# PARACONSISTENCY: TOWARDS A TENTATIVE INTERPRETATION

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Manuscript received: 1998.9.13.
Final version: 1999.10.15.

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BIBLID [0495-4548 (2001) 16: 40; p. 119-145]

ABSTRACT: In this expository paper, we examine some philosophical and technical issues brought by paraconsistency (such as, motivations for developing a paraconsistent logic, the nature of this logic, and its application to set theory). We also suggest a way of accommodating these issues by considering some problems in the philosophy of logic from a new perspective.

Keywords: paraconsistent logic, paraconsistency, paraconsistent set theory.

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> The mathematician will have to take account not only of those theories that come near to reality but also, as in geometry, of all logically possible theories, and he must always be careful to obtain a complete survey of the consequences implied by the system of axioms laid down.
>
> David Hilbert: 1901, 'Mathematical Problems'.

## *Introduction*

Within the contemporary philosophical landscape, it is now generally acknowledged the crucial role that logic has performed. Several philosophical domains, ranging from the theory of knowledge and the philosophy of science, to metaphysics and ethics, have been drastically changed by the introduction of the conceptual machinery supplied by logical theories.

A more delicate question, however, is raised when one considers the question: what to say about our *philosophical* understanding of logic itself, and in particular of non-classical ones? One of the main purposes of the present paper is to defend a particular view about this issue; one that fares well (or so we hope) with other interpretative questions about logic (such as the nature of logic and its application to science). Our main strategy to do so will be to consider in some detail a specific case: that resulting from paraconsistent logic, the logic of inconsistent but non-trivial systems (for some information on the history of this logic, see Arruda (1980), D'Ottaviano (1990), and da Costa, Béziau and Bueno (1995*b*); see also da Costa, Béziau and Bueno (1995*a*), and da Costa and Bueno (1996)). Among the various non-classical logics this is perhaps one of the most unusual, at least if we consider the classical Aristotelian setting: as is well known, within paraconsistent logic the principle of non-contradiction is not always valid. Formally, there is no difficulty in characterising this possibility; things however change dramatically when we have to *make sense* of what is happening. So this is a case calling for interpretation. But the question naturally arises: what, at this level, is an interpretation?

After briefly answering this question, we shall provide a particular interpretation of paraconsistent logic, and indicate how it comes to grips with some philosophical perplexities that emerge along the way. We shall discuss some motivations for paraconsistent logic, the use of the latter in set theory, the role of the distinction between pure and applied logics in the understanding of paraconsistency, and the application of paraconsistent logic in the study of Russell's set, of paraconsistent Boolean algebras and the theory of syllogism.

A last word of warning. We conceive the present work as examining in outline several (philosophical and technical) issues related to paraconsistency. We do not intend to develop their analysis here, being simply concerned with the presentation of a rather general setting. After the formulation of this setting, each of these issues shall be considered in detail in future papers.

## *1. Logic, mathematics, paraconsistency*

If one wishes to understand the meaning and nature of logic, it is important to make plain, from the outset, that nowadays it is a field of knowledge at the same level as mathematics. It is divided thus (as will be examined in section 3) into two domains: a pure one and an applied one. From the pure viewpoint, it studies certain abstract structures, such as formal languages, models and Turing machines. Let us present some remarks on these three kinds of structures (of course, there are many others as well, but we shall not be concerned with them here).

A formal language is an abstract structure that codifies certain aspects of common languages. From a logical-algebraic viewpoint, it is a kind of free algebra. In fact, a theory of formal languages can be developed algebraically: algebraic techniques can then be applied, and the standard logical concepts have an algebraic version. For instance, within classical logic, a *theory* becomes a *filter*, a *consistent theory* turns into a *proper filter*, a *complete theory* into a *ultra-filter*, Gödel's incompleteness theorem into the existence of certain *filters* that are not *ultra-filters* etc. From the point of view of pure logic, the algebraic method is both more general and convenient than the one based on formal languages. It is usually claimed that, with Frege, logic was removed from the algebraic framework that Boole had given to it, and that this was a progress. Presently, it is known that this is false, and algebraic logic shows this quite clearly. Based on algebra, it is possible to classify and to study not only classical logic, but also non-classical ones, that are currently proliferating at a considerable rate: intuitionistic logic, many-valued logic, fuzzy logic, paraconsistent logic, non-alethic logic, non-linear logic, substructural logic, probabilistic logic, quantum logic etc.

General model theory is concerned not only with models of classical formal languages, such as the classical first-order predicate calculus, which is the most well known, but also with models of heterodox languages: non-classical, Boolean-valued, models of classical languages; classical models of intuitionistic theories (Kripke models, for instance); quantum models of languages related to quantum mechanics (see Takeuti 1981*a* and 1981*b*); classical models of paraconsistent theories and paraconsistent models of such theories and so on. Thus, it is possible to prove that Brouwer-Heyting first-order intuitionistic logic is complete according to the classical Kripke semantics, though it is incomplete, and not completable, in connection to an intuitionistic one (Gödel).

Nowadays, there are several distinct *semantics* constructed, just as the classical one, within standard set theory, such as, to mention just a few of them, paraconsistent, quantum or Boolean semantics. These semantics can be generalised to a general valuation theory, which combines algebraic, topological and set theoretical ideas. The models obtained through forcing (Cohen) and

Gödel's theory of constructive sets fall within the general theory of models, and those who are not acquainted with such themes do not have a notion of the current state of the evolution of logic. If we recall that classical model theory has already given us several topics that one cannot disregard (prime models, saturated models, categoricity in potency and Morley's theorem, omission of types, elimination of quantifiers, real closed fields, classification theory (Shellah) etc.), one immediately notices the enormous richness of general model theory or mathematical semantics. (For further comments on semantics, see da Costa, Bueno and Béziau 1995.)

The theory of machines, or recursion theory, has been so developed in the last years that it is no longer possible to follow its literature in all of its details. To the traditional themes -such as, recursive functions, Rice's theorem, arithmetical hierarchy, analytical hierarchy, Post languages etc.-, many others have been added, making this area of logic still richer. Some have tried, for instance, to extend the notion of calculability through Turing machines or recursive functions, as is the case of Smale and his collaborators.

Everything that was just recollected supplies evidence for the fact that the development of pure logic is the same as that of pure mathematics. Understanding its nature and meaning is equivalent to understanding, in general, the meaning and nature of pure mathematics. It is enough to notice here that its progress, at least in principle, is made on an *a priori* and abstract level; experience (thought of in a comprehensive sense), both related to common life as well as to science, has only a heuristic value.

Regarding applied logic, just as with applied mathematics, things are quite distinct. Logic, for instance, thought of as the science of the valid forms of inference, is placed within this domain, that is, within *applied* logic. The problem, in this case, consists in discovering abstract structures that reflect the real mechanisms of deductive inferences in a certain domain. Thus, one can be concerned with inferences found in ordinary life, in both traditional and constructive mathematics, as well as in quantum mechanics and natural sciences.

As opposed to its pure counterpart, applied logic is not articulated in an abstract and *a priori* level. On the contrary, it somehow depends on experience (in a comprehensive sense) and on pragmatic factors as well (theoretical simplicity, intuitiveness, capacity of systematisation and so on). For this reason, the *constructive* study of *constructive* mathematical thought does not fit with classical logic schemes; in other words, the categories and processes of classical logic (principle of excluded middle, classical method of *reductio ad absurdum* etc.) cannot reflect the mechanisms underlying constructive thinking. Hence the existence of various *constructive* logics (Brouwer-Heyting, Griss,...).

Analogously, standard quantum mechanics (as it will be argued in section 3.2) seems to lead to non-classical logics, provided that one wishes to account adequately for what happens at the quantum domain (modular logics, orthomodular logics, Kochen-Specker structures (see Kochen and Specker 1967 etc.).

The numerous, but interconnected, topics presented in this paper were chosen in order to clarify some aspects of the nature of paraconsistent logic, its meaning and significance. It should be clear from the outset that everything that is said concerning logic in general obviously also holds for paraconsistent logic. In particular, the latter can be viewed both as a pure subject as well as an applied one. In the first case, just as the rest of mathematics itself, it is concerned with conceptual structures defined and investigated in an *a priori* way. In the second case, thought of as an applied discipline, it depends on experience and is dependent on pragmatic constraints.

The division between pure and applied logic within a paraconsistent domain is extremely important, allowing in particular the better examination of some problems. As we shall see in section 3.1, some specialists criticise certain paraconsistent systems for the fact that in these systems the law of substitution of equivalents does not hold; but they would prefer to have this as a valid law. However, from the perspective of pure logic, such a critique would be similar to that made by an algebraist who wishes that only commutative groups be studied... From the applied standpoint, nevertheless, such a discussion might be relevant, provided that one is taking into account certain applications, for instance, to the domain of computer science. Unfortunately, though, this is frequently not what happens, being just as if one has access to a platonic, true logic, adopted as a standard of comparison between all the alternative logical systems under consideration. On the contrary, when one is concerned with an issue of applied paraconsistent logic (for instance, to expert systems), it makes sense to determine whether a certain property, such as the one just mentioned, is or is not to be met by the logical system being outlined. Furthermore, it is possible to ask whether in some expert systems, in order to handle contradictory bits of information, it is appropriate that the underlying logic, of a paraconsistent kind, has a second negation, which behaves classically.

Having presented these general remarks concerning logic, mathematics and paraconsistency, before considering, in section 3, some aspects of the theoretical status of the latter, we shall briefly examine some of the main motivations for its introduction.

## *2. Motivation: paraconsistency and set theory*

Talking about the axiomatisation of physical theories, Hilbert wrote the words above, that we have taken as our motto; words that indeed guide any axiomatic research. To some extent, it is possible to say that paraconsistent logic has appeared as the result of applying this Hilbertian norm to the axiomatisation of set theory.

Indeed, Cantor's naive theory was based mainly on two fundamental principles: the postulate of extensionality (if the sets $x$ and $y$ have the same elements, then they are equal), and the postulate of separation or comprehension (every property determines a set, composed of the objects that have this property). The latter postulate, in the standard (first-order) language of set theory, becomes the following formula (or scheme of formulas):

$$(1) \quad \exists y \forall x (x \in y \leftrightarrow F(x))$$

Now, it is enough that one replaces the formula $F(x)$, in (1), for $x \notin x$ in order to derive Russell's paradox. That is, the principle of separation (1) is inconsistent. Thus, if one adds (1) to first-order logic, conceived as the logic of set theoretic language, a trivial theory is obtained.

There are also other paradoxes, such as Curry's and Moh Schaw-Kwei's, that indicate that (1) is trivial or, more precisely, trivialises the language of set theory, if the underlying logic is classical, even ignoring negation. In other words, classical positive logic is incompatible with (1); the same holds also for several other logics, such as the intuitionistic one.

Classical set theories are distinguished by the restrictions that are imposed on (1), to the effect of avoiding paradoxes. In order that the theory thus obtained does not become too weak, some further axioms, besides extensionality and separation (with due restrictions), are added, depending on the particular case in question.

Thus, for instance, in Zermelo-Fraenkel (ZF), separation is formulated in the following way:

$$(2) \quad \exists y \forall x (x \in y \leftrightarrow (F(x) \wedge x \in z)),$$

where the variables are subject to obvious conditions. In ZF, then, $F(x)$ determines the subset of the elements of the set $z$ that have the property $F$ (or satisfy the formula $F(x)$). In the Kelly-Morse system, on the other hand, separation is as follows:

$$(3) \quad \exists y \forall x (x \in y \leftrightarrow (F(x) \wedge \exists z (x \in z))).$$

And, finally, in Quine's NF the notion of stratification is employed, and the scheme of separation has the form:

(4) $\exists y \forall x (x \in y \leftrightarrow F(x))$,

provided that the formula $F(x)$ be stratifiable (besides the standard conditions regarding the variables).

However, adopting Hilbert's motto, we can ask whether it would be possible to examine the problem from a distinct viewpoint: what is needed in order to maintain the scheme (1) without restrictions (with no regard to the conditions on the variables)? The answer is immediate: one should change the underlying logic, so that (1) does not inevitably lead to trivialisation. The separation scheme, without 'big' restrictions, leads to contradictions. Hence, such a logic has to be a paraconsistent one.

It was slowly verified that there are infinitely many ways to make the classical restrictions to the separation scheme weaker, each of them corresponding to distinct categories of paraconsistent logics. Furthermore, extremely feeble logics have been formulated, and based on them it is possible to employ, without trivialisation, the scheme (1). Some set theories, in which the forms (2), (3) and (4) of separation are either combined or adopted in isolation, have also been constructed.

An important point is that several paraconsistent set theories contain the classical one, in Zermelo-Fraenkel's, Kelly-Morse's or Quine's formulations. Hence, paraconsistency goes beyond the classical domain, and allows, among other things, the reconstruction of traditional mathematics (see da Costa, Béziau and Bueno (1998), da Costa (1986), da Costa (1999), da Costa, Bueno and Volkov (1999), and Mortensen (1995)). It is quite fair then to claim that paraconsistent theories extend the classical ones, just as Poncelet's imaginary geometry comprises the standard 'actual' geometry.

Moreover, we should stress a difficulty found in the very foundations of logic. Classical elementary logic (it would, in fact, be enough to consider only part of its positive part) *and* the separation postulate are both evident; we are even bound to claim that they are equally evident or intuitive. However, they are mutually incompatible, and constitute thus a case of incompatible evidences -this generates a difficulty from the viewpoint of classical logic.

Without presenting detailed philosophical analyses, we shall just note that classical theories adopt a particular line of approach, and paraconsistent theories, another one. All this is in perfect agreement with our quotation of Hilbert: one should explore all the possibilities. And we stress, such an exploration contributes to a better comprehension of the classical position itself: a clearer understanding of negation; the possibility of the discourse, even if one partially rejects the principle of non-contradiction; a proof that this principle is at least partially true, and so on.

It would be natural to think that, being based on somewhat different motivations and presenting distinct features, paraconsistent logic and the classical one might somehow dissent as far as their theoretical status are concerned. Things however may not be this way -an issue to which we now turn.

## *3. Paraconsistency: remarks on its theoretical status*

### *3.1. Pure logic, applied logic and paraconsistency*

Logic is usually considered as an *a priori* and analytic domain; it is taken to be independent of experience, and its laws are thought of as compatible with any contingent state of affairs that might happen. This view, however disseminated, is by no means undisputed; indeed, as Heisenberg stressed a long time ago:

> (...) if one wishes to speak about the atomic particles themselves, one must either use the mathematical scheme as the only supplement to natural language or one must combine it with a language that makes use of a modified logic or of no well-defined logic at all (Heisenberg 1958, p. 46).

And Schrödinger has also noticed:

> As our mental eye penetrates into smaller and smaller distances and shorter and shorter times, we find nature behaving so entirely differently from what we observe in visible and palpable bodies of our surrounding that *no* model shaped after our large-scale experiences can ever be true (Schrödinger 1952).

Both remarks are symptomatic of a striking fact: quantum mechanics unavoidably leads to logical settings distinct from the classical ones. As far as we know, and as we shall argue for in the next section, it seems that there is a quantum logic considerably diverse to that found in our traditional logical framework. Nevertheless, as is well known, all the argumentation concerning the logical foundations of quantum physics is not developed in *a priori* lines; instead, experiments, such as Gerlach's and Stern's on the spin of particles, as well as quantum laws, such as Heisenberg's principle, should be taken into account -and these are the experiences and laws that made us reconsider the basis of the underlying logic of physics.

Intuitionistic logic, for its part, is one of the possible adequate ways employed in order to systematise constructive thinking in mathematics. Classical logic by no means reflects the constructive activity of the mathematicians, for it depends on the implicit assumption that they work in domains composed by objects *already given*, whose existence their constructive work is not concerned with.

Thus, quantum and intuitionistic logics supply evidence for the thesis that logic, in its applications, is dependent on the particular features of the domain that it organises. It is plain that we are referring here to applied logical systems, and not to pure logic. The pure logician, of course, can elaborate and scrutinise any system, independently from the experience. However, regarding their applications, there is the inter-connection between the logical dimension and the domain of application, which is based specially on pragmatic considerations, though further aspects are also relevant for the individualisation of the appropriate logic within this context, such as heuristic reasoning and the nature of the domain studied.

Concerning the analyticity of logic, this seems doubtful even within the boundaries of classical logic. This leads us to themes such as the independence of the axiom of choice and of the continuum hypothesis, which do not easily fit in the category of the analytic statements (nor is the formulation of Zermelo's axiom analytic, nor even further questions linked to the controversies and problems that it has given rise to). Higher-order logic itself -logic of higher-order and set theory- commits us to axioms of an existential trait; elementary logic, furthermore, has what we could call *synthetic features*, related to its semantics, which involves topics on set theory of a non-analytic nature.

Something similar also holds for the semantics of quantum physics, which cannot be based on the standard semantic, set theoretic notions. As Manin claims:

> New quantum physics has shown us models of entities with quite different behavior. Even 'sets' of photons in a looking-glass box, or of electrons in a nickel piece are much less Cantorian than the 'set' of grains of sand. In general, a highly probabilistic 'physical infinity' looks considerably more complicated and interesting than a plain infinity of 'things' (Manin 1974, p. 36).

On the other hand, realist conceptions *à la* Frege and Gödel, according to which logic supplies the most general features of the universe, only seem to be defensible on largely speculative grounds (Tarski, for instance, considered them as kinds of superstition or of mysticism). Nowadays, given the proliferation of heterodox logical theories, especially the existence of infinite paraconsistent logics containing a considerable part of traditional logic, the defence of an extreme realist view becomes a difficult task.

These remarks have an important motivation: to justify, though indirectly, some theses about paraconsistent logic. In fact, when a paraconsistent system is built as a pure theory, its main features, in general, are not obviously open in principle to criticism, unless to show that it is trivial. For instance, it is usual to criticise certain paraconsistent propositional logics for not having relations of congruence involving all the connectives (in particular, it is not the case that $\vdash \alpha \leftrightarrow \beta$ entails that $\vdash \neg \alpha \leftrightarrow \neg \beta$). Instead of these logics, some specialists propose distinct ones, which present natural relations of congruence, but which satisfy the law of non-contradiction $\neg (\alpha \wedge \neg \alpha)$, a law that, of course, does not hold in the former ones. Now, strictly speaking, such a discussion seems to be either meaningless, or purely speculative, for logics without congruence relations were formulated as pure logics, and within this domain (as well as in pure mathematics), freedom is enormous -all the alternatives should be, in principle, liable to exploration. These kinds of criticism, however, do not miss the mark in the domain of applied logic, where it is possible to employ pragmatic arguments and concrete motives (which depend on the experience, in a comprehensive sense).

In summary, we think that an exclusively philosophical argumentation, taken for itself, does not solve technical questions either in pure logic, or in applied. However, considerations grounded on other reasons, related to the domains under investigation, are the ones that matter in applied logic.

Incidentally, to some extent, one of the most important applications of a logic consists in the construction of mathematics; thus, a logic which is not strong enough for us to obtain considerable parts of classical mathematics faces a cumbersome difficulty. This point was already noticed by Hilbert himself, when he claimed (though perhaps being a bit hasty in his generalisation regarding the role of Aristotelian laws of logic in the construction of mathematics):

> But we cannot relinquish the use either of the principle of excluded middle or of any other law of Aristotelian logic expressed in our axioms, since the construction of analysis is impossible without them (Hilbert 1927, p. 471).

All that we have been arguing for thus far fits the case of paraconsistent systems, which, we note *en passant*, have found immense applications in artificial intelligence, computer science and the foundations of empirical sciences (see, e.g., in a huge literature, Subrahmanian (1987), Blair and Subrahmanian (1987) and (1988), and Kifer and Subrahmanian (1992)).

### *3.2. Logic, quantum mechanics and paraconsistency*

A striking feature of twentieth century research on logic consists in the fact that several logics, different from the classical one, have been created (consider, for instance, intuitionistic logic, many-valued logic, quantum logic etc.). Hence a new problem has emerged: how to justify, in each particular case, the legitimacy of the employment of one of these logics? A typical example of this issue can be found in quantum mechanics, in which the use of traditional logic is by no means unproblematic.

This circumstance is one that leads to a split of logic into the two parts mentioned above. To insist once again on this point, we have on the one hand, pure logic, which is developed analogously to pure mathematics, in principle in an *a priori* and abstract way, and on the other hand, applied logic, employed for instance in ordinary inferences, in constructive thinking and in quantum mechanics. In pure logic, one finds topics such as, analytical hierarchy, arithmetical hierarchy, saturated models, polyadic algebra, cylindrical algebra, Martin's axiom and topology, forcing, Boolean-valued models etc.

In applied logic, a central theme is the one related to the logic of quantum mechanics: what is the natural underlying logic for the pre-formal theory about this physical domain? In order to answer to this question, it is necessary to circumvent two basic difficulties. (1) As an overwhelming cluster of evidence seems to indicate, quantum logic, according to von Neumann's and Birkhoff's pioneering work, violates the distributive law of classical logic:

$$\alpha \wedge (\beta \vee \gamma) \leftrightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$$

(2) Elementary particles break the usual theory of identity or, properly speaking, the notion of identity (or of equality) does not seem to be suitably applicable to them: as Schrödinger, Heisenberg, Weyl (cf. Weyl 1963) and other physicists have pointed out, it is simply meaningless to talk about particles being equal or different.

It is possible to have an idea, though in outline and rather schematic, of the first difficulty by considering a (quite simplified) example.

In standard quantum mechanics, every electron *e* has an angular momentum, or spin, in the *x* direction, whose value is always +1/2 or -1/2; that is, if the momentum of *e* in the *x* direction is denoted by $e_x$, then $e_x = +1/2 \vee e_x = -1/2$. On the other hand, given the so-called Heinsenberg's principle, it is not possible to measure the angular momentum of *e* in two (distinct) directions simultaneously.

Let then $x$ and $y$ be two distinct directions, and let us suppose that one has measured the momentum of $e$ in the direction $x$ and that $e_x = +1/2$; hence, $e_x = +1/2$ is true. However, as has been said, $e_y = +1/2 \vee e_y = -1/2$ is always true (in any instant). Accordingly, one can deduce that the conjunction

(I) $e_x = +1/2 \wedge (e_y = +1/2 \vee e_y = -1/2)$

is also true. From (I), given the distributivity of the conjunction in relation to the disjunction, it follows:

(II) $e_x = +1/2 \wedge (e_y = +1/2 \vee e_y = -1/2) \leftrightarrow (e_x = +1/2 \wedge e_y = +1/2) \vee$
$\vee (e_x = +1/2 \wedge e_y = -1/2)$

As was seen, the left component of the biconditional is true; however, given that it is not possible to measure simultaneously the moment of $e$ in distinct directions $x$ and $y$, the right component either is false or simply meaningless. Thus, the application of classical logic leads to difficulties.

There are several possible ways to try to maintain traditional logic and to overcome the problem presented by (II). Nevertheless, thus far none of them has received unanimous acceptance. (It simply does not work if one proposes to change standard quantum mechanics for another theory, for it is of the former that we are talking about; it is also not enough to note that it is the measurement that 'creates' the spin's value, and thus the proposition '$e_y = +1/2 \vee e_y = -1/2$' is not true nor false, for such a remark is against classical logic etc.)

The issue concerning the possibility of applying the category of equality to elementary particles is really delicate, and its solution does not seem to be simple. Both traditional set theory and the mathematics constructed within it presuppose the theory of equality. It follows, therefore, that a collection of electrons, for instance, does not constitute a set in the classical sense.

To sum up, there are considerable hindrances facing the possibility of applying classical logic to quantum mechanics.

Even the semantics of this theory gives rise to difficulties, given that the standard semantic methods are elaborated within traditional set theory. Such a situation is already considered even in good logic textbooks, such as Manin's:

> Analyzing quantum mechanical phenomena reveals a profound divergence between the internal logical structures of the macroworld and the microworld. Although explanations of these differences by means of natural language and natural logic are agonizingly difficult and, in the last analysis, always leave one feeling unsatisfied, these attempts to explain continue. The development of these foundations of physics in the twentieth century has taught us a serious lesson. Creating and understanding these foun-

> dations turned out to have very little to do with epistemological abstractions which were of such importance to the twentieth century critics of the foundations of mathematics: finiteness, consistency, constructibility, and, in general, the Cartesian notion of intuitive clarity. Instead, completely unforeseen principles moved into the spotlight: complementarity, and nonclassical, probabilistic truth function. The electron is infinite, capricious, and free, and does not at all share our love for algorithm (Manin 1977, pp. 82-83).

All this discussion supplies evidence for the thesis that logic, at least concerning its applications, is not bounded to entirely *a priori* constraints. Indeed, the criteria guiding its applications are the same that direct the applications of any mathematical theory, for instance, they are similar to those corresponding to pure geometry.

Finally, we would like to point out that a new kind of quantum logic has been proposed by Dalla Chiara and Giuntini: paraconsistent quantum logics (cf. Dalla Chiara and Giuntini 1989). These are weak forms of quantum logic, in which the non-contradiction and the excluded middle principles do not hold. As the authors argue, these logics can be seen as a 'logical abstraction' from the class of all *effects* in the operational approach to quantum mechanics, having also some interesting applications to this domain.

## *4. Paraconsistency: some technical applications*

Having briefly examined some theoretical traits of paraconsistent logic, in what follows we shall consider some technical developments within the paraconsistent framework. Our main point now, is to indicate some striking features that one finds when such a logic is employed as the underlying logic of mathematical reasoning. On the one hand, though this might not be *that* surprising, given its main traits as a particular non classical logic in which the principle of non-contradiction is somehow restricted, some fairly unusual results (at least regarding our classical intuition) are obtained -but of a remarkable interest. On the other hand, and in straight connection to this point, despite the nature of such results, the pattern and the kind of reasoning involved in order to reach them are quite standard, indeed fully similar to current mathematical practice. Though one may study within a paraconsistent framework the properties of 'contradictory' objects, it is simply not the case that 'anything goes'. One just cannot prove anything one wishes about them. In spite of being contradictory, such objects, as it were, are *not* trivial. Just as in the case if the standard ones studied within the classical branches of mathematics, this kind of object has the same independence from our thoughts and desires: some properties hold of them, and some absolutely not.

So, on the methodological side, a paraconsistent mathematics is by no means threatening, or at least it is as frightening as classical mathematics can be; and on the heuristic edge, some interesting new results are obtained. What more can we expect of a mathematical framework? (This is obviously a rhetorical question; but for those who have taken it seriously, we remark once again that even regarding the application issue such a framework fares rather well, particularly within artificial intelligence and, more generally, in order to consider inconsistent bits of information.)

Moreover, with a paraconsistent logic we can formulate definitions and reason in the presence of contradictions, without ruling them out (as is the case with the use of classical logics to systematise sets of consistent beliefs about a domain). It is possible to present here an analogy with geometry: the introduction of improper elements (Desargues, Poncelet,...), or of imaginary or ideal elements (Poncelet, Plücker, Klein,...) in geometry gives rise to no intrinsic difficulty for its development; and this happens even though ordinary space intuition does not hold any longer.

However, instead of arguing in rather *a priori* lines for these points, and in order to present some further details about them, we shall consider a particular, concrete case, relevant for our purposes, formulating and proving then some results on a celebrated 'contradictory object': Russell's set. In section 4.2, a paraconsistent Boolean algebra shall be constructed, and finally, in section 4.3, we will briefly present a semantic analysis of a paraconsistent logic.

### *4.1. Russell's set and paraconsistency*

We shall be working within a paraconsistent set theory, whose details will be suppressed (some of these details, such as those concerning the restrictions to be included in the separation scheme, can be found in da Costa (1964), and da Costa (1986); see also da Costa, Béziau and Bueno 1998). It suffices to note that, within this paraconsistent set theory, there are sets such as Russell's, the power set (the set of all subsets of a given set), and the union set of another set; furthermore, the rules of intuitionistic or classical positive logic as well as the principle of the excluded middle hold. Just as in the classical case, there are infinitely many paraconsistent set theories; in some of them, e.g., two sets cannot be simultaneously equal and different, but in others they can. Moreover, in some such theories we can derive Burali-Forti's and Cantor's paradoxes (see da Costa, Béziau and Bueno (1998); for details concerning paraconsistent mathematics, see, e.g., da Costa (1999), Mortensen (1995), and da Costa, Bueno and Volkov (1999)).

*Definition 1.* (*Russell's set*) $R = \{x: x \notin x\}$.

*Theorem 1.* $R \in R \wedge R \notin R$.

*Proof.* Given the definition of $R$, $x \in R \leftrightarrow x \notin x$. Hence, replacing $x$ for $R$: $R \in R \leftrightarrow R \notin R$. However, if $R \in R$, it follows that $R \notin R$, and if $R \notin R$, then $R \notin R$. Therefore, by the excluded middle, $R \notin R$. Similarly, one may prove that $R \in R$.

*Theorem 2.* $y \in \{x\} \leftrightarrow y = x$.

*Proof.* It is an immediate consequence of the definition of $\{x\}$.

*Theorem 3.* $x \in R \rightarrow \{x\} \in R$.

*Proof.* It is the case that either $\{x\} \notin \{x\}$, or $\{x\} \in \{x\}$. In the first case, $\{x\} \in R$, by the definition of $R$. In the second, $\{x\} = x$, and, given the hypothesis, $\{x\} \in R$.

*Theorem 4.* $x, y \in R \rightarrow \{x, y\} \in R$.

*Proof.* We have that either $\{x, y\} \notin \{x, y\}$ or $\{x, y\} \in \{x, y\}$. In the first hypothesis, $\{x, y\} \in R$. In the second, it follows that either $\{x, y\} = x$, or $\{x, y\} = y$, and once again, given the hypothesis, we have that $\{x, y\} \in R$.

*Theorem 5.* $\{\{x, R\}\} \in R$.

*Proof.* Indeed, either $\{\{x, R\}\} \in \{\{x, R\}\}$, or $\{\{x, R\}\} \notin \{\{x, R\}\}$. In the second case, it is obvious that $\{\{x, R\}\} \in R$. In the first, by theorem 2, it follows that $\{\{x, R\}\} = \{x, R\}$, and thus, $x = R = \{x, R\}$; therefore, $x = R$, and given that $R \in R$, by theorem 4, $\{x, R\} \in R$. Consequently, by theorem 3, $\{\{x, R\}\} \in R$.

*Theorem 6.* (Arruda and Batens 1982) $\cup R = V$, where $V = \{x: x = x\}$.

*Proof.* It is enough to prove that, for every $x$, $x \in \cup R$. Let us suppose that (1) $\{x, R\} \notin \{x, R\}$; hence, $\{x, R\} \in R$ and, by the definition of union, $x \in \cup R$. On the other hand, if (2) $\{x, R\} \in \{x,R\}$, then either $\{x, R\} = x$, or $\{x, R\} = R$. If $\{x, R\} = R$, it follows that $x \in \cup R$. If $\{x, R\} = x$, one has that $\{\{x, R\}\} = \{x\}$, and given that $\{\{x, R\}\} \in R$ (theorem 5), it follows that $\{x\} \in R$; accordingly, $x \in \cup R$.

*Remark.* So, a set theory with Russell's set has in general a universal class. (A classical set theory of the ZF kind with universal class was developed in

Church (1974); in da Costa (1986) this was extended to a paraconsistent set theory.)

*Definition 2.* $\wp(x)$ denotes the *power set* of $x$.

*Theorem 7.* (Arruda) $\ldots \subset \wp(\wp(R)) \subset \wp(R) \subset R$.

*Proof.* If $x \in \wp(R)$, $x \subset R$. Now, either $x \notin x$, or $x \in x$. If $x \notin x$, $x \in R$; if $x \in x$, given that $x \subset R$, it follows that $x \in R$. Therefore, $\wp(R) \subset R$.

Furthermore, if $x \in \wp(\wp(R))$, then $x \subset \wp(R)$, and by the preceding result, $x \subset R$; hence, $x \in \wp(R)$. Thus, $\wp(\wp(R)) \subset \wp(R) \subset R$. One can now easily complete this proof.

Additional results concerning $R$ are the following ones:

*Theorem 8.* $\varnothing \in R$, $\{\varnothing\} \in R$, $\{\{\varnothing\}\} \in R, \ldots$

*Theorem 9.* $\exists x\,(x \notin R)$.

*Theorem 10.* $x, y \in R \rightarrow \langle x, y \rangle \in R$.

*Theorem 11.* $x \subset R \rightarrow x \in R$.

*Theorem 12.* $R \times R \subset R$.

Given theorem 5, it is possible to demonstrate that $R$ is, as it were, a 'internal model' of the set theory in which we work. Moreover, given that $\cup R = V$, it follows that the existence of $R$ implies the existence of infinite sets.

The properties of $R$ are by no means arbitrary. Thus, it is not possible to prove everything with regard to $R$, without also proving, at the same time, that some of the classical, standard set theories are inconsistent (see da Costa 1986, and also da Costa 1964).

Besides $R$, it is not difficult to introduce and to study Russell's relations:

$$\langle x_1, x_2, \ldots, x_n \rangle \in R_{n,i} \leftrightarrow \langle x_1, x_2, \ldots, x_n \rangle \notin x_i .$$

It is easy to prove that:

*Theorem 13.* $R_{n,i} \in R_{n,i} \wedge R_{n,i} \notin R_{n,i}$.

*Theorem 14.* $V \times V \times \ldots \times V = \cup R_{n,i}$, where the product on the left has $n$ terms.

It is plain that $R_{1,1}$ is $R$, when we make $\langle x \rangle = x$.

### *4.2. A paraconsistent Boolean algebra*

Within various paraconsistent set theories of a certain kind, it is possible to consider intuitively a set as an ordered pair, in the classical sense, of sets that are part of a universe-set $V$. Thus, a set $X$ is the pair $<X_1, X_2>$, where:

(1) $x \in X$ if, and only if, $x \in X_1$;
(2) $x \notin X$ if, and only if, $x \in X_2$;
(3) $x \in X$ and $x \notin X$ is equivalent to $x \in X_1$ and $x \in X_2$.

Given that the principle of the excluded middle is maintained in certain paraconsistent set theories, it should be the case that $X_1 \cup X_2 = V$. If $X_1 \cap X_2 = \varnothing$, a *classical* set is obtained.

Let us consider then the collection of the sets just constructed on $V$, which shall be denoted by $\mathcal{V}$. An element of $\mathcal{V}$ is called a *paraconsistent set,* or a p-set. In what follows we shall outline an algebra of p-sets $\mathcal{V}$. We will suppose that the p-sets are embedded in a classical set theory, for instance, ZF.

*Definition 1.* (*Union*) If $X = <X_1, X_2>$ and $Y = <Y_1, Y_2>$, then $X \cup Y = <X_1 \cup Y_1, X_2 \cap Y_2>$.

*Definition 2.* We shall denote by *1* the pair $<V, \varnothing>$, and by *0*, $<\varnothing, V>$.

*Theorem 1.* All the following identities hold:

$X \cup X = X$;
$X \cup Y = Y \cup X$;
$(X \cup Y) \cup Z = X \cup (Y \cup Z)$;
$1 \cup X = 1$;
$0 \cup X = X$.

*Definition 3.* (*Intersection*) If $X = <X_1, X_2>$ and $Y = <Y_1, Y_2>$, then $X \cap Y = <X_1 \cap Y_1, X_2 \cup Y_2>$.

*Theorem 2.* The following are some of the properties of the intersection:

$X \cap X = X$;
$X \cap Y = Y \cap X$;
$(X \cap Y) \cap Z = X \cap (Y \cap Z)$;
$1 \cap X = X$;
$0 \cap X = 0$.

*Definition 4.* (*Complement*) If $X = \langle X_1, X_2 \rangle$, then $X' = \langle X_2, X_1 \rangle$.

*Definition 5.* (*Inclusion*) If $X = \langle X_1, X_2 \rangle$ and $Y = \langle Y_1, Y_2 \rangle$, then $X \subset Y$ means that $X_1 \subset Y_1$ and $Y_2 \subset X_2$.

*Theorem 3.* The following identities hold:

$X'' = X$;
$1' = 0$;
$0' = 1$;
$X \cup X' \subset 1$;
$0 \subset X \cap X'$.

*Theorem 4.* These are some of the properties of the inclusion:

$X \subset X$;
if $X \subset Y$ and $Y \subset X$, then $X = Y$;
if $X \subset Y$ and $Y \subset Z$, then $X \subset Z$;
$0 \subset X$;
$X \subset X \cup X'$;
$X \cap X' \subset X$;
$X \subset 1$.

*Definition 6.* The structure $P = \langle V, \cap, \cup, ', 0, 1 \rangle$ is called a *paraconsistent Boolean algebra.*

Through the employment of this structure it is possible to formalise several paraconsistent patterns of reasoning, just as with classical Boolean algebras one can put in algebraic terms various classical inferences. Moreover, one can verify that the paraconsistent logical mechanism considered here does not exclude classical logic, but extends it in some sense; though under another viewpoint, it can be embedded into traditional logical structures. Of course, such remarks are valid for particular categories of paraconsistent structures; however, they are of extreme relevance in order to corroborate the fact that both paraconsistent logic as well as paraconsistent mathematics, as far as we understand them, do not destroy either the traditional logic, or standard mathematics, but only complement them and, in certain cases, extend them.

The structure of the paraconsistent Boolean algebra clearly is richer than the classical one. Thus, for instance, one can introduce two operators, $\alpha_1$ and $\alpha_2$, such that, given a p-set $X$, $\alpha_1(X) = X_1$ and $\alpha_2(X) = X_2$, where $X_1$ and $X_2$ are in another Boolean algebra, the classical algebra of the subsets of $V$ etc.

When the structure $P$, in definition 6, is such that, for every $X = <X_1, X_2>$, it is the case that $X_1 \cap X_2 = \varnothing$, one obtains a Boolean algebra that essentially is the usual algebra of the subsets of $V$.

In this way, it is possible to construct a general theory of paraconsistent structures (algebraic, topological, of order etc.), obtaining thus a generalisation of the traditional theory of structures, such as Bourbaki's (cf. Bourbaki 1968). Moreover, paraconsistent structures, such as those described in this section, have been applied to several areas, such as computer science, artificial intelligence and logic programming (see, e.g., Subrahmanian (1987), Blair and Subrahmanian (1987) and (1988), and Kifer and Subrahmanian (1992)). This provides a significant motivation for their study.

### *4.3. Semantic analysis of a paraconsistent logic*

It is not difficult to elaborate a monadic quantificational paraconsistent logic (the extension to the polyadic case does not present a considerable challenge) with a semantics based on Zermelo-Fraenkel set theory (ZF). The construction is analogous to the case of fuzzy set theory, which is usually built within ZF.

We shall call $M$ the logic to be constructed here. Its language is that of the uniform monadic classical quantificational calculus (there is only one individual variable $x$), with individual constants, but without identity. The predicate symbols are denoted by Latin capital letters and the individual constants by Latin small ones.

A structure for $M$ is a semantic construction of the following kind:

$$S = <V', P', Q', R', \ldots, a', b', c', \ldots>,$$

where $V$, the universe of the structure, is a non-empty set, $P' = <P_1, P_2>$, $P_1 \subset V$, $P_2 \subset V$, and $P_1 \cup P_2 = V$; $Q' = <Q_1, Q_2>$, $Q_1 \subset V$, $Q_2 \subset V$, and $Q_1 \cup Q_2 = V$ etc., and $a', b', c', \ldots$ are elements of $V$. $P'$, for instance, is a (paraconsistent) predicate; if an element $k \in V$ is such that $k \in P_1$, $k$ satisfy $P$; if $k \in P_2$, then $k$ does not satisfy $P$. It is obvious that if $k$ belongs simultaneously to $P_1$ and to $P_2$, $k$ satisfies and does not satisfy $P$.

An interpretation of $M$ in $S$ assigns to each predicate symbol of $M$ a paraconsistent predicate of $S$, and to each individual constant one of the elements $a', b', c', \ldots$, as is usual. One defines diagram language $M(S)$ in the standard way, and any interpretation can be extended to all the introduced names (in Shoenfield's sense; see Shoenfield 1967).

Let us now define a valuation associated with an interpretation. In order to do so, we shall present some notations:

Given a formula $F$, we shall denote by $F^*$ the formula obtained from $F$ in the following way: (1) one eliminates $\rightarrow$ and $\leftrightarrow$ through the usual definitions, in terms either of $\neg$ and $\vee$, or of $\neg$ and $\wedge$; (2) every negation is transported to the 'inner part' of the formula, so that it affects only atomic subformulas or negations of such subformulas of $F$.

If $I$ is an interpretation of $M$ in $S$, $v_I$ or, simply, $v$, is the associated valuation. It can be defined thus, where $F$ is a sentence of $M(S)$:

(1) $v(F) = v(F^*)$;
(2) $v(G \vee H) = 1 \Leftrightarrow v(G) = 1$ or $v(H) = 1$;
(3) $v(G \wedge H) = 1 \Leftrightarrow v(G) = v(H) = 1$;
(4) $v(\neg^{2n+1} P(k)) = v(\neg P(k))$;
(5) $v(\neg^{2n} P(k)) = v(P(k))$;
(6) $v(P(k)) = 1 \Leftrightarrow k \in P_1$;
(7) $v(P(k)) = 0 \Leftrightarrow k \notin P_1$;
(8) $v(\neg P(k)) = 1 \Leftrightarrow k \in P_2$;
(9) $v(\neg P(k)) = 0 \Leftrightarrow k \notin P_2$;
(10) $v(\forall x\, G(x)) = 1 \Leftrightarrow v\,(G(k)) = 1$, for every name or constant $k$;
(11) $v(\forall x\, G(x)) = 0 \Leftrightarrow v\,(G(k)) = 1$, for some name or constant $k$;
(12) '$v(\exists x\, G(x)) = 1$' is defined in the usual way;
(13) '$v(\exists x\, G(x)) = 0$' is also defined in the usual way.

As is plain, in this definition, one supposes that $k$ denotes a name or an individual constant, that $I$ assigns $P$ to $P'$ etc.

Thus, one defines in $M$: $\models_M F$ if, for every interpretation $I$ and valuation $v_I$, $v_I(F) = 1$.

Given a formula $F^*$, one denotes by $F^{*0}$ the following formula: one replaces, in $F^*$, every occurrence of $P$ for $P_1$ (new predicate symbol) and of $\neg P$ for $P_2$ (new predicate symbol, other than $P_1$) etc. Hence for every predicate symbol $P$ of $M$, we associate two new predicates, $P_1$ and $P_2$; for Q, we associate $Q_1$ and $Q_2$ etc.

Let us add to $M$ a new implication symbol, $\supset$, semantically characterised by the classical condition: $v(G \supset H) = 1$ if, and only if, $v(G) = 0$ or $v(H) = 1$. Moreover, let us also add to $M$ the new predicate symbols $P_1$ and $P_2$, $Q_1$ and $Q_2$,..., admitting that in no formula are there occurrences of $\supset$ within the scope of negations. Then, an axiomatic for $M$ is the following:

(1) A system of postulates for classical positive uniform calculus, relative to $\supset$, $\wedge$, $\vee$, $\forall$, and $\exists$.

(2) $P_1 \vee P_2$, $Q_1 \vee Q_2$,...
(3) $F^{*0} / F$, for every formula in which there is no occurrence of $\supset$.

It follows that: $\vdash_M F \Leftrightarrow \vDash_M F$.

We briefly state some theorems of $M$:

$\vdash_M F \wedge \neg F \rightarrow G$
$\vdash_M \neg (F \wedge \neg F)$
$\vdash_M F \vee (F \supset G)$
$\vdash_M \forall x \neg (Px \wedge \neg Px)$
$\vdash_M (F \vee \neg F)$
$\vdash_M \forall x (Px \vee \neg Px)$

However, the rule and the sentences below do *not* hold in $M$:

$F, F \rightarrow G / G$
$Pa \wedge \neg Pa \supset G$
$Pa \wedge \neg Pa$

The predicates $R$ such that $R_1 k \wedge R_2 k$ are satisfied by *no* $k$ in $V$ are called *classical*. In this case, $v(Ra \wedge \neg Ra) = 0$, for every $a$ in $V$, and $R$ has a classical behavior.

As just presented, $M$ consists in a starting point in order to develop, for instance, a paraconsistent syllogistic -just as the one that shall be described in the next section. Furthermore, it can also be employed as the foundations for a syllogistic whose nature was outlined by N.A. Vasil'év, one of the forerunners of paraconsistent logic. (For an exposition of his views and references to his works, see Arruda (1984).)

*5. A case study: syllogism and paraconsistency*

In this section, an application of the general framework supplied by paraconsistent logic will be made to that which is perhaps one of the most ancestral domains of traditional logic: the theory of syllogism. The main point consists in addressing the issue regarding the employment of a paraconsistent logic in order to articulate such a theory, and examining which of the traditional inferences still hold.

After briefly reviewing, in section 5.1, some aspects of classical syllogistic, in section 5.2, we shall concisely present some possible answers to this issue.

### *5.1. Classical syllogistic*

Within traditional logic, which arose from various sources through a process of modification and adaptation of Aristotelian ideas, categorical syllogism has a relevant role. It turned out that the traditional system of doctrines is not coherent, and here we shall interpret it in one of its possible ways.

There are four types of categorical propositions which constitute the core of traditional logic: (1) universal affirmative ones: (*A*) every *a* is *b* (every man is mortal); (2) particular affirmative ones: (*I*) some *a* is *b* (some man is mortal); (3) universal negative ones: (*E*) no *a* is *b* (no man is mortal); and (4) particular negative ones: (*O*) some *a* is not *b* (some man is not mortal). In propositions *A*, *I*, *E*, *O*, *a* and *b* are terms, linked by the copula (verb); *a* is the subject and *b*, the predicate. Following Lukasiewicz's proposal, we may symbolise such propositions thus: *Aab*, *Iab*, *Eab* and *Oab*. (For Lukasiewicz's interpretation of Aristotle, see his 1971.) We shall suppose that the terms that appear in them are not empty, that is, the classes related to them have elements (this seems to be Aristotle's position), and also that such classes are not singular, i.e., that they do not have just one member (therefore, no term is a proper name).

Following the standard account, we shall interpret categorical propositions within classical monadic first-order predicate calculus, in the following way:

| | |
|---|---|
| $Aab$ | $\forall x\,(a(x) \rightarrow b(x))$ |
| $Iab$ | $\exists x\,(a(x) \wedge b(x))$ |
| $Eab$ | $\forall x\,(a(x) \rightarrow \neg\, b(x))$ |
| $Oab$ | $\exists x\,(a(x) \wedge \neg\, b(x))$ |

Given this interpretation, it is possible to examine several parts of traditional logic, determining the validity of both formulas as well as inferences.

Regarding categorical propositions, traditional logic is concerned with four main themes: (1) the square of oppositions; (2) the theory of conversion; (3) immediate inferences; and (4) the theory of categorical syllogism. We shall briefly consider each of them.

*Opposition.* As it is known, though this figure is not to be found in Aristotle's works, the propositions *A*, *I*, *E* and *O* are arranged in the vertices of a square, where *A* and *O*, and *E* and *I* are called contradictories, *A* and *E*, contraries etc. One can show, for instance, that two contradictory propositions are not both truth, nor both false (one is the negation of the other) etc.

*Conversion.* The conversion consists in changing either the position of the terms in a categorical proposition, or its quantity (the fact of being universal or particular), or its quality (affirmative or negative), or the nature of its terms (from positive ones, for instance, *animal*, to negative ones, *non-animal*, and vice-versa). There are three basic kinds of conversion: the simple one, the one based on accident, and obversion; through their combination, new types of conversion are obtained. The logician is mainly concerned with the determination of the validity of such operations: when they lead from true propositions to true ones. For instance, with recourse to conversion by accident, one may conclude that 'Some animals are human' from the proposition 'Every human is an animal'. In order to express obversion within monadic predicate calculus, we assume that to every predicate $p(x)$ there is an associated one, $p'(x)$, such that $\forall x\,(\neg\, p(x) \leftrightarrow p'(x))$.

*Immediate inferences.* These kinds of inferences have only one premise. The theories of opposition and of conversion supply criteria to test the validity of them. Thus, based on conversion theory, from *Aab* it is possible to conclude *Iba*, as may be shown within the monadic predicate calculus. A further example is the following one: from *Aab* one may conclude *Iab*, by subordination, a relation examined in the theory of opposition, and a valid inference within monadic calculus.

*Categorical syllogism.* A categorical syllogism or, for short, simply syllogism, consists in an inference with two premises and one conclusion, both of them being categorical propositions. It is known that there are 256 possible syllogisms, distributed in four figures, each of them having certain valid modes. In our case, there are six valid modes to each figure, adding up to twenty-four valid syllogisms. Traditionally such syllogisms have special names. For the first figure: Barbara, Celarent, Darii, Ferio, Barbari and Celaront; for the second figure: Cesare, Camestres, Festino, Baroco, Cesaro and Camestros; third figure: Darapti, Disamis, Datisi, Felapton, Bocardo and Ferison; and fourth figure: Bramantip, Camenes, Dimaris, Fesapo, Fresison and Camenos. The vowel and the consonants in these names have certain meanings which are not relevant for our present purposes. (For further historical details on traditional syllogistic, cf. Kneale and Kneale 1988, pp. 23-112.)

### *5.2. Paraconsistent syllogistic*

Similarly to the case of traditional syllogistic, which was interpreted within classical monadic predicate calculus, it is possible to develop a paraconsistent syllogistic. It is based on, for instance, the monadic calculus corresponding to the paraconsistent predicate logic $C^*_1$. In order to reach that, it suffices that one translate the propositions *A*, *I*, *E* and *O* into $C^*_1$: the translations are formally the same as the ones just presented in the last section, which were based on the classical setting.

There are two brief remarks to be made within this context. (1) The valid positive deductions in $C^*_0$, the classical predicate calculus, are also valid in $C^*_1$; that is, when no explicit negation is involved, the *positive* deductions of $C^*_0$ and $C^*_1$ are the same. (2) In $C^*_1$ one can find 'paraconsistent' predicates, such that, for instance, there are elements that satisfy the predicate and, at the same time, do not satisfy it; i.e., for some predicate $p$ the following holds:

$$\exists x\,(p(x) \wedge \neg\, p(x)).$$

Thus, based on arguments rather similar to the ones found in the classical case, it is possible to verify the validity of inferences, and one changes accordingly the theories of opposition, conversion, immediate inferences and syllogism. (Each predicate within the universe of discourse has three parts: of the elements that satisfy it, of those that do not satisfy it, and of those that simultaneously satisfy it and do not satisfy it. Simple graphics supply then evidence for the validity, or for the invalidity, of certain inferences and conversions.)

Based on this approach, one can prove the following result. In the paraconsistent logic $C^*_1$, all modes of the first and of the third figures of the syllogism are valid; none of the second is valid; and of the fourth, just Bramantip and Dimaris modes are valid.

It is worth mentioning that $C^*_1$ has a strong negation, of a classical trend, and if such negation is adopted in the interpretation of syllogistic reasoning, the classical theory is obtained.

As is known, Lukasiewicz has axiomatised the theory of categorical syllogism, based on the classical propositional calculus and admitting as specific axioms certain categorical propositions, as well as some appropriate definitions. Based on the paraconsistent propositional calculus, for instance, the calculus $C_1$ (cf. da Costa 1974), it is also possible to formulate an axiomatics for paraconsistent syllogistic, articulated in parallel lines to the theory just outlined. Moreover, we should note that there are further extensions or modifications of the Aristotelian syllogistic that also admit paraconsistent versions, such as Hamilton's, De Morgan's and Gergone's.

## *6. Concluding remarks*

From the previous remarks, several conclusions that outline a particular view of paraconsistent logic and, in general, of contemporary logic might be drawn.

We shall just briefly point out some of them here, leaving their development for further works.

(1) Paraconsistent logic, as opposed to the classical one, despite being a logic which allows us to examine the properties of 'contradictory objects', such as Russell's set, does not lead us to trivialisation, and moreover it is simply *not* the case that these objects have every imaginable feature. To some extent, they behave just as normally as other standard classical objects.

(2) The tentative points suggested here shall indicate that paraconsistent logic is philosophically neutral, in the same sense that, for instance, mathematics is. The latter, just as the former, cannot justify by itself any metaphysical or, in general, 'speculative' position. (It goes without saying, however, that logic and mathematics as well as the activity of logicians and mathematicians are subject to philosophical interpretations.)

(3) In this regard, we would like to stress however that one *cannot* prove that 'speculative' philosophical interpretations of paraconsistent logic *cannot be true* (though it might be also difficult to show that they are). Our interpretation, nevertheless, not being committed to such 'speculative' approaches, seems to be philosophically more acceptable.

(4) Once the distinction between pure and applied logic is made, it seems natural to claim that the latter is not restricted exclusively to *a priori* considerations, but depends on the particular features of the domain to which it is applied (or on the propositional way of representing the latter). As von Neumann claimed:

> The basic idea is that the system of logics which one uses should be derived from aggregate experiences relative to the main application which one wishes to make -logics should be inspired by experience. (von Neumann [1937], p. 2)

*Acknowledgements*

We wish to thank Steven French for comments on an earlier version of this paper. Thanks are also due to two anonymous referees for their helpful suggestions.

## *BIBLIOGRAPHY*


Arruda, A.I.: 1980, 'A Survey of Paraconsistent Logic', in Arruda, Chuaqui, and da Costa (eds.) (1980), pp. 1-41.

Arruda, A.I.: 1984, 'N.A. Vasil'év: A Forerunner of Paraconsistent Logic', *Philosophia Naturalis* 21, 472-491.

Arruda, A.I., and Batens, D.: 1982, 'Russell's Set versus the Universal Set in Paraconsistent Set Theories', *Logique et Analyse* 25, 121-136.

Arruda, A., Chuaqui, R., and da Costa, N.C.A. (eds.): 1980, *Mathematical Logic in Latin America*, Amsterdam, North-Holland.

Batens, A. *et al.* (eds): 1999, *Proceedings of the World Congress of Paraconsistency*, Dordrecht, Kluwer Academic Publishers, forthcoming.

Beltrametti, E., and van Fraassen, B.C. (eds.): 1981, *Current Issues in Quantum Logic*, New York, Plenum Press.

Blair, H.A., and Subrahmanian, V.S.: 1987, 'Paraconsistent Logic Programing', *Proc. 7th Intl. Conf. on Foundations of Software Technology and Theoretical Computer Science*. Lecture Notes in Computer Science, vol. 310, Berlin, Springer-Verlag, pp. 340-360.

Blair, H.A., and Subrahmanian, V.S.: 1988, 'Paraconsistent Foundations for Logic Programming', *The Journal of Non-Classical Logic* 5, 45-73.

Bourbaki, N.: 1968, *Theory of Sets*, Boston, Mass., Addison-Wesley.

Browder, F.E. (ed.): 1974, *Mathematical Developments Arising from Hilbert Problems* (Proceedings of the Symposium in Pure Mathematics, *XXVIII*), Providence, American Mathematical Society.

Church, A.: 1974, 'Set Theory with a Universal Set', in Henkin (ed.) (1974), pp. 297-308.

da Costa, N.C.A.: 1964, 'Sur un Système Inconsistant de la Théorie des Ensembles', *Comptes Rendus Hebdomadaires des Séances de l' Académie des Sciences*, Paris, 258, 3144-3147.

da Costa, N.C.A.: 1974, 'On the Theory of Inconsistent Formal Systems', *Notre Dame Journal of Formal Logic* 15, 497-510.

da Costa, N.C.A.: 1986, 'On Paraconsistent Set Theory', *Logique et Analyse* 115, 361-371.

da Costa, N.C.A.: 1999, 'Paraconsistent Mathematics', forthcoming in Batens *et al.* (eds) [1999].

da Costa, N.C.A., Béziau, J.-Y., and Bueno, O.: 1995*a*, 'Aspects of Paraconsistent Logic', *Bulletin of the Interest Group in Pure and Applied Logics* 3, 597-614.

da Costa, N.C.A., Béziau, J.-Y., and Bueno, O.: 1995*b*, 'Paraconsistent Logic in a Historical Perspective', *Logique et Analyse* 150-151-152, 111-125.

da Costa, N.C.A., Béziau, J.-Y., and Bueno, O.: 1998, *Elements of Paraconsistent Set Theory* [in Portuguese], Campinas, Coleção CLE.

da Costa, N.C.A., and Bueno, O.: 1996, 'Consistency, Paraconsistency and Truth (Logic, the Whole Logic and Nothing but the Logic)', *Ideas y Valores* 100, 48-60.

da Costa, N.C.A., Bueno, O., and Béziau, J.-Y.: 1995, 'What is Semantics? A Brief Note on a Huge Question', *Sorites - Electronic Quarterly of Analytical Philosophy* 3, 43-47.

da Costa, N.C.A., Bueno, O., and Volkov, A.: 1999, 'Outline of a Paraconsistent Category Theory', University of São Paulo, California State University and University of Paraná, forthcoming.

Dalla Chiara, M.L., and Giuntini, R.: 1989, 'Paraconsistent Quantum Logics', *Foundations of Physics* 19, 891-904.

D'Ottaviano, I.: 1990, 'On the Development of Paraconsistent Logic and da Costa's Work', *The Journal of Non-Classical Logic* 7, 89-152.

Heisenberg, W.: 1958, *Physics and Philosophy*, London, Allen & Unwin.

Henkin, L. (ed.): 1974, *Proceedings of the Tarski Symposium*, Providence, American Mathematical Society.

Hilbert, D.: 1927, 'The Foundations of Mathematics'. English translation of the original 1927 German paper reprinted in van Heijenoort (ed.) (1967), pp. 464-479.

Kifer, M., and Subrahmanian, V.S.: 1992, 'Theory of Generalized Annoted Logic Programming and its Applications', *Journal of Logic Programming 12*, 335-367.

Kneale, W., and Kneale, M.: 1988, *The Development of Logic*, Oxford, Clarendon Press. First published in 1962.

Kochen, S., and Specker, E.: 1967, 'The Problem of Hidden Variables in Quantum Mechanics', *Journal of Mathematics and Mechanics* 17, 59-87.

Lukasiewicz, J.: 1971, 'On the Principle of Contradiction in Aristotle', *Review of Metaphysics* 24, 485-509.

Manin, Yu.I.: 1974, 'Problems of Present Day Mathematics - Foundations', in Browder (ed.) (1974).

Manin, Yu.I.: 1977, *A Course in Mathematical Logic*, New York, Springer-Verlag.

Mortensen, C.: 1995, *Inconsistent Mathematics*, Dordrecht, Kluwer.

Schrödinger, E.: 1952, *Science and Humanism*, Cambridge, Cambridge University Press.

Shoenfield, J.R.: 1967, *Mathematical Logic*, Reading, Mass., Addison-Wesley Publishing Company.

Subrahmanian, V.S.: 1987, 'On the Semantics of Quantitative Logic Programs', *Proc. 4th IEEE Symp. on Logic Programing*, San Francisco, Computer Society Press, pp. 173-182.

Takeuti, G.: 1981*a*, 'Quantum Set Theory', in Beltrametti and van Fraassen (eds.) (1981).

Takeuti, G.: 1981*b*, *Two Applications of Logic to Mathematics*, Princeton, Princeton University Press.

van Heijenoort, J. (ed.): 1967, *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, Cambridge, Mass., Harvard University Press.

von Neumann, J.: [1937], 'On Alternative Systems of Logics', unpublished manuscript, von Neumann Archives, Library of Congress, Washington, D.C.

Weyl, H.: 1963, *Philosophy of Mathematics and Natural Science*, New York, Atheneum.



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