A PRESERVATION THEOREM FOR EQUALITY-FREE HORN SENTENCES†

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ABSTRACT: We prove the following preservation theorem for the Horn fragment of Equality-free Logic:

Theorem 0.1. For any sentence $\sigma \in L$, the following are equivalent:

i) σ is preserved under H_S, H_S^{-1} and P_R.

ii) σ is logically equivalent to an equality-free Horn sentence.

Keywords: model theory, equality-free logic, Horn sentence, preservation theorems.

CONTENTS

1. Introduction

This paper is a contribution to the Model Theory of languages without equality. *Equality-free logic* is the fragment of first-order logic composed by all the formulas that do not contain the equality symbol. In recent times, research in the areas of Algebraic Logic and Computer Science has attracted the attention on some fragments of equality-free logic. In this article we prove a preservation theorem for the Horn fragment of this logic. Let L be a first-order language. It is said that a formula $\phi \in L$ is a *basic Horn formula* if ϕ is a disjunction of formulas

$$\chi_1 \vee \dots \vee \chi_n$$

where at most one of the formulas χ_i is atomic, the rest being negations of atomic formulas. A *Horn formula* $\phi \in L$ is built up from basic Horn formulas by means of the connective $\wedge$ and the quantifiers $\forall$, $\exists$. H.J. Keisler proved, assuming the continuum hypothesis, that a first-order sentence is logically equivalent to a Horn sentence if and only if it is preserved under reduced products, see [13], and then F. Galvin eliminated this hypothesis, see [9]. There are two more proofs of this theorem, one was obtained by R. Mansfield, using boolean-valued models, see [14]. The other is due to S. Shelah, see [19]; he indicates that the theorem could be obtained by the technique he introduced to prove that two elementary equivalent models have isomorphic ultrapowers. We prove the following result:

Theorem. For any sentence $\sigma \in L$, the following are equivalent:

i) σ is preserved under $\mathbf{H}_S$, $\mathbf{H}_S^{-1}$ and $\mathbf{P}_R$.

ii) σ is logically equivalent to an equality-free Horn sentence.

Our proof follows the main lines of Keisler and Galvin proof. Independently, G.C. Nelson has given a proof of the same result in [16], using the indications of S. Shelah. The notion of *quasi-isomorphism* which he uses is equivalent to our notion of *relative correspondence*. Some other recent works on equality-free logic are [4], [5], [6] [7] and [8].

The following notation will be used in this work. From now on L will be a similarity type with at least one relation symbol. We denote by L the set of first-order formulas of type L and by L_0 the set of quantifier-free formulas of L. Given a class Σ of formulas, let us denote by Σ^- the class of equality-free formulas of Σ, that is, the class of all formulas of Σ that do not contain the equality symbol. Given L-structures $\mathcal{A}$ and $\mathcal{B}$ and a class Σ of formulas, we write $\mathcal{A} \equiv_\Sigma \mathcal{B}$ and $\mathcal{A} \equiv_\Sigma^- \mathcal{B}$ to mean that $\mathcal{A}$ and $\mathcal{B}$ satisfy exactly the same sentences Σ and Σ^-, respectively. In case that Σ is one of the classes L, L_0 we will abbreviate the expression using $\equiv^-$ and $\equiv_0^-$, respectively. For any L-structure $\mathcal{A}$ and any set $B \subseteq A$, we denote by $L(B)$ the similarity type obtained from L by adding a new constant symbol for each element of B and we denote by $\mathcal{A}_B$ the natural expansion of $\mathcal{A}$ to $L(B)$, where every new constant denotes its corresponding element. For the sake of clarity we use the same symbol for the constant and for the element that is denoted by the constant. $|A|$ denotes the power of the set A. Given a structure $\mathcal{A}$, for any $B \subseteq A$, $\langle B \rangle$ denotes the substructure of $\mathcal{A}$ generated by B.

Three notions are important in the proof of the preservation theorem for equality-free languages. These three notions are: strict homomorphism, relative correspondence and filter-product. The paper concentrates on the study of filter-products, notion that plays in equality-free logic the same role that reduced products play in logic with equality. The other two notions have been studied in [1], [2], [17], [16], [5]. Now we introduce and give a characterization of the concept of strict homomorphism.

Definition 1. *If $\mathcal{A}$ and $\mathcal{B}$ are L-structures, it is said that an homomorphism $h: \mathcal{A} \to \mathcal{B}$ is* strict *if for any n-adic relation symbol $R \in L$ and any $a_1,\ldots,a_n \in A$,*

$$<a_1,\ldots,a_n> \in R^{\mathcal{A}} \Leftrightarrow <h(a_1),\ldots,h(a_n)> \in R^{\mathcal{B}}.$$

Proposition 1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be L-structures and $h: \mathcal{A} \to \mathcal{B}$ an homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Then the following are equivalent:*

i) *h is a strict homomorphism.*

ii) *For any atomic formula $\phi \in L^{-}$, $\phi = \phi(x_1,\ldots,x_n)$, and for any sequence $a_1,\ldots,a_n$ of elements of A,*

$$\mathcal{A} \models \phi[a_1,\ldots,a_n] \quad \text{iff} \quad \mathcal{B} \models \phi[h(a_1),\ldots,h(a_n)].$$

ii) *For any formula $\phi \in L^{-}$, $\phi = \phi(x_1,\ldots,x_n)$, and for any sequence $a_1,\ldots,a_n$ of elements of A,*

$$\mathcal{A} \models \phi[a_1,\ldots,a_n] \quad \text{iff} \quad \mathcal{B} \models \phi[h(a_1),\ldots,h(a_n)].$$

Given a class K of L-structures we define the following classes of L-structures: $\mathbf{P}_{\mathrm{R}_\kappa}(K)$, the class of all reduced products, over proper κ-complete filters, of systems of members of K; $\mathbf{H}_\mathrm{S}(K)$, the class of all strict homomorphic images of members of K and $\mathbf{H}_\mathrm{S}^{-1}(K)$, the class of all strict homomorphic counter-images of members of K. We suppose that every one of the above classes is closed under isomorphic images and that the reduced products are of non-empty systems.

Now we introduce a notion that plays in equality-free languages the same role that the isomorphism relation plays in logic with equality.

Definition 2. *Let $\mathcal{A}$ and $\mathcal{B}$ be L-structures. A relation $R \subseteq A \times B$ is a relative correspondence between $\mathcal{A}$ and $\mathcal{B}$ if* $\mathrm{dom}(R) = A$, $\mathrm{rg}(R) = B$ and

(1) for any constant $c \in L$, $c^{\mathcal{A}} R c^{\mathcal{B}}$,

(2) for any n-adic function symbol $f \in L$, any $a_1,\ldots,a_n \in A$ and any $b_1,\ldots,b_n \in B$ such that $a_i R b_i$ for each $i = 1,\ldots,n$,

$$f^{\mathcal{A}}(a_1,\ldots,a_n) R f^{\mathcal{B}}(b_1,\ldots,b_n),$$

(3) for any n-adic function symbol $S \in L$, any $a_1,\dots,a_n \in A$ and any $b_1,\dots,b_n \in B$ such that $a_i R b_i$ for each $i = 1,\dots,n$,

$$<a_1,\dots,a_n> \in S^{\mathcal{A}} \Leftrightarrow <b_1,\dots,b_n> \in S^{\mathcal{B}}.$$

We say that two L-structures $\mathcal{A}$ and $\mathcal{B}$ are *relatives* (or that $\mathcal{A}$ is *relative to* $\mathcal{B}$), in symbols $\mathcal{A} \sim \mathcal{B}$, if there is a relative correspondence between them. The notion of relative correspondence was introduced by G. Zubieta in [20], using condition iv) of next Proposition 1.2 as the defining one, but only for relational structures, and independently by W. Blok and D. Pigozzi in [1], using condition ii) of Proposition 1.2 as the defining one but for the special case of logical matrices. The word *relative* was introduced by the last two authors. The equivalences between ii), iii) and iv) already appear in [1] for the special case of logical matrices. Another equivalence of this concept is the notion of quasi-isomorphism in [17] and [16]. The proof of the next proposition can be found in [5].

Proposition 1.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be L-structures. The following are equivalent:*

i) $\mathcal{A} \sim \mathcal{B}$

ii) *There are $n \in \omega$ and L-structures $C_0,\dots, C_n$ such that $\mathcal{A} = C_0$, $\mathcal{B} = C_n$ and for any $i < n$, $C_{i+1} \in \mathrm{H_S}(C_i)$ or $C_{i+1} \in \mathrm{H}_S^{-1}(C_i)$.*

iii) *$\mathcal{A}$, $\mathcal{B} \in \mathrm{H_S}(C)$, for some C.*

iv) *$\mathcal{A}$, $\mathcal{B} \in \mathrm{H}_S^{-1}(C)$, for some C.*

v) *There are enumerations of A and B, $\bar{a} = (a_i : i \in I)$ and $\bar{b} = (b_i : i \in I)$ respectively, such that $(\mathcal{A}, \bar{a}) \equiv_0^- (\mathcal{B}, \bar{b})$.*

vi) *There are enumerations of A and B, $\bar{a} = (a_i : i \in I)$ and $\bar{b} = (b_i : i \in I)$ respectively, such that $(\mathcal{A}, \bar{a}) \equiv^- (\mathcal{B}, \bar{b})$.*

In equality-free logic the reduced product operators are not the most natural ones because there is no need to consider quotients modulo the relation associated to the filter. We now introduce some operators that play in equality-free logic the same role that reduced products play in logic with equality. They have been considered before by J. Monk [15] and W. Blok and D. Pigozzi [2]. Now we need some more notation. If $(\mathcal{A}_i)_{i\in I}$ is a family of L-structures, $\prod_{i\in I}\mathcal{A}_i$ is the direct product of the family and for any filter F over I, $\prod_{i\in I}\mathcal{A}_i/F$ is the reduced product modulo F. We denote by $\mathcal{A}^I$ the direct power of $\mathcal{A}$, and given an ultrafilter U over I, we denote by $\mathcal{A}^U$ the ultrapower of $\mathcal{A}$ modulo U.

Definition 3. *Let I be a non-empty set, $(\mathcal{A}_i)_{i\in I}$ a family of L-structures and F a filter over I. We define the* filter-product *of the family $(\mathcal{A}_i)_{i\in I}$ modulo F, that we denote by $\prod^F_{i\in I} \mathcal{A}_i$, as follows:*

- *The domain of $\prod^F_{i\in I} \mathcal{A}_i$ is $\prod_{i\in I} A_i$.*
- *For any constant $c \in L$, $c^{\prod^F_{i\in I} \mathcal{A}_i} = \langle c^{\mathcal{A}_i} : i \in I\rangle$.*
- *For any n-adic function symbol $f \in L$ and any $a_1,\dots, a_n \in \prod_{i\in I} A_i$,*
  $$f^{\prod^F_{i\in I} \mathcal{A}_i}(a_1,\dots, a_n) = \langle f^{\mathcal{A}_i}(a_1(i),\dots, a_n(i)) : i \in I\rangle.$$
- *For any n-adic relation symbol $R \in L$ and any $a_1,\dots, a_n \in \prod_{i\in I} A_i$,*
  $$\langle a_1,\dots, a_n\rangle \in R^{\prod^F_{i\in I} \mathcal{A}_i} \quad \textit{iff} \quad \{i \in I : \langle a_1(i),\dots, a_n(i)\rangle \in R^{\mathcal{A}_i}\} \in F.$$

Observe that the relation $\sim_F$ defined on $\prod_{i\in I} A_i$ by:

$$a \sim_F b \qquad \textit{iff} \quad \{i \in I : a(i) = b(i)\} \in F,$$

for any $a,b \in \prod_{i\in I} A_i$, is a congruence relation of $\prod^F_{i\in I} \mathcal{A}_i$ and the reduced product $\prod_{i\in I} \mathcal{A}_i/F$ is precisely the quotient $\prod^F_{i\in I} \mathcal{A}_i/\sim_F$. Therefore, $\prod^F_{i\in I} \mathcal{A}_i \in$ $\in \mathbf{H}_S^{-1}(\prod_{i\in I} \mathcal{A}_i/F)$ and so $\prod^F_{i\in I} \mathcal{A}_i$ and $\prod_{i\in I} \mathcal{A}_i/F$ are relatives.

2. Equality-free Horn sentences

To start this section we give a preservation theorem for equality-free sentences. The proof can be found in [5].

Theorem 2.1. *Let K be a class of L-structures. The following are equivalent:*

i) K is axiomatizable by a set of equality-free sentences.

ii) K is closed under ultraproducts, $\mathbf{H}_S$ and $\mathbf{H}_S^{-1}$ and for any L-structure $\mathcal{A}$ the following holds: if some ultrapower of $\mathcal{A}$ lies in K, then $\mathcal{A} \in K$.

Corollary 2.2. *Let $T \cup \{\sigma\}$ be a set of sentences of L. Then:*

i) T is axiomatizable by a set of equality-free sentences iff T is preserved under $\mathbf{H}_S$ and $\mathbf{H}_S^{-1}$.

ii) σ is logically equivalent to an equality-free sentence iff σ is preserved under $\mathbf{H}_S$ and $\mathbf{H}_S^{-1}$.

In order to prove the preservation result for Horn sentences in L^-, we first show the following lemma:

Proposition 2.3. *Let $\mathcal{B}$ be an L-structure, I a non-empty set and $(\mathcal{A}_i)_{i\in I}$ a family of L-structures. The following are equivalent:*

i) $\mathcal{B} \sim \prod_{i\in I}\mathcal{A}_i/F$, for some proper filter F over I.

ii) $\mathcal{B} \sim \prod^F_{i\in I} \mathcal{A}_i$, for some proper filter F over I.

There are enumerations of $\prod_{i\in I}A_i$ and B, $\bar{a} = (a_j : j \in J)$ and $\bar{b} = (b_j : j \in J)$ respectively, such that for any basic Horn formula $\phi(x_1,\ldots, x_n) \in L^-$ and any $a_{j_1},\ldots, a_{j_n} \in \prod_{i\in I}A_i$, if for every $i \in I$

$$\mathcal{A}_i \models \phi[a_{j_1}(i),\ldots, a_{j_n}(i)],$$

then,

$$\mathcal{B} \models \phi[b_{j_1},\ldots, b_{j_n}].$$

Proof. i) $\Leftrightarrow$ ii) is clear. ii) $\Rightarrow$ iii) Assume that $\mathcal{B} \sim \prod^F_{i\in I}\mathcal{A}_i$. By Proposition 1.2, there are enumerations of $\prod_{i\in I}A_i$ and B, $\bar{a} = (a_j : j \in J)$ and $\bar{b} = (b_j : j \in J)$ respectively, such that

$$(\textstyle\prod^F_{i\in I} \mathcal{A}_i, \bar{a}) \equiv^- (\mathcal{B}, \bar{b}).$$

Suppose that $\phi(x_1,\ldots, x_n)$ is a basic Horn formula of L^- and $a_{j_1},\ldots, a_{j_n}$ elements of $\prod_{i\in I}A_i$ such that for every $i \in I$

$$\mathcal{A}_i \models \phi[a_{j_1}(i),\ldots, a_{j_n}(i)].$$

Then, since ϕ is a Horn formula,

$$\prod_{i\in I} \mathcal{A}_i/F \models \phi[[a_{j_1}]_F,\ldots, [a_{j_n}]_F].$$

and since ϕ is equality-free,

$$\textstyle\prod^F_{i\in I} \mathcal{A}_i \models \phi[a_{j_1},\ldots, a_{j_n}].$$

Therefore,

$$\mathcal{B} \models \phi[b_{j_1},\ldots, b_{j_n}].$$

Thus, condition iii) holds.

iii) $\Rightarrow$ ii) Suppose that there are enumerations of $\prod_{i\in I}A_i$ and B, $\overline{a} = (a_j : j \in J)$ and $\mathcal{B} = (b_j : j \in J)$ respectively, such that condition iii) holds. Given a sequence $\overline{d} = a_{j_1},\ldots, a_{j_n} \in \prod_{i\in I}A_i$ we will denote by $\overline{d}(i)$ the sequence $a_{j_1}(i),\ldots, a_{j_n}(i)$ and by $\overline{b}_{\overline{d}}$ the corresponding sequence of B, $b_{j_1},\ldots, b_{j_n}$. Now we define for any $n \in \omega$, any sequence $\overline{d} = a_{j_1},\ldots, a_{j_n} \in \prod_{i\in I}A_i$ and any equality-free atomic formula $\phi = \phi(x_1,\ldots, x_n)$, the set

$$S_{\phi\overline{d}} = \{i \in I : \mathcal{A}_i \models \phi\,[\overline{d}(i)]\}$$

and the set $R = \{S_{\phi\overline{d}} : \mathcal{B} \models \phi\,[\overline{b}_{\overline{d}}]\}$. We can distinguish two cases. Case I: $R = \emptyset$. In this case, let $F = \{I\}$. It is easy to check that

$$(\Pi^F_{i\in I}\,\mathcal{A}_i, \overline{a}) \equiv_{\overline{0}} (\mathcal{B}, \overline{b}).$$

Case II: $R \neq \emptyset$. Observe that R has the finite intersection property: suppose that $S_{\phi_1\overline{d}_1},\ldots, S_{\phi_k\overline{d}_k} \in R$ and assume that for any $i \neq j$, $1 \leq i \leq k$, $1 \leq j \leq k$, the set of variables that occur in ϕ_i and the set of variables that occur in ϕ_j are disjoint. Then,

$$\mathcal{B} \models \phi_1 \wedge \ldots \wedge \phi_k\,[\overline{b}_{\overline{d}_1},\ldots, \overline{b}_{\overline{d}_k}].$$

Suppose also, searching for a contradiction, that $S_{\phi_1\overline{d}_1} \cap \ldots \cap S_{\phi_k\overline{d}_k} = \emptyset$. Then, for any $i \in I$,

$$\mathcal{A}_i \models \neg\phi_1 \vee \ldots \vee \neg\phi_k\,[\overline{d}_1(i),\ldots, \overline{d}_k(i)].$$

And since $\neg\phi_1 \vee \ldots \vee \neg\phi_k$ is an equality-free basic Horn formula, by iii),

$$\mathcal{B} \models \neg\phi_1 \vee \ldots \vee \neg\phi_k\,[\overline{b}_{\overline{d}_1},\ldots, \overline{b}_{\overline{d}_k}],$$

which is absurd. Hence we can conclude that R has the finite intersection property.

Let now F be the filter over I generated by R. Since R has the finite intersection property, F is proper. We show that

$$(\Pi^F_{i\in I}\,\mathcal{A}_i, \overline{a}) \equiv_{\overline{0}} (\mathcal{B}, \overline{b}).$$

Let $\psi = \psi(x_1,\ldots, x_n)$ be an equality-free atomic formula and $\overline{c} = a_{j_1},\ldots, a_{j_n} \in \Pi_{i\in I}A_i$. Suppose that

$$\Pi^F_{i\in I}\,\mathcal{A}_i \models \psi[\overline{c}].$$

Then,

$$X = \{i \in I : \mathcal{A}_i \models \psi[\bar{c}(i)]\} \in F$$

and consequently, since F is the filter over I generated by R, there are $S_{\phi_1 \bar{d}_1}, \ldots, S_{\phi_k \bar{d}_k} \in R$ such that

$$X \supseteq S_{\phi_1 \bar{d}_1} \cap \ldots \cap S_{\phi_k \bar{d}_k}.$$

Assume that for any $i \neq j$, $1 \leq i \leq k$, $1 \leq j \leq k$, the set of variables that occur in ϕ_i and the set of variables that occur in ϕ_j are disjoint. Then, for any $i \in I$,

$$\mathcal{A}_i \models \phi_1 \wedge \ldots \wedge \phi_k \rightarrow \psi[\bar{d}_1(i), \ldots, \bar{d}_k(i), \bar{c}(i)].$$

On the one hand, since $\phi_1 \wedge \ldots \wedge \phi_k \rightarrow \psi$ is equivalent to an equality-free basic Horn formula and condition iii) holds,

$$\mathcal{B} \models \phi_1 \wedge \ldots \wedge \phi_k \rightarrow \psi[\bar{b}_{\bar{d}_1}, \ldots, \bar{b}_{\bar{d}_k}, \bar{b}_{\bar{c}}].$$

And on the other hand, since $S_{\phi_1 \bar{d}_1}, \ldots, S_{\phi_k \bar{d}_k} \in R$,

$$\mathcal{B} \models \phi_1 \wedge \ldots \wedge \phi_k [\bar{b}_{\bar{d}_1}, \ldots, \bar{b}_{\bar{d}_k}].$$

Hence, $\mathcal{B} \models \psi[\bar{b}_{\bar{c}}]$. Conversely, if $\mathcal{B} \models \psi[\bar{b}_{\bar{c}}]$, then $S_{\psi \bar{c}} \in R \subseteq F$. And since ψ is an equality-free atomic formula,

$$\prod_{i \in I}^{F} \mathcal{A}_i \models \psi[\bar{c}].$$

Hence, we can conclude that

$$(\prod_{i \in I}^{F} \mathcal{A}_i, \bar{a}) \equiv_{\bar{0}} (\mathcal{B}, \bar{b}).$$

By Proposition 1.2, $\mathcal{B} \sim \prod_{i \in I}^{F} \mathcal{A}_i$. □

Let us now introduce some terminology. Given an infinite cardinal κ and a set I of power κ, the expression "for κ-almost all $i \in I$" will mean "for all but fewer than κ elements of I". If we assume that $2^\kappa = \kappa^+$, the next proposition gives us a sufficient condition for a saturated structure of power κ^+ to be relative to a reduced product of some set of structures. We will use this fact later on to prove the desired preservation theorem.

Proposition 2.4. *Let κ be a cardinal such that $2^\kappa = \kappa^+$ and assume that $|L| + \aleph_0 \leq \kappa$. Let $\mathcal{B}$ be a saturated L-structure of power κ^+, I a set of power κ and $(\mathcal{A}_i)_{i\in I}$ a family of L-structures of power $\leq \kappa^+$. Suppose that for any Horn sentence $\sigma \in L^-$, if for κ-almost all $i \in I$*

$$\mathcal{A}_i \models \sigma,$$

then

$$\mathcal{B} \models \sigma.$$

Then, $\mathcal{B} \in \mathbf{H}_S^{-1}\mathbf{H}_S\mathbf{P}_R(\{\mathcal{A}_i : i \in I\})$.

Proof. Let $\mathcal{A} = \prod_{i\in I}\mathcal{A}_i$. Since $2^\kappa = \kappa^+$, we have that $|A| \leq \kappa^+$. Let $(a'_\xi : \xi \in \kappa^+)$ and $(b'_\xi : \xi \in \kappa^+)$ be enumerations of A and B respectively. By transfinite induction we define two enumerations, $(a_\xi : \xi \in \kappa^+)$ and $(b_\xi : \xi \in \kappa^+)$, of A and B respectively, such that for any $\nu \in \kappa^+$, $(a_\xi : \xi \in \nu)$ and $(b_\xi : \xi \in \nu)$ satisfy the following condition: for any Horn formula $\phi(x_1,\dots, x_n) \in L^-$, any $\xi_1,\dots, \xi_n \in \nu$ and any $a_{\xi_1},\dots, a_{\xi_n} \in A$,

$$\text{if for } \kappa\text{-almost all } i \in I,\ \mathcal{A}_i \models \phi\,[a_{\xi_1}(i),\dots, a_{\xi_n}(i)],\ \text{then } \mathcal{B} \models \phi\,[b_{\xi_1},\dots, b_{\xi_n}]. \qquad (1)$$

Suppose inductively that we have defined $(a_\xi : \xi \in \nu)$ and $(b_\xi : \xi \in \nu)$ such that satisfy condition (1). We can distinguish two cases: case $\nu = \mu + 2k$ and case $\nu = \mu + 2k + 1$, where μ is a limit or 0, and $\kappa \in \omega$.

Case $\nu = \mu + 2k$. Let $a_\nu = a'_{\mu+\kappa}$. Let p be the set of all the formulas $\phi(x, b_{\xi_1},\dots, b_{\xi_n}) \in L(B)$ such that $\phi(x, y_1,\dots, y_n) \in L$ is an equality-free Horn formula, $\xi_1,\dots, \xi_n \in \nu$ and for κ-almost all $i \in I$,

$$\mathcal{A}_i \models \phi\,[a_\nu(i), a_{\xi_1}(i),\dots, a_{\xi_n}(i)].$$

If $\phi(x, b_{\xi_1},\dots, b_{\xi_n}) \in p$, then for κ-almost all $i \in I$,

$$\mathcal{A}_i \models \exists x\phi\,[a_{\xi_1}(i),\dots, a_{\xi_n}(i)].$$

Therefore, as the set of Horn formulas of L^- is closed under $\exists$, by inductive hypothesis,

$$\mathcal{B} \models \exists x\phi\,[b_{\xi_1},\dots, b_{\xi_n}].$$

And since the set of Horn formulas of L^- is closed under $\wedge$, p is a 1-type over the set $(b_\xi : \xi \in \nu)$ in $\mathcal{B}$. Consequently, since $\mathcal{B}$ is saturated, $\mathcal{B}$ satisfies p. Let b_ν be such a realization of p.

Case $\nu = \mu + 2k + 1$. Let $b_\nu = b'_{\mu+\kappa}$. Let q be the set of all the formulas $\phi(x, a_{\xi_1}, \ldots, a_{\xi_n}) \in L(A)$ such that $\phi(x, y_1, \ldots, y_n) \in L$ is an equality-free Horn formula, $\xi_1, \ldots, \xi_n \in \nu$ and

$$\mathcal{B} \models \neg\phi\,[b_\nu, b_{\xi_1}, \ldots, b_{\xi_n}].$$

For any equality-free Horn formula $\phi(x, y_1, \ldots, y_n) \in L$, if

$$\mathcal{B} \models \neg\phi\,[b_\nu, b_{\xi_1}, \ldots, b_{\xi_n}],$$

then $\mathcal{B} \models \neg\forall x\phi\,[b_{\xi_1}, \ldots, b_{\xi_n}]$. We define for any $\phi = \phi(x, a_{\xi_1}, \ldots, a_{\xi_n}) \in q$ the set

$$I_\phi = \{i \in I : \mathcal{A}_i \not\models \forall x\phi\,[a_{\xi_1}(i), \ldots, a_{\xi_n}(i)]\}.$$

Since the set of Horn formulas of L^- is closed under $\forall$, by inductive hypothesis, $|I_\phi| = \kappa$. Therefore, by Lemma 6.1.6. of [4], since $|q| \leq \kappa$, there is $(J_\phi : \phi \in q)$ such that:

a) for any $\phi \in q$, $J_\phi \subseteq I_\phi$ and $|J_\phi| = \kappa$.
b) for any $\phi, \phi' \in q$, if $\phi \neq \phi'$ then $J_\phi \cap J_{\phi'} = \varnothing$.

We now define $a_\nu \in A$ in the following way:

$$a_\nu(i) = \begin{cases} a_\nu(i) \in A_i \text{ s. t. } \mathcal{A}_i \models \neg\phi\,[a_\nu(i), a_{\xi_1}(i), \ldots, a_{\xi_n}(i)], & \text{if } i \in J_\phi, \\ \text{arbitrary}, & \text{otherwise}, \end{cases}$$

for any $i \in I$.

Once we have finished the construction, $(a'_\xi : \xi \in \kappa^+)$ and $(b'_\xi : \xi \in \kappa^+)$ have the desired property: for any $\nu \in \kappa^+$, $(a_\xi : \xi \in \nu)$ and $(b_\xi : \xi \in \nu)$ satisfy condition (1). Therefore, these enumerations satisfy also condition iii) of Proposition 2.3. Hence we can conclude that $\mathcal{B} \in \mathrm{H}_\mathrm{S}^{-1}\mathrm{H}_\mathrm{S}\mathrm{P}_\mathrm{R}(\{\mathcal{A}_i : i \in I\})$. □

Now, assuming the continuum hypothesis (CH), we will prove the preservation theorem for Horn sentences. Later we will show how to eliminate this strong hypothesis.

Theorem 2.5 (CH). *For any sentence $\sigma \in L$, the following are equivalent:*

i) σ is preserved under $\mathbf{H}_S$, $\mathbf{H}_S^{-1}$ *and* $\mathbf{P}_R$.

ii) σ is logically equivalent to an equality-free Horn sentence.

Proof. ii) $\Rightarrow$ i) is clear. i) $\Rightarrow$ ii) Suppose that σ is preserved under $\mathbf{H}_S$, $\mathbf{H}_S^{-1}$ and $\mathbf{P}_R$. We can assume that L is finite (we consider only the symbols that occur in σ). If σ is inconsistent it is clear. Otherwise, let Σ be the following set of sentences

$$\Sigma = \{\psi \in L^- : \psi \text{ is a Horn sentence and } \models \sigma \rightarrow \psi\}.$$

Clearly $\Sigma \neq \emptyset$. We prove that there is $\psi \in \Sigma$ such that $\models \psi \rightarrow \sigma$. By compactness, since Σ is closed under $\wedge$, it is enough to show that $\Sigma \models \sigma$. Since σ is preserved under $\mathbf{H}_S$, $\mathbf{H}_S^{-1}$, by Corollary 2.2, σ is logically equivalent to an equality-free sentence. Therefore, in order to prove that $\Sigma \models \sigma$, since any finite model is equivalent in equality-free logic to an infinite model, it suffices to show that, for any infinite L-structure $\mathcal{A}$, if $\mathcal{A} \models \Sigma$, then $\mathcal{A} \models \sigma$. Moreover, since L is finite, by the Löwenheim-Skolem Theorem, we can restrict ourselves to infinite countable models. Suppose that $\mathcal{A}$ is an infinite countable L-structure such that $\mathcal{A} \models \Sigma$. We will show that $\mathcal{A} \models \sigma$. Let $\mathcal{B}$ be an elementary extension of $\mathcal{A}$ of power $\aleph_1$ and saturated. Since L is finite and $|A| < 2^{\aleph_0} = \aleph_1$, the existence of such extension is guaranteed by Lemma 5.1.4 of [4].

Let now Ψ be the set

$$\Psi = \{\psi \in L^- : \psi \text{ is a Horn sentence and } \neg\psi \wedge \sigma \text{ is consistent}\}.$$

Clearly $\Psi \neq \emptyset$. We choose, for any $\psi \in \Psi$ a countable model $\mathcal{A}_\psi$ of $\neg\psi \wedge \sigma$. Let $I = \omega \times \Psi$ and for any $i = \langle n, \psi\rangle \in I$, $\mathcal{A}_i = \mathcal{A}_\psi$. Now we show that for any Horn sentence $\psi \in L^-$, if for κ-almost all $i \in I$, $\mathcal{A}_i \models \psi$, then $\mathcal{B} \models \psi$. Suppose that $\psi \in L^-$ is a Horn sentence such that for ψ-almost all $i \in I$, $\mathcal{A}_i \models \psi$. Then $\psi \notin \Psi$, otherwise for any $n \in \omega$ and any $i = \langle n, \psi\rangle \in I$, $\mathcal{A}_i \models \neg\psi$, contradicting the fact that for κ-almost all $i \in I$, $\mathcal{A}_i \models \psi$. Therefore, $\psi \in \Sigma$ and since $\mathcal{A} \preceq \mathcal{B}$, $\mathcal{B} \models \Sigma$ and consequently $\mathcal{B} \models \psi$. Hence, by Proposition 2.4, $\mathcal{B} \in \mathbf{H}_S^{-1}\mathbf{H}_S\mathbf{P}_R(\{\mathcal{A}_i : i \in I\})$. Since σ is preserved under $\mathbf{H}_S$, $\mathbf{H}_S^{-1}$ and $\mathbf{P}_R$ and for any $i \in I$, $\mathcal{A}_i \models \sigma$, $\mathcal{B} \models \sigma$ and thus, $\mathcal{A} \models \sigma$. We can conclude that $\Sigma \models \sigma$. Then, σ is logically equivalent to an equality-free Horn sentence. □

Now, let us eliminate the continuum hypothesis of Theorem 2.5. We assume that L is finite and we assign suitable Gödel numbers to the symbols and expressions of L. For a definition of arithmetical predicate and arithmetical statement and a reference to find the following theorems see Chapter 6.2. of [4].

Theorem 2.6. *Let* Γ *be an arithmetical statement,*

$$\textit{If } \mathrm{ZF} + \mathrm{CH} \vdash \Gamma, \textit{ then } \mathrm{ZF} \vdash \Gamma.$$

Observe that the predicates "σ is an equality-free sentence of L" and "σ is an equality-free Horn sentence of L" are recursive and the predicate "σ is logically equivalent to an equality-free Horn sentence of L" is arithmetical.

Proposition 2.7. *The predicate "σ is preserved under* $\mathrm{H_S}$ *and* $\mathrm{H_S^{-1}}$*" is arithmetical.*

Proof. By Corollary 2.2, because the predicate "σ is an equality-free sentence of L" is recursive. □

Theorem 2.8 (Galvin). *The predicate "σ is preserved under* $\mathrm{P_R}$*" is arithmetical.*

Corollary 2.9. *The predicate "σ is preserved under* $\mathrm{H_S}$, $\mathrm{H_S^{-1}}$ *and* $\mathrm{P_R}$*" is arithmetical.*

Proof. By Proposition 2.7 and Theorem 2.8. □

Corollary 2.10. *The statement "σ is logically equivalent to an equality-free Horn sentence of L if and only if σ is preserved under* $\mathrm{H_S}$, $\mathrm{H_S^{-1}}$ *and* $\mathrm{P_R}$*" is arithmetical.*

Theorem 2.11. For any sentence $\sigma \in L$, the following are equivalent:

i) σ *is preserved under* $\mathrm{H_S}$, $\mathrm{H_S^{-1}}$ *and* $\mathrm{P_R}$.

ii) σ *is logically equivalent to an equality-free Horn sentence.*

Proof. By Corollary 2.10 and Theorems 2.5 and 2.6. □

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Notes

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