frac{\text{Spin}(9)}{\text{Spin}(7)\otimes\text{Spin}(2)}$ harmonic variables [32–34].

Let us stress that the conditions (5.36) are stronger than the ones imposed by (4.22) with $q, p = 1, \ldots, 16$. These latter would imply $w_q^{(p)} \in \text{SO}(16)$ only, while (5.36) results in $w_q^{(p)} \in \text{Spin}(9) \subset \text{SO}(16)$.

Another useful observation is that the second equation in (5.36) with $J = 8, 9$ can be written in the form

$$\Psi_{qp} = 2\bar{w}_{qA} \mathcal{U}^{AB} \bar{w}_{pB} \,, \qquad \bar{\Psi}_{qp} = 2w_q{}^A \bar{\mathcal{U}}_{AB} w_p{}^B \,, \tag{5.39}$$

where the complex symmetric matrices $\mathcal{U}_{AB}$ and $\bar{\mathcal{U}}^{AB} = (\mathcal{U}_{AB})^*$ obey

$$\bar{\mathcal{U}}_{AC}\mathcal{U}^{CB} = \delta_A{}^B \,. \tag{5.40}$$

These matrices can be identified with the charge conjugation matrix of SO(7), which is symmetric and can be chosen to be the unity matrix. Then $\mathcal{U}_{AB} = \delta_{AB} = \mathcal{U}^{AB}$ and (5.39) simplifies to

$$\not{\Psi}_{qp} = 2\bar{w}_{qA}\bar{w}_{pA} \,, \qquad \bar{\not{\Psi}}_{qp} = 2w_q{}^A w_p{}^A \,, \tag{5.41}$$

which gives more reasons to state that $\bar{w}_{qA}$ is a square root of the complex nilpotent matrix $\not{\Psi}_{qp}$.

The covariant harmonic derivatives preserving all the constraints on the $\frac{\text{Spin}(9)}{\text{Spin}(7)\otimes\text{U}(1)}$ harmonic variables, eqs. (4.21), (4.22), (5.36) and (5.41), have the form

$$\mathbb{D}^{\check{J}} = \frac{1}{2}U_I\frac{\partial}{\partial U_I^{\check{J}}} - U_I^{\check{J}}\frac{\partial}{\partial \bar{U}_I} + \frac{i}{2}\tilde{\sigma}^{\check{J}AB}\bar{w}_{qB}\frac{\partial}{\partial w_q^A} \,, \tag{5.42}$$

$$\bar{\mathbb{D}}^{\check{J}} = \frac{1}{2}\bar{U}_I\frac{\partial}{\partial U_I^{\check{J}}} - U_I^{\check{J}}\frac{\partial}{\partial U_I} + \frac{i}{2}\sigma^{\check{J}}_{AB}w_q^B\frac{\partial}{\partial \bar{w}_{qA}} \,, \tag{5.43}$$

$$\mathbb{D}^{(0)} = U_I\frac{\partial}{\partial U_I} - \bar{U}_I\frac{\partial}{\partial \bar{U}_I} + \frac{1}{2}\left(\bar{w}_{qA}\frac{\partial}{\partial \bar{w}_{qA}} - w_q^A\frac{\partial}{\partial w_q^A}\right) \,, \tag{5.44}$$

$$\mathbb{D}^{\check{I}\check{J}} = \frac{1}{2}\left(U_K^{\check{I}}\frac{\partial}{\partial U_K^{\check{J}}} - U_K^{\check{J}}\frac{\partial}{\partial U_K^{\check{I}}}\right) + \frac{i}{2}\sigma^{\check{I}\check{J}}{}_B{}^A\left(w_q^B\frac{\partial}{\partial w_q^A} - \bar{w}_{qA}\frac{\partial}{\partial \bar{w}_{qB}}\right) \,, \tag{5.45}$$

where $\sigma^{\check{J}}_{AB} = \tilde{\sigma}^{\check{J}AB}$ are SO(7) Clebsch-Gordan coefficients and $\sigma^{\check{I}\check{J}} = \sigma^{[\check{I}}\tilde{\sigma}^{\check{J}]}$.

As far as we have chosen 7d charge conjugation matrix to be equal to unity matrix, the contraction of two spinor subindices is allowed (see (5.41)), and we can write all the SO(7) spinor indices as subindices. However we find convenient to keep some part of these as superindices as an indication of the origin of variables which carry them as well as with the aim to keep as manifest as possible the similarity of our 11D SUGRA formalism to the 10D SYM case.

5.5.2 Analytic on-shell superfields from constrained on-shell superfields

Using the above described harmonic variables it is not difficult to find a projection of the symmetric traceless tensor superfield H_{IJ} which, as a result of (5.30), obeys an analyticity equation. Indeed, let us define a complex superfield

$$\Phi = H^{IJ}U_IU_J \,. \tag{5.46}$$

Multiplying (5.30) on U_IU_J we find that this superfield satisfies

$$D_q^+\Phi = i\not{\Psi}_{qp}\,\Psi_p^J U^J \,, \tag{5.47}$$

which, in the light of (5.33), implies $(\bar{\not{\Psi}}\not{\Psi})_{pq}D_q^+\Phi = 0$. Using the factorization of the projector (5.35) and eqs. (4.22) we find that this is equivalent to the analyticity condition

$$\bar{D}_A^+\Phi = 0 \,, \qquad \bar{D}_A^+ = \bar{w}_{qA}D_q^+ \,. \tag{5.48}$$

Besides (5.48), the analytic superfield (5.46) obeys

$$\mathbb{D}^{\check{J}}\Phi = 0\,, \tag{5.49}$$

$$\mathbb{D}^{\check{I}\check{J}}\Phi = 0\,, \tag{5.50}$$

$$\mathbb{D}^{(0)}\Phi = 2\Phi\,. \tag{5.51}$$

As in the case of 10D SYM, using the harmonic derivative (5.43) and the remaining parts of (5.47) and (5.30) we can obtain the expression for all other components of the constrained superfield H_{IJ} in terms of Φ and its complex conjugate $\bar{\Phi}$.

Passing to the analytic coordinate basis of the on-shell superspace (5.23), which is described by $D = 11$ $\mathcal{N} = 8$ version of (4.42), one can check that a superfield Φ obeying eq. (5.48) depends on η_A but not on its c.c. $\bar{\eta}^A = (\eta_A)^*$. This is reflected by the name analytic superfield which we have attributed to such Φ.

Notice that the complex fermionic variable η_A is almost identical with the one used in the description of $\mathcal{N} = 8$ 4D supergravity; the word 'almost' here refers to the fact that only SO(7) subgroup of SU(8) acts on its index $_A$.[7] The decomposition of our analytic superfield in η_A,[8]

$$\Phi(\rho^{\#}, v_{\alpha q}^{-}; w, \bar{w}; \eta_A) = \phi^{(+2)} + \eta_A\psi^{(+3/2)A} + \ldots + (\eta)^{\wedge 7\, A}\psi_A^{(-3/2)} + (\eta)^{\wedge 8}\phi^{(-2)} \tag{5.52}$$

looks very much the same as chiral superfield (1.11) describing the linearized $\mathcal{N} = 8$ supergravity. However, all the fields in its decomposition depend on a different set of variables:

$$\phi^{(+2)} = \phi^{(+2)}(\rho^{\#}, v_{\alpha q}^{-}; w, \bar{w})\,, \qquad \psi^{(+3/2)A} = \psi^{(+3/2)A}(\rho^{\#}, v_{\alpha q}^{-}; w, \bar{w})\,, \quad \text{etc.} \tag{5.53}$$

versus $\phi^{+2} = \phi^{+2}(\lambda, \bar{\lambda})$ with complex two component $\lambda = (\bar{\lambda})^*$ in D=4.

The set of 24=1+9+14 bosonic variables our on-shell fields depend on includes 'energy' $\rho^{\#}$, spinor harmonic variables $v_{\alpha q}^{-}$, which are considered as homogeneous coordinates of the celestial sphere $\mathbb{S}^9$ realized as a coset of Lorentz group $\frac{\mathrm{Spin}(1,10)}{[\mathrm{SO}(1,1)\times\mathrm{Spin}(9)]\otimes K_9}$ (2.29), and a set of $\frac{\mathrm{Spin}(9)}{\mathrm{Spin}(7)\otimes\mathrm{Spin}(2)}$ internal harmonic variables $w_q{}^A, \bar{w}_{Aq}$ (5.38).

The signs and numerical superscribes of the component fields in (5.52) and (5.53) indicate their charges under U(1) symmetry transformations acting on η_A, $\bar{w}$ and w. These can be easily calculated in the assumption that the overall charge of the superfield is equal to +2.[9]

The above statements about charges of variables and superfields under $\mathrm{U}(1) \subset \mathrm{SO}(9) \subset \mathrm{SO}(1,10)$ can be formulated as a differential equation (cf. (1.5))

$$\hat{h}^{(11D)}\Phi(\rho^{\#}, v_{\alpha q}^{-}; w, \bar{w}; \eta_A) = 2\Phi(\rho^{\#}, v_{\alpha q}^{-}; w, \bar{w}; \eta_A)\,, \tag{5.54}$$

[7]Probably, to observe the SU(8) symmetry, one has to consider $(w, \bar{w})$ as parametrizing the coset $\mathrm{SO}(16)/[\mathrm{SU}(8) \otimes H]$ with some $H \subset \mathrm{SO}(16)$. (This in its turn would require to consider $\mathcal{U}^{AB}$ in (5.39) to be an independent spin-tensor coordinate). Thus a hidden SO(16) symmetry of 11D SUGRA might be relevant in this problem. It is tempting to speculate that E_8 hidden symmetry might also happen to be useful in this context.

[8]To streamline the presentation at this stage we prefer to pass to the Fourier image of the superfields with respect to $x^{=}$ (actually $x_L^{=} = x^{=} + 2i\eta_A\bar{\eta}^A$) coordinate ((5.21) *vs* (5.22)).

[9] This assumption is suggested by the origin of the analytic superfield in SO($D-2$) tensor, $\Phi = H_{IJ}U_IU_J$.

JHEP05(2018)103

where

$$\hat{h}^{(11D)} := \frac{1}{2}\bar{w}_{qA}\frac{\partial}{\partial \bar{w}_{qA}} - \frac{1}{2}w_q^A\frac{\partial}{\partial w_q^A} + \frac{1}{2}\eta_A\frac{\partial}{\partial \eta_A} \tag{5.55}$$

is the 11D counterpart of the helicity operator (1.10). On the analytic superfields it coincides with the harmonic covariant derivative $D^{(0)}$ (5.44),

$$\hat{h}^{(11D)} = D^{(0)}|_{\text{on analytic superfields}}\,, \tag{5.56}$$

so that eq. (5.55) actually coincide with (5.51).

In the above presented description by analytic superfield the 44+84=128 bosonic fields of the on-shell 11D supergravity ($h_{IJ} = h_{((IJ))}$ and $A_{IJK} = A_{[IJK]}$ in the SO(9)=SO(D–2) covariant notation) are split onto **1+28+70+28+1** representations of SO(7)=SO(D–4),

$$e_\mu^a, A_{\mu\nu\rho} \leftrightarrow (h_{((IJ))}\,,\, A_{[IJK]}) \;=\; (\phi^{(-4)}, \phi^{(-2)AB}, \phi^{ABCD}, \phi^{(+2)}_{AB}, \phi^{(+4)})\,, \tag{5.57}$$

and 128 fermionic fields splits on **1+28+70+28+1**

$$\psi_\mu^\alpha \leftrightarrow (\Psi_{Iq}\,|\,\gamma^I_{qp}\Psi_{Ip} = 0) \;=\; (\psi_A^{(-3)}, \psi^{(-1)}_{ABC}, \psi^{(+1)ABC}, \psi^{(+3)A})\,. \tag{5.58}$$

5.6 Supersymmetry transformation of the analytic superfields

As in the case of 10D SYM, we can find that rigid 11D supersymmetry acts on the analytic on-shell superfield of 11D supergravity by

$$\begin{aligned}\Phi'(\rho^\#, v_q^-; w, \bar{w}; \eta_A) &= \exp\{-4\rho^\#\bar{\epsilon}^{-A}\eta_A\}\Phi(\rho^\#, v_q^-; w, \bar{w}; \eta_A - \epsilon_A^-)\\ &= \exp\left\{-4\rho^\#\bar{\epsilon}^{-A}\eta_A - \epsilon_A^-\frac{\partial}{\partial\eta_A}\right\}\,\Phi(\rho^\#, v_q^-; w, \bar{w}; \eta_A)\,,\end{aligned} \tag{5.59}$$

where

$$\bar{\epsilon}^{-A} = \epsilon_q^- w_q^A = \epsilon^\alpha v_{\alpha q}^- w_q^A\,, \qquad \epsilon_A^- = \epsilon_q^- \bar{w}_{qA} = \epsilon^\alpha v_{\alpha q}^- \bar{w}_{qA} \tag{5.60}$$

and ϵ^α is constant fermionic parameter.

The supersymmetry generator defined by $\Phi' = e^{-\epsilon^\alpha Q_\alpha}\Phi$,

$$Q_\alpha = \tilde{q}_\alpha + \hat{q}_\alpha = 4\rho^\# v_{\alpha q}^-\eta_A^- w_q^A + v_{\alpha q}^-\bar{w}_{qA}\frac{\partial}{\partial\eta_A^-} \tag{5.61}$$

is given by the sum of the algebraic part $\tilde{q}_\alpha$ and of the differentail operator $\hat{q}_\alpha$,

$$\tilde{q}_\alpha = 4\rho^\# v_{\alpha q}^- w_q^A\eta_A\,, \tag{5.62}$$

$$\hat{q}_\alpha = v_{\alpha q}^-\bar{w}_{qA}\frac{\partial}{\partial\eta_A}\,. \tag{5.63}$$

However, in distinction to the D=4 case, to split the parameter of rigid supersymmetry ϵ^α on the parts corresponding to $\tilde{q}_\alpha$ and $\hat{q}_\alpha$ we need to use the composite complex spinor harmonic variables $v_q^{+\alpha}w_q^A$ and $v_q^{+\alpha}\bar{w}_{qA}$ (while in D=4 the splitting appears automatically because $\tilde{q}_\alpha^{(D=4)}$ and $\hat{q}_{\dot{\alpha}}^{(D=4)}$ carry different type of Weyl spinor indices).

To show that the algebra of supersymmetry generators (5.61) is closed on the momentum,

$$\{Q_\alpha, Q_\beta\} = 4\rho^\# v_{\alpha q}^- v_{\beta q}^- = 2p_a\Gamma^a_{\alpha\beta}\,, \tag{5.64}$$

we have to use (4.21), $2w^A_{(q}\bar{w}_{p)A} = \delta_{qp}$.

6 Analytic superamplitudes in D=10 and D=11

In this section and below, to avoid doubling of the formulae, we will tend to write the universal equations describing simultaneously D=10 and D=11 case whenever it is possible. (See section 1.1 for the universal description of our index notation).

6.1 Properties of analytic superamplitudes

The simplest superamplitudes are multiparticle counterparts of the analytic superfields (4.57) and (5.52)

$$\mathcal{A}_n \, \delta^D \left(\sum_i^n k_{ai} \right) = \mathcal{A}_n(\{\rho_i^{\#}, v_{\alpha q i}^{-}; w_i, \bar{w}_i; \eta_{Ai}\}) \, \delta^D \left(\sum_i^n \rho_i^{\#} u_{ai}^{=} \right) . \tag{6.1}$$

They do not carry indices, are Lorentz invariant, invariant under $\prod_{i=1}^{n} [\mathrm{SO}(1,1)_i \otimes \mathrm{SO}(D-2)_i \otimes \mathrm{SO}(D-4)_i]$ and covariant under $\prod_i \mathrm{SO}(2)_i = \mathrm{U}(1)_i$ symmetry transformations.

The Lorentz group $\mathrm{SO}(1, D-1)$ acts nontrivially on spinor harmonic variables $v_{\alpha q i}^{-}$ only, $\mathrm{SO}(D-2)_i$ act on $v_{\alpha q i}^{-}$ and on the internal harmonic variables $(w_{qi}^{A}, \bar{w}_{Aqi})$, $\mathrm{SO}(1,1)_i$ act on $v_{\alpha q i}^{-}$ and on the fermionic $\eta_{Ai} = \eta_{Ai}^{-}$, and $\mathrm{SO}(D-4)_i$ transform w_{qi}^{A} and $\bar{w}_{Aqi}, \eta_{Ai}$ in $\mathbf{4s}$ and $\overline{\mathbf{4s}}$, respectively. Finally, $\mathrm{SO}(2)_i = \mathrm{U}(1)_i$ symmetries act nontrivially on $w_{qi}^{A}, \bar{w}_{Aqi}, \eta_{Ai}$ with the same value of i, and on the amplitude which carries the charge $s = \mathcal{N}/4$ (+2 for 11D SUGRA and +1 for 10D SYM) with respect to each of the $\mathrm{U}(1)_i$ group.

The gauge symmetry $\prod_i \mathrm{SO}(1,1)_i \otimes \mathrm{SO}(D-2)_i \otimes \mathrm{SO}(D-4)_i$ make possible to identify each set of harmonic variables, $(v_{\alpha q i}^{-}, v_{\alpha \dot{q} i}^{+})$ and $(w_{qi}^{A}, \bar{w}_{Aqi})$, with generalized homogeneous coordinates of the cosets:

$$\{(v_{\alpha q i}^{-}, v_{\alpha \dot{q} i}^{+})\} = \left(\frac{\mathrm{Spin}(1, D-1)}{[\mathrm{SO}(1,1) \otimes \mathrm{Spin}(D-2)] \otimes K_{(D-2)}} \right)_i = \mathbb{S}_i^{(D-2)} \tag{6.2}$$

and[10]

$$\{w_{qi}^{A}, \bar{w}_{Aqi}\} = \left(\frac{\mathrm{SO}(D-2)}{\mathrm{SO}(D-4) \otimes \mathrm{U}(1)} \right)_i . \tag{6.3}$$

Notice that here, in distinction to section 2 devoted to D=4 case, we prefer to write explicitly the momentum preserving delta function $\delta^D \left(\sum_i^n k_{ai} \right) = \delta^D \left(\sum_i^n \rho_i^{\#} u_{ai}^{=} \right)$ and denote by $\mathcal{A}$ the amplitudes with arguments obeying the overall momentum conservation

$$\sum_i^n k_{ai} = \sum_i^n \rho_i^{\#} u_{ai}^{=} = 0 \, , \qquad u_{ai}^{=} \Gamma^a_{\alpha\beta} = 2 v_{\alpha q \, i}^{-} v_{\beta q \, i}^{-} \, . \tag{6.4}$$

The representations of the variables and superamplitude with respect to the symmetry groups are summarized in table 1, where the parameter $s = \mathcal{N}/4$ distinguish the cases of D=10 SYM (s=1) and D=11 SUGRA (s=2).

[10] $K_{(D-2)i}$ symmetry (2.31) implies an independence of the amplitude on the complementary $v_{\alpha \dot{q} i}^{+}$ harmonics. This is reflected by the list of arguments of the amplitude in (6.1). Let us recall that for $D = 11$ case $\dot{q} = q = 1, \ldots, 16$, while for D=10 $\dot{q} = 1, \ldots, 8$ and $q = 1, \ldots, 8$ are indices of different spinor representations of SO(8).

variables/representations	SO(1,1)$_i$ weight	SO(D-2)$_i$ repr.	SO(D-4)$_i$ repr.	SO(2)$_i$=U(1)$_i$ charge	Spin(1,D-1)
$\mathcal{A}_n(\{\rho^{\#}_i, v_i^-; w_i, \bar{w}_i; \eta_i\})$				$+s$	
$\rho^{\#}_{(i)}$	+2				
$v^-_{\alpha q(i)}$	−1	**8s**			**16s**
$w_{q(i)}^{\ A}$		**8s**	**4s**	−1/2	
$\bar{w}_{qA(i)}$		**8s**	$\overline{\mathbf{4s}}$	+1/2	
$\eta_{A(i)} \equiv \eta^-_{A(i)}$	-1		$\overline{\mathbf{4s}}$	+1/2	
U_I		**7+s**		+1	
$\bar{U}_I$		**7+s**		−1	

Table 1. SO(D-4) and SO(D-2) representations, SO(1,1) weights and U(1) charges of the analytic superamplitude and its arguments; $s=1$ for D=10 SYM and $s=2$ for 11D SUGRA. U_I and $\bar{U}_I$ are bilinears of w and $\bar{w}$ as defined in (4.25) for D=10 and (5.41) for D=11.

This table also indicates that the simplest superamplitudes (6.1) are Lorentz scalars, have charges $+s$ with respect to all the U(1)$_i$ symmetry groups and are inert under all other bosonic symmetry transformations $\mathrm{SO}(1,1)_i\otimes\mathrm{SO}(D-2)_i\otimes\mathrm{SO}(D-4)_i$. As we will discuss below, the analytic superamplitudes also obey a set of equations with harmonic covariant derivatives which provide us with counterparts of the 4D helicity constraints (1.5).

More complicated superamplitudes, which do carry the nontrivial representations of $\mathrm{SO}(D-4)_i$ and different charges under $\mathrm{SO}(2)_i=\mathrm{U}(1)_i$ can be obtained by acting on the analytic superamplitude (6.1) by fermionic covariant derivatives $D^+_{A(i)}$ and by harmonic covariant derivatives.

6.2 From constrained to analytic superamplitudes. 10D SYM

Let us discuss the relation of the above described analytic superamplitude (6.1) with the constrained superamplitude formalism [15, 16].

The basic constrained superamplitude of 10D SYM theory

$$\mathcal{A}_{I_1\ldots I_n}(k_1,\theta^-_1;\ldots;k_n,\theta^-_n)\,\delta^D\left(\sum_i^n k_{ai}\right)=\mathcal{A}_{I_1\ldots I_n}(\{\rho^{\#}_i, v^-_{\alpha qi};\theta^-_{qi}\})\,\delta^D\left(\sum_i^n \rho^{\#}_i u^{=}_{ai}\right), \qquad (6.5)$$

carry n vector indices of SO(8)$_i$ groups. It obeys the equations (see [16] for details)

$$D^{+j}_q\mathcal{A}^{(n)}_{I_1\ldots I_j\ldots I_n}=2\rho^{\#}_j\,\gamma^{I_j}_{q\dot{q}_j}\mathcal{A}^{(n)}_{I_1\ldots I_{j-1}\dot{q}_j I_{j+1}\ldots I_n}\,, \qquad (6.6)$$

where

$$D^{+j}_q=\partial^{+j}_q+2\rho^{\#}_j\theta^-_{qj}\,, \qquad \partial^{+j}_q:=\frac{\partial}{\partial\theta^-_{qj}}\,. \qquad (6.7)$$

To express the analytic superamplitude (6.1) through the constrained superamplitude (6.5), let us first contract the $SO(8)_i$ vector indices of this latter with the complex null-vectors $U_{I_i\, i}$ of the corresponding internal frames (6.3),

$$\tilde{\mathcal{A}}_n(\{\rho_i^{\#}, v_{\alpha qi}^{-}; w_i, \bar{w}_i; \theta_{qi}^{-}\}) = U_{I_1 1} \ldots U_{I_n n}\, \mathcal{A}_{I_1 \ldots I_n}(\{\rho_i^{\#}, v_{\alpha qi}^{-}; \theta_{qi}^{-}\}), \tag{6.8}$$

$$\not{U}_{q\dot{p}\, i} := \gamma^I_{q\dot{p}} U_{I\, i} = 2\bar{w}_{qAi} w^A_{\dot{p}i}\,. \tag{6.9}$$

Using (6.9) and the properties (4.22) of the internal harmonics (6.3), one can easily check that

$$\bar{D}_A^{+j} \tilde{\mathcal{A}}_n(\{\rho_i^{\#}, v_{\alpha qi}^{-}; w_i, \bar{w}_i; \theta_{qi}^{-}\}) = 0 \qquad \forall j = 1, \ldots, n\,. \tag{6.10}$$

In these equations

$$\bar{D}_A^{+j} = \bar{w}_{qAj} D_q^{+j} = \frac{\partial}{\partial \bar{\eta}_j^{-A}} + 2\rho_j^{\#} \eta_{Aj}^{-}\,, \qquad \eta_{Aj}^{-} = \theta_{qj}^{-} \bar{w}_{qAj}\,, \qquad \bar{\eta}_j^{-A} = \theta_{qj}^{-} w_{qj}^{A}\,. \tag{6.11}$$

Our analytic 10D SYM superamplitude is related to (6.8) by

$$\mathcal{A}_n(\{\rho_i^{\#}, v_{\alpha qi}^{-}; w_i, \bar{w}_i; \eta_{Ai}\}) = e^{-2\sum_j \rho_j^{\#} \eta_{Bj}^{-} \bar{\eta}_j^{-B}}\, \tilde{\mathcal{A}}_n(\{\rho_i^{\#}, v_{\alpha qi}^{-}; w_i, \bar{w}_i;\ \eta_{Ai}^{-} w_{qi}^{A} + \bar{\eta}_i^{-A} \bar{w}_{qAi}\})\,. \tag{6.12}$$

Indeed, one can easily check that

$$\bar{D}_A^{+(j)} e^{2\sum_j \rho_j^{\#} \eta_{Bj}^{-} \bar{\eta}_j^{-B}} = e^{2\sum_j \rho_j^{\#} \eta_{Bj}^{-} \bar{\eta}_j^{-B}} \frac{\partial}{\partial \bar{\eta}_j^{-A}}$$

so that $\mathcal{A}_n$ of (6.12) is $\bar{\eta}_i^{-A}$-independent due to (6.10).

Resuming, the analytic superamplitude (6.1) is expressed in terms of constrained superamplitude (6.5) by contracting its SO(8) vector indices $_{I_i}$ with appropriate null-vectors $U_{I_i i}$ constructed from internal harmonics as in (4.25):

$$\begin{aligned} &\mathcal{A}_n(\{\rho_i^{\#}, v_{\alpha qi}^{-}; w_i, \bar{w}_i; \eta_{Ai}\}) \\ &\quad = e^{-2\sum_j \rho_j^{\#} \eta_{Bj}^{-} \bar{\eta}_j^{-B}}\, U_{I_1 1} \ldots U_{I_n n}\, \mathcal{A}_{I_1 \ldots I_n}(\{\rho_i^{\#}, v_{\alpha qi}^{-};\ \eta_{Ai}^{-} w_{qi}^{A} + \bar{\eta}_i^{-A} \bar{w}_{qAi}\}). \end{aligned} \tag{6.13}$$

It is not difficult to check that this amplitude also obeys

$$D_j^{\check{J}} \mathcal{A}_n(\{\rho_i^{\#}, v_{\alpha qi}^{-}; w_i, \bar{w}_i; \eta_{Ai}\}) = 0\,, \qquad j = 1, \ldots, n\,, \tag{6.14}$$

$$D_j^{\check{J}\check{K}} \mathcal{A}_n(\{\rho_i^{\#}, v_{\alpha qi}^{-}; w_i, \bar{w}_i; \eta_{Ai}\}) = 0\,, \qquad j = 1, \ldots, n\,, \tag{6.15}$$

$$D_j^{(0)} \mathcal{A}_n(\{\rho_i^{\#}, v_{\alpha qi}^{-}; w_i, \bar{w}_i; \eta_{Ai}\}) = \mathcal{A}_n(\{\rho_i^{\#}, v_{\alpha qi}^{-}; w_i, \bar{w}_i; \eta_{Ai}\})\,, \quad j = 1, \ldots, n\,, \tag{6.16}$$

with the derivative defined as in (4.47)–(4.50), (4.31)–(4.33), but for j-th internal harmonic variables. Eqs. (6.14)–(6.16) can be considered as counterparts of the D=4 super-helicity constraints (1.13).

6.3 Analytic superamplitudes of 11D SUGRA from constrained superamplitudes

Eq. (6.12) with $q = 1, \dots, 2\mathcal{N}$ and $A = 1, \dots, \mathcal{N}$ describes also the relation of the constrained and analytic superamplitudes of 11D SUGRA if we set $\mathcal{N} = 8$ and

$$\tilde{\mathcal{A}}_n(\{\rho^{\#}_{(i)}, v^-_{\alpha q(i)}; w_i, \bar{w}_i; \theta^-_{qi}\}) = U_{I_1 1} U_{J_1 1} \dots U_{I_n n} U_{J_n n}\, \mathcal{A}_{((I_1 J_1))\dots((I_n J_n))}(\{\rho^{\#}_i, v^-_{\alpha q i}\})\,. \quad (6.17)$$

Here the basic superamplitude of the constrained superfield formalism, $\mathcal{A}_{((I_1 J_1))\dots((I_n J_n))}$ symmetric and traceless on each pair of SO(9)$_i$ vector indices enclosed in doubled brackets, obeys the equation [15, 16]

$$D^{+(j)}_{q_j} \mathcal{A}^{(n)}_{((I_1 J_1))\dots((I_j J_j))\dots((I_n J_n))} = i\gamma_{(I_j|q_j p_j}\, \mathcal{A}^{(n)}_{((I_1 J_1))\dots|J_j)p_j\dots((I_n J_n))}\,. \quad (6.18)$$

The r.h.s. of this equation contains γ-traceless $\mathcal{A}^{(n)}_{((I_1 J_1))\dots((I_{j-1} J_{j-1}))\, J_j p_j\, ((I_{j+1} J_{j+1}))\dots((I_n J_n))}$,

$$\gamma^{J_j}_{q p_j}\, \mathcal{A}^{(n)}_{((I_1 J_1))\dots\, J_j p_j\, \dots((I_n J_n))} = 0\,. \quad (6.19)$$

Finally, $U_{I_1 i}$ in the r.h.s. of (6.17) is expressed through bilinear of the internal harmonics by

$$\mathcal{U}^{(i)}_{qp} := \gamma^I_{qp} U_{I\,i} = 2\bar{w}_{qAi}\bar{w}_{pAi} \quad (6.20)$$

(see (5.41)).

Due to (6.18) and (6.19), $\tilde{\mathcal{A}}_n$ of (6.17) obeys (6.10) and the 11D superamplitude (6.12) is analytic, i.e. it depends on η^-_{Ai} but is independent of its complex conjugate $\bar{\eta}^{-A}_i$.

The analytic amplitude (6.17) also obeys the equations

$$\begin{aligned}
&D^{\check{J}}_j \mathcal{A}_n(\{\rho^{\#}_i, v^-_{\alpha q i}; w_i, \bar{w}_i; \eta_{Ai}\}) = 0\,, && j = 1, \dots, n\,, && (6.21)\\
&D^{\check{J}\check{K}}_j \mathcal{A}_n(\{\rho^{\#}_i, v^-_{\alpha q i}; w_i, \bar{w}_i; \eta_{Ai}\}) = 0\,, && j = 1, \dots, n\,, && (6.22)\\
&D^{(0)}_j \mathcal{A}_n(\{\rho^{\#}_i, v^-_{\alpha q i}; w_i, \bar{w}_i; \eta_{Ai}\}) = 2\mathcal{A}_n(\{\rho^{\#}_i, v^-_{\alpha q i}; w_i, \bar{w}_i; \eta_{Ai}\})\,, && j = 1, \dots, n\,, && (6.23)
\end{aligned}$$

with $\check{J}, \check{K} = 1, \dots, 7$.

6.4 Supersymmetry transformations of the analytic superamplitudes

The supersymmetry acts on our analytical superamplitudes as

$$\begin{aligned}
\mathcal{A}'_n(\{\rho^{\#}_i, v^-_{\alpha q i}; w^A_{qi}, \bar{w}_{qAi}; \eta_{Ai}\}) &= e^{-\epsilon^\alpha(\tilde{q}_\alpha + \hat{q}_\alpha)}\, \mathcal{A}_n(\{\rho^{\#}_i, v^-_i; w_i, \bar{w}_i; \eta_{Ai}\})\\
&= e^{-\epsilon^\alpha \tilde{q}_\alpha}\, \mathcal{A}_n(\{\rho^{\#}_i, v^-_i; w_i, \bar{w}_i; \eta_{Ai} - \epsilon^-_{Ai}\})\,, \quad (6.24)
\end{aligned}$$

where (see (5.62) and (5.63))

$$\epsilon^\alpha \tilde{q}_\alpha = 4\sum_{i=1}^n \rho^{\#}_i \bar{\epsilon}^{A-}_i \eta_{Ai}\,, \qquad \epsilon^\alpha \hat{q}_\alpha = \sum_{i=1}^n \epsilon^-_{Ai}\frac{\partial}{\partial \eta_{Ai}} \quad (6.25)$$

and (see (5.60))

$$\epsilon^-_{Ai} = \epsilon^\alpha \bar{v}^-_{\alpha Ai} = \epsilon^\alpha v^-_{\alpha q i}\bar{w}_{qAi}\,, \qquad \bar{\epsilon}^{A-}_i = \epsilon^\alpha v^{-A}_{\alpha i} = \epsilon^\alpha v^-_{\alpha q i} w^A_{qi}\,. \quad (6.26)$$

As in the case of the on-shell superfields, the supersymmetry generator acting on superamplitude splits

$$Q_\alpha = \tilde{q}_\alpha + \hat{q}_\alpha \tag{6.27}$$

onto the purely algebraic part and the differential operator

$$\tilde{q}_\alpha = 4\sum_{i=1}^{n} \rho_i^{\#} v_{\alpha i}^{-A} \eta_{Ai}^{-} = 4\sum_{i=1}^{n} \rho_i^{\#} v_{\alpha qi}^{-} w_{qi}^{A} \eta_{Ai}\,, \tag{6.28}$$

$$\hat{q}_\alpha = \sum_{i=1}^{n} v_{\alpha Ai}^{-} \frac{\partial}{\partial \eta_{Ai}} = \sum_{i=1}^{n} v_{\alpha qi}^{-} \bar{w}_{qAi} \frac{\partial}{\partial \eta_{Ai}}\,. \tag{6.29}$$

It is easy to check (using $2w_{(q|i}{}^{A}\bar{w}_{|p)Ai} = \delta_{qp}$ (4.21)) that the generators (6.27) obey the supersymmetry algebra, and actually anti-commute as far as the momentum is conserved:

$$\{Q_\alpha, Q_\beta\} = 4\sum_{i=1}^{n} \rho_i^{\#} v_{\alpha qi}^{-} v_{\beta qi}^{-} = 2\Gamma^{a}_{\alpha\beta} \sum_{i=1}^{n} p_{ai} = 0\,. \tag{6.30}$$

6.5 Supermomentum in D=10 and D=11

Although we have succeeded in writing the supersymmetry generator in terms of complex η^{-}_{Ai} and its derivative, the simplest way to write a supersymmetric invariant linear combination of fermionic variables uses the real fermionic θ^{-}_{qi} of the constrained superfield formalism (see (4.42)). Indeed, the real fermionic spinor

$$q_\alpha := \sum_{i=1}^{n} \rho_i^{\#} v_{\alpha qi}^{-} \theta_{qi}^{-}\,, \tag{6.31}$$

which can be called *supermomentum*, is transformed into the momentum by supersymmetry

$$\delta_\epsilon q_\alpha = \epsilon^\beta \sum_{i=1}^{n} \rho_i^{\#} v_{\alpha qi}^{-} v_{\beta qi}^{-} = \frac{1}{2}\Gamma^{a}_{\alpha\beta}\epsilon^\beta \sum_{i=1}^{n} p_{ai} \tag{6.32}$$

and, hence is supersymmetric invariant when momentum is conserved,

$$\delta_\epsilon q_\alpha = 0 \qquad \text{when} \qquad \sum_{i=1}^{n} p_{ai} = 0\,. \tag{6.33}$$

7 Convenient parametrization of spinor harmonics (convenient gauge fixing of the auxiliary gauge symmetries)

7.1 Reference spinor frame and minimal parametrization of spinor harmonics

It looks convenient to fix the gauge with respect to the defining gauge symmetries of the spinor frame variables $[\mathrm{SO}(1,1)_i \otimes \mathrm{SO}(D-2)_i] \otimes\!\!\!\!\!\!\times K_{(D-2)\,i}$ by setting

$$v_{\alpha qi}^{-} = v_{\alpha q}^{-} + \frac{1}{2} K_i^{=I} \gamma^{I}_{q\dot{p}} v_{\alpha\dot{p}}^{+}\,, \qquad v_{\alpha\dot{q}i}^{+} = v_{\alpha\dot{q}}^{+}\,. \tag{7.1}$$

JHEP05(2018)103

Here $(v^-_{\alpha q}, v^+_{\alpha\dot{q}})$ is an auxiliary reference spinor frame the components of which can be identified with homogeneous coordinates of an auxiliary coset (or reference coset) $\frac{SO(1,D-1)}{[SO(1,1)\otimes SO(D-2)]\otimes K_{D-2}} = \mathbb{S}^{D-2}$. Clearly, the reference spinor frame can be chosen arbitrary, in a way convenient for the problem under consideration.

Eq. (7.1) provides the explicit parametrization of the spinor harmonics $(v^-_{\alpha qi}, v^+_{\alpha\dot{q}i})$ describing a celestial sphere of i-th D-dimensional observer by single $SO(D-2)$ vector $K_i^{=I}$. This is manifestly invariant under one set of $[SO(1,1)\otimes SO(D-2)] \otimes K_{D-2}$ gauge symmetries acting on the reference spinor frame variables.

Eq. (7.1) lead to the following expressions for vector harmonics in terms of reference vector frame

$$u^{=}_{ai} = u^{=}_a + K_i^{=I} u^I_a + \frac{1}{4}(\vec{K}_i^{=})^2 u^{\#}_a \,, \tag{7.2}$$

$$u^I_{ai} = u^I_a + \frac{1}{2}K_i^{=I}\, u^{\#}_a \,, \tag{7.3}$$

$$u^{\#}_{ai} = u^{\#}_a \,. \tag{7.4}$$

The momentum of i-th particle is expressed through $K_i^{=I}$ and density $\rho_i^{\#}$ by

$$k_{ai} = \rho_i^{\#} u^{=}_{ai} = \rho_i^{\#}\left(u^{=}_a + K_i^{=I} u^I_a + \frac{1}{4}(\vec{K}_i^{=})^2 u^{\#}_a\right) . \tag{7.5}$$

Thus, in the gauge (7.1), the n-point amplitude (6.1) is a function of energies $\rho_i^{\#}$, of $SO(D-2)$ vectors K_i^I, of the fermionic η_{Ai} and also of the constrained complex bosonic $w_i, \bar{w}_i$ variables,

$$\mathcal{A}_n = \mathcal{A}_n(\{\rho_i^{\#}, K_i^{=I}; w_i, \bar{w}_i; \eta_{Ai}\}) \,. \tag{7.6}$$

This latter dependence will be specified below.

7.2 Generic parametrization of spinor harmonic variables and $K^{\#I} = 0$ gauge

A generic parametrization of the spinor harmonics (2.16) is

$$v^-_{\alpha qi} = e^{-\alpha_i}\mathcal{O}_{iqp}\left(v^-_{\alpha p} + \frac{1}{2}K_i^{=I}\gamma^I_{p\dot{q}} v^+_{\alpha\dot{q}}\right) , \tag{7.7}$$

$$v^+_{\alpha\dot{q}i} = \mathcal{O}_{i\dot{q}\dot{p}}\, e^{\alpha_i} v^+_{\alpha\dot{p}} + \frac{1}{2}K_i^{\#I} v^-_{\alpha pi}\gamma^I_{p\dot{q}} \,, \tag{7.8}$$

where the 'physical' degrees of freedom are carried by $SO(D-2)$ vector $K_i^{=I}$ parametrizing the celestial sphere $\mathbb{S}^{(D-2)}$ (through a kind of stereographic projection). Besides this, the r.h.s. of eqs. (7.7), (7.8) contain α_i, which is the parameter of $SO(1,1)$, $\mathcal{O}_{iqp}$ and $\mathcal{O}_{i\dot{q}\dot{p}}$, which are the $Spin(D-2)$ matrices,[11] and $K_i^{\#I}$, which parameterizes the $K_{(D-2)}$ symmetry transformations. All these transformations are used as identification relations on the set of spinor harmonic variables. This is tantamount to saying that they are the gauge symmetry

[11] Notice that for both D=10 and D=11 the $Spin(D-2)$ valued matrices $\mathcal{O}_{qpi}$ obey also $\mathcal{O}_{qp_1 i}\mathcal{O}_{pp_1 i} = \delta_{qp}$; this is to say $Spin(D-2) \subset SO(2\mathcal{N})$ for these cases.

of the spinor frame construction. We can fix their values arbitrarily thus providing an explicit parametrization of the coset (6.2). A particular choice $\alpha_i = 0 = K_i^{\#I}$, $\mathcal{O}_{iqp} = \delta_{qp}$ gives us the simple expressions (7.1) and (7.2).

In $D = 10$ case we have to complete (7.7), (7.8) with

$$v_{\dot{q}i}^{-\alpha} = e^{-\alpha_i}\mathcal{O}_{i\dot{q}\dot{p}}\left(v_{\dot{p}}^{-\alpha} - \frac{1}{2}K_i^{=I}v_p^{+\alpha}\gamma_{p\dot{p}}^I\right), \tag{7.9}$$

$$v_{qi}^{+\alpha} = e^{\,\alpha_i}\mathcal{O}_{iqp}v_p^{+\alpha} - \frac{1}{2}K_i^{\#I}\gamma_{q\dot{p}}^I v_{\dot{p}i}^{-\alpha}, \tag{7.10}$$

while in $D = 11$, where $\dot{q} = q$, these equations are equivalent to (7.7) and (7.8).

Eq. (7.7) and (7.8) imply

$$v_{\dot{p}i}^{-\alpha}v_{\alpha qj}^{-} = \frac{1}{2}e^{-\alpha_i-\alpha_j}K_{ji}^{=I}(\mathcal{O}_j\gamma^I\mathcal{O}_i^T)_{q\dot{p}}, \tag{7.11}$$

where

$$K_{ji}^{=I} = K_j^{=I} - K_i^{=I}. \tag{7.12}$$

In the gauge (7.1) this simplifies to

$$v_{\dot{p}i}^{-\alpha}v_{\alpha qj}^{-} = \frac{1}{2}K_{ji}^{=I}\gamma_{q\dot{p}}^I \tag{7.13}$$

and becomes antisymmetric in i, j. This latter fact suggests to search the 10D (and 11D) counterparts of the 4D expression $< ij >$ (1.17) on the basis of (7.13).

The complete parametrization of the vector frame variables corresponding to (7.7), (7.8) is given by

$$u_{ai}^{=} = e^{-2\alpha_i}\left(u_a^{=} + \frac{1}{4}(\vec{K}_i^{=})^2u_a^{\#} + K_i^{=I}u_a^I\right) \tag{7.14}$$

and quite complicated expressions for $u_{ai}^{\#}$ and u_{ai}^I. The light-like momentum of i-th particle has the form of (7.5), but with a redefined $\rho_i^{\#}$,

$$k_{ai} = \tilde{\rho}_i^{\#}\left(u_a^{=} + K_i^{=I}u_a^I + \frac{1}{4}(\vec{K}_i^{=})^2u_a^{\#}\right), \tag{7.15}$$

$$\tilde{\rho}_i^{\#} = e^{-2\alpha_i}\rho_i^{\#}. \tag{7.16}$$

The expressions for $u_{ai}^{\#}$ and u_{ai}^I simplify essentially if we use the $K_{(D-2)i}$ symmetry to fix the gauge

$$K_i^{\#I} = 0 \tag{7.17}$$

in which case

$$u_{ai}^{\#} = e^{+2\alpha_{(i)}}u_a^{\#}, \qquad u_{ai}^I = \mathcal{O}_i^{IJ}\left(u_a^J + \frac{1}{2}u_a^{\#}K_i^{=J}\right), \tag{7.18}$$

$$\gamma_{q\dot{p}}^I\mathcal{O}_i^{IJ} = \gamma_{p\dot{q}}^J\mathcal{O}_{pqi}\mathcal{O}_{\dot{q}\dot{p}i}. \tag{7.19}$$

The spinor frame parametrization with $K_i^{\#I} = 0$ (7.17) is given by the same eqs. (7.7) and (7.9), while eqs. (7.8) and (7.10) simplify essentially:

$$v_{\alpha\dot{q}i}^{+} = e^{\,\alpha_i}\mathcal{O}_{i\dot{q}\dot{p}}v_{\alpha\dot{p}}^{+}, \qquad v_{qi}^{+\alpha} = \mathcal{O}_{iqp}e^{\,\alpha_i}v_p^{+\alpha}. \tag{7.20}$$

7.3 Internal harmonics and reference internal frame

As $\mathrm{SO}(D-2)_i$ auxiliary gauge symmetry acts not only on i-th set of spinor harmonics but also on i-th set of internal harmonics $(\bar{w}_{qA\,i}, w_{q\,i}^{\;A})$, the introduction of the reference spinor frame in (7.7), (7.8) should be accompanied by the introduction of the reference 'internal frame'. This is described by the set of harmonic variables $(\bar{w}_{qA}, w_q{}^A)$ parametrizing the coset $\left(\frac{\mathrm{SO}(D-2)}{\mathrm{SO}(D-4)\otimes\mathrm{SO}(2)}\right)$ (reference coset). The i-th internal harmonic variables can be decomposed on this reference frame,

$$\bar{w}_{qA\,i} = \mathcal{O}_{qp\,i}\bar{w}_{pB}\, e^{-i\beta_i}\,\mathcal{U}^{\dagger\,B}_{A\,i}\,, \qquad w_{q\,i}^{\;A} = \mathcal{O}_{qp\,i}w_p{}^B\, e^{+i\beta_i}\mathcal{U}_{B\,i}^{A}\,, \tag{7.21}$$

$$\mathcal{U}_{B\,i}^{A} \in \mathrm{SO}(D-4) \subset \mathrm{SU}(\mathcal{N})\,. \tag{7.22}$$

In the case of D=10, we must also introduce the internal reference frame with c-spinor SO(8) indices $(\bar{w}_{\dot{q}A}, w_{\dot{q}}{}^A)$ and relate it to i-th internal harmonics by

$$\bar{w}_{\dot{q}A\,i} = \mathcal{O}_{\dot{q}\dot{p}\,i}\bar{w}_{\dot{p}B}\, e^{i\beta_i}\,\mathcal{U}^{\dagger\,B}_{A\,i}\,, \qquad w_{\dot{q}\,i}^{\;A} = \mathcal{O}_{\dot{q}\dot{p}\,i}w_{\dot{p}}{}^B\, e^{-i\beta_i}\mathcal{U}_{B\,i}^{A}\,. \tag{7.23}$$

The $\mathrm{Spin}(D-2)$ valued matrices $\mathcal{O}_{qp\,i}$ in (7.21) and $\mathcal{O}_{\dot{q}\dot{p}\,i}$ in (7.23) are bridges between $\mathrm{SO}(D-2)_i$ acting on i-th spinor frame and $\mathrm{SO}(D-2)$ acting on the reference spinor frame. In other words, the first index of $\mathcal{O}_{qp\,i}$ $(\mathcal{O}_{\dot{q}\dot{p}\,i})$ matrix is transformed by $\mathrm{SO}(D-2)_i$ and the second- by $\mathrm{SO}(D-2)$ group. One can also consider them as compensators for $\mathrm{SO}(D-2)_i$ auxiliary gauge symmetry.

Similarly, the unitary $\mathrm{Spin}(D-4)$ valued matrices $\mathcal{U}_{B\,i}^{A}$ (7.22) are bridges between $\mathrm{Spin}(D-4)_i$ and $\mathrm{Spin}(D-4) \subset \mathrm{SU}(\mathcal{N})$ groups, and the phase factor $e^{i\beta_i}$ serves as a bridge between $\mathrm{U}(1)_i$ and $\mathrm{U}(1)$ acting on the reference 'internal frame'. Notice the opposite phases $e^{-i\beta_i}$ and $e^{+i\beta_i}$ in the expressions for $\bar{w}_{qA\,i}$ and $\bar{w}_{\dot{q}A\,i}$ of 10D case. These are needed to make charged the complex null-vectors U_{Ii}, $\bar{U}_{Ii}$ which are related with reference internal vector frame by

$$U_{Ii} = e^{-2i\beta_i}U_J\mathcal{O}_i^{JI}\,, \qquad \bar{U}_{Ii} = e^{+2i\beta_i}\bar{U}_J\mathcal{O}_i^{JI}\,, \tag{7.24}$$

where $\mathcal{O}_i^{JI}$ is $\mathrm{SO}(D-2)$ matrix related to $\mathcal{O}_{qpi}$ and $\mathcal{O}_{\dot{p}\dot{q}i}$ from (7.21) and (7.23) by (7.19).

Generically $\mathcal{O}_{qp\,i}$ and $\mathcal{O}_{\dot{q}\dot{p}\,i}$ in (7.21) and (7.23) can be different from the matrices denoted by the same symbols in (7.7) and (7.8). We however impose the condition that they are the same.

Actually this implies that we do not have n independent sets of internal harmonics, but only one reference internal frame, and that the derivatives with respect to j-th internal harmonics does not live inert its i-th cousin, for example

$$D_j^{\check{J}}\bar{w}_{\dot{q}A\,i} = -\frac{i}{2}e^{\beta_{ij}}\mathcal{U}^{\dagger}{}_{Aij}^{\;B}\sigma^{\check{J}}_{BC}w_{\dot{q}j}^{\;C} \equiv e^{\beta_{ij}}\mathcal{U}^{\dagger}{}_{Aij}^{\;B}D_j^{\check{J}}\bar{w}_{\dot{q}B\,j}\,, \qquad \beta_{ij} := \beta_i - \beta_j\,, \tag{7.25}$$

but

$$D_j^{\check{J}}\bar{w}_{qA\,i} = 0\,, \qquad D_j^{\check{J}}w_{\dot{q}\,i}^{\;A} = 0\,, \tag{7.26}$$

and

$$D_{\check{j}}^{\check{J}} U_{I\,i} = 0\,. \tag{7.27}$$

This implies that the analytic superamplitude of D=10 SYM obeying (6.14) have to be constructed with the use of $\bar{w}_{qA\,i}, w_{\dot{q}\,i}^{A}$ and $U_{I\,i}$ variables only. Similarly, the analytic 11D superamplitudes (6.21) are constructed with the use of $\bar{w}_{qA\,i}$ and $U_{I\,i}$ variables.

To motivate this identification, let us recall that the only role of the internal harmonics is to split the real fermionic variables θ^{-}_{qi} on a pair of complex conjugate η^{-}_{Ai} and $\bar{\eta}^{-A}_{i}$ thus introducing a complex structure (see discussion in section 4.6). Our choice implies that we induce all the complex structures, for all $i = 1, \ldots, n$, from a single complex structure. This latter is introduced with the reference internal frame $(\bar{w}_{qA}, w_q^{A})$ which serves as a compensator for Spin$(D-2)$ gauge symmetry of the reference spinor frame.

7.4 Complex spinor frames and reference complex spinor frame

The identification of all the sets of internal harmonics through (7.21) and (7.23) automatically implies that the SO$(D-2)$ symmetry transformations of the reference spinor frame acts also on the reference internal frame. This allows to introduce a complex reference spinor frame (cf. (4.69))

$$v^{-}_{\alpha A} := v^{-}_{\alpha q}\bar{w}_{qA}\,, \quad \bar{v}^{-A}_{\alpha} := v^{-}_{\alpha p} w_p{}^{A}\,, \quad v^{+}_{\alpha A} := v^{+}_{\alpha\dot{p}}\bar{w}_{\dot{p}A}\,, \quad \bar{v}^{+A}_{\alpha} := v^{+}_{\alpha\dot{p}} w_{\dot{p}}{}^{A}\,, \tag{7.28}$$

$$v_A^{-\alpha} := v_{\dot{q}}^{-\alpha}\bar{w}_{\dot{q}A}, \quad \bar{v}^{-A\alpha} := v_{\dot{q}}^{-\alpha} w_{\dot{q}}{}^{A}, \quad v_A^{+\alpha} := v_q^{+\alpha}\bar{w}_{qA}, \quad \bar{v}^{+A\alpha} := v_q^{+\alpha} w_q{}^{A} \tag{7.29}$$

and to express the complex spinor harmonics

$$v^{-}_{\alpha A\,i} := v^{-}_{\alpha q\,i}\bar{w}_{qA\,i}\,, \quad \bar{v}^{-A}_{\alpha i} := v^{-}_{\alpha qi} w_{qi}{}^{A}\,, \quad v^{+}_{\alpha Ai} := v^{+}_{\alpha\dot{q}i}\bar{w}_{\dot{q}Ai}\,, \quad \bar{v}^{+A}_{\alpha i} := v^{+}_{\alpha\dot{q}i} w_{\dot{q}i}^{A}\,, \tag{7.30}$$

$$v_{Ai}^{-\alpha} := v_{\dot{q}i}^{-\alpha}\bar{w}_{\dot{q}Ai}\,, \quad \bar{v}_i^{-A\alpha} := v_{\dot{q}\,i}^{-\alpha} w_{\dot{q}i}^{A}\,, \quad v_{Ai}^{+\alpha} := v_{qi}^{+\alpha}\bar{w}_{qiA}\,, \quad \bar{v}_i^{+\alpha A} := v_{qi}^{+\alpha} w_{qi}^{A} \tag{7.31}$$

in terms of that.

In particular, one finds

$$v^{-}_{\alpha Ai} = e^{-\alpha_i - i\beta_i}\mathcal{U}^{\dagger\,B}_{Ai}\left(v^{-}_{\alpha B} + \frac{1}{2}K_i^{=I}U_I v^{+}_{\alpha B} + \frac{i}{2}K_i^{=I}U_I^{\check{J}}\sigma^{\check{J}}_{BC}\bar{v}^{+C}_{\alpha}\right)\,, \tag{7.32}$$

$$\bar{v}_i^{-\alpha A} = e^{-\alpha_i - i\beta_i}\left(\bar{v}^{-\alpha B} - \frac{1}{2}K_i^{=I}U_I\bar{v}^{+\alpha B} - \frac{i}{2}K_i^{=I}U_I^{\check{J}} v_C^{+\alpha}\tilde{\sigma}^{\check{J}CB}\right)\mathcal{U}^{A}_{Bi}\,, \tag{7.33}$$

and

$$\bar{v}^{-A}_{\alpha i} = e^{-\alpha_i + i\beta_i}\left(v_\alpha^{-B} + \frac{1}{2}K_i^{=I}\bar{U}_I v_\alpha^{+B} + \frac{i}{2}K_i^{=I}U_I^{\check{J}}\tilde{\sigma}^{\check{J}BC}\bar{v}^{+}_{\alpha C}\right)\mathcal{U}^{A}_{Bi}\,, \tag{7.34}$$

$$v_{Ai}^{-\alpha} = e^{-\alpha_i + i\beta_i}\mathcal{U}^{\dagger\,B}_{Ai}\left(v_B^{-\alpha} - \frac{1}{2}K_i^{=I}\bar{U}_I v_B^{+\alpha} + \frac{i}{2}K_i^{=I}U_I^{\check{J}}\sigma^{\check{J}}_{BC}\bar{v}^{+C\alpha}\right)\,. \tag{7.35}$$

The complex spinor harmonics (7.28) and (7.29) obey

$$\begin{aligned} &v_A^{-\alpha}v_\alpha^{-B} = 0\,, \qquad v_A^{+\alpha}v_\alpha^{+B} = 0, \qquad v^{\pm A\alpha}v_\alpha^{\pm B} = 0\,, \qquad v_A^{\pm\alpha}v^{\pm}_{\alpha B} = 0\,, \\ &v_A^{+\alpha}v_\alpha^{-B} = \delta_A{}^{B}\,, \qquad v_A^{-\alpha}v_\alpha^{+B} = \delta_A{}^{B}\,. \end{aligned} \tag{7.36}$$

The product of harmonics from different frames, say i-th and j-th, can be calculated using (7.32), (7.34) and (7.36). Clearly for $j = i$ the relation of the form of (7.36) are reproduced for i-th set of complex spinor harmonics.

In particular, eqs. (7.32) and (7.33) imply

$$\begin{aligned} < i^{-B} j^{-}_{A} >:= \bar{v}_i^{-\alpha B} v^{-}_{\alpha Aj} &= \mathcal{U}_{Ei}^{B} e^{-\alpha_i - i\beta_i} \frac{1}{2} K^{=I}_{\bar{j}i} U_I \, \mathcal{U}^{\dagger}{}_{Aj}^{E} e^{-\alpha_j - i\beta_j} \\ &= \frac{1}{2} K^{=I}_{\bar{j}i} U_I \, \mathcal{U}^{\dagger}{}_{Aji}^{B} e^{-\alpha_i - \alpha_j - i\beta_i - i\beta_j} \, . \end{aligned} \tag{7.37}$$

The expression in the first line of (7.37) is convenient to calculate products of bracket matrices, while the second is more compact due to the use of notation

$$\mathcal{U}^{\dagger}{}_{Aji}^{B} := \mathcal{U}^{\dagger}{}_{Aj}^{C} \, \mathcal{U}_{Ci}^{B} = \mathcal{U}_{Aij}^{B} \, . \tag{7.38}$$

When deriving eqs. (7.32)–(7.37) the following consequences of (4.21), (4.24) and (4.25) are useful

$$\bar{w}_{qA} \gamma^I_{q\dot{p}} = \bar{w}_{\dot{p}A} U_I + i \sigma^{\check{J}}_{AB} w^B_{\dot{p}} U^{\check{J}}_I \, , \qquad w^A_q \gamma^I_{q\dot{p}} = w^A_{\dot{p}} \bar{U}_I + i \tilde{\sigma}^{\check{J}AB} \bar{w}_{\dot{p}B} U^{\check{J}}_I \, , \tag{7.39}$$

$$\gamma^I_{q\dot{p}} w_{\dot{p}}{}^A = U_I w_q{}^A + i \bar{w}_{qB} \tilde{\sigma}^{\check{J}BA} U^{\check{J}}_I \, , \qquad \gamma^I_{q\dot{p}} \bar{w}_{\dot{p}A} = \bar{w}_{qA} \bar{U}_I + i w_q{}^B \sigma^{\check{J}}_{BA} U^{\check{J}}_I \, . \tag{7.40}$$

One can also calculate the expressions for $< i^{-}_{B} j^{-A} >:= v^{-\alpha}_{Bi} \bar{v}^{-A}_{\alpha j}$. However, in our perspective the contraction (7.37) is much more interesting as far as it obeys

$$D^{\check{J}}_l < i^{-B} j^{-}_{A} >= 0 \, , \qquad \forall l = 1, \ldots, n \, . \tag{7.41}$$

The expressions for complimentary harmonics in terms of complex reference spinor frame simplify essentially in the gauge (7.17), $K^{\#I}_i = 0$, where (7.20) and (7.21), (7.23) result in

$$v^{+}_{\alpha Ai} = \mathcal{U}^{\dagger}{}_{Ai}^{B} v^{+}_{\alpha B} e^{\alpha_i + i\beta_i} \, , \qquad \bar{v}^{+A\alpha}_i = \bar{v}^{+B\alpha} \mathcal{U}^{A}_{Bi} e^{\alpha_i + i\beta_i} \, , \tag{7.42}$$

and

$$\bar{v}^{+A}_{\alpha i} = \bar{v}^{+B}_{\alpha} \mathcal{U}^{A}_{Bi} e^{\alpha_i - i\beta_i} \, , \qquad v^{+\alpha}_{iA} = \mathcal{U}^{\dagger}{}_{Ai}^{B} v^{+\alpha}_{B} e^{\alpha_i - i\beta_i} \, . \tag{7.43}$$

This allows to find

$$< i^{+B} j^{-}_{A} >:= \bar{v}_i^{+\alpha B} v^{-}_{\alpha Aj} = \mathcal{U}^{\dagger}{}_{Aji}^{B} e^{-\alpha_{ji} - i\beta_{ji}} \, , \tag{7.44}$$

$$< i^{-B} j^{+}_{A} >:= \bar{v}_i^{-\alpha B} v^{+}_{\alpha Aj} = \mathcal{U}^{\dagger}{}_{Aji}^{B} e^{+\alpha_{ji} + i\beta_{ji}} = \mathcal{U}^{B}_{Aij} e^{-\alpha_{ij} - i\beta_{ij}} \, . \tag{7.45}$$

Let us stress that these are gauge fixed expressions: when $K^{\#I} \neq 0$ the r.h.s.-s will acquire the contributions proportional to (7.11).

Using the bridges e^{α_i}, $e^{i\beta_i}$ and $\mathcal{U}_{Ai}^{B}$ we can transform the complex fermionic variable η^{-}_{Ai} carrying SU(4)$_i$ index, U(1)$_i$ charge and SO(1,1)$_i$ weight to

$$\tilde{\eta}^{-}_{Ai} := e^{\alpha_i + i\beta_i} \mathcal{U}_{Ai}^{B} \, \eta^{-}_{Bi} \, , \tag{7.46}$$

which is inert under $\mathrm{SU}(4)_i \otimes \mathrm{U}(1)_i \otimes \mathrm{SO}(1,1)_i$ but transforms nontrivially under the gauge symmetry $\mathrm{SU}(4)\otimes\mathrm{U}(1)\otimes\mathrm{SO}(1,1)$ of the reference complex spinor frame. The advantage of such variables is that $\tilde{\eta}^-_{Ai} + \tilde{\eta}^-_{Aj}$ is covariant for any values of i and j. The expressions in the r.h.s.-s of eqs. (7.44) and (7.45), as well as of their counterparts with $j=0$, corresponding to the reference complex spinor frame,

$$< i^{+B}\, {}^{-}_{\cdot A} > := \bar{v}_i^{+\alpha B} v^-_{\alpha A} = \mathcal{U}_{Ai}^{B} e^{\alpha_i + i\beta_i} , \qquad < i^{-B}\, {}^{+}_{\cdot A} > := \bar{v}_i^{-\alpha B} v^+_{\alpha A} = \mathcal{U}_{Ai}^{B} e^{-\alpha_i - i\beta_i} , \tag{7.47}$$

$$< .^{+B} j^-_A > := \bar{v}^{+\alpha B} v^-_{\alpha Aj} = \mathcal{U}^{\dagger B}_{Aj} e^{-\alpha_j - i\beta_j} , \quad < .^{-B} j^+_A > := \bar{v}^{-\alpha B} v^+_{\alpha Aj} = \mathcal{U}^{\dagger B}_{Aj} e^{+\alpha_j + i\beta_j} , \tag{7.48}$$

can be used as covariant counterparts of the above $\mathrm{SU}(4)_i \otimes \mathrm{U}(1)_i \otimes \mathrm{SO}(1,1)_i$ bridges. They will be useful below in the discussion on 3-point superamplitude of 10D SYM.

In particular, it will be important that

$$< i^{+B}\, {}^{-}_{\cdot C} > < .^{-C} i^+_A > = \delta_A{}^B e^{2\alpha_i + 2i\beta_i} , \qquad < .^{-B} i^+_C > < i^{+C}\, {}^{-}_{\cdot A} > = \delta_A{}^B e^{+2\alpha_i + 2i\beta_i} , \tag{7.49}$$

$$< i^{-B}\, {}^{+}_{\cdot C} > < .^{+C} i^-_A > = \delta_A{}^B e^{-2\alpha_i - 2i\beta_i} , \qquad < .^{+B} i^-_C > < i^{-C}\, {}^{+}_{\cdot A} > = \delta_A{}^B e^{-2\alpha_i - 2i\beta_i} , \tag{7.50}$$

and

$$\det < i^{-B}\, {}^{+}_{\cdot A} > = e^{-4\alpha_i - 4i\beta_i} , \qquad \det < i^{+B}\, {}^{-}_{\cdot A} > = e^{4\alpha_i + 4i\beta_i} , \tag{7.51}$$

represent the scale and phase factors corresponding to i-th particle.

8 3-point analytic superamplitudes in 10D and 11D

8.1 Three particle kinematics and supermomentum

Let us study 3-particle kinematics in the vector frame formalism. With (2.2) we can write the momentum conservation as

$$\rho_1^{\#} u_1^{=a} + \rho_2^{\#} u_2^{=a} + \rho_3^{\#} u_3^{=a} = 0 . \tag{8.1}$$

Then, using (7.2)–(7.4) we split (8.1) into

$$\rho_1^{\#} + \rho_2^{\#} + \rho_3^{\#} = 0 , \tag{8.2}$$

$$\rho_1^{\#} K_1^{=I} + \rho_2^{\#} K_2^{=I} + \rho_3^{\#} K_3^{=I} = 0 , \tag{8.3}$$

$$\rho_1^{\#} (\vec{K}_1)^2 + \rho_2^{\#} (\vec{K}_2)^2 + \rho_3^{\#} (\vec{K}_3)^2 = 0 . \tag{8.4}$$

Eq. (8.2) makes (8.3) equivalent to

$$\frac{K_{32}^{=I}}{\rho_1^{\#}} = \frac{K_{13}^{=I}}{\rho_2^{\#}} = \frac{K_{21}^{=I}}{\rho_3^{\#}} , \tag{8.5}$$

where $K_{ji}^{=I} \equiv K_{[ji]}^{=I} = K_j^{=I} - K_i^{=I}$ (7.12). Using (8.5) and (8.2) we find that (8.4) implies

$$(\vec{K}_{21})^2 = 0 \qquad \Rightarrow \qquad (\vec{K}_{13})^2 = 0 , \qquad (\vec{K}_{32})^2 = 0 . \tag{8.6}$$

The solution of eqs. (8.6) for real vectors $K^{=I}_{ji}$ are trivial. Thus a nontrivial on-shell 3-particle amplitude can be defined only for complexified $K^{=I}_{ji}$ which implies that the light-like momenta k^a_i of the scattered particles are complex.

The general solution of the momentum conservation conditions can be written in terms of say $K^{=I}_1$ and complex null vector $\mathcal{K}^I$ as

$$K^{=I}_2 = K^{=I}_1 + \mathcal{K}^I, \qquad K^{=I}_3 = K^{=I}_1 + \mathcal{K}^I \frac{\rho^{\#}_2}{\rho^{\#}_1 + \rho^{\#}_2}, \qquad \mathcal{K}^I\mathcal{K}^I = 0. \tag{8.7}$$

Notice that, to make the above equations valid for arbitrary parametrization (7.14), it is sufficient just to rescale the scalar densities as in (7.16)

$$\rho^{\#}_i \longrightarrow \tilde{\rho}^{\#}_i = e^{-2\alpha_i}\rho^{\#}_i \,. \tag{8.8}$$

In particular,

$$\frac{K^{=I}_{32}}{\tilde{\rho}^{\#}_1} = \frac{K^{=I}_{21}}{\tilde{\rho}^{\#}_3} = \frac{K^{=I}_{13}}{\tilde{\rho}^{\#}_2} =: \mathcal{K}^{==I}\,, \qquad \mathcal{K}^{==I}\mathcal{K}^{==I} = 0 \tag{8.9}$$

are valid for a generic parametrization of the spinor harmonics.

8.2 3-points analytical superamplitudes in 10D SYM and 11D SUGRA

A suggestion about the structure of 10D and 11D tree superamplitudes may be gained from the observation that, when the external momenta belong to a 4d subspace of the D-dimensional space, they should reproduce the known answer for 4-dimensional tree superamplitudes of $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA, respectively. Due to the momentum conservation, this is always the case for a three point amplitude and superamplitude.

In this section we find the gauge fixed form of the 10D 3-point superamplitude in the gauge (7.17) and also present its 11D cousin. We also describe the first stages in search for covariant form of the three point superamplitudes, which, although have not allowed to succeed yet, might be suggestive for further study.

8.2.1 3-points analytical superamplitude of 10D SYM. Gauge fixed form

We chose as D=4 reference point the anti-MHV superamplitude of $\mathcal{N}=4$ SYM (1.15). As we show in appendix A, using an explicit parametrization of 4D helicity spinors in terms of reference spinor frame we can write it in the following form (see (A.38))

$$\begin{aligned}\mathcal{A}^{\overline{\rm MHV}}(1,2,3) &= \mathbb{K}^{==}\, e^{-2i(\beta_1+\beta_2+\beta_3)}\,\delta^4\left(\tilde{\rho}^{\#}_1\tilde{\eta}^-_{1A} + \tilde{\rho}^{\#}_2\tilde{\eta}^-_{2A} + \tilde{\rho}^{\#}_3\tilde{\eta}^-_{3A}\right)\\ &= -\frac{\mathbb{K}^{=}_{21}}{(\tilde{\rho}^{\#}_1 + \tilde{\rho}^{\#}_2)} e^{-2i(\beta_1+\beta_2+\beta_3)}\,\delta^4\left(\tilde{\rho}^{\#}_1\tilde{\eta}^-_{[13]A} + \tilde{\rho}^{\#}_2\tilde{\eta}^-_{[23]A}\right).\end{aligned} \tag{8.10}$$

Here $\tilde{\eta}^-_{Ai} = \eta_{Ai}/\sqrt{\tilde{\rho}^{\#}_i}$ and $\tilde{\rho}^{\#}_i$ are D=4 counterparts of the rescaled 10D variables (7.46) and (8.8) (see (A.35) and (A.25) in appendix A), $\mathbb{K}^{==} = \mathbb{K}^{=}_{21}/\rho^{\#}_3$ (see (A.27)) and $\mathbb{K}^{=}_{21}$ is a complex number, which can be associated through $\mathbb{K}^{=}_{21} = K^{=1}_{21} + iK^{=2}_{21}$ with a real

2-component vector $(K_{21}^{=1}, K_{21}^{=2})$, the D=4 counterpart of the generic $(D-2)$-vector $K_{21}^{=I}$ in (7.12). It is tempting to identify $\mathbb{K}_{21}^{=}$ with $K_{21}^{=I} U_I$ of the previous section:

$$^{D=4}\ \mathbb{K}_{21}^{=} \quad \longleftrightarrow \quad K_{21}^{=I} U_I \ ^{D=10\ (\text{and } D=11)} . \tag{8.11}$$

The argument of the fermionic delta function in (8.10) also has the straightforward 10D counterpart $\tilde{\rho}_1^{\#} \tilde{\eta}_{A1}^{-} + \tilde{\rho}_2^{\#} \tilde{\eta}_{A2}^{-} + \tilde{\rho}_3^{\#} \tilde{\eta}_{A3}^{-}$ where $\tilde{\eta}_{Ai}^{-}$ and $\tilde{\rho}_i^{\#}$ are defined in (7.46) and (8.8) in such a way that all of them carry indices, charges and weights with respect to the same $\mathrm{SO}(1,1) \otimes \mathrm{SU}(4) \otimes \mathrm{U}(1)$ acting on the reference complex frame,

$$\tilde{\eta}_{Ai}^{-} := e^{\alpha_i + i\beta_i} \mathcal{U}_{Ai}^{\ B}\, \eta_{Bi}^{-} , \qquad \tilde{\rho}_i^{\#} = e^{-2\alpha_i} \rho_i^{\#} . \tag{8.12}$$

Thus, the straightforward generalization of (8.10) to the case of $D = 10$ SYM theory reads

$$\begin{aligned} \mathcal{A}_3^{D=10\ \text{SYM}} &= -\frac{K_{21}^{=I} U_I}{2(\tilde{\rho}_1^{\#} + \tilde{\rho}_2^{\#})}\, e^{-2i(\beta_1+\beta_2+\beta_3)}\, \delta^4 \left(\tilde{\rho}_1^{\#} \tilde{\eta}_{[13]A}^{-} + \tilde{\rho}_2^{\#} \tilde{\eta}_{[23]A}^{-} \right) \\ &= \frac{1}{2} \mathcal{K}^{==I} U_I\, e^{-2i(\beta_1+\beta_2+\beta_3)}\, \delta^4 \left(\tilde{\rho}_1^{\#} \tilde{\eta}_{1A}^{-} + \tilde{\rho}_2^{\#} \tilde{\eta}_{2A}^{-} + \tilde{\rho}_3^{\#} \tilde{\eta}_{3A}^{-} \right) , \end{aligned} \tag{8.13}$$

where the complex null-vector $\mathcal{K}^{==I}$ is defined in (8.9).

The multiplier $e^{-2i(\beta_1+\beta_2+\beta_3)}$ makes the superamplitude invariant under U(1) symmetry acting on the reference internal frame variables and supplies it instead with charges +1 with respect to all $\mathrm{U}(1)_i$ groups, $i = 1, 2, 3$, related to scattered particles. All other variables in (8.13) are redefined in such a way that they are inert under $\prod_i^3 [\mathrm{SO}(D-2)_i \otimes \mathrm{SO}(1,1)_i \otimes \mathrm{U}(1)_i \otimes \mathrm{SO}(D-4)_i]$ and are transformed only by $\mathrm{SO}(D-2) \otimes \mathrm{SO}(1,1) \otimes \mathrm{U}(1) \otimes \mathrm{SO}(D-4)$ acting on the reference spinor frame and reference internal frame.

8.2.2 Searching for a gauge covariant form of the 3-points superamplitude

Let us try to search for a covariant expression for amplitude which, upon gauge fixing, reproduce (8.13). Again, a guideline can be found in 4D expression (1.15). Counterparts of $< ij >$ blocks are given by the matrices $\sqrt{\rho_i^{\#} \rho_j^{\#}} < i^{-B} j_A^{-} >$ with $< i^{-B} j_A^{-} >$ defined in (7.37) so that a possible 10D cousin of the denominator in (1.15) is given by the trace of the product of three such matrices,

$$\begin{aligned} &\rho_1^{\#} \rho_2^{\#} \rho_3^{\#} < 1^{-A} 2_B^{-} >< 2^{-B} 3_C^{-} >< 3^{-C} 1_A^{-} >= \\ &\quad = \frac{1}{2^3} \tilde{\rho}_1^{\#} \tilde{\rho}_2^{\#} \tilde{\rho}_3^{\#}\ K_{21}^{=I} U_I\, K_{32}^{=J} U_J\, K_{21}^{=K} U_K\, e^{-2i(\beta_1+\beta_2+\beta_3)} \\ &\quad = \frac{1}{2^3} (\tilde{\rho}_1^{\#} \tilde{\rho}_2^{\#} \tilde{\rho}_3^{\#})^2\, (\mathcal{K}^{==I} U_I)^3\, e^{-2i(\beta_1+\beta_2+\beta_3)} . \end{aligned} \tag{8.14}$$

The next problem is to search for a counterpart of $\eta_{A1} < 23 >$ expression in the argument of fermionic delta function in (1.15). Here the straightforward generalization $\propto \eta_{B1}^{-} < 2^{-B} 3_A^{-} >$ does not work: it is not covariant under $\mathrm{SU}(4)_1$ and $\mathrm{SU}(4)_2$. The covariance may be restored by using the matrices (7.45): the matrix

$$\eta_{C1}^{-} < 1^{-C} 2_B^{+} >< 2^{-B} 3_A^{-} >= \tilde{\eta}_{B1}^{-} e^{-2\alpha_1 - 2i\beta_1} \mathcal{U}_{A3}^{\dagger\, B} e^{-\alpha_3 - i\beta_3} \tag{8.15}$$

is transformed in $(1,1,4)$ of $SU(4)_1 \otimes SU(4)_2 \otimes SU(4)_3$. However its nontrivial weights $(-2,0,-1)$ and charges $(+1,0,+1/2)$, indicated by multipliers $e^{-2\alpha_1 - 2i\beta_1}$ and $e^{-\alpha_3 - i\beta_3}$, do not allow to sum it with its η_2^- and η_3^- counterparts without breaking the gauge symmetries. To compensate the above multipliers one can use the matrices (7.49). In such a way we arrive at the expression

$$
\begin{aligned}
\ll \eta_1^- 2^- 3^- \gg_A := \eta^-_{B1} < 1^{-B} 2^+_C >< 2^{-C} 3^-_D >< 3^{+D} {}_{\cdot E}^{-} >< {}_{\cdot}^{-E} 1^+_F >< 1^{+F} {}_{\cdot A}^{-} > = \\
= \frac{1}{2} \tilde{\eta}^-_{iA} K_{32}^{=I} U_I = \frac{1}{2} \tilde{\rho}_1^{\#} \tilde{\eta}_i^- \mathcal{K}^{==I} U_I \, ,
\end{aligned} \tag{8.16}
$$

which is invariant under $\prod_{i=1}^{3} SU(4)_i \otimes U(1)_i \otimes SO(1,1)_i$ and carries the nontrivial representations of only $SU(4) \otimes U(1) \otimes SO(1,1)$ group acting on the reference complex spinor frame. Then, as $\ll \eta_i^- j^- k^- \gg_A$ with i,j,k given by arbitrary permutation of 123 has the same transformation properties, we can sum them and write the 10D counterpart of the fermionic delta function in (1.15),

$$
\begin{aligned}
&\delta^4 \left((\rho_1^{\#} \rho_2^{\#} \rho_3^{\#})^{1/2} \left(\ll \eta_1^- 2^- 3^- \gg_A + \ll \eta_2^- 3^- 1^- \gg_A + \ll \eta_3^- 1^- 2^- \gg_A \right) \right) = \\
&= (\rho_1^{\#} \rho_2^{\#} \rho_3^{\#})^2 \delta^4 \left(\ll \eta_1^- 2^- 3^- \gg_A + \ll \eta_2^- 3^- 1^- \gg_A + \ll \eta_3^- 1^- 2^- \gg_A \right) = \\
&= \frac{(\mathcal{K}^{==I} U_I)^4}{2^4} (\rho_1^{\#} \rho_2^{\#} \rho_3^{\#})^2 \, \delta^4 \left(\tilde{\rho}_1^{\#} \tilde{\eta}_{1A}^- + \tilde{\rho}_2^{\#} \tilde{\eta}_{2A}^- + \tilde{\rho}_3^{\#} \tilde{\eta}_{3A}^- \right) = \\
&= \frac{(\mathcal{K}^{==I} U_I)^4}{2^4} (\tilde{\rho}_1^{\#} \tilde{\rho}_2^{\#} \tilde{\rho}_3^{\#})^2 \, \frac{e^{-4i(\beta_1 + \beta_2 + \beta_3)}}{\prod_{i=1}^3 \det < i^{-B} {}_{\cdot A}^{+} >} \, \delta^4 \left(\tilde{\rho}_1^{\#} \tilde{\eta}_{1A}^- + \tilde{\rho}_2^{\#} \tilde{\eta}_{2A}^- + \tilde{\rho}_3^{\#} \tilde{\eta}_{3A}^- \right) ,
\end{aligned} \tag{8.17}
$$

where in the last lines we have used (8.16) and (7.51).

Then, the covariant candidate amplitude is given by (8.17) divided by the product of (8.14) and multiplied by three determinants (7.51),

$$
\begin{aligned}
&\mathcal{A}_3^{D=10\,\mathrm{SYM}} \stackrel{?}{=} \\
&\frac{\delta^4 \left((\rho_1^{\#} \rho_2^{\#} \rho_3^{\#})^{1/2} \left(\ll \eta_1^- 2^- 3^- \gg_A + \ll \eta_2^- 3^- 1^- \gg_A + \ll \eta_3^- 1^- 2^- \gg_A \right) \right)}{\rho_1^{\#} \rho_2^{\#} \rho_3^{\#} < 1^{-A} 2^-_B >< 2^{-B} 3^-_C >< 3^{-C} 1^-_A >} \prod_{i=1}^{3} \det < i^{-B} {}_{\cdot A}^{+} > \\
&\qquad = \rho_1^{\#} \rho_2^{\#} \rho_3^{\#} \, \delta^4 \left(\ll \eta_1^- 2^- 3^- \gg_A + \ll \eta_2^- 3^- 1^- \gg_A + \ll \eta_3^- 1^- 2^- \gg_A \right) \times \\
&\qquad \times \frac{\det < 1^{-B} {}_{\cdot A}^{+} > \det < 2^{-C} {}_{\cdot D}^{+} > \det < 3^{-E} {}_{\cdot F}^{+} >}{< 1^{-A} 2^-_B >< 2^{-B} 3^-_C >< 3^{-C} 1^-_A >} \, .
\end{aligned} \tag{8.18}
$$

One can easily check that in the gauge (7.17) (see (7.9), (7.10) with explicit parametrization (7.32), (7.33), (7.42)) this expressions reduces to (8.13).

However, the main problem of the above covariant expression (besides that it depends explicitly on reference complex spinor frame) is that apparently it does not obey (6.14),

$$
D_j^{\check{J}} \mathcal{A}_3^{D=10\,\mathrm{SYM\ of\ eq.\ (8.18)}} (\{ \rho_i^{\#}, v_{\alpha q i}^-; w_i, \bar{w}_i; \eta_{Ai} \}) \neq 0 \, . \tag{8.19}
$$

Indeed, it is constructed with the use of blocks (7.45) and (7.47) and, if we consider the complex spinor frames as composed from spinor and internal harmonics as in (7.30) and

JHEP05(2018)103

use (7.25), we find, for instance,

$$D^{\check{J}} < i^{-B} j_A^{+} > = -\frac{i}{2} e^{2i\beta_j} (\mathcal{U}_j^{\dagger} \sigma^{\check{J}} \mathcal{U}_j^{\dagger T})_{AC} < i^{-B} j_C^{+} > \neq 0\,,$$
$$D^{\check{J}} < i^{-B} \cdot_A^{+} > = -\frac{i}{2} \sigma^{\check{J}}_{AC} < i^{-B} \cdot_C^{+} > \neq 0\,, \tag{8.20}$$

Thus we should find either a different covariant representation for the gauge fixed amplitude (8.13), or a way to relax/to modify the condition (6.14) for the analytic superamplitudes.

Alternatively, one can use the gauge fixed form of the 3-point superamplitude as a basis of a gauge fixed superamplitude formalism. In it all the K_{8i} symmetries acting on i-th spinor frame variables are gauge fixed by the conditions (7.17). This gauge fixing is performed with respect to a symmetry acting on the auxiliary variables, complementary spinor harmonics. The gauge fixed expressions (7.42) and (7.43), as well as the expressions for the physically relevant spinor harmonics (7.30)–(7.29), use the reference spinor frame. This makes our gauge fixed superamplitude formalism manifestly Lorentz covariant. Such a role of reference spinor frame is in consonance with the original idea of introducing Lorentz harmonics to covariantize the light-cone gauge [51].

8.2.3 Analytical 3-point superamplitude of D = 11 supergravity

Similarly, the form of 3-point $\mathcal{N}=8$ 4D supergravity superamplitude, which is essentially the square of the $\mathcal{N}=4$ 4D SYM one (see e.g. [11, 12]), suggests the following gauge fixed expression for the basic 3-point superamplitude of 11D supergravity,

$$\mathcal{A}_3^{D=11\ \text{SUGRA}} = \left(\frac{K_{21}^{=I} U_I}{2(\tilde{\rho}_1^{\#} + \tilde{\rho}_2^{\#})} \right)^2 e^{-4i(\beta_1+\beta_2+\beta_3)}\, \delta^8 \left(\tilde{\rho}_1^{\#} \tilde{\eta}^{-}_{[13]A} + \tilde{\rho}_2^{\#} \tilde{\eta}^{-}_{[23]A} \right)$$
$$= \left(\frac{1}{2} \mathcal{K}^{==I} U_I \right)^2 e^{-2i(\beta_1+\beta_2+\beta_3)}\, \delta^8 \left(\tilde{\rho}_1^{\#} \tilde{\eta}^{-}_{1A} + \tilde{\rho}_2^{\#} \tilde{\eta}^{-}_{2A} + \tilde{\rho}_3^{\#} \tilde{\eta}^{-}_{3A} \right)\,, \tag{8.21}$$

Eq. (8.21) can be obtained by gauge fixing from

$$(\rho_1^{\#} \rho_2^{\#} \rho_3^{\#})^2\, \delta^8 \left(\ll \eta_1^{-} 2^{-} 3^{-} \gg_A + \ll \eta_2^{-} 3^{-} 1^{-} \gg_A + \ll \eta_3^{-} 1^{-} 2^{-} \gg_A \right) \times$$
$$\times \left(\frac{\det < 1^{-B} \cdot_A^{+} > \ \det < 2^{-C} \cdot_D^{+} > \ \det < 3^{-E} \cdot_F^{+} >}{< 1^{-A} 2_B^{-} >< 2^{-B} 3_C^{-} >< 3^{-C} 1_A^{-} >} \right)^2 . \tag{8.22}$$

However, as (8.18) in the case of 10D SYM, this expression does not obey eq. (6.14), so that we should find either the reason to relax/to modify these equations, or to search for a different covariant expression reproducing (8.21) upon gauge fixing.

Another interesting possibility is to use the gauge fixed spinor frame variables, obeying (7.17) for all sets of spinor harmonics. As we have already stressed above, in distinction with light-cone gauge, such a gauge fixed superamplitude formalism possesses manifest Lorentz invariance and supersymmetry. This possibility is also under study now.

9 Conclusion and discussion

In this paper we have constructed the basis of the analytic superfield formalism to calculate (super)amplitudes of 10D SYM and 11D SUGRA theories. This is alternative to the constrained superamplitude formalism of [15, 16] and also to the 'Clifford superfield' approach of [14]. The fact that it has more similarities with D=4 superamplitude calculus of $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA promises to allow us to use more efficiently the D=4 suggestions for its further development. In particular, such a suggestion was used to find the gauge fixed form of the 3-point analytic superamplitude of 10D SYM and 11D SUGRA, eqs. (8.13) and (8.21).

We have begun by solving the equations of the constrained on-shell superfield formalism of 10D SYM and 11D SUGRA [15, 16, 21] in terms of single analytic superfield depending holomorphically on $\mathcal{N}=4$ and $\mathcal{N}=8$ complex coordinates, respectively. These $\mathcal{N}$ complex coordinates η^-_A are related to $2\mathcal{N}$ real fermionic coordinates θ^-_q of the constrained superfield formalism by complex rectangular matrix $\bar{w}_{qA}$ $(=(w^A_q)^*)$. This and its conjugate $w^A_q = (\bar{w}_{qA})^*$ obey some constraints which allow us to consider them as homogeneous coordinates of the coset $\frac{\mathrm{Spin}(D-2)}{\mathrm{Spin}(D-4)\otimes \mathrm{U}(1)}$ and to call them internal harmonic variables.

Similarly, the constrained n-point superamplitudes of the $\prod_{i=1}^{n} \mathrm{SO}(D-2)_i$ covariant constrained superfield formalism can be expressed in terms of analytic superamplitudes which depend, besides the n sets of 10D or 11D spinor helicity variables, also on n sets $(\bar{w}_{qA\,i}, w^A_{q\,i})$ of $\frac{\mathrm{Spin}(D-2)_i}{\mathrm{Spin}(D-4)_i\otimes \mathrm{Spin}(2)_i}$ internal harmonic variables. The sets of 10D and 11D spinor helicity variables include Lorentz harmonics or spinor frame variables $v^-_{\alpha q\, i}$ which, after the constraints and gauge symmetries are taken into account, parametrize the celestial sphere $\mathbb{S}^{(D-2)}$. Together with scalar densities $\rho^{\#}_i$, they describe the light-like momenta and the "polarizations" ($\mathrm{SO}(D-2)_i$ small group representations) of the scattered particles. The constrained superamplitudes, which depend on these spinor helicity variables and $(2\mathcal{N})$-component real fermionic variables θ^-_{qi}, carry indices of the small groups $\mathrm{SO}(D-2)_i$. In contrast, the analytic superamplitudes do not carry indices but only charges $s=\mathcal{N}/4$ of $\mathrm{U}(1)_i$ which act on the internal frame variables $(\bar{w}_{qA\,i}, w^A_{q\,i})$ and on the complex fermionic $\eta^-_{Ai} = \theta^-_{qi}\bar{w}_{qA\,i}$. They may be constructed from the basic constrained superamplitudes by contracting their $\mathrm{SO}(D-2)_i$ vector indices with complex null vectors U_{Ii} constructed from bilinear combinations of $(\bar{w}_i, w_i)$.

The dependence of the analytic superamplitudes on internal harmonics is restricted by the equations in terms of harmonic covariant derivatives which reflect the fact that the original constrained superamplitudes are independent of $(\bar{w}_i, w_i)$. Moreover, the internal harmonics $(\bar{w}_i, w_i)$ are pure gauge with respect to the $\mathrm{SO}(D-2)_i$ symmetry which acts also on the spinor harmonics $(v^-_{\alpha q\, i}, v^+_{\alpha\dot{q}\, i})$. We have shown that internal harmonics can be defined in such a way that analytic superamplitudes actually depend only on complex spinor harmonics $(v^{\mp}_{\alpha A\, i}, \bar{v}^{A\mp}_{\alpha\, i})$ (7.30) parametrizing the coset $\frac{\mathrm{Spin}(1,D-1)}{[\mathrm{SO}(1,1)\otimes \mathrm{Spin}(D-4)\otimes \mathrm{U}(1)]\circledS K_{D-2}}$,

$$\{(v^{\mp}_{\alpha A\, i}, \bar{v}^{A\mp}_{\alpha\, i})\} = \frac{\mathrm{Spin}(1,D-1)}{[\mathrm{SO}(1,1)\otimes \mathrm{Spin}(D-4)\otimes \mathrm{U}(1)]\circledS K_{D-2}}\,. \tag{9.1}$$

However, we find convenient to consider these complex spinor harmonics to be composed from the real spinor harmonics, parametrizing the coset isomorphic to the celestial sphere $\mathbb{S}^{(D-2)} = \frac{\text{Spin}(1,D-1)}{[\text{SO}(1,1)\otimes\text{Spin}(D-2)]\otimes K_{D-2}}$ (6.2), and the above mentioned internal harmonics $(\bar{w}_i, w_i)$, in spite of that these latter are pure gauge with respect to $\text{Spin}(D-2)_i$ symmetry (see (7.21) and (7.7)).

We have found a parametrization of the spinor frame variables and of the internal frame which is especially convenient for the analysis of the analytic superamplitudes. This has allowed us to establish the correspondence of higher dimensional quantities with basic building blocks of 4D superamplitudes and to use it to find the expressions for analytic 3-point superamplitudes of D=10 SYM and D=11 SUGRA theories. These are the necessary basic ingredients for calculation of the n-point superamplitudes with the use of on-shell recurrent relations, the problem which we intend to address in a forthcoming paper.

The first stages in this direction should include a better understanding of the structure of the 3-point analytic superamplitudes, in particular the search for its more convenient, parametrization independent form, as well as the derivation of the BCFW-type recurrent relations for the analytic superamplitudes. These should be more closely related to the relations for D=4 superamplitudes [7, 11] than the BCFW-type recurrent relations for real constrained 11D and 10D superamplitudes presented in [15, 16].

In particular, one expects the BCFW deformations used in such recurrent relations to have an intrinsic complex structure, similar to the one in D=4 equations of [7]. As we show in appendix B, starting from BCFW deformations of spinor frame and fermionic variables in [15], which are essentially real, this is indeed the case. The resulting BCFW-like deformations of the complex spinor frame variables (7.30) and of the complex fermionic variables (4.42)

$$\widehat{v^{-}_{\alpha A(n)}} = v^{-}_{\alpha A(n)} + z\, v^{-}_{\alpha A(1)} \sqrt{\rho^{\#}_{1}/\rho^{\#}_{n}}\,, \qquad \widehat{\bar{v}^{A-}_{\alpha(n)}} = \bar{v}^{A-}_{\alpha(n)}\,, \tag{9.2}$$

$$\widehat{v^{-}_{\alpha A(1)}} = v^{-}_{\alpha A(1)}\,, \qquad \widehat{\bar{v}^{A-}_{\alpha(1)}} = \bar{v}^{A-}_{\alpha(1)} - z\, \bar{v}^{A-}_{\alpha(n)} \sqrt{\rho^{\#}_{n}/\rho^{\#}_{1}}\,, \tag{9.3}$$

$$\widehat{\eta^{-}_{A\,n}} = \eta^{-}_{A\,n} + z\, \eta^{-}_{A\,1} \sqrt{\rho^{\#}_{1}/\rho^{\#}_{n}}\,, \qquad \widehat{\eta^{-}_{A\,1}} = \eta^{-}_{A\,1} \tag{9.4}$$

have the structure quite similar to that of the 4D super-BCFW deformations from [11] (see (B.1)–(B.3) in appendix B).

Thus presently their exist three alternative superamplitude formalisms for 10D SYM, two of which have been also generalized for the case of 11D supergravity. These are Clifford superfield approach of [14], constrained superamplitude approach of [15, 16] and the analytic superamplitude formalism of the present paper. As discussed in [16], and also briefly commented in section 4.6.1, the one particle counterparts of all three types of superamplitudes can be obtained by different ways of covariant quantization of 10D and 11D massless superparticles. In short, the separation point is how to deal with the Poisson brackets of the fermionic second class constraints, (4.72).

The formalism of [14] and the analytic superfield approach of the present paper imply 'solving' the constraints by passing to the Dirac brackets (4.73) and quantizing these. In such a way we obtain the Clifford algebra like anticommutation relation (4.74) for 8 (16 in D=11 case) real fermionic variables $\hat{\theta}_q^-$. To arrive at the one-particle counterpart of the superamplitudes from [14], one should consider the superparticle 'wavefunction' to be dependent on the whole set of Clifford algebra valued variables $\hat{\theta}_q^-$, i.e. to be a 'Clifford superfield'. In contrast, to obtain an analytic superfield as superparticle wavefunction, we need to split 8 real $\hat{\theta}_q^-$ on 4 complex η_A^- and its complex conjugate $\bar{\eta}^{-A}$, which obey the Heisenberg-like algebra. This implies that $\bar{\eta}^{-A}$ can be considered as creation operator or complex momentum conjugate to the annihilation operator η_A^-. Then in the η_A^--coordinate (or holomorphic) representation the superparticle quantum state vector depends on η_A^-, but not on $\bar{\eta}^{-A}$. In other words, it will be described by analytic superfield, the one-particle counterpart of our analytic superamplitudes.

From this perspective, one can arrive at doubts in consistency of the Clifford superamplitude approach of [14]. Indeed, in terms of complex fermionic variables the above described appearance of an unconstrained Clifford superfield in superparticle quantization requires to allow the wavefunction to depend on both coordinate η_A^- and momentum $\bar{\eta}^{-A}$ variables in an arbitrary manner. Then such a Clifford superfield wavefunction is not allowed in quantum mechanics in its generic form and some conditions need to be imposed to restrict its dependence on $\bar{\eta}^{-A}$ and/or η_A^-. The analytic superfields and superamplitudes can be obtained on this way: by imposing on Clifford superfields/superamplitudes just the conditions to be independent of $\bar{\eta}^{-A}$.

The constrained superfields, the one-particle counterparts of the constrained superamplitudes, appear as a result of superparticle quantization if, instead of passing to Dirac brackets (4.73), we realize the fermionic second class constrains as differential operators $D_q^+ = \frac{\partial}{\partial \theta_q^-} + \ldots$ obeying the quantum counterpart (3.4) of (4.72). The 'imposing' of the quantum second class constraint is then achieved by considering a θ_q^--dependent multicomponent state vectors Ψ_Q $(= (\Psi_{\dot{q}}, W^I)$ in D=10) and requiring them to obey a set of linear differential equations $D_q^+ \Psi_Q = \Delta_{qQP} \Psi_P$ ((3.7) and (3.8) in D=10; see [16] for details of this procedure). The advantages of this approach is the use of Grassmann fermionic coordinates (rather than Clifford algebra valued ones) as well as its manifest covariance under the 'small group' SO(8) (SO(9)) symmetry. The disadvantage is that superfields and superamplitudes are subject to the above mentioned set of quite complicated equations, which have no clear counterpart in D=4 case. This makes the calculations in the constrained superamplitude framework quite involving (in comparative terms) and creates difficulties for the (straightforward) use of the experience gained in D=4. Also the decomposition of constrained superfields on components looks quite non-minimal: in the 10D case, 9 components of constrained superfield, all nonvanishing, are constructed of two fields describing the on-shell degrees of freedom of SYM, bosonic w^I and fermionic $\psi_{\dot{q}}$, appearing already in first two terms of the decomposition.

In contrast, the components of the analytic superfields include different components of $w^I = (\phi^{(+)}, \phi^{AB}, \phi^{(-)})$ and $\psi_{\dot{q}} = (\psi^{+1/2\,A}, \psi_A^{-1/2})$ only ones. Thus the great advantage of the analytic superamplitude formalism is its minimality. It is also much more similar to the on-shell superfield and superamplitude description used for maximal D=4 SYM and SUGRA theories. In particular, this similarity helped us to find the gauge fixed expression for the 3-point analytic superamplitudes of 10D SYM and 11D SUGRA. The price to be paid for these advantages is the harmonic superspace type realization of the SO(8) (SO(9)) symmetry and, consequently, dependence on additional set of harmonic variables $\bar{w}_{qA}, w_q^A$ parametrizing Spin(8)/[SU(4)$\otimes$U(1)] coset. Presently the analytic superemplitude formalism is under further development which, as we hope, will result in a significant progress in 10D and 11D amplitude calculations.

An alternative direction we are also working out is to use the structure of the analytic 3-point superamplitude for deriving the expression for its cousin from the real constrained superamplitude formalism [15, 16], and to use the interplay of the constrained and analytic superamplitude approaches for their mutual development.

It will be also interesting to reproduce the analytic superamplitudes from an appropriate formulation of the ambitwistor string [60–62]. Notice that, although original ambitwistor string model [60] had been of NSR-type and had been formulated in D=10, quite soon [63] it was appreciated its relation with null-superstring [50] (see [64, 65] for related results and [66] for more references on null-string) and with twistor string [6, 66–68]. This suggested its existence in spacetime of arbitrary dimension, including D=11 and D=4, and the last possibility was intensively elaborated in [69–73]. An approach to derive the analytic superamplitudes from the Green-Schwarz type spinor moving frame formulation of D=10 and D=11 ambitwistor superstring [63] looks promising and we plan to address it in the future publications.

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A On D=4 spinor helicity formalism

In D=4 Spin(1,3) = SL(2,$\mathbb{C}$) and the spinor frame or Lorentz harmonic variables $v_\alpha^\pm = (v_{\dot\alpha}^\pm)^*$ [49] are restricted by the only condition $v^{-\alpha}v_\alpha^+ = 1$,

$$(v_\alpha^+, v_\alpha^-) \in \mathrm{SL}(2,\mathbb{C}) \quad \Leftrightarrow \quad v^{-\alpha}v_\alpha^+ = 1\,. \tag{A.1}$$

In a theory invariant under $[\mathrm{SO}(1,1)\otimes \mathrm{SO}(2)] \otimes\!\!\!\!\times \mathbb{K}_2$ transformations

$$v^+_\alpha \mapsto e^{a+ib}(v^+_\alpha + k^\# v^-_\alpha)\,, \qquad \bar{v}^+_{\dot\alpha} \mapsto e^{a-ib}(\bar{v}^+_{\dot\alpha} + \bar{k}^\# \bar{v}^-_{\dot\alpha})\,, \tag{A.2}$$
$$v^-_\alpha \mapsto e^{-a-ib} v^-_\alpha\,, \qquad \bar{v}^-_{\dot\alpha} \mapsto e^{-a+ib} \bar{v}^-_{\dot\alpha}\,, \tag{A.3}$$

the set of such harmonic variables parametrize the sphere $\mathbb{S}^2$ [17, 18],

$$\{(v^+_\alpha, v^-_\alpha)\} = \frac{\mathrm{Spin}(1,3)}{[\mathrm{SO}(1,1)\otimes \mathrm{Spin}(2)] \otimes\!\!\!\!\times \mathbb{K}_2} = \frac{\mathrm{SL}(2,\mathbb{C})}{[\mathrm{SO}(1,1)\otimes \mathrm{U}(1)] \otimes\!\!\!\!\times \mathbb{K}_2} = \mathbb{S}^2\,. \tag{A.4}$$

When the spinor frame is associated with a light-like momenta by the generalized Cartan-Penrose relation

$$p_{\alpha\dot\alpha} = \rho^\# v^-_\alpha \bar{v}^-_{\dot\alpha} \tag{A.5}$$

(cf. (1.4)), $\mathbb{S}^2$ in (A.4) is the celestial sphere.

In the scattering problem we can associate the spinor frame to each of n light-like momenta and to express the corresponding helicity spinors of (1.4) in terms of the spinor harmonics

$$\lambda_{\alpha(i)} = \sqrt{\rho^\#_{(i)}}\, v^-_{\alpha(i)}\,, \qquad \bar\lambda_{\dot\alpha(i)} = \sqrt{\rho^\#_{(i)}}\, \bar{v}^-_{\dot\alpha(i)}\,, \qquad p_{\alpha\dot\alpha(i)} = \rho^\#_{(i)} v^-_{\alpha(i)} \bar{v}^-_{\dot\alpha(i)}\,. \tag{A.6}$$

As we have used only $v^-_{\alpha(i)}$, the complementary spinor harmonic $v^+_{\alpha(i)}$ remains arbitrary up to the constraint (A.1),

$$v^{-\alpha}_{(i)} v^+_{\alpha(i)} = 1\,. \tag{A.7}$$

Actually, this is the statement of $\mathbb{K}_2$ symmetry (parametrized by $k^\#$ and $\bar{k}^\#$ in (A.2), (A.3)), which can be used as an identification relation on the set of harmonic variables (as indicated in (A.4)), and in this sense is the gauge symmetry. We can fix these $\mathbb{K}_{2(i)}$ gauge symmetries by identifying (up to a complex multipliers) all the complementary spinors of the spinor frames associated to the momenta of the scattered particles

$$(v^+_{(i)} v^+_{(j)}) \equiv v^{+\alpha}_{(i)} v^+_{\alpha(j)} = 0 \qquad \Leftrightarrow \qquad v^+_{\alpha(i)} \propto v^+_{\alpha(j)} \qquad \forall\, i,j = 1,\ldots,n\,. \tag{A.8}$$

It is convenient to reformulate this statement by introducing an auxiliary spinor frame $(v^\pm_\alpha)$, which is not associated to any of the scattered particles, and to state that any of the spinor frames $(v^\pm_{\alpha(i)})$ is related to that by (cf. (A.2), (A.3))

$$v^+_{\alpha(i)} = e^{\alpha_i + i\beta_i} v^+_\alpha\,, \qquad \bar{v}^+_{\dot\alpha(i)} = e^{\alpha_i - i\beta_i} \bar{v}^+_{\dot\alpha}\,, \tag{A.9}$$
$$v^-_{\alpha(i)} = e^{-\alpha_i - i\beta_i}(v^-_\alpha + \mathbb{K}^=_i v^+_\alpha)\,, \qquad \bar{v}^-_{\dot\alpha(i)} = e^{-\alpha_i + i\beta_i}(\bar{v}^-_{\dot\alpha} + \bar{\mathbb{K}}^=_i \bar{v}^+_{\dot\alpha})\,. \tag{A.10}$$

In this gauge the contractions of the spinors from different frames read

$$< v^-_{(i)} v^-_{(j)} > \equiv v^{-\alpha}_{(i)} v^-_{\alpha(j)} = e^{-(\alpha_i+\alpha_j) - i(\beta_i+\beta_j)} \mathbb{K}^=_{ji}\,,$$
$$< v^-_{(i)} v^+_{(j)} > = e^{(\alpha_j - \alpha_i) + i(\beta_j - \beta_i)}\,, \tag{A.11}$$
$$[\bar{v}^-_{(i)} \bar{v}^-_{(j)}] \equiv \bar{v}^{-\dot\alpha}_{(i)} \bar{v}^-_{\dot\alpha(j)} = e^{-(\alpha_i+\alpha_j) + i(\beta_i+\beta_j)} \bar{\mathbb{K}}^=_{ji}\,,$$
$$[\bar{v}^-_{(i)} \bar{v}^+_{(j)}] = e^{(\alpha_j - \alpha_i) - i(\beta_j - \beta_i)}\,, \tag{A.12}$$

JHEP05(2018)103

where

$$\mathbb{K}^{=}_{ji} := \mathbb{K}^{=}_{j} - \mathbb{K}^{=}_{i} \,. \tag{A.13}$$

Of course, we can use the $\mathrm{SO}(1,1)_i \times \mathrm{SO}(2)_i$ gauge symmetries to fix also $\alpha_i = 0$ and $\beta_i = 0$ $\forall i = 1, \ldots, n$, but the multipliers β_i might be useful as they actually indicate the helicity of the field or amplitude, while α_i can be 'eaten' by the 'energy' variables $\rho^{\#}_i$. Indeed, the i-th light-like momentum (A.6) can be now written as

$$\begin{aligned} p_{\alpha\dot{\alpha}(i)} &= \widetilde{\rho}^{\#}_{(i)}(v^-_{\alpha} + \mathbb{K}^{=}_{(i)} v^+_{\alpha})\,(\bar{v}^-_{\dot{\alpha}} + \bar{\mathbb{K}}^{=}_{(i)} \bar{v}^+_{\dot{\alpha}}) \\ &= \widetilde{\rho}^{\#}_{(i)} u^{=}_{\alpha\dot{\alpha}} + \widetilde{\rho}^{\#}_{(i)} \mathbb{K}^{=}_{(i)} u^{+-}_{\alpha\dot{\alpha}} + \widetilde{\rho}^{\#}_{(i)} \bar{\mathbb{K}}^{=}_{(i)} u^{-+}_{\alpha\dot{\alpha}} + \widetilde{\rho}^{\#}_{(i)} \mathbb{K}^{=}_{(i)} \bar{\mathbb{K}}^{=}_{(i)} u^{\#}_{\alpha\dot{\alpha}} \,, \end{aligned} \tag{A.14}$$

where $\tilde{\rho}^{\#}_i = e^{-2\alpha_i} \rho^{\#}_i$ (8.8) and

$$u^{=}_{\alpha\dot{\alpha}} = v^-_{\alpha} \bar{v}^-_{\dot{\alpha}} \,, \qquad u^{\#}_{\alpha\dot{\alpha}} = v^+_{\alpha} \bar{v}^+_{\dot{\alpha}} \,, \qquad u^{\pm\mp}_{\alpha\dot{\alpha}} = v^{\pm}_{\alpha} \bar{v}^{\mp}_{\dot{\alpha}} \,, \tag{A.15}$$

are two real and two complex conjugate ($u^{+-}_a = (u^{-+}_a)^*$) vectors of Newman-Penrose light-like tetrade (see [43, 44] and refs. therein).

Using the complementary harmonics $v^+_{\alpha i}$, $\bar{v}^+_{\dot{\alpha} i}$ of the auxiliary frame as reference spinors, we can identify polarization vectors with the i-th frame counterparts of the above described complex null-vectors u^{-+}_a and $u^{+-}_a = (u^{-+}_a)^*$:

$$\varepsilon^{(+)}_{\alpha\dot{\alpha}(i)} = u^{-+}_{\alpha\dot{\alpha}(i)} \equiv v^-_{\alpha(i)} \bar{v}^+_{\dot{\alpha}(i)} \,, \qquad \varepsilon^{(-)}_{\alpha\dot{\alpha}(i)} = u^{+-}_{\alpha\dot{\alpha}(i)} \equiv v^+_{\alpha(i)} \bar{v}^-_{\dot{\alpha}(i)} \,. \tag{A.16}$$

In the gauge (A.8) these identification implies that

$$\varepsilon^{(+)}_{(i)} \cdot \varepsilon^{(+)}_{(j)} := \frac{1}{2} \varepsilon^{(+)}_{\alpha\dot{\alpha}(i)} \varepsilon^{(+)\alpha\dot{\alpha}}_{(j)} = 0 \,. \tag{A.17}$$

Using (A.14) and (A.16) we can easily find

$$\varepsilon^{(+)}_{(i)} k_{(j)} = \frac{\rho^{\#}_{(i)}}{2} (v^-_{(j)} v^-_{(i)})(\bar{v}^-_{(j)} \bar{v}^+_{(i)}) \,, \qquad \varepsilon^{(+)}_{(i)} \varepsilon^{(-)}_{(j)} = -\frac{1}{2} (v^-_{(i)} v^+_{(j)})(\bar{v}^-_{(j)} \bar{v}^+_{(i)}) \,, \tag{A.18}$$

and then, for instance,

$$\begin{aligned} (\varepsilon^{(+)}_{(1)} k_{(2)})\, (\varepsilon^{(+)}_{(2)} \varepsilon^{(-)}_{(3)}) &= -\frac{\rho^{\#}_{(2)}}{4} (v^-_{(2)} v^-_{(1)})\, (v^-_{(2)} v^+_{(3)})\, (\bar{v}^-_{(2)} \bar{v}^+_{(1)})\, (\bar{v}^-_{(3)} \bar{v}^+_{(2)}) \\ &= -\frac{\tilde{\rho}^{\#}_{(2)}}{4}\, K^{=}_{21}\, e^{2i(\beta_3 - \beta_2 - \beta_1)}, \end{aligned} \tag{A.19}$$

This allows us to calculate 3-gluon amplitude of $\mathcal{N} = 4$ 4D SYM,

$$\begin{aligned} \mathcal{M}(1^+, 2^+, 3^-) &= g \epsilon^{(+)a}_{(1)} \epsilon^{(+)b}_{(2)} \epsilon^{(-)c}_{(3)} t_{abc}(k_1, k_2, k_3) \\ &= g(\varepsilon^{(+)}_{(1)} k_{(2)}\, \varepsilon^{(+)}_{(2)} \varepsilon^{(-)}_{(3)} + \varepsilon^{(+)}_{(2)} k_{(3)}\, \varepsilon^{(-)}_{(3)} \varepsilon^{(+)}_{(1)} + \varepsilon^{(-)}_{(3)} k_{(1)}\, \varepsilon^{(+)}_{(1)} \varepsilon^{(+)}_{(2)}) = \\ &= -\frac{g}{4}\, e^{2i(\beta_3 - \beta_2 - \beta_1)} \left(\tilde{\rho}^{\#}_{(2)} K^{=}_{21} + \tilde{\rho}^{\#}_{(3)} K^{=}_{32} \right) \\ &= \frac{g}{4}\, \tilde{\rho}^{\#}_{(3)} K^{=}_{21}\, e^{2i(\beta_3 - \beta_2 - \beta_1)} \end{aligned} \tag{A.20}$$

(see [58, 59] for the definition of t_{abc} tensor). Notice that the last term in the second line of this equation vanishes as a result of (A.17) and that at the last stage of transformations of this equation we have used the consequence of the momentum conservation in 3-particle process which we are going to discuss now.

A.1 Momentum conservation in a 3-point 4D amplitude

In our notation the momentum conservation in the 3-particle process is expressed by

$$\rho^{\#}_{(1)} v^{-}_{\alpha(1)} \bar{v}^{-}_{\dot{\alpha}(1)} + \rho^{\#}_{(2)} v^{-}_{\alpha(2)} \bar{v}^{-}_{\dot{\alpha}(2)} + \rho^{\#}_{(3)} v^{-}_{\alpha(3)} \bar{v}^{-}_{\dot{\alpha}(3)} = 0 . \tag{A.21}$$

This implies

$$\tilde{\rho}^{\#}_{1} + \tilde{\rho}^{\#}_{2} + \tilde{\rho}^{\#}_{3} = 0 , \tag{A.22}$$

$$\tilde{\rho}^{\#}_{1} K^{=}_{1} + \tilde{\rho}^{\#}_{2} K^{=}_{2} + \tilde{\rho}^{\#}_{3} K^{=}_{3} = 0 ,$$

$$\Rightarrow \quad K^{=}_{32} = \frac{\tilde{\rho}^{\#}_{1}}{\tilde{\rho}^{\#}_{3}} K^{=}_{21} = \frac{\tilde{\rho}^{\#}_{1}}{\tilde{\rho}^{\#}_{2}} K^{=}_{13} \quad \Rightarrow \quad K^{=}_{13} = \frac{\tilde{\rho}^{\#}_{2}}{\tilde{\rho}^{\#}_{3}} K^{=}_{21} \tag{A.23}$$

as well as

$$K^{=}_{32} \bar{K}^{=}_{32} := (K^{=}_{3} - K^{=}_{2})(\bar{K}^{=}_{3} - \bar{K}^{=}_{2}) = 0 . \tag{A.24}$$

Here we have used the notation (A.13) and

$$\tilde{\rho}^{\#}_{i} := e^{-2\alpha_i} \rho^{\#}_{i} . \tag{A.25}$$

The solution of eq. (A.24) is nontrivial only if $(\bar{K}^{=}_{(32)})^* \neq K^{=}_{(32)}$. In this case one of two branches of the general solution is described by

$$\bar{K}^{=}_{3} = \bar{K}^{=}_{2} = \bar{K}^{=}_{1} \tag{A.26}$$

while $K^{=}_{(1,2,3)}$ can be different but obeying (A.23) with $\tilde{\rho}^{\#}_{(1,2,3)}$ restricted by (A.22). From now on we will denote these complex nonvanishing $K^{=}_{(1,2,3)}$ restricted by 3-particle kinematics by $\mathbb{K}^{=}_{(1,2,3)}$. We will also use the solution of (A.23) in terms of complex non-vanishing $\mathbb{K}^{==}$

$$\frac{\mathbb{K}^{=}_{32}}{\tilde{\rho}^{\#}_{1}} = \frac{\mathbb{K}^{=}_{21}}{\tilde{\rho}^{\#}_{3}} = \frac{\mathbb{K}^{=}_{13}}{\tilde{\rho}^{\#}_{2}} =: \mathbb{K}^{==} . \tag{A.27}$$

Eq. (A.26) implies

$$\bar{v}^{-}_{\dot{\alpha}(1)} \propto \bar{v}^{-}_{\dot{\alpha}(2)} \propto \bar{v}^{-}_{\dot{\alpha}(3)} \tag{A.28}$$

while $v^{-}_{\alpha(1)}$, $v^{-}_{\alpha(2)}$ and $v^{-}_{\alpha(3)}$ are different.

A.2 3-gluon amplitude and superamplitude in maximal D=4 SYM

The standard expression for the 3-point amplitude in D=4 SYM is written in terms of

$$< ij > = < \lambda_i \lambda_j > = \lambda^{\alpha}_{i} \lambda_{\alpha j} = \sqrt{\rho^{\#}_{i} \rho^{\#}_{j}} < v^{-}_{i} v^{-}_{j} >$$
$$= \sqrt{\tilde{\rho}^{\#}_{i} \tilde{\rho}^{\#}_{j}} e^{-i(\beta_i + \beta_j)} \mathbb{K}^{=}_{ji} . \tag{A.29}$$

If we were trying to guess the corresponding expression starting from (A.20), the β_i dependence indicates that this should be (up to a coefficient)

$$\mathcal{M}(1^+, 2^+, 3^-) = \frac{< 12 >^3}{< 23 >< 31 >} \equiv \frac{< 12 >^4}{< 12 >< 23 >< 31 >} . \tag{A.30}$$

Using (A.29) and (A.23) one can easily check that this expression indeed reproduce (A.20),

$$\frac{<12>^4}{<12><23><31>} = \tilde{\rho}_3^{\#}\mathbb{K}^{=}_{21}\, e^{2i(\beta_3-\beta_2-\beta_1)} = (\tilde{\rho}_3^{\#})^2\mathbb{K}^{==}\, e^{2i(\beta_3-\beta_2-\beta_1)} . \tag{A.31}$$

In our notation the anti-MHV ($\overline{\text{MHV}}$) type superamplitude reads (see (1.15))

$$\mathcal{A}^{\overline{\text{MHV}}}(1,2,3) = \frac{1}{<12><23><31>}\, \delta^4\left(\eta_1 <23> + \eta_2 <31> + \eta_3 <12>\right), \tag{A.32}$$

while the MHV amplitude is

$$\begin{aligned}\mathcal{A}^{\text{MHV}}(1,2,3) &= \frac{1}{[12]\,[23]\,[31]}\, \delta^8\left(\bar\lambda_{\dot\alpha 1}\eta_{A1} + \bar\lambda_{\dot\alpha 2}\eta_{A2} + \bar\lambda_{\dot\alpha 3}\eta_{A3}\right) = \\ &= \frac{1}{[12]\,[23]\,[31]}\,\frac{1}{2^4}\prod_{A=1}^{4}\sum_{i,j=1}^{3}[ij]\;\eta_{Ai}\eta_{Aj} .\end{aligned} \tag{A.33}$$

The covariance of δ function under the phase transformations of the bosonic spinors holds when the fermionic variables η_{Ai} have the same phase transformation property as $\lambda_{\alpha i}$. This reflects its origin in Penrose-Ferber incidence type relation $\eta_{Ai} = \theta^{\alpha}_{Ai}\lambda_{\alpha i}$ [45] which in terms of our Lorentz harmonic notation reads $\eta_{Ai} = \sqrt{\rho_i^{\#}}\eta^-_{Ai} := \sqrt{\rho_i^{\#}}\theta^{\alpha}_{Ai}v^-_{\alpha i}$.

Notice also that the indices A of all the fermionic coordinates are transformed by the same SU(4), which is the R-symmetry group of $\mathcal{N}=4$ D=4 SYM.

Using (A.10), (A.9) and (A.22), (A.27), we can write the Grassmann delta function of (A.33) in the form

$$\begin{aligned}&\delta^4\left(\eta_1 <23> + \eta_2 <31> + \eta_3 <12>\right) = \\ &= \left(\tilde\rho_1^{\#}\tilde\rho_2^{\#}\tilde\rho_3^{\#}\right)^2 e^{-4i(\beta_1+\beta_2+\beta_3)}\,\delta^4\left(\tilde\eta^-_{1A}\mathbb{K}^{=}_{32} + \tilde\eta^-_{2A}\mathbb{K}^{=}_{13} + \tilde\eta^-_{3A}\mathbb{K}^{=}_{21}\right) \\ &= \left(\tilde\rho_1^{\#}\tilde\rho_2^{\#}\tilde\rho_3^{\#}\right)^2 (\mathbb{K}^{==})^4\, e^{-4i(\beta_1+\beta_2+\beta_3)}\;\delta^4\left(\tilde\rho_1^{\#}\eta^-_{1A} + \tilde\rho_2^{\#}\tilde\eta^-_{2A} + \tilde\rho_3^{\#}\tilde\eta^-_{3A}\right) \\ &= \frac{\left(\tilde\rho_1^{\#}\tilde\rho_2^{\#}\right)^2\;(\mathbb{K}^{=}_{21})^4}{(\tilde\rho_3^{\#})^2}\; e^{-4i(\beta_1+\beta_2+\beta_3)}\delta^4\left(\tilde\rho_1^{\#}\eta^-_{[13]A} + \tilde\rho_2^{\#}\tilde\eta^-_{[23]A}\right),\end{aligned} \tag{A.34}$$

where[12]

$$\tilde\eta^-_{Ai} := e^{\alpha_i + i\beta_i}\eta^-_{Ai}\,, \qquad \tilde\eta^-_{A[ji]} = \tilde\eta^-_{Aj} - \tilde\eta^-_{Ai}\,, \tag{A.35}$$

$\tilde\rho_i^{\#}$ is defined in (A.25).

Similarly, the fermionic delta function in (1.16) can be written as

$$\begin{aligned}&\delta^8\left(\bar\lambda_{\dot\alpha 1}\eta_{A1} + \bar\lambda_{\dot\alpha 2}\eta_{A2} + \bar\lambda_{\dot\alpha 3}\eta_{A3}\right) = \delta^8\left(\rho^{\#}_{(1)}\bar v^-_{\dot\alpha(1)}\eta^-_{A(1)} + \rho^{\#}_{(2)}\bar v^-_{\dot\alpha 2}\eta^-_{A2} + \rho^{\#}_3\bar v^-_{\dot\alpha 3}\eta^-_{A3}\right) \\ &= \delta^8\left(\bar v^-_{\dot\alpha}\left(\tilde\rho_1^{\#}\tilde\eta^-_{A[13]} + \tilde\rho_2^{\#}\tilde\eta^-_{A[23]}\right) + \bar v^+_{\dot\alpha}\left(\bar{\mathbb{K}}^{=}_1\tilde\rho_1^{\#}\tilde\eta^-_{A[13]} + \bar{\mathbb{K}}^{=}_2\tilde\rho_2^{\#}\tilde\eta^-_{A[23]}\right)\right) \\ &= \delta^8\left(\left(\bar v^-_{\dot\alpha} + \bar{\mathbb{K}}^{=}_1\bar v^+_{\dot\alpha}\right)\tilde\rho_1^{\#}\tilde\eta^-_{A[13]} + \left(\bar v^-_{\dot\alpha} + \bar{\mathbb{K}}^{=}_2\bar v^+_{\dot\alpha}\right)\tilde\rho_2^{\#}\tilde\eta^-_{A[23]}\right).\end{aligned} \tag{A.36}$$

[12] One can check that $\tilde\eta^-_{Ai} = \theta^{\alpha}_{Ai}(v^-_{\alpha} + \mathbb{K}^{=}_i v^+_{\alpha})$ which makes transparent that all $\tilde\eta^-_{Ai}$ are transformed by the common $\mathrm{U}(1)\otimes\mathrm{SO}(1,1)$ group, but are inert under all the $\mathrm{U}(1)_j\otimes\mathrm{SO}(1,1)_j$ gauge symmetries, including the one with $i=j$.

In this notation, the multiplier in the $\overline{\text{MHV}}$ superamplitude (1.15) reads

$$\frac{1}{<12><23><31>} = \frac{e^{2i(\beta_1+\beta_2+\beta_3)}}{\tilde{\rho}^{\#}_{(1)}\tilde{\rho}^{\#}_{(2)}\tilde{\rho}^{\#}_{(3)}} \frac{1}{\mathbb{K}^{=}_{21}\mathbb{K}^{=}_{32}\mathbb{K}^{=}_{13}} = \frac{e^{2i(\beta_1+\beta_2+\beta_3)}}{(\tilde{\rho}^{\#}_{1}\tilde{\rho}^{\#}_{2})^2} \frac{\tilde{\rho}^{\#}_{3}}{(\mathbb{K}^{=}_{21})^3}$$
$$= \frac{e^{2i(\beta_1+\beta_2+\beta_3)}}{(\tilde{\rho}^{\#}_{1}\tilde{\rho}^{\#}_{2}\tilde{\rho}^{\#}_{3})^2} \frac{1}{(\mathbb{K}^{==})^3} . \tag{A.37}$$

Using (A.37) and (A.34), we can write the 3-point anti-MHV superamplitude (1.15) in the form

$$\mathcal{A}^{\overline{\text{MHV}}}(1,2,3) = (\mathbb{K}^{==})\, e^{-2i(\beta_1+\beta_2+\beta_3)}\, \delta^4\left(\tilde{\rho}^{\#}_{1}\tilde{\eta}^{-}_{1A} + \tilde{\rho}^{\#}_{2}\tilde{\eta}^{-}_{2A} + \tilde{\rho}^{\#}_{3}\tilde{\eta}^{-}_{3A}\right) ,$$
$$= -\frac{\mathbb{K}^{=}_{21}}{(\tilde{\rho}^{\#}_{1} + \tilde{\rho}^{\#}_{2})} e^{-2i(\beta_1+\beta_2+\beta_3)}\, \delta^4\left(\tilde{\rho}^{\#}_{1}\tilde{\eta}^{-}_{[13]A} + \tilde{\rho}^{\#}_{2}\tilde{\eta}^{-}_{[23]A}\right) . \tag{A.38}$$

B BCFW-like deformations of complex frame and complex fermionic variables

An important tool to reconstruct tree $D = 4$ (super)amplitudes from the basic 3-point (super)amplitude is given by BCFW recurrent relation [7] and their superfield generalization [11]. The counterparts of these latter 4D relations for constrained superamplitudes of 11D SUGRA and 10D SYM have been presented in [15, 16]. They use the real BCFW deformations of real bosonic and fermionic variables of the constrained superamplitude formalism. In contrast, in the case of the BCFW-type recurrent relations for analytic superamplitudes (which are still to be derived), one expects the BCFW deformations used in such recurrent relations to have an intrinsic complex structure, similar to the one of the D=4 relations [7, 11]

$$\lambda^A_{(n)} \mapsto \widehat{\lambda^A_{(n)}} = \lambda^A_{(n)} + z\lambda^A_{(1)}, \qquad \bar{\lambda}^{\dot{A}}_{(n)} \mapsto \widehat{\bar{\lambda}^{\dot{A}}_{(n)}} = \bar{\lambda}^{\dot{A}}_{(n)}, \tag{B.1}$$

$$\lambda^A_{(1)} \mapsto \widehat{\lambda^A_{(1)}} = \lambda^A_{(1)} , \qquad \bar{\lambda}^{\dot{A}}_{(1)} \mapsto \widehat{\bar{\lambda}^{\dot{A}}_{(1)}} = \bar{\lambda}^{\dot{A}}_{(1)} - z\bar{\lambda}^{\dot{A}}_{(n)}, \tag{B.2}$$

$$\widehat{\eta_{A\,n}} = \eta_{A\,n} + z\,\eta^{-}_{A\,1} \ , \qquad \widehat{\eta_{A\,1}} = \eta_{A\,1} . \tag{B.3}$$

Let us show how this can be reached starting from the BCFW deformations of real spinor frame variables [15, 16]

$$\widehat{v^{-}_{\alpha q(n)}} = v^{-}_{\alpha q(n)} + z\sqrt{\frac{\rho^{\#}_{(1)}}{\rho^{\#}_{(n)}}}\, v^{-}_{\alpha p(1)}\, \mathbb{M}_{pq} , \tag{B.4}$$

$$\widehat{v^{-}_{\alpha q(1)}} = v^{-}_{\alpha q(1)} - z\sqrt{\frac{\rho^{\#}_{(n)}}{\rho^{\#}_{(1)}}}\, \mathbb{M}_{qp}\, v^{-}_{\alpha p(n)} , \tag{B.5}$$

and of the real fermionic variables

$$\widehat{\theta^-_{p(n)}} = \theta^-_{p(n)} + z\, \theta^-_{q(1)}\, \mathbb{M}_{qp} \sqrt{\frac{\rho^\#_{(1)}}{\rho^\#_{(n)}}}\,, \tag{B.6}$$

$$\widehat{\theta^-_{q(1)}} = \theta^-_{q(1)} - z \sqrt{\frac{\rho^\#_{(n)}}{\rho^\#_{(1)}}}\, \mathbb{M}_{qp}\, \theta^-_{p(n)}\,. \tag{B.7}$$

Here $\alpha = 1, \ldots, 4\mathcal{N}$ and $q, p = 1, \ldots, 4\mathcal{N}$ (we should set $\mathcal{N} = 8$ and 4 for 11D SUGRA and 10D SYM, respectively) and z is an arbitrary number. In principle this can be considered to be real $z \in \mathbb{R}$ [14], although $z \in \mathbb{C}$ is neither forbidden and actually more convenient in amplitude calculations.

The above shift of spinor moving frame variables results in shifting the momentum of the first and of the n-th particle,

$$\widehat{k^a_{(1)}} = k^a_{(1)} - zq^a\,, \qquad \widehat{k^a_{(n)}} = k^a_{(n)} + zq^a\,, \tag{B.8}$$

on a light-like vector q^a orthogonal to both $k^a_{(1)}$ and $k^a_{(n)}$,

$$q_a q^a = 0\,, \qquad q_a k^a_{(1)} = 0\,, \qquad q_a k^a_{(n)} = 0\,, \tag{B.9}$$

provided we choose

$$\mathbb{M}_{qp} = -\frac{1}{\sqrt{\rho^\#_{(1)}\rho^\#_{(n)}(u^=_{(1)} u^=_{(n)})}}\, (v^-_{q(1)}\, \not{q}\, v^-_{p(n)})\,, \tag{B.10}$$

$$\not{q}^{\alpha\beta} := q^a \tilde{\Gamma}^{\alpha\beta}_a\,, \qquad \not{q}_{\alpha\beta} := q_a \Gamma^a_{\alpha\beta}\,. \tag{B.11}$$

The light-likeness of q^a (B.9) implies the nilpotency of the matrix $\mathbb{M}$,

$$\mathbb{M}_{rp}\mathbb{M}_{rq} = 0\,, \qquad \mathbb{M}_{qr}\mathbb{M}_{pr} = 0\,. \tag{B.12}$$

We can also write the expression for light-like complex vector in terms of deformation matrix,

$$q^a = \frac{1}{\mathcal{N}} \sqrt{\rho^\#_1 \rho^\#_n}\; v^-_{q(1)} \tilde{\Gamma}^a \mathbb{M}_{qp} v^-_{p(n)}\,. \tag{B.13}$$

The nilpotency condition (B.12) guarantees that the shifted spinor moving frame variables obey the characteristic constraints, eqs. (2.33) with shifted light-like momenta $k_{(1)}$ and $k_{(n)}$ (B.8) or, equivalently, (2.22) with shifted light-like $u^{=a}_{(1)}$ and $u^{=a}_{(n)}$,

$$\widehat{u^{=a}_{(1)}} = u^{=a}_{(1)} - \frac{zq^a}{\rho^\#_{(1)}}\,, \qquad \widehat{u^{=a}_{(n)}} = u^{=a}_{(n)} + \frac{zq^a}{\rho^\#_{(n)}}\,. \tag{B.14}$$

Notice that (B.4) and (B.5) imply

$$\widehat{k^a_1} + \widehat{k^a_n} = k^a_1 + k^a_n\,. \tag{B.15}$$

The complex structure similar to the one of D=4 BCFW deformations can be reproduced after passing to the complex spinor harmonics (7.30)—(7.33) composed from the spinor harmonics and the internal harmonic variables. The internal harmonics can be used to solve the nilpotency conditions (B.12) for the matrix $\mathbb{M}_{qp}$ in (B.4)–(B.7). The solution

$$\mathbb{M}_{qp} = \bar{w}_{qA\,1} \mathfrak{M}^A{}_B w^B_{pn}\,, \tag{B.16}$$

with an arbitrary hermitian $\mathcal{N} \times \mathcal{N}$ matrix $\mathfrak{M}^A{}_B$, results in the following deformation of the complex spinor frame variables (7.28) and of the complex fermionic variables:

$$\widehat{v^-_{\alpha A(n)}} = v^-_{\alpha q(A)} + z\, v^-_{\alpha B(1)}\, \mathfrak{M}^B{}_A \sqrt{\rho^\#_{(1)}/\rho^\#_{(n)}}\,, \qquad \widehat{\bar{v}^{A-}_{\alpha(n)}} = \bar{v}^{A-}_{\alpha(n)}\,, \tag{B.17}$$

$$\widehat{v^-_{\alpha A(1)}} = v^-_{\alpha A(1)}\,, \qquad \widehat{\bar{v}^{A-}_{\alpha(1)}} = \bar{v}^{A-}_{\alpha(1)} - z\, \mathfrak{M}^A{}_B\, \bar{v}^{B-}_{\alpha(n)} \sqrt{\rho^\#_{(n)}/\rho^\#_{(1)}} \tag{B.18}$$

and

$$\widehat{\eta^-_{A\,n}} = \eta^-_{A\,n} + z\, \eta^-_{B\,1}\, \mathfrak{M}^B{}_A \sqrt{\rho^\#_{(1)}/\rho^\#_{(n)}}\,, \qquad \widehat{\eta^-_{A\,1}} = \eta^-_{A\,1}\,. \tag{B.19}$$

These are already quite similar to the 4D super-BCFW transformations (B.1), (B.2), (B.3). To make the similarity even closer, we can choose $\mathfrak{M}^B{}_A = \delta^B{}_A$. In such a way we arrive at (9.2), (9.3), and (9.4).

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