Putnam's indispensability argument revisited, reassessed, revived*

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ABSTRACT: Crucial to Hilary Putnam's realism in the philosophy of mathematics is to maintain the objectivity of mathematics without the commitment to the existence of mathematical objects. Putnam's indispensability argument was devised as part of this conception. In this paper, I reconstruct and reassess Putnam's argument for the indispensability of mathematics, and distinguish it from the more familiar, Quinean version of the argument. Although I argue that Putnam's approach ultimately fails, I develop an alternative way of implementing his form of realism about mathematics that, by using different resources than those Putnam invokes, avoids the difficulties faced by his view.

Keywords: Indispensability, Ontological Commitment, Objectivity, Platonism, Modalism, Putnam.

RESUMEN: Es esencial para el realismo de Putnam en filosofía de la matemática el poder mantener la objetividad de la matemática sin comprometerse con la existencia de objetos matemáticos. La versión de Putnam del argumento de la indispensabilidad se concibió desde esta concepción. En este artículo reconstruyo y re-evalúo la versión del argumento de Putnam, distinguiéndolo de la versión quineana. Muestro que la propuesta de Putnam fracasa, y desarrollo una forma alternativa de articular esta forma de realismo matemático. La alternativa propuesta utiliza recursos diferentes a los de Putnam y evita así las dificultades que la propuesta de Putnam enfrenta.

Palabras clave: indispensabilidad, compromiso ontológico, objetividad, platonismo, modalismo, Putnam.

1. Introduction

Hilary Putnam's indispensability argument (1971/1979; 2012*a*), as opposed to the version of the argument implicitly found in W.V.O. Quine (1960), was not meant to establish the commitment to the existence of mathematical objects. It aimed to provide support for a realist understanding of mathematics, preserving the objectivity of mathematical practice without taking mathematical objects to exist. In Putnam's hands, and anticipating much of the current discussion regarding the indispensability of mathematics, the argument acknowledges the expressive, inferential, and (to a certain extent) explanatory role that mathematical theories —suitably interpreted— can play in scientific practice. This is achieved,

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in part, by emphasizing the importance that reference to mathematical objects plays in the formulation of scientific theories, while resisting throughout the commitment to platonism.

To avoid this commitment, Putnam recommends the adoption of a modal-structural interpretation of mathematical theories: the perspective of mathematics as modal logic (Putnam 1967/1979). On this interpretation, mathematical theories are interpreted in a modal language, and instead of stating, for instance, that an infinitude of prime numbers exist, they only express that if there were structures satisfying the axioms of Peano arithmetic, it would be true in those structures that there are infinitely many prime numbers (and, moreover, it is possible that there are such structures). With these modal translations in place, one avoids the commitment to the existence of mathematical objects, while preserving the truth of mathematical statements, despite the indispensable role of the relevant mathematics. (For a detailed development of the modal-structural interpretation, see Hellman (1989) and (1996); I will return to it below.)

In this paper, I reconstruct and reassess Putnam's indispensability argument, and argue that it ultimately fails. As will become clear, in the context of set theory, the modal-structural interpretation has commitments that require the introduction of mathematical objects, and it is unclear how these commitments can be avoided in light of the resources explicitly invoked by Putnam. As a result, Putnam's argument, as formulated, ends up leading to the same platonist conclusion found in Quine's —something that Putnam explicitly intended to avoid, albeit unsuccessfully in the end.

Despite that, I argue, not everything is lost. I develop an alternative way of implementing the project of keeping the objectivity of mathematics without mathematical objects, by using different devices than those employed by Putnam. In particular, (a) I argue that mathematical discourse has modal content quite independently of the translation into a modal language (a translation that is recommended by the modal-structural interpretation but which should be resisted for the reasons just mentioned); (b) I identify the role played by such modal content in the application of mathematics, including in explanatory contexts, with suitable interpretations of mathematical theories, which Putnam was concerned with; (c) I highlight the importance of using ontologically neutral quantifiers (see Azzouni (2004) and Bueno (2005)) as a strategy to avoid commitment to mathematical objects, and (d) I argue that neutral quantifiers offer precisely the sort of neutrality in ontological issues that shapes mathematical practice and that underlies Putnam's goal of resisting platonism.

The resulting view achieves what Putnam intended —in a nutshell, objectivity without objects— although *via* a distinct, and hopefully more successful, route. (This paper is a companion to Bueno (2013*a*), and develops in greater detail a line of argument that was only sketched at the end of that work.)

2. *The indispensability argument: Quine versus Putnam*

The indispensability argument, as used in the philosophy of mathematics, has a peculiar status. It is attributed to Quine, although Quine himself never presented the argument in an explicit form. He clearly felt the force of considerations to the effect that if reference to certain entities, such as classes, is indispensable to our best scientific theories, we cannot avoid ontological commitment to these entities (see Quine 1960). Despite that, he never explicitly formulated the argument. The argument was then presented, in greater detail, by Putnam (1971/1979), who attributed it to Quine, but did *not* fully endorse it. In the end, Mark Colyvan advanced what is now the argument's canonical formulation, but in a version which is not explicitly found in either Quine's or Putnam's writings. (For additional discussion, see Liggins (2008) and Bueno (2013*a*).)

In Colyvan's formulation (2001, 11), the argument goes as follows:

— (P_1) We ought to have ontological commitment to all and only the objects that are indispensable to our best scientific theories.
— (P_2) Mathematical objects are indispensable to our best scientific theories [in the sense that reference to such objects is indispensable to the theories in question].

Therefore, we ought to have ontological commitment to mathematical objects.

Articulated in this way, the argument has the advantage of being logically valid. The truth of the conclusion is then guaranteed by the truth of the premises. The issue then becomes whether the premises are true. The first premise, (P_1), is an articulation of Quine's criterion of ontological commitment, which adjusts such commitment exactly to those items that are indispensable to our best theories of the world. On the one hand, one need not be ontologically committed to what is not, strictly speaking, required by such theories. If the theories in question can be formulated and used in predictive and explanatory contexts *without* any reference to certain objects, then there is no requirement that the relevant objects be among the ontological commitments of the theories. Reference to such objects can be ultimately avoided, perhaps by paraphrasing away any talk of them. On the other hand, *all* objects that turn out to be indispensable to the best theories of the world are among the theories' ontological commitments. In this sense, a kind of quantificational completeness is involved here in the sense that no object whose reference can be dispensed with is among the ontological commitments of the theories.

The indispensability argument's second premise, (P_2), highlights the indispensable need to quantify over mathematical objects in order to formulate, articulate, and use our best theories of the world. According to this second premise, versions of such theories (in physics, chemistry, biology, etc.) that do not quantify over mathematical entities are not as effective, attractive, or adequate as those that do. By not invoking mathematics, these theories would be unable to function properly in their roles of expressing certain relations among events in the world, making predictions about relevant phenomena, accommodating them, or explaining their behavior. In other words, (P_2) emphasizes the functions played by mathematics in scientific theorizing and the need for quantification over mathematical objects and structures that emerges from scientific practice.

As these considerations suggest, the indispensability argument crucially relies on the conditions under which quantification over certain objects is *indispensable*. But what exactly are these conditions? I highlight three ways in which quantification over certain objects cannot be dispensed with, which correspond to three distinct roles played by such objects in scientific theorizing:

(a) *Expressive* role: Quantification over certain objects is, in many cases, made in order to express certain facts (situations, possibilities, etc.). If such expressions cannot be formulated without reference to the objects in question, the quantification is expressively indispensable.

(b) *Predictive role*: Objects are often quantified over in predictive contexts in which they are invoked to make predictions about certain phenomena. If such predictions cannot be obtained without reference to the relevant objects, the quantification is predictively indispensable.

(c) *Explanatory* role: Objects are frequently quantified over to explain certain phenomena. If the explanations in question cannot be implemented without reference to such objects, the quantification is explanatorily indispensable.

In the indispensability argument, it is in terms of these three roles that mathematics is considered as being indispensable to our best theories of the world. Quantification over mathematical objects is used to express, predict, and explain a variety of features of the world. In this way, it is indispensable, and the indispensability argument articulates and explicitly integrates these multiple roles played by mathematical objects in scientific research.

Needless to say, some of these roles are controversial. For instance, there is no agreement that mathematics plays an explanatory role in the sciences (see, for instance, Lange (2017) for a defense that it does, and Bueno and French (2012) and (2018), for the opposing view). The controversial nature of these roles is part and parcel of the debates over the indispensability of mathematics.

It was noted above that Quine (1960) used the indispensability argument as a reason to support ontological commitment to classes. On his view, these are the only abstract, mathematical objects that are indispensable to science. After all, in light of suitable set-theoretic reductions, other mathematical objects (such as functions, numbers, spaces) can be formulated in terms of classes alone. As a result, they are ultimately dispensable. In the end, Quine was a grudging platonist, who believed in the existence of a single kind of abstract (that is, causally inert, non-spatiotemporal) mathematical entity: classes. If he could have it his way, he would rather be a nominalist (see, for instance, Goodman and Quine (1947) for an attempt to develop a constructive nominalism, which could not be made to work in the end).

In contrast, Putnam has never been a platonist. At least he systematically tried to resist platonism, except perhaps in contexts in which he was presenting Quine's indispensability argument. In these contexts, he *seems* to be endorsing a platonist view. But, I argue, this is not Putnam's considered view. It is *Quine's*. Putnam is only *dialectically* presenting a view that is platonist, but not endorsing it. As he notes:

> My 'indispensability' argument [as opposed to Quine's] was an argument for the objectivity of mathematics in a realist sense —that is, for the idea that mathematical truth must not be identified with provability. Quine's indispensability argument was an argument for 'reluctant Platonism', which he himself characterized as accepting the existence of 'intangible objects' (numbers and sets). (Putnam 2012, 183)

The contrast between Quine's indispensability argument and Putnam's is very clear: they have different conclusions: the commitment to mathematical entities, according to the former, the support for the objectivity of mathematics, according to the latter.

In his 1971 work in which the indispensability argument is explicitly discussed, Putnam notes:

> So far I have been developing an argument for realism along roughly the following lines: quantification over mathematical entities is indispensable for science, both formal and physical; therefore we should accept such quantification; but this commits us to accepting the existence of the mathematical entities in question. *This type of argument stems, of course, from Quine*, who has for years stressed both the indispensability of quantification over mathematical entities and the intellectual dishonesty of denying the existence of what one daily presupposes. (Putnam 1971/1979, 347; italics added)

It is important to note that Putnam is *not endorsing* the platonist version of Quine's indispensability argument. He is just presenting Quine's formulation; nothing more. His remark to the effect that he has "been developing an argument for realism" is not an expression of the endorsement of the existence of mathematical entities, but rather the recognition that the argument that he is presenting has not been explicitly formulated by Quine. Hence additional development is needed to make the argument explicit, which is precisely what Putnam is doing. However, he also acknowledges that the type of argument that he is developing "stems, of course, from Quine". For this reason, this passage should not be interpreted as Putnam adopting Quine's indispensability argument, but simply his articulation and explicit presentation of an argument that originated in Quine's work.

Instead, Putnam's own commitments lie in maintaining a *realist* interpretation of mathematics. But, in his hands, this should be carefully distinguished from platonism. In fact, I suggest that, for Putnam, a realist account of mathematics has the following four features:

(R_1) Mathematical statements always have a truth-value.
(R_2) The truth of mathematical statements does not require the existence of (abstract) mathematical objects, that is, it does not demand platonism.
(R_3) The objectivity of mathematics should be secured without the existence of mathematical objects.
(R_4) Mathematical truth is distinct from provability.

Taken together, these four features specify a particular conception of mathematics and its cognitive status. Each feature is crucial to Putnam's approach. First, with regard to (R_1), mathematical statements need to have a truth-value so that a form of realism about mathematics can be secured. This contrasts with a constructivist approach, according to which certain mathematical statements (for instance, those involving infinities) may not have a truth-value, or may not have them until certain constructions are implemented. It is the independence of the particular truth-value that a mathematical statement may have from the methods of construction, or from the means of securing reference to mathematical objects, that characterizes a central component of the realist picture. Platonists typically satisfy this component by emphasizing that the existence of mathematical objects is independent of any methods one may devise to construct or describe them. Thus, the form of anti-realism that results from constructivism, according to which the truth of mathematical statements is dependent from the methods that are used to establish it, is one of Putnam's targets.

Second, regarding (R_2), despite the emphasis on the fact that mathematical statements are truth apt and always have a truth-value, platonism is still rejected. The truth of mathematical statements does not require a platonist ontology. After all, there is a suitable interpretation of such statements in which they come out true independently of the existence of mathematical objects. (The details of this interpretation, and the role of modality in it, will be examined below.)

Third, with respect to (R_3), as part of the realist approach that Putnam favors, the objectivity of mathematics needs to be secured and preserved. This is also implemented *via* a suitable interpretation of mathematical discourse, according to which true mathematical statements, particularly those that are taken to be so in classical mathematics, are considered to be true in the interpretation that is offered. However, this interpretation does not rely on platonism. As will become clear, mathematical objectivity is achieved independently of platonist commitments. In the end, on Putnam's view, the objectivity of mathematics does not depend on the existence of mathematical objects.

Finally, concerning (R_4), also in contrast with a constructivist approach, provability and mathematical truth are distinct. On the one hand, the notion of truth in mathematics does not presuppose platonism; on the other, provability is part of the way in which mathematical statements are typically established, but it should not be taken to be the defining feature of mathematical truth. After all, on a realist interpretation, some true mathematical statements may not be provable. Once again, this feature is secured in platonism by insisting that mathematical objects exist independently of one's ability to prove anything about them. Putnam is suspicious of this trait of platonism, and articulates a form of realism about mathematics that distinguishes truth from provability.

This raises the issue of what notion of truth is ultimately invoked by this form of mathematical realism. It seems to me that a deflationary conception should be enough for the realist about mathematics. Putnam contrasts truth as correspondence, according to which truth is a particular correspondence with reality, with a deflationary view, according to which there is nothing more to truth than instances of *T*-sentences of the form "'*P*' is true if, and only if, *P*" (Putnam 1983, xiii-xv, and references therein). Interestingly, Putnam links the correspondence view with platonism, given that the view understands

> such notions as 'refers to' and 'corresponds to' by associating these notions with Platonic objects ('correspondences') —either unique objects or else whole batches of objects. (Putnam 1983, xiii)

However, the deflationary view does not depend on such correspondences. According to Putnam, the deflationary proposal (which he calls 'disquotational view'):

> [i]s at home in a larger view on which our understanding of our first language comes about through the internalization of *assertibility conditions* and not through the learning of truth conditions in the realist sense. (Putnam 1983, xiv; italics in the original)

Thus, the deflationary view avoids the commitment to platonist notions (such as a correspondence relation) as well as truth conditions in the way the realist characterizes them. Instead, according to Putnam, the view requires the introduction of a robust, internalized notion of assertibility, given that no correspondence is involved. (This is ultimately part of Putnam's internal realist picture; see also Putnam (1981).)

The important point is that nothing as strong as a conception of truth as correspondence is needed. The realist needs to ensure the independence of one's beliefs about the world from what goes on in it. This is crucial to the objectivity of the resulting account. In order to achieve that, the notion of truth is invoked, given that in terms of it the relevant objectivity can be formulated: what is objective is what is true independently of one's beliefs. And a deflationary truth is enough for this purpose. Suppose that *P* is a true statement such that its truth-value remains what it is whether one believes that *P* or not. Given the deflationary conception, one can derive *P* from "It is true that *P*", which, in light of *P*'s content, is an expression of the independence of *P* from one's beliefs about *P*. Thus, for a statement of this sort, the truth of something is independent from the beliefs on may have about it.

Furthermore, as is well known, Putnam is critical of metaphysical conceptions of realism, and his considerations against platonism and the correspondence theory of truth are part of the same critical attitude that opposes certain metaphysical excesses. In fact, it is precisely what Putnam perceived as an incoherence in metaphysical realism, articulated in his model-theoretic argument, that provides one of the main motivations for the eventual development of internal realism (see, in particular, Putnam (1983), Chapter 1, which was originally published in 1980, and Putnam (1981)). On his view, some of the metaphysical views he opposes are blatantly incoherent. For instance, metaphysical realism, Putnam argues, ultimately leads to the absurd conclusion that every consistent theory about the world is true. This is, of course, one of the points of Putnam's model-theoretic argument (for a discussion, see, for instance, Lewis (1984) and van Fraassen (1997) and (2008)). Although internal realism and mathematical realism, as formulated above in terms of (R_1)-(R_4), are strictly independent views, they share the same philosophical motivation *vis-à-vis* both the rejection of metaphysics and the aim of developing a more coherent alternative.

Given that Putnam links a correspondence conception of truth with platonism, and in light of his goal of avoiding a platonist ontology, it is not surprising that the deflationary view of truth is then adopted throughout. This encompasses mathematical contexts and their modal interpretations. As a result, mathematical truths, including the truth of modal statements in mathematics, are understood in deflationary terms. For Putnam, assertibility conditions are internalized, to assert "'*P*' is true" just is to assert *P*. This move, however, does not invoke platonist truth conditions, which associate reference and correspondence with abstract objects (an association that Putnam intends to resist, as will become clear shortly). On Putnam's view, at least on the way I understand it, truth as correspondence, in mathematics and beyond, simply drops out of the picture altogether. (My thanks go to an anonymous reviewer from pressing me on this issue.)

3. *Mathematical modalism*

In order to resist a platonist, metaphysically inflationary, account of mathematics, Putnam advances a particular form of modalism about mathematics. On this view, mathematical statements are interpreted in a modal second-order language (Putnam 1967/1979, 43-59;

Putnam 1979/1994, 507-508; Hellman 1989; and Hellman 1996). Consider a statement to the effect that there are infinitely many prime numbers. On the modalist interpretation, it is translated into two statements:

(1) If there were structures satisfying the axioms of Peano arithmetic, it would be true in those structures that there are infinitely many prime numbers.
(2) It is possible that there are structures satisfying the axioms of Peano arithmetic.

Statement (2) is required so that (1) will not be vacuously satisfied. If (2) were false, (1) would be true even if the negation of the consequent were in place. In that case, the translation scheme would clearly fail.

As a result of the modal interpretation, or so the argument goes, it is possible to articulate an interpretation of mathematics in which the content of mathematical statements is preserved (in this sense, verbal agreement with platonism is maintained). And by applying the modal translations to each line of a proof, entire proofs of mathematical results are similarly preserved. However, the modalist insists, no commitment to mathematical objects is found, since only the possibility of suitable mathematical structures is quantified over.

In light of mathematical modalism, the four features of Putnam's mathematical realism discussed above can be maintained:

— (R_1) *Mathematical statements* have *truth-values*: According to modalism, each modal translation is true as long as the original (non-modalized) statement is true. The modal translations, after all, are truth-preserving.
— (R_2) The *truth* of mathematical statements *does not require the existence of mathematical objects*: As noted, given the modal translations, modal statements are true due to the possibility of certain structures, not the existence of mathematical entities. In this way, platonism is avoided.
— (R_3) The view secures the *objectivity of mathematics without* the existence of *mathematical objects*: Once the modal translations are implemented, whether certain statements hold or not in the relevant structures is not up to who considers such statements. It is a matter of what holds, or fails to hold, in the structures in question. This secures the objectivity of mathematics while avoiding any commitment to mathematical ontology, given feature (R_2) just discussed.
— (R_4) *Mathematical truth is distinct from provability:* On this view, whether a mathematical statement is true or not should not be equated with whether it is provable or not. Rather, mathematical truth emerges from what holds, or fails to hold, in suitable possible structures. Mathematical truth and provability, thus, come apart.

For these reasons, mathematical modalism ultimately allows Putnam to articulate the form of mathematical realism he favors, clearly satisfying each of the four features of the view. We now have all of the ingredients needed to formulate the indispensability argument that stems from Putnam's work.

4. *Putnam's indispensability argument*

In order to reconstruct Putnam's indispensability argument, it is crucial to understand that the argument was meant to provides support for a particular realist interpretation of mathematics, namely, the one that satisfies conditions (R_1)-(R_4), above. (An earlier formulation of this argument was presented in Bueno (2013*a*).)

— (P_1') All theories that quantify over objects that are indispensable to our best theories of the world ought to be interpreted realistically.
— (P_2') Mathematical theories quantify over objects that are indispensable to our best theories of the world (this includes purely mathematical theories).

Therefore, mathematical theories ought to be interpreted realistically.

In light of mathematical modalism, we have here only an argument for a realist interpretation of mathematics, not considerations in favor of platonism. For the reasons discussed above, interpreting mathematics realistically *via* modal operators is meant to avoid commitment to the existence of mathematical entities; the existence of such objects should not play any role either in the argument or in mathematical modalism.

However, Putnam's nominalization strategy of mathematics *via* modal logic faces a significant limitation: it is not clear that set theory can be successfully nominalized in this way. In the case of ZFC (Zermelo-Fraenkel set theory with the axiom of choice), the modal translations would require the truth of statements such as:

— (S_1) If there were structures satisfying the axioms of ZFC, it would be true in such structures that every set is well ordered.
— (S_2) It is possible that there are structures satisfying the axioms of ZFC.

But what are the grounds for the possibility of such structures? On the model-theoretic approach, the possibility of set-theoretic structures is understood in terms of the existence of models of set theory. These models, in turn, obtain if inaccessible cardinals are in place. Inaccessible cardinals cannot be reached from smaller cardinals by applications of the usual cardinal arithmetic operations: it is in this sense that they are inaccessible. Furthermore, the existence of inaccessible cardinals cannot be proved in ZFC (assuming ZFC's consistency), nor can it be proved, using methods formalizable in ZFC, that the existence of inaccessible cardinals is consistent with ZFC (see Jech 2003, 167-168). So, the introduction of inaccessible cardinals is a substantial assumption. But once such cardinals have been posited, it follows that if κ an inaccessible cardinal, V_κ is a model of ZFC (Jech 2003, 167-168). In other words, in the familiar model-theoretic context, the possibility of structures satisfying the axioms of ZFC is formulated in terms of inaccessible cardinals.

What grounds do we have to think that there are such cardinals? Note that it is not just the possibility of inaccessible cardinals that is at stake, but their actuality. After all, as just noted, the model-theoretic understanding of the possibility of set-theoretic structures depends on the existence of inaccessible cardinals, so that one can guarantee the existence of models of ZFC in terms of which the possibility of set-theoretic structures can be justified. But this move leads the modal interpretation straight back into platonism (assuming the usual, ontologically committing, interpretation of the quantifiers).

One alternative would be to adopt a primitive notion of possibility and resist the model-theoretic re-conceptualization of possibility in terms of the existence of suitable set-theoretic models. However, this leaves open the original issue of what grounds the possibility of structures satisfying the axioms of ZFC. Until this issue is settled, the modal translation scheme cannot be implemented for set theory.

Why should we care about this difficulty? It could be argued that if set theory cannot be fully nominalized, so much the worse for set theory. For centuries, mathematical theorizing has been conducted entirely independently (in fact, in the complete absence) of set theory: the development of the theory would not take place until the end of the nineteenth century and the beginning of the twentieth. Moreover, set-theoretic reductions can fail to provide illuminating reconstructions of particular mathematical theories. Consider, for instance, the development of graph theory entirely within set theory without invoking the usual representations of graphs in terms of arrows (the result is clearly an extraordinarily awkward and cumbersome theory) or recall the formulations of category theory that are done entirely independently of set theory (and how certain concerns about understanding categories in terms of sets will then vanish). As a result, or so the argument goes, leaving set theory behind may not be such a huge problem after all.

Although there is something to be said for the sentiment behind this objection, it is clear that set theory does play an important role in contemporary mathematical practice. Sets are typically invoked to formulate a number of mathematical theories, from linear algebra to analysis, from algebraic geometry to differential equations. And in light of the widespread use of set theory in mathematics, a nominalization strategy that does not accommodate it is, at best, seriously incomplete and, at worst, just plainly inadequate. However, for the reasons discussed above, until a proper nominalization of set theory is implemented, it is unclear that mathematical modalism can be made to work properly.

One could complain that this argument goes too far. After all, the same kind of argument provided against models of ZFC could be used to challenge that models of Peano arithmetic (PA) are grounded at all. How are models of PA (infinite models of arithmetic) actually grounded? If one considers ω, the first infinite cardinal, the argument goes, perhaps one could take it to be, from the perspective of PA, a kind of inaccessible cardinal, in the sense that it cannot be reached from smaller cardinals by applying the usual cardinal arithmetic operations. So, it is not only models of ZFC that are ungrounded, the same goes for models of PA. But then the argument, as presented above, just exceeds what is reasonable. (I owe this objection to an anonymous reviewer.)

In response, there is an important difference between the ZFC argument and the PA argument, since one could, at least in principle, ground the relevant models of PA in the world, by mapping each natural number to a physical object, assuming, for the sake of argument, that there are enough things in the world, or by mapping each number to a point in space-time, provided that they are, for the sake of argument, taken to be physical objects. However, it is not plausible to consider, even in principle, that there is such a mapping in the case of models of ZFC, for, as argued, they assume inaccessible cardinals. This requires that there be just way too many things in the world, far more than it would be plausible to expect.

This highlights a significant difference between the models of PA and those of ZFC: what these models demand is importantly different, and the demands made by the models of ZFC are significantly higher than those of PA. This means that the critique found in the

ZFC argument has a motivation that the corresponding critique in the PA argument lacks: in light of the difference between the commitments of ZFC and PA, the inaccessible cardinals required by the former are genuinely inaccessible and are importantly different from the "kind of" inaccessible cardinals hypothesized in the objection against models of PA. While there is an issue about the grounds for inaccessible cardinals in ZFC, the same issue is not present in the case of PA.

One final point. It should be noted that the addition of the modal operators to implement the modal translations yields what is arguably a very artificial maneuver. One fails to take mathematical discourse literally, since each mathematical statement needs to be rewritten for primarily philosophical reasons, and a parallel discourse is thereby created in lieu of an engagement with actual mathematical practice (Bueno 2013*b*). Instead of attempting to understand how the quantificational resources of mathematical language are employed, such quantificational apparatus is embedded into modal translations, which are nowhere to be found in actual mathematical practice. For those who take as one of the main goals of a philosophical understanding of mathematics to make sense of relevant aspects of mathematical practice, the introduction of modal translations is a step in the wrong direction, as it takes one away from reaching this goal. In contrast, an approach that avoids the introduction of translation schemes (modal or otherwise), and that takes mathematical discourse literally would be much closer to what is in principle required to accommodate mathematical practice. In what follows, I consider such an approach.

5. *An alternative*

In light of the difficulties raised to mathematical modalism, the issue emerges as to whether an alternative approach can be developed that satisfies the four features of mathematical realism discussed above, while avoiding the troubles that mathematical modalism faces. The view I consider identifies and explores three significant traits of mathematics: (a) its inherent modal character; (b) the role of modality in the application of mathematics, and (c) the significance of ontologically neutral quantifiers. After discussing each of these traits, I examine their implication to mathematical realism as understood by Putnam, and argue that the resulting view, which relies on traits (a)-(c), satisfies all of the features of mathematical realism formulated above (see conditions (R_1)-(R_4)).

5.1. Modal content of mathematics

I noted the important role that Putnam assigns to modal translations of mathematical statements in his search for a way of preserving mathematical objectivity without the commitment to mathematical objects. I also indicated some costs associated with the resulting modal translations, since they undermine the possibility of taking mathematical discourse literally and engender a parallel discourse to mathematical practice. The result, I noted, is an approach that is foreign to the practice and that fails to illuminate the way in which the relevant quantificational apparatus is in fact used. To understand significant aspects of mathematical practice, it is crucial to take the practice in its own way rather than transform it into something else that is couched in terms that have very little to do with the practice itself.

But the form of modalism about mathematics that Putnam defends touches upon an important trait of mathematical discourse: its inherent modal character. As it turns out, mathematical discourse has modal content quite independently of any translation into a modal language. To a certain extent, with the introduction of modal operators, the use of modal translations can be thought of as indirectly acknowledging the presence of this aspect of mathematical discourse, since the modal content of the language is made explicit. But this maneuver also unnecessarily reifies that content by identifying it with the modal operators that are explicitly introduced.

As it turns out, the relevant modal content of mathematics is both more widespread and subtler than the explicit modal translations can capture. It is more widespread in that the modal content is not restricted to, nor can it be typically found in, the logical form of actual mathematical statements. These statements, as used by mathematicians, are typically not overtly modal: their logical form is not usually of necessity or possibility statements, but rather they usually are couched in a non-modal language. This fact yields the (misleading) appearance that mathematics is not modal after all. The modal content is also subtler than the introduction of modal operators would suggest since mathematical statement express possibilities and impossibilities even if their logical form is not explicitly modal.

Consider, for instance, the modal content of *reductio* proofs. They establish the impossibility of certain mathematical configurations, on pain of inconsistency. For example, in his work on number theory, Euclid established that there are infinitely many prime numbers (a prime number is a whole number that is divisible without remainder only by 1 and itself). One of the familiar arguments to establish this result relies on *reductio ad absurdum*. It is assumed, for *reductio*, that there are only finitely many prime numbers, which are listed in the sequence: $p_1, p_2, ..., p_n$. The proof proceeds by providing a construction to the effect that a prime number that is not on that list *can always be obtained.* Let $P = (p_1 \cdot p_2 ... p_n) + 1$ and suppose that p is a prime number that divides P. Note that p is different from each primes $p_1, p_2, ..., p_n$. If this were not the case, it would be possible to divide the difference $(P - (p_1 \cdot p_2 ... p_n)) = 1$. But this is not possible. Thus, p is a new prime not on the original list of primes $p_1, p_2, ..., p_n$. However, this contradicts the assumption that the original list comprised all prime numbers. (For a discussion, see Krantz 2011, 42-43.)

Note the modal content involved already in the very structure of the proof: the argument yields the *possibility* of systematically obtaining a prime number not on the initial allegedly finite list of primes. In this sense, mathematical constructions have a significant modal content: they provide a way to establish the *possibility* of identifying certain mathematical objects. In this case, as noted, what follows from such a construction is a contradiction, given the assumption that that there is a finite list of all prime numbers. But if the assumption of the finiteness of primes leads to an absurdity, given the characterization of such numbers, their infinity *must* hold (assuming, as usual, classical logic). As a result, the very statement of the theorem also has modal content, given the *impossibility* of exhibiting all prime numbers in a finite list.

Of course, there is nothing special about this example, which is entirely typical of cases involving mathematical constructions, whether in the context of *reductio* proofs or not. The modal content of mathematics is, in fact, ubiquitous: it is found, just as in the example above, in the formulation of mathematical theorems, in the articulation of mathematical constructions, and in the implementation of mathematical proofs. In the end, there is no need to introduce modal operators as suggested by Putnam's modal interpretation of mathematics: mathematics, in its theorems, constructions, and proofs, already has a modal content of its own.

It may be objected that this is misleading. If we were to take seriously the modal content of mathematics, we would need to add suitable modal operators to the relevant discourse to indicate precisely where the relevant modalities are introduced. But this would simply bring us back to Putnam's formulation of mathematics in terms of modal logic (which was developed further, as noted above, by Hellman). How is the alternative I suggest any different?

In response, I resist the assumption that it is only by explicitly adding modal operators that the relevant modalities are introduced. This assumption gets the situation backwards. The modality is already part of mathematics, given the content of the relevant statements. That there is an infinitude of prime numbers is the expression of an *impossibility*, namely, that it is *not possible*, given what prime numbers are and given classical logic, that there is a finite list of all the primes. Thus, the modality is already part of the content of the statement even if the statement itself is not explicitly formulated as a modal one. It is because of the underlying modal content that one can, in principle, add a modal operator to the relevant statement and preserve its truth-value. But to add this modal operator is to provide an artificial reconstruction of the discourse and, as a result, not to take it literally. Hence, this reconstruction is not strictly needed, given the modal content of the statement, and adds an artificial layer of interpretation to mathematical discourse.

To argue otherwise is to assume that there is something special about the formal rendition of a mathematical statement. But the formal rendition only works because the statement under consideration is already modal. This means, as noted, that the addition of modal operators is ultimately unnecessary, and the form of modalism I favor highlights this point. (My thanks go to an anonymous reviewer for pressing this issue.)

5.2. Modality and the application of mathematics

The modal content of mathematics plays a role not only in situations in which pure mathematics is involved, but also in contexts in which the application of various theories and models is at issue. This includes, of course, explanatory contexts, in which suitable interpretations of mathematical theories are crucially invoked.

Consider, for instance, the use of Hilbert spaces in the foundations of non-relativist quantum mechanics. What prompted, in 1932, John von Neumann to introduce such spaces in the formulation of quantum theory was precisely the realization that the modal content provided by Hilbert spaces could be put to an effective use. Until then, matrix and wave mechanics, the two main formulations of quantum mechanics at that point, could not be consistently shown to be equivalent, given that the underlying mathematical spaces invoked by each theory were clearly non-isomorphic: one being discrete, the other continuous. However, von Neumann noted, if one considers the space of functions defined over each of the corresponding spaces, that is, over the respective spaces of matrix and wave mechanics, two Hilbert spaces are obtained. They can, in turn, be shown to be isomorphic (von Neumann 1932). Together with the expressive properties of Hilbert spaces that allow for the representation of quantum states (among other features of the relevant physical systems), this motivates the idea of taking Hilbert spaces as the underlying mathematical framework to formulate quantum mechanics. Such spaces have a suitable structure that allows for the *possibility* of representing a number of features of quantum mechanics (from quantum states to probabilities). And by invoking this framework, the mathematical equivalence between matrix and wave mechanics can then be established (for further details and references, see Bueno and French 2018).

In the end, the Hilbert space formalism provides a significant surplus structure, that is, additional relations, properties, and objects, which are offered by this mathematical theory. The surplus structure may initially seem to be idle, but with suitable interpretations, it ends up playing an important role. In fact, it is this surplus structure that allows von Neumann to establish the mathematical equivalence between matrix and wave mechanics, by specifying an isomorphism between the relevant Hilbert spaces.

These considerations illustrate the connection between surplus structure and the modal content of mathematics: it is in terms of such a surplus that certain applied results *can be* obtained, since they allow for the possibility of representing certain physical situations in terms of suitably interpreted mathematical relations. In the end, many cases of the application of mathematics rely on the modal content provided by surplus structure along these lines (see Bueno and French (2018) for additional discussion of this issue in the context of applied mathematics).

5.3. Ontologically neutral quantifiers

As opposed to the introduction of modal operators recommended by Putnam's modal interpretation, a better strategy to avoid commitment to mathematical objects is articulated by the use of ontologically neutral quantifiers (Azzouni 2004; Bueno 2005).

The central idea is that quantification does not require the existence of the objects that are quantified over (even those that are indispensable to our best theories of the world). After all, universal quantification can be thought of as quantification over all of the domain, independently of the existence of the objects that are quantified over. In turn, existential quantification is quantification over some of the domain, with again no assumption regarding the existence of the relevant objects.

In order to express the existence of something, one invokes an *existence predicate*, a predicate that specifies sufficient conditions for something to exist. If necessary conditions were also introduced, it would be difficult to avoid begging the question against certain views. For instance, the capacity of entering into causal relations can be thought of as being sufficient for the existence of objects: nonexistent objects are not causally active. However, if this condition were also taken to be necessary for the existence of something, clearly one would thereby beg the question against platonist views, for abstract objects, despite being causally inert, are taken to exist on these views.

Note that quantification (even indispensable quantification) is often recognized as not being ontologically committing. If it is claimed that "Some sets are too big to exist", one is not thereby asserting the existence of sets that do not exist, which would be clearly inconsistent. Rather, one is noting that, among the sets, some, those that are significantly big, do not exist. Clearly, this recognizes the possibility of quantifying over objects that do not exist.

Similarly, if one asks physicists whether, due to the role played by vectors in Hilbert spaces in the formulation of quantum mechanics, they believe in the existence of such vectors, they will typically deny such commitment. Their commitment (if any) is with the relevant physical objects, their properties and relations, rather than with the underlying mathematical setting.

It may be argued that even if one invokes ontologically neutral quantifiers, this would not fully avoid commitment to mathematical objects. After all, mathematical practice also involves singular reference. Some mathematical statements involve singular terms, such as the irrational number π or the complex number *i*, and if these terms successfully refer, the corresponding objects exist, independently of the use of neutral quantification.

In response, note that the objection assumes that reference, in particular singular reference, requires the existence of the objects that are referred to. So, just by referring to π or *i*, one would thereby refer to an existing object and, as a result, existing abstract entities would be invoked. But why should one assume that successful reference requires the existence of the objects that are referred to? Similarly, why should one assume that successful quantification requires the existence of the objects that are quantified over? The answer, in both instances, is that there is no need to make such an assumption (see Azzouni 2004). It is often the case that one successfully refers to objects that do not exist, such as Sherlock Holmes or Lady Macbeth, just as one successfully quantifies over objects that do not exist, such as ghosts or witches. In either case, existence should not be required for successful reference or quantification. Otherwise, contradictions would emerge.

Consider, for instance, the sentence:

(a) Sherlock Holmes does not exist.

If reference to Sherlock Holmes required its existence, (a) would be contradictory: the sentence denies the existence of the object that the singular term would demand. Similarly, if quantification required the existence of what is quantified over, the sentence

(b) There are some sets that do not exist.

would be contradictory as well. The sentence denies the existence of the objects that the quantification would demand. Since neither (a) nor (b) is contradictory, in fact both seem to be plainly true, neither reference nor quantification requires existence in the end. (My thanks go to an anonymous reviewer for pressing me on this issue.)

As a result, it is possible to avoid commitment to mathematical objects despite granting that quantification over them is indeed indispensable to our best theories of the world. This provides a strategy to resist ontological commitment to mathematical entities without the need to introduce modal operators (or other devices to rewrite mathematical theories), along the lines recommended by Putnam's approach. Note that the objectivity of mathematics is still preserved: it is not up to us what follows from certain mathematical principles and a given logic (all formulated via ontologically neutral quantifiers). I will return to this point below.

5.4. Putnam's mathematical realism revisited

Once introduced, neutral quantifiers offer precisely the sort of neutrality in ontological issues that shapes mathematical practice and which can be used to realize Putnam's goal of resisting platonism while preserving the objectivity of mathematics. All four features of mathematical realism, as conceptualized by Putnam, are satisfied without the difficulties that the particular implementation of his view faces. I will consider them briefly in turn.

(i) *Mathematical statements have truth-values*: Ontologically neutral quantifiers do not change the truth-value of mathematical statements. A statement to the effect that there are infinitely many prime numbers will come out true when ontologically neutral quantifiers are used. A finitude of primes, we noted above, given the way prime numbers have been formulated and given a logic (which, in mathematical practice, is typically classical), leads to a contradiction. The infinitude of primes is obtained inferentially from the relevant conception of prime numbers.

(ii) *The truth of mathematical statements does not require the existence of mathematical objects*: Ontologically neutral quantifiers avoid the commitment to mathematical entities. Nothing in mathematical practice requires the existence of mathematical objects, particularly in the metaphysical sense articulated by platonists, in which existence is often understood in terms of ontological independence (mathematical objects would exist even if no one ever thought of them or described them). If we consider the proof of the infinitude of prime numbers mentioned above, nowhere in the proof do we find a commitment to such a metaphysical conception of existence. In particular, the truth of the relevant mathematical statement (the infinitude of primes) is a function of the way in which the objects in question (prime numbers) have been formulated and conceptualized.

Of course, there is no way of referring to mathematical objects (prime numbers or other entites) but *via* some description or other. However, this fact does not support a platonist interpretation of mathematics, which adds to the conceptualization of mathematical objects found in mathematical practice a particular metaphysics, namely, mathematical entities are thought of as being abstract and as existing independently of any description or reference to them. Nothing in the practice *requires* such additions (although they are consistent with the practice). Given that platonism is not required, in light of ontologically neutral quantifiers, it can be resisted.

(iii) The view secures the *objectivity of mathematics without* the existence of *mathematical objects*: Once certain mathematical principles are introduced, and a logic is adopted, it is not up to us whether certain statements hold or not. It is a matter of what follows or not from the principles in question. Whether all sets are well-ordered or not depends on whether the axiom of choice is assumed or not. If it is, then they all are; otherwise, they are not. One cannot simply stipulate such a result: it follows (or not) in light of the relevant assumption. This provides all of the objectivity that is needed in mathematics, without any commitment to a platonist ontology, given the use of neutral quantifiers.

(iv) *Mathematical truth is distinct from provability*: Whether a mathematical statement is true or not cannot be equated with whether it is provable or not. Consider, for instance, the Gödel sentence: its truth is manifest given the content of the sentence, even though it is not probable from the relevant principles. But being able to prove certain mathematical results provides the most common form of epistemic access to the truth of these results. Despite that, mathematical truth is not characterized by provability, given that some true mathematical statements cannot be proved. Truth and provability do come apart.

Finally, ontologically neutral quantifiers can be used without difficulty in set theory. After all, such quantifiers bear no ties to any particular mathematical theory. As a result, the difficulty of making sense of the possibility of certain set-theoretic structures, which, as noted above, challenged the modal interpretation of mathematics, does not emerge. The account I propose does not require the introduction of modal operators. Furthermore, there is no need to rewrite mathematical theories *via* such operators, given that mathematics already has modal content, and instead of modal operators, ontologically neutral quantifiers are used to avoid commitment to the existence of mathematical entities. In this way, such quantifiers allow one to articulate the form of mathematical realism that Putnam favors without platonism and without the problems that undermined his view.

6. *Conclusion*

The proposal sketched here, which combines the primitive modal content of mathematics with ontologically neutral quantifiers, realizes Putnam's goal of securing the objectivity of mathematics without the commitment to mathematical objects. It explores a different, and hopefully more successful, path than Putnam's, but preserves the crucial features of his approach. Although more needs to be said, I hope enough was said to suggest that this is a path well worth exploring further.

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