

IKERLANAK

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by

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2019

Working Paper Series: IL. 113/19

Departamento de Fundamentos del Análisis Económico I
Ekonomi Analisiaren Oinarriak I Saila



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The Nucleolus, the Kernel, and the Bargaining Set: An Update*

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June 2019

Abstract

One of the many important contributions in David Schmeidler's distinguished career was the introduction of the nucleolus. This paper is an update on the nucleolus and its two related supersolutions, i.e., the kernel and the bargaining set.

JEL classification numbers: C71, C72.

Keywords: nucleolus, kernel, bargaining set.

*We thank Antoine Billot and Tzachi Gilboa for kindly inviting us to participate in the special issue of *La Revue Economique* in honor of David Schmeidler. This paper is an outgrowth of four lectures delivered by Serrano at the Hebrew University of Jerusalem, 10th Summer School in Economic Theory, in June 1999. Part of section 5 is based on Shimomura (2015). We thank Zeky Murra and Xu Zhang for terrific research assistance. Iñarra acknowledges research support from the Spanish Government grant ECO2015-67519-P, and Shimomura from Grant-in-Aid for Scientific Research (A)18H03641 and (C)19K01558.

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1 Introduction

One of David Schmeidler's many important contributions in his distinguished career was the introduction of the nucleolus, one of the central single-valued solution concepts in cooperative game theory. This paper is an updated survey on the nucleolus and its two related supersolutions, i.e., the kernel and the bargaining set. As a first approach to these concepts, we refer the reader to the great survey by Maschler (1992); see also the relevant chapters in Peleg and Sudholter (2003). Building on the notes of four lectures on the nucleolus and the kernel delivered by one of the authors at the Hebrew University of Jerusalem in 1999, we have updated Maschler's survey by adding more recent contributions to the literature. Following a similar structure, we have also added a new section that covers the bargaining set.

The nucleolus has a number of desirable properties, including nonemptiness, uniqueness, core selection, and consistency. The first way to understand it is based on an egalitarian principle among coalitions. However, by going over the axioms that characterize it, what comes across as important is its connection with coalitional stability, as formalized in the notion of the core. Indeed, if one likes a single-valued version of core stability that always yields a prediction, one should consider the nucleolus as a recommendation. The kernel, which contains the nucleolus, is based on the idea of "bilateral equilibrium" for every pair of players. And the bargaining set, which contains the kernel, checks for the credibility of objections coming from coalitions. In this paper, section 2 presents preliminaries, section 3 is devoted to the nucleolus, section 4 to the kernel, and section 5 to the bargaining set.

2 Preliminaries

We begin by defining a nontransferable utility (NTU) game in coalitional form (N, V) , where $N = \{1, 2, \dots, n\}$ is a finite set of players, and for every $S \subseteq N, S \neq \emptyset, V(S) \subseteq \mathbb{R}^{|S|}, V(S) \neq \emptyset$. In the sequel and abusing notation, we shall identify the cardinality of a set with its lower case representation. Thus, $|S| = s$.

Two subclasses of NTU games are:

1. Pure bargaining problems: The pair (U, d) is a pure bargaining problem, where $U = (V(N)) \subseteq \mathbb{R}^n$ is the feasible utility set and $d \in \mathbb{R}^n$ is the disagreement or threat point. Therefore, for all $S \neq N, V(S) \subseteq \prod_{i \in S} V(\{i\})$. That is, intermediate size coalitions are powerless.
2. Transferable utility (TU) games: For every $S \subseteq N$, there exists a real number $v(S)$ such that

$$V(S) = \{x \in \mathbb{R}^s : \sum_{i \in S} x_i \leq v(S)\}.$$

That is, there exists a numeraire good that allows transfers of utility from player to player at a one-to-one rate. For this subclass of problems, we shall use the following notational conventions.

1. Denote by 2^N the set of all subsets of N and let $v(S) : 2^N \mapsto \mathbb{R}$. Then, we speak of the pair (N, v) as the TU game.
2. Given a vector $x \in \mathbb{R}^n$, we denote by $x(S) = \sum_{i \in S} x_i$ and by x^S the projection of x to the subspace corresponding to the players in S .

3. We shall denote by $X(N, v)$ the efficient frontier of the game (N, v) , also referred to as its preimputations, i.e.,

$$X(N, v) = \{x \in \mathbb{R}^n : x(N) = v(N)\}.$$

4. We shall denote by $X_0(N, v)$ the individually rational and efficient payoff set in the game (N, v) , also referred to as its imputations, i.e.,

$$X_0(N, v) = \{x \in X(N, v) : x_i \geq v(\{i\}) \quad \forall i \in N\}.$$

We make one last preliminary observation. What we will call nucleolus and kernel here is what the literature has called prenucleolus and prekernel. These are the true solution concepts, which do not impose individual rationality (as opposed to nucleolus and kernel properly speaking). In many economic applications, the distinction is irrelevant, because the prenucleolus and prekernel will turn out to be individually rational.

3 The Nucleolus

3.1 Definition and Properties

The nucleolus is a solution concept for the class of TU games (some generalizations to the class of NTU games exist, but they are somewhat problematic). It is an object that is hard to define and analyze, but with very nice properties.

Consider $x \in X(N, v)$. For each such x and for each $S \in 2^N \setminus \{N, \emptyset\}$, define the excess of coalition S at x as: $e_S(x) = x(S) - v(S)$. We shall take

this number as an index of the “welfare” of coalition S at x . Define the vector $e(x) \in \mathbb{R}^{2^N-2}$ as $e(x) = (e_S(x))_{S \in 2^N \setminus \{N, \emptyset\}}$. This is the vector of all excesses of the different coalitions at x . Let the vector $e^*(x)$ be a permutation of the entries of $e(x)$ arranged in increasing order.

We shall say that $e(x)$ is leximin superior to $e(y)$ [$e(x) \succ_{lexm} e(y)$] if $e^*(x)$ is lexicographically superior to $e^*(y)$, i.e., if there exists $t'+1 \in \{1, 2, \dots, 2^n-2\}$ such that $e_t^*(x) = e_t^*(y)$ for $t = 1, 2, \dots, t'$ and $e_{t'+1}^*(x) > e_{t'+1}^*(y)$.

Definition (Schmeidler, 1969): The (pre)nucleolus of the game (N, v) is

$$\text{nc}(N, v) = \{x \in X(N, v) : \nexists y \in X(N, v), \quad e(y) \succ_{lexm} e(x)\}.$$

The nucleolus maximizes recursively the “welfare” of the worst treated coalitions. One can understand it as an application of the Rawlsian social welfare function to a society where each coalition’s welfare is evaluated independently.

Example 3.1: Let $N = \{1, 2, 3\}$ and consider the following TU game: $v(N) = 42$, $v(\{1, 2\}) = 20$, $v(\{1, 3\}) = 30$, $v(\{2, 3\}) = 40$, $v(\{i\}) = 0 \quad \forall i \in N$. Let us begin by considering the equal split vector $x = (14, 14, 14)$. Note that

$$e^*(x) = (-12, -2, 8, 14, 14, 14).$$

Here, the worst treated coalition is $\{2, 3\}$. If you were a planner concerned with maximizing the “welfare” of the worst treated coalition, you would like to transfer utility from player 1 to players 2 and 3. For example, consider the vector $y = (4, 24, 14)$, where 10 units have been transferred from player 1 to player 2. Note that

$$e^*(y) = (-12, -2, 4, 8, 14, 24).$$

Thus, we have actually gone in the wrong direction (x is better than y for this planner). However, from x , it seems to make more sense to transfer units from 1 to 3. Consider the vector $z = (4, 14, 24)$, whose associated

$$e^*(z) = (-2, -2, -2, 4, 14, 24).$$

It turns out that $z = \text{nc}(N, v)$.

Next we show some properties of the nucleolus.

- (1) Individual rationality: $\text{nc}(N, v) \in X_0(N, v)$ if the game (N, v) is superadditive [i.e., for every $S, T \subseteq N, S \cap T = \emptyset$, we have that $v(S \cup T) \geq v(S) + v(T)$.]

Proof of (1): Suppose not. Let $x \in \text{nc}(N, v)$ and suppose there exists $j \in N$ such that $x_j < v(\{j\})$ [call i the player for whom the individual excess is smallest: $x_i - v(\{i\}) = \min_{j \in N} \{x_j - v(\{j\})\}$].

First, if \mathcal{D}_1 is the collection of coalitions S whose excess is the smallest at x , it must be the case that $i \in S$ for every $S \in \mathcal{D}_1$. To see this, suppose not: there exists $S \in \mathcal{D}_1$ and $i \notin S$. Then,

$$x(S \cup \{i\}) - v(S \cup \{i\}) \leq x_i + x(S) - v(S) - v(\{i\}) < x(S) - v(S),$$

a contradiction. Thus, $i \in S$ for every $S \in \mathcal{D}_1$.

Now consider the allocation y : $y_i = x_i + \epsilon$, and for all $j \neq i$, $y_j = x_j - \epsilon/(n-1)$. It should be clear that for every $S \in \mathcal{D}_1$, $e_S(y) > e_S(x)$. Further, choosing $\epsilon > 0$ arbitrarily small, we have that $e_S(y) < e_T(y)$ whenever $S \in \mathcal{D}_1$ and $T \notin \mathcal{D}_1$. Thus, $e(y) \succ_{lxm} e(x)$, which is a contradiction. Q.E.D.

(2) Nonemptiness: $\text{nc}(N, v) \neq \emptyset$.

Proof of (2): (We present a proof assuming that (N, v) is superadditive. If not, one must show first that the nucleolus lives in a compact set.) Consider the set $X_0(N, v) = Y_0$. Consider the problem $\max_{x \in Y_0} \min_S \{x(S) - v(S)\}$. Note that the function $\min_S x(S) - v(S)$ is continuous in x and that Y_0 is compact. Since the function is continuous and Y_0 is compact, the set of maximizers is non-empty and compact. Denote this by $Y_1 \subseteq Y_0$.

Now write the problem $\max_{x \in Y_1} \min_S^2 x(S) - v(S)$, where we denote by \min^2 the second worst treated coalitions. By the same arguments, the new set of maximizers Y_2 is non-empty and compact. Since we have a finite number of coalitions, this process can be repeated only a finite number of times. Then, by induction, $\text{nc}(N, v) \neq \emptyset$. Q.E.D.

(3) Uniqueness: $\text{nc}(N, v)$ is a singleton.

Proof of (3): Suppose x and $y \in \text{nc}(N, v)$, $x \neq y$. Then, $e^*(x) = e^*(y)$. Denote the list of proper coalitions by $S_1, S_2, \dots, S_m, m = 2^n - 2$ as arranged in $e^*(x)$. Then, since $x \neq y$, there must exist k to be the first in the order for which $e_k^*(x) \neq e_{S_k}(y)$. Further, it must be the case that $e_k^*(x) < e_{S_k}(y)$. Also, for every $l > k$ in the order, $e_l^*(x) \geq e_k^*(x)$ and $e_{S_l}(y) \geq e_k^*(x)$.

Then, consider the allocation $z = (x + y)/2$. Note that $e_h^*(z) = e_h^*(x)$ for $h < k$, $e_k^*(z) > e_k^*(x)$ and $e_l^*(z) > e_k^*(x)$. Thus, $e(z) \succ_{lxm} e(x)$, which is a contradiction. Q.E.D.

(4) Core selection: Denoting the core by $\mathcal{C}(N, v)$, we have $\text{nc}(N, v) \in \mathcal{C}(N, v)$

whenever $\mathcal{C}(N, v) \neq \emptyset$.

The proof is easy and left as an exercise.

(5) Consistency: Let $x = \text{nc}(N, v)$ and define the Davis-Maschler reduced game (S, v_{xS}) for coalition S as follows: let $v_{xS}(S) = v(N) - x(N \setminus S)$; and for every $T \subset S, T \neq \emptyset$, $v_{xS}(T) = \max_{Q \subseteq N \setminus S} \{v(T \cup Q) - x(Q)\}$. Then, $x^S = \text{nc}(S, v_{xS})$ for every $S \subseteq N$. This is a property of internal consistency of a solution. Remarkably, this exact property is shared by other solution concepts (such as the core, and as we shall see, the bargaining set and the kernel).¹

(6) Covariance: Let (N, v) be a game, $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}^n$. Construct the game (N, w) , where for every $S \subseteq N$, $w(S) = \alpha v(S) + \beta(S)$. Then, $\text{nc}(N, w) = \alpha \text{nc}(N, v) + \beta$. That is, the nucleolus is invariant to positive affine transformations that preserve the TU property of the game.

(7) Anonymity: Let $\pi : N \mapsto N$ be a bijection. Then, $\text{nc}(\pi(N, v)) = \pi(\text{nc}(N, v))$. That is, players' names do not matter.

Two interesting characterizations of the nucleolus have been found. One is based on balanced collections of coalitions and the other on well-known axioms.

A collection of coalitions $M = \{S_1, \dots, S_k\}$ is balanced if one can find coefficients $\lambda_1, \dots, \lambda_k$ associated with each coalition, $0 \leq \lambda_{k'} \leq 1$ for $1 \leq k' \leq k$ such that for all $i \in N$, $\sum_{i \in S_{k'} \in M} \lambda_{k'} = 1$.

¹See Thomson (1990) and Driessen (1991) for surveys on this property.

Fix a payoff vector $x \in X(N, v)$. Let $\mathcal{D}_1(x)$ be the collection of coalitions with the smallest excess at x ; let $\mathcal{D}_2(x)$ be the collection of coalitions with the second smallest excess at x , and so on.

Theorem (Kohlberg, 1971): Fix an arbitrary TU game (N, v) . A payoff vector $x = \text{nc}(N, v)$ if and only if for all $k \geq 1$, $\cup_{k'=1}^k \mathcal{D}_{k'}(x)$ are balanced collections.

In particular, it follows from this theorem that, for every player, $\mathcal{D}_1(\text{nc}(N, v))$ must contain at least one coalition that includes this player and at least one coalition that excludes him.

The other major characterization is based on consistency:

Theorem (Sobolev, 1975): Consider the class of all TU games. There exists a unique nonempty single-valued solution satisfying covariance, anonymity, and consistency: it is $\text{nc}(N, v)$.

It is interesting to compare the axioms used by Sobolev to those behind other solutions. For example, the Shapley value satisfies all except consistency (although if one replaces the Davis/Maschler reduced game with the one introduced by Hart&Mas-Colell, the Shapley value is characterized with the other four axioms). On the other hand, the core satisfies all but non-emptiness and single-valuedness. In the class of games where the core is nonempty (balanced games), the only difference between the core and the nucleolus is whether one wants to insist on single-valuedness. (See Potters (1991) for other related axiomatizations of the nucleolus.)

3.2 Some Applications and Bargaining

There are by now numerous applications of the nucleolus to different problems. In most cases, its use has been motivated as an alternative to the Shapley value, the other single-valued solution to TU Games. These applications include cost allocation problems (as in the Birmingham airport runways, or in different network problems) and surplus sharing problems, one of whose particular cases is bankruptcy. In this subsection we shall concentrate on the application of the nucleolus to bankruptcy problems.

The following example of bankruptcy is taken from the Talmud. Let E be the estate to be divided, and d_1 , d_2 and d_3 the claims of three creditors against the estate E .

[Enter Table 1 here]

For centuries, the underlying general principle behind these numbers was unclear. Jewish scholars argued that when $E = 100$, the money was too little to go around; in this case, they argued, every creditor is going to be paid so little that it makes sense to have equal division. If $E = 300$, the estate was exactly half of the sum of the claims. Thus, it makes sense to apply the proportional solution and each creditor gets exactly half of her claim. The disturbing fact was that the case $E = 200$ was attributed to an error in transcription.

A separate problem, also found in the Talmud, is described as the “contested garment” problem. Two men were arguing who owned a garment. One of them said it was his; the other said that half was his. The Talmud gives a clear solution to this problem: the part of the estate conceded by a claimant is awarded to the other. The rest of the estate, which is contested

by both, should be split in half. Formally, given a two-agent bankruptcy problem $(E, (d_1, d_2))$, the CG rule is the following:

$$\mathcal{CG}(E, (d_1, d_2)) = \left(\begin{array}{l} \max\{0, E - d_2\} + \frac{E - \max\{0, E - d_1\} - \max\{0, E - d_2\}}{2} \\ \max\{0, E - d_1\} + \frac{E - \max\{0, E - d_1\} - \max\{0, E - d_2\}}{2} \end{array} \right)$$

In the example in Table 1, this rule assigns the split $(0.75, 0.25)$, which, to make matters worse, is different from equal or proportional split. What rule, therefore, did the Talmud have in mind to solve bankruptcy problems?

Suppose one is interested in introducing a consistency property in these problems. After all, we understood how the writers of the Talmud wanted to solve two-person problems (the contested garment CG rule), but we have no clue for more than two creditors. The following seems a sensible formulation of consistency:

Let (E, d) be an n -person bankruptcy problem, where $0 \leq E \leq d(N)$ and $d_i \geq 0$. A bankruptcy rule is a function f that assigns to each problem (E, d) a split of the estate $f(E, d)$, i.e., $\sum_{i \in N} f_i(E, d) = E$ and for all $i \in N$, $0 \leq f_i(E, d) \leq d_i$.

A bankruptcy rule f is CG-consistent whenever we have the following: if $f(E, d) = x$ is the n -creditor split, then for every pair i, j , we have that $(x_i, x_j) = \mathcal{CG}(x_i + x_j, (d_i, d_j))$.

Define now a coalitional game associated with the bankruptcy problem: $(N, v_{E,d})$, where N is the set of creditors and for every $S \subseteq N$, $v_{E,d}(S) = \max\{0, E - d(N \setminus S)\}$.

Theorem (Aumann and Maschler, 1985): There exists a unique rule which is CG-consistent. It is $f(E, d) = \text{nc}(N, v_{E,d})$.

Proof: Step 1: We prove that, given a bankruptcy problem (E, d) , there

exists a unique CG-consistent rule. Suppose not: assume there exist two splits x and y of the estate E that are CG-consistent. That is, for all $i, j \in N$, $(x_i, x_j) = \mathcal{CG}(x_i + x_j, (d_i, d_j))$ and $(y_i, y_j) = \mathcal{CG}(y_i + y_j, (d_i, d_j))$. Note that the CG rule is monotonic, i.e., $\mathcal{CG}(E, (d_i, d_j)) \leq \mathcal{CG}(E', (d_i, d_j))$ whenever $E \leq E'$.

Because $x \neq y$ and $x(N) = y(N) = E$, there exist i and j such that $x_i > y_i$ and $x_j < y_j$. Without loss of generality, suppose $x_i + x_j \geq y_i + y_j$. But then, consistency and monotonicity of the CG rule imply that $(x_i, x_j) \geq (y_i, y_j)$, a contradiction.

Step 2: For notational simplicity, denote the game $(N, v_{E,d})$ by (N, w) . Let $x = \text{nc}(N, w)$. By consistency of the nucleolus, we know that for all two-player coalitions $S = \{i, j\}$, $(x_i, x_j) = \text{nc}(S, w_{xS})$. By definition of the nucleolus of this two-player game, we have

$$x_i - w_{xS}(\{i\}) = x_j - w_{xS}(\{j\}).$$

This can be expressed as:

$$x_i = w_{xS}(\{i\}) + \frac{x_i + x_j - w_{xS}(\{i\}) - w_{xS}(\{j\})}{2}.$$

Therefore, we need to check only that

$$w_{xS}(\{i\}) = \max\{0, x_i + x_j - d_j\}.$$

To see this, note first that $x \in \mathcal{C}(N, w)$, which implies the core inequalities for all one-person and $(n-1)$ -person coalitions, that for all $k \in N$, $0 \leq x_k \leq d_k$.

By the definition of the Davis-Maschler reduced game, we have that

$$w_{xS}(\{i\}) = \max_{Q \subseteq N \setminus S} w(\{i\} \cup Q) - x(Q) = w(\{i\} \cup Q^*) - x(Q^*).$$

By the definition of the game $(N, v_{E,d})$, we can write that the last expression equals

$$\begin{aligned} & \max\{0, E - d_j - d(N \setminus S \setminus Q^*)\} - x(Q^*) = \\ & = \max\{-x(Q^*), x_i + x_j - d_j + x(N \setminus S \setminus Q^*) - d(N \setminus S \setminus Q^*)\} \end{aligned}$$

Note that $w_{xS}(\{i\}) \geq 0$ since creditor i always has the option of using $Q^* = \emptyset$. Therefore, the possible values of $w_{xS}(\{i\})$ are:

- When $Q^* = \emptyset$,

$$w_{xS}(\{i\}) = \max\{0, x_i + x_j - d_j + x(N \setminus S) - d(N \setminus S)\} \geq 0.$$

- When $\emptyset \subset Q^* \subset N \setminus S$,

$$w_{xS}(\{i\}) = \max\{-x(Q^*), x_i + x_j - d_j + x(N \setminus S \setminus Q^*) - d(N \setminus S \setminus Q^*)\} \geq 0.$$

- When $Q^* = N \setminus S$,

$$w_{xS}(\{i\}) = \max\{-x(N \setminus S), x_i + x_j - d_j\} \geq 0.$$

Because $0 \leq x \leq d$, it follows that, without loss of generality, we can think that either $Q^* = \emptyset$ or $Q^* = N \setminus S$. But then the result follows by considering all possible cases for where the maximum takes place. Q.E.D.

It is also interesting to investigate what kind of noncooperative bargaining procedures may lead to the nucleolus. Consider the following ones, defined for the class of bankruptcy problems.

Let us start with bilateral bankruptcy problems $(E, (d_i, d_j))$. Let player i make a proposal x , $0 \leq x \leq d$, $x_i + x_j = E$. If player j accepts, the proposal

is implemented. If not, a fair coin is tossed. With probability $1/2$, a player will get his best possible outcome and with probability $1/2$ his worst possible outcome. That is, player i will get either $\min\{E, d_i\}$ or $E - \min\{E, d_j\}$ with equal probability.

Claim: This game's unique subgame perfect equilibrium outcome is the CG rule allocation.

Moving to n -player settings, consider now the game $G_1(E, d)$, in which the CG rule is used to “settle” bilateral disputes. In the game $G_1(E, d)$, let player 1 be one with the highest claim in the multilateral bankruptcy problem (E, d) . Player 1 makes a proposal x , $0 \leq x \leq d$, $x(N) = E$. Following the natural protocol, player $i = 2, \dots, n$ must respond sequentially. If player 2 accepts, he receives x_2 and leaves the game. If he rejects, he receives his share z_2 from the CG rule applied to the problem $(x_1 + x_2, (d_1, d_2))$ and leaves the game. Let w^i be player 1's interim share right after he has dealt with player i . Thus, $w^1 = x_1$ and $w^2 = w^1 + x_2 - \max\{x_2, z_2\}$.

In general, $w^i = w^{i-1} + x_i - \max\{x_i, z_i\}$. If player i accepts, he receives x_i and leaves the game. If he rejects, he receives the share z_i from $\mathcal{CG}(w^{i-1} + x_i, (d_1, d_i))$ and leaves the game. Player 1 ends up with a share w^n and the game ends.

Theorem (Serrano, 1995): The unique subgame perfect equilibrium outcome of the game $G_1(E, d)$ is $\text{nc}(N, v_{E,d})$.

3.3 More Recent Literature

Maschler's (1992) comprehensive survey allows us to focus on contributions that have appeared thereafter. The nucleolus has generated a large literature,

with more than 2000 citations since the publication of Schmeidler's original paper in 1969. We concentrate here on contributions around the following topics:

(i) Computation and related results: First, we report on work that uses the Kohlberg criterion (Kohlberg (1971)). Recall $\mathcal{D}_1(x)$ is the collection of coalitions with the smallest excess at x ; $\mathcal{D}_2(x)$ is the collection of coalitions with the second smallest excess at x , and so on. As is well known, the Kohlberg criterion determines which imputation is the nucleolus of a game. For that purpose, one should compute the families of coalitions of smallest excess at x , second smallest, third smallest, etc. and also check the balancedness of the successive unions of these sets of coalitions. Taking advantage of the specific structure of some classes of games, the criterion has proven to be a useful tool for simplifying the procedures to compute its nucleolus.

Convex games: Convex games were introduced in Shapley (1971). A game (N, v) is convex if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subset N$. Convex games are balanced games. Many interesting economic situations can be modeled as convex games, among them the bankruptcy games (Aumann and Maschler (1985)) we already talked about.

Let (N, v) be a game and nc its nucleolus. Let $\mathcal{D}_1(nc(N, v))$ denote the family of proper coalitions of N with minimal excess at $nc(N, v)$.

Next, consider some specific collections of coalitions, referred to as partitions and antipartitions. An antipartition \mathcal{A} is a family of coalitions $\{N \setminus S\}$ such that the family \mathcal{P} of coalitions $\{S\}$ is a partition of N . Both families \mathcal{P} and \mathcal{A} are balanced collections of coalitions. In an antipartition, as a balanced collection, all its coalitions receive the same weight. By using the

definition of excess of a coalition, together with the property of efficiency, it is not difficult to see that their excesses are the following:

If \mathcal{P} is a partition,

$$e(\mathcal{P}) = \frac{\sum_{S \in \mathcal{P}} v(S) - v(N)}{|\mathcal{P}|}$$

If \mathcal{A} is an antipartition,

$$e(\mathcal{A}) = \frac{\sum_{S \in \mathcal{A}} v(S) - (|\mathcal{A}| - 1)v(N)}{|\mathcal{A}|}.$$

Observe that the computation of the excess of a partition or antipartition depends only on the worth of its coalitions.

Arin and Iñarra (1998) shows that, for any convex game (N, v) , the family $\mathcal{D}_1(\text{nc}(N, v))$ contains either a partition or an antipartition of N . This result, along with the facts that the nucleolus satisfies the D-M reduced game property and that the reduced games of convex games are themselves convex, allows to define a procedure that serves to compute the nucleolus for convex games:

- (i) Given a convex game (N, v) , calculate the excesses of the partitions and antipartitions, selecting the one with the highest excess.
- (ii) Determine $\text{nc}(S) = v(S) - e(S, \text{nc})$ for all S of the selected family.
- (iii) Define the reduced games for all S , $|S| \geq 2$.
- (iv) Repeat the procedure starting from (i) until a unique payoff for each player in the game is obtained.

We illustrate the procedure with the following example:

Example 3.2: (A 5-person convex game) (Arin and Iñarra, 1996):

- $v(N) = v(\{1, 2, 3, 4, 5\}) = 8$,
- The worth of each 4-player coalition is 4:

$$\begin{aligned} v(\{1, 2, 3, 4\}) &= v(\{1, 2, 3, 5\}) = v(\{1, 3, 4, 5\}) = \\ &= v(\{2, 3, 4, 5\}) = v(\{1, 2, 4, 5\}) = 4 \end{aligned}$$

- Out of the ten 3-player coalitions, $v(\{1, 2, 3\}) = 4$ and

$$v(\{3, 4, 5\}) = v(\{1, 2, 4\}) = v(\{1, 4, 5\}) = v(\{2, 4, 5\}) = 1,$$

- Out of the ten 2-player coalitions, $v(\{1, 2\}) = v(\{4, 5\}) = 1$,
- and $v(S) = 0$ otherwise.

It can be verified that $\mathcal{D}_1(\text{nc}) = \langle \{1, 2, 3\}, \{4\}, \{5\} \rangle$, and that $\text{nc}(\{1, 2, 3\}) = \frac{16}{3}$, $\text{nc}(4) = \frac{4}{3}$, $\text{nc}(5) = \frac{4}{3}$. The resulting D-M reduced game for the only nonsingleton coalition T in the partition is $v_{\text{nc},T}(\{1, 2\}) = v_{\text{nc},T}(\{1, 3\}) = v_{\text{nc},T}(\{2, 3\}) = \frac{4}{3}$, $v_{\text{nc},T}(T) = \frac{16}{3}$, and $v_{\text{nc},T}(S) = 0$ otherwise. In this reduced game, the application of step (i) yields the partition of all singletons. Therefore, we end up with $\text{nc}_1 = \text{nc}_2 = \text{nc}_3 = \frac{16}{9}$ and $\text{nc}_4 = \text{nc}_5 = \frac{4}{3}$.

For large games, with a large number of partitions and antipartitions, the procedure may be impractical. However, if the partitions and antipartitions can be selected from a small family of relevant coalitions, the procedure becomes more efficient. This is the case for the following two subclasses of convex games:

Airport profit games: Littlechild and Thompson (1977) models the problem of cost allocation arising from the construction of a landing strip for the

Birmingham airport as an airport game. In this game, agents are located in the nodes of a line-graph and use the common facility, the landing strip, starting from its first node until some other node is reached, so that only coalitions of agents in consecutive nodes are considered. Littlechild (1974) presents a simple algorithm for the computation of the nucleolus for this class of games. However, in many situations, each agent’s benefit from using a common facility should be taken into account. Then, given an ordering of the agents in the nodes of a line-graph, the worth of a coalition is defined as the maximum revenue minus cost attainable by the coalition, *i.e.*, $v(S) = \max\{b(R) - C(R) : R \subseteq S \text{ for each } S \subseteq N\}$, where $b(R)$ is the total profit of the members of coalition R . The structure of this convex game allows Bránzei *et al.* (2006) to present an algorithm based on the maximal excesses of coalitions, considering a sequence of airport problems, starting from the original one, where each problem is a “reduction” of the preceding one.

Highway games: this is a class of convex games which arise from situations where there is a resource that agents jointly use. Suppose that the resource can be presented as an ordered set of several segments, where each segment has a fixed cost and each agent requires the use of consecutive segments. Think, for instance, on a highway in which any driver can enter at any point of the segment and get out at any other point, as long as the segments are consecutive. Kuipers *et al.* (2000) introduces that subclass of convex games and presents a procedure to calculate the nucleolus based on Arin and Iñarra’s (1998) procedure. Because of the structure of these games, it is possible to discard some irrelevant partitions or antipartitions beforehand, speeding up

the procedure.

Veto-rich games: A veto-rich game (N, v) with n as a veto player is a non-negative game such that $v(S) = 0$ if $n \notin S$. Only singletons and coalitions that contain player n can be essential in this game. A veto-rich game has a nonempty core if and only if $v(S) \leq v(N)$ for all $S \subseteq N$. Many economic situations can be modelled as veto games. Examples include information market games with one possessor of information and many demanders (see Muto *et al.* (1989)) and production economies with only one landowner and landless peasants (see Shapley and Shubik (1967)). Arin and Feltkam (1997) exploits the special properties of veto-rich games in which only singletons and coalitions that contain the veto player are essential to compute the nucleolus.² For these games, the $\mathcal{D}_1(\text{nc})$ contains a partition in which one coalition contains the veto player and the rest are singletons. The reduced game for the unique coalition S , with $|S| > 1$, is itself a veto-rich game. Hence, a procedure to calculate the nucleolus similar to the one presented in Arin and Iñarra (1998), where only partitions are considered, can be applied.

Γ -Component Additive Games: In standard cooperative game theory, it is assumed that any coalition of players may form. However, in many situations where the number of players is large, communication is limited. Then it seems reasonable to assume that only players who can communicate with each other are able to cooperate. Myerson (1977) introduces graphs to model communication channels between players. In these situations, players are located in the nodes of a graph and the edges between nodes indicate the communication between players. An example of these games are Γ -

²They prove the coincidence of the nucleolus and the kernel.

component additive games introduced by Potters and Reijnierse (1995).³ In these games, players are located in the nodes of a tree graph and the gains from cooperation are derived from the existence of a path that links the players in each set.

Let (N, A) be an undirected graph, where A is a set of edges joining each pair of nodes $i, j \in N$, *i.e.*, every $a_{ij} \in A$ is said to join the nodes i and j . A path between any two nodes is a sequence of edges belonging to A , joining both nodes. A graph $\Gamma = (N, A)$, is said to be connected if there exists a path between every pair of nodes. An undirected graph is a tree if each pair of nodes is connected by exactly one path.

Let $A(S) = \{a_{ij} \in A : i, j \in S\}$. A coalition $S \subset N$ is connected if the (sub)graph $(S, A(S))$ is a tree. Let $C(N)$ be the set of all connected coalitions. $C \subseteq S$ is a component in S if and only if $(C, A(C))$ is a tree and there does not exist a set C^* such that $C \subset C^* \subseteq S$ and $(C^*, A(C^*))$ is a tree. The set of all components of S is called $K(S)$. A Γ -component additive game is a pair (N, v) such that $v(S) = \sum_{K \in K(S)} v(K)$. Grafe *et al.* (1995) presents a procedure where in each step only the excesses of partitions are considered. By using the Kohlberg' criterion, the paper shows that the outcome of the procedure is the nucleolus.

A line of work distinct from the Kohlberg criterion is based on linear programming. The nucleolus of any coalitional game can be computed by successively solving a sequence of linear programs (see Maschler *et al.*, 1979). But if one considers a generic TU game, the first difficulty involves simply

³Potters and Reijnierse (1995) proves the coincidence between the kernel and the nucleolus.

listing the characteristic function. Clearly, this approach does not give us an efficient algorithm to obtain the nucleolus. An efficient algorithm is now commonly understood as one that takes at most a time bounded by a polynomial of the input size. The first polynomial-time algorithm for the nucleolus in a special cost game on trees was derived by Megiddo (1978). In general, the results on efficiently computing the nucleolus depend on the specific structure of the games (see Faigle *et al.* 2001; Kuipers *et al.* 2000).

An important class to consider is that of assignment games. In a bilateral assignment market, a product that comes in indivisible units is exchanged for money, and each participant either supplies or demands exactly one unit. The units need not be alike and the same unit may have different values for different participants. Assuming that side payments are allowed, Shapley and Shubik (1971) defines the assignment game as a cooperative model for this bilateral market and proves the nonemptiness of its core. Llerena *et al.* (2015) characterizes the nucleolus for assignment games, by means of consistency with respect to Owen's reduced game (a variant of DM's) and symmetry of maximum complaints of the two sides of the market (which requires that, at each solution outcome, the most dissatisfied buyer has the same complaint as the most dissatisfied seller). Solymosi and Raghavan (1994) presents an algorithm that determines the nucleolus of an assignment game.

Finally, we point out that, in contrast to the Shapley value, the nucleolus of a TU game is only dependent on the worth of few coalitions. Most of these worths can be decreased without changing the nucleolus. Therefore, the computation of the nucleolus can be simplified, if we can say beforehand

which coalitions are important and which ones are not. Reijnierse and Poters(1998) shows that there exists a collection of at most $2(n - 1)$ coalitions that determines the nucleolus, but how to identify these coalitions is the problem in general games. They show, though, how the computation of the nucleolus of monotonic simple games can be successfully simplified.

(ii) Cost allocation problems: In a cost allocation problem, there is a group of agents that is willing to implement a common project and to allocate its cost among them. To describe a cost allocation problem we have to indicate the set of agents involved and, for each possible group of agents, the cost of the project. Notice that a cost allocation problem presented in this way is the same as a TU game, although cost allocation problems deal with costs instead of benefits and the interpretations of concepts and results in both contexts are dual of one another. A cost game is a TU game, usually denoted by (N, c) , in which $c(S)$ is the minimal costs the members of coalition S have to incur when they cooperate, with $c(\emptyset) = 0$.

Fiestras-Janeiro *et al.* (2012) provides a survey of applications dealing with solutions to cost allocation problems arising in the real world. They concentrate on contributions that they classify into three specific areas:

Transportation costs. Engevall *et al.* (1998, 2004) analyze a cost allocation problem that arises in the distribution planning at a gas and oil company in Sweden. In that problem, the total distribution cost of a specific tour had to be divided among the customers that were visited. They formulated the problem as a travelling salesman game and as a vehicle-routing game. They compared the current tariff applied by the company to other rules like the nucleolus and the Shapley value using real data of the gas and oil com-

pany. Another interesting application concerns the toll design for highways in Spain, which can be found in Kuipers *et al.* (2000).

Natural and environmental problems: Lejano and Davos (1995) considers a water reuse project in Southern California and compute the Shapley value and the nucleolus, among other solutions. Okada and Mikami (1992) applies cooperative game theory tools to allocate costs arising in collective problems of pollution reduction.

Power industry problems. Tsukamoto and Iyoda (1996) analyzes the usage of cooperative games for fixed cost allocation to wheeling transactions in a power system. They use the nucleolus for this purpose and compare it with other allocating methods. Stamtzis and Erlich (2004) follows the same line and analyzes the cost allocation problem for the fixed cost of a power system through the core, the nucleolus, and the Shapley value. Bjorndal *et al.* (2005) studies the problem of allocating the transmission-embedded cost among the transactions. The paper proposes a new method for allocating the embedded costs combining some conventional usage-based methods with the ideas underlying the nucleolus.

(iii) The Shapley value and the nucleolus: The Shapley value and the nucleolus are the most used single-valued cooperative solutions. One of the main reasons for the attractiveness of the Shapley value lies in the fact that it respects the principle of monotonicity, *i.e.* if a game (N, w) is obtained from game (N, v) by increasing the worth of a single coalition S , then the members of S receive a payoff in game (N, w) that is not smaller than in game (N, v) . Another important desideratum is that of coalitional stability, *i.e.*, that the solution be a member of the core when this is nonempty. However, in the

class of balanced games, these two principles are not in general compatible. Young (1985) proves that for $|N| \geq 5$, no core allocation rule is coalitionally monotonic. Hence, while the Shapley value does not respect core stability, the nucleolus fails to satisfy coalitional monotonicity (Hokari (2000) shows this even for the class of 4-player convex games).

The nucleolus is based on the notion of coalitional excesses. In contrast, in the per-capita nucleolus (Grotte (1970)) the excesses are divided by the cardinality of the coalition so that each member receives the same amount of the surplus. The per-capita nucleolus is a single-valued solution that satisfies monotonicity in the aggregate, but it does not satisfy coalitional monotonicity. Arin and Katsev (2016a) introduces a solution called Surplus Distributor nucleolus, SD-nucleolus, in which nonegalitarian divisions of the excesses of the coalitions are considered, showing that in the classes of convex and veto-rich games the SD-nucleolus is the only known core-solution concept that satisfies coalitional monotonicity.

Another way to overcome the monotonicity/core stability incompatibility is to study classes of games in which the nucleolus coincides with the Shapley value. This is the case for the following classes of games: liability games, clique games and appointment games. In *liability games* (Csóka and Herings (2017)) a firm has liabilities to a group of creditors and an asset value to distribute. The question is how to divide the asset among the creditors and the firm. These liability problems induce a class of TU games, liability games, which are superadditive and constant-sum. The paper analyzes the (pre)nucleolus of these games. For the case of a solvent firm, the nucleolus coincides with the Shapley value. In this case, the firm receives what remains

after all creditors have been paid their liabilities in full. If the firm is insolvent with only two positive liabilities, that coincidence also holds. In *appointment games*, Chun *et al.* (2016) considers the following situation: suppose that starting from home, a traveler makes a scheduled visit to a group of sponsors and returns home. If a sponsor in the route cancels her appointment, the traveler returns home and waits for the next appointment. These authors are interested in finding a way of dividing the total traveling cost among sponsors in the appointment problem by applying solutions developed in cooperative game theory and show that the Shapley value and nucleolus (or prenucleolus) coincide. Trudeau and Vidal-Puga (2017) introduce the *clique games*, a new class of cooperative games described as follows: the set of agents is divided into cliques that cover it. A coalition creates value when it contains many agents belonging to the same clique, with the value increasing linearly with the number of agents in the same clique. Agents may belong to more than one clique, but the intersection of two cliques contains at most one agent. Finally, if two agents are not in the same clique, there exists at most one way to “connect” them through a chain of connected cliques. Again, there is coincidence between the nucleolus and the Shapley value. The authors provide several examples of clique games, chief among them, the minimum cost spanning tree problems.

(iv) Sequential games that lead to the nucleolus: One way to justify a solution to cooperative games is to provide a dynamic process that leads the players to the proposed solution. Serrano (1993) offers a three-agent strategic justification of the nucleolus for superadditive TU games and shows that it is impossible to extend his result to more than three agents. The extension

requires restricting attention to a subclass of games, such as bankruptcy games and surplus-sharing problems, as done in Serrano (1995), which we have already discussed.

Arin *et al.* (2009) model airport problems in which a group of agents jointly use a public facility with different needs for it. If the facility can satisfy the need of an agent, then it can also satisfy an agent with smaller need at no extra cost. These authors introduce an n -stage sequential game along the lines of the game in Dagan *et al.* (1997) and show that their game strategically justifies the nucleolus for the cost-sharing problem. At the beginning of their game, agent n announces a contributions vector z . If z is unanimously accepted, then that is the outcome; otherwise, bilateral negotiations take place. The other agents reply sequentially according to a specified order. Acceptors contribute their components of z . Each rejector j negotiates her contribution with agent n by invoking the standard rule to solve the negotiation between herself and agent n . The authors show that their game has a unique Nash equilibrium outcome, which is, moreover, the nucleolus contributions vector.

Hu *et al.* (2012) considers a 3-stage sequential game in which the agent with the largest need is the responder, and all the other agents are proposers. These authors show that for each such problem, there is a unique subgame perfect equilibrium outcome of that game and it coincides with the nucleolus.

For simple games, Montero (2006) proposes the nucleolus as a power index, showing that, unlike other power indices, it is self-confirming, i.e., it can be obtained as an equilibrium of a noncooperative game in which the index itself is used as the probability vector assigned to the players. Specifically,

the non cooperative game is described as follows: At every round, nature uses a probability distribution to select a player who proposes a coalition S that includes her/him and a division of $v(S)$. A player not in S remains a singleton and receives zero. The rest of players in S accept or reject sequentially the proposal. If all players in S accept, the game ends. If one or several players reject, the game proceeds to the next period, in which nature selects a new proposer, using the same probability distribution. There is a fixed discount factor in each round, and if no proposal is ever accepted, all players receive 0. The author proves that, if the initial probability distribution is the nucleolus, then it coincides with the equilibrium expected payoffs.

(v) Other related works: We conclude by mentioning some works that introduce new solution concepts inspired by the nucleolus. They either satisfy some desirable properties that the nucleolus does not have, or adapt the nucleolus to different contexts beyond games in characteristic function.

Sudhölter (1997) introduces the modiclus in an attempt to treat all coalitions equally. In his work, the excesses of the coalitions that define the nucleolus are replaced with the differences of excesses between pairs of coalitions. The modiclus is obtained by lexicographically minimizing the nonincreasingly-ordered vector of differences of excesses. The difference of excesses between two coalitions S and T is regarded as their mutual envy, and the modiclus attempts to minimize it. This solution shares common properties with the nucleolus, but differs from it in that, for any TU game, the modiclus may not select a core element, even if the core is nonempty. We note, though, that it selects a core element in convex games. Ruiz *et al.* (1996) proposes the least square nucleolus, based on the minimization

of the variance of coalitional excesses. Jackson (2005) introduces a solution analog to the nucleolus called the networkolus for network games. The networkolus allocates the worth generated among players, which depends not only on their identities, but also on how they are connected to each other. Alvarez-Mozos and Ehlers (2017) provide a natural extension of the nucleolus to coalitional games with externalities. In these contexts, the excess of an embedded coalition is measured as the difference between the worth of the embedded coalition and what the coalition gets in the allocation. Then, for each allocation, the excesses of all embedded coalitions are rearranged in nonincreasing order, and the nucleolus is the set of allocations that lexicographically minimize the rearranged excesses of all embedded coalitions. Finally, Klauke (2002) investigates extensions to NTU games.

4 The Kernel

4.1 Definition and properties

The kernel is a set-valued solution concept that has been criticized for being based on interpersonal comparisons of utility. As we shall see, this criticism is not entirely valid. Its mathematical properties are well-known for the class of TU Games, but generalizations to the class of NTU games have been proposed. We will focus on one of them below. We begin by presenting the kernel of TU Games.

Let (N, v) be a TU game. For each pair of players $k, l \in N$, define the

surplus of player k against l at the payoff vector x as follows:

$$s_{kl}(x) = \max_{k \in S, l \notin S} \{v(S) - x(S)\}.$$

If x determines the starting pay rate to players, this is the maximum utility increase that player k can expect to get by departing from x without the consent of player l .

Definition (Davis and Maschler, 1965): The (pre)kernel of the game (N, v) is the set

$$\mathcal{K}(N, v) = \{x \in X(N, v) : s_{kl}(x) = s_{lk}(x) \quad \forall k, l \in N\}.$$

Thus, at a kernel payoff, all players are in a sort of “bilateral equilibrium”, in the sense that the threats to each other are equalized. The definition seems to involve interpersonal utility comparisons. We will revisit this issue soon, though. First, we present some properties of the kernel.

- (1) Individual rationality: If the game (N, v) is superadditive, $\mathcal{K}(N, v) \in X_0(N, v)$.

Proof of (1): Let $x \in \mathcal{K}(N, v)$ and suppose that $x_i < v(\{i\})$. Recall that the set $\mathcal{D}_1(x)$ is the set of coalitions that, at x , receive the smallest excess.

We begin by showing that $i \in S$ if $S \in \mathcal{D}_1(x)$. Consider first any coalition $T \subset N \setminus \{i\}$. Then, $T \notin \mathcal{D}_1(x)$ because:

$$\begin{aligned} e_T(x) &= x(T) - v(T) > x(T \cup \{i\}) - v(T) - v(\{i\}) \geq \\ &\geq x(T \cup \{i\}) - v(T \cup \{i\}) = e_{T \cup \{i\}}(x). \end{aligned}$$

(A weak form of superadditivity – 0 monotonicity [i.e., $v(T) + v(\{i\}) \leq v(T \cup \{i\})$ when $i \notin T$]– was used in the last inequality). Moreover, $T = N \setminus \{i\} \notin \mathcal{D}_1(x)$ either, because:

$$\begin{aligned} e_{N \setminus \{i\}}(x) &= x(N \setminus \{i\}) - v(N \setminus \{i\}) > \\ &> x(N) - v(N \setminus \{i\}) - v(\{i\}) \geq x(N) - v(N) = 0 \end{aligned}$$

whereas for example $e_{\{i\}}(x) < 0$.

Therefore, if $S \in \mathcal{D}_1(x)$, $i \in S$. Since $N \setminus \{i\} \notin \mathcal{D}_1(x)$, there exists $j \in N$ such that there exists a coalition $S \in \mathcal{D}_1(x)$ such that $j \notin S$. But then, $s_{ij}(x) > s_{ji}(x)$, which is a contradiction. Q.E.D.

- (2) Nonemptiness: For all TU games (N, v) , $\mathcal{K}(N, v) \neq \emptyset$.

Proof of (2): Several proofs are available. One uses that the relation $s_{kl}(x) > s_{lk}(x)$ is transitive and appeals to the KKM lemma. A second proof shows that $\text{nc}(N, v) \in \mathcal{K}(N, v)$. (Recall the observation we made after Kohlberg’s theorem).

- (3) Core bisection: If $x \in \mathcal{K}(N, v) \cap \mathcal{C}(N, v)$, for any $i, j \in N$, $i \neq j$, and fixing $x^{N \setminus \{i, j\}}$, the point (x_i, x_j) bisects the intersection of the core and the transfer line between players i and j (the “bargaining range” between players i and j). Thus, it would appear that the kernel does not depend on interpersonal utility comparisons, at least those kernel payoffs that are also in the core.
- (4) Consistency. To see this, note that the maximization involved in the definition of the surplus can be separated into two stages, one using

players out of the reduced game and a second one, using players in the reduced game.

- (5) Converse consistency. Let $x \in X(N, v)$. If for every two-player reduced game (S, v_{xS}) we have that $x^S \in \mathcal{K}(S, v_{xS})$, then $x \in \mathcal{K}(N, v)$.
- (6) Covariance: Let (N, v) be a TU game, $\alpha > 0$, and $\beta \in \mathbb{R}^N$. Construct the TU game (N, w) , where $w(S) = \alpha v(S) + \beta(S)$ for every $S \subset N$. If $x \in \mathcal{K}(N, v)$, then $\alpha x + \beta \in \mathcal{K}(N, w)$.
- (7) Equal treatment: If $x \in \mathcal{K}(N, v)$, $x_i = x_j$ whenever players i and j are substitutes. This means that for every $S \subseteq N \setminus \{i, j\}$, $v(S \cup \{i\}) = v(S \cup \{j\})$.

Theorem (Peleg, 1986): Over the class of all TU games, there exists a unique solution satisfying nonemptiness, efficiency, covariance, equal treatment, consistency, and converse consistency. It is $\mathcal{K}(N, v)$.

4.2 Reinterpretation of the Kernel and an Extension to NTU Games

Next, developing further the comments we made after the core bisection property, we follow Serrano (1997) in reinterpreting the kernel in a way that makes it independent of interpersonal utility comparisons. Recall the basic equations of the kernel:

$$s_{kl}(x) = s_{lk}(x),$$

which can be rewritten as:

$$v_{x\{kl\}}(\{k\}) - x_k = v_{x\{kl\}}(\{l\}) - x_l,$$

or:

$$x_k = \frac{1}{2}v_{x\{kl\}}(\{k\}) + \frac{1}{2}[x_k + x_l - v_{x\{kl\}}(\{l\})].$$

That is, first reinterpret the two-player reduced game as a bargaining problem between players k and l , where the “pie” to be divided is $x_k + x_l$ and where the threat point is $(v_{x\{kl\}}(\{k\}), v_{x\{kl\}}(\{l\}))$. The kernel is then the set of payoffs where every pair of players splits in half this pie (when modified by the outside options embodied in the threat point). This is a generalization of the insight of Maschler, Peleg, and Shapley (1979), in seeing the kernel as payoffs where certain bilateral bargaining ranges are split in half. The advantage is that this is a fact inherently linked to the definition of the kernel, and quite independent of the core.

Let us see a couple of examples:

Example 4.1: Consider the TU game (N, v) , where $N = \{1, 2, 3\}$, and $v(\{i\}) = 0 \forall i \in N$, $v(\{1, 2\}) = 4$, $v(\{1, 3\}) = 3$, $v(\{2, 3\}) = 2$, $v(N) = 6$. Then, $\mathcal{K}(N, v) = (3, 2, 1)$. Figure 1 represents the reduced problems faced by the pairs of players $(1, 2)$, $(1, 3)$ and $(2, 3)$. The payoff in the kernel splits in half the “available surplus” to each pair determined by the threat point, which is a function of the payoff awarded to the third player.

[Enter Figure 1 here]

Example 4.2: Consider again Example 3.1 in the nucleolus section: a 3-player 0-normalized game where $v(\{1, 2\}) = 20$, $v(\{1, 3\}) = 30$, $v(\{2, 3\}) = 40$, $v(N) = 42$. Again, the kernel is a singleton: $\mathcal{K}(N, v) = (4, 14, 24)$. Figure 2 represents the three two-player reduced problems, where the threat points now lie outside of the feasible set.

[Enter Figure 2 here]

In fact, this reinterpretation of the kernel lends itself easily to a generalization to NTU games.

Consider the class of smooth NTU games, where the boundary of $V(N)$ admits a differentiable representation $g(x) = 0$. Denote by $g_i(x)$ the partial derivative of g with respect to x_i at the payoff vector x .

Let us introduce two more properties:

- (8) Scale invariance: Consider an NTU game (N, V) , let $\alpha \in \mathbb{R}_{++}^n$ and $\beta \in \mathbb{R}^n$. Transform player i 's payoff function from x_i into $\alpha_i x_i + \beta_i$ and call the resulting NTU game (N, W) . A solution σ satisfies scale invariance if $\sigma(N, W) = \alpha\sigma(N, V) + \beta$.
- (9) Local independence: Let (N, V) and (N, V') be two games with all other things equal except $V(N) \neq V'(N)$. Let x be efficient in both games, i.e., $g(x) = 0$ and $g'(x) = 0$. Furthermore, let the gradient of $g(x)$ be parallel to the gradient of $g'(x)$. A solution σ satisfies local independence if, whenever $x \in \sigma(N, v)$, $x \in \sigma(N, V')$.

Theorem (Serrano and Shimomura, 1998):⁴ Over the class of smooth NTU games, there exists a unique solution satisfying nonemptiness for two-player games, efficiency, scale invariance, equal treatment in TU Games, local independence, consistency, and converse consistency. It is the kernel:

$$\mathcal{K}(N, v) = \{x \in V(N) : g(x) = 0 \quad \text{and}$$

$$\forall k \neq l, g_k(x)[V_{x\{kl\}}(\{k\}) - x_k] = g_l(x)[V_{x\{kl\}}(\{l\}) - x_l]\}$$

⁴Although Serrano and Shimomura (1998) use the term ‘‘Nash set,’’ we opt to use here the term ‘‘kernel.’’

That is, at a payoff of the NTU kernel, the elasticity of the payoff difference relative to the threat point for each pair of players is 1. Defining $dx_l/dx_k = g_k(\cdot)/g_l(\cdot)$, we have:

$$\frac{dx_l}{dx_k}(x) \frac{x_k - V_{x\{kl\}}}{x_l - V_{x\{kl\}}} = 1.$$

A graphic way to represent this is that the relevant bargaining range is split in half. By the relevant range, we mean the segment of the tangent plane truncated at the coordinates of the threat point. Figure 3, panel A represents this when the threat point is in the feasible set, and figure 3, panel B does when it is outside of it.

[Enter Figure 3 here]

Example 4.3: Consider the following two person nonconvex pure bargaining problem. Suppose two bargainers are negotiating over how to split two dollars and the consent of both is needed to split any pie. Suppose player i 's utility function for $i = 1, 2$ is the following: $u(x_i) = x_i$ if i 's share $x_i \leq 1$, while $u(x_i) = 4x_i - 3$ otherwise.⁵ Then, \mathcal{K} consists of three points for this case:

$$\mathcal{K}(N, V) = \{(1, 1), (5/2, 5/8), (5/8, 5/2)\}.$$

(See Figure 4). That is, three possible splits of the pie are prescribed: equal division (the problem is symmetric) and two others where the risk-loving agent receives 11/8, while the risk-neutral one gets 5/8.

[Enter Figure 4 here]

The NTU kernel coincides with the kernel for TU games, and with the Nash solution when applied to convex pure bargaining problems. Clearly,

⁵The reader should disregard the kink in the utility function. A smooth version of this example can easily be written.

much more research is needed on this solution concept. Although existence is a problem in the general class of NTU games, it will be interesting to uncover restricted classes of games where the kernel is nonempty. Also, it should be tested in applications as a natural generalization of the Nash solution to contexts where convexity is not assumed and where coalitions play a role. Finally, the equivalence question is not trivial (it is known that the kernel is not contained in some versions of the bargaining set).

4.3 More Recent Literature

We organize this more recent literature around a number of topics:

(i) **Pairwise “equilibrium”**: Chang (1991) studies the bisection property of the kernel in games with coalition structures. Although the kernel and the prekernel coincide in the class of 0-monotonic games, the equivalence breaks down if we relax this assumption mildly, as shown in Chang and Hsiao (1993). Kikuta (1997) shows that prekernel and kernel coincide for the reasonable part of the game, and Chang and Lian (2002) provides other sufficient conditions for this coincidence. Rochfort (1984) shows how to get to the kernel with the related idea of symmetrical pairwise bargained equilibria in assignment games, extended in Moldovanu (1990) to the NTU domain, and in Tejada and Rafels (2010) for the case of multilateral bargaining, the latter leading to a set different than the kernel. Chang and Hu (2017) provides a related noncooperative interpretation of the kernel, much along the lines of Serrano (1997), where pairs of players chosen at random bargain over their bilateral surplus, with the Davis-Maschler reduced game offering outside option payoffs. Orshan and Sudholter (2012) drop the symmetry requirement

in the split of bilateral surpluses and consider asymmetric kernels. In a related contribution, Arin and Katsev (2016b) define the surplus-distributor kernel. Solymosi (2015) shows that, in permutation games, a subclass of totally-balanced games, the prekernel is contained in the least core, again reconciling the ideas of bilateral equilibrium and coalitional stability.

(ii) Other characterizations: Calvo and Gutierrez (1996) provides an alternative characterization of the kernel on the basis of stability properties. Hokari and Kubres (2003) characterize the aspirations kernel, when choosing aspirations is part of the cooperative game. Khmelnitskaya and Sudholter (2013) characterize the prekernel and prenucleolus of games with communication structures, and Katsev and Yanovskaya (2013) for games of restricted cooperation. Kleppe *et al.* (2016) characterize the class of symmetrically-weighted solutions, which contains the kernel.

(iii) Large games and other classes of TU games: Shapley (1992) studies kernels of replicated TU games and economies, and finds nonconvergence to the set of competitive allocations. Einy, Monderer, and Moreno (1998) considers the kernel, least core, and bargaining sets of large games with a countable set of players, showing that the least core is a nonempty subset of the space of all countably additive measures, that the intersection of the prekernel and the least core is nonempty, and that the Aumann-Davis-Maschler and the Mas-Colell bargaining sets contain the set of all countably additive payoff measures in the prekernel.

Granot and Granot (1992) looks at kernels of network problems, trees, and maximum flow graphs, as well as assignment games, and their connection with the core and the nucleolus. Granot *et al.* (1996) studies tree games,

shows that the kernel coincides with the nucleolus, and provides an algorithm to compute it. Faigle *et al.* (1998) also provides an efficient algorithm for computation, and Faigle *et al.* (2001) does so for the efficient computation of the intersection of kernel and least core. Meinhardt (2006) computes the entire prekernel based on linear-programming methods; see also Meinhardt (2007), with a computation based on the indirect-function method, and Meinhardt (2014) for a monograph outlining these and other results about the kernel. Potters and Reijnierse (1995) extends the results of Maschler, Peleg, and Shapley (1972) from convex games to Γ -component additive games (graph restricted), and Getán-Oliván *et al.* (2015) to almost-convex games; Arin and Katsev (2013) provides an alternative proof of the coincidence of kernel and nucleolus of convex games. Kishimoto and Watanabe (2017) studies the kernel of a patent licencing game and shows it to be a singleton, thereby always offering a prediction on optimal licenses. For fuzzy TU games, Liu *et al.* (2018) and Huang *et al.* (2019) study fuzzy kernels and their connections with the fuzzy Aumann-Davis-Maschler bargaining set and fuzzy Mas-Colell bargaining set, respectively.

(iv) NTU games: Billera and McLean (1994) suggest an alternative extension of the kernel to NTU games, based on convex analysis. One way to overcome the difficulties of establishing nonemptiness of the NTU prekernel is to suggest the average prekernel, which consists of the set of efficient payoffs where the average (over all other players) surplus of each player is zero; interestingly, the average prekernel coincides with the prekernel in TU games. The average prekernel was proposed in Orshan and Zarzuelo (2000), albeit under the name of bilateral consistent prekernel. A nonemptiness

result of the average prekernel for boundary-separating games is proved in Orshan, Valenciano, and Zarzuelo (2003), and an axiomatization is proposed in Serrano and Shimomura (2006).

5 The Bargaining Set

5.1 Definition and Properties

Consider the following question: Is it possible that all players cooperate in a coalitional game where the core is empty? One answer to this question may be in the affirmative. The reason is that we do not have to take into account every single “improvement” of a coalition. Suppose that payoffs are given to all individuals of a group. Even if a subgroup can make its members better off by working together, the “improvement” they plan to carry out may be disturbed by a counter offer from another subgroup.

Aumann and Maschler (1961,1964) and Davis and Maschler (1962,1963) interpret a proposal of “improvement” not only as an elimination of dissatisfaction of a subcoalition for the grand coalition, but also as an “objection” from a player to another player.⁶ They present the following concept of “counterobjection.”⁷

Now, suppose that payoff profile x is given to an n -player coalitional game

⁶We refer to the first formalization of these ideas as the Aumann-Davis-Maschler (ADM) bargaining set.

⁷There are several verbal explanations found in their original papers and in Osborne and Rubinstein (1994, P.282). There are subtle differences between them. They look almost the same, but the differences become clearer in mathematical expressions, which we will discuss later.

(N, v) . Suppose further that one of the players, i , is informing another player, j , of the following: “I am forming coalition S that does not contain you; I plan to produce a payoff profile y for S that gives me a greater payoff than my payoff at x and makes the other members of S at least as happy as at x .” The pair (S, y) is the description of an objection from player i to player j given x . The formal definition is as follows:

Definition: Let (N, v) be a TU game, $i, j \in N$, and x a payoff profile on N . Then an *ADM objection of i against j at x* is a pair (S, y) of coalition S and $y \in X(S, v)$ such that $i \in S, j \notin S, y_i > x_i$, and $y_{-i} \geq x_{-i}$.

In other words, the interpretation is that player i protests against player j that “If I form coalition S without you and create payoff list y , then I can improve myself without making the other members of S worse off than at the status quo x .” Observe that the core of a TU game can be redefined as the set of payoff profiles that do not cause any objection of any player against any other player.

Notice that this type of improvement requires that the player making an objection should be strictly better off without hurting any other member of the same coalition. The statements “the player forming a coalition being strictly better off without making any of the other members worse off” and “all members of a coalition being made strictly better off” are equivalent in a TU game. However, they are different for more general cases such as an NTU game. In addition, this definition makes the implications of Davis/Maschler consistency clearer, which is to be discussed later.⁸

⁸In fact, there are quite a few papers in which an “objection of i against j at x ” is

Now suppose that player j can form coalition T excluding player i and achieve a new payoff profile z which gives to each member of T a payoff at least as high as at x and, to each member belonging to both S and T , a payoff at least as high as at y . Aumann, Davis, and Maschler named such a pair (T, z) a counterobjection from player j to objection (S, y) given payoff profile x . The formal definition is:

Definition: Let (N, v) be a TU game, $i, j \in N$, x a payoff profile on N , and (S, y) an objection of i against j at x . Then, an *ADM counterobjection from j to (S, y) at x* is a pair (T, z) of coalition T and $z \in X(T, v)$ such that $i \notin T$, $j \in T$, and $(z^{S \cap T}, z^{T \setminus S}) \geq (y^{S \cap T}, x^{T \setminus S})$.

The interpretation is that player j , who had been objected to by player i , can say: “If I form coalition T and produce payoffs z , then I can make all the members of T who are also in S at least as happy as with y and the other members of T at least as happy as with x .”

In a situation where such objections and counterobjections are proposed, an objection is said to be “justified” if it has no counterobjection. We consider the set of (pre)imputations with no “justified” objection, and call this the “Aumann-Davis-Maschler (pre)bargaining set”. The formal definition is as follows:

Definition (Aumann and Maschler, 1964/Davis and Maschler, 1963/Aumann and Dreze, 1974): The *ADM prebargaining set of the game (N, v)* is the set

$$\mathbf{M}_1(N, v) = \{x \in X(N, v) : \text{For each ADM objection of}$$

defined by strict improvement of all members of the coalition (i.e., $y_k > x_k$ for all $k \in S$).

$i \in N$ against $j \in N$ at x , there is an ADM counterobjection}.

The *ADM bargaining set of the game* (N, v) is the set

$$\mathbb{M}_1^{(i)}(N, v) = \{x \in X_0(N, v) : \text{For each ADM objection of } \\ i \in N \text{ against } j \in N \text{ at } x, \text{ there is an ADM counterobjection}\}$$

Both mappings \mathbb{M}_1 and $\mathbb{M}_1^{(i)}$ are solutions on the class \mathcal{G} of all TU games. Aumann and Maschler (1961, 1964) define \mathbb{M}_1 as one of several versions of the bargaining set. Davis and Maschler (1962, 1963) propose $\mathbb{M}_1^{(i)}$ by imposing individual rationality on \mathbb{M}_1 (Davis and Maschler, 1962, P. 4, footnote 2). Aumann and Dreze (1974, P. 227, footnote 8) say that the bargaining set they investigate is $\mathbb{M}_1^{(i)}$, but this claim is not correct because they do not impose individual rationality. Properly speaking, the bargaining set they discuss is \mathbb{M}_1 defined above.

Example 5.1: Let $N = \{1, 2\}$ and consider 2-person TU games. Define the set of efficient payoff profiles with no excesses of the two players by

$$\bar{L}(\{1, 2\}, v) = \{x \in X(\{1, 2\}, v) : (x_1, x_2) \leq (v(\{1\}), v(\{2\}))\}$$

Then

$$\mathbb{M}_1(\{1, 2\}, v) = X_0(\{1, 2\}, v) \cup \bar{L}(\{1, 2\}, v).$$

$$\mathbb{M}_1^{(i)}(\{1, 2\}, v) = X_0(\{1, 2\}, v).$$

Remember the question about the possibility of cooperation when the core is empty. The answer by Aumann, Davis, and Maschler is that, even if

the payoff profile is not in the core, players actually never make an objection against any other player as long as it is in the ADM bargaining set. That is, each player is aware that, given that payoff profile, every single objection he can make is to be stopped by a counterobjection.

A payoff profile of the ADM bargaining set is therefore considered stable. Notice that a “justified” objection is a special case of an objection. Then, there is no justified objection to a payoff profile if there is no objection. Hence, a payoff profile of the core is contained in the ADM bargaining set. Furthermore, the ADM bargaining set has been proved to be nonempty for any TU game (Davis and Maschler (1962, 1963); Peleg (1963)). Therefore, the ADM bargaining set expresses the possibilities of players’ cooperation and payoff distribution, even for a TU game where the core is empty.

However, the ADM bargaining set may include a large number of – sometimes undesirable– payoff profiles, so that selections therein can be of interest. The kernel and the nucleolus are thus proposed as “refined” sub-solutions of the ADM bargaining set, which satisfy the following inclusion relations:

1. For every TU game, the kernel is included in the ADM bargaining set.
2. For every TU game, the nucleolus is contained in the kernel.
3. For every balanced TU game, the nucleolus is contained in the core, which is itself contained in the ADM bargaining set.

The above results state the inclusions for the individually rational versions of the solutions. More fundamental results are presented in the proposition below:

Proposition: For each TU game (N, v) ,

$$\mathcal{C}(N, v) \subset \mathbb{M}_1(N, v)$$

$$nc(N, v) \in \mathcal{K}(N, v) \subset \mathbb{M}_1(N, v)$$

For each balanced TU game (N, v) ,

$$nc(N, v) \in \mathcal{K}(N, v) \cap \mathcal{C}(N, v) \subset \mathbb{M}_1(N, v)$$

Thanks to these inclusion relations, it is straightforward that the existence and uniqueness of the nucleolus, proven by Schmeidler (1969), warrants the nonemptiness of the kernel and the ADM bargaining set. Sobolev (1975) proves the existence and uniqueness of the prenucleolus with help of the theorem of Schmeidler (1969) (see Yanovskaya (2002)), so that the nonemptiness of the prekernel and the ADM prebargaining set is guaranteed. Aumann writes in his memoir of Maschler in Aumann *et al.* (2008, P.6) about the ADM bargaining set, kernel, and nucleolus as follows:

That paper [Aumann and Maschler (1964)] has been cited many hundreds of times; it became one of my—and no doubt Mike’s—most popular works. Mike’s stubbornness really paid off. Moreover, the paper led to a very large literature, it was truly seminal. Later offshoots—one might say descendants—of that original concept were the Maschler–Davis Bargaining set M_1^i , for which there is an existence theorem (with a beautiful, highly nontrivial proof), and which is altogether more pleasant to work

with, as well as the Kernel and Schmeidler's Nucleolus; taken together, these concepts constitute one of the richest, and yes, most elegant chapters of game theory, with a great many applications yielding beautiful insights.

Next, we elaborate on some properties of the ADM prebargaining set.

(1) Kernel inclusion: For all TU games (N, v) , $\mathcal{K}(N, v) \subset \mathbb{M}_1(N, v)$.

Proof of (1): Let $x \in \mathcal{K}(N, v)$. Then, $x \in X(N, v)$. Choose two distinct players $i, j \in N$ and suppose that there is an objection (S^*, y^*) of i against j at x . Then, $i \in S^*, j \notin S^*, v(S^*) - x(S^*) > 0$ and $\max\{v(S) - x(S) | i \in S, j \notin S\} = \max\{v(T) - x(T) | j \in T, i \notin T\}$. Hence, there is a coalition T^* such that

$j \in T^*, j \notin T^*$ and $v(S^*) - x(S^*) \leq v(T^*) - x(T^*)$. If $S^* \cap T^* \neq \emptyset$, $v(S^*) - x(S^* \setminus T^*) \leq v(T^*) - x(T^* \setminus S^*)$, so that there is an ADM counterobjection (T^*, z^*) such that $z_k^* = x_k$ for each $k \in T^* \setminus S^*$ and $z_l^* \geq y_l^*$ for each $l \in S^* \cap T^*$. If $S^* \cap T^* = \emptyset$, $0 < v(T^*) - x(T^*)$, so that there is an ADM counterobjection (T^*, z^*) such that $z_l^* > x_l$ for each $l \in T^*$. Thus, $x \in \mathbb{M}_1(N, v)$. Hence, $\mathcal{K}(N, v) \subset \mathbb{M}_1(N, v)$. *Q.E.D.*

(2) Nonemptiness: For all TU games (N, v) , $\mathbb{M}_1(N, v) \neq \emptyset$.

Proof of (2): Recall $nc(N, v) \in \mathcal{K}(N, v)$. Since $\mathcal{K}(N, v) \subset \mathbb{M}_1(N, v)$, $nc(N, v) \in \mathbb{M}_1(N, v)$. Thus, $\mathbb{M}_1(N, v) \neq \emptyset$. *Q.E.D.*

(3) Consistency: Let (N, v) be a TU game, and $x \in \mathbb{M}_1(N, v)$. Then, $x^B \in \mathbb{M}_1(B, v_{xB})$ for each $B \subset N$ with $|B| \geq 2$.

Proof of (3): The following proof is borrowed from Aumann and Dreze (1974, Theorem 7). Let $x \in \mathbb{M}_1(N, v)$ and $B \subset N$ with $|B| \geq 2$. Define the Davis/Maschler reduced game (B, v_{xB}) . Since $v_{xB}(B) = v(N) - x(N \setminus B)$ and $x \in \mathbb{M}_1(N, v) \subset X(N, v)$, it follows that $x(B) = v_{xB}(B)$. Let $i, j \in B$. Suppose that there is an objection (S, y) , $S \subset B$, of i against j at x^B in (B, v_{xB}) . Then, $y(S) = v_{xB}(S)$, $y_i > x_i$, and $y_{-i} \geq x_{-i}^B$. By the definition of v_{xB} , there is $Q \subset N \setminus B$ such that $v_{xB}(S) = v(S \cup Q) - x(Q)$. Since $i \in S$ and $j \notin S$, $i \in S \cup Q$ and $j \notin S \cup Q$. Hence, $(S \cup Q, (y, x^Q))$ is an objection of i against j at x in (N, v) . Since $x \in \mathbb{M}_1(N, v)$, there is an ADM counterobjection $(T \cup R, z)$ with $T \subset B$ and $R \subset N \setminus B$ from j to $(S \cup Q, (y, x^Q))$ at x in (N, v) such that $v(T \cup R) = z(T \cup R)$ and

$$z = (z^{S \cap T}, z^{T \setminus S}, z^{Q \cap R}, z^{R \setminus Q}) \geq (y^{S \cap T}, x^{T \setminus S}, x^{Q \cap R}, x^{R \setminus Q})$$

Note that z can be chosen so that $v(T \cup R) = z(S \cap T) + z(T \setminus S) + x(Q \cap R) + x(R \setminus Q)$ and

$$z = (z^{S \cap T}, z^{T \setminus S}, x^{Q \cap R}, x^{R \setminus Q}), (z^{S \cap T}, z^{T \setminus S}) \geq (y^{S \cap T}, x^{T \setminus S})$$

Since $x(Q \cap R) + x(R \setminus Q) = x(R)$, it follows that $z(T) = z(S \cap T) + z(T \setminus S) = v(T \cup R) - x(R)$. Consider the reduced game (B, v_{xB}) again. Since $T \subset B$ and $R \subset N \setminus B$, there exists $R^* \subset N \setminus B$ such that

$$v_{xB}(T) = v(T \cup R^*) - x(R^*) \geq v(T \cup R) - x(R) = z(T)$$

Hence, there exists $z^* \in X(T, v_{xB})$ such that $z^* \geq z^T = (z^{S \cap T}, z^{T \setminus S}) \geq (y^{S \cap T}, x^{T \setminus S})$. Thus, (T, z^*) is an ADM counterobjection from j to (S, y) at x^B in (B, v_{xB}) , so that $x \in \mathbb{M}_1(B, v_{xB})$

Since $v(T) = z(S \cap T) + z(Q \cap T) + z(T \setminus (S \cup Q)) \geq y(T) + x(Q \cap T) + x(T \setminus (S \cup Q))$, we can choose counterobjection (T, z^*) from j to $(S \cup Q, (y, x^Q))$ at x in (N, v) such that

$$z^{*S \cap T} \geq y^T \text{ and } (z^{*Q \cap T}, z^{*T \setminus (S \cup Q)}) = (x^{Q \cap T}, x^{T \setminus (S \cup Q)})$$

We then have shown that $(T \cap B, z^{*B})$ is an ADM counterobjection from j to (S, y) at x^B in (B, v_{xB}) . Hence, $x^B \in \mathbb{M}_1(B, v_{xB})$. *Q.E.D.*

As a corollary to *consistency*, we have the following:

- (3') Bilateral consistency: Let (N, v) be a TU game, and $x \in \mathbb{M}_1(N, v)$. Then, $x^P \in \mathbb{M}_1(P, v_{xP})$ for each $P \subset N$ with $|P| = 2$.
- (4) Converse consistency: Let (N, v) be a TU game, and $x \in X(N, v)$. If $x^P \in \mathbb{M}_1(P, v_{xP})$ for every two-player reduced game (P, v_{xP}) , then $x \in \mathbb{M}_1(N, v)$.

Proof of (4): Let $x \in X(N, v)$, and suppose that $x^P \in \mathbb{M}_1(P, v_{xP})$ for every two-player reduced game (P, v_{xP}) . In the game (N, v) , choose two distinct players $i, j \in N$. Let (S, y) be an objection of player i to another player j at x . Then, there is $Q \subset N \setminus \{i, j\}$ such that $v(\{i\} \cup Q) - x(Q) > x_i$. Notice that $x^{\{i, j\}} \in \mathbb{M}_1(\{i, j\}, v_{x\{i, j\}})$. Hence, $(x_i, x_j) \geq (v_{x\{i, j\}}(\{i\}), v_{x\{i, j\}}(\{j\}))$ or $(x_i, x_j) \leq (v_{x\{i, j\}}(\{i\}), v_{x\{i, j\}}(\{j\}))$.

By the definition of $v_{x\{i,j\}}$, $v_{x\{i,j\}}(\{i\}) \geq v(\{i\} \cup Q) - x(Q)$. Then, $v_{x\{i,j\}}(\{i\}) > x_i$. Thus, $x_j \leq v_{x\{i,j\}}(\{j\})$. By the definition of $v_{x\{i,j\}}$, there is $R \subset N \setminus \{i, j\}$ such that $x_j \leq v(\{j\} \cup R) - x(R)$, i.e., $x_j + x(R) \leq v(\{j\} \cup R)$. Define $z_j = v(\{j\} \cup R) - x(R)$, then $z_j \geq x_j$ and $z_j + x(R) = v(\{j\} \cup R)$. Notice that $i \notin \{j\} \cup R$ and $j \in \{j\} \cup R$. Hence, $(\{j\} \cup R, (z_j, x^Q))$ is an ADM counterobjection from j to the objection (S, y) at x . Thus, for each objection of i to j at x , there is a counterobjection from j at x . Therefore, $x \in \mathbb{M}_1(N, v)$. *Q.E.D.*

- (5) Covariance: Let (N, v) be a TU game, $\alpha > 0$ and $\beta \in \mathbb{R}^N$. Construct the TU game (N, w) , where $w(S) = \alpha v(S) + \beta(S)$ for every $S \subset N$. If $x \in \mathbb{M}_1(N, v)$, then $\alpha x + \beta \in \mathbb{M}_1(N, w)$.

We next prove a result that, as far as we know, is new to the literature:

Theorem: Over the class of all TU games, there exists a unique solution σ satisfying nonemptiness, (bilateral) consistency, converse consistency, and the “both on the same boat” property, i.e., $\sigma(P, v) = X_0(P, v) \cup \bar{L}(P, v)$ for each 2-person TU game (P, v) . It is \mathbb{M}_1 .

Proof: The solution \mathbb{M}_1 is a solution on \mathcal{G} which satisfies nonemptiness, bilateral consistency, converse consistency, and $\mathbb{M}_1(P, v) = X_0(P, v) \cup \bar{L}(P, v)$ for each 2-person TU game (P, v) .

Suppose now that σ is a solution satisfying nonemptiness, consistency, converse consistency, and $\sigma(P, v) = X_0(P, v) \cup \bar{L}(P, v)$ for each 2-person TU game (P, v) . Thus, for each 2-person TU game (P, v) , $\sigma(P, v) = \mathbb{M}_1(P, v)$.

Let (N, v) be a TU game with $|N| \geq 3$. We prove $\mathbb{M}_1(N, v) \subset \sigma(N, v)$. Let $x \in \mathbb{M}_1(N, v)$. By the bilateral consistency of \mathbb{M}_1 , for each $P \subset N$

with $|P| = 2$, $x^P \in \mathbf{M}_1(P, v_{xP}) = X_0(P, v_{xP}) \cup \bar{L}(P, v_{xP}) = \sigma(P, v_{xP})$. By the converse consistency of σ , $x \in \sigma(N, v)$. Thus, $\mathbf{M}_1(N, v) \subset \sigma(N, v)$.

We next prove $\sigma(N, v) \subset \mathbf{M}_1(N, v)$. Let $x \in \sigma(N, v)$. By the bilateral consistency of σ , then for each $P \subset N$ with $|P| = 2$, $x^P \in \sigma(P, v_{xP}) = X_0(P, v_{xP}) \cup \bar{L}(P, v_{xP}) = \mathbf{M}_1(P, v_{xP})$. By the converse consistency of \mathbf{M}_1 , $x \in \mathbf{M}_1(N, v)$. Thus, $\sigma(N, v) \subset \mathbf{M}_1(N, v)$. Hence, for each TU game (N, v) with $|N| \geq 3$, $\sigma(N, v) = \mathbf{M}_1(N, v)$.

It therefore follows that $\sigma = \mathbf{M}_1$ over \mathcal{G} . *Q.E.D.*

The following is essentially the same as Theorem 5.14 in Peleg (1986):

Theorem (Peleg, 1986): Over the class of all TU games, there exists a unique solution σ satisfying efficiency, (bilateral) consistency, converse consistency, and individual rationality for two-player games. It is the core \mathcal{C} .

Notice that we impose efficiency on solutions on \mathcal{G} . Then, the requirement of individual rationality for each 2-person TU game coincides with the axiom Peleg (1986) calls “unanimity”. We refer to the condition “ $\sigma(P, v) = X_0(P, v) \cup \bar{L}(P, v)$ for each 2-person TU game (P, v) ” as the *both on the same boat* axiom, by which we mean “cooperation for better, for worse, for richer, for poorer, in sickness and in health,” the promised vows in marriage.

In addition, Aumann and Maschler (1985, P.210) defines the *standard solution of 2-person TU game* $(\{1, 2\}, v)$ by

$$\begin{aligned} & \sigma(\{1, 2\}, v) \\ = & (v(\{1\}) + \frac{1}{2}(v(\{1, 2\}) - v(\{1\}) - v(\{2\})), v(\{2\}) + \frac{1}{2}(v(\{1, 2\}) - v(\{1\}) - v(\{2\}))) \end{aligned}$$

in order to show the bilateral consistency and converse consistency of the

prekernel. Peleg (1986, Remark 4.4, Theorem 4.5) axiomatizes the prekernel as the only solution satisfying consistency and converse consistency such that the values of 2-person games coincide with those of the standard solution. We can therefore regard the core, the ADM prebargaining set, and the prekernel, respectively as “extensions” of the two-person rules of “unanimity,” “both on the same boat,” and the standard solution, where the extension to multi-person games satisfies consistency and converse consistency. On the other hand, Peleg (1986, Remark 4.6) points out that the prenucleolus does not satisfy converse consistency (recall that it satisfies consistency).

5.2 Modified Versions

We discuss next some problems with the ADM bargaining set, which remain unsolved (They are pointed out by Zhou (1994, Section 3)).

The first problem is as follows: Suppose that player i forms a coalition to make an objection against another player j . Then, in order for player j to counterobject to player i , is it effective even if player j forms a coalition containing no members of the objecting coalition? In particular, according to the definition, when two players can form disjoint coalitions, and can make objections, none of them are justified. Then, the original recommendation passes the definition of the ADM bargaining set.

In fact, Myerson (1991, P. 453) imposes the constraint “ $S \cap T \neq \emptyset$ ” on the objecting coalition S and the counterobjecting coalition T in the definition of the ADM bargaining set. However, it is assumed that “ $i \in S \ \& \ j \notin S$ ” and “ $j \in T \ \& \ i \notin T$,” so that “ $S \cap T \neq \emptyset$ ” implies “ $S \setminus T \neq \emptyset \ \& \ T \setminus S \neq \emptyset$.” Then, the constraint Myerson imposes on the ADM bargaining set means “ $S \cap T \neq \emptyset$ ”

& $S \setminus T \neq \emptyset$ & $T \setminus S \neq \emptyset$,” which is called the *proper intersection condition* and introduced by Zhou (1994) to define a new bargaining set. Zhou (1994, Example 3.2) points out that the bargaining set in which this constraint is imposed does not generally contain either the kernel or the nucleolus. As a result, both the kernel and the nucleolus lose the role of refinements of the bargaining set. We will discuss more details on this point later.

The second problem is: Why are objections and counterobjections defined with different kinds of inequalities? We refer to improvements of members from payoff profile x to payoff profile y in a coalition defined by $y \gg x$, $y \geq x$, and $y \geq x$ & $y \neq x$ as *strict improvement*, *weak improvement*, and *Pareto improvement*, respectively. Strict improvement implies Pareto improvement, and Pareto improvement implies weak improvement. A weak improvement that is not a Pareto improvement means $y = x$, namely *the status quo*. As the improvement for objections is made weaker or that for counterobjections is made stronger, the number of justified objections increases and the set of payoff profiles with no justified objections becomes smaller.

Consider whether an objection can be countered using the status quo as the counterobjection. Suppose that player i makes an objection against player j at the status quo by forming coalition S and player j makes a counterobjection by forming coalition T . According to the ADM definition, it may be possible that player j disturbs player i 's plans to form coalition S if player j gives at least one member of S the same level of payoff as offered by i . We however wonder whether player j can prevent player i from forming an objecting coalition by giving himself and possibly some other players exactly the same level of payoffs as at the status quo. Then, it seems

that the validity or plausibility of the definition of counterobjections relying on weak improvements depends on whether the counterobjecting coalition includes a part of the objecting coalition.

The third problem is: Which unit proposes an objection and a counterobjection, a player or a coalition? To define the core, an objection is assumed to be made from a coalition to the status quo. To define the ADM bargaining set, as we have seen, an objection is made by a player against another player given the status quo, and the counterobjecting coalition does not contain the leader, who is the player making the objection. Hence, the justifications of an objection depend on who is designated as the leader of the coalition, even if the coalition and the proposed payoff profile of the objection is fixed. Mas-Colell (1989) answers this problem by defining an objection “from a coalition to the status quo,” a counterobjection “from another coalition to the objection raised against the status quo,” and proposes a modification of the ADM bargaining set as a result. The formal definitions are as follows:

Definition: Let (N, v) be a TU game, and x a payoff profile. Then a *Pareto objection to x* is a pair (S, y) of coalition S and $y \in X(S, v)$ such that $y \geq x^S$ & $y \neq x^S$.

It is interpreted that coalition S protests in public against their payoffs x^S that “If we, coalition S , produce payoff list y , then we can improve upon some of the members of S without making the other members worse off than at the status quo x^S .” The core of a TU game can be redefined as “the set of payoff profiles to which there is no Pareto objection.”

Definition: Let (N, v) be a TU game, x a payoff profile on N , and (S, y) an objection to x . Then a *Pareto counterobjection to (S, y) at x* is a pair

(T, z) of coalition T and $z \in X(T, v)$ such that $(z^{S \cap T}, z^{T \setminus S}) \geq (y^{S \cap T}, x^{T \setminus S})$
 $\& (z^{S \cap T}, z^{T \setminus S}) \neq (y^{S \cap T}, x^{T \setminus S})$.

The interpretation is that coalition T replies: “If we, coalition T , make a list z of payoffs, then we can make all the members of T who are also in S at least as happy as with y , the other members of T at least as happy as with x , and at least one member strictly better off than at $(y^{S \cap T}, x^{T \setminus S})$.”

If we replace Pareto improvement in the definition of counterobjection by weak improvement (as in the definition of ADM counterobjection), then we can define another type of counterobjection.

Definition: Let (N, v) be a TU game, x a payoff profile on N , and (S, y) an objection to x . Then a *weak counterobjection to (S, y) at x* is a pair (T, z) of coalition T and $z \in X(T, v)$ such that $(z^{S \cap T}, z^{T \setminus S}) \geq (y^{S \cap T}, x^{T \setminus S})$.

We are ready to introduce the bargaining sets defined by the above types of objections and counterobjections.

Definition (Mas-Colell, 1989; Vohra, 1991):

The *weak Mas-Colell prebargaining set of the game (N, v)* is the set

$$\overline{MB}(N, v) = \{x \in X(N, v) : \text{For each Pareto objection to } x, \\ \text{there is a weak counterobjection at } x\}$$

The *Mas-Colell prebargaining set of the game (N, v)* is the set

$$MB(N, v) = \{x \in X(N, v) : \text{For each Pareto objection to } x, \\ \text{there is a Pareto counterobjection at } x\}$$

The *Mas-Colell bargaining set of the game (N, v)* is the set

$$MB_0(N, v) = \{x \in X_0(N, v) : \text{For each Pareto objection to } x,$$

there is a Pareto counterobjection at $x\}$

The mappings \overline{MB} , MB and MB_0 are solutions on \mathcal{G} . Mas-Colell's original (1989) is MB , for which individual rationality is not required. Mas-Colell (1989, P.138) points out that, for all TU games, MB includes the prekernel, which contains the prenucleolus, so that $MB(N, v)$ is nonempty. Vohra (1991) proposes MB_0 by imposing individual rationality on MB , and shows the nonemptiness of MB_0 by proving that MB_0 includes the kernel for TU games satisfying zero-monotonicity (Vohra, 1991, Proposition 3.2). Mas-Colell (1989, P.139) notes that MB and the ADM prebargaining set \mathbf{M}_1 are not comparable, but it is easily shown that \mathbf{M}_1 is a subsolution of \overline{MB} for all TU games.

Proposition: For each TU game (N, v) ,

$$\mathbf{M}_1(N, v) \subset \overline{MB}(N, v)$$

Proof: Let $x \in \mathbf{M}_1(N, v)$. Then, $x \in X(N, v)$. Let (S, y) be a Pareto objection to x . Then, $y \geq x^S$ and there is a player $i \in S$ such that $y_i > x_i$. Since $S \neq N$, there is a player $j \notin S$. Then, (S, y) is an ADM objection of i against j at x . Since $x \in \mathbf{M}_1(N, v)$, there exists an ADM counterobjection (T, z) to (S, y) at x . Then, $(z^{S \cap T}, z^{T \setminus S}) \geq (y^{S \cap T}, x^{T \setminus S})$. This means that (T, z) is a weak counterobjection to (S, y) at x . Hence, $x \in \overline{MB}(N, v)$. *Q.E.D.*

We illustrate these bargaining sets of two-player games in the following example:

Example 5.2: Let $N = \{1, 2\}$ and consider 2-person TU games. Define the set of efficient payoff profiles with negative excesses of the two players by

$$L(\{1, 2\}, v) = \{x \in X(\{1, 2\}, v) : (x_1, x_2) \ll (v(\{1\}), v(\{2\}))\}$$

Then

$$\overline{MB}(\{1, 2\}, v) = X_0(\{1, 2\}, v) \cup \overline{L}(\{1, 2\}, v).$$

$$MB(\{1, 2\}, v) = X_0(\{1, 2\}, v) \cup L(\{1, 2\}, v).$$

$$MB_0(\{1, 2\}, v) = X_0(\{1, 2\}, v).$$

Notice that $\overline{MB}(\{1, 2\}, v)$ and $MB_0(\{1, 2\}, v)$ are closed sets, and $MB(\{1, 2\}, v)$ is not a closed set when $v(\{1\}) + v(\{2\}) > v(\{1, 2\})$.

Zhou (1994) modifies the Mas-Colell Bargaining Set by imposing the proper intersection condition on a counterobjection, using strict Pareto objections and weak counterobjections. This leads to the *Zhou bargaining set* of the game (N, v) , i.e., $ZB(N, v)$. Note how all objections from individual players are justified, so that individual rationality is automatically implied. Zhou (1994, Theorem 2.5) proves nonemptiness of his bargaining set for general games with coalition structures. As a corollary, the nonemptiness of ZB follows from a weak form of superadditivity called grand coalition superadditivity, which means that $\sum_{B \in \mathcal{P}} v(B) \leq v(N)$ for every partition \mathcal{P} of N . The following inclusion relations are straightforward from the definitions:

$$\mathcal{C}(N, v) \subset ZB(N, v) \subset MB(N, v) \subset \overline{MB}(N, v).$$

The counterobjections defined by weak improvements make Zhou's bargaining set nonempty for general games and the mapping ZB upper hemicontinuous in v given N .

Shimomura (1997) defines two modifications of the Zhou bargaining set. In the first, counterobjections are defined by strict improvements, leading to the *strict Zhou bargaining set* ZB^* .⁹ In the second, we say that coalition T is dominant over S whenever any strict objection from S can be strictly countered in the Zhou style by T . This leads to the *steady bargaining set* $SB(N, v)$, which is the set of $x \in X(N, v)$ such that for each coalition S , there is a coalition T that is dominant over S at x . Shimomura (1997, Theorem 2) proves nonemptiness of SB under grand coalition superadditivity. As a corollary, nonemptiness of ZB^* under grand coalition superadditivity follows. The following inclusions hold:

$$\mathcal{C}(N, v) \subset SB(N, v) \subset ZB^*(N, v) \subset ZB(N, v).$$

If a game satisfies balancedness, which implies grand coalition superadditivity, then the nonempty core, SB , ZB^* , and ZB all contain the nucleolus. However, if a game satisfies grand coalition superadditivity but does not satisfy balancedness, then ZB may not contain the nucleolus, and hence SB may not contain the nucleolus either. Consider the following example:

Example 5.3: $N = \{1, 2, 3, 4, 5\}$, $v(\{i\}) = 0$ for every $i \in N$, $v(\{1, 2\}) = v(\{3, 4\}) = v(\{3, 5\}) = v(\{4, 5\}) = 2.1$, $v(N) = 5$, and the other $v(S)$ are gen-

⁹Shimomura (1997) modifies the Mas-Colell bargaining set and the Zhou bargaining set by defining both objections and counterobjections with strict improvements. Izquierdo and Rafels (2018) call them the Mas-Colell bargaining set *à la Shimomura* and the Zhou bargaining set *à la Shimomura*, respectively.

erated so that v is the minimal superadditive game compatible with the given worths: The nucleolus, which is also the prenucleolus, is $x^* = (1, 1, 1, 1, 1)$. Since v is the minimal superadditive game, we have $v(\{1, i\}) = v(\{2, i\}) = 0$, $v(\{1, i, j\}) = v(\{2, i, j\}) = 2.1$ for each $i, j \in \{3, 4, 5\}$, and $v(\{1, 3, 4, 5\}) = v(\{2, 3, 4, 5\}) = 2.1$. Since $v(\{1, 2\}) - x^*(\{1, 2\}) > 0$, there is a strict objection from coalition $\{1, 2\}$ to x^* . Suppose that coalition $\{1, 2\}$ makes the strict objection $(\{1, 2\}, (1.05, 1.05))$. Then, there is no coalition that can make even a weak Zhou counterobjection at x^* . Thus, $x^* \notin ZB(N, v)$. It also follows $x^* \notin SB(N, v)$. Note that x^* is contained in the ADM bargaining set of (N, v) but none of the players $\{3, 4, 5\}$ can counterobject the ADM objection $(\{1, 2\}, (1.05, 1.05))$ of player 1 at x^* by forming coalition T such that $1 \notin T, 2 \in T$, and $\{3, 4, 5\} \cap T \neq \emptyset$. Finally, $\mathcal{K}(N, v) = \{x | x_1 = x_2 \in [\frac{37}{40}, \frac{21}{20}], x_3 = x_4 = x_5 \in [\frac{29}{30}, \frac{21}{20}], x(N) = 5\}$, and $x^* \in \mathcal{K}(N, v)$.

Thus, this example shows that the constraint imposed by Myerson (1991, P. 453) results in that restricted version of the ADM bargaining set losing the containment of both the nucleolus and the kernel. The example also shows that neither ZB nor SB include the nucleolus in general. In this sense, applying stricter standards to “reasonable” justified objections sometimes conflicts with the egalitarian social welfare principle behind the nucleolus. Hence, we may say that these two bargaining sets are not quite in the same family as the ADM bargaining set and the Mas-Colell bargaining set.

5.3 Further Topics

(i) **Coincidence of bargaining sets with the core.** It is well known that, in two-person and three-person TU games, the ADM bargaining set $\mathcal{M}_1^{(i)}$ and the core \mathcal{C} coincide if the game is balanced. Solymosi (2002) proves that, in four-person balanced games, $\mathcal{M}_1^{(i)}$ and the core also coincide. On the other hand, Maschler (1976) presents an example of a five-person totally-balanced game whose ADM bargaining set contains many points besides the core. Izquierdo and Rafels (2012b) proves that, in any TU game (N, v) , there exists the number $k^*(v)$ such that the ADM bargaining set coincides with the core if the worth $v(N)$ of the grand coalition is greater than or equal to $k^*(v)$, and establishes the formula to compute the number $k^*(v)$. In addition, the paper shows by way of example that the existence of such a critical number is not shared by other variants, such as the Mas-Colell prebargaining set MB , the Mas-Colell bargaining set MB_0 , and the Mas-Colell bargaining set (*à la Shimomura*) MB^* (for the definition of MB^* , see footnote 9).

For the class of convex games, Maschler *et al.* (1972) establishes that $\mathcal{C} = \mathcal{M}_1^{(i)}$, and Dutta *et al.* (1989) proves that $\mathcal{C} = MB_0$. One can show that the Mas-Colell bargaining set MB_0 is included in the Mas-Colell bargaining set (*à la Shimomura*) MB^* , so that then, $\mathcal{C} \subset MB_0 \subset MB^*$. Izquierdo and Rafels (2012a) proves that $\mathcal{C} = MB^*$ for all convex games, which is a stronger result than Dutta *et al.*'s. Recall that $\mathcal{C} \subset SB \subset ZB^* \subset MB^*$. Then, the coincidence result of Izquierdo and Rafels (2012a) implies that all these inclusions are actually equalities on the class of convex games, i.e., $\mathcal{C} = SB = ZB^* = MB_0 = MB^*$.

Conversely, Izquierdo and Rafels also characterize the domain of convex

games by the coincidence with the core. Namely, the class of convex games is the only class of zero-monotonic games on which the Mas-Colell bargaining set (*à la Shimomura*) MB^* coincides with the core \mathcal{C} (Izquierdo and Rafels (2012a)), and the only class of zero-monotonic and grand coalition superadditive games on which the steady bargaining set SB coincides with the core \mathcal{C} (Izquierdo and Rafels (2018)). Since SB is the minimal supersolution of the core among the bargaining sets we have investigated, the small difference between \mathcal{C} and SB makes their characterization theorem remarkably striking.

(ii) The consistent bargaining set. One may wonder what happens if “counterobjections to counterobjections” are taken into account, further counterobjections to such counterobjections, and so on and so forth. Dutta *et al.* (1989) proposes such a modification and, accordingly, defines a variant of the Mas-Colell prebargaining set named the *consistent bargaining set*, for which the following facts can be noted: (i) it is a supersolution of the core and a subsolution of the Mas-Colell prebargaining set; (ii) it coincides with the Mas-Colell prebargaining set for all three-person superadditive TU games; (iii) as shown by example, there is a four-person superadditive TU game where the consistent bargaining set is empty; (iv) there are TU games where it violates individual rationality (suggesting that, perhaps, it should be called the consistent prebargaining set); (v) it does not generally satisfy consistency with respect to the DM reduced game; and (vi) it is not known whether it contains the nucleolus or whether it has a nonempty intersection with the kernel when the core is empty, and its inclusion relations with other versions of the bargaining set, such as Zhou’s or Shimomura’s, are also unknown.

(iii) Equivalence and convergence in economic environments. Shap-

ley and Shubik (1984) shows that the ADM bargaining set is approximately Walrasian in replica sequences of exchange economies with smooth quasilinear preferences. Anderson (1998) extends the Shapley-Shubik result to nonreplica sequences of exchange economies with smooth preferences, which need not be quasilinear. On the other hand, Shapley (1992) constructs a replica sequence of economies with Leontief preferences and finds an element of the kernel that does not converge to the set of Walrasian allocations. The kernel is contained in the ADM bargaining set, which is in turn contained in the Mas-Colell bargaining set. Hence, this example is also a nonconvergence example for these two bargaining sets.

Mas-Colell (1989) considers an exchange economy with a continuum of consumers, and proves that his prebargaining set coincides with the set of Walrasian allocations. His result is stronger than Aumann's (1964) core equivalence theorem in continuum environments.

Although Mas-Colell defines his prebargaining set of coalitional games by Pareto objections and Pareto counterobjections, he formulates his prebargaining set of exchange economies by Pareto objections and strict counterobjections. Anderson, Trockel, and Zhou (1997) investigates whether the Mas-Colell bargaining set and the Zhou bargaining set, which are defined by Pareto objections and counterobjections, of replicated exchange economies converge to the set of Walrasian allocations. Anderson *et al.* (1997) constructs an example in which neither the Mas-Colell bargaining set nor the Zhou bargaining set converge to the set of Walrasian allocations (For more details about the difference between Mas-Colell(1989) and Anderson **et al.** (1997) , see Hara (2005, P.548-549)).

(iv) NTU games. We say that an NTU game is “nonleveled” if the feasible payoff set for every coalition is strongly comprehensive. Vohra (1991) is the first work to find a class of nonleveled NTU games for which the Mas-Colell bargaining set is nonempty, and names them “weakly balanced games.” He also defines the subclass of “weakly TU balanced games” Shimomura (1995) presents a sufficient condition on nonleveled NTU games to obtain the nonemptiness of the strict Zhou bargaining set with coalition structures, and calls it the “noncrossing condition.” Chang and Chen (2006) defines “subbalanced games” and proves, through “weak balancedness” and “subbalancedness,” the nonemptiness of the Mas-Colell bargaining set for nonleveled weakly TU games, and that of the Zhou bargaining set for nonleveled weakly TU games satisfying grand coalition superadditivity. Furthermore, Peleg and Sudholter (2005) presents an example of a nonleveled superadditive NTU game for which the Mas-Colell bargaining set is empty. In addition, Holzman (2001) investigates inclusion relations between the ADM and the Mas-Colell bargaining sets, proving that the former is included in the latter for each nonleveled superadditive NTU game, also showing by way of example that the ADM bargaining set may not be included even in the closure of the Mas-Colell bargaining set if the game does not satisfy nonleveledness.

(v) Bargaining and bargaining sets. It is not trivial to propose noncooperative bargaining foundations of the bargaining set. The reason is that the outcomes prescribed by an objection and a counterobjection may not be simultaneously feasible, which must be a minimal requirement of any implementing mechanism. After showing that implementation in Nash equilibrium is impossible, Serrano and Vohra (2002a) proposes a multi-stage game that

implements the ADM bargaining set of exchange economies in subgame perfect equilibrium. In their game, after a proposal is made, depending on the identity of the rejector, if any, a different outcome is specified. Also, after a proposal is put on the table and all the responders agree, the proposer has one extra chance to either ratify the proposal or reject it. While the first feature is key to solving the feasibility difficulty outlined above, the second feature takes care of the related problem of inappropriate proposals or responses (“inappropriate” when judged by the logic of the bargaining set). A modification of the procedure, found in Serrano and Vohra (2002b), implements the Mas-Colell bargaining set. Einy and Wettstein (1999) proposes a different game, but where violations of feasibility take place, and Pérez-Castrillo and Wettstein (2000) solves the feasibility issue by adding extra players.

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Table 1

	$d_1 = 300$	$d_2 = 200$	$d_3 = 100$
$E = 100$	33.3	33.3	33.3
$E = 200$	75	75	50
$E = 300$	150	100	50

Figure 1

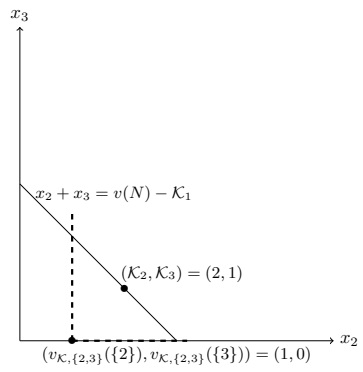
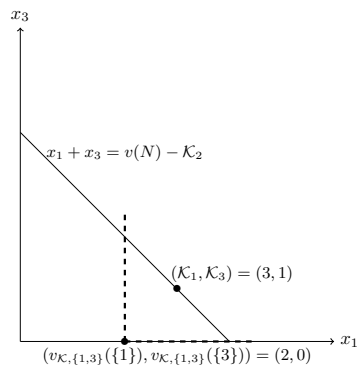
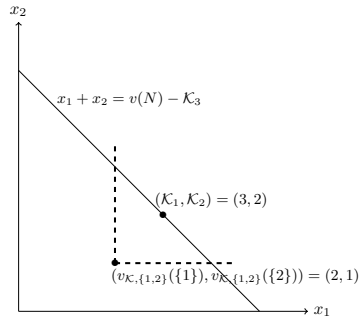


Figure 2

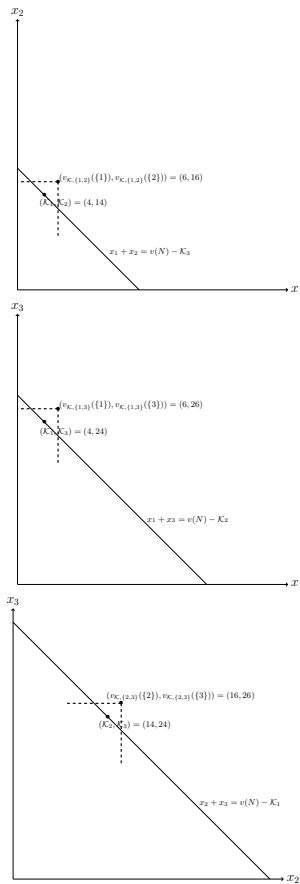
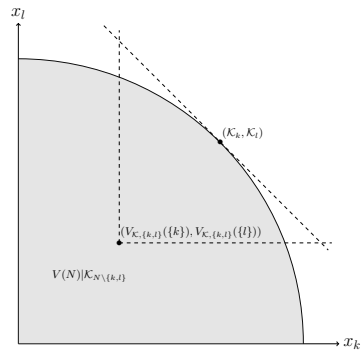


Figure 3

Panel A



Panel B

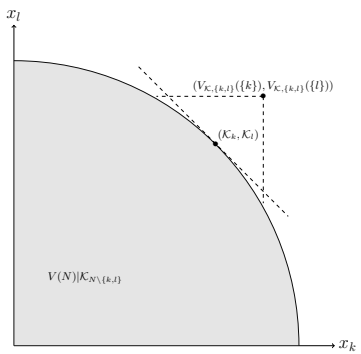


Figure 4

