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On Some Sufficiency-Type Stability and Linear State-Feedback Stabilization Conditions for a Class of Multirate Discrete-Time Systems

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Abstract: This paper presents and discusses the stability of a discrete multirate sampling system whose sets of sampling rates (or sampling periods) are the integer multiple of those operating on all the preceding substates. Each of such substates is associated with a particular sampling rate. The sufficiency-type stability conditions are derived based on simple conditions on the norm, spectral radius and numerical radius of the matrix of the dynamics of a system parameterized at the largest sampling period.

Keywords: discrete system; multirate sampling; single rate sampling; numerical radius; spectral radius; stability

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1. Introduction

Multirate sampling has been proven to be a powerful tool to accommodate the relevant signals (state variables, measured output components or control inputs) with respect to the needed faster or slow data sampling requirements in terms of data acquisition or influence of the dynamics, so as to achieve the expected system performance. It has been also used to attenuate the influence of additive noise. Furthermore, it has been of interest to accommodate certain controls that do not require the same sampling rates, as for instance the aileron and rudder deflection maneuvers in aircraft. See, for instance, References [1–3] and some of the references therein. In those works, it was proposed to have a global discrete model for systems having several sampling rates to facilitate the analytical description and the potential design tools. The basic underlying idea for the use of multirate sampling is that, in general, not all the signals being processed require identical appropriate sampling rates. On the other hand, there is a need in some applications to alleviate either the computational efforts or the memory storage efforts. In certain applications, there is a need to accommodate the available sensor information with respect to the various natural sampling rates of the state and output variables and control signals. In [4], the controllability and the closed-loop pole assignment have been formulated under multirate sampling. In particular, the controllability property has been proven to be achievable under weaker assumptions than the general ones at faster input sampling rates related to the state/output sampling rates. Such an issue is of specific interest if the discrete dynamics is modelled for the largest sampling period while the controls at the faster sampling rates are generated via appropriate design gains from each preceding control input at the largest sampling period. This technique is also of interest in model matching and related adaptive control designs since the controllability is equivalent to the free spectrum-assignment in the linear time-invariant case. See, for instance, References [3,4]. It turns out that the intuition dictates that different controller gains can be used, if necessary, for the various inputs generated at all the sampling intervals contained in the larger sampling period parameterizing the state evolution. In addition, this benefit increases in the case when the various input components operate at distinct fast rates. On the other hand, a matching mechanism has been investigated in [5] for networked control systems under multirate sampling while the fault-tolerant design for a class of batch networked problems has been investigated in [6] under actuator faults and external disturbances under multirate sampling and quantization effects. Furthermore, impulsive-type control in complex systems has been studied in [7] through the use of cognitive maps and multirate sampling. Further studies have been performed in [8,9] and [10–13] and the references therein concerning multirate and non-uniform sampling, the design of observers and the eventual presence of either noisy disturbances or a combination with asynchronous inter-sample output predictions. Fast and slow optimal controllers have been successfully developed in [14] integrating a composite combined controller for a fast subsystem and a slow subsystem resulting from a decomposition analysis of the whole system describing a heavy water reactor. On the other hand, it is well-known that the sampling period is a critical parameter for the evolution of the transients, the stability and bandwidth constraints, the inter-sample rippling overshoots and the signal adaptation and accommodation with respect to sampling in a wide variety of control and signal processing problems. See, for instance, References [15–18] and the references therein. In particular, sliding mode controllers have been investigated in [15] in the discrete stochastic case under the design of observers. The existence conditions and a stability analysis of the observer are given, as well, in that paper. The properties under discretization and the eventual presence of delays of the popular Beverton–Holt equation for the evolution of certain species that reproduce by eggs has been studied in [16,17] under the carrying capacity properties of the environment. The eventual design of such a gain as an eventual controller to govern the evolution equation has been examined. The strategy has been proven to be useful in some problems where such a design is feasible, for instance in aquaculture exploitation factories. The main objective of the paper is the study of a class of linear discrete systems that have coupled dynamics at different sampling rates and their global stability and asymptotic stability properties, which are obtained based on stability and convergence matrix tools rather than from the Lyapunov theory of frequency tools. The whole system matrix is assumed to be decomposable into subsystems such that each one runs at its own sampling rate and each sampling rate is an integer multiple of those associated with the preceding subsystems in the whole system matrix. It is found that the properties can be better achievable, in general under weaker constraints, for parameterizations derived at slower sampling rates. The paper is organized as follows. Section 2 describes an unforced linear and time-invariant discrete model, which has two coupled substates at two different sampling rates, the faster one being an integer submultiple of the slowest one. It is implicitly assumed that the matrix of the dynamics of the whole system can be decomposed in such a way. This implies that that the dynamics influence of each subsystem is assumed to be driven by its substate at its own sampling period (described by the corresponding diagonal entries of the matrix of dynamics) and the combined action of the remaining subsystems coupled to it through the nonzero off-diagonal entries of the same row block. It is assumed that the two subsystems have sampling periods that satisfy an integer multiplicity constraint. Some illustrative examples are given. Section 3 generalizes the model for the case of $p \geq 2$ sampling rates, which follow the design rule that all are, respectively, the integer multiple and integer submultiple of the two nearest neighbors. Therefore, it is assumed that the subsystems satisfy an integer multiplicity constraint, each one being an integer multiple of all the previous ones in the whole dynamics decomposition. Compared to the existing background literature cited in the references and related references therein, the primary modelling attention is devoted to describing the dynamics of the whole tandem of coupled subsystems at the largest sampling rate even if the system is unforced. On the other hand, the characterization and investigation of the spectral radius, the numerical radius and the ℓ_2-norm of the matrix of the dynamics [19–21] are the focus in Sections 2 and 3, so that the system matrix is convergent if the above parameters are strictly bounded by unity, or at least those respective values are non-strictly bounded by unity. This characterizes the stability properties in the discrete context. Remember that the above three parameters satisfy, in the given order, a relation of the type of less than or equal to for any complex or real square matrix. In those cases, the dynamic system is either globally asymptotically stable or, at least, globally stable as a result. In such a way, the stability is investigated focusing the problem on the properties of the matrices that describe the dynamics rather than the use of Lyapunov theory tools. On the other hand, Section 4 relies on the closed-loop system under multirate sampling and linear state-feedback by the appropriate synthesis of the controller gains by extending the unforced class of systems of Sections 2 and 3 to the use of sampled controls under multirate sampling. In this case, one describes the fast and slow modes of the dynamics affected by the various input rates. As a result, either closed-loop model matching to a prescribed suited stable dynamics is achieved or, at least, the matching of the closed-loop eigenvalues of the matrix of the dynamics to prescribed stable allocations is achieved. The main underlying developed idea is that the spectrum assignment is achievable for the largest sampling period under multirate sampling in some cases in which it is not achievable under single-rate sampling. Some illustrative examples are also described. It can be pointed out that the use of multi-stage methods is also common as an efficiency optimization tool to improve the computational performances in other fields like, for instance, its use to improve the efficiency of numerical integration [22,23]. A comparison of the proposed stability analysis technique could be performed from the discussion in [24,25], and some of the references therein, related to the Lyapunov-type stability method. A possible extension of the proposed method by using non-uniform sampling rates in a multirate sampling disposal could be got by combining results with the methodology for adaptive sampling of [26].

Notation:

$\boldsymbol{Z}_{0+} = \boldsymbol{Z}_+ \cup \{0\}$; $\boldsymbol{Z}_+ = \{z \in \boldsymbol{Z} : z > 0\}$; $\overline{n} = \{1, 2, \ldots, n\}$.

$\boldsymbol{R}_{0+} = \boldsymbol{R}_+ \cup \{0\}$; $\boldsymbol{R}_+ = \{r \in \boldsymbol{R} : r > 0\}$.

I_n is the $n - th$ identity matrix.

$M_{m\times n}$ denotes that the matrix M has order $m \times n$.

$\overline{z} = \{1, 2, \ldots, z\}$.

$z = \dot{q}$ means that z is an integer multiple of q, and $z \neq \dot{q}$ means that z is not an integer multiple of q.

The ℓ_2 (or spectral)-norm is denoted by $\|.\|_2$, and $\lambda_{max}(.)$ is the maximum eigenvalue of the $(.)$-real symmetric and $\lambda_{min}(.)$ its minimum eigenvalue. The ℓ_1-norm is denoted by $\|.\|_1$, and the ℓ_∞-norm is denoted by $\|.\|_\infty$.

The spectral radius of a complex square matrix X is $r_X = \underset{\lambda \in sp(X)}{max} |\lambda| = inf\|X\|$, where $sp(X)$ is the spectrum of X, the infimum is taken over the whole set of matrix vector-induced norms and $\|X\|_2 = r^{1/2}_{X^TX} = \lambda^{1/2}_{max}(X^TX)$. If X is Hermitian, then $\|X\|_2 = r_X$.

The numerical radius of a complex square matrix X is $\omega_X = \underset{\lambda \in W_X}{max} |\lambda|$, where $W_X = \{x^*Xx : x \in \boldsymbol{C}^n\}$ is the numerical range of X. It holds that $r_X \leq \omega_X \leq \|X\|$ for any matrix vector-induced norm $\|X\|$, and the inequalities are equalities if X is Hermitian. Note that X is convergent if $r_X < 1$, if $\omega_X < 1$ and if $\|X\| < 1$.

A positive definite (semidefinite) real square matrix M is denoted as $M \succ 0 (\succeq 0)$. A negative definite (semidefinite) real square matrix M is denoted as $M \succ 0 (\succeq 0)$. $M \succ (\succeq) N$ means $M - N \succ (\succeq) 0$.

The Kronecker product of A and B is denoted by $A \otimes B$. Assume that the algebraic equation $A_{m\times n} X_{n\times r} B_{r\times s} = C_{m\times s}$ is solvable in $X = [x_1, x_2, \cdots, x_n]^T$, whose n columns x_i are vectors of dimension r, for a given triple (A, B, C) of real or complex matrices of the indicated orders. The solution(s) X can be calculated via the vectorization of the equation:

$$[A_{m\times n} \otimes B^*_{s\times r}]_{ms\times nr} [vec(X)]_{nr\times 1} = [vec(C)]_{ms\times 1}$$

where the vector $vec(X) = \left[x_1^T, x_2^T, \cdots, x_n^T\right]^T$ consists of the rows of X arranged in its natural order so that it contains all the entries of the unknown matrix X. Note that the Rouché–Frobenius theorem from linear algebra establishes that $AXB = C$ is solvable in X iff $rank(A \otimes B^*) = rank(A \otimes B^*,\ vec(C))$.

2. A Multirate Sampling System with Two Sampling Rates

A discrete linear multirate sampling system with two sampling periods (or rates) $T_1 = T/q$ and $T_2 = T$ for some basic sampling period $T \in \boldsymbol{R}_+$ and $q(\geq 2) \in \boldsymbol{Z}_+$ is supposed to be described at the smaller sampling period T_1 by:

$$x_{k+1} = G_k x_k \tag{1}$$

$\forall k \in \boldsymbol{Z}_{0+}$ where $x_0 = x(0) \in \boldsymbol{R}^n$. It is assumed that the state is partitioned into two substates as $x_k = \left(x_{1k}^T, x_{2k}^T\right)^T$, where $x_{1k} \in R^{n_1}$ is ran by the smaller sampling period T_1 and x_{2k} is governed by the larger sampling period T_2, where $x_{2k} \in R^{n_2}$; $\forall k \in \boldsymbol{Z}_{0+}$ with $n = n_1 + n_2$, and:

$$G_k = \begin{bmatrix} G_{11k} & G_{12k} \\ G_{21k} & G_{22k} \end{bmatrix}; \forall k \in \boldsymbol{Z}_{0+} \tag{2}$$

The system of (1) and (2) is described as governed by the larger sampling period T as follows:

$$x_{k+1} = M_k x_k; \tag{3}$$

$\forall k \in \boldsymbol{Z}_{0+}$, where

$$M_k = \begin{bmatrix} M_{11k} & M_{12k} \\ M_{21k} & M_{22k} \end{bmatrix}; \forall k \in \boldsymbol{Z}_{0+} \tag{4}$$

Assumption 1. *The system of (3) and (4) is characterized as follows when governed by the smaller sampling period T_1:*

$$M_{k/q} = \begin{cases} \begin{bmatrix} A_{11} & A_{12} \\ 0 & I_{n_2} \end{bmatrix} \text{if } k \neq \dot{q} \\ \\ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{if } k = \dot{q} \end{cases} ; \forall k \in \boldsymbol{Z}_{0+} \tag{5}$$

where the integer subscript k runs for the larger sampling period T, that is at sampling instants ran by the larger sampling period $t_k = k(qT_1) = kT$ for $k \in \boldsymbol{Z}_{0+}$. At these sampling instants, the second parameterization of (5) works. The whole set of sampling instants ran by the smaller sampling period is $t_k(i) = t_k + iT_1 \in [t_k, t_{k+1})$ for $i \in \overline{q-1} \cup \{0\}$ with $t_k(0) = t_k$ for all $k \in \boldsymbol{Z}_{0+}$. Note that for $i \in \overline{q-1}$, i.e., if $i \neq 0$, then $t_k(i) \neq t_k(0)(= t_k)$ is not of the form $(kq)T$, then the parameterization of the matrix of dynamics is given by the first expression in (5). We simplify the notation and description throughout the paper by running the system dynamics for the large sampling period at sampling instants $t_k = kT$; $k \in \boldsymbol{Z}_{0+}$. Thus, the linear time-varying parameterization of (1) for the smaller sampling period $T_1 = T/q$ corresponds, equivalently, to (4), subject to (5), with $G_{11(k/q)} = A_{11}$, $G_{12(k/q)} = A_{12}$, $G_{21(k/q)} = 0$ if $k \neq \dot{q}$ and $G_{22(k/q)} = A_{21}$ if $k = \dot{q}$; and $G_{22(k/q)} = I_{n_2}$ if $k \neq \dot{q}$ and $G_{22(k/q)} = A_{22}$ if $k = \dot{q}$. Note that (5) reflects that the dynamics of the subsystem parameterized at the larger sampling period (k is an integer multiple of q) does not change for the sampling instants corresponding to the smaller sampling period (k is not an integer multiple of q).

Simple illustrative example: A simple illustrative example of order two, under Assumption 1, of a system of two coupled scalar states x_k(fast), w_k(slow) with $q = 2$ under arbitrary non-zero initial conditions x_0, y_0 is for the smaller sampling period T_1:

$$x_{k+1} = ax_k + bw_k; w_{k+1} = w_k$$

$$x_{k+2} = ax_{k+1} + bw_{k+1} = a^2x_k + abw_k + bw_{k+1} = a^2x_k + (a+1)bw_k$$

$$w_{k+2} = cx_{k+1} + dw_{k+1} = cax_k + cbw_k + dw_{k+1} = cax_k + (cb+d)w_k$$

$\forall k \in \mathbf{Z}_{0+}$, and M_k of Assumption 1 becomes $M_1 = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$; $M_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Note that there are two coupled states; one of them $x_{(.)}$ evolves at the faster sampling rate. For such a sampling rate, the second state $w_{(.)}$ remains unaltered. At the larger (or slower) sampling rate, the state that runs at the fast sampling rate is again updated, while the second one running at the larger sampling rate is updated, as well. This figures out Assumption 1 in a simple illustrative form. A parameterization for the larger sampling period $T = 2T_1$ is $x_{k+1} = M'x_k$ where $M' = \begin{bmatrix} a^2 & ab+b \\ ca & cb+d \end{bmatrix}$ if the sampling index is reassigned as $k \leftarrow 2k$. We can easily recognize: (a) a unit $(2,2)$ entry of M_1 meaning that the slow state becomes unaltered when the smaller sampling period does not equalize (or it is an integer multiple) of the largest one; (b) a square of the (1,1) entry of M' meaning that the dynamics of the small sampling period runs twice as the slow sampling period operates; and (c) a compacted "controllability-like" entry $ab + b$ reflecting a row controllability vector $[b, ab]$ if two consecutive forcing inputs w_k and w_{k+1} (in fact, the slow sate as an external input to the fast one) are identical and act on the fast sampled state. The underlying idea is that the slow state forces the fast one through the coupling dynamics, and this is reflected in a controllability-like matrix taking account of this action through each q consecutive samples of the slow sampling period, which provide with identical values.

Note that if a continuous-time system is discretized with a zero-order and hold device (Z.O.H.) while injecting a piecewise constant input (taking constant values in each inter-sample period), then the signals of the sampling system are identical to those of the continuous one at sampling instants. However, the obtained discrete-time one has no information in the inter-sample period unless the information is again re-taken from the continuous framework (via the modified z-transform, or via the direct solution for all time of the differential equations under a piecewise constant control). However, this "intermediate" information is not known if the continuous-time system is unknown (i.e., if just the discrete model is known) or if the input is more general than piecewise constant, which eventual jumps at sampling instants. On the other hand, note that the whole information recovery from a discrete-time system for all time is not feasible, in general, since it requires the use of a cardinal, or ideal, filter, which is not physically realizable.

The system parameterization based on the larger sampling period is given by the subsequent result:

Proposition 1. *The matrix of dynamics Equation (4) results in being time-invariant under Assumption 1 for the larger sampling period T and given by*:

$$M_{11} = A_{11}^q; M_{12} = \sum_{j=0}^{q-1} A_{11}^j A_{12} + A_{12} \tag{6.a}$$

$$M_{21} = A_{21}A_{11}^{q-1}; A_{12} = \sum_{j=0}^{q-2} A_{21}A_{11}^j A_{12} + A_{22} \tag{6.b}$$

Proof. It follows from the following calculations:

$$\begin{aligned} M &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & I_{n_2} \end{bmatrix}^{q-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & I_{n_2} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & I_{n_2} \end{bmatrix}^{q-2} \\ &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} A_{11}^{q-1} & \sum_{j=0}^{q-2} A_{11}^j A_{12} \\ 0 & I_{n_2} \end{bmatrix} = \begin{bmatrix} A_{11}^q & \sum_{j=0}^{q-1} A_{11}^j A_{12} + A_{12} \\ A_{21}A_{11}^{q-1} & \sum_{j=0}^{q-2} A_{21}A_{11}^j A_{12} + A_{22} \end{bmatrix} \end{aligned} \tag{7}$$

□

A stability result is given in the following result:

Theorem 1. *The multirate sampling system (1), subject to Assumption 1, is globally asymptotically stable if* A_{11}^{q} *is symmetric,* $\left[A_{21},\ A_{12}^{T}\right]^{T} \in Ker\left[A_{11}^{q-1^{T}},\ -\left(I_{n_1}+\sum_{j=0}^{q-1}A_{11}^{j}\right)\right]$, $A_{22}^{T} = A_{22}+\sum_{j=0}^{q-2}\left(A_{21}A_{11}^{j}A_{12}-A_{12}^{T}A_{11}^{j^{T}}A_{21}^{T}\right)$ *and, furthermore, any of the conditions below hold*:

(i) $r_{11}=r_{A_{11}}<1, r_{22}=r_{A_{22}}<1-r_{11}^{q}$,

$$max(\|A_{12}\|_2,\|A_{21}\|_2) < \max_{\delta_0\in \boldsymbol{R}_0} min\left[\left(1-r_{11}^{q}-r_{22}\right)/\left(1+r_{11}^{q-1}+\frac{1-r_{11}^{q}}{1-r_{11}}+\frac{1-r_{11}^{q-1}}{1-r_{11}}\delta_0\right),\delta_0\right]$$

(ii) $max(\|A_{12}\|_2,\|A_{21}\|_2)\in(\delta_1,\delta_2)$, where:

$$\delta_1=\frac{\sqrt{b^2-4ac}-b}{2a};\ \delta_2=\frac{\sqrt{b^2-4ac}+b}{2a}$$

$$a=\frac{1-r_{11}^{q}}{1-r_{11}};\ b=1+r_{11}^{q-1}+\frac{1-r_{11}^{q}}{1-r_{11}};\ c=r_{11}^{q}+r_{22}-1$$

Proof. Note that the constraints on the block matrices in M imply that M is symmetric since $M_{11}=M_{11}^{T}=A_{11}^{q}$, $M_{22}=M_{22}^{T}$ and $M_{21}^{T}=M_{12}$. Then, the spectral radius of M coincides with its spectral norm. Note from (6) that for any subordinated matrix norms and some $\varepsilon_0,\varepsilon(\geq\varepsilon_0)\in\boldsymbol{R}_+$, the following relationships hold if r_M is the spectral radius of M and $r_{11}=r_{A_{11}}$ and $r_{22}=r_{A_{22}}$ are the respective spectral radii of A_{11} and A_{22}, which are also identical to their spectral norms since they are symmetric and the norm infimums are taken on all the set of matrix subordinated norms:

$$\begin{aligned}
&\|M\|\leq\|M_{11}\|+\|M_{12}\|+\|M_{21}\|+\|M_{22}\|\\
&=\|A_{11}^{q}\|+\|\textstyle\sum_{j=0}^{q-1}A_{11}^{j}A_{12}+A_{12}\|+\|A_{21}A_{11}^{q-1}\|+\|\sum_{j=0}^{q-2}A_{21}A_{11}^{j}A_{12}+A_{22}\|\\
&\leq inf\|A_{11}^{q}\|+inf\|\textstyle\sum_{j=0}^{q-1}A_{11}^{j}A_{12}+A_{12}\|+inf\|A_{21}A_{11}^{q-1}\|+inf\|\sum_{j=0}^{q-2}A_{21}A_{11}^{j}A_{12}+A_{22}\|+\varepsilon_0\\
&\leq r_{11}^{q}+\left(1+\tfrac{1-r_{11}^{q}}{1-r_{11}}\right)\|A_{12}\|_2+r_{11}^{q-1}\|A_{21}\|_2+\tfrac{1-r_{11}^{q-1}}{1-r_{11}}\|A_{12}\|_2\|A_{21}\|_2+r_{22}+\varepsilon\\
&\leq r_{11}^{q}+\left(1+\tfrac{1-r_{11}^{q}}{1-r_{11}}\right)\delta+r_{11}^{q-1}\delta+\tfrac{1-r_{11}^{q-1}}{1-r_{11}}\delta^2+r_{22}+\varepsilon\\
&=r_{11}^{q}+\left(1+r_{11}^{q-1}+\tfrac{1-r_{11}^{q}}{1-r_{11}}+\tfrac{1-r_{11}^{q-1}}{1-r_{11}}\delta\right)\delta+r_{22}+\varepsilon
\end{aligned}\tag{8}$$

since the spectral radius of a matrix is the infimum over the set of subordinated norms. It holds that there exists a subordinated matrix norm $\|\,\|_a$ such that ε can be taken arbitrarily close to zero so that:

$$\|M\|_a\leq r_M\leq m=r_{11}^{q}+\left(1+r_{11}^{q-1}+\frac{1-r_{11}^{q}}{1-r_{11}}+\frac{1-r_{11}^{q-1}}{1-r_{11}}\delta\right)\delta+r_{22}$$

where $\delta\geq max(inf\|A_{12}\|,inf\|A_{21}\|)$. Thus, the system is globally asymptotically stable if M is convergent, which is guaranteed if $m<1$, that is if:

$$\delta<\frac{1-r_{11}^{q}-r_{22}}{\left(1+r_{11}^{q-1}+\frac{1-r_{11}^{q}}{1-r_{11}}+\frac{1-r_{11}^{q-1}}{1-r_{11}}\delta\right)}\tag{9}$$

which is guaranteed if, for some non-negative small enough real constant δ_0, it holds that:

$$\delta<min\left[\frac{1-r_{11}^{q}-r_{22}}{\left(1+r_{11}^{q-1}+\frac{1-r_{11}^{q}}{1-r_{11}}+\frac{1-r_{11}^{q-1}}{1-r_{11}}\delta_0\right)},\delta_0\right]\tag{10}$$

provided that $r_{11} < 1$ and $r_{22} < 1 - r_{11}^q$. Property (i) has been proven. To prove Property (ii), note that Property (i) is guaranteed if $p(\delta) = a\delta^2 + b\delta + c < 0$. Note also that $p(\delta)$ is a convex parabola so that $p(\delta) < 0$ if $\delta \in (\delta_1, \delta_2)$ where δ_1 and δ_2 are the two zeros of $p(\delta)$, which exist and are real since $ac < 0$, so that $\delta_1, \delta_2 \in \boldsymbol{R}_+$. □

Remark 1. *Weaker conditions for global asymptotic stability than those of Theorem 1 are obtained by replacing the spectral radii by some subordinated matrix norm $\|.\|$ (such as the ℓ_2, or spectral, norm) in the stability constraints to guarantee that $\|M\| < 1$. Note that if the norm is guaranteed to be strictly bounded by unity, then the symmetry condition to manipulate the spectral radius is not needed. On the other hand, parallel conditions for non-asymptotic global stability are obtained by replacing the spectral radii by subordinated norms and the strict inequalities by non-strict ones to guarantee that $\|M\| \leq 1$.*

Note that Theorem 1 implies that the block diagonal matrices of M are both assumed to be convergent and the off-diagonal ones, which represent the coupled dynamics between both multirate self-dynamics, are sufficiently weak related to the above convergence radii. The subsequent two results relax those constraints at the expense of considering extra conditions on the negative semi-definiteness of the coupled dynamics or the positivity of the whole matrix dynamics. The subsequent result does not assumes that M is symmetric, and it is based on its numerical radius.

Corollary 1. *The multirate sampling system (1), subject to Assumption 1, is globally asymptotically stable if*:

$$max\left(\omega_{A_{11}^q}, \omega_{\sum_{j=0}^{q-2} A_{21}A_{11}^jA_{12}+A_{22}}\right) < 1 - \frac{\left\|\sum_{j=0}^{q-1} A_{11}^jA_{12} + A_{12} + A_{21}A_{11}^{q-1}\right\|_2}{2}$$

Proof. It follows from (7) since $\omega_M \leq max(\omega_{M_{11}}, \omega_{M_{22}}) + \frac{1}{2}\|M_{12} + M_{21}\|_2$ and $\omega_M < 1$ under the given conditions. □

Corollary 2. *Assume that $M = M_0 + \widetilde{M}$ where $M_0 = Block\ Diag(M_{11} + M_{11a},\ M_{22} + M_{22a})$ and $\widetilde{M} = \begin{bmatrix} -M_{11a} & M_{12} \\ M_{21} & -M_{22a} \end{bmatrix} \preceq 0$. Then, $\omega_M \leq max(\omega_{M_{11}+M_{11a}},\ \omega_{M_{22}+M_{22a}}) - \lambda_{min}\left(-\widetilde{M}\right)$, and multirate sampling system (1), subject to Assumption 1, is globally asymptotically stable if $max(\omega_{M_{11}+M_{11a}},\ \omega_{M_{22}+M_{22a}}) < 1 + \lambda_{min}\left(-\widetilde{M}\right)$.*

Proof. Note that:

$$\begin{aligned} \omega_M = max\left(x^T\left(M_0 + \widetilde{M}\right)x : \|x\| = 1\right) = max\left(x^TM_0x - x^T\left(-\widetilde{M}\right)x : \|x\| = 1\right) \\ \leq max\left(x^TM_0x : \|x\| = 1\right) + max\left(-x^T\left(-\widetilde{M}\right)x : \|x\| = 1\right) = \omega_{M_0} + min\left(-x^T\widetilde{M}x : \|x\| = 1\right) \\ = \omega_{M_0} - \lambda_{min}\left(-\widetilde{M}\right) \end{aligned} \tag{11}$$

which is less than unity under the given conditions. □

Example 1. *Assume that $n_2 = n_1$, $A_{12} = -\left(\sum_{j=0}^{q-1} A_{11}^j + I_{n_1}\right)^{-1} A_{21}A_{11}^{q-1}$ and $A_{22} = -\sum_{j=0}^{q-2} A_{21}A_{11}^jA_{12}$. Then, the multirate sampling system under Assumption 1 is globally asymptotically stable if $\omega_{A_{11}^q} < 1$ from Corollary 1.*

Example 2. *Assume that $A_{22} = -\sum_{j=0}^{q-2} A_{21}A_{11}^jA_{12}$. Then, from Corollary 1 and $\omega_{11} = \omega_{A_{11}} \leq a$, the multirate sampling system under Assumption 1 is globally asymptotically stable if $\|A_{11}\|_2 \leq a < 1$ and $\frac{2-a-a^q}{1-a}\|A_{12}\|_2 + \|A_{21}\|_2a^{q-1} < 2(1 - a^q)$.*

3. A Multirate Sampling System with p Sampling Rates

It is now assumed that the system of (3) and (4) is generalized as follows for any number of multirate sampling periods under certain conditions described in detail:

Assumption 2. *(**a**) The multirate sampling system is subject to $p+1$ distinct sampling periods $\{T_1,\ T_2,\ \ldots,\ T_p = T\}$ such that $T_{i+1}/T_i = q_i$ where $q_i(\geq 2) \in \mathbf{Z}_+$, i.e., each one is an integer multiple of all its preceding ones (then, $T_i = \dot{T}_{j-1}$; $\forall j \in \overline{i}$); $\forall i \in \overline{p}$.*

*(**b**) The sampling period T_i runs a set of n_i variables grouped in a vector $x_i \in \mathbf{R}^{n_i}$; $\forall i \in \overline{p}$ so that the combined system (3) of state vector $x_k = \left(x_{1k}^T, x_{2k}^T, \ldots, x_{p,k}^T\right)^T$; $\forall k \in \mathbf{Z}_{0+}$ is described by a square real matrix:*

$$M_k = \begin{bmatrix} M_{11k} & M_{12k} & \cdots & M_{1,p+1,k} \\ M_{21k} & M_{22k} & \cdots & M_{2,p+1,k} \\ \cdots & \cdots & \cdots & \cdots \\ M_{p+1,1k} & M_{p+1,2k} & \cdots & M_{p+1,p+1,k} \end{bmatrix}; \forall k \in \mathbf{Z}_{0+} \tag{12}$$

of order $n = \sum_{i=1}^{p+1} n_i$ at each sampling instant of the base (smaller) sampling period T_1 with $M_{ijk} \in \mathbf{R}^{n_i \times n_j}$; $\forall (i,j) \in \overline{p} \times \overline{p}$, $\forall k \in \mathbf{Z}_{0+}$.

*(**c**)* $M_k = G_1 = \begin{bmatrix} A_{11} & A_{12} \cdots A_{1p} \\ 0 & Block\ Diag\left(I_{n_2},\ \ldots,\ I_{n_p}\right) \end{bmatrix}; \forall k \notin \dot{q}_1,$

$$M_k = G_2 = \begin{bmatrix} A_{11} & A_{12} & A_{13} \cdots\cdots\cdots\cdots\cdots & A_{1p} \\ A_{21} & A_{22} & A_{23} \cdots\cdots\cdots\cdots\cdots & A_{2p} \\ 0 & 0 & Block\ Diag\left(I_{n_3},\ \ldots,\ I_{n_p}\right) & \end{bmatrix}; \forall k \in \dot{q}_1 \cap \overline{\dot{q}_2},$$

$$M_k = G_{p-1} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I_{n_p} \end{bmatrix}; \forall k \in \dot{q}_{p-2} \cap \overline{\dot{q}_{p-1}}, \tag{13}$$

$$M_k = G_p = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix}; \forall k \in \dot{q}_{p-1},$$

where $A_{ij} \in \mathbf{R}^{n_i \times n_j}$; $\forall (i,j) \in \overline{p} \times \overline{p}$ with $q_i = T_{i+1}/T_i \in \mathbf{Z}_+$; $\forall i \in \overline{p-1}$, and $\overline{\dot{z}}$ denotes the complementary set of integer numbers to $\dot{z}$, that is the set of integers that are not multiples of $z \in \mathbf{Z}$.

Note that Assumption 2 refers to the fact that the sampling periods are each an integer multiple of the preceding one, while they are supposed to operate sequentially. This sequential operation is such that each subsystem associated with the sampling period T_{i+1} only updates its components as the subsystem associated with the preceding sampling period T_i has performed a cycle of q_i samples run at this rate, equivalently, a cycle of $\overline{q}_{i1} = \prod_{j=1}^{i} q_i$ samples ran at the base sampling period T_1. Note that we have the following direct relationships:

$$T_{i+1} = q_i T_i = \overline{q}_{ik} T_k = \overline{q}_{i1} T_1; \forall i \in \overline{p-1} \tag{14}$$

where $\overline{q}_{ik} = \prod_{j=k}^{i} \left[q_j\right]; \forall i,k \in \overline{p}$. In particular, $T = T_p = qT_1$, where $q = \overline{q}_{p1} = \prod_{j=1}^{p-1} \left[q_j\right]$. The following result is direct:

Proposition 2. *The multirate sampling system built satisfying Assumption 2 becomes a discrete time-invariant system of the form*:

$$x_{k+1} = Ax_k;\, x_0 = \theta \in \boldsymbol{R}^n,\ k(= qk_1) \in \mathbf{Z}_+;\ \forall k_1 \in \mathbf{Z}_{0+} \tag{15}$$

when governed at samples being integer multiple of q, where:

$$A = \prod_{j=2}^{p}\left[G_j^{q_{j-1}}\right] := (G_p)^{q_{p-1}} \bullet (G_{p-1})^{q_{p-2}} \bullet \cdots \bullet (G_2)^{q_1} \tag{16}$$

with the above matrix product being defined to the left.

The following result follows directly from Proposition 2.

Theorem 2. *The multirate sampling system satisfying Assumption 2 has the following properties*:

(*i*) *It is globally stable if* $\prod_{j=2}^{p}\|G_j\|_2^{q_{j-1}} \le 1$, *and it is globally asymptotically stable if* $\prod_{j=2}^{p}\|G_j\|_2^{q_{j-1}} < 1$.

(*ii*) *It is globally stable if* $\sum_{j=2}^{p} q_{j-1} ln\lambda_{max}^{1/2}\left(G_j^T G_j\right) \le 0$, *and it is globally asymptotically stable if* $\sum_{j=2}^{p} q_{j-1} ln\lambda_{max}^{1/2}\left(G_j^T G_j\right) < 0$.

Proof. It follows from (16) that $\left(\prod_{j=2}^{p}\|G_j\|_2^{q_{j-1}} \le 1\right) \Rightarrow (\|A\|_2 \le 1)$ and that $\left(\prod_{j=2}^{p}\|G_j\|_2^{q_{j-1}} < 1\right) \Rightarrow (\|A\|_2 < 1)$. In the first case, the sequence $\left\{\|x_{qk}\|_2\right\}_{k\in\mathbf{Z}_{0+}}$ is bounded from (15)–(16) for any given finite $\|x_0\|_2$. Since $Card\{x_\ell : \ell \in [kq, (k+1)q), k \in \mathbf{Z}_{0+}\} = q = \prod_{j=1}^{p-1}[q_j] < \infty$ and $\left\{\|x_{qk}\|_2\right\}_{k\in\mathbf{Z}_{0+}} < \infty$, if $\prod_{j=2}^{p}\|G_j\|_2^{q_{j-1}} \le 1$, then it follows from (15) that $\|x_k\|_2 \le \max_{1\le j\le q-1}\left(1,\ \|A\|_2^j\right) \sup_{j\in\mathbf{Z}_{0+}}\|x_{qj}\|_2 < \infty;\ \forall k \in \mathbf{Z}_{0+}$. On the other hand, if $\prod_{j=2}^{p}\|G_j\|_2^{q_{j-1}} < 1$, then $\left\{\|x_{qk}\|_2\right\}_{k\in\mathbf{Z}_{0+}} \to 0$. This implies from the finiteness of the above cardinal q that $\{\|x_k\|_2\}_{k\in\mathbf{Z}_{0+}} \to 0$ since:

$$\lim_{k\to\infty}\sup\left(\|x_k\|_2 - \max_{1\le j\le q-1}\left(1, \|A\|_2^j\right)\sup_{k\in\mathbf{Z}_{0+}}\|x_{qk}\|_2\right) = \lim_{k\to\infty}\sup\|x_k\|_2 \le 0 \tag{17}$$

since $\lim_{k\to\infty}\sup\left(\sup_{k\in\mathbf{Z}_{0+}}\|x_{qk}\|_2\right) = 0$ and since $\left\{\|x_{qk}\|_2\right\}_{k\in\mathbf{Z}_{0+}} \to 0$. Property (i) is proven. The proof of Property (ii) is similar to that of Property (i) by expressing the corresponding above conditions on products of ℓ_2-norms as the sum of their logarithms. □

A reformulation in terms of the numerical radius follows:

Theorem 3. *The multirate sampling system satisfying Assumption 2 has the following properties*:

(*i*) *It is globally asymptotically stable if* $W = a^{-1}(I+Z)^{-1/2}\left(\prod_{j=2}^{p}\left[G_j^{q_{j-1}}\right]\right)(I-Z)^{-1/2}$ *is a (non-necessarily strict) contraction for some real* $a \in (0,\ 1)$ *and some symmetric strict contraction* Z. *A sufficient condition for the global asymptotic stability is that* W^2 *is a strict contraction.*

(*ii*) *It is globally stable if* $W = (I+Z)^{-1/2}\left(\prod_{j=2}^{p}\left[G_j^{q_{j-1}}\right]\right)(I-Z)^{-1/2}$ *is a (non-necessarily strict) contraction for some real* $a \in (0,\ 1)$ *and some symmetric strict contraction* Z.

Proof. It is obvious that $\omega(A) \le a$ for any given $a \in \boldsymbol{R}_{0+}$ if $A = aA_a$ and $\omega(A_a) \le 1$, which holds if and only if [21] $A_a = (I+Z)^{1/2}W(I-Z)^{1/2}$ for some existing (non-necessarily strict) contractions W and Z (i.e., $W^TW \preceq I$, $Z^TZ \preceq I$) with Z symmetric (Hermitian if A is complex). Note that if $Z^2 = Z^TZ \prec I$, i.e., Z is a strict contraction, then W is unique since $(I+Z)$ and $(I-Z)$ are non-singular since:

$$(I+Z)(I-Z) = I - Z^2 + Z - Z = I - Z^2 \succ 0 \tag{18}$$

so that $(I+Z)$ and $(I-Z)$ are non-singular. Thus, one has from (16),

$$A = \prod_{j=2}^{p}\left[G_j^{q_{j-1}}\right] = aA_a = a(I+Z)^{1/2}W(I-Z)^{1/2} \tag{19}$$

which gives the result since $(I+Z)$ and $(I-Z)$ are non-singular and $\rho(A) \le \omega(A) = a\omega(A_d) \le a < 1$ implies that A is a convergent matrix. On the other hand, note that $\rho(A^2) \le \omega(A^2) \le \omega^2(A) \le a^2 < 1$ implying that A is convergent is also guaranteed if $W^2 = a^{-2}(I+Z)\left(\prod_{j=2}^{p}\left[G_j^{2q_{j-1}}\right]\right)(I-Z)$ is a contraction for some symmetric strict contraction Z since $A^2 = \prod_{j=2}^{p}\left[G_j^{2q_{j-1}}\right] = a^2A_a^2 = a^2(I+Z)W^2(I-Z)$. It follows from (15) that $\left\{\|x_{qk}\|_2\right\}_{k\in\mathbf{Z}_{0+}} \to 0$ and from the finiteness of q that $\{\|x_k\|_2\}_{k\in\mathbf{Z}_{0+}} \to 0$ as a result for any given bounded initial conditions; see the proof of Theorem 2. Property (i) has been proven. The proof of Property (ii) is close with $a=1$ implying the uniform boundedness of the state sequence from (15) for any given finite initial conditions.

Remark 2. *Note from (16) and (13) that all the matrices, G_p excepted, whose left product conforms to the matrix of dynamics A governing the multirate sampling system at the largest sampling rate cannot be because of their structure spectral and numerical radius less than one [19–21]. Therefore, the global asymptotic stability of the system has to be guaranteed by an appropriate constraint on the numerical or spectral radius, being sufficiently small related to unity, of G_p being able to compensate their corresponding values of $(G_{p-1})^{q_{p-2}}\bullet\cdots\bullet(G_2)^{q_1}$.*

To characterize the tolerance of the stability to parametrical disturbances, we rewrite (16) as:

$$\begin{gathered} A_\delta = A + \widetilde{A}_\delta = \prod_{j=2}^{p}\left[\left(G_j+\widetilde{G}_j\right)^{q_{j-1}}\right] \\ := \left(G_p+\widetilde{G}_p\right)^{q_{p-1}}\bullet\left(G_{p-1}+\widetilde{G}_{p-1}\right)^{q_{p-2}}\bullet\cdots\bullet\left(G_2+\widetilde{G}_2\right)^{q_1} \end{gathered} \tag{20}$$

where the superscripted tildes stand for the various parametrical disturbances assumed to satisfy $\|\widetilde{G}_j\|_2 \le \delta \le \delta_0 < 1$ for $j = 2,3,\ldots,p$. Thus, one gets from (20) that:

$$\begin{gathered} \|A_\delta\|_2 \le \prod_{j=1}^{p-1}\left[\sum_{i=0}^{q_{p-j}}\begin{pmatrix} q_{p-j} \\ i \end{pmatrix}\|G_{p-j+1}\|_2^{q_{p-j-i}}\delta^i\right] \\ \|A\|_2 \le \prod_{j=1}^{p-1}\|G_{p-j+1}\|_2^{q_{p-j}} \\ \|\widetilde{A}_\delta\|_2 \le \prod_{j=1}^{p-1}\left[\begin{pmatrix} q_{p-j} \\ 0 \end{pmatrix}\|G_{p-j+1}\|_2^{q_{p-j}} + \begin{pmatrix} q_{p-j} \\ 1 \end{pmatrix}\|G_{p-j+1}\|_2^{q_{p-j-1}}\delta + o(\delta^2)\right] \\ = \sum_{j=1}^{p-1}\sum_{\ell(\neq j)=1}^{p-1}\begin{pmatrix} q_{p-j} \\ 0 \end{pmatrix}\begin{pmatrix} q_{p-\ell} \\ 1 \end{pmatrix}\|G_{p-j+1}\|_2^{q_{p-j}}\|G_{p-\ell+1}\|_2^{q_{p-\ell-1}}\delta + o(\delta^2) \\ = \sum_{j=1}^{p-1}\sum_{\ell(\neq j)=1}^{p-1} q_{p-\ell}\|G_{p-j+1}\|_2^{q_{p-j}}\|G_{p-\ell+1}\|_2^{q_{p-\ell-1}}\delta + o(\delta^2) \\ \le \left(\sum_{j=1}^{p-1}\sum_{\ell(\neq j)=1}^{p-1} q_{p-\ell}\|G_{p-j+1}\|_2^{q_{p-j}}\|G_{p-\ell+1}\|_2^{q_{p-\ell-1}} + K_0\right)\delta_0 \end{gathered} \tag{21}$$

for some non-negative real constant $K_0 < +\infty$ since $\delta_0 < 1$. Note that:

$$r(A_\delta) \le \omega(A_\delta) \le \omega(A) + \omega\left(\widetilde{A}_\delta\right) \le \omega(A) + \|\widetilde{A}_d\|_2 \le \|A\|_2 + \|\widetilde{A}_d\|_2 < 1 \tag{22}$$

If:

$$\omega\left(\widetilde{A}_\delta\right) \le \|\widetilde{A}_d\|_2 < 1 - \|A\|_2 \le 1 - \omega(A) \tag{23}$$

then, the following stability result holds:

Theorem 4. *Assume that* $\|A\|_2 < 1$ *and* $\|\widetilde{G}_j\|_2 \leq \delta \leq \delta_0 < 1$ *for* $j = 2, 3, \ldots, p$. *The multirate sampling system that satisfies Assumption 2 is globally asymptotically stable if*:

$$\delta_0 \leq \frac{1 - \|A\|_2}{\sum_{j=1}^{p-1} \sum_{\ell(\neq j)=1}^{p-1} q_{p-\ell} \|G_{p-j+1}\|_2^{q_{p-j}} \|G_{p-\ell+1}\|_2^{q_{p-\ell-1}} + K_0}$$

or if:

$$\delta_0 \leq \frac{1 - \omega(A)}{\sum_{j=1}^{p-1} \sum_{\ell(\neq j)=1}^{p-1} q_{p-\ell} \|G_{p-j+1}\|_2^{q_{p-j}} \|G_{p-\ell+1}\|_2^{q_{p-\ell-1}} + K_0}$$

for some real constant $K_0 \geq \delta_0^{-1}(|\|A_\delta\|_2 - \|A\|_2|)$.

4. Multirate Input Sampling with Combined Pole-Placement and Closed-Loop Stability Analysis via Linear State-Feedback

Throughout this section, one considers a forced multirate sampling control that extends that satisfying Assumption 1 to the forced case. The generalization to a forced system under Assumption 2 with more than two sampling rates is direct at the expense of a more involved presentation and mathematical derivations. Consider a controlled time-invariant multirate sampling system with two sampling periods $T_1 = T/q$ and T for some $q(\geq 2) \in \mathbf{Z}_+$ given by:

$$\begin{aligned} x_{k+1} &= Ax_k + By_k + Eu_k \\ y_{k+q} &= Dy_k + Cx_k + Fu_k \end{aligned} \tag{24}$$

under initial conditions x_0 and y_0; $\forall k \in \mathbf{Z}_{0+}$, which runs the smaller sampling period T_1, where $x_k \in \mathbf{R}^{n_1}$ and $y_k \in \mathbf{R}^{n_2}$ are the respective state vectors and $u_k \in \mathbf{R}^m$ is the feedback control, governed at the smaller sampling period T_1, which is generated as follows:

$$u_{k+i} = K_i x_k + G_i y_k \tag{25}$$

for some control gains $K_i \in \mathbf{R}^{m \times n_1}$ and $G_i \in \mathbf{R}^{m \times n_2}$ for $i = 0, 1, \cdots, q-1$ and any $k \in \mathbf{Z}_{0+}$. Since $y_{k+i} = y_k$ $i = 0, 1, \cdots, q-1$, the replacement of (25) in (24) leads to:

$$x_{k+q} = \left(A^q + \sum_{i=0}^{q-1} A^{q-i-1} E K_i\right) x_k + \sum_{i=0}^{q-1} A^{q-i-1} (B + EG_i) y_k \tag{26a}$$

$$y_{k+q} = (C + FK_0)x_k + (D + FG_0)y_k \tag{26b}$$

The dynamics of (26) is governed by $z_{k+1} = \overline{A}_c z_k$ with $z_k = \left(x_k^T, y_k^T\right)^T \in \mathbf{R}^n$, $n = n_1 + n_2$; $\forall k \in \mathbf{Z}_{0+}$ for any given initial conditions $z_0 = \left(x_0^T, y_0^T\right)^T$ for the largest sampling period T with $\overline{A}_c = \overline{A} + \overline{BK}$, where one gets from Assumption 1:

$$\overline{A} = \begin{bmatrix} A^q & \sum_{i=0}^{q-1} A^{q-i-1} B \\ C & D \end{bmatrix}$$

$$\overline{B} = \begin{bmatrix} E & AE & \cdots & A^{n-1}E \\ 0 & 0 & & \cdots F \end{bmatrix} \tag{27}$$

$$\overline{K} = \begin{bmatrix} K_{q-1} & G_{q-1} \\ K_{q-2} & G_{q-2} \\ \vdots & \vdots \\ K_0 & G_0 \end{bmatrix}$$

Note that the matrix of dynamics $\overline{A}$ reflects the class of systems under Assumption 1 for two sampling rates and (see also the given simple illustrative example given jointly with Assumption 1) in the following sense. For the slow sampling rate system modelling, the matrix power of the dynamics of the fast sampling rate, being equal to the ratio of sampling rates, appears explicitly in the $(1,1)$ block matrix. A sum of terms appears also explicitly in the $(1,2)$-block matrix being expanded from a controllability matrix taking into account the influence of q consecutive identical slow substates in the fast sampling substate through dynamic coupling. On the other hand, the control matrix $\overline{B}$ reflects the influence of the fast and slow inputs in the closed-loop dynamics. The descriptions could also be easily extended for a decomposition into subsystems under more than two sampling rates under Assumption 2.

Some results are now discussed concerning the spectrum assignability and the stability of the closed-loop multirate sampling system of (26) and (27) through an appropriate synthesis of the controller gains under a controllability assumption. The following first result is concerned with prefixed stable spectrum assignability, while the second one refers to closed-loop stabilization under the weaker assumption of the stabilizability of the controlled open-loop system.

Proposition 3. *Assume that $\left(\overline{A},\overline{B}\right)$ is a controllable pair, i.e., rank$\left[\overline{B},\overline{AB},\cdots\overline{A}^{n-1}\overline{B}\right]=n$, equivalently, rank$\left[\lambda I_n-\overline{A},\overline{B}\right]=n;\forall\lambda\in sp\left(\overline{A}\right)$, according to the Popov–Belevitch–Hautus rank controllability test. Then, the spectrum of $\overline{A}_c$ can be assigned to any prescribed positions, in particular to prescribed stable allocations, by the appropriate choice of the matrix of gain controllers $\overline{K}$. If the assignment is made to any allocations within the unit complex circle, thus $\overline{A}_c$ is convergent, and then, the system is globally asymptotically stable.*

Proof. It is obvious since controllability is equivalent to the spectrum assignability of the closed-loop eigenvalues through linear state-feedback. □

Proposition 4. *Assume that $\left(\overline{A},\overline{B}\right)$ is stabilizable, i.e., rank$\left[\lambda I_n-\overline{A},\overline{B}\right]=n;\ \forall\lambda\in sp\left(\overline{A}\right)\cup\{\vartheta\in\mathbf{C}:|\vartheta|\geq 1\}$ according to the Popov–Belevitch–Hautus rank stabilizability test. Then, the spectrum of $\overline{A}_c$ can be allocated in stable positions via the choice of $\overline{K}$ so that $\overline{A}_c$ is convergent, and the system is globally asymptotically stable as a result.*

Proof. It is obvious since stabilizability implies that the unstable and critically unstable open-loop modes can be re-allocated to stable positions through linear state-feedback. □

Assume that the control law (25) is modified as follows:

$$u_{k+i}=K_i x_{k+i}+G_i y_k \tag{28}$$

for $i=0,1,\cdots,q-1$ and any $k\in\mathbf{Z}_{0+}$. The combination of (24) and (28) yields

$$x_{k+i}=A^i x_k+\sum_{j=0}^{i-1}A^{i-j-1}\left(By_k+Eu_{k+j}\right) \tag{29}$$

and the matrix defining the closed-loop dynamics of order $n=n_1+n_2$ is defined by $\overline{A}_c=\overline{A}_{c0}+\widetilde{\overline{A}}_{c0}$, where:

$$\overline{A}_{c0}=\begin{bmatrix}\overline{A}_{c0_{11}} & 0\\ 0 & \overline{A}_{c0_{22}}\end{bmatrix}=\begin{bmatrix}\prod_{i=0}^{q-1}[A+EK_i] & 0\\ 0 & D+FG_0\end{bmatrix} \tag{30}$$

$$\widetilde{\overline{A}}_{c0}=\begin{bmatrix}0 & \sum_{i=0}^{q-1}\left(\prod_{j=i+1}^{q-1}\left[A+EK_j\right]\right)(B+EG_i)\\ C+FK_0 & 0\end{bmatrix}$$

It can happen that the multirate sampling system does not fulfil the controllability conditions of Proposition 3 or those of the stabilizability of Proposition 4. It is now discussed how to proceed in those cases under weaker "a priori" constraints on the multirate sampling system. Concerning the closed-loop dynamics defined by the matrix $\overline{A}_c$, note the following features, which have to be made compatible for the stabilization of the multirate sampling system via linear state-feedback:

Feature 1: Assume that the anti-diagonal part $\widetilde{\overline{A}}_{c0}$ of $\overline{A}_c$ is suited to be prefixed, if possible, to suitable prefixed sub-matrices by the appropriate synthesis of the controller gains resulting in $\widetilde{\overline{A}}_{c0} = \begin{bmatrix} 0 & \widetilde{\overline{A}}_{c0_{12}} \\ \widetilde{\overline{A}}_{c0_{21}} & 0 \end{bmatrix}$. Thus, according to the vectorization by using the appropriate Kronecker products [14,15,20], to solve:

$$C + FK_0 = \widetilde{\overline{A}}_{c0_{21}};\ \sum_{i=0}^{q-1}\left(\prod_{j=i+1}^{q-1}[A+EK_j]\right)(B+EG_i) = \widetilde{\overline{A}}_{c0_{12}} \tag{31}$$

Equivalently, we have:

$$(F_{n_2\times m}\otimes I_{n_1})_{n_1n_2\times n_1m}[vec(K_0)]_{n_1m\times 1} = \left[vec\left(\widetilde{\overline{A}}_{co_{21}} - C\right)\right]_{n_1n_2\times 1} \tag{32}$$

$$\left(\Sigma_{i=0}^{q-1}\left(\Pi_{j=i+1}^{q-1}[A+EK_j]\right)E_{n_1\times m}\otimes I_{n_2}\right)_{n_1n_2\times n_2m}[vec(G_i)]_{mn_2\times 1} = \left[vec\left(\widetilde{\overline{A}}_{co_{12}} - \Sigma_{i=0}^{q-1}\left(\Pi_{j=i+1}^{q-1}[A+EK_j]\right)B\right)\right]_{n_1n_2\times 1}$$

Proposition 5. *Assume that $m \geq max(n_1,n_2)/q$ and that there exist a prefixed goal anti-diagonal matrix with $(1,2)$ block matrix $\widetilde{\overline{A}}_{c021}$ and controller gains K_i, $\forall i \in \overline{q-1}\cup\{0\}$ such that*:

$$rank(F\otimes I_{n_1}) = rank\left[(F\otimes I_{n_1}),\left[vec\left(\widetilde{\overline{A}}_{co_{21}} - C\right)\right]\right] \leq n_1 min(n_2, qm) \tag{33}$$

$$\begin{gathered} rank\left(\Pi_{j=1}^{q-1}[A+EK_j]E\otimes I_{n_2},\Pi_{j=2}^{q-1}[A+EK_j]E\otimes I_{n_2},\cdots,[A+EK_{q-1}]E\otimes I_{n_2}\right) \\ = rank\left[\Pi_{j=1}^{q-1}[A+EK_j]E\otimes I_{n_2},\Pi_{j=2}^{q-1}[A+EK_j]E\otimes I_{n_2},\cdots,[A+EK_{q-1}]E\otimes I_{n_2}, vec\left(\widetilde{\overline{A}}_{co_{12}} - \Sigma_{i=0}^{q-1}\left(\Pi_{j=i+1}^{q-1}[A+EK_j]\right)B\right)\right] \\ \leq n_2\ min(n_1, qm) \end{gathered} \tag{34}$$

Thus, there exist controller gains G_i; $\forall i \in \overline{q-1}\cup\{0\}$ parameterizing the modified control law (28) such that (31) and (32) are solvable. Furthermore:

If (C,F) is controllable, then $sp\left(\widetilde{\overline{A}}_{c0_{21}}\right)$ is freely-assignable from the selection of the controller gain K_0.

If there exists some prefixed goal block matrix $\widetilde{\overline{A}}^*_{c0_{21}}$ in the anti-diagonal matrix $\widetilde{\overline{A}}_{c0_{21}}$ such that:

$$rank(F\otimes I_{n_1}) = rank\left[F\otimes I_{n_1}, vec\left(\widetilde{\overline{A}}_{c0_{21}} - C\right)\right] \tag{35}$$

then there exists K_0 such that $C + FK_0 = \widetilde{\overline{A}}^*_{c0_{21}}$.

Feature 2: Assume that the spectrum of the diagonal part $\overline{A}_{c0}$ of $\overline{A}_c$ is suited to be prefixed, if possible, to suitable stable values by the appropriate synthesis of the controller gains. We have the following elementary related result:

Proposition 6. *Assume that (A,E) and (D,F) are controllable pairs. Then, for any given set of n complex numbers $\{\lambda_1,\lambda_2,\cdots,\lambda_n\}$ satisfying $|\lambda_i| < 1$, there exist controller gains $K_i = K$ for $i \in \overline{q-1}\cup\{0\}$ and G_0 parameterizing (28) such that the spectrum of $\overline{A}_{c0}$ in (30) can be fixed to $sp\overline{A}_{co} = \left\{\lambda_1^q,\lambda_2^q,\cdots,\lambda_{n_1}^q,\lambda_{n_1+1},\lambda_{n_1+2},\cdots,\lambda_n\right\}$.*

If only (A,E) is controllable, or if only (D,F) is controllable, then the respective spectra of the diagonal block matrices of $\overline{A}_{c0}$ can be pre-assigned.

Feature 3: If (A, E) is not controllable, it is possible to select the (1, 1)-block matrix of $\overline{A}_{c0}$ for some prefixed goal value $\overline{A}^*_{c0_{11}}$ such that the algebraic equation:

$$\left(\prod_{i=0}^{q_1-1}[A+EK_i]\right)(A+EK_{q_1})\left(\prod_{i=q_1+1}^{q-1}[A+EK_i]\right)=\overline{A}^*_{c0_{11}} \tag{36}$$

is solvable in K_{q_1} for some $0 \le q_1 \le q-2$. If $q_1 = q-2$ (respectively,$q_1 = 0$), then the last (respectively, the first) left-hand-side term of matrix products in (35) is the n_1-th identity. The following result is direct:

Proposition 7. *Assume that, for some given* $\overline{A}^*_{c011}$*, some given set of controller gains* $\{K_0, \ldots, K_{q_1-1}, K_{q_1+1}, \ldots, K_{q-1}\}$ *and integer* $0 \le q_1 \le q-1$*, the rank condition below holds*:

$$\begin{aligned} &rank\left[\left(\prod_{i=0}^{q_1-1}[A+EK_i]\right)E\otimes\left(\prod_{i=q_1+1}^{q-1}[A+EK_i]\right)^T\right] \\ &= rank\left[\left(\prod_{i=0}^{q_1-1}[A+EK_i]\right)E\otimes\left(\prod_{i=q_1+1}^{q-1}[A+EK_i]\right)^T, vec\left(\overline{A}^*_{c011}-\left(\prod_{i=0}^{q_1-1}[A+EK_i]\right)A\left(\prod_{i=q_1}^{q-1}[A+EK_i]\right)^T\right)\right] \end{aligned} \tag{37}$$

where $\prod_{i=q_1}^{q-1}[A+EK_i] \to I_{n_1}$ if $q_1 = q-1$ and $\prod_{i=0}^{q_1-1}[A+EK_i] \to I_{n_1}$ if $q_1 = 0$. Then, (36) is solvable in K_{q_1} from the equivalent vectorized algebraic equation:

$$\left[\left(\prod_{i=0}^{q_1-1}[A+EK_i]\right)E\otimes\left(\prod_{i=q_1+1}^{q-1}[A+EK_i]\right)^T\right]vec(K_{q_1}) = vec\left(\overline{A}^*_{c011}-\left(\prod_{i=0}^{q_1-1}[A+EK_i]\right)A\left(\prod_{i=q_1}^{q-1}[A+EK_i]\right)^T\right) \tag{38}$$

The closed-loop synthesis objective is that the whole matrix $\overline{A}_c$ be convergent via the controller synthesis so that the closed-loop system results in being globally asymptotically stable as a result. A way for that is that its diagonal part be stable while the anti-diagonal part has a sufficiently small norm related to the spectral radius or the norm of the diagonal one.

Some guidelines for a controller synthesis methodology combining the above results are now described:

Step 1: If condition (37) of Proposition 7 is fulfilled for some $1 \le q_1 \le q_1 - 1$ and some prefixed targeted matrix of stable spectrum $\overline{A}^*_{c0_{11}}$ for $\overline{A}_{c0_{11}}$, then solve (38), equivalent to (36), to calculate some compatible controller gain K_{q_1} for given controller gains K_i for $i \ne q_1(\ne 0)$.

Step 2: If (D, F) is controllable, then calculate the controller gain G_0, according to Proposition 6, to prefix $\overline{A}_{c0_{22}}$ to some prefixed value $\overline{A}^*_{c0_{22}}$ with a stable spectrum.

Step 3: If the condition (33) of Proposition 5 holds for some prefixed $\tilde{\overline{A}}^*_{c0_{21}}$ value for $\tilde{\overline{A}}_{c0_{21}}$, then synthesize the controller gain K_0 so that the first equation of (32) holds.

Step 4: Assume that the condition (34) of Proposition 5 is modified to re-allocate the left-hand-side term in G_0 (already calculated in Step 2) to its right part and the resulting solvability rank condition holds for some prefixed $\tilde{\overline{A}}^*_{c0_{12}}$ value for $\tilde{\overline{A}}_{c0_{12}}$. Then, synthesize the controller gains G_i for $i \in \overline{q-1}$ so that the second modified equation of (32), associated with the above modification of the condition (34) holds.

Steps 1–2 prefix the two diagonal matrix blocks of the diagonal part $\overline{A}_{c0}$ of $\overline{A}_c$ to stable spectra of by the synthesis of $K_{q_1}(q_1 \ne 0)$ and G_0 provided that (37) folds and (D, F) is controllable.

Steps 1–2 prefix the anti-diagonal part $\tilde{\overline{A}}_{c0}$ of $\overline{A}_c$ to prescribed values, which are suitable if they have sufficiently small norms.

If (A, E) is controllable and there exists $K = K_0 = K_i$; $i \in \overline{q-1}$ for K_0 satisfying the condition (34), then $\overline{A}_{c0_{11}}$ can be fully matched, rather than just its spectrum, equalized to a stable targeted value $\overline{A}^*_{c0_{11}}$ through the choices of the controller gains G_i for $i \in \overline{q-1} \cup \{0\}$, and then, the whole $\overline{A}_{c0}$ can be prefixed to a stable matrix $\overline{A}^*_{c0}$ if (D, F), provided that it is controllable (Step 2).

The following results hold concerning the spectral radius and ℓ_2-norms of $\overline{A}_c = \overline{A}_{c0} + \tilde{\overline{A}}_{c0}$, the second one being a direct conclusion from the sub-additive property of norms:

Assertion 1. *The inequality* $r_{\overline{A}_c} \leq min\left(r_{\overline{A}_{c0}} + \|\widetilde{\overline{A}}_{c0}\|_2, r_{\widetilde{\overline{A}}_{c0}} + \|\overline{A}_{c0}\|_2\right)$ *holds.*

Proof. Assume that the assertion is false. Then,

$$2\left(\|\overline{A}_{c0}\|_2 + \|\widetilde{\overline{A}}_{c0}\|_2\right) \geq 2\|\overline{A}_c\|_2 \geq 2r_{\overline{A}_c}$$

$$> 2max\left(r_{\overline{A}_{c0}} + \|\widetilde{\overline{A}}_{c0}\|_2, r_{\widetilde{\overline{A}}_{c0}} + \|\overline{A}_{c0}\|_2\right) \geq r_{\overline{A}_{c0}} + r_{\widetilde{\overline{A}}_{c0}} + \|\overline{A}_{c0}\|_2 + \|\widetilde{\overline{A}}_{c0}\|_2 \tag{39}$$

which yields the contradiction:

$$\|\overline{A}_{c0}\|_2 + \|\widetilde{\overline{A}}_{c0}\|_2 \geq r_{\overline{A}_c} > \|\overline{A}_{c0}\|_2 + \|\widetilde{\overline{A}}_{c0}\|_2 \tag{40}$$

Hence, the assertion is true. □

Assertion 2. $\|\overline{A}_c\|_2 < 1$ *if* $\|\overline{A}_{c0}\|_2 < 1$ *and* $\|\widetilde{\overline{A}}_{c0}\|_2 < 1 - \|\overline{A}_{c0}\|_2$ *hold.*

Assertions 1 and 2 can be combined with the stabilizing controller synthesis guidelines by remembering that the spectral radius of a square matrix is the maximum absolute value of its eigenvalues and it is less than or equal to any matrix norm.

Example 3. *Consider the subsequent discrete system with two fast and slow sampling periods* T_1 *and* $T = 2T_1$ *and two coupled fast and slow scalar substates* x *and* y:

$$x_{k+1} = ax_k + gy_k + bu_k \tag{41}$$

$$y_{k+1} = \alpha y_k + \beta x_k + \gamma u_k \tag{42}$$

for any initial conditions x_0 and y_0, where $b \neq 0$ and $k', k \in \mathbf{Z}_{0+}$ run the solution sequences $\{x_{k'}\}$ and $\{y_k\}$. Take particular values k and k' where the matching of sampling instants $kT = k'T_1$ holds. Then, $(k+1)T = (k'+2)T_1$, so that, for the slow sampling period, we have:

$$x_{k+1} = a^2 x_k + (a+1)gy_k + abu_k + bu_{k'+1} \tag{43}$$

It follows from (42) and (43) that the Popov–Belevitch–Hautus controllability and stabilizability tests are performed via the matrix:

$$T(z) = \begin{bmatrix} z - a^2 & (a+1)g & b & ab \\ \beta & z - \alpha & 0 & \gamma \end{bmatrix} \tag{44}$$

where z is the one-step discrete advance operator for the slow sampling period T, that is $v_{k+1} = zv_k$ for a sequence $\{v_k\}$. Note that if only a single sampling rate is used, i.e., $T = T_1$, then the matrix to perform such tests is from (41) and (42):

$$T_s(z) = \begin{bmatrix} z - a & g & b \\ \beta & z - \alpha & \gamma \end{bmatrix} \tag{45}$$

The single rate necessary and sufficient controllability condition is $rankT_s(z) = 2$ for any $z \in \mathbf{C}$, and that of stabilizability is $rankT_s(z) = 2$ for any $z \in \mathbf{C}$ with $|z| \geq 1$.

The multirate necessary and sufficient controllability condition for the slow sampling rate $T = 2T_1$ is $rankT(z) = 2$ for any $z \in \mathbf{C}$ and that of stabilizability is $rankT(z) = 2$ for any $z \in \mathbf{C}$ with $|z| \geq 1$. Note that the stabilizability tests are positive if the controllability ones are positive, but the converse is not true. Observe also the following facts:

(1) $2 \geq rankT(z) \geq rankT_s(z)$ for any $z \in \mathbf{C}$.

(2) If $min(a, \gamma) > 0$, $g = \beta = 0$ and $\alpha = a$, then $rankT_s(\alpha) = \underset{z \in \mathbf{C}}{min} rankT_s(z) = 1$ for any $\alpha \in \mathbf{R}$, so that controllability and stabilizability fail in the single rate case.

(3) If $min(a, \gamma) > 0$, $g = \beta = 0$ and $\alpha = a^2$, then $rankT(\alpha) = \underset{z \in \mathbf{C}}{min} rankT(z) = 2$ for any $\alpha \in \mathbf{R}$, so that controllability holds, and then, the stabilizability holds, as well, in the multirate case with $T = 2T_1$ for any $T_1 > 0$.

(4) If $\beta = \gamma = 0$, $g \neq 0$ and $\alpha = a$, then $rankT_s(\alpha) = \underset{z \in \mathbf{C}}{min} rankT_s(z) = 1$ for any $\alpha \in \mathbf{R}$. If $\alpha = a^2$, then $rankT(\alpha) = \underset{z \in \mathbf{C}}{min} rankT(z) = 1$ for any $\alpha \in \mathbf{R}$; $\forall z \in \mathbf{C}$. However, note that if $\alpha = a \neq 1$, then $\alpha \neq a^2$,$z - \alpha \neq z - a^2$; $\forall z \in \mathbf{C}$. Therefore, $\underset{z \in \mathbf{C}}{min} rankT_s(z) = 1$, and the single rate system is not controllable and is not stabilizable either if $|a| \geq 1$, while $\underset{z \in \mathbf{C}}{min} rankT(z) = 2$ and the multirate sampling system is controllable and stabilizable. The above propositions and controller synthesis method easily yield some control gains useful for the achievement of closed-loop stabilization.

Example 4. *Consider a more general example case of higher state order than Example 3 with $n = n_1 + n_2$ with $min(n_1, n_2) \geq 2$ and the replacement of the parameterizing scalars by matrices of the appropriate orders as follows $a \to A$, $b \to B$, $g \to G$, $\alpha \to \Phi$, $\beta \to \Gamma$, $\gamma \to \Psi$. The sampling rates fulfil the constraint $T = pT_1$, $p \geq \mu - 1$, with μ being the degree of the minimal polynomial of A, which as such satisfies the constraint $1 \leq \mu \leq n_1$. The testing matrices of (44) and (45) now become*:

$$T(z) = \begin{bmatrix} zI_{n_1} - A^p & (A + I_{n_1})G & B\ AB \cdots & A^{p-1}B \\ \Gamma & zI_{n_2} - \Phi & 0 & \Psi \end{bmatrix}; T_s(z) = \begin{bmatrix} zI_{n_1} - A & G & B \\ \Gamma & zI_{n_2} - \Phi & \Psi \end{bmatrix} \quad (46)$$

Note that:

(1) If (A, B) and (Φ, Γ) are controllable, then $rank\left[B, AB, \cdots, A^{\mu-1}B\right] = n_1$ and $rank[zI_{n_2} - \Phi, \Gamma] = n_2$. Thus, the multirate sampling system is controllable even if $\Psi = (A + I_{n_1})G = 0$ since $rankT(z) = n$; $\forall z \in \mathbf{C}$, so that the subsystem under the slow sampling rate is not controllable through the input sequence. In this case, for $\Psi = G = 0$ and (A, B) and (Φ, Γ) being controllable pairs, that is so that $rank[zI_{n_1} - A, B] = n_1$ and $rank[zI_{n_2} - \Phi, \Gamma] = n_2$ (i.e., both subsystems are controllable), $T_s(z) = \begin{bmatrix} zI_{n_1} - A & 0 & B \\ \Gamma & zI_{n_2} - \Phi & 0 \end{bmatrix}$, whose rank can be defective for some $z \in \mathbf{C}$ for certain parameterizations if Φ and A have some common eigenvalues.

(2) $minrankT(z) \geq minrankT_s(z)$; $\forall z \in \mathbf{C}$ so that even in some cases that $T(z)$ is rank-defective for some $z \in \mathbf{C}$, but is has some stable uncontrollable eigenvalues, it can happen that the multirate sampling system is stabilizable even if the single rate one is not stabilizable.

Example 5. *Consider the transfer function $G(s) = \frac{1}{s+1} + \frac{\pi}{(s+0.02)^2 + \pi^2}$. Since it has no zero-pole cancellations, any minimal (i.e., third-order) state-space realization $\left(c^T, A, b\right)$ is controllable and observable and, in particular, the pair (A, b) is controllable. It has two complex conjugate poles $s_{1,2} = -0.02 \pm \mathbf{i}\pi$. Therefore, since $|Ims_1 - Ims_2| = 2\pi = \frac{2\pi z}{T}$ for any sampling rate $T = z \in \mathbf{Z}_+$, the controllability property becomes lost under discretization through a zero-order sampling and hold device at any sampling period $T \in US_T \equiv \mathbf{Z}_+$. However, if a faster input sampling rate is used according to a sampling rate selection $T \in S_T = \{1/z : z(\geq 2) \in \mathbf{Z}_+\}$, then the resulting discrete-time system keeps controllability from that of the continuous-time counterpart since $US_T \cap S_T = \varnothing$, and then, arbitrary spectrum closed-loop assignment is achievable for any $T \in S_T$. Furthermore, it turns out that the pairs $\left(e^{AT}, \int_0^T e^{A(t-\tau)} bd\tau\right)$ are uncontrollable for any $T \in US_T$ and controllable for any $T \in ST$.*

Note that, in the case that there is an input delay in the model dynamics, the use of extended models is possible with delayed replicas of the state vector to incorporate such delays before proceeding to the controller gain synthesis. See, for instance [16,17].

5. Concluding Remarks

This paper has studied the stability and stabilization of a class of multirate sampling discrete-time systems, which were decomposable into several subsystems whose dynamics run at different sampling rates, which are the integer multiple of those associated with the preceding subsystems in the whole matrix of dynamics. It has been assumed that the dynamics of each subsystem change as jointly driven by its own sampling period, described by the corresponding diagonal entries of the matrix of dynamics and the action of the remaining subsystems coupled to it through the nonzero off-diagonal matrix blocks of the same group of row block matrices. The stability has been investigated through the characterization and computation of the spectral radius, numerical radius and the spectral norm of the whole system matrix, so that it is guaranteed to be convergent if their values are strictly less than one, or at least, the above characteristic parameters have respective values bounded by unity from above. In those cases, the dynamic system is guaranteed to be, respectively, either globally asymptotically stable or, at least, globally stable as a result. Later on, some further studies were performed related to the synthesis of the closed-loop systems under multirate sampling and linear state-feedback by the appropriate design of the controller gains. For this purpose, the fast and slow modes of the system dynamics were described as being influenced by the various sampling rates of the input components. Two basic features have been investigated, namely: (a) the closed-loop model matching to a prescribed suited stable dynamics is achievable; (b) the matching of the closed-loop eigenvalues to prescribed stable allocations of the matrix of dynamics is achievable. The main underlying idea developed is that the spectrum assignment is achievable for the largest sampling period under multirate sampling in some cases when it is not achievable under single-rate sampling. Some illustrative examples have been also described.

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