# Two versions of minimal intuitionism with the CAP. A note

Gemma ROBLES and José M. MÉNDEZ

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ABSTRACT. Two versions of minimal intuitionism are defined restricting Contraction. Both are defined by means of a falsity constant F. The first one follows the historical trend, the second is the result of imposing special constraints on F. Relational ternary semantics are provided.

Keywords: Intuitionistic logic, Contraction Axiom, Converse Ackermann Property, Constructive Falsity.

## *1. Motivation*

As it is known, minimal intuitionism can be viewed as a definitional extension of positive intuitionistic logic. The idea is to add to the positive language a propositional falsity constant F along with the definition $\neg A =_{df} A \rightarrow F$. Then, given the intuitionistic positive theorems,

a. $[A \rightarrow (B \rightarrow C)] \rightarrow [B \rightarrow (A \rightarrow C)]$

b. $A \rightarrow [(A \rightarrow B) \rightarrow B]$

c. $(A \rightarrow B) \rightarrow [(B \rightarrow C) \rightarrow (A \rightarrow C)]$

d. $[A \rightarrow (A \rightarrow B)] \rightarrow (A \rightarrow B)$

e. $[A \rightarrow (B \rightarrow C)] \rightarrow [(A \rightarrow B) \rightarrow (B \rightarrow C)]$

we have, for example (and to limit ourselves to conditional-negation theorems) the following minimal intuitionistic theses:

a’. $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$

b’. $A \rightarrow \neg\neg A$

c’. $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$

d’. $(A \rightarrow \neg A) \rightarrow \neg A$

e’. $(A \rightarrow \neg B) \rightarrow [(A \rightarrow B) \rightarrow \neg A]$

Minimal intuitionism departs (from this syntactical point of view) from full intuitionistic logic in the absence of, e. g.,

f. $A \to (\neg A \to B)$

or

g. $(A \wedge \neg A) \to B$

though, we note,

h. $A \to (\neg A \to \neg B)$

or

i. $(A \wedge \neg A) \to \neg B$

are present.

On the other hand, let us focus on the CAP. A positive logic with a falsity constant F has the Converse Ackermann Property (CAP) if all the formulas of the form $(A \to B) \to C$ are unprovable whenever C contains neither $\to$ nor F. The CAP can intuitively be interpreted as the non-derivability of necessitive propositions from non-necessitive ones (A is *necessitive* if A is of the form NB, that is, if A is equivalent to $(B \to B) \to B$ (Anderson and Belnap 1975, §4.3)).

The question about which systems do possess the CAP is first proposed in Anderson and Belnap (1975, §8.12). In Méndez (1987) it is answered for implicative and positive logics. Syntactically speaking, the solution consists in restricting *contraction* ((d) above) and *assertion* ((b) above) to the case in which B is an implicative formula (A is implicative iff A is of the form $B \to C$). Thus, logics with the CAP are contractionless logics. Actually, they are the natural bridge between strict contractionless logics and logics with contraction.

But what about negation in these logics? That is, which kind(s) of negation(s) is (are) compatible with the CAP? We briefly note the results we are aware of. In Méndez (1988) a sort of semiclassical negation, in Kamide (2002), Kamide (2003) a so-called "strong negation" is added to the positive logics of Méndez (1987). Now, the aim of this paper is to define minimal negation in the positive intuitionistic logic with the CAP of Méndez (1987).

As the title makes clear, we consider two possibilities. The first one follows the historical trend commented above: we definitionally extend positive intuitionistic logic with the CAP, $I^0_{+}$, with F. This gives us the logic $I^0_{m1}$. Now let us try to motivate the second one. As (a)-(e) above are not provable in $I^0_{+}$, (a')-(e') are not provable either. Nevertheless, (a')-(e') are, of course, not only minimally acceptable but also desirable indeed. Moreover, they are CAP-compatible. In consequence, we shall show how to impose additional constraints on F to make these formulas valid, which give us the second possibility. This gives us the logic $I^0_{m2}$.

The logics we present here have not been, to our knowledge, defined in the literature. Nevertheless, it is not hard to derive them from some published work, as we show below.

## *2. The logics* $I^0_{m1}$, $I^0_{m2}$

Positive intuitionistic logic with the CAP, $I^0_{+}$, can be axiomatized as follows (see Méndez (1987)).

Axioms:

A1. $A \rightarrow A$

A2. $A \rightarrow (B \rightarrow A)$

A3. $(A \rightarrow B) \rightarrow [(B \rightarrow C) \rightarrow (A \rightarrow C)]$

A4. $A \rightarrow [[A \rightarrow (B \rightarrow C)] \rightarrow (B \rightarrow C)]$

A5. $[A \rightarrow [A \rightarrow (B \rightarrow C)]] \rightarrow [A \rightarrow (B \rightarrow C)]$

A6. $(A \wedge B) \rightarrow A$ / $(A \wedge B) \rightarrow B$

A7. $[(A \rightarrow B) \wedge (A \rightarrow C)] \rightarrow [A \rightarrow (B \wedge C)]$

A8. $A \rightarrow (A \vee B)$ / $B \rightarrow (A \vee B)$

A9. $[(A \rightarrow C) \wedge (B \rightarrow C)] \rightarrow [(A \vee B) \rightarrow C]$

A10. $[A \wedge (B \vee C)] \rightarrow [(A \wedge B) \vee C]$

Rules:

*Modus ponens* (MP): if $A \rightarrow B$ and $A$, then $B$.

*Adjunction* (Ad): if $A$ and $B$, then $A \wedge B$.

Now, $I^0_{m1}$ is a definitional extension of $I^0_{+}$ with the propositional falsity constant F along with the definition $\neg A =_{df} A \rightarrow F$. For instance, the following theorems belong to $I^0_{m1}$:

T1. $\neg F$

T2. $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$

T3. $\neg B \rightarrow [(A \rightarrow B) \rightarrow \neg A]$

T4. $F \rightarrow \neg A$

T5. $\neg A \rightarrow (A \rightarrow \neg B)$

T6. $A \rightarrow (\neg A \rightarrow \neg B)$

T7. $(\neg A \wedge \neg B) \leftrightarrow \neg(A \vee B)$

T8. $(\neg A \vee \neg B) \rightarrow \neg(A \wedge B)$

T9. $\neg(A \wedge B) \rightarrow (A \rightarrow \neg B)$

T10. $(\neg A \vee \neg B) \rightarrow (A \rightarrow \neg B)$

T11. $(A \vee \neg B) \rightarrow (\neg A \rightarrow \neg B)$

T12. $(A \wedge \neg A) \rightarrow \neg B$

T13. $[(A \vee \neg B) \wedge \neg A] \rightarrow \neg B$

On the other hand, $I^0_{m2}$ is the result of adding to $I^0_{m1}$ the axioms:

A11. $A \rightarrow [(A \rightarrow F) \rightarrow F]$

i.e., $A \rightarrow \neg\neg A$, and

A12. $[A \rightarrow (A \rightarrow F)] \rightarrow (A \rightarrow F)$

i.e., $(A \rightarrow \neg A) \rightarrow \neg A$.

In addition to T1-T13 of $I^0_{m1}$, we have, for example, the following theorems:

T14. $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$

T15. $(A \rightarrow B) \rightarrow [(A \rightarrow \neg B) \rightarrow \neg A]$

T16. $(A \rightarrow \neg B) \rightarrow [(A \rightarrow B) \rightarrow \neg A]$

T17. $(A \wedge B) \rightarrow \neg(A \rightarrow \neg B)$

T18. $\neg(A \wedge \neg A)$

T19. $\neg\neg\neg A \rightarrow \neg A$

T20. $\neg\neg(A \vee \neg A)$

## *3. Converse Ackermann Property*

It is proved in Salto, Méndez and Robles (2001) that $LC^0_m$ has the CAP. $I^0_{m2}$ is (syntactically) included in $LC^0_m$. Therefore, $I^0_{m1}$ and $I^0_{m2}$ have the CAP.

## *4. Semantics for* $I^0_+$

Given a pair <K, R> where K is a non-empty set and R a ternary relation on K, let us define the binary relation ≤, the quaternary relation $R^2$ and the five element relation $R^3$ by, for every $a, b, c, d \in K$,

d1. $a \leq b$ iff $(\exists x \in K)\ Rxab$

d2. $R^2abcd$ iff $(\exists x \in K)(Rabx$ and $Rxcd)$

d3. $R^3abcde$ iff $(\exists x \exists y \in K)(Rabx$ and $Rxcy$ and $Ryde)$

An $I^0_+$ model is a triple <K, R, > where K is a non-empty set, R is a ternary relation on K satisfying the following conditions for every $a, b, c, d \in K$:

P1. $a \leq a$

P2. $a \leq b$ and $Rbcd \Rightarrow Racd$

P3. $R^2abcd \Rightarrow (\exists x \in K)(Racx$ and $Rxbd)$

P4. $R^2abcd \Rightarrow R^2bacd$

P5. $R^2abcd \Rightarrow R^3abbcd$

P6. $Rabc \Rightarrow a \leq c$

Finally, is a valuation relation from K to the sentences of $I^0_+$ satisfying the following conditions for all formulas p, A, B and a point $a$ in K:

i. $a$ p and $a \leq b \Rightarrow b$ p

ii. $a$ $A \vee B$ iff $a$ A or $a$ B

iii. $a$ $A \wedge B$ iff $a$ A and $a$ B

iv. $a$ $A \rightarrow B$ iff for all $b, c, e \in K$, (R$abc$ and $b$ A) $\Rightarrow c$ B

A is valid in $I^0_+$ iff $a$ A for all $a \in K$ in all models.

Remark: the postulates

P7. $R^2abcd \Rightarrow (\exists x \in K)(Rbcx$ and $Raxd)$

P8. R$abc$ $b \leq c$

are derivable.

It is not difficult to prove along the lines of [3] that a formula A is valid iff A is a $I^0_+$ theorem.

## 5. *Semantics for* $I^0_{m1}$

A $I^0_{m1}$ model is a quadruple <K, S, R, > where S is a non-empty subset of K and <K, R, > is a $I^0_+$ model such that the relation satisfies in addition the clauses

v. $a \leq b$ and $a$ F $\Rightarrow b$ F

vi. $a$ F iff $a \notin S$

A formula A is $I^0_{m1}$ *valid* iff $a$ A for all $a \in K$ in all models. Semantic consistency (semantic soundness of $I^0_{m1}$ relative to the semantics of $I^0_{m1}$ models) is immediately derived from that of $I^0_+$. (Note that being S non-empty, F is not valid).

## 6. *Semantics for* $I^0_{m2}$

A $I^0_{m2}$ model is similar to a $I^0_{m1}$ model but with the addition of the postulates

P9. (R$abc$ and $c \in S$) $\Rightarrow (\exists x \in S)$ R$bax$

P10. (R$abc$ and $c \in S$) $\Rightarrow (\exists x \in K)(\exists y \in S)$(R$abx$ and R$xby$).

A formula A is $I^0_{m2}$ valid iff *a* A for all $a \in K$ in all models. Semantic consistency is left to the reader (the validity of A11 is proved with P9; the validity of A12 with P10).

## 7. *Completeness of* $I^0_{m1}$ *and* $I^0_{m2}$

As noted in §3, $I^0_{m1}$ and $I^0_{m2}$ are sublogics of $LC^0_m$. In Salto, Méndez and Robles (2001) it is proved the completeness of $LC^0_m$ with respect to an extension of the semantics here provided for $I^0_{m2}$. Now, it is not difficult to prove completeness theorems for $I^0_{m1}$ and $I^0_{m2}$ using appropriate restrictions of the theorems and lemmas there employed (just let aside any references in Salto, Méndez and Robles (2001) to P7-P10 in the case of $I^0_{m1}$ and to P10 in the case of $I^0_{m2}$).

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## *REFERENCES*

Anderson, A.R., and Belnap, N. (1975). *Entailment: The Logic of Relevance and Necessity,* vol. 1. Princeton University Press.

Kamide, N. (2002). "A canonical model construction for substructural logics with strong negation", *Reports on Mathematical Logic*, 36, 95-116.

——— (2003). "Normal substructural logics with strong negation", *Journal of Philosophical Logic*, 32, 589-612.

Méndez, J.M. (1987). "A Routley-Meyer semantics for Converse Ackermann Property", *Journal of Philosophical Logic*, 16, 69-76.

——— (1988). "Converse Ackermann Property and semiclassical negation", *Studia Logica*, 47, 159-168.

Salto, F., J.M. Méndez, and G. Robles (2001). "Restricting the contraction axiom in Dummett's LC: a sublogic of LC with the Converse Ackermann Property, the logic $LC^0$", *Bulletin of the section of Logic*, 30, 3, 139-146.

**Gemma Robles** is a PhD candidate for the Universidad de Salamanca. Her doctoral dissertation is expected to be read at the beginning of 2005. Her publications include over twelve items on contractionless logics and semantics for different concepts of negation in *Bulletin of the Section of Logic*, *Reports on Mathematical Logic* and *Teorema*. She works with Francisco Salto and José M. Méndez in the research project directed by the latter.

**Address:** Departamento de Filosofía y Lógica y Filosofía de la Ciencia. Campus Unamuno. Edificio FES. Universidad de Salamanca, 37007 Salamanca. E-mail: gemm@usal.es.

**José M. MÉNDEZ** is Professor of Logic in the Universidad de Salamanca. He has been working ever since the eighties on relevance logics and different implicative systems. His results are published in *Journal of Philosophical Logic*, *Notre Dame Journal of Formal Logic*, *Journal of Symbolic Logic*, *Bulletin of the Section of Logic*, *Studia Logica*, *Logique et Analyse* and other research journals and contributed and edited books. From the mid-nineties onward he has published several works with Francisco Salto and/or Gemma Robles on contractionless and akin logics. His research projects and interests cover many non-classical logics and semantics.

**ADDRESS:** Departamento de Filosofía y Lógica y Filosofía de la Ciencia. Campus Unamuno. Edificio FES. Universidad de Salamanca, 37007 Salamanca. E-mail: sefus@usal.