TWO EXTENSIONS OF LEWIS' S3 WITH PEIRCE LAW†

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ABSTRACT: We define two extensions of Lewis' S3 with two versions of Peirce's Law. We prove that both of them have the Ackermann Property.

Keywords: Peirce's Law, modal logic, non-classical logics.

CONTENTS

1. Introduction

Peirce's Law is indeed a peculiar implicational thesis which is also of capital interest in general propositional logic. In (Salto, Méndez 1999) we have discussed the effect of Peirce's Law on some modal, intuitionistic and relevance logics. The aim of this note is to present two results posing two problems. Let us refer by PC ["Peirce classical"] and by PS5 [Peirce S5"] to respectively, [the symbol $\rightarrow$ will denote here strict implication]

(PC) $[(A\rightarrow B)\rightarrow A]\rightarrow A$

and

(PS5) $[[(A\rightarrow B)\rightarrow C]\rightarrow(A\rightarrow B)]\rightarrow(A\rightarrow B)$

Now, let $S3P_1$ ($S3P_2$) be the result of adding PS5 (PC) to Lewis'S3. We shall prove:

(i) $S3P_1$ and $S3P_2$ are different logics, the first one included in the second.
(ii) $S3P_1$ is not included in (nor includes) Lewis' S4.
(iii) $S3P_1$ is included in (but does not include) Lewis' S5.
(iv) $S3P_2$ is not included in (nor includes) Lewis' S4 or Lewis' S5.
(v) Both systems [$S3P_1$ and $S3P_2$] do have the Ackermann Property.

The possible interest of $S3P_1$ and $S3P_2$ is to be founded (among other reasons) as follows. Anderson and Belnap have reasoned at length that the Ackermann Property is necessary in any logic deserving to be called a logic of entailment (see (Anderson, Belnap 1975)). Now, though S4 and S5 do not have this property [the S4 axiom $B\rightarrow(A\rightarrow A)$ being an inmediate counterexample], it is however certainly predicable (see §5) of both $S3P_1$ and $S3P_2$ [therefore, also of S3]. Then, granting as Lewis did (see (Lewis, Langford 1959)) the admissibility of paradoxes of strict implication [i.e., leaving aside questions of relevance], $S3P_1$ and $S3P_2$ are two strong logics of strict implication. Hence, it is natural to pose the problems:

(vi) Is $S3P_1$ ($S3P_2$) an admissible implicational logic in Lewis' sense?
(vii) Which are the semantics for $S3P_1$ and $S3P_2$?

2. Lewis' S3

Lewis' S3 can be axiomatized with (see (Lewis, Langford 1959), (Méndez 1988), (Salto, Méndez 1999)):

Axioms:

A1. $A\rightarrow A$
A2. $(A\rightarrow B)\rightarrow[(B\rightarrow C)\rightarrow(A\rightarrow C)]$
A3. $(B\rightarrow C)\rightarrow[(A\rightarrow B)\rightarrow(A\rightarrow C)]$
A4. $[A\rightarrow(A\rightarrow B)]\rightarrow(A\rightarrow B)$
A5. $(A\rightarrow B)\rightarrow[(C\rightarrow D)\rightarrow(A\rightarrow B)]$
A6. $A\rightarrow(A\vee B)$
A7. $B\rightarrow(A\vee B)$
A8. $[(A\rightarrow C)\wedge(B\rightarrow C)]\rightarrow[(A\vee B)\rightarrow C]$
A9. $(A\wedge B)\rightarrow A$
A10. $(A\wedge B)\rightarrow B$

A11. [(A→B)∧(A→C)]→[A→(B∧C)]
A12. (A→¬A)→¬A
A13. (¬A→B)→(¬B→A)
A14. A→¬¬A

Rules of derivation:

Modus ponens: if ⊢A and ⊢A→B, then ⊢B.
Adjunction: if ⊢A and ⊢B, then ⊢A∧B.

3. $S3P_1$ and $S3P_2$

As noted above, $S3P_1$ is the result of adding

(PS5) [[(A→B)→C]→(A→B)]→(A→B)

to S3, and $S3P_2$ the result of adding

(PC) [(A→B)→A]→A

to S3.

4. Peirce's Law and Lewis' S3

Consider the following set of matrices (I):

→	0	1	2	3
0	2	2	2	2
1	0	2	0	2
*2	0	0	2	2
*3	0	0	0	2

∧	0	1	2	3
0	0	0	0	0
1	0	1	0	1
*2	0	0	2	2
*3	0	1	2	3

∨	0	1	2	3
0	0	1	2	3
1	1	1	3	3
*2	2	3	2	3
*3	3	3	3	3

	¬
0	3
1	2
*2	1
*3	0

where designated values are starred. This set verifies $S3P_1$ but falsifies the S4 axiom B→(A→A) [v(A)=2, v(B)=3] and also PC [v(A)=1, v(B)=0].

On the other hand, the set of matrices (II), where designated values are starred:

→	0	1	2	3
0	2	2	2	2
*1	0	2	0	2
*2	0	2	2	2
*3	0	0	0	2

∧	0	1	2	3
0	0	0	0	0
*1	0	2	0	1
*2	0	2	2	2
*3	0	1	2	3

∨	0	1	2	3
0	0	1	2	3
*1	1	1	1	3
*2	2	1	2	3
*3	3	3	3	3

	¬
0	3
*1	2
*2	1
*3	0

verifies $S3P_2$ but falsifies the S4 axiom $B\rightarrow(A\rightarrow A)$ [v(A)=2, v(B)=1]. From these facts we inmediately have:

(a) $S3P_1$ and $S3P_2$ are different logics, the first one included in the second.
(b) $S3P_1$ does not include S4.
(c) $S3P_2$ does not include S4.

Now, it is known that S4 is S3 plus the S4 axiom $B\rightarrow(A\rightarrow A)$ and S5 is S4 plus PS5. It is also known that S3, S4 and S5 are different logics, S3 being included in S4, the latter in its turn included in S5, from which PC is not derivable. Then, we inmediately have:

(d) $S3P_1$ does not include S5.
(e) $S3P_2$ does not include S5.
(f) $S3P_1$ is not included in S4.
(g) $S3P_1$ is included in S5.
(h) $S3P_2$ is not included in S4.
(i) $S3P_2$ is not included in S5.

Thus we have, as promised, propositions (i)-(v) stated in the introduction.

5. Ackermann Property

A logic L has the Ackermann Property (AP) just in case $A\rightarrow(B\rightarrow C)$ is not a theorem of L when A is a propositional variable. According to Anderson and Belnap, a logic L is not a logic of implication if AP is not predicable of L (see (Anderson, Belnap 1975)). In this sense, the sets of matrices I and II verify, respectively, $S3P_1$ and $S3P_2$. Now, let $A\rightarrow(B\rightarrow C)$ be any wff where A is a wff in which neither $\rightarrow$ nor $\neg$ appear. Assign all variables in A the value 1. Then, $v(A\rightarrow(B\rightarrow C))=0$ no matter the value of B and C. Thus, $S3P_1$ and $S3P_2$ both have the AP.

Notes

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