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Approximation of Fixed Points of C^*-Algebra-Multi-Valued Contractive Mappings by the Mann and Ishikawa Processes in Convex C^*-Algebra-Valued Metric Spaces

Azadeh Ghanifard [1,†], Hashem Parvaneh Masiha [1,†] and Manuel De La Sen [2,*,†]

1 Faculty of Mathematics, K. N. Toosi University of Technology, Tehran 16569, Iran; a.ghanifard@email.kntu.ac.ir (A.G.); masiha@kntu.ac.ir (H.P.M.)
2 Institute of Research and Development of Processes University of the Basque Country, 48940 Leioa, Spain
* Correspondence: manuel.delasen@ehu.eus
† These authors contributed equally to this work.

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Abstract: The aim of the present paper is to state and prove some convergence theorems for the Mann and Ishikawa iteration schemes involving C^*-algebra-multi-valued contractive mappings in the setting of convex C^*-algebra-valued metric spaces. The convergence theorems of the proposed iterations to a common fixed point of finite and infinite family of such mappings are also established.

Keywords: convex C^*-algebra-valued metric space; C^*-algebra-multi-valued contractive mapping; Mann iteration; Ishikawa iteration; fixed point

1. Introduction and Priliminaries

Let $\mathcal{A}$ be a unital algebra(over the field $\mathbb{R}$ or $\mathbb{C}$) with the unit element $1_{\mathcal{A}}$ and the zero element $0_{\mathcal{A}}$. A conjugate linear map $* : \mathcal{A} \to \mathcal{A}$ is an involution on $\mathcal{A}$ if $a^{**} = a$ and $(ab)^* = b^*a^*$, for any $a, b \in \mathcal{A}$. Moreover, if there exists a complete submultiplicative norm $\|.\|$ on a $*$-algebra $(\mathcal{A}, *)$ such that $\|a\| = \|a^*\|$, for all $a \in \mathcal{A}$, then $\mathcal{A}$ is called a Banach $*$-algebra. A C^*-algebra $\mathcal{A}$ is a Banach $*$-algebra such that $\|aa^*\| = \|a\|^2$, for each $a \in \mathcal{A}$.

Set $\mathcal{A}_h = \{a \in \mathcal{A} : a = a^*\}$. An element $a \in \mathcal{A}$ is called a positive element and is denoted by $a \succeq 0_{\mathcal{A}}$, if $a \in \mathcal{A}_h$ and $\sigma(a) \subset [0, +\infty)$, where $\sigma(a) = \{\mu \in \mathbb{R}; \mu 1_{\mathcal{A}} - a \notin \mathcal{A}^{-1}\}$ is the spectrum of a. Using positive elements, there exists a natural partial ordering $\preceq$ on $\mathcal{A}$ as follows:

$$a \preceq b \iff b - a \succeq 0_{\mathcal{A}}.$$

By $\mathcal{A}_+$ we denote the set of all positive elements of $\mathcal{A}$.

For further details and results on C^*-algebras, refer to [1–3]. In particular, we will use the following lemmas:

Lemma 1. ([1]) *Let $\mathcal{A}$ be a C^*-algebra. Then:*

(i) $\mathcal{A}_+ = \{a^*a : a \in \mathcal{A}\}$;
(ii) *Let $c \in \mathcal{A}$. If $a, b \in \mathcal{A}_h$ with $a \preceq b$, then $c^*ac \preceq c^*bc$.*
(iii) *For any $a, b \in \mathcal{A}_h$, if $0_{\mathcal{A}} \preceq a \preceq b$, then $0 \leq \|a\| \leq \|b\|$.*

Lemma 2. ([1]) *Let $\mathcal{A}$ be a C^*-algebra and $h, k \in \mathcal{A}$ with $h \succeq 0_{\mathcal{A}}$ and $k \succeq 0_{\mathcal{A}}$. Then $h + k \succeq 0_{\mathcal{A}}$.*

C^*-algebras are now an important tool in the theory of unitary representations of locally compact groups and are also used in algebraic formulations of quantum mechanics. Based on the notion and properties of C^*-algebras, several researchers introduced the notion of C^*-algebra-valued metric spaces as a generalization of the metric spaces and established some fixed point theorems satisfying the contractive or expansive conditions on such spaces. In 2014, Ma et al. [4] introduced the following concept of C^*-algebra-valued metric:

Definition 1. ([4]) *Let M be a nonempty set. Suppose the mapping $\rho : M \times M \to \mathcal{A}$ satisfies the following conditions for each $x, y, z \in M$:*

(1) $\rho(x,y) \succeq 0_{\mathcal{A}}$ *and* $\rho(x,y) = 0_{\mathcal{A}} \Leftrightarrow x = y$;
(2) $\rho(x,y) = \rho(y,x)$;
(3) $\rho(x,y) \preceq \rho(x,z) + \rho(z,y)$.

Then ρ is called a C^*-algebra-valued metric on M and $(M, \mathcal{A}, \rho)$ is called a C^*-algebra-valued metric space.

Definition 2. ([4]) *Let $(M, \mathcal{A}, \rho)$ be a C^*-algebra-valued metric space, $\{x_n\} \subset M$ and $x \in M$.*

(i) *If for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $\|\rho(x_n, x)\| < \varepsilon$, then $\{x_n\}$ is said to be convergent to x with respect to $\mathcal{A}$ and we say x is the limit of $\{x_n\}$. we denote it by $\lim_{n\to\infty} x_n = x$.*
(ii) *If for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $\|\rho(x_n, x_m)\| < \varepsilon$, then $\{x_n\}$ is said to be a Cauchy sequence with respect to $\mathcal{A}$.*
(iii) *The triple $(M, \mathcal{A}, \rho)$ is called a complete C^*-algebra-valued metric space if every Cauchy sequence is convergent with respect to $\mathcal{A}$.*

They also defined the concept of C^*-algebra-valued contractive mapping as follows:

Definition 3. ([4]) *Let $(M, \mathcal{A}, \rho)$ be a C^*-algebra-valued metric space. A mapping $T : M \to M$ is said to be a C^*-algebra-valued contractive mapping on M, if there exists an element $k \in \mathcal{A}$ with $\|k\| < 1$ such that for any $x, y \in M$:*

$$\rho(Tx, Ty) \preceq k^* \rho(x,y) k,$$

As a main result, they proved the following theorem:

Theorem 1. ([4]) *Let $(M, \mathcal{A}, \rho)$ be a complete C^*-algebra-valued metric space and T be a C^*-algebra-valued contractive mapping on M. Then T has a unique fixed point in M.*

The theory of multi-valued mappings is a branch of mathematics which has been developed intensively in the last years. This theory has applications in control theory, convex optimization, differential inclusions and economics.

In the study of fixed points for multi-valued mappings two type of distances are generally used. One is the Pompeiu–Hausdorff distance which for any two bounded subsets A_1 and A_2 of a metric space (M, d) is defined by

$$H(A_1, A_2) = \max\{\sup_{a \in A_1} d(a, A_2), \sup_{b \in A_2} d(A_1, b)\}$$

where $d(a, B) = \inf\{d(a,b); b \in B\}$. The another is the δ-distance which for any subsets A_1, A_2, mentioned above, is defined by

$$\delta(A_1, A_2) = \sup\{d(a,b); a \in A_1, b \in A_2\}$$

There are many works in fixed point theory which have utilized Hausdorff distance or δ-distance. See for instance [5–10].

Now, in order to state our results, we need to define the distance between two subsets in C^*-algebra-valued metric space:

Definition 4. *Let $(M, \mathcal{A}, \rho)$ be a C^*-algebra-valued metric space. A subset A of M is called bounded if*

$$\sup\{\|\rho(x,y)\| : x, y \in A\} < \infty.$$

We shall denote by $B(M)$ the family of nonempty bounded subsets of M.

Definition 5. *Let $(M, \mathcal{A}, \rho)$ be a C^*-algebra-valued metric space. The distance between two subsets $A_1, A_2 \in B(M)$ is defined by*

$$D(A_1, A_2) = (\max\{\sup_{a \in A_1} \|\rho(a, A_2)\|, \sup_{b \in A_2} \|\rho(A_1, b)\|\})1_{\mathcal{A}},$$

where

$$\rho(a, B) = (\inf\{\|\rho(a,b)\| : b \in B\})1_{\mathcal{A}}.$$

So, similar to definition 1.5, we can give the concept of contractivity for multi-valued maps in C^*-algebra-valued metric spaces:

Definition 6. *Let $(M, \mathcal{A}, \rho)$ be a C^*-algebra-valued metric space. A multi-valued mapping $T : M \to B(M)$ is called a C^*-algebra-multi-valued contractive mapping, if there exists a $k \in \mathcal{A}$ with $\|k\| < 1$ such that*

$$D(Tx, Ty) \preceq k^* \rho(x,y) k,$$

for all $x, y \in M$.

A point $x \in M$ is called a fixed point of the mapping T if $x \in Tx$. The set of all fixed points of T is denoted by $F(T)$.

Example 1. *Let $M = \mathbb{R}$ and $\mathcal{A} = M_2(\mathbb{R})$ (the set of all 2×2 matrix on $(\mathbb{R})$ with the norm $\|A\| = \max_{i,j} |a_{ij}|$, where a_{ij} are the entries of the matrix $A \in M_2(\mathbb{R})$ and the involution given by $A^* = A^T$. Define $\rho : M \times M \to \mathcal{A}$ by*

$$\rho(x,y) = \mathrm{diag}(|\mathrm{x} - \mathrm{y}|, |\mathrm{x} - \mathrm{y}|),$$

where $\mathrm{diag}(|\mathrm{x} - \mathrm{y}|, |\mathrm{x} - \mathrm{y}|)$ *is a diagonal matrix of order 2 with the two diagonal entries $|x - y|$. Clearly, $(M, \mathcal{A}, \rho)$ is a C^*-algebra valued metric space. We consider the following partial ordering on $\mathcal{A}$:*

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \preceq \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \Leftrightarrow \; a_i \leq b_i \text{ for i} = 1, 2, 3, 4.$$

Let $T : M \to B(M)$ *be defined by* $T(x) = [\frac{x}{5}, \frac{x}{3}]$*, for all* $x \in M$*. Then*

$$\begin{aligned} D(Tx, Ty) &= (\max\{\sup_{a\in Tx} \|\rho(a, Ty)\|, \sup_{b\in Ty} \|\rho(Tx, b)\|\})1_{\mathcal{A}} \\ &= (\max\{\sup_{a\in Tx} \inf_{b\in Ty} |a-b|, \sup_{b\in Ty} \inf_{a\in Tx} |a-b|\})1_{\mathcal{A}} \\ &= (\max\{|\frac{x}{5} - \frac{y}{5}|, |\frac{x}{3} - \frac{y}{3}|\})1_{\mathcal{A}} \\ &= (|\frac{x}{3} - \frac{y}{3}|)1_{\mathcal{A}} \\ &= \mathrm{diag}(|\frac{\mathrm{x}}{3} - \frac{\mathrm{y}}{3}|, |\frac{\mathrm{x}}{3} - \frac{\mathrm{y}}{3}|) \end{aligned}$$

So, T *is a contraction on* M *with* $k = \mathrm{diag}(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Example 2. *In the above example ,* C^**-algebra valued metric* ρ *can be defined as* $\rho(x, y) = diag(|x-y|, \alpha|x-y|)$ *where* $0 \leq \alpha < 1$*. Also, it can be considered as a triangular matrix of order 2 with the entries* $|x-y|$, $|x-y|$, $|sinx - siny|$ *and* 0.

In 1970, Takahashi [11] introduced the following notion of convexity in metric spaces which is a generalization of convexity in normed spaces:

Definition 7. ([11]) *Let* (M, d) *be a metric space and* $I = [0, 1]$*. A mapping* $W : M \times M \times I \to M$ *is said to be a convex structure on* M *if*

$$d(u, W(x, y, t)) \leq td(u, x) + (1-t)d(u, y),$$

for each $x, y, u \in M$ *and all* $t \in I$.

He generalized some fixed point theorems previously proved in Banach space. Since then, many authors have studied fixed point theorems in convex metric spaces. see for instance [12–17].

By help of positive elements in C^*-algebra $\mathcal{A}$, one can easily transfer this concept to C^*-algebra-valued metric spaces:

Definition 8. *Let* $(M, \mathcal{A}, \rho)$ *be a* C^**-algebra-valued metric space and* $I = [0, 1]$*. A convex structure on* M *is a mapping* $W : M \times M \times I \to M$ *which satisfies the following condition for each* $x, y, u \in M$ *and* $t \in I$*:*

$$\rho(u, W(x, y, t)) \preceq t\rho(u, x) + (1-t)\rho(u, y).$$

A C^**-algebra-valued metric space* $(M, \mathcal{A}, \rho)$ *together with a convex structure* W *is called a convex* C^**-algebra-valued metric space and is denoted by* $(M, \mathcal{A}, W, \rho)$.

A subset C *of* M *is called convex if* $W(x, y, t) \in C$*, for all* $x, y \in C$ *and* $t \in I$.

Example 3. *Let* $M = \mathbb{R}^2$ *and* $\mathcal{A} = M_2(\mathbb{R})$ *and* $\preceq$ *be a partial ordering on* $\mathcal{A}$ *given by*

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \preceq \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \Leftrightarrow a_i \leq b_i \text{ for i} = 1, 2, 3, 4.$$

Define $\rho : M \times M \to \mathcal{A}$ *by*

$$\rho((x_1, x_2), (y_1, y_2)) = \mathrm{diag}(|\mathrm{x}_1 - \mathrm{y}_1|, |\mathrm{x}_2 - \mathrm{y}_2|),$$

and $W : M \times M \times [0,1] \to M$ *by*

$$W((x_1, x_2), (y_1, y_2), \alpha) = (\alpha x_1 + (1-\alpha)y_1, \alpha x_2 + (1-\alpha)y_2),$$

for all $(x_1, x_2), (y_1, y_2) \in M$ *and* $\alpha \in [0,1]$. *Then* $(M, \mathcal{A}, W, \rho)$ *is a convex* C^**-algebra-valued metric space.*

Example 4. *Let* $M = M_2(\mathbb{R})$ *and* $\mathcal{A} = \mathbb{R}^2$. *Suppose* $\rho : M \times M \to \mathcal{A}$ *is defined by*

$$\rho(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}) = (\Sigma_{i=1}^4 |a_i - b_i|, 0)$$

and $W : M \times M \times [0,1] \to M$ *is defined by* $W(A, B, t) = tA + (1-t)B$, *for each* $A, B \in M$ *and* $t \in [0,1]$. *Then* $(M, \mathcal{A}, W, \rho)$ *is a convex* C^**-algebra-valued metric space.*

2. Main Results

The main result of this paper is given by the following theorem:

Theorem 2. *Let* $(M, \mathcal{A}, W, \rho)$ *be a complete convex* C^**-algebra-valued metric space and* D *be a nonempty convex subset of* M. *Suppose that* $T : D \to CB(D)$ *is a* C^**-algebra-multi-valued contractive mapping with constant* a *such that* $F(T) \neq \emptyset$ *and* $Tp = \{p\}$, *for all* $p \in F(T)$. *Let* $\{x_n\}$ *be the Mann iterative scheme defined by*

$$x_{n+1} = W(y_n, x_n, \alpha_n),$$

where $y_n \in Tx_n$ *and* $\alpha_n \in [0,1]$. *Then* $\{x_n\}$ *converges to a fixed point of* T *if* $\lim_{n\to\infty} \rho(x_n, F(T)) = 0_{\mathcal{A}}$.

Proof. Take $p \in F(T)$. Then

$$\begin{aligned} \rho(x_{n+1}, p) &= \rho(W(y_n, x_n, \alpha_n), p) \\ &\preceq \alpha_n \rho(y_n, p) + (1-\alpha_n)\rho(x_n, p). \end{aligned}$$

So

$$\begin{aligned} \|\rho(x_{n+1}, p)\| &\leq \alpha_n \|\rho(y_n, p)\| + (1-\alpha_n)\|\rho(x_n, p)\| \\ &= \alpha_n \|\rho(y_n, Tp)\| + (1-\alpha_n)\|\rho(x_n, p)\| \\ &\leq \alpha_n \|\rho(Tx_n, Tp)\| + (1-\alpha_n)\|\rho(x_n, p)\| \\ &\leq \alpha_n \|a^*\| \|\rho(x_n, p)\| \|a\| + (1-\alpha_n)\|\rho(x_n, p)\| \\ &= \alpha_n \|a\|^2 \|\rho(x_n, p)\| + (1-\alpha_n)\|\rho(x_n, p)\| \\ &< \alpha_n \|\rho(x_n, p)\| + (1-\alpha_n)\|\rho(x_n, p)\| = \|\rho(x_n, p)\|. \end{aligned}$$

Notice that the above strict inequality holds only when $x_n \neq p$, for each $n \in \mathbb{N}$. In fact, if $x_k = p$ for some $k \in \mathbb{N}$, then $x_n = p$ for all $n \geq k$ and so $\{x_n\}$ converges to p in a finite number of iterations which proves our theorem.

Now, we prove that $\{x_n\}$ is a Cauchy sequence. $\lim_{n\to\infty} \rho(x_n, F(T)) = 0_{\mathcal{A}}$ implies that for $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$\|\rho(x_n, F(T))\| \leq \frac{\varepsilon}{3}$$

for all $n \geq n_1$. This implies that there exists $q_1 \in F(T)$ such that

$$\|\rho(x_n, q_1)\| \leq \frac{\varepsilon}{2}$$

for all $n \geq n_1$. Hence

$$\rho(x_{n+m}, x_n) \preceq \rho(x_{n+m}, q_1) + \rho(q_1, x_n).$$

It implies that

$$\begin{aligned}
\|\rho(x_{n+m}, x_n)\| &\leq \|\rho(x_{n+m}, q_1)\| + \|\rho(q_1, x_n)\| \\
&< \|\rho(x_n, q_1)\| + \|\rho(q_1, x_n)\| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

for $x_n \neq q_1$(by using the same argumentation as above). Therefore $\{x_n\}$ is a Cauchy sequence with respect to $\mathcal{A}$. By the completeness of $(M, \mathcal{A}, W, \rho)$, $\{x_n\}$ is convergent. Thus, there exists $p^* \in M$ such that $\lim_{n\to\infty} x_n = p^*$. we will show that p^* is a fixed point of T.

Let $\varepsilon' > 0$. Since $\lim_{n\to\infty} x_n = p^*$, there exists $n_2 \in \mathbb{N}$ such that

$$\|\rho(x_n, p^*)\| \leq \frac{\varepsilon'}{4},$$

for each $n \geq n_2$. Further, $\lim_{n\to\infty} \rho(x_n, F(T)) = 0_{\mathcal{A}}$ implies that there exists natural number $n_3 \geq n_2$ such that for any $n \geq n_3$,

$$\|\rho(x_n, F(T))\| \leq \frac{\varepsilon'}{12},$$

and consequently there exists $q_2 \in F(T)$ such that for each $n \geq n_3$,

$$\|\rho(x_n, q_2)\| \leq \frac{\varepsilon'}{8}.$$

Therefore

$$\begin{aligned}
\rho(Tp^*, p^*) \preceq\ & \rho(Tp^*, q_2) + \rho(q_2, Tx_{n_3}) + \rho(Tx_{n_3}, q_2) + \rho(q_2, x_{n_3}) \\
& + \rho(x_{n_3}, p^*).
\end{aligned}$$

This yields

$$\begin{aligned}
\|\rho(Tp^*, p^*)\| &\leq \|\rho(Tp^*, q_2)\| + \|\rho(q_2, Tx_{n_3})\| + \|\rho(Tx_{n_3}, q_2)\| \\
&\quad + \|\rho(q_2, x_{n_3})\| + \|\rho(x_{n_3}, p^*)\| \\
&\leq \|D(Tp^*, Tq_2)\| + 2\|D(Tq_2, Tx_{n_3})\| + \|\rho(q_2, x_{n_3})\| + \|\rho(x_{n_3}, p^*)\| \\
&\leq \|a^*\|\|\rho(p^*, q_2)\|\|a\| + 2\|a^*\|\|\rho(q_2, x_{n_3})\|\|a\| + \|\rho(q_2, x_{n_3})\| + \|\rho(x_{n_3}, p^*)\| \\
&= \|a\|^2\|\rho(p^*, q_2)\| + 2\|a\|^2\|\rho(q_2, x_{n_3})\| + \|\rho(q_2, x_{n_3})\| + \|\rho(x_{n_3}, p^*)\| \\
&< \|\rho(p^*, q_2)\| + 2\|\rho(q_2, x_{n_3})\| + \|\rho(q_2, x_{n_3})\| + \|\rho(x_{n_3}, p^*)\| \\
&\leq \|\rho(p^*, x_{n_3})\| + \|\rho(x_{n_3}, q_2)\| + 2\|\rho(q_2, x_{n_3})\| + \|\rho(q_2, x_{n_3})\| + \|\rho(x_{n_3}, p^*)\| \\
&= 2\|\rho(x_{n_3}, p^*)\| + 4\|\rho(x_{n_3}, q_2)\| \\
&\leq \frac{\varepsilon'}{2} + \frac{\varepsilon'}{2} = \varepsilon'.
\end{aligned}$$

where $x_{n_3} \neq q_2$. Thus we have $\rho(Tp^*, p^*) = 0_{\mathcal{A}}$ and so $p^* \in Tp^*$. This completes the proof. □

The following result can be easily established from above theorem:

Corollary 1. *Let D be a nonempty convex subset of a complete convex C^*-algebra-valued metric space $(M, \mathcal{A}, W, \rho)$. Suppose that $T : D \to CB(D)$ is a C^*-algebra-multi-valued contractive mapping with constant a for which $F(T) \neq \emptyset$ and $Tp = \{p\}$, for all $p \in F(T)$. Let $\{x_n\}$ be the Ishikawa iterative scheme defined by*

$$x_{n+1} = W(z_n, x_n, \alpha_n),$$

$$y_n = W(z'_n, x_n, \beta_n)$$

where $z_n \in Ty_n$, $z'_n \in Tx_n$ and $\alpha_n, \beta_n \in [0, 1]$. Then $\{x_n\}$ converges to the fixed point of T if $\lim_{n\to\infty} \rho(x_n, F(T)) = 0_{\mathcal{A}}$

Next, we consider two multi-valued mappings T and S with the given contractive condition and prove the convergence of proposed iteration process to a common fixed point of them:

Theorem 3. *Let $(M, \mathcal{A}, W, \rho)$ be a complete convex C^*-algebra-valued metric space and D be a nonempty convex subset of M. Let $S, T : D \to CB(D)$ be two multi-valued mappings which satisfy the condition*

$$D(Tx, Sy) \preceq a^*\rho(x, y)a,$$

for all $x, y \in D$ with $a \in \mathcal{A}$ and $\|a\| < 1$. Suppose that $F = F(T) \cap F(S) \neq \emptyset$ and $Tp = \{p\} = Sp$, for any $p \in F$. Then the sequence of Ishikawa iterates defined by

$$x_{n+1} = W(z_n, x_n, \alpha_n),$$

$$y_n = W(z'_n, x_n, \beta_n)$$

where $z_n \in Sy_n$, $z'_n \in Tx_n$ and $\alpha_n, \beta_n \in [0, 1]$, converges to a point in F if $\lim_{n\to\infty} \rho(x_n, F) = 0_{\mathcal{A}}$.

Proof. Take $p \in F$. As in the proof of Theorem 2, suppose that $x_n \neq p$, for all $n \in \mathbb{N}$. Then

$$\begin{aligned}\rho(y_n, p) &= \rho(W(z'_n, x_n, \beta_n), p)\\ &\preceq \beta_n\rho(z'_n, p) + (1 - \beta_n)\rho(x_n, p)\end{aligned}$$

which implies

$$\begin{aligned}\|\rho(y_n, p)\| &\leq \beta_n\|\rho(z'_n, p)\| + (1 - \beta_n)\|\rho(x_n, p)\|\\ &\leq \beta_n\|D(Tx_n, Sp)\| + (1 - \beta_n)\|\rho(x_n, p)\|\\ &\leq \beta_n\|a\|^2\|\rho(x_n, p)\| + (1 - \beta_n)\|\rho(x_n, p)\|\\ &< \|\rho(x_n, p)\|\end{aligned}$$

and

$$\begin{aligned}\|\rho(x_{n+1}, p)\| &\leq \alpha_n\|\rho(z_n, p)\| + (1 - \alpha_n)\|\rho(x_n, p)\|\\ &\leq \alpha_n\|D(Sy_n, Tp)\| + (1 - \alpha_n)\|\rho(x_n, p)\|\\ &< \alpha_n\|\rho(y_n, p)\| + (1 - \alpha_n)\|\rho(x_n, p)\|\\ &< \|\rho(x_n, p)\|,\end{aligned}$$

where $y_n \neq p$, for all $n \in \mathbb{N}$. As in the proof of theorem 2, one can show that $\{x_n\}$ is a Cauchy sequence with respect to $\mathcal{A}$ and by the completeness of $(M, \mathcal{A}, W, \rho)$, it converges to some $p^* \in M$. Again, with a similar process in the proof of Theorem 2, we conclude that $p^* \in F(T) \cap F(S)$ and this complete the proof of theorem. □

Finally we extend our results for finite and infinite family of C^*-algebra-multi-valued contractive mappings. Since the idea is similar to the one given in above theorems, we just only state the result without the proof.

Theorem 4. *Suppose that D is a nonempty convex subset of a complete convex C^*-algebra-valued metric space $(M, \mathcal{A}, W, \rho)$ and $\{T_i : D \to CB(D) \, : \, i = 1, ..., m\}$ be a finite family of C^*-algebra-multi-valued contractive mappings such that $F = \cap_{i=1}^{m} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$, for any $p \in F$ and $i = 1, 2, ..., m$. Consider the iterative process defined by*

$$\begin{aligned} y_{1n} &= W(z_{1n}, x_n, \alpha_{1n}), \\ y_{2n} &= W(z_{2n}, x_n, \alpha_{2n}), \\ &\cdots \\ y_{(m-1)n} &= W(z_{(m-1)n}, x_n, \alpha_{(m-1)n}), \\ x_{n+1} &= W(z_{mn}, x_n, \alpha_{mn}), \end{aligned}$$

where $\alpha_{in} \in [0, 1]$ and $z_{in} \in T_i(y_{(i-1)n})$ $(y_{0n} = x_n)$, for all $n \in \mathbb{N}$ and $i = 1, 2, \ldots, m$. Then $\{x_n\}$ converges to a point in F if $\lim_{n\to\infty} \rho(x_n, F) = 0_{\mathcal{A}}$.

Remark 1. *In above theorem, we can also consider the following iterative scheme for a finite family $\{T_i\}_{i=1}^{m}$:*

$$x_n = W(x_{n-1}, y_n, \alpha_n)$$

where $y_n \in T_n(x_n)$ and $T_n = T_n(mod\ m)$, for all $n \in \mathbb{N}$.

Theorem 5. *Suppose that D is a nonempty convex subset of a complete convex C^*-algebra-valued metric space $(M, \mathcal{A}, W, \rho)$ and $\{T_i : D \to CB(D) \, : \, i = 1, 2, \ldots\}$ is an infinite family of C^*-algebra-multi-valued contractive mappings such that $F := \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$, for any $p \in F$ and $i = 1, 2, \ldots$. Consider the iterative process defined by*

$$\begin{aligned} x_{n+1} &= W(z_n^{'}, x_n, \alpha_n) \\ y_n &= W(z_n, x_n, \beta_n) \end{aligned}$$

where $z_n^{'} \in T_n(y_n)$, $z_n \in T_n(x_n)$ and $\{\alpha_n\}, \{\beta_n\} \in [0, 1]$. Then $\{x_n\}$ converges to a point in F if $\lim_{n\to\infty} \rho(x_n, F) = 0_{\mathcal{A}}$.

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