SYSTEMS WITH THE CONVERSE ACKERMANN PROPERTY

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ABSTRACT

A system S has the "converse Ackermann property" (C.A.P.) if (A → B) → C is unprovable in S whenever C is a propositional variable. In this paper we define the fragments with the C.A. P. of some well-know propositional systems in the spectrum between the <u>minimal</u> and <u>classical logic</u>. In the first part we succesively study the implicative and positive fragments and the full calculi. In the second, we prove by a matrix method that each one of the systems has the C.A.P. Thus, we think the problem proposed in Anderson & Belnap (1975) § 8.12 has been solved.

1.1 <u>Implicative fragments</u>

The axioms of each system are instances of some subset of the following set of axioms:

A1. ((A → A) → B) → B

A2. (A → (A → B)) → (A → B)

A3. (B → C) → ((A → B) → (A → C))

A4. (A → B) → ((B → C) → (A → C))

A5. A → A

A6. A → ((A → B) → B)

A7. A → (A → A)

A8. A → (A → A)

A9. B → (A → A)

A10. B → (A → A)

A11. ((A → B) → A) → A

A12. A → (B → A)

Where B is an <u>implicative formula</u> in A1, A2, A6, A9 and A11; and A is an <u>implicative formula</u> in A7 and A11. A formula is <u>implicative</u> if it is of the form B → C where B and C are any formula. The systems are axiomatized as follows:

C.T°: A2, A3, A4 and A5

C.E°: A1, A2 and A4

C.EM°: C.E° plus A7

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C.R°: A2, A3, A5 and A6

C.RM°: C.R° plus A8

C.S3°: C.E° plus A9

C.S4°: C.E° plus A10

C.S5°: C.S4° plus A11

C.I°: C.E° plus A12

C.C°: C.I° plus A11

These systems are implicative fragments with the C.A.P. of (respectively) Ticket entailment (T), Entailment (E), Entailment-Mingle (E.M.), Relevance Logic (R), Relevance-Mingle (R.M.) (See Anderson and Belnap 1975) S3, S4 and S5 of Lewis, Intuitionistic Logic (I) and Classical Logic (C).

Following Anderson and Belnap (1975, § 8.11) we consider as minimal logic (C.M.L.) the system axiomatizad by A3, A4 and A5. If we add to C.M.L. the schema

$$\text{(i)}\ (A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C)$$

or the schema

$$\text{(ii)}\ (A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$$

we get implicative Ticket entailment (C.T.) in which (iii) below is a theorem. If we replace (i) or (ii) in C.T. by

$$\text{(iii)}\ (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$$

we get a subsystem of C.T. (C.Tm) as shown by the following matrix (2 and 3 are designated values)

→	0	1	2	3
0	3	3	3	3
1	0	2	3	3
2	0	0	3	3
3	0	0	0	3

that verifies C.Tm but falsifies (i) and (ii) only when

$$v(A) = v(B) = v(C) = 1$$

Now, C.M.L. certainly has the C.A.P. but Anderson and Belnap have shown that the C.A.P. is not predicable of C.T. In fact, the C.A.P. is not even a property of C.T.m because a formula of the form $(A \rightarrow B) \rightarrow p$, namely:

$$\{\{((A \rightarrow (A \rightarrow A)) \rightarrow ((A \rightarrow A) \rightarrow (A \rightarrow A))\} \rightarrow A\} \rightarrow A$$

is a C.Tm-thesis, as the reader may prove.

Observe that, in consecuence, none of the systems we have defined includes C.Tm though each of them has restricted versions of (i), (ii) and (iii): (i) and (iii) are theses when C is an implicative formula, (ii) is this when B is an implicative formula.

Note, finally, in A11, A and B must be implicative formulae because if we let B be a propositional variable we can prove (ii) with the aid of A4.

1.2 Conjuntion, Disjunction

We add to each one of the implicative systems in § 1.1 the axiom-schemata:

A13. $A \to (A \vee B)$

A14. $B \to (A \vee B)$

A15. $((A \to C) \& (B \to C)) \to ((A \vee B) \to C)$

A16. $(A \& B) \to A$

A17. $(A \& B) \to B$

A18. $((A \to B) \& (A \to C)) \to (A \to (B \& C))$

A19. $(A \& (B \vee C)) \to ((A \& B) \vee C)$

and the rule: if A and B, then A & B (Adjunction) we note that $A \to (B \to (A \& B))$

inmediately follows in $I^\circ_+ (C^\circ_+)$ (It is not a thesis in the remainig systems) Therefore, A19 is not independent in I°_+, C°_+.

1.3 Negation

In the treatment of negation for the systems defined in § .2 we must distinguish those with the Ackerman property from those without it: $S3^\circ_+$, EM°_+, E°_+ and T°_+ are in the first class, the remaining systems in the second. The reason for this distinction is that if we add, as is presumable,

A20. $(\sim A \to B) \to (\sim B \to A)$

A21. $A \to \sim\sim A$

A22. $(A \to \sim A) \to \sim A$ (A is an implicative formula)

as schemata for negation, any system without the Ackerman property (that is, with a thesis of the form $p \to (B \to C)$) will collapse into a system without the C.A.P., as shown by the following deduction

1. $p \to (B \to C)$
2. $\sim p \to (B \to C)$
3. $\sim (B \to C) \to p$
4. $((B \to C) \to \sim(B \to C)) \to \sim (B \to C)$
5. $((B \to C) \to \sim(B \to C)) \to p$

where (5) is of the form $(A \to B) \to p$. So, though we can undoubtebly add A20, A21 and A23 to $S3^{\circ}_{+}$, EM°_{+}, E°_{+} and T°_{+} (See §2), we can add these schemata to none of the remaining systems. Thus, if we want an unified "concept" of negation for systems with the C.A.P., we are left, we think, with two possibilities: one is a "semi-classical" type of negation axiomatizable by A20 and A21; the other a semi-intuitionistic" one that can be axiomatized by

A23. $(A \to \sim B) \to (B \to \sim A)$

and

A24. $(A \to \sim A) \to \sim A$

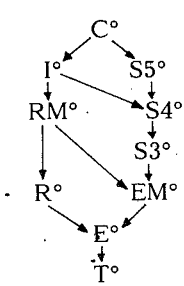
We note that

$$A \to (\sim A \to B)$$

is derivable in $I^{\circ}_{+}(C^{\circ}_{+})$ plus the semi-classical concept of negation (Not in the other systems) but it cannot be added to the same systems if the negation is the "semi-intuitionistic" one: I°_{+} and C°_{+} would collapse into systems without the C.A.P.

2. Matrices

The systems in §1.3 can deductively be arranged (exactly as its unrestricted counterparts) in the following diagram where the arrow (→) stands for set inclusion



for the diagram to hold, it is of course understood that only one of the types of negation (no matter which) is added to each of the positive fragments.

On the other hand, it is easy to see that these systems are different from each other: T°, for instance, is a subsystem of T but A1 is not a thesis of T; consecuently, T° does not include E°, etc.

We can now prove that C°_{+} both with semi-classical and semi-intuitionistic negation has the C.A.P. Thus, all positive systems with one or the other type of negation have the C.A.P. Further, we can prove that $S3^{\circ}_{+}$ plus A20, A21 and A22 has the C.A.P.; therefore EM°_{+}, E°_{+} and T°_{+} with A20, A21 and A22 have also the C.A.P.

The set of matrices (2 is the designated value)

→	0	1	2
0	2	0	2
1	2	2	2
2	0	0	2

&	0	1	2
0	0	1	0
1	1	1	1
2	0	1	2

v	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

	∿
0	0
1	2
2	1

verifies C°_{+} plus the semi-classical type of negation; the same set for →, & v and

	∿
0	2
1	2
2	0

verifies C°_{+} plus the semi-intuitionistic one.

The set (with 2 and 3 as designated values)

→	0	1	2	3
0	2	2	2	2
1	1	2	2	2
2	1	1	2	2
3	1	1	1	2

&	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

v	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

	∿
0	3
1	2
2	1
3	0

verifies $S3^{\circ}_{+}$ plus A20, A21 and A22.

Now, if (A → B) → C is any formula in which C is a propositional variable, assign C the value 1 and 0 for the first and second set of matrices respectively. Then, the value of (A → B) → C is 0 and 1 respectively, no matter the value of A → B.

REFERENCES

ANDERSON, A.R. and BELNAP, N.D.,Jr.: Entailment: The Logic of Relevance and Necessity. Princeton University Press, Princeton N.J. 1975.

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