Classification and Physical Aspects of Constant Mean Curvature Rotational Surfaces

Final Degree Dissertation
Degree in Mathematics

Idoia Castillo Sintes

Supervisor:
Óscar Jesús Garay Bengoechea

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Contents

Introduction

Physically, Constant Mean Curvature (CMC) surfaces in $\mathbb{R}^3$ can be described in terms of soap films. In fact, compact CMC surfaces can be seen as a model for soap bubbles, while minimal surfaces (or surfaces with zero mean curvature) model soap films in equilibrium with arbitrary boundary [7],[8]. CMC surfaces have a long history in Physics and Mathematics and there is a huge literature on the subject. The aim of this Degree's Final Dissertation is to review some fundamental facts on CMC surfaces with rotational symmetry. We will use modern tools to prove the classical Delaunay's classification of CMC rotational surfaces and the classical characterization of CMC surfaces in terms of the area functional (both date back to 1791 [2]). They are complemented with a recent classification of rotational surfaces with prescribed curvature function [6] and a recent variational characterization of rotational CMC surfaces as liquid bridges [9].
Surfaces of revolution with constant mean curvature in $\mathbb{R}^3$ where completely characterized over a hundred years ago by C. Delaunay in 1841 [2]. By using the geometric properties of conics and their evolutes, Delaunay derived a non-linear ordinary differential equation involving the radius of curvature of the planar curve sweeping a CMC rotational surface. He also showed that this ODE arises geometrically by rolling the focus of a conic along a straight line without slippage (the roulette of a conic). Thus, as Delaunay's theorem says, roulettes of conics are the meridians of CMC surfaces of revolution.
In Chapter 2 we prove Delaunay's theorem by using a more modern approach, following Delaunay's guidelines. In Section 2.1. we first find a parametrization for the roulettes of conics. Although there are several parametrizations in the literature, most of them are expressed in terms of elliptic functions [2],[3],[5], we have chosen to parametrize the roulettes of conics in terms of the parametrizations of the conics themselves [1]. This yields more or less simple expressions for the geometric features of the corresponding surfaces (including their mean curvature). We also derive a direct parametrization for surfaces of revolution having roulettes of conics as profile curves (Delaunay's surfaces). Using the above parametrization, it is shown that Delaunay's surfaces have constant mean curvature. Then, in Section 2.2. the converse of this fact is proved. In fact, we prove that CMC surfaces (suitably parametrized) satisfy the ODE equation discovered

by Delaunay (Proposition 2.2.1. and Remark 2.2.1.), what allows us to conclude the Delaunay's theorem (Theorem 2.2.2.): CMC surfaces of revolution are Delaunay's surfaces. Apart from the elementary cases of planes, spheres and cylinders, there are three cases of Delaunay surfaces, catenoids, unduloids and nodoids, corresponding to the choice of the conic as a parabola, an ellipse, or a hyperbola, respectively (see Chapter 2). The main references used in this chapter have been [1],[2],[3],[5],[7].
In Chapter 3, we use Kenmotsu's approach [6] to obtain in Section 3.1. a parametrization of all surfaces of revolution in $\mathbb{R}^3$ having prescribed mean curvature function $H(s)$ (not necessarily constant). Then, in Section 3.2., the above expression is used in combination with Remark 2.2.1., to obtain another proof that CMC rotation surfaces are Delaunay's surfaces. Our basic references for this chapter have been [3] and [6].
Finally, in Chapter 4 a variational characterization and several physical aspects of CMC surfaces are analysed. In fact, M. Sturm in an appendix to Delaunay's work [2] characterized Delaunay's surfaces variationally. Indeed, they were characterized as the solution to an isoperimetric problem in calculus of variations, since they can be seen as surfaces of revolution having a minimal lateral area for a fixed volume [3],[7]. This isoperimetric variational approach is discussed in Section 4.1. where we follow [7] and [8] to prove Theorem 4.1.1. that says that minimal surfaces ($H = 0$) expressed as a graph are critical for the area functional. We also prove Theorem 4.1.1. that says that CMC surfaces are critical for the area functional for admissible variations preserving the algebraic volume. We notice in Section 4.1. that minimal surfaces ($H = 0$) do not necessarily minimize the area (see Example 4.1.1.). A short physical motivation is given in the preamble to this Chapter 4. Finally, focusing on rotational surfaces, we see in Section 4.2. that CMC surfaces of revolution can be regarded as models for liquid bridges between two vertical plates (Theorem 4.2.1.). The main references for this chapter have been [4],[7],[8] and [9].
All figures in this work have been obtained with Wolfram Mathematica 11.2.

Chapter 1

Preliminaries

Along this work we will mostly be using techniques of Differential Geometry, Differential Calculus and a few basic techniques of Calculus of Variations. While the latter are described in Chapter 4, here we review some fundamental definitions and fix notation. For this, we have used Rafael López's work mentioned in the bibliography [7].

Definition 1.0.1. An *immersed (or parametrized) surface* in $\mathbb{R}^3$ is a map $\mathbf{x} : U \longrightarrow \mathbb{R}^3$ of class C^∞, where $U \subset \mathbb{R}^2$ is an open subset, such that the differential $d\mathbf{x}_q : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is injective for all $q \in U$. The image $S = \mathbf{x}(U)$ of $\mathbf{x}$ is called the *trace* of the immersed surface, where

$$S = \{(x, y, z) \in \mathbb{R}^3 | (x, y, z) = \mathbf{x}(u, v), (u, v) \in U\}.$$

Now, consider the surface $S \subset \mathbb{R}^3$, or $\mathbf{x} = \mathbf{x}(u, v) : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ a differentiable map such that $S = \mathbf{x}(U)$ is a surface of $\mathbb{R}^3$. For each $p \in S$ the tangent plane T_pS is formed by all velocity vectors of curves passing through p at $T_pS = \{\alpha'(0); \alpha : I \longrightarrow S, \alpha(0) = p\}$. Fix $\mathbf{N}(p)$ a unit vector orthogonal to $T_p(s)$. Consider all planes P containing $\mathbf{N}(p)$, which are transverse to S at p. Then $P \bigcap S$ is a planar curve containing p, called a *normal section*. Take the orientation on the curve such that the normal vector to this curve is $\mathbf{N}(p)$. Each plane P is determined by a tangent direction $v \in T_pS$. We parametrize $P = P_v$, where $v \in \mathbb{S}^1(p) = \{v \in T_pS; \|v\| = 1\}$. Denote $\alpha_v = P_v \bigcap S$ and let us use a parametrization of α_v that satisfies $\alpha_v(0) = p$ and $\alpha'_v(0) = v$. Then $k_v(p) = k_{\alpha_v}(0)$, where k is the curvature of α_v at 0. By the compactness of $\mathbb{S}^1(p)$, there exists some $v_1, v_2 \in T_pS$ such that

$$k_1(p) = k_{v_1}(0) = max\{k_v(0); v \in \mathbb{S}^1(p)\},$$

$$k_2(p) = k_{v_2}(0) = min\{k_v(0); v \in \mathbb{S}^1(p)\}.$$

Definition 1.0.2. The numbers $k_i(p)$ are the *principal curvatures* of S at p, and v_i are the *principal directions* for $i = 1, 2$.

The principal directions at each point are orthogonal (this means that $k_1(p) \perp k_2(p)$ for any $p \in S$). Moreover, if we change the normal vector $\mathbf{N}$ to $\widehat{\mathbf{N}} = -\mathbf{N}$, the signs of the principal curvatures change ($\widehat{k_1}(p) = -k_1(p)$ and $\widehat{k_2}(p) = -k_2(p)$).
Now we can define the curvature of S at p as a type of "average" of the principal curvatures, for instance, geometrical or arithmetic average.

Definition 1.0.3. The *Gauss curvature* $K(p)$ and the *mean curvature* $H(p)$ are defined respectively as

$$K(p) = k_1(p)k_2(p) \quad \text{and} \quad H(p) = \frac{k_1(p) + k_2(p)}{2}.$$

All concepts are invariant by rigid motions of space, except perhaps, by a sign. In fact, the change from $\mathbf{N}$ to $\widehat{\mathbf{N}} = -\mathbf{N}$ implies that H changes of sign ($\widehat{H}(p) = -H(p)$) but K does not change ($\widehat{K}(p) = K(p)$).

Definition 1.0.4. A *minimal surface* is a surface whose mean curvature vanishes on the surface ($H(p) = 0$ for any $p \in S$).

Definition 1.0.5. An *orientation* (or a *Gauss map*) on a surface S is a differentiable map $\mathbf{N} : S \longrightarrow \mathbb{R}^3$ such that $\|\mathbf{N}(p)\| = 1$ and $\mathbf{N}(p) \perp T_pS$ for each $p \in S$.

We can restrict the image of $\mathbf{N}$ and write $\mathbb{S}^2$ instead of $\mathbb{R}^3$ since the norm of all vectors $\mathbf{N}(p)$ is one.
Any surface is locally orientable, that is, given a point $p \in S$, there exists a neighbourhood V of p at S such that V is an orientable surface. If $\mathbf{x} : U \longrightarrow \mathbb{R}^3$ is a local parametrization of the surface around p, we define

$$\mathbf{N}(\mathbf{x}(u,v)) = \mathbf{N}(u,v) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}(u,v).$$

Here $\times$ is the cross or vector product and the subscripts u and v denote the corresponding derivatives. Therefore $\mathbf{x}(U) \subset S$ is an open set of S oriented by $\mathbf{N} \circ \mathbf{x}$. We also point out that closed immersed surfaces (compact with no boundary) are orientable thanks to the existence of an interior domain of the surface. Since all the surfaces are locally graphs, we can always find a local parametrization for each point of the surface.
Consider a Gauss map $\mathbf{N} : S \longrightarrow \mathbb{S}^2$ and the differentiable map $d\mathbf{N}_p : T_pS \longrightarrow T_{\mathbf{N}(p)}\mathbb{S}^2 \cong T_pS$. This map is defined by

$$d\mathbf{N}_p(v) = \left(\frac{d}{dt}\mathbf{N}(\alpha(t))\right)\Big|_{t=0}$$

where $\alpha : I \longrightarrow S$ is a curve on S such that $\alpha(0) = p$ and $\alpha'(0) = v$. Then $d\mathbf{N}_p(v)$ is an endomorphism, which is self-adjoint, that is,

$$< d\mathbf{N}_p(u), v > = < u, d\mathbf{N}_p(v) >, \quad u, v \in T_pS,$$

or equivalently, the bilinear form $\sigma_p : T_pS \times T_pS \longrightarrow \mathbb{R}$ given by

$$\sigma_p(u,v) = - < d\mathbf{N}_p(u), v >$$

is symmetric. Both $d\mathbf{N}_p$ and σ_p are diagonalizable.

Definition 1.0.6. The map $-d\mathbf{N}_p : T_pS \longrightarrow T_pS$ is the *Weingarten map* and $\sigma_p : T_pS \times T_pS \longrightarrow \mathbb{R}$ given by $\sigma_p(u,v) = - < d\mathbf{N}_p(u), v >$ is the *second fundamental form.*

Moreover, it is known that the Weingarten map is diagonalizable and

Theorem 1.0.1. *The eigenvalues of $-d\boldsymbol{N}_p$ are precisely the principal curvatures k_1 and k_2.*

Thus, the matrix associated to $-d\mathbf{N}_p$ can be diagonalized as

$$\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix},$$

and, therefore,

$$H(p) = -\frac{1}{2} trace(d\mathbf{N}_p) = -\frac{1}{2}(k_1(p) + k_2(p)).$$

Theorem 1.0.2. *In local coordinates $\boldsymbol{x} = \boldsymbol{x}(u,v)$, the mean curvature H is given by the formula*

$$H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}, \tag{1.1}$$

where

$$\begin{array}{lll} E = < \boldsymbol{x}_u, \boldsymbol{x}_u >, & F = < \boldsymbol{x}_u, \boldsymbol{x}_v >, & G = < \boldsymbol{x}_v, \boldsymbol{x}_v >, \\ e = < \boldsymbol{x}_{uu}, \boldsymbol{N} >, & f = < \boldsymbol{x}_{uv}, \boldsymbol{N} >, & g = < \boldsymbol{x}_{vv}, \boldsymbol{N} > . \end{array}$$

Roughly speaking, a surface of revolution is swept out by rotating a curve of the plane OXY around the OX axis. More precisely,

Definition 1.0.7. Let $\alpha(s) = (x(s), y(s))$ be a curve in the plane $z = 0$ in $\mathbb{R}^3$. Call I to the open subset of $\mathbb{R}$ where α is defined and suppose that $y(s) > 0$ in I. Then, we can define a *surface of revolution* S_α in $\mathbb{R}^3$, whose *generatrix* is α and whose rotation axis is the x axis, as

$$S_\alpha = \{(x(s), y(s)\cos\theta, y(s)\sin\theta) \in \mathbb{R}^3 \mid s \in I,\ 0 \le \theta \le 2\pi\}. \tag{1.2}$$

Then, the map $\mathbf{x}(s,\theta) = (x(s), y(s)\cos\theta, y(s)\sin\theta)$ can be used to parametrize the surface.
In general, we will assume that α is an arc-length parametrized curve, that is, $(x'(s))^2 + (y'(s))^2 = 1$. For the surface of revolution, by differentiating

the given parametric relation (1.2) with respect to the parameters s and θ, we get

$$\begin{aligned}
\mathbf{x}_s(s,\theta) &= (x'(s), y'(s)\cos\theta, y'(s)\sin\theta), \\
\mathbf{x}_\theta(s,\theta) &= (0, -y(s)\sin\theta, y(s)\cos\theta). \\
\mathbf{x}_{ss} &= (x''(s), y''(s)\cos\theta, y''(s)\sin\theta) \\
\mathbf{x}_{s\theta} &= (0, -y'(s)\sin\theta, y'(s)\cos\theta) \\
\mathbf{x}_{\theta\theta} &= (0, -y(s)\cos\theta, -y(s)\sin\theta). \\
\mathbf{x}_s \times \mathbf{x}_\theta &= (y(s)y'(s), -y(s)x'(s)\cos\theta, -y(s)x'(s)\sin\theta) \\
\|\mathbf{x}_s \times \mathbf{x}_\theta\| &= \sqrt{y(s)^2y'(s)^2 + y(s)^2x'(s)^2} = \sqrt{y(s)^2} = y(s)
\end{aligned}$$

If we compute the unit normal vector, we get

$$\mathbf{N} = (y'(s), -x'(s)\cos\theta, -x'(s)\sin\theta)$$

By using the definitions in Theorem 1.0.2, we obtain the expressions for the coefficients of the First and Second Fundamental Forms

$$\begin{aligned}
E &= x'(s)^2 + y'(s)^2(\cos^2\theta + \sin^2\theta) = x'(s)^2 + y'(s)^2 = 1, \\
F &= -y'(s)\cos\theta y(s)\sin\theta + y'(s)\sin\theta y(s)\cos\theta = 0, \\
G &= y(s)^2\sin^2\theta + y(s)^2\cos^2\theta = y(s)^2, \\
e &= y'(s)x''(s) - x'(s)y''(s)\cos^2\theta - x'(s)y''(s)\sin^2\theta \\
&= y'(s)x''(s) - x'(s)y''(s), \\
f &= x'(s)y'(s)\sin\theta\cos\theta - x'(s)y'(s)\sin\theta\cos\theta = 0, \\
g &= y(s)x'(s)\cos^2\theta + y(s)x'(s)\sin^2\theta = y(s)x'(s).
\end{aligned}$$

Next, the mean curvature $H(s)$ of the surface can be easily calculated using the relation given in (1.1). After some simplifications, we obtain

$$\begin{aligned}
H(s) &= \frac{y(s)^2x''(s)y'(s) - y(s)^2y''(s)x'(s) + x'(s)y(s)}{2y(s)^2} \\
&= \frac{y(s)x''(s)y'(s) - y(s)y''(s)x'(s) + x'(s)}{2y(s)}. \qquad (1.3)
\end{aligned}$$

Chapter 2

Delaunay surfaces

In this chapter we describe the classical construction of CMC surfaces due to Delaunay [5] and state Delaunay's theorem (Theorem 2.2.2.). As it has been said in the introduction, this is achieved by first finding a suitable parametrization of Delaunay's surfaces to show that they are CMC surfaces, and then by characterizing CMC rotation surfaces (Proposition 2.2.1.) in terms of an ODE discovered by Delaunay (Remark 2.2.1.).

2.1 Roulettes of conics, Delaunay's surfaces and their parametrizations

Remember that *conics* are those curves that we get when we intersect a cone and a plane. Then, we have the following definitions:

Definition 2.1.1. Given a curve that rolls over another fixed curve without slipping, then a point of the moving curve describes a curve that is called *roulette*.

Definition 2.1.2. Surfaces of revolution whose profile curve is the roulette of the focus of a conic rolling over a line, and whose axis of revolution is the given line, are called *Delaunay surfaces*.

We will find a parametrization of the trace of the focus $F' = (F_1(t), F_2(t))$ of a conic when it rolls on a straight line. Consider a point $P = (P_1, P_2)$ of the conic, T the tangent line of the conic that goes through P, the focus of the conic F, R the perpendicular line to T that goes through F and Q, where Q is the intersection between T and R.

Our goal now is to parametrize the curve that describes the trace of the focus $F = (F_1, F_2)$. Apart from the elementary curves of spheres and cylinders, there are three classes of Delaunay surfaces: catenoids, unduloids and nodoids corresponding to the choice of conics as parabolas, ellipses or hyperbolas, respectively. To fix ideas, let us start by assuming that the conic

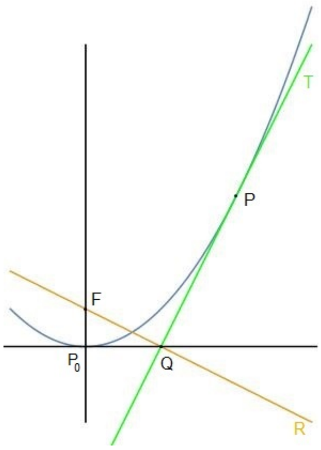


Figure 2.1: Initial position of a roulette of a parabola.

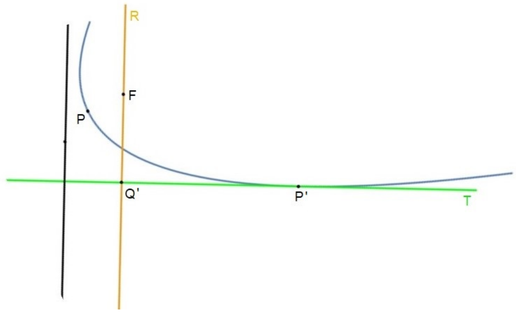


Figure 2.2: 2nd position.

is a parabola. We can see in Figure 2.2. that the first coordinate of the focus and the first coordinate of $Q' = (Q_1, Q_2)$ are the same,

$$F_1(t) = Q_1 = s(t) - |\overrightarrow{P'Q'}| = s(t) - |\overrightarrow{PQ}|; \tag{2.1}$$

where $s(t)$ is the arclength parabola from its vertex P_0 to P, and $|\overrightarrow{PQ}|$ is the length from P to Q, or, in other words, the length from Q_1 to P_1. Instead of rolling the conic over a line we could interchange roles, fix the conic and use the tangent line at each point of the conic.
A parametrization of the parabola is given by $\alpha(t) = \big(2b\sinh t, b\sinh^2 t\big)$, where $b > 0$ and $t \in [t_1, t_2]$. The arc length for the parabola from $t_0 = 0$ to t is given by

$$s(t) = \int_{t_0}^{t} |\alpha'(u)|du = b(t + \sinh t\cosh t).$$

The line T is parametrized as

$$T: \quad (2b\sinh t, b\sinh^2 t) + \lambda(2b\cosh t, 2b\sinh t\cosh t)$$

since it goes through the point $P = \alpha(t)$ of the parabola and has the direction vector $\alpha'(t)$. Then, considering that Q is the intersection between the lines $y = 0$ and T, equalling the second coordinate of T with zero, we get that $Q = (b\sinh t, 0)$. It follows that $\overrightarrow{PQ} = (-b\sinh t, -b\sinh^2 t)$ and the length of the segment $\overline{PQ}$ is

$$|\overrightarrow{PQ}| = b\sinh t\cosh t.$$

Then,

$$F_1(t) = s(t) - |\overrightarrow{PQ}| = bt.$$

Now, we compute $F_2 = |\overrightarrow{FQ}|$, where $|\overrightarrow{FQ}|$ is the length of the segment that goes from F to Q. The segment R is parametrized using the fact that goes through Q and that its direction vector is perpendicular to $\alpha'(t)$:

$$R: \quad (b\sinh t, 0) + \lambda(-2b\sinh t\cosh t, 2b\cosh t).$$

Then, since F is the intersection between R and $x = 0$, we calculate it by equalling the first coordinate of R with zero, and we get $F = (0, b)$. It follows that $\overrightarrow{FQ} = (b\sinh t, -b)$ and $F_2(t) = |\overrightarrow{FQ}| = b\cosh t$, thus the parametrization of the focus is given by

$$A(t) = (F_1(t), F_2(t)) = (bt, b\cosh t).$$

Then $A(t)$ gives a parametrization of the catenary, whose surface of revolution is the catenoid (see figures 2.3. and 2.4.).

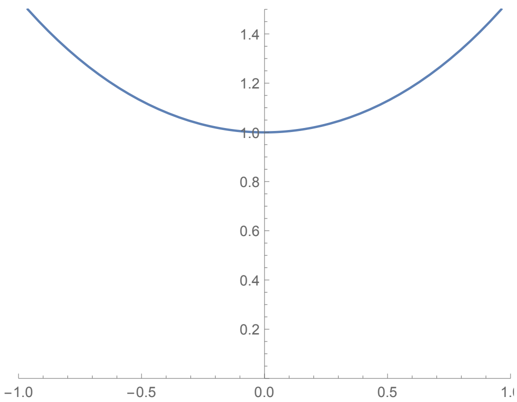


Figure 2.3: Catenary.

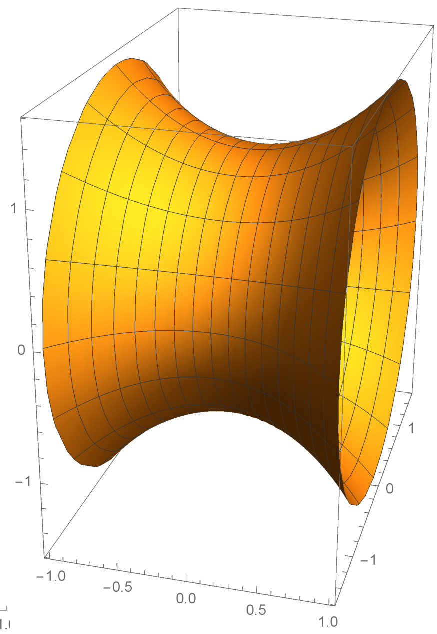


Figure 2.4: Catenoid.

For the ellipse, parametrized by $\beta(t) = (a\cos t, b\sin t)$, with $b < a$, $c = \sqrt{a^2 - b^2}$ and $t \in [t_1, t_2]$, the arc length from t_0 to t is

$$s(t) = \int_{t_0}^{t} |\beta'(z)|dz = \int_{t_0}^{t} \sqrt{a^2 - c^2\cos^2 z}dz.$$

Since the ellipse has two foci, in this case two curves are generated. Taking

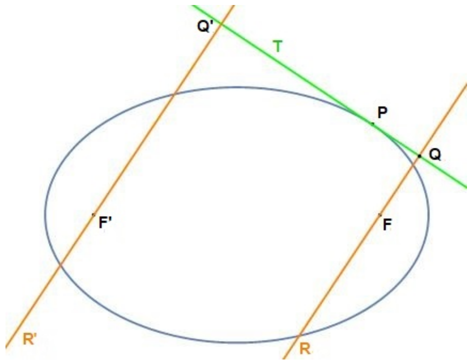


Figure 2.5: Ellipse.

F, the closest focus to the tangent, the length of the segment that goes from P to Q is

$$|\overrightarrow{PQ}| = \frac{c\sin t(a - c\cos t)}{\sqrt{a^2 - c^2\cos^2 t}}.$$

Thus

$$F_1 = s(t) - |\overrightarrow{PQ}| = \int_{t_0}^{t} \sqrt{a^2 - c^2\cos^2 z}dz - \frac{c\sin t(a - c\cos t)}{\sqrt{a^2 - c^2\cos^2 t}}.$$

Now, F_2 corresponds to the length of the segment that goes from F to Q, thus,

$$F_2 = |\overrightarrow{FQ}| = \frac{b(a - c\cos t)}{\sqrt{a^2 - c^2\cos^2 t}}.$$

Hence, $B_1(t) = (F_1, F_2)$ is therefore the parametrization of the roulette generated by the focus of the ellipse. Choosing the other focus F', it follows after computing the length of $\overline{PQ'}$ that the first coordinate is

$$F_1 = \int_{t_0}^{t} \sqrt{a^2 - c^2\cos^2 z}dz - \frac{c\sin t(a + c\cos t)}{\sqrt{a^2 - c^2\cos^2 t}},$$

and the second coordinate is the length of the segment from F' to Q', thus

$$F_2 = \frac{b(a + c\cos t)}{\sqrt{a^2 - c^2\cos^2 t}},$$

and $B_2(t) = (F_1, F_2)$ is, therefore, the parametrization of the roulette generated by the focus F' of the ellipse. The roulette of the focus of an ellipse is called the undulary.

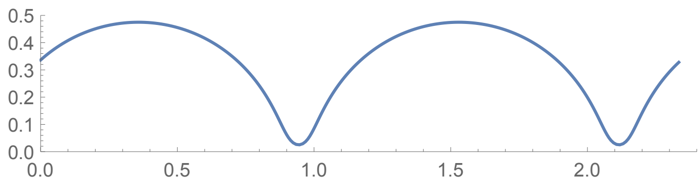


Figure 2.6: Undulary.

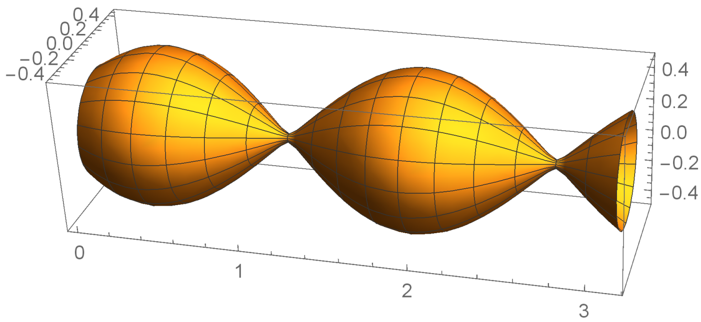


Figure 2.7: Unduloid.

Now, we consider the hyperbola parametrized by $\gamma(t) = (a\cosh t, b\sinh t)$ with $a, b > 0$, $c = \sqrt{a^2 + b^2}$ and $t \in [t_1, t_2]$. The arc length from t_0 to t is

$$s(t) = \int_{t_0}^{t} |\gamma'(z)|dz = \int_{t_0}^{t} \sqrt{c^2\cosh^2 z - a^2}dz.$$

First, we consider F, the closest focus to the tangent. By computing the length of the segment $\overline{PQ}$ it then follows that the first coordinate of the trace of the focus is

$$F_1 = s(t) - |\overrightarrow{PQ}| = \int_{t_0}^{t} \sqrt{c^2\cosh^2 z - a^2}dz - \frac{c\sinh t(c\cosh t - a)}{\sqrt{c^2\cosh^2 t - a^2}},$$

and the second coordinate is given by the length of $\overline{FQ}$, namely,

$$F_2 = |\overrightarrow{FQ}| = \frac{b(c\cosh t - a)}{\sqrt{c^2\cosh^2 t - a^2}}.$$

$C_1(t) = (F_1, F_2)$ is therefore the parametrization of the roulette generated by the focus F of the hyperbola.
Taking the focus F' instead, and computing the length of the segment $\overline{PQ'}$ or the first coordinate of the focus is

$$F_1 = s(t) - |\overrightarrow{PQ'}| = \int_{t_0}^{t} \sqrt{c^2\cosh^2 z - a^2}dz - \frac{c\sinh t(c\cosh t + a)}{\sqrt{c^2\cosh^2 t - a^2}},$$

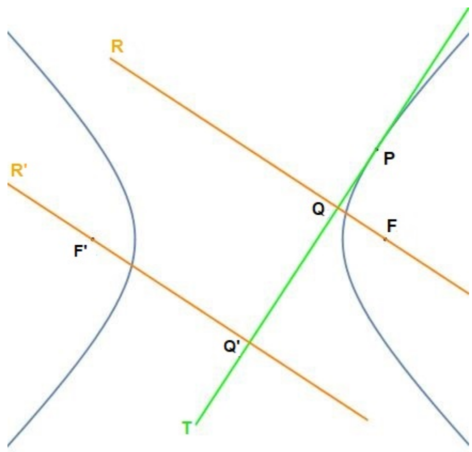


Figure 2.8: Hyperbola.

and the second coordinate is the length of $\overline{F'Q'}$, thus,

$$F_2 = |\overrightarrow{F'Q'}| = \frac{b(c\cosh t + a)}{\sqrt{c^2\cosh^2 t - a^2}}.$$

$C_2(t) = (F_1, F_2)$ is therefore the parametrization of the roulette generated by the focus F'. The roulette of the focus of a hyperbola is called the nodary (see images 2.9. and 2.10.).

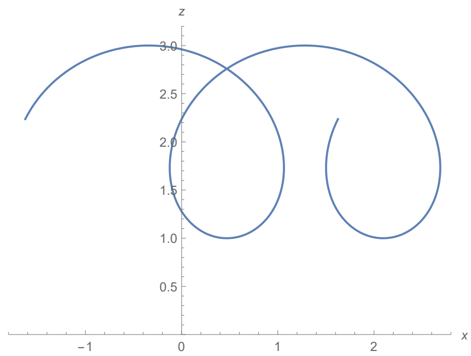


Figure 2.9: Nodary.

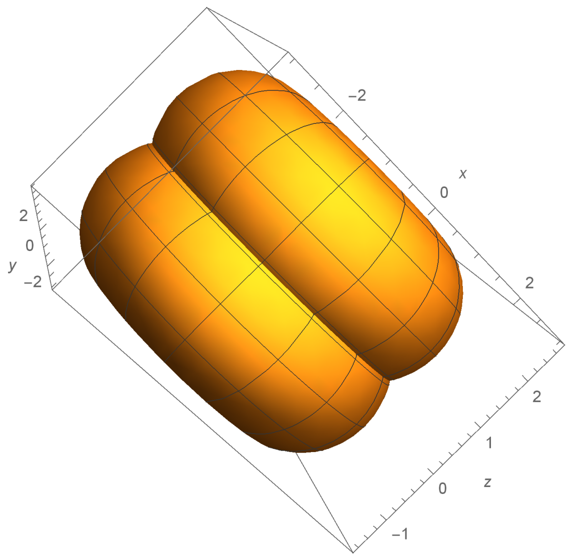


Figure 2.10: Nodoid.

Then, the surfaces of revolution generated by the curves $A(t)$, $B_1(t)$, $B_2(t)$, $C_1(t)$ and $C_2(t)$ obtained in the previous lines admit the following parametrization:

$$\mathbf{x}(t,v) = (F_1(t), F_2(t)\cos v, F_2(t)\sin v)\,, \tag{2.2}$$

which correspond respectively with

- Catenoids, if the profile curve is $A(t)$.
- Unduloids, if the profile curve is either $B_1(t)$ or $B_2(t)$.
- Nodoids, if the profile curve is either $C_1(t)$ or $C_2(t)$.

Moreover, using the above parametrization (2.2) and Theorem 1.0.2, it can be seen after a direct long calculation that $H = 0$ for catenoids, $H = \dfrac{1}{2a}$ with $a > 0$ for unduloids and $H = -\dfrac{1}{2a}$ with $a > 0$ for nodoids. In other words, Delaunay surfaces have constant mean curvature.

2.2 Delaunay's theorem

Basically, Delaunay surfaces are the only surfaces of revolution with constant mean curvature. This result was obtained by Delaunay in [2] using geometric properties of roulettes. On the other hand, the following remark will be fundamental in our approach to Delaunay's result.

Remark 2.2.1. Using again the geometric properties of conics and roulettes, it has been shown in [2] that if $\alpha(s) = (x(s), y(s))$ is an arc-length parametrization of the roulette of the focus of a given conic β, then,

- β is a parabola if and only if $\dfrac{dx}{ds}(s) = \dfrac{a}{y(s)}$ and $\alpha(s)$ is a catenary (S_α is a catenoid).
- β is an ellipse if and only if $\dfrac{dx}{ds}(s) = \pm\dfrac{y^2(s)+b^2}{2ay(s)}$ and $\alpha(s)$ is an undulary (S_α is an unduloid).
- β is a hyperbola if and only if $\dfrac{dx}{ds}(s) = \pm\dfrac{y^2(s)-b^2}{2ay(s)}$ and $\alpha(s)$ is a nodary (S_α is a nodoid).

Here a and b are real constants and $a > 0$.

Now, leaving aside the most basic cases of planes and cylinders, we can prove the following result:

Proposition 2.2.1. *Assume that $\alpha(s) = (x(s), y(s))$ is an arc-length parametrized planar curve and denote by S_α the surface of revolution obtained by revolving α around the x-axis. Assume that S_α is not a cylinder nor a plane. Then,*

- *S_α is minimal, that is, $H = 0$, if and only if, $\frac{dx}{ds}(s)\, y(s) = c$, for some $c \in \mathbb{R}$.*
- *S_α has non-zero mean curvature $H = \frac{1}{2a}$, for $a \in \mathbb{R} \setminus \{0\}$, if and only if, $\frac{dx}{ds}(s)y(s) + A = \frac{1}{2a}y^2(s)$, with $A \in \mathbb{R}$. Moreover, S_α is a sphere if and only if $A = 0$.*

Proof. From (1.3) and since α is arc-length parametrized, we get the following system of differential equations:

$$\begin{cases} 2H(s)y(s) - y(s)x''(s)y'(s) + y(s)y''(s)x'(s) - x'(s) = 0 \\ x'(s)^2 + y'(s)^2 = 1 \end{cases}$$

Solving this system we will obtain an explicit formula for the generatrix curve of a general surface of revolution.
Among the proof we will use some equalities that are obtained from the fact that α is arc-length parametrized:

$$\begin{aligned} x'(s)^2 + y'(s)^2 = 1 \quad &\Rightarrow \quad 2x'(x)x''(s) + 2y'(s)y''(s) = 0 && (2.3) \\ &\Rightarrow \quad -x'(s)x''(s) = y'(s)y''(s) && (2.4) \end{aligned}$$

and
$$x'(s)^2 + y'(s)^2 = 1 \Rightarrow -x'(s)^2 = y'(s)^2 - 1.$$
Multiplying the first equation of the system by $x'(s)$ we get
$$2H(s)y(s)x'(s) - y(s)x''(s)y'(s)x'(s) + y(s)y''(s)x'(s)^2 - x'(s)^2 = 0.$$
Using the equation (2.4), the term $-y(s)x''(s)y'(s)x'(s)$ becomes $y(s)y'(s)^2y''(s)$; then,
$$\begin{aligned} 0 &= 2H(s)y(s)x'(s) + y(s)y'(s)^2y''(s) + y(s)y''(s)x'(s)^2 - x'(s)^2, \\ 0 &= 2H(s)y(s)x'(s) + y(s)y''(s)(y'(s)^2 + x'(s)^2) - x'(s)^2, \\ 0 &= 2H(s)y(s)x'(s) + y(s)y''(s) - x'(s)^2, \\ 0 &= 2H(s)y(s)x'(s) + y(s)y''(s) + y'(s)^2 - 1. \end{aligned}$$
Since $(y(s)y'(s))' = y'(s)y'(s) + y(s)y''(s)$ we can rewrite the equation as
$$2H(s)y(s)x'(s) + (y(s)y'(s))' - 1 = 0 \tag{2.5}$$
Now we multiply the first equation of the initial system by $y'(s)$:
$$2H(s)y(s)y'(s) - y(s)x''(s)y'(s)^2 + y(s)y''(s)x'(s)y'(s) - x'(s)y'(s) = 0.$$
Using (2.4), the term $y(s)y''(s)x'(s)y'(s)$ becomes $-y(s)x'(s)^2x''(s)$, so
$$2H(s)y(s)y'(s) - y(s)x''(s)(y'(s)^2 + x'(s)^2) - x'(s)y'(s) = 0.$$
For the second equation of the initial system,
$$2H(s)y(s)y'(s) - y(s)x''(s) - x'(s)y'(s) = 0.$$
Lastly, since $(y(s)x'(s))' = y'(s)x'(s) + y(s)x''(s)$, we get
$$2H(s)y(s)y'(s) - (y(s)x'(s))' = 0. \tag{2.6}$$
Let us prove the first point. Assume that $H = 0$. Then, from (2.6) we know that $(x(s)y'(s))' = 0$, and this means that $y(s)x'(s) = c$ for some $c \in \mathbb{R}$.
For the other implication, assume that $x(s)y'(s) = c$ is constant. Then (2.6) becomes $2H(s)y(s)y'(s) = 0$ or $H(s)(y^2(s))' = 0$. Then two things can happen: either $H(s) = 0$ or $(y^2(s))' = 0$. Let us see that the second option implies that the surface is a cylinder. If $(y^2(s))' = 0$, then $y^2(s) = C$ is constant, and it follows that $y(s) = K$ is also constant. Then, $y'(s) = 0$ and from $x'(s)^2 + y'(s)^2 = 1$, we get that $x'(s)^2 = 1$ or $x'(s) = \pm 1$, thus, $x(s) = \pm s + B$, so y is constant for any x and the generated surface is a cylinder.
Now we will prove the second point. Firstly, assume that $H = \dfrac{1}{2a}$. Then, equation (2.6) becomes $(Hy(s)^2 - y(s)x'(s))' = 0$, thus, $Hy(s)^2 - y(s)x'(s) =$

A with $A \in \mathbb{R}$ is constant and considering that $H = \dfrac{1}{2a}$, it follows $x'(s)y(s) + A = \dfrac{1}{2a}y^2(s)$.
Notice that if $A = 0$, then we easily obtain the sphere. In fact, if $A = 0$, we have

$$x'(s)y(s) = \frac{1}{2a}y(s)^2 \quad \Rightarrow \quad x'(s) = \frac{1}{2a}y(s) \tag{2.7}$$

Moreover, the equation $x'(s)^2 + y'(s)^2 = 1$ implies that there exists $\theta(s)$ such that

$$x'(s) = \sin\theta(s) \tag{2.8}$$
$$y'(s) = \cos\theta(s) \tag{2.9}$$

Differentiating (2.7) and using (2.8) and (2.9), we have

$$x''(s) = \frac{1}{2a}y'(s) \quad \Rightarrow \quad \theta'(s)\cos\theta(s) = \frac{1}{2a}\cos\theta(s) \quad \Rightarrow \quad \theta'(s) = \frac{1}{2a}$$
$$\Rightarrow \quad \theta(s) = \frac{1}{2a}s + b.$$

Then,

$$\begin{aligned} x(s) &= -2a\cos\left(\frac{1}{2a}s + b\right) + c_1, \\ y(s) &= 2a\sin\left(\frac{1}{2a}s + b\right) + c_2. \end{aligned}$$

Since the equation

$$(x(s) - c_1)^2 + (y(s) - c_2)^2 = 4a^2$$

is satisfied, $\alpha(s) = (x(s), y(s))$ is a circle and S_α is a sphere.
For the other implication, suppose that $x'(s)y(s) + A = \dfrac{1}{2a}y^2(s)$ is satisfied. Deriving this equation, we get $\dfrac{1}{a}y(s)y'(s) = (y(s)x'(s))'$ and using (2.6) we derive $\dfrac{1}{a}y(s)y'(s) = 2H(s)y(s)y'(s)$, thus, $\left(2H(s) - \dfrac{1}{a}\right)y(s)y'(s) = 0$ or $\left(2H(s) - \dfrac{1}{a}\right)\left(\dfrac{1}{2}y^2(s)\right)' = 0$. Then two things can happen: either $2H(s) - \dfrac{1}{a} = 0$ (and it follows that $H = \dfrac{1}{2a}$) or $(\frac{1}{2}y^2(s))' = 0$. We have shown before that the second option implies that the surface of revolution is a cylinder, so the proposition follows. □

Combining the above Proposition 2.2.1 with Remark 2.2.1 we have the following theorem.

$$E = 1 + f'(u)^2, \quad F = 0, \qquad G = f(u)^2,$$
$$e = \frac{-f''}{\sqrt{1+f'^2}}, \quad f = 0, \quad g = \frac{f}{\sqrt{1+f'^2}},$$

Theorem 2.2.2. *(Delaunay's theorem) An arc-length parametrized curve α generates a surface S_α with constant mean curvature when we rotate it around the x axis, if and only if, α is the roulette of a focus of a conic.*

Finally, for later use in Section 4.2., we obtain another version of Remark 2.2.1, when the profile curve is described as a curve parametrized as a graph $\alpha(u) = (u, f(u))$. This can always be achieved locally. Let us consider a surface of revolution given by the parametrization

$$\mathbf{x}(u,v) = (u, f(u)\cos v, f(u)\sin v),$$

where u belongs to an open interval I of the real line, $f(u)$ is a real-valued smooth function, v belongs to the interval $(0, 2\pi)$ and $f(u) > 0$ for any $u \in I$. By differentiating the parametric relation $\mathbf{x}(u,v)$, with respect to the parameters u and v, we get

$$\mathbf{x}_u(u,v) = (1, f'(u)\cos v, f'(u)\sin v),$$
$$\mathbf{x}_v(u,v) = (0, -f(u)\sin v, f(u)\cos v).$$

We obtain the expressions for the coefficients of the First and Second fundamental Forms with the unit normal $\mathbf{N} = \dfrac{1}{\sqrt{1+f'^2}}(f', -\cos v, -\sin v)$. Next, with the aid of the above calculated coefficients values, the mean curvature H of the surface can be easily calculated using the relation in Theorem 1.0.2. After some simplifications, we obtain

$$H = \frac{-ff'' + 1 + f'^2}{2f\sqrt{(1+f'^2)^3}}. \tag{2.10}$$

The following theorem is a well-known consequence of the equation above.

Theorem 2.2.3. *Assume that a surface of revolution S other than a cylinder is parametrized by*

$$\boldsymbol{x}(u,v) = (u, f(u)\cos v, f(u)\sin v).$$

- *Then $H = 0$ if and only if it is part of a catenoid.*
- *$H = \pm\dfrac{1}{2a}$ is constant if and only if the function $f(u)$ satisfies*

$$f^2 \pm \frac{2af}{\sqrt{1+f'^2}} = \pm b^2, \tag{2.11}$$

where a and b are positive constants.

Proof. Let us consider the equation of the mean curvature of a surface of revolution given in (2.10). Suppose that $H = c/2$ is constant. Then,

- If $c = 0$: the equation for the curvature's equation becomes $1 + f'^2 - ff'' = 0$, and it can be shown that its general solution is $f(v) = \frac{1}{e}\cosh(cu+d)$, which means that $\alpha(u)$ is a catenary. Then, if a surface of revolution is minimal ($H = 0$), it is part of a catenoid. The opposite direction of the proof follows directly.

- If $c \neq 0$: we get the differential equation $1 + f'^2 - ff'' = cf(1 + f'^2)^{3/2}$. If $c = -\frac{1}{a}$ with $a > 0$, we can rewrite the equation as $\frac{a(1+f'^2) - ff''a}{(1+f'^2)^{3/2}} + f = 0$. If f is constant, $(x(u), y(u)) = (u, f(u))$ gives an horizontal line and S is a cylinder. Assume f is not constant. Then multiplying it by $2f'$, it becomes

$$\frac{2af'(1+f'^2) - 2ff'f''a}{(1+f'^2)^{3/2}} + 2ff' = 0 \quad \Rightarrow \quad \frac{d}{du}\left[\frac{2af}{\sqrt{1+f'^2}} + f^2\right] = 0.$$

Thus,

$$\frac{2af}{\sqrt{1+f'^2}} + f^2 = \pm b^2,$$

and repeating the process for $c = \frac{1}{a}$ we get

$$\frac{2af}{\sqrt{1+f'^2}} - f^2 = \pm b^2. \tag{2.12}$$

For the opposite implication of the theorem we just need to follow the steps that we have done from the end to the beginning.

□

Notice that equation (2.12) can be rewritten as

$$\frac{1}{\sqrt{1+f'^2}} = \frac{f^2 \pm b^2}{2af}, \tag{2.13}$$

and our profile curve is $\alpha(u) = (u, f(u))$, so, the arc-length parameter is

$$s(u) = \int_0^u \sqrt{1+f'^2(t)}dt.$$

Deriving it we get

$$\frac{ds}{du} = \sqrt{1+f'^2(u)}, \quad \frac{du}{ds} = \frac{1}{\sqrt{1+f'^2(u)}},$$

and substituting it in (2.13) we get

$$\frac{dx}{ds} = \frac{du}{ds} = \frac{f^2 \pm b^2}{2af} = \frac{y(s)^2 \pm b^2}{2ay(s)}.$$

From Remark 2.2.1 we know that

$$\frac{dx}{ds} = \frac{y(s)^2 + b^2}{2ay(s)} \quad \text{or} \quad f^2 \pm \frac{2af}{\sqrt{1+f'^2}} = -b^2 \tag{2.14}$$

corresponds to the undulary (and rotating it we get the unduloid) and

$$\frac{dx}{ds} = \frac{y(s)^2 - b^2}{2ay(s)} \quad \text{or} \quad f^2 \pm \frac{2af}{\sqrt{1+f'^2}} = b^2 \tag{2.15}$$

gives the nodary (and rotating it we get the nodoid). It is an amazing fact discovered by Delaunay that this differential equation (2.11) arises geometrically. There is a geometric construction which produces the differential equation above and, consequently, all surfaces of revolution of constant mean curvature.

Chapter 3

Surfaces of revolution with prescribed mean curvature

In the previous chapters we have seen how to obtain the mean curvature for any surface of revolution and what Delaunay surfaces are. Now we will analyze another problem: the following theorem tells us how we can find all surfaces of revolution whose mean curvature is a prescribed smooth function.

3.1 Kenmotsu's approach

Here, we will use Kenmotsu's approach to the problem [6] and we will obtain another proof of Delaunay's theorem when H is constant.

Theorem 3.1.1. *(Kenmotsu's solution) Given a continuous function $H = H(s)$ with $s \in I$, the generatrix curve of a surface of revolution with mean curvature $H(s)$ is given by*

$$\begin{aligned}\alpha(s, H(s), a, b, c) \quad = \quad & (\int_0^s \frac{(G(t)+b)F'(t) - (F(t)-a)G'(t)}{\sqrt{(F(t)-a)^2 + (G(t)+b)^2}} dt + c, \\ & \sqrt{(F(s)-a)^2 + (G(s)+b)^2})\end{aligned}$$

where

$$F(s) := \int_0^s \sin\left(2\int_0^u H(t)dt\right) du, \qquad G(s) := \int_0^s \cos\left(2\int_0^u H(t)dt\right) du$$

and $a, b, c \in \mathbb{R}$.

Proof. Introducing the complex notation, we consider the expression (2.5)$-i$(2.6), where $i = \sqrt{-1}$ is the imaginary unity number. This expression is

$$2H(s)y(s)x'(s) + (y(s)y'(s))' - 1 - 2iH(s)y(s)y'(s) + i(y(s)x'(s))' = 0.$$

If we define in an appropriate way

$$Z(s) := y(s)y'(s) + i(y(s)x'(s)), \tag{3.1}$$

this function satisfies, for the equation above, the ordinary complex differential equation of order one

$$Z'(s) - 2iH(s)Z(s) - 1 = 0. \tag{3.2}$$

Now, our objective is to solve the equation above. Using the integrating factor $u(s) = \exp\left(-\int 2iH(s)\, ds\right)$ we obtain that the general solution of (3.2)

$$Z(s) = \left(\int_0^s \exp\left(-\int_0^u 2iH(t)dt\right) du + K\right) \exp\left(\int_0^s 2iH(u)du\right). \tag{3.3}$$

Now we have

$$\begin{aligned} \exp\left(\int_0^u 2iH(t)dt\right) &= \cos\left(2\int_0^u H(t)dt\right) + i\sin\left(2\int_0^u H(t)dt\right), \\ \exp\left(-\int_0^u 2iH(t)dt\right) &= \cos\left(2\int_0^u H(t)dt\right) - i\sin\left(2\int_0^u H(t)dt\right). \end{aligned}$$

Then, the solution of the differential equation is

$$\begin{aligned} Z(s) &= \left(\int_0^s \left[\cos\left(2\int_0^u H(t)dt\right) - i\sin\left(2\int_0^u H(t)dt\right)\right] du + K\right) \\ &\quad \times \left(\cos\left(2\int_0^u H(t)dt\right) + \sin\left(2\int_0^u H(t)dt\right)\right). \end{aligned}$$

If we define the functions F and G as it says in the theorem

$$F(s) = \int_0^s \sin\left(2\int_0^u H(t)dt\right) du, \quad G(s) = \int_0^s \cos\left(2\int_0^u H(t)dt\right) du,$$

we can rewrite $Z(s)$ as

$$Z(s) = (G(s) - iF(s) + b + ai)(G'(s) + iF'(s)),$$

where $K = b + ai$. Now we will write $Z(s)$ in a more convenient way. Since $1 = -i^2 = -ii$, if we multiply $Z(s)$ with $-ii$:

$$Z(s) = -ii(-i(F(s) - a) + G(s) + b)(G'(s) + iF'(s)),$$

we multiply the first factor by i and the second one by $-i$:

$$\begin{aligned} Z(s) &= i(-i(F(s) - a) + G(s) + b)(-i)(G'(s) + iF'(s)) \\ &= (F(s) - a + i(G(s) + b))(F'(s) - iG'(s)). \end{aligned}$$

Thus, finally we get

$$Z(s) = ((F(s) - a) + i(G(s) + b)) \left(F'(s) - iG'(s)\right). \tag{3.4}$$

From this expression we can get the generatrix curve $\alpha(s) = (x(s), y(s))$ of the surface of revolution that we were looking for. We will start from the calculus of $y(s)$. For this, we calculate $Z(s)$'s modulus using its definition in (3.1) and we get

$$\begin{aligned} \|Z(s)\| &= \|y(s)y'(s) + iy(s)x'(s)\| \\ &= \sqrt{y(s)^2 y'(s)^2 + y(s)^2 x'(s)^2} = \sqrt{y(s)^2} = y(s). \end{aligned}$$

On the other hand, from the expression obtained in (3.4) we have

$$y(s) = \|Z(s)\| = \| \left((F(s) - a) + i(G(s) + b)\right) \left(F'(s) - iG'(s)\right) \|. \tag{3.5}$$

Since the modulus of the product of two complex numbers is the product of the modulus and

$$\|F'(s) - iG'(s)\| = \sqrt{\cos^2\left(2\int H(s)ds\right) + \sin^2\left(2\int H(s)ds\right)} = 1,$$

we get that

$$\begin{aligned} y(s) &= \| \left((F(s) - a) + i(G(s) + b)\right) \| \| \left(F'(s) - iG'(s)\right) \| \\ &= \sqrt{(F(s) - a)^2 + (G(s) + b)^2}, \end{aligned}$$

then, we get that

$$\boxed{y(s) = \sqrt{(F(s) - a)^2 + (G(s) + b)^2}.}$$

Now, we will calculate $x(s)$. Using the definition of $Z(s)$ in (3.1), if we subtract its conjugate, we have

$$Z(s) - \overline{Z(s)} = y(s)y'(s) + iy(s)x'(s) - (y(s)y'(s) - iy(s)x'(s)) = 2iy(s)x'(s).$$

Then,

$$x'(s) = \frac{Z(s) - \overline{Z(s)}}{2iy(s)}. \tag{3.6}$$

The numerator is just 2 times the imaginary part of (3.4), in the denominator we substitute $y(s)$ by the formula that we have just obtained

$$x'(s) = \frac{-2i(F(s) - a)G'(s) + 2i(G(s) + b)F'(s)}{2i\sqrt{(F(s) - a)^2 + (G(s) + b)^2}},$$

and integrating we finally get

$$\boxed{x(s) = \int_0^s \frac{(G(t) + b)F'(t) - (F(t) - a)G'(t)}{\sqrt{(F(t) - a)^2 + (G(t) + b)^2}} dt + c.}$$

□

3.2 Finding CMC rotational surfaces using Kenmotsu's theorem

In this section we will see that if we choose H to be constant in Kenmotsu's theorem's proof, we get the plane, the cylinder, the sphere, the catenoid, the unduloid and the nodoid. In other words, we recover Delaunay's surfaces.

- **Zero constant mean curvature**

When $H(s) = 0$, the formula (3.2) in Kenmotsu's theorem's proof gives $Z'(s) = 1$ and the solution

$$Z(s) = s + C = s + c_1 + ic_2,$$

for some complex number $C = c_1 + ic_2$. This gives us

$$y(s) = |Z(s)| = \sqrt{(s+c_1)^2 + c_2^2}, \tag{3.7}$$

$$x'(s) = \frac{ImZ}{y} = \frac{c_2}{\sqrt{(s+c_1)^2 + c_2^2}}.$$

By integrating x we obtain $x(s) = c_2 arcsinh\left(\frac{s+c_1}{c_2}\right)$ hence $s + c_1 = c_2 \sinh\left(\frac{x}{c_2}\right)$. Substituting into equation (3.7) we obtain

$$y(s) = \sqrt{c_2^2 \sinh^2\left(\frac{x}{c_2}\right) + c_2^2} = c_2 \cosh\left(\frac{x}{c_2}\right).$$

Clearly, this is a parametrization of a catenary.

- **Non zero constant mean curvature**

If $H \neq 0$ then from (3.3) $Z(s)$ becomes

$$\begin{aligned} Z(s) &= \left(\frac{1}{2iH}(1 - e^{-2iHs}) + C\right) e^{2iHs} \\ &= \frac{1}{2iH}\left((1 + 2iHC) - e^{-2iHs}\right) e^{2iHs} \\ &= \frac{Be^{i(2Hs+\theta)} - 1}{2iH}, \end{aligned}$$

where $Be^{i\theta} = 1 + 2iHC$ for some $B, \theta \in \mathbb{R}$ and $C \in \mathbb{C}$ is an arbitrary constant. Using the fact that $y(s) > 0$ we have by translation of the arclength and by restricting our attention to $H > 0$

$$\begin{aligned} y(s) &= |Z(s)| = \frac{1}{2H}\sqrt{1 + B^2 + 2B\sin 2Hs}, \\ x'(s) &= \frac{ImZ(s)}{y(s)} = \frac{1 + B\sin 2Hs}{\sqrt{1 + B^2 + 2B\sin 2Hs}}. \end{aligned}$$

Hence the solution is the one-parameter family of surfaces of revolution having constant mean curvature H given by

$$\alpha(s,H,B)=\left(\int_0^s \frac{1+B\sin 2Ht}{\sqrt{1+B^2+2B\sin 2Ht}}dt, \frac{1}{2H}\sqrt{1+B^2+2B\sin 2Hs}\right) \tag{3.8}$$

for any $B\in\mathbb{R}$ and $H>0$. Let us see which surfaces of revolution are generated by the generatrix α for different values of B. When $H(s)=H$ is constant and $B=0$ we obtain the cylinder. Substituting these in the formula above (3.8), we get the generatrix

$$\alpha(s,H,0)=\left(s,\frac{1}{2H}\right)$$

which is the straight-line $y=\dfrac{1}{2H}$ parallel to the x axis. So, its surface of revolution is the cylinder.

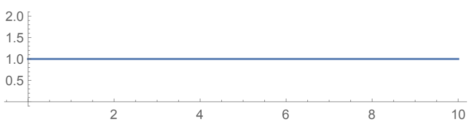


Figure 3.1: Horizontal line, $H=0.5$.

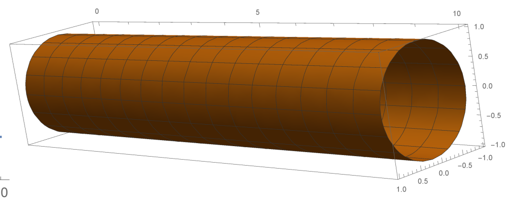


Figure 3.2: Cylinder, $H=0.5$.

When $H(s)=H$ doesn't depend on s and $B=1$ we obtain the sphere. First, let us substitute $B=1$ in formula (3.8) of the generatrix curve.

$$\begin{aligned}\alpha(s,H,1) &= \left(\int_0^s \frac{1+\sin(2Ht)}{\sqrt{2\sin(2Ht)+2}}dt, \frac{1}{2H}\sqrt{2\sin(2Hs)+2}\right)\\ &= \left(\frac{1}{\sqrt{2}}\int_0^s \sqrt{1+\sin(2Ht)}dt, \frac{1}{\sqrt{2}H}\sqrt{\sin(2Hs)+1}\right).\end{aligned}$$

Using the formula of the double angle and trigonometric identities, we have

$$\sin(2Ht)+1=2\sin(Ht)\cos(Ht)+\sin^2(Ht)+\cos^2(Ht)=(\cos(Ht)+\sin(Ht))^2.$$

Thus,

$$\sqrt{\sin(2Ht)+1}=\cos(Ht)+\sin(Ht).$$

Using this equality, the first coordinate of the curve becomes

$$\begin{aligned}x(s) &= \frac{1}{\sqrt{2}}\int_0^s(\cos(Ht)+\sin(Ht))dt=\frac{1}{\sqrt{2}}\left[\frac{1}{H}\sin(Ht)-\frac{1}{H}\cos(Ht)\right]_0^s\\ &= \frac{1}{\sqrt{2}H}(\sin(Hs)-\cos(Hs)+1).\end{aligned}$$

For the second coordinate, we have

$$y(s) = \frac{1}{\sqrt{2}H}(\cos(Hs) + \sin(Hs)).$$

So,

$$\alpha(s, H, 1) = \left(\frac{1}{\sqrt{2}H}(\sin(Hs) - \cos(Hs) + 1), \frac{1}{\sqrt{2}H}(\cos(Hs) + \sin(Hs)) \right). \tag{3.9}$$

Let us see that the curve $\alpha(s, H, 1) = (x(s), y(s))$ inscribes half circumference

$$\begin{aligned}
&\left(x(s) - \frac{1}{\sqrt{2}H} \right)^2 + y(s)^2 = \\
= \;& \left(\frac{1}{\sqrt{2}H} \sin(Hs) - \frac{1}{\sqrt{2}H} \cos(Hs) \right)^2 + \left(\frac{1}{\sqrt{2}H} \cos(Hs) + \frac{1}{\sqrt{2}H} \sin(Hs) \right)^2 \\
= \;& \frac{1}{2H^2} (\sin(Hs) - \cos(Hs))^2 + \frac{1}{2H^2} (\sin(Hs) + \cos(Hs))^2 \\
= \;& \frac{1}{2H^2}(\sin^2(Hs) - 2\sin(Hs)\cos(Hs) + \cos^2(Hs) \\
& + \sin^2(Hs) + 2\sin(Hs)\cos(Hs) + \cos^2(Hs)) \\
= \;& \frac{1}{2H^2}(2 - 2\sin(Hs)\cos(Hs) + 2\sin(Hs)\cos(Hs)) = \frac{1}{H^2}.
\end{aligned}$$

This means that the generatrix α is part of a circumference of radius $\frac{1}{H}$ with centre $(\frac{1}{\sqrt{2}H}, 0)$. If we rotate α around the x-axis we obtain the sphere as we wanted.

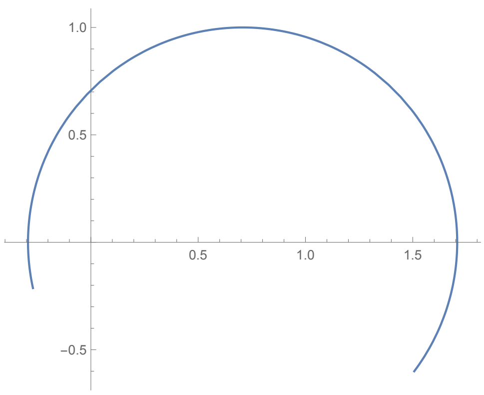


Figure 3.3: Semicircle, $H = 1$.

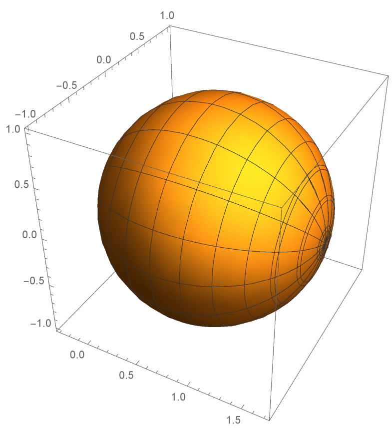


Figure 3.4: Sphere, $H = 1$.

When $H(s) = H$ doesn't depend on s and $0 < B < 1$ we get the undulary and the unduloid. In fact, using the parametrization (3.8), after long direct computations, it can be checked that $\alpha(s, H, B) = (x(s), y(s))$ is arc-length parametrized $(x'(s)^2 + y'(s)^2 = 1)$. Hence, calling $H = \frac{1}{2a}$, $b^2 = \frac{1-B^2}{4H^2}$ we have

$$\frac{dx}{ds} = \frac{y^2 + b^2}{2ay}\,.$$

Thus, Remark 2.2.1 gives us that $\alpha(s, H, B) = (x(s), y(s))$ is a undulary.

When $H(s) = H$ is constant and $B > 1$ we get the nodary and the nodoid. Again, we start from the parametrization (3.8). Now, with the same notation as before and by long direct computations, we can also check that $\alpha(s, H, B) = (x(s), y(s))$ is arc-length parametrized and that calling $H = \frac{1}{2a}$, $b^2 = \frac{B^2-1}{4H^2}$ we have

$$\frac{dx}{ds} = \frac{y^2 - b^2}{2ay}\,.$$

Again, the Remark 2.2.1 gives us that $\alpha(s, H, B) = (x(s), y(s))$ is a nodary.

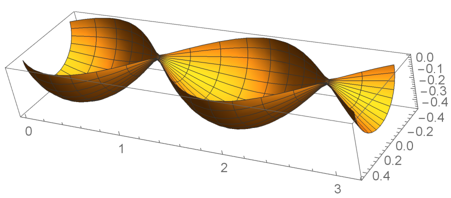


Figure 3.5: Half unduloid, $H = 2$, $B = 0.9$.

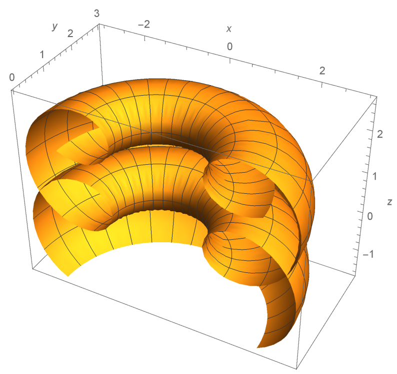


Figure 3.6: Half nodoid, $H = 0.5$, $B = 2$.

Chapter 4

Variational characterization of CMC surfaces

In this section we will see following [7] that CMC surfaces are solutions of a variational problem since they minimize the area among those surfaces with the same boundary and enclosed volume. Physically, CMC surfaces correspond with the following physical situation. We deposit an amount of liquid on a planar substrate and assume that there are no chemical and physical reactions between liquid, air and solid. We also neglect the gravitational forces. We denote by L, A and S the liquid, air and solid phases, and by S_{IJ} the interface between the I and J phases for $I, J \in \{L, A, S\}$. In mechanical equilibrium, the liquid drop attains its shape when the following equation holds

$$P_L(p) - P_A(p) = 2\gamma H(p) \qquad \textit{(Laplace)}$$

for each $p \in S_{LA}$. Here P_L and P_A are the pressures in the liquid and air. The constant γ is the surface tension coefficient of the liquid and H is the mean curvature of the interface S_{LA}. The coefficient γ is determined by chemical and physical properties of the liquid and it measures the intermolecular forces that exist in the liquid which are necessary to move the molecules from inside to the S_{LA} interface. If the pressures in both sides of the interface are constant (that is, if $P_L(p) = P_L$ and $P_A(p) = P_A$), then the interface is a surface with constant mean curvature:

$$P_L - P_A = \gamma 2H(p) \quad \Rightarrow \quad H(p) = \frac{P_L - P_A}{2\gamma} = constant$$

In our system, the only force acting on the interface is the surface tension. This is proportional to the area of this interface. Then the energy is proportional to the area of S_{LA}. We remark that the volume of the drop remains constant. If we perturb the drop, the liquid tries to reduce its energy (proportional to the area of S_{LA}) and when this occurs, this interface has constant mean curvature. Thus, using a particular version of the Principle of minimum energy, we can say that the shapes of (small) liquid drops are modelled by CMC surfaces.

4.1 Isoperimetric approach

CMC surfaces can also be described by using a variational principle. This was first made by Sturm [2] who noticed that CMC surfaces are connected to the *Isoperimetric Problem.* The isoperimetric problem is connected to the classical Dido's problem in $\mathbb{R}^2$. This problem consists on finding a closed curve which encloses the maximum area for a given perimeter. The answer for this problem is the circumference. There is a dual equivalent problem that consists on finding the closed curve that has the shortest perimeter for a given area. The answer is again the circumference. This class of problems are called isoperimetric problems.
From the mathematical viewpoint, CMC surfaces can be introduced by the isoperimetric problem extended to $\mathbb{R}^3$: among all compact surfaces in Euclidean space enclosing the same volume, which is the one with smaller area? For minimal surfaces, the analogous problem is the so-called minimizing area: characterize those surfaces which have least area among all surfaces with the same boundary. In both cases, a surface which is a minimum for the area must be a critical surface for the area functional: $A'(0) = 0$.
To achieve our goal, we will need to introduce some formalisms.

Definition 4.1.1. Let U be an open set of $\mathbb{R}^2$ and $\mathbf{x} : U \longrightarrow \mathbf{x}(U)$ be a parametrized surface or $\mathbb{R}^3$. Assume that $\gamma : I \longrightarrow U$ is a piecewise differentiable closed simple curve in U. Denote by $\Gamma = \gamma(I)$ the trace of γ and by D its interior domain ($\overline{D} = \Gamma \cup D$ and $\partial D = \Gamma$). Then $\mathbf{x}(\overline{D})$ is called a *parametrized surface with boundary* $\mathbf{x}(\Gamma)$.

For simplicity, a parametrized surface with boundary will be called a surface $S = \mathbf{x}(\overline{D})$ with boundary $\partial S = \mathbf{x}(\Gamma)$. Let us consider a compact surface S with possible non-empty boundary ∂S. A *boundary preserving variation* or *admissible variation* of $\mathbf{x}$ is a differentiable map $g : U \times (-\epsilon, \epsilon) \longrightarrow \mathbb{R}^3$ such that

(i) for each t, $g^t : U \longrightarrow \mathbb{R}^3$ given by $g^t(p) = g(p, t)$ is a surface with boundary.

(ii) $g(p, 0) = \mathbf{x}(p)$, that is, $g^0 = \mathbf{x}$.

(iii) $g(p, t) = \mathbf{x}(p)$ for any $t \in (-\epsilon, \epsilon)$ and $p \in \Gamma$. This means that the variation fixes the boundary.

We define the area and algebraic volume functionals of the variation $A, V : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}$, as

$$A(t) = area(g^t), \quad V(t) = volume(g^t),$$

or

$$A(t) = \textstyle\int_{S_t} 1 dS_t, \quad V(t) = -\frac{1}{3}\int_{S_t} \langle \mathbf{x}_v, \mathbf{N}_v \rangle dS_t.$$

For a geometric justification of the second formula see [7, 8]. We want to see which is the surface that minimizes the area (more generally, which is critical for the area) for a fixed volume, that is, we will focus in the variations where $V(t) = V(0)$ for any t and we will find those immersions $\mathbf{x}$, such that $A'(0) = 0$ for any such variation. The following theorem gives us the result that we are searching for.

Theorem 4.1.1. *Assume S is a compact surface with boundary ∂S. Then*

(i) *S is minimal ($H = 0$) if and only if $A'(0) = 0$ for any admissible variation.*

(ii) *S has constant mean curvature $H \neq 0$ if and only if it is a critical point of the area for any admissible preserving volume variation.*

Proof. We will show the theorem for the case that the surface is a graph. Notice that this is not a big restriction since all surfaces are locally graphs. Consider S given as a graph of a function f defined on $U \subset \mathbb{R}^2$ and parametrized as $\mathbf{x}(x, y) = (x, y, f(x, y))$ for any $(x, y) \in U$. Consider a variation S_t as graphs on U where

- $g : U \times (-\epsilon, \epsilon) \longrightarrow \mathbb{R}$, with $S_t = g(U, t)$ and $g(x, y, 0) = f(x, y)$. This means that at $t = 0$, we have the original graph f.
- $g(x, y, t) = f(x, y)$ for any $(x, y) \in \partial D$. With this condition, the variation preserves the boundary of S.

The area of S_t is

$$A(t) = \int_{S_t} 1 dS_t = \int_U \sqrt{1 + g_x^2 + g_y^2} dxdy.$$

We differentiate with respect to t:

$$A'(t) = \int_U \frac{2g_x g_{xt} + 2g_y g_{yt}}{2\sqrt{1 + g_x^2 + g_y^2}} dxdt = \int_U \frac{g_x g_{xt} + g_y g_{yt}}{\sqrt{1 + g_x^2 + g_y^2}} dxdy,$$

Now, let $t = 0$. Since $g(x, y, 0) = f(x, y)$, $g_x(x, y, 0) = f_x(x, y)$, $g_y(x, y, 0) = f_y(x, y)$ etc. and for $\bigtriangledown f = (f_x, f_y)$ we get $\| \bigtriangledown f \| = \sqrt{f_x^2 + f_y^2}$, we obtain

$$A'(0) = \int_U \frac{f_x g_{xt} + f_y g_{yt}}{\sqrt{1 + f_x^2 + f_y^2}} (x, y, 0) dxdy = \int_U \frac{f_x g_{xt} + f_y g_{yt}}{\sqrt{1 + \|\nabla f\|^2}} (x, y, 0) dxdy.$$

Denote

$$T(f) = \frac{\nabla f}{\sqrt{1+\|\nabla f\|^2}} = \left(\frac{f_x}{\sqrt{1+f_x^2+f_y^2}}, \frac{f_y}{\sqrt{1+f_x^2+f_y^2}}\right).$$

The divergence satisfies

$$\begin{aligned}
div(g_t T) &= g_t divT + < \bigtriangledown g_t, T > \\
&= g_t divT + < (g_{tx}, g_{ty}), T > \\
&= g_t divT + \frac{g_{tx} f_x}{\sqrt{1+f_x^2+f_y^2}} + \frac{g_{ty} f_y}{\sqrt{1+f_x^2+f_y^2}} \\
&= g_t divT + \frac{f_x g_{xt} + f_y g_{yt}}{\sqrt{1+\|\nabla f\|^2}}.
\end{aligned}$$

Combining this with the above expression for $A'(0)$,

$$\begin{aligned}
A'(0) &= \int_U div(g_t T)(x,y,0)dxdy - \int_U g_t divT(f)(x,y,0)dxdy \\
&= \int_{\partial U} g_t \langle T(f), \overrightarrow{n}\rangle(x,y,0)ds - \int_U g_t divT(f)(x,y,0)dxdy,
\end{aligned}$$

where in the last identity we have used the divergence theorem, see [7] page 80, and where $\overrightarrow{n}$ is the outer unit normal vector to ∂U. The first integral vanishes because $g(x,y,t) = f(x,y)$ doesn't depend on t for any $(x,y) \in \partial U$, and so, $g_t(x,y,0) = 0$ on ∂U. Recall that the divergence is defined as $divF = \nabla \cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (F_1, F_2) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$. Then

$$\begin{aligned}
&A'(0) = \\
&= -\int_U g_t divT(f)(x,y,0)dxdy = -\int_U g_t div\left(\frac{(f_x,f_y)}{\sqrt{1+f_x^2+f_y^2}}\right)(x,y,0)dxdy \\
&= -\int_U g_t \left(\frac{\partial}{\partial x}\left(\frac{f_x}{\sqrt{1+f_x^2+f_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{f_y}{\sqrt{1+f_x^2+f_y^2}}\right)\right)(x,y,0)dxdy
\end{aligned}$$

$$
\begin{aligned}
&= -\int_U g_t \left(\frac{f_{xx}\sqrt{1+f_x^2+f_y^2} - f_x \dfrac{f_x f_{xx} + f_y f_{yx}}{\sqrt{1+f_x^2+f_y^2}}}{1+f_x^2+f_y^2} \right)(x,y,0)dxdy \\
&\quad - \int_u g_t \left(\frac{f_{yy}\sqrt{1+f_x^2+f_y^2} - f_y \dfrac{f_x f_{xy} + f_y f_{yy}}{\sqrt{1+f_x^2+f_y^2}}}{1+f_x^2+f_y^2} \right)(x,y,0)dxdy \\
&= -\int_U g_t \frac{f_{xx} + f_x^2 f_{xx} + f_y^2 f_{xx} - f_x^2 f_{xx} - f_x f_y f_{yx}}{(1+f_x^2+f_y^2)^{3/2}}(x,y,0)dxdy \\
&\quad - \int_U g_t \frac{f_{yy} + f_x^2 f_{yy} + f_y^2 f_{yy} - f_x f_y f_{xy} - f_y^2 f_{yy}}{(1+f_x^2+f_y^2)^{3/2}}(x,y,0)dxdy \\
&= -\int_U g_t \left(\frac{f_{xx} + f_y^2 f_{xx} - 2 f_x f_y f_{xy} + f_{yy} + f_x^2 f_{yy}}{(1+f_x^2+f_y^2)^{3/2}} \right)(x,y,0)dxdy.
\end{aligned}
$$

On the other hand, we can compute the mean curvature of S following Theorem 1.0.2 and using the parametrization $\mathbf{x}(x,y) = (x,y,f(x,y))$:

$$
H(x,y) = \frac{1}{2} \frac{f_{xx} + f_y^2 f_{xx} - 2 f_x f_y f_{xy} + f_{yy} + f_x^2 f_{yy}}{(1+f_x^2+f_y^2)^{3/2}}.
$$

As we can see,

$$
A'(0) = -\int_U (2H) g_t(x,y,0)dxdy. \tag{4.1}
$$

What we have shown so far is true for any surface. This formula is called *the First Variational Formula for the area.*

Let us prove (i):

If $H = 0$, then $A'(0) = 0$ for any g, and we have finished one of the implications of the proof. Assume now that $A'(0) = 0$ for any preserving volume variation of S. We consider an appropriate variation g given by:

$$
g(x,y,t) = f(x,y) + tHp(x,y),
$$

where $p(x,y) > 0$ on U and $p(x,y) = 0$ for $(x,y) \in \partial U$. Then $g_t(x,y,t) = f_t(x,y) + Hp(x,y) = Hp(x,y)$ implies

$$
0 = A'(0) = -\int_U (2H) Hp(x,y,0)dxdy = -2\int_U H^2 p(x,y,0)dxdy
$$

and so, $\int_S H^2 p \, dS = 0$ and since p is positive, $H = 0$.

As a conclusion, we have proved that $H = 0$ on the surface if and only if the

surface is a critical point of the functional area. In other words, minimal surfaces are critical for the area.
Finally, let us prove (ii):
If we assume that the variation preserves the volume, we have the constraint $V(t) = \int_U g dxdy = constant$. If $t = 0$ is a critical point of $A(t)$, a version of Lagrange multiplier's theorem [8] says that there exists a constant $\lambda \in \mathbb{R}$ such that

$$A'(0) + \lambda V'(0) = 0.$$

Now, since $V(t) = \int_U g(x, y, t)dxdy$,

$$V'(t) = \frac{\partial}{\partial t}\int_U g(x,y,t)dxdy = \int_U g_t(x,y,t)dxdy$$

and

$$V'(0) = \int_U g_t(x,y,0)dxdy. \tag{4.2}$$

Then

$$A'(0) = -\lambda V'(0) = -\int_U \lambda g_t(x,y,0)dxdy.$$

And, for the first variational formula, (4.1),

$$\begin{aligned} 0 &= A'(0) + A'(0) = -\int_U 2Hg_t(x,y,0)dxdy - \int_U \lambda g_t(x,y,0)dxdy \\ &= -\int_U g_t(2H+\lambda)(x,y,0)dxdy. \end{aligned}$$

If this happens for any g, for appropriate variations g, we have $2H + \lambda = 0$, that is, $H = -\dfrac{\lambda}{2}$ is constant.
For the other implication, assume that H is constant. Then, (4.1) becomes

$$A'(0) = -\int_U (2H)g_t(x,y,0)dxdy = -2H\int_U g_t(x,y,0)dxdy$$

and, since the variation preserves the volume,

$$V'(0) = \int_U g_t(x,y,0)dxdy = 0.$$

Thus,

$$A'(0) = -2H\int_U g_t(x,y,0)dxdy = -2HV'(0) = 0.$$

□

We remark that the algebraic volume changes when we translate the surface, however, this does not affect the above result, since the volumes of surfaces of a translated admissible variation will be the same as those of the

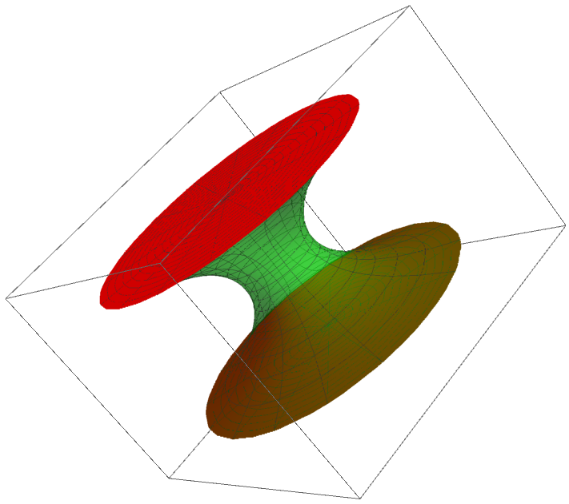


Figure 4.1: S_d (green) and $\widehat{S}$ (red), for $d = 2$.

original variation but a constant ([7], page 14).
Observe also that the proof of Theorem 5.0.1. means that a surface is minimal if and only if it is critical of the area for any admissible variation. In particular, if a surface minimizes the area for admissible variations, then $H = 0$, but the converse is not true. The following example illustrates this point.

Example 4.1.1. Let $a > 0$ and fix $b = \cosh d$. Let

$$\begin{aligned} S_d &= \{(x, y, z) \in S_\alpha \mid |z| < d\}, \\ \widehat{S} &= \{x^2 + y^2 < b^2,\ z = \pm d\}. \end{aligned}$$

The surface S_d is part of the catenoid S_α with $-d < z < d$, and $\widehat{S}$ is the union of two discs with radius b. Both surfaces are minimal and have the same boundary. Let γ be that boundary:

$$\partial S_d = \partial \widehat{S} = \gamma = \{x^2 + y^2 = b,\ z = \pm d\}.$$

Let us calculate the area of both surfaces. The area of $\widehat{S}$ encloses the area of both discs, so,

$$A(\widehat{S}) = 2\pi b^2 = 2\pi \cosh^2 d.$$

The area of S_d is given by

$$A(S_d) = \iint_{S_d} 1 \, dA_{\mathbf{x}^d},$$

where $\mathbf{x}^d : U_d \to \mathbb{R}^3$ and $U_d = (-d,d) \times (0, 2\pi)$, given by $\mathbf{x}^d(u,v) = \mathbf{x}(u,v) = (\cosh u \cos v, \cosh u \sin v, u)$ is a parametrization of S_d. Then,

$$A(S_d) = \iint_{U_d} \sqrt{EG - F^2} dudv = \int_0^{2\pi} \int_{-d}^{d} (EG - F^2)^{1/2} dudv = 2\pi \int_{-d}^{d} \cosh^2 u du.$$

Here E, F and G are the coefficients of the first fundamental form. Integrating by parts, we get

$$\begin{aligned} t &= \cosh u, \quad & dt &= \sinh u du, \\ ds &= \cosh u du, \quad & s &= \sinh u. \end{aligned}$$

$$\int \cosh^2 u du = \cosh u \sinh u - \int \sinh^2 u du = \cosh u \sinh u - \int (\cosh^2 u - 1) du$$

Thus,

$$\int \cosh^2 u du = \frac{1}{2} (\cosh u \sinh u + u)$$

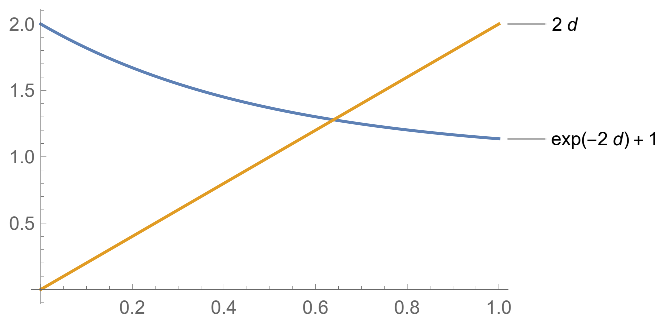
and

$$A(S_d) = 2\pi \left[\frac{u + \sinh u \cosh u}{2} \right]_{-d}^{d} = 2\pi (d + \sinh d \cosh d)$$

Hence $A(\widehat{S}) < A(S_d)$ happens when

$$\begin{aligned} \cosh^2 d < d + \sinh d \cosh d \quad &\Rightarrow \quad \frac{e^{2d} + e^{-2d} + 2}{4} < d + \frac{e^{2d} - e^{-2d}}{4} \\ &\Rightarrow \quad e^{-2d} + 1 < 2d. \end{aligned}$$

Let d_0 be the intersection of the curves $y = e^{-2d} + 1$ and $y = 2d$.



If $d < d_0$, $A(\widehat{S}) > A(S_d)$, whereas if $d > d_0$, $A(\widehat{S}) < A(S_d)$. Then, if $d > d_0$, S_d is not the surface with minimum area, so S_d doesn't minimize the area of all surfaces with γ as a boundary.

Remark 4.1.1. *(Dual Isoperimetric Problem).* Surfaces of revolution with CMC can be obtained as surfaces which are maximum for the enclosed volume for variations with constant area [4].

4.2 Mathematical models for the liquid bridge between two plates

In this section, we investigate the shape of the tear meniscus that forms around a contact lens following the study of Thanuja Paragoda in Chapter 3 of [9]. Since the tear film's thickness is much smaller than the radius of the cornea and the contact lens, one may neglect the curvature of both the contact lens and the cornea, and treat them as flat surfaces. Thus, to make this problem amenable to analysis, we consider a meniscus of a liquid bridge that forms between two vertical plates.
In other words, we consider a liquid drop, which is trapped between two vertical plates, and model the profile curve of the drop using a Calculus of Variations approach. We will obtain a formula for the profile curve that is formed between the liquid-air interface by minimizing the total potential energy of the drop while imposing a volume constraint. According to [9], the total potential energy of a liquid is mainly composed of three different energy forms.

(i) Surface energy of a liquid surface, which is proportional to the surface area of the liquid-air interface (free surface).

(ii) Wetting energy that arises due to the contact area of solid-liquid interface.

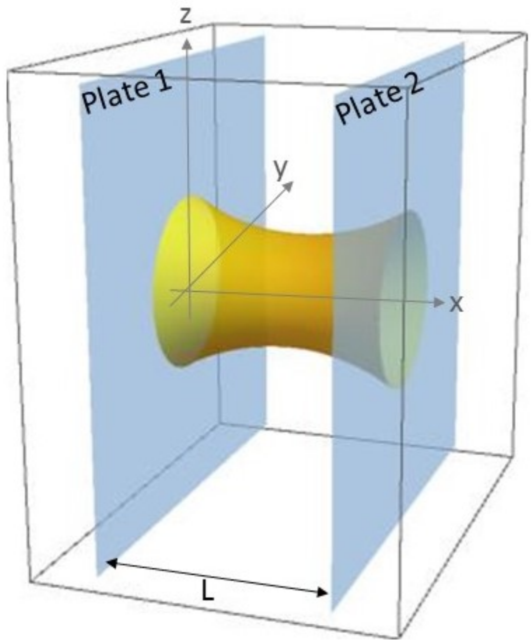
(iii) And, gravitation potential energy, which we will neglect here.

We consider a rotationally-symmetric liquid drop, and hence the latter energy type is neglected on our analysis.
To simplify, we consider a liquid drop which is trapped between two vertical plates, and the profile of the drop has the equation $z = f(x)$ with respect to the configuration of the Cartesian coordinate system, which is on the left plate (plate 1). Note that the continuity of the drop implies that $f(x) > 0$ on the interval $[0, L]$, and assume the shape of the solid-liquid contact area on the plates to be circles with radii $f(0)$ and $f(L)$, respectively. The relative adhesion coefficients of the liquid with the plates 1 and 2 are β_1 and β_2.

Thus, under the absence of gravity, the total energy E of a rotationally symmetric liquid drop may be written in the following form [9]

$$E = \int_0^L 2\pi\gamma f(x)\sqrt{1 + f'^2(x)}dx - \gamma\beta_1\pi f(0)^2 - \gamma\beta_2\pi f(L)^2,$$

where γ denotes the surface energy per unit area of the liquid; β_i is the relative adhesion coefficient between the i^th wall and the liquid. The integral term represents the surface energy of the liquid drop, and the last two terms



denote the wetting energy of the drop. We wish to minimize the energy E subjected to the volume constraint

$$V_0 = \int_0^L \pi f(x)^2 dx,$$

where V_0 denotes the volume of the liquid drop. Thus, by using a version of the Lagrange multiplier's theorem [4], the new energy functional $\overline{E}$ that includes the volume constraint V_0 is

$$\begin{aligned}\overline{E} &= \int_0^L 2\pi\gamma f(x)\sqrt{1+f'^2(xf)}dx - \gamma\beta_1\pi f(0)^2 \\ &\quad -\gamma\beta_2\pi f(L)^2 + \lambda\left(\int_0^L \pi f(x)^2 dx - V_0\right).\end{aligned}$$

Here, the Lagrange multiplier λ is an unknown constant. We consider the variation of $\overline{E}$ ($\delta\overline{E}$) with respect to the drop radius (capillary surface height) $f(x)$ and the meniscus height at the end points: $f(0)$ and $f(L)$. Then, from formula (5) of [4] in page 56, we have that the variation of the energy is given by

$$\begin{aligned}\delta\overline{E} &= 2\pi\gamma\int_0^L \sqrt{1+f'^2(x)}\delta f dx + 2\pi\gamma\int_0^L f\frac{f'}{\sqrt{1+f'^2}}\delta f_x dx \\ &\quad -2\pi\gamma\beta_1 f(0)\delta f(L) - 2\pi\gamma\beta_2 f(L)\delta f(L) + 2\pi\lambda\int_0^L f(x)\delta f dx,\end{aligned}$$

which simplifies to

$$\begin{aligned}\delta\overline{E} &= -2\pi\gamma f(0)\left(\beta_1 + \frac{f'(0)}{\sqrt{1+f'^2(0)}}\right)\delta f(0)\\ &\quad +2\pi\gamma f(L)\left(\frac{f'(L)}{\sqrt{1+f'^2(L)}} - \beta_2\right)\delta f(L)\\ &\quad +2\pi\gamma\int_0^L\left(\frac{\lambda}{\gamma}f + \sqrt{1+f'^2(x)} - \frac{d}{dx}\left(\frac{ff'}{\sqrt{1+f'^2(x)}}\right)\right)\delta f(x)dx.\end{aligned}$$

The necessary condition for the energy minimization is that $\delta\overline{E} = 0$. Thus, we have the following system of equations that reads

$$\frac{\lambda}{\gamma}f + \sqrt{1+f'^2(x)} - \frac{d}{dx}\left(\frac{ff'}{\sqrt{1+f'^2(x)}}\right) = 0 \quad in\ [0, L]\,, \tag{4.3}$$

$$\beta_1 + \frac{f'(0)}{\sqrt{1+f'^2(0)}} = 0 \quad at\ x = 0, \tag{4.4}$$

$$\frac{f'(L)}{\sqrt{1+f'^2(L)}} - \beta_2 = 0 \quad at\ x = L. \tag{4.5}$$

Let the value of the contact angles of the liquid meniscus with the plates be θ_1 and θ_2, and assume they are rotationally invariant. Hence, we have the same contact angle values along the periphery of the contact circles. We observe that $f'(0) = -\cot\theta_1$ and $f'(L) = \cot\theta_2$. Then, the relative adhesion coefficients β_1 and β_2 may be expressed as

$$\beta_1 = \cos\theta_1 \quad \text{and} \quad \beta_2 = \cos\theta_2.$$

Finally, simplifying (4.3) results in

$$\frac{\lambda}{\gamma} = \frac{ff'' - f'^2 - 1}{f(1+f'^2)^{3/2}}, \quad \forall x \in [0, L]\,.$$

This equation represents the liquid surface of the drop in terms of its profile curve $f(x)$, and we further observe that the right hand side of this equation relates to the mean curvature of the liquid surface. In fact, since Lagrange multiplier λ and the surface energy per unit area of the liquid γ are constants, using (2.10) the above equation leads us to an equation of the form

$$\frac{\lambda}{\gamma} = \frac{ff'' - f'^2 - 1}{f(1+f'^2)^{3/2}} = 2H. \tag{4.6}$$

Notice that we have just proved that the rotational bridge surface is a CMC surface. This result can be formulated as [9]

Theorem 4.2.1. *Rotational liquid bridges between vertical walls, which minimize the surface and wetting type energies represent rotationally CMC surfaces with $H \neq 0$.*

Remark 4.2.1. Multiplying (4.6) by f', rearranging terms, and integrating with respect to the x variable, one may obtain

$$
\begin{aligned}
2Hff' + \frac{f'((1+f'^2) - ff'')}{\sqrt{(1+f'^2)^3}} = 0 \quad &\Leftrightarrow \quad \left(\frac{f}{\sqrt{1+f'^2}}\right)' + H(f^2)' = 0 \\
\Leftrightarrow \quad \left(\frac{f}{\sqrt{1+f'^2}} + Hf^2\right)' = 0 \quad &\Leftrightarrow \quad \frac{f}{\sqrt{1+f'^2}} + Hf^2 = C_1 \\
\Leftrightarrow \quad f^2 + \frac{f}{H\sqrt{1+f'^2}} = C_2, &
\end{aligned}
$$

where C_1, C_2 are constants. Then, from the differential equation (2.14) using the positive sign and choosing $H = \dfrac{1}{2a}$ and $-b^2 = C_2$ one can see using Remark 2.2.1 that f is an undulary.

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