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On Some New Multivalued Results in the Metric Spaces of Perov's Type

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Abstract: The purpose of this paper is to present some new fixed point results in the generalized metric spaces of Perov's sense under a contractive condition of Hardy–Rogers type. The data dependence of the fixed point set, the well-posedness of the fixed point problem and the Ulam–Hyers stability are also studied.

Keywords: multivalued Hardy–Rogers type contractive operator; fixed point; common fixed point; well-posedness; Ulam–Hyers stability; data dependence

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1. Introduction and Preliminaries

In 1964 Perov [1] extended the known Banach theorem from 1922 on spaces endowed with vector valued metrics. The main motivation for introduction of metric spaces in the sense of the Perov type is the problem of solving a system of ordinary differential equations.

Let $\mathbb{R}^m$ be known the Euclidean space, then $\mathbb{R}^m$ is an ordered vector space by the cone

$$C = \{(c_1, \ldots, c_m) : c_i \geq 0 \text{ for all } i = 1, \ldots, m\}.$$

For more details see [2–7].

The concept of vector-valued metric was introduced by Perov [1] as follows:

Let X be a nonempty set. A mapping $\widetilde{d} : X \times X \to \mathbb{R}^m$ is called a vector-valued metric on X if the following properties are satisfied:

(1) $\widetilde{d}(x,y) \succeq 0$ and $\widetilde{d}(x,y) = 0$ if and only if $x = y$,
(2) $\widetilde{d}(x,y) = \widetilde{d}(y,x)$,
(3) $\widetilde{d}(x,y) \preceq \widetilde{d}(x,z) + \widetilde{d}(z,y)$,

for all $x, y, z \in X$, where $0 = (\underbrace{0, \ldots, 0}_{m})^T$.

A generalized metric space in Perov's sense is the pair $(X, \widetilde{d})$.

According to ([8], proposition 2.1), it follows that the generalized metric in Perov's sense $\widetilde{d}$ has a form $\widetilde{d} = (d_1, \ldots, d_m)^T$ where each $d_i : X \times X \to [0, \infty), i \in \{1, \ldots, m\}$ is a pseudometric (i.e., $d_i(x,x) = 0$, $d_i(x,y) = d_i(y,x)$ and $d_i(x,y) \leq d_i(x,z) + d_i(z,y)$). This means that $(X, \widetilde{d})$ is a generalized metric space in Perov's sense if and only if

$$\widetilde{d}(x,y) = (d_1(x,y), \ldots, d_m(x,y))^T, \text{ for all } x, y \in X.$$

Let $\mathbb{M}_{m,m}(\mathbb{R}_+)$ be the family of all square matrices of order m with positive elements. We denote the zero and unit matrix by Θ and I respectively. We use the symbol A^T for transpose matrix of A. A matrix A converges to zero if $A^n \to \Theta$ as $n \to \infty$.

We will use the following known result, see for example [9].

Theorem 1. *Let $A \in \mathbb{M}_{m,m}(\mathbb{R}_+)$. The following properties are equivalent:*

(i) *A matrix A converges to Θ as $n \to \infty$;*
(ii) *If $\lambda \in \mathbb{C}$ such that $det(A - \lambda I) = 0$ then $|\lambda| < 1$;*
(iii) *The matrix $I - A$ is regular and $(I - A)^{-1} = I + A + A^2 + \cdots$.*

Let us note that Perov's metric is a very particular case of the so-called K-metric (see [4] and the references therein), which in turn was rediscovered by Huang and Zhang [3] under the name of cone metric.

In 1973, Hardy and Rogers [10] gave a generalization of Reich fixed point theorem. Since then, many authors used different Hardy–Rogers contractive type conditions in order to obtain fixed point results.

Let (X, d) be a complete metric space. We will use the following notations:

$P(X)$*—is the set of all nonempty subsets of* X;

$P_{cl}(X)$*—is the set of all nonempty closed subsets of* X;

$P_{cp}(X)$*—is the set of all nonempty compact subsets of* X;

$D : P(X) \times P(X) \to \mathbb{R}_+, D(A,B) = inf\{d(a,b) : a \in A, b \in B\}$*-is the gap functional.*

$\rho : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \rho(A,B) = \sup\{D(a,B) : a \in A\}$*-is the excess functional.*

$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, H(A,B) = \max\{\rho(A,B); \rho(B,A)\}$*-is the Pompeiu–Hausdorff functional.*

Let $T : X \to P(X)$ be a multivalued operator and $Y \subseteq X$. Then:

$f : X \to Y$ *is a selection for* $T : X \to P(Y)$ if $f(x) \in T(x)$, for each $x \in X$;

$Graph(T) := \{(x,y) \in X \times Y \mid y \in T(x)\}$*-is the graphic of* T;

$Fix(T) := \{x \in X \mid x \in T(x)\}$*-is the set of the fixed points of* T;

We also denote by $\mathbb{N}$ the set of all natural numbers and by $\mathbb{N}^* := \mathbb{N} \cup \{0\}$.

Let $(X, \widetilde{d})$ be a generalized metric space in Perov's sense. Here, if $u, v \in \mathbb{R}^m$, $u := (u_1, \ldots, u_m)$, $v := (v_1, \ldots, v_m)$, then by $u \preceq v$ we mean $u_j \leq v_j$, for each $j \in \{1, \ldots, m\}$, while $u \prec v$ mean $u \preceq v$ and $u_j \neq v_j$, for all $j \in \{1, \ldots, m\}$. With $\widetilde{B}(x,a)$ we denote the open ball centered at $x \in X$ with radius a, i.e.,

$$\widetilde{B}(x,a) := \{y \in X | \widetilde{d}(x,y) \prec a\},$$

where $x := (x_1, \ldots, x_m)$ and $a := (a_1, \ldots, a_m)$.

For the following notations see [11–13].

Definition 1. *Let (X, d) be a metric space. A mapping $T : X \to P(X)$ is a multi-valued weakly Picard operator (or MWP) if for each $y \in X$ and each $z \in T(y)$ there exists a sequence $(y_n)_{n \in \mathbb{N}}$ such that:*

(i) *$y_0 = y$, $y_1 = z$;*
(ii) *$y_{n+1} \in T(y_n)$, for each $n \in \mathbb{N}$;*
(iii) *the sequence $(y_n)_{n \in \mathbb{N}}$ is convergent and its limit is a fixed point of T.*

Remark 1. *A sequence* $(y_n)_{n\in\mathbb{N}}$ *satisfying the conditions* (i) *and* (ii) *in Remark 1 is called a sequence of successive approximations of T starting from* $(y,z) \in Graph(T)$.

If $T : X \to P(X)$ is an MWP operator, then we define $T^\infty : Graph(T) \to P(Fix(T))$ by the formula $T^\infty(y,z) := \{t \in Fix(T) \mid$ there exists a sequence of successive approximations of T starting from (y,z) that converges to $t\}$.

Definition 2. *Let (X,d) be a metric space and* $T : X \to P(X)$ *be an MWP operator. Then T is called c-multivalued weakly Picard operator (briefly c-MWP operator) if and only if there exists a selection* f^∞ *of* T^∞ *such that:*

$$d(x, f^\infty(y,z)) \le cd(y,z), \text{ for all } (y,z) \in Graph(T).$$

About of weakly Picard operators see example [12,13].

Also, for Ulam stability of some functional equations see [11,14–20].

The definition of Ulam–Hyers stability for multivalued operators is given in [11] as follows.

Definition 3. *Let (X,d) be a metric space and* $T : X \to P(X)$ *be a multivalued operator. By definition, the fixed point equation*

$$x \in T(x) \tag{1}$$

is Ulam–Hyers stable if there exists a real number $c > 0$ *such that: for each* $\varepsilon > 0$ *and each solution* y^* *of the inequation*

$$D(y, T(y)) \le \varepsilon \tag{2}$$

there exists a solution x^* *of Equation (1) such that*

$$d(y^*, x^*) \le c\varepsilon.$$

Remark 2. *([11]) If T is a multivalued c-weakly Picard operator, then the fixed point Equation (1) is Ulam–Hyers stable.*

The purpose of this paper is to present some multivalued fixed point results in generalized metric spaces in Perov's sense using a contractive condition of Hardy–Rogers type. The data dependence of the fixed point set, the well-posedness of the fixed point problem and the Ulam–Hyers stability are also studied.

2. Main Results

First let us define some important notions for the case of generalized metric space in Perov's sense.

$\widetilde{D} : P(X) \times P(X) \to \mathbb{R}^m_+$, $\widetilde{D}(A,B) := (D_1(A,B), \dots, D_m(A,B))^T$ for given $m \in \mathbb{N}$—*is the gap generalized functional.*

$\widetilde{\rho} : P(X) \times P(X) \to \mathbb{R}^m_+ \bigcup\{+\infty\}$, $\widetilde{\rho}(A,B) := (\rho_1(A,B), \dots, \rho_m(A,B))^T$, for given $m \in \mathbb{N}$—*the excess generalized functional.*

$\widetilde{H} : P(X) \times P(X) \to \mathbb{R}^m_+ \cup \{+\infty\}$, $\widetilde{H}(A,B) := (H_1(A,B), \dots, H_m(A,B)))^T$, for given $m \in \mathbb{N}$—*the Pompeiu–Hausdorff generalized functional.*

Obvious, D_i, ρ_i and H_i, for $i \in \{1, \dots, m\}$ are pseudometrics.

Lemma 1. *Let* $(X, \widetilde{d})$ *be a generalized metric space in Perov's sense,* $A, B \subseteq X$ *and* $q > 1$. *Then for any* $a \in A$ *there exists* $b \in B$ *such that:*

$$\widetilde{d}(a,b) \preceq q\widetilde{H}(A,B).$$

Proof. Since, $\widetilde{d}(a,b) \preceq q\widetilde{H}(A,B)$ if and only if $\widetilde{d_i}(a,b) \preceq q\widetilde{H_i}(A,B)$ for $i \in \{1,2,\dots,m\}$, where $\widetilde{d}(a,b) = (d_1(a,b), \dots, d_m(a,b))^T$, $\widetilde{H}(A,B) = (H_1(A,B), \dots, H_m(A,B))^T$, and d_i and H_i are pseudometrics for each $i \in \{1, \ldots, m\}$, we as in the standard metric spaces obtain $\widetilde{d}_i(a,b) \preceq q\widetilde{H}_i(A,B)$. $\square$

Lemma 2. *Let $(X, \widetilde{d})$ be a generalized metric space in Perov's sense. Then $\widetilde{D}(\{x\}, A) = 0_{m\times 1}$ if and only if $x \in \bar{A}$.*

Proof. We must prove that $\bar{A} = \{x \in X | \widetilde{D}(\{x\}, A) = 0_{m\times 1}\}$. Then, if $\widetilde{D}(\{x\}, A) = 0_{m\times 1}$ that means $D_i(\{x\}, A) = 0$ with $0 \in \mathbb{R}$ for each $i \in \{1, 2, \ldots, m\}$. This is further equivalent as in the case of standard metric spaces. $\square$

Lemma 3. *Let $A \in \mathbb{M}_{m,m}(\mathbb{R}_+)$ be a matrix converges to zero. Then there exists $Q > 1$ such that for every $q \in (1, Q)$ we have that qA is converges to zero.*

Proof. Since A is a matrix converges to zero, we have spectral radius $\rho(A) < 1$. Since $q\rho(A) = \rho(qA) < 1$ we can choose $Q := \frac{1}{\rho(A)} > 1$ and hence, the conclusion follows. $\square$

Let us give the definition of multivalued Hardy–Rogers type operators on generalized metric space in Perov's sense.

Definition 4. *Let $(X, \widetilde{d})$ be a generalized metric space in Perov's sense and $T : X \to P(X)$ be a given multivalued operator. If there exist $A, B, C \in \mathbb{M}_{m,m}(\mathbb{R}_+)$ such that*

$$\widetilde{H}(T(x), T(y)) \preceq A\widetilde{d}(x,y) + B[\widetilde{D}(x, T(x)) + \widetilde{D}(y, T(y))] + C[\widetilde{D}(x, T(y)) + \widetilde{D}(y, T(x))],$$

for all $x, y \in \mathbb{R}$, we say that T is a Hardy–Rogers type operator.

The following theorem is one of the main results.

Theorem 2. *Let $(X, \widetilde{d})$ be a complete generalized metric space in Perov' sense, $T : X \to P_{cl}(X)$ be a multivalued Hardy–Rogers type operator. If there exist the matrices $A, B, C \in \mathbb{M}_{m,m}(\mathbb{R}_+)$ such that:*

(i) $I - q(B + C)$ *is nonsingular and* $(I - q(B + C))^{-1} \in \mathbb{M}_{m,m}(\mathbb{R}_+)$, *for* $q \in (1, Q)$;

(ii) $M = (I - q(B + C))^{-1} q(A + B + C)$ *converges to* Θ.

Then T is a multivalued weakly Picard operator.

Proof. Let $x_0 \in X$ and $x_1 \in T(x_0)$. If $x_0 = x_1$ we obtain the desired conclusion. Let $x_0 \neq x_1$, $x_1 \in T(x_0)$ and $q \in (1, Q)$, where Q is defined as in Lemma 3. Then, by Lemma 1 there exists $x_2 \in T(x_1)$ such that:

$$\begin{aligned}
\widetilde{d}(x_1, x_2) &\preceq q\widetilde{H}(T(x_0), T(x_1)) \\
&\preceq qA\widetilde{d}(x_0, x_1) + qB[\widetilde{D}(x_0, T(x_0)) + \widetilde{D}(x_1, T(x_1))] + qC[\widetilde{D}(x_0, T(x_1)) + \widetilde{D}(x_1, T(x_0))] \\
&\preceq qA\widetilde{d}(x_0, x_1) + qB[\widetilde{d}(x_0, x_1) + \widetilde{d}(x_1, x_2)] + qC[\widetilde{d}(x_0, x_2) + \widetilde{d}(x_1, x_1)] \\
&= q(A + B)\widetilde{d}(x_0, x_1) + qB(\widetilde{d}(x_1, x_2)) + qC[\widetilde{d}(x_0, x_1) + \widetilde{d}(x_1, x_2)] \\
&= q(A + B + C)\widetilde{d}(x_0, x_1) + q(B + C)\widetilde{d}(x_1, x_2).
\end{aligned}$$

Then we have: $[I - q(B + C)]\widetilde{d}(x_1, x_2) \preceq q(A + B + C)\widetilde{d}(x_0, x_1)$.
We get the inequality

$$\widetilde{d}(x_1, x_2) \preceq [I - q(B + C)]^{-1} q(A + B + C)\widetilde{d}(x_0, x_1) = M\widetilde{d}(x_0, x_1). \tag{3}$$

For the next step we have

$$\begin{aligned}\widetilde{d}(x_2,x_3) &\preceq q\widetilde{H}(T(x_1),T(x_2)) \preceq qA\widetilde{d}(x_1,x_2)+qB[\widetilde{D}(x_1,T(x_1))+\widetilde{D}(x_2,T(x_2))]\\ &+ qC[\widetilde{D}(x_1,T(x_2))+\widetilde{D}(x_2,T(x_1))]\\ &= qA\widetilde{d}(x_1,x_2)+qB[\widetilde{d}(x_1,x_2)+\widetilde{d}(x_2,x_3)]+qC[\widetilde{d}(x_1,x_3)+\widetilde{d}(x_2,x_2)]\\ &= q(A+B)\widetilde{d}(x_1,x_2)+qB(\widetilde{d}(x_2,x_3))+qC[\widetilde{d}(x_1,x_2)+\widetilde{d}(x_2,x_3)]\\ &= q(A+B+C)\widetilde{d}(x_1,x_2)+q(B+C)\widetilde{d}(x_2,x_3).\end{aligned}$$

Then we have $[I-q(B+C)]\widetilde{d}(x_2,x_3) \preceq q(A+B+C)\widetilde{d}(x_1,x_2)$.
Using Equation (3) we obtain the inequality

$$\widetilde{d}(x_2,x_3) \preceq [I-q(B+C)]^{-1}q(A+B+C)\widetilde{d}(x_1,x_2) = M\widetilde{d}(x_1,x_2) \preceq M^2\widetilde{d}(x_0,x_1). \tag{4}$$

Continuing this process we shall obtain a sequence $(x_n)_{n\in\mathbb{N}} \in X$, with $x_n \in T(x_{n-1})$ such that

$$\widetilde{d}(x_n,x_{n+1}) \preceq M^n\widetilde{d}(x_0,x_1), \tag{5}$$

with $M \in \mathbb{M}_{m,m}(\mathbb{R}_+)$ and $n \in \mathbb{N}$.

We will prove next that $(x_n)_{n\in\mathbb{N}}$ is Cauchy, by estimating $\widetilde{d}(x_n,x_m)$, for every $m,n \in \mathbb{N}$ with $m > n$.

$$\begin{aligned}\widetilde{d}(x_n,x_m) &\preceq \widetilde{d}(x_n,x_{n+1})+\widetilde{d}(x_{n+1},x_{n+2})+\cdots+\widetilde{d}(x_{m-1},x_m)\\ &\preceq M^n(\widetilde{d}(x_0,x_1))+M^{n+1}(\widetilde{d}(x_0,x_1))+\cdots+M^{m-1}(\widetilde{d}(x_0,x_1))\\ &\preceq M^n(I+M+M^2+\cdots+M^{m-n-1}+\cdots)\widetilde{d}(x_0,x_1)\\ &\preceq M^n(I-M)^{-1}\widetilde{d}(x_0,x_1)).\end{aligned}$$

Note that $(I-M)$ is nonsingular since M is converges to Θ. This implies that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Since $(X,\widetilde{d})$ is complete we get that there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Thus we have:

$$\widetilde{D}(x^*,T(x^*)) = (D_1(x^*,T(x^*)),\ldots,D_m(x^*,T(x^*))).$$

Further, for all $i \in \{1,\ldots,m\}$: $D_i(x^*,T(x^*)) \leq d_i(x^*,x_{n+1})+D_i(x_{n+1},T(x^*))$.
We obtain that:

$$\begin{aligned}\widetilde{D}(x^*,T(x^*))) &\preceq \widetilde{d}((x^*,x_{n+1}))+\widetilde{H}(T(x_n),T(x^*))\\ &\preceq \widetilde{d}((x^*,x_{n+1}))+qA\widetilde{d}((x_n,x^*))+qB[\widetilde{D}(x_n,T(x_n))+\widetilde{D}(x^*,T(x^*))]\\ &+ qC[\widetilde{D}(x_n,T(x^*))+\widetilde{D}(x^*,T(x_n))]\\ &\preceq q(A+C)\widetilde{d}((x_n,x^*))+qB[\widetilde{d}(x_n,x_{n+1})+\widetilde{D}(x^*,T(x^*))]+qC\widetilde{d}(x^*,x_{n+1}).\end{aligned}$$

Then we get: $\widetilde{D}(x^*,T(x^*)) \preceq (I-qB)^{-1}[q(A+C)\widetilde{d}((x_n,x^*))+qB\widetilde{d}(x_n,x_{n+1})+qC\widetilde{d}(x^*,x_{n+1})]$.
Letting $n \to \infty$ we get that $\widetilde{D}(x^*,T(x^*)) = 0_{m\times 1}$, then $D_i(x^*,T(x^*)) = 0$ with $0 \in \mathbb{R}$, for any $i \in \{1,2,\ldots,m\}$. By Lemma 2 we have $x^* \in \overline{T(x^*)}$. Hence $x^* \in T(x^*)$. Then T is an MWP operator. □

Our next result relates to the uniqueness of a fixed point for multivalued Hardy–Rogers type mapping in the context of a generalized metric spaces of Perov's type.

Theorem 3. *Let $(X,\widetilde{d})$ be a generalized metric space in Perov's sense and $T : X \to P_{cl}(X)$ be a multivalued Hardy–Rogers type operator. If there exists the matrices $A,B,C \in \mathbb{M}_{m,m}(\mathbb{R}_+)$ such that all the conditions of Theorem 2 satisfied and, additionally, $I-q(A+2C)$ is nonsingular and $[I-q(A+2C)]^{-1} \in \mathbb{M}_{m,m}(\mathbb{R}_+)$, $q \in (1,Q)$, then T has a unique fixed point x^*.*

Proof. The existence of the fixed point is assured by Theorem 2. For uniqueness we suppose that there exists $y^* \in X$ such that $y^* \in T(y^*)$ and $y^* \neq x^*$.

Let $q \in (1, Q)$, where Q is defined as in Lemma 3. Then we have:

$$\widetilde{d}(x^*, y^*) \preceq q\widetilde{H}(T(x^*), T(y^*)) \preceq qA\widetilde{d}(x^*, y^*) + qB[\widetilde{D}(x^*, T(x^*)) + \widetilde{D}(y^*, T(y^*))]$$

$$+qC[\widetilde{D}(x^*, T(y^*)) + \widetilde{D}(y^*, T(x^*))] \preceq qA\widetilde{d}(x^*, y^*) + 2qC\widetilde{d}(x^*, y^*).$$

This implies that $[I - q(A + 2C)]\widetilde{d}(x^*, y^*) \preceq 0_{m\times 1}$. Since $I - q(A + 2C) \neq \Theta$ we get that $\widetilde{d}(x^*, y^*) = 0_{m\times 1}$ that means $d_i(x^*, y^*) = 0$ with $0 \in \mathbb{R}$, for any $i \in \{1, 2, \ldots, m\}$. Then $x^* = y^*$. □

The result we now state is an immediate set of consequences of Theorem 2.

Theorem 4. *Let $(X, \widetilde{d})$ be a complete generalized metric space in Perov's sense, $T : X \to P_{cl}(X)$ be a multivalued Hardy–Rogers type operator. Suppose that all the hypothesis of Theorem 2 are fulfilled. Then the following statements are true:*

(1) $Fix(T) \neq \emptyset$.

(2) *There exists a sequence $(x_n)_{n\in\mathbb{N}} \in X$ such that $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$ and converge to a fixed point of T.*

(3) *One has the estimation $\widetilde{d}(x_n, x^*) \preceq (I - q(B + C))^{-1}[q(A + B + C)]^n\widetilde{d}(x_0, x_1)$, where $x^* \in Fix(T)$.*

Proof. For proof of Equations (1) and (2) see the proof of Theorem 2. If in

$$\widetilde{d}(x_n, x_m) \preceq M^n(I - M)^{-1}\widetilde{d}(x_0, x_1)),$$

m tends to $+\infty$ we get Equation (3), where $M = (I - q(B + C))^{-1}q(A + B + C)$. □

Example 1. *Let $X = [0, +\infty)$ endowed with the generalized metric $\widetilde{d} : X \times X \to \mathbb{R}^2$ defined by $\widetilde{d}(x, y) = \begin{pmatrix} |x-y| \\ |x-y| \end{pmatrix}$. Let $T : \mathbb{R} \to P_{cl}(\mathbb{R})$ be an operator given by:*

$$T(x) = \begin{cases} \{1, \frac{x}{4}\}, & \text{for } x \in \mathbb{R}, \text{ with } x > 1; \\ \{0, \frac{x}{3}\}, & \text{for } x \in \mathbb{R}, \text{ with } 0 < x \leq 1; \\ \{\frac{1}{5}, \frac{1}{4}\}, & \text{otherwise}. \end{cases}$$

Next we prove that weakly Hardy–Rogers type condition is true. Let $A = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$.

Case 1. *For $x > 1$ we have:*

$$\widetilde{H}(T(x), T(y)) = \begin{pmatrix} H_1(T(x), T(y)) \\ H_2(T(x), T(y)) \end{pmatrix} = \begin{pmatrix} \max\{\sup\limits_{x\in T(x)} d(x, T(y)), \sup\limits_{y\in T(y)} d(y, T(x))\} \\ \max\{\sup\limits_{x\in T(x)} d(x, T(y)), \sup\limits_{y\in T(y)} d(y, T(x))\} \end{pmatrix} =$$

$$= \begin{pmatrix} \max\{\inf\{|\frac{x}{4}|, |\frac{x}{4} - \frac{y}{4}|\}, \inf\{|\frac{y}{4}|, |\frac{y}{4} - \frac{x}{4}|\}\} \\ \max\{\inf\{|\frac{x}{4}|, |\frac{x}{4} - \frac{y}{4}|\}, \inf\{|\frac{y}{4}|, |\frac{y}{4} - \frac{x}{4}|\}\} \end{pmatrix} = \begin{pmatrix} |\frac{x}{4} - \frac{y}{4}| \\ |\frac{x}{4} - \frac{y}{4}| \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} |x-y| \\ |x-y| \end{pmatrix} \preceq \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} |x-y| \\ |x-y| \end{pmatrix} = A\widetilde{d}(x, y).$$

Case 2. *For $0 < x \leq 1$ we have:*

$$\widetilde{H}(T(x),T(y)) = \begin{pmatrix} H_1(T(x),T(y)) \\ H_2(T(x),T(y)) \end{pmatrix} = \begin{pmatrix} \max\{\sup\limits_{x\in T(x)} d(x,T(y)),\ \sup\limits_{y\in T(y)} d(y,T(x))\} \\ \max\{\sup\limits_{x\in T(x)} d(x,T(y)),\ \sup\limits_{y\in T(y)} d(y,T(x))\} \end{pmatrix} =$$

$$= \begin{pmatrix} \max\{\inf\{|\frac{x}{3}|,|\frac{x}{3}-\frac{y}{3}|\},\inf\{|\frac{y}{3}|,|\frac{y}{3}-\frac{x}{3}|\}\} \\ \max\{\inf\{|\frac{x}{3}|,|\frac{x}{3}-\frac{y}{3}|\},\inf\{|\frac{y}{3}|,|\frac{y}{3}-\frac{x}{3}|\}\} \end{pmatrix} = \begin{pmatrix} |\frac{x}{3}-\frac{y}{3}| \\ |\frac{x}{3}-\frac{y}{3}| \end{pmatrix} \preceq$$

$$\preceq \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} |x-y| \\ |x-y| \end{pmatrix} = A\widetilde{d}(x,y).$$

Case 3. *For other choices of x we have:*

$$\widetilde{H}(T(x),T(y)) = \begin{pmatrix} H_1(T(x),T(y)) \\ H_2(T(x),T(y)) \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \end{pmatrix} \preceq \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} |x-y| \\ |x-y| \end{pmatrix} = A\widetilde{d}(x,y).$$

Thus, the weakly Hardy–Rogers type condition is accomplished for $A = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ and $B = C = \Theta$ or $B + C = \Theta$. Since all the hypothesis of the Theorem 2 hold we get that T has fixed points on X. Then T is an MWP operator.

Next, let us give a common fixed point result.

Theorem 5. *Let $(X, \widetilde{d})$ be a complete generalized metric space in Perov's sense and let $T, G : X \to P_{cl}(X)$ be two multivalued Hardy–Rogers type operators. There exists the matrices $A, B, C \in \mathbb{M}_{m,m}(\mathbb{R}_+)$ such that:*

(i) $I - q(B+C)$ *is nonsingular and* $(I - q(B+C))^{-1} \in \mathbb{M}_{m,m}(\mathbb{R}_+)$*, for* $q \in (1, Q)$*;*
(ii) $I - q(A+2C)$ *is nonsingular and* $[I - q(A+2C)]^{-1} \in \mathbb{M}_{m,m}(\mathbb{R}_+)$*;*
(iii) $M = (I - q(B+C))^{-1} q(A+B+C)$ *converges to* Θ.

Then:

(1) T *and* G *have a common fixed point* $x^* \in X$*;*
(2) x^* *is a unique common fixed point of* T *and* G.

Proof. (1) Let $x_0 \in X$ and $x_1 \in T(x_0)$. If $x_0 = x_1$ we obtain the desired conclusion. Let $x_0 \neq x_1$, $x_1 \in T(x_0)$ and $q \in (1, Q)$, where Q is defined by Lemma 3. Then, by Lemma 1, there exists $x_2 \in T(x_1)$ such that we construct $(x_n)_{n\in\mathbb{N}}$ the sequence of successive approximations for T and G, defined by:

$$x_{2n+1} \in T(x_{2n}), n = 0, 1, ...$$

$$x_{2n+2} \in G(x_{2n+1}), n = 0, 1, \ldots$$

Then we have

$$\begin{aligned}
\widetilde{d}(x_{2n}, x_{2n+1}) &\preceq q\widetilde{H}(G(x_{2n-1}), T(x_{2n})) \\
&\preceq qA\widetilde{D}(x_{2n-1}, T(x_{2n}) + qB[\widetilde{D}(x_{2n}, T(x_{2n})) + \widetilde{D}(x_{2n-1}, G(x_{2n-1}))] \\
&+ qC[\widetilde{D}(x_{2n}, G(x_{2n-1})) + \widetilde{D}(x_{2n-1}, T(x_{2n}))] \\
&= qA\widetilde{d}(x_{2n-1}, x_{2n}) + qB[\widetilde{d}(x_{2n}, x_{2n+1}) + \widetilde{d}(x_{2n-1}, x_{2n})] + qC\widetilde{d}(x_{2n-1}, x_{2n+1}) \\
&\preceq qA\widetilde{d}(x_{2n-1}, x_{2n}) + qB[\widetilde{d}(x_{2n}, x_{2n+1}) + \widetilde{d}(x_{2n-1}, x_{2n})] \\
&+ qC[\widetilde{d}(x_{2n-1}, x_{2n}) + \widetilde{d}(x_{2n}, x_{2n+1})].
\end{aligned}$$

Then, letting $q \to 1$ we have:

$$\widetilde{d}(x_{2n}, x_{2n+1}) \preceq (I-(B+C))^{-1}(A+B+C)\widetilde{d}(x_{2n-1}, x_{2n}) = M\widetilde{d}(x_{2n-1}, x_{2n}).$$

Continuing the process we get

$$\begin{aligned}
\widetilde{d}(x_{2n+1}, x_{2n+2}) &\preceq q\widetilde{H}(T(x_{2n}), G(x_{2n+1})) \\
&\preceq qA\widetilde{D}(x_{2n}, T(x_{2n+1}) + qB[\widetilde{D}(x_{2n}, T(x_{2n})) + \widetilde{D}(x_{2n+1}, G(x_{2n+1}))] \\
&+ qC[\widetilde{d}(x_{2n}, G(x_{2n+1})) + \widetilde{d}(x_{2n+1}, T(x_{2n}))] \\
&= qA\widetilde{d}(x_{2n}, x_{2n+1}) + B[\widetilde{d}(x_{2n}, x_{2n+1}) + \widetilde{d}(x_{2n+1}, x_{2n+2})] + qC\widetilde{d}(x_{2n}, x_{2n+2}) \\
&\preceq qA\widetilde{d}(x_{2n}, x_{2n+1}) + qB[\widetilde{d}(x_{2n}, x_{2n+1}) + q\widetilde{d}(x_{2n+1}, x_{2n+2})] \\
&+ qC[\widetilde{d}(x_{2n}, x_{2n+1}) + \widetilde{d}(x_{2n+1}, x_{2n+2})].
\end{aligned}$$

Then we have:

$$\widetilde{d}(x_{2n+1}, x_{2n+2}) \preceq (I-q(B+C))^{-1}q(A+B+C)\widetilde{d}(x_{2n}, x_{2n+1}) = M\widetilde{d}(x_{2n}, x_{2n+1}).$$

Further we obtain that $\widetilde{d}(x_n, x_{n+1}) \preceq M^n\widetilde{d}(x_0, x_1)$ for each $n \in \mathbb{N}$.

Following the same steps like in the proof of the Theorem 2 we estimate $\widetilde{d}(x_n, x_m)$, for every $m, n \in \mathbb{N}$ with $m > n$.

$$\begin{aligned}
\widetilde{d}(x_n, x_m) &\preceq \widetilde{d}(x_n, x_{n+1}) + \widetilde{d}(x_{n+1}, x_{n+2}) + \cdots + \widetilde{d}(x_{m-1}, x_m) \\
&\preceq M^n(\widetilde{d}(x_0, x_1)) + M^{n+1}(\widetilde{d}(x_0, x_1)) + \cdots + M^{m-1}(\widetilde{d}(x_0, x_1)) \\
&\preceq M^n(I + M + M^2 + \cdots + M^{m-n-1} + \cdots)\widetilde{d}(x_0, x_1) \\
&\preceq M^n(I-M)^{-1}\widetilde{d}(x_0, x_1)).
\end{aligned}$$

Note that $(I-M)$ is nonsingular since M is convergent to Θ. This implies that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. From the fact that $(X, \widetilde{d})$ is complete we have that there exists $x^* \in X$ such that $\widetilde{d}(x_n, x^*) \to 0_{m\times 1}$ as $n \to \infty$. Then $\widetilde{d}_i(x_n, x^*) \to 0$ as $n \to \infty$ with $0 \in \mathbb{R}$, for any $i \in \{1, 2, \ldots, m\}$ as $n \to \infty$.

Next we prove that x^* is a fixed point for T by esteeming $\widetilde{D}(T(x^*), x^*)$. Then we obtain:

$$\begin{aligned}
\widetilde{D}(T(x^*), x^*) &\preceq q\widetilde{D}(T(x^*), x_{2n+2}) + \widetilde{d}(x_{2n+2}, x^*) \\
&\preceq \widetilde{H}(T(x^*), G(x_{2n+1})) + \widetilde{d}(x_{2n+2}, x^*) \\
&\preceq qA\widetilde{d}(x^*, x_{2n+1}) + qB[\widetilde{D}(x^*, T(x^*)) + \widetilde{D}(x_{2n+1}, G(x_{2n+1}))] \\
&+ qC[\widetilde{D}(x^*, G(x_{2n+1})) + \widetilde{D}(x_{2n+1}, T(x^*))] + \widetilde{d}(x_{2n+2}, x^*)) \\
&\preceq (I-q(B+C))^{-1}[qA\widetilde{d}(x^*, x_{2n+1}) + qB\widetilde{d}(x_{2n+1}, x_{2n+2}) \\
&+ qC\widetilde{d}(x^*, x_{2n+2}) + qC\widetilde{d}(x_{2n+1}, x^*) + \widetilde{d}(x_{2n+2}, x^*)].
\end{aligned}$$

Letting $n \to \infty$ we get that $\widetilde{D}(T(x^*), x^*) = 0_{m\times 1}$, that means $D_i(T(x^*), x^*) = 0$ as $n \to \infty$, with $0 \in \mathbb{R}$, for any $i \in \{1, \ldots, m\}$. By Lemma 2 obtain that $x^* \in \overline{T(x^*)}$. Hence $x^* \in T(x^*)$, since $T(x^*)$ is closed. Similarly, we estimate $\widetilde{D}(G(x^*), x^*)$ and we found that x^* is fixed point for G. Then x^* is a common fixed point for the operators T and G.

(2) We assume that there exist $y^* \in X$ another common fixed point of T and G. Then we have:

$$\widetilde{d}(y^*, x^*) \preceq q\widetilde{H}(T(y^*), G(x^*)) \preceq qA\widetilde{d}(y^*, x^*) + qB[\widetilde{d}(y^*, T(y^*)) + q\widetilde{d}(x^*, G(x^*))]$$

$$+qC[\widetilde{D}(y^*, G(x^*)) + \widetilde{D}(x^*, T(y^*))] \preceq q(A+2C)\widetilde{d}(y^*, x^*).$$

Then we have: $(I - q(A + 2C))\widetilde{d}(y^*, x^*) \preceq 0_{m\times 1}$. By the hypothesis (ii) we obtain that $\widetilde{d}(y^*, x^*) \preceq 0_{m\times 1}$. Then $d_i(y^*, x^*) \preceq 0$, for $0 \in \mathbb{R}$ and $i \in \{1, \dots, m\}$. Result that $y^* = x^*$. Then x^* is the unique common fixed point for T and G. □

3. Ulam–Hyers Stability, Well-Posedness and Data Dependence of Fixed Point Problems

First, let us present the extension of Ulam–Hyers stability for fixed point inclusions for the case of multivalued operators on generalized metric space in Perov's sense.

Definition 5. *Let $(X, \widetilde{d})$ be a generalized metric space in Perov's sense and $T : X \to P(X)$ be an operator. By definition, the fixed point equation*

$$x \in T(x) \tag{6}$$

is Ulam–Hyers stable if there exists a real positive matrix $N \in \mathbb{M}_{m,m}(\mathbb{R}+)$ such that: for each $\varepsilon > 0$ and each solution y^* of the inequation

$$\widetilde{D}(y, T(y)) \preceq \varepsilon I_{m\times 1} \tag{7}$$

there exists a solution x^* of Equation (6) such that

$$\widetilde{d}(y^*, x^*) \preceq N\varepsilon I_{m\times 1}.$$

Definition 6. *The fixed point Equation (6) is well-posed if $x^* \in Fix(T)$ and $x_n \in X$, $n \in \mathbb{N}$, such that $\widetilde{D}(x_n, T(x_n)) \to 0_{m\times 1}$ as $n \to \infty$, then $x_n \to x^*$ as $n \to \infty$.*

Theorem 6. *Let $(X, \widetilde{d})$ be a generalized metric space in Perov's sense and $T : X \to P_{cl}(X)$ be a multivalued Hardy–Rogers type operator defined in Definition (4). Then, for every nonsingular matrix $I - q(A + B + 2C)$ such that $N = [I - q(A + B + 2C)]^{-1} \in \mathbb{M}_{m,m}(\mathbb{R}_+)$, for $q \in (1, Q)$, the fixed point Equation (6) is Ulam–Hyers stable.*

Proof. Since $T : X \to P_{cl}(X)$ is an MWP operator we get that $\{x^*\} \in Fix(T)$. Let $\varepsilon > 0$ and $w^* \in X$ be a solution of Equation (6), i.e., $\widetilde{D}(w^*, T(w^*)) \leq \varepsilon I_{m\times 1}$.

For $q \in (1, Q)$, where Q is defined by Lemma 3 we get:

$$\begin{aligned}
\widetilde{d}(x^*, w^*) &= \widetilde{D}(T(x^*), w^*) \\
&\preceq \widetilde{H}(T(x^*), T(w^*)) + \widetilde{D}(T(w^*), w^*) \\
&\preceq qA\widetilde{d}(x^*, w^*) + qB[\widetilde{D}(x^*, T(w^*)) + \widetilde{d}(w^*, w^*)] + qC[\widetilde{D}(x^*, T(w^*)) + \widetilde{D}(w^*, T(x^*))] \\
&+ \varepsilon I_{m\times 1} \\
&= q(A + B + 2C)\widetilde{d}(x^*, w^*) + \varepsilon I_{m\times 1}.
\end{aligned}$$

Then we have: $\widetilde{d}(x^*, w^*) \preceq [I - q(A + B + 2C)]^{-1}\varepsilon I_{m\times 1}$.

Using previous notation we obtain $\widetilde{d}(x^*, w^*) \preceq N\varepsilon I_{m\times 1}$.

Then, the fixed point Equation (6) is Ulam–Hyers stable. □

Let us give the following results which assure the well-posedness with respect to the generalized metric $\widetilde{d}$.

Theorem 7. *Let $(X, \widetilde{d})$ be a generalized metric space in Perov's sense and $T : X \to P_{cl}(X)$ be a multivalued Hardy–Rogers type operator defined in Definition (4). Then, for every matrix nonsingular $I - q(A + 2C)$ with $q \in (1, Q)$, such that the matrix $N = [I - q(A + 2C)]^{-1}q(I + B + C) \in \mathbb{M}_{m,m}(\mathbb{R}_+)$ is a matrix convergent to Θ, for every matrices $A, B, C \in \mathbb{M}_{m,m}(\mathbb{R}_+)$ the fixed point Equation (6) is well-posed.*

Proof. Let $x^* \in Fix(T)$ and let $(x_{n\in\mathbb{N}}) \in X$ such that $\widetilde{D}(x_n, T(x_n)) \to 0_{m\times 1}$ as $n \to \infty$. Then for $v_n \in T(x_n)$ such that $\widetilde{d}(x_n, v_n) = \widetilde{D}(x_n, T(x_n))$, $n \in \mathbb{N}$ and $q \in (1, Q)$, where Q is defined by Lemma 3, we have:

$$\begin{aligned} \widetilde{d}(x_n, x^*) &\preceq \widetilde{d}(x_n, v_n) + \widetilde{d}(v_n, x^*) \\ &\preceq \widetilde{d}(x_n, v_n) + \widetilde{D}(v_n, T(x^*)) \\ &\preceq \widetilde{D}(x_n, T(x_n)) + \widetilde{H}(T(x_n), T(x^*)) \\ &\preceq \widetilde{D}(x_n, T(x_n)) + qA\widetilde{d}(x_n, x^*) + qB[\widetilde{D}(x_n, T(x_n)) + \widetilde{D}(x^*, T(x^*))] \\ &+ qC[\widetilde{D}(x_n, T(x^*)) + \widetilde{D}(x^*, T(x_n))]. \end{aligned}$$

Then we have the inequality:

$$\begin{aligned} [I - q(A + C)]\widetilde{d}(x_n, x^*) &\preceq (I + qB)\widetilde{D}(x_n, T(x_n)) + qC\widetilde{D}(x^*, T(x_n)) \\ &\preceq (I + qB)\widetilde{D}(x_n, T(x_n)) + qC[\widetilde{d}(x_n, x^*) + \widetilde{D}(x_n, T(x_n))]. \end{aligned}$$

Then, by $[I - q(A + 2C)]\widetilde{d}(x_n, x^*) \preceq (I + q(B + C))\widetilde{D}(x_n, T(x_n))$ we obtain:

$$\widetilde{d}(x_n, x^*) \preceq [I - q(A + 2C)]^{-1}(I + q(B + C))\widetilde{D}(x_n, T(x_n)).$$

Letting $n \to \infty$ in the above inequality we obtain: $\widetilde{d}(x_n, x^*) \to 0_{m\times 1}$. That means $d_i(x_n, x^*) \to 0$ as $n \to \infty$ with $0 \in \mathbb{R}$, for each $i \in \{1, \ldots, m\}$. Then $x_n \to x^*$ as $n \to \infty$. □

The following result is a well-posedness result for the common fixed point problem.

Theorem 8. *Let $(X, \widetilde{d})$ be a generalized metric space in Perov's sense and $T, G : X \to P_{cl}(X)$ be a multivalued Hardy–Rogers type operator defined in Definition (4). Then, if there exists a matrix nonsingular $I - q(A + 2C)$ such that $(I - q(A + 2C))^{-1} \in \mathbb{M}_{m,m}(\mathbb{R}_+)$, with $q \in (1, Q)$, for every matrices $A, B, C \in \mathbb{M}_{m,m}(\mathbb{R}_+)$, then the fixed point problem of T and G is well-posed.*

Proof. From the Theorem 5 we know that T and G have a unique common fixed point $x^* \in X$. Let $(x_n)_{n\in\mathbb{N}} \in X$ be a sequence such that $\widetilde{D}(T(x_n), x_n) \to 0_{m\times 1}$ or $\widetilde{D}(G(x_n), x_n) \to 0_{m\times 1}$. Let $v_n \in T(x_n)$ such that $\widetilde{d}(x_n, v_n) = \widetilde{D}(x_n, G(x_n))$.

For $q \in (1, Q)$, where Q is defined by Lemma 3, we have:

$$\widetilde{d}(x_n, x^*) \preceq \widetilde{d}(x_n, v_n) + \widetilde{d}(v_n, x^*) \preceq \widetilde{D}(x_n, G(x_n)) + q\widetilde{H}(G(x_n), T(x^*)).$$

$$\begin{aligned} \widetilde{H}(G(x_n), T(x^*)) &\preceq A\widetilde{d}(x_n, x^*) + B[\widetilde{D}(x_n, G(x_n)) + \widetilde{D}(x^*, T(x^*))] + C[\widetilde{D}(x_n, T(x^*)) + \widetilde{D}(x^*, G(x_n))] \\ &\preceq A\widetilde{d}(x_n, x^*) + B[\widetilde{D}(x_n, G(x_n)) + \widetilde{d}(x^*, x^*)] + C[\widetilde{d}(x_n, x^*) + \widetilde{D}(x^*, G(x_n))]. \end{aligned}$$

Using the triangle inequality we have:

$$\begin{aligned} \widetilde{d}(x_n, x^*) &\preceq \widetilde{D}(x_n, G(x_n)) + \widetilde{D}(G(x_n), x^*) \\ &\preceq \widetilde{D}(x_n, G(x_n)) + qA\widetilde{d}(x_n, x^*) + qB[\widetilde{D}(x_n, G(x_n)) + \widetilde{d}(x^*, x^*)] \\ &+ qC[\widetilde{d}(x_n, x^*) + \widetilde{D}(x^*, G(x_n))]. \end{aligned}$$

Then we get: $\widetilde{d}(x_n, x^*) \preceq (I - q(A + 2C))^{-1}q(I + B + C)\widetilde{D}(x_n, G(x_n))$.

Letting $n \to \infty$ in the above inequality, we obtain $\widetilde{d}(x_n, x^*) \to 0_{m\times 1}$, that means $d_i(x_n, x^*)) \to 0$ as $n \to \infty$ with $0 \in \mathbb{R}$, for each $i \in \{1, \ldots, m\}$. Then $x_n \to x_0$ as $n \to \infty$. □

Next, let us give a data dependence result.

Theorem 9. *Let $(X, \widetilde{d})$ be a generalized metric space in Perov's sense and $T_1, T_2 : X \to P_{cl}(X)$ be multivalued operators which satisfy the following conditions:*

(i) for $A, B, C, M \in \mathbb{M}_{m,m}(\mathbb{R}_+)$ with $M = [I - q(B + C)]^{-1} q(A + B + C)$ a matrix convergent to Θ such that, for every $x, y \in X$ with $i \in \{1, 2\}$ and $q \in (1, Q)$, we have:
$\widetilde{H}(T_i(x), T_i(y)) \preceq qA\widetilde{d}(x, y) + qB[\widetilde{D}(x, T_i(x)) + \widetilde{D}(y, T_i(y))] + qC[\widetilde{d}(x, T_i(y)) + \widetilde{d}(y, T_i(x))]$;

(ii) there exists $\eta > 0$ such that $\widetilde{H}(T_1(x), T_2(x)) \preceq (I - M)^{-1} \eta I_{m \times 1}$, for all $x \in X$.

Then for $x_1^* \in T_1(x_1^*)$ there exist $x_2^* \in T_2(x_2^*)$ such that $\widetilde{d}(x_1^*, x_2^*) \preceq (I - M)^{-1} \eta I_{m \times 1}$; (respectively for $x_2^* \in T_2(x_2^*)$ there exist $x_1^* \in T_1(x_1^*)$ such that $\widetilde{d}(x_2^*, x_1^*) \preceq (I - M)^{-1} \eta I_{m \times 1}$)

Proof. As in the proof of Theorem 2 we construct a sequence of successive approximations $(x_n)_{n \in \mathbb{N}} \in X$ of T_2 with $x_0 := x_1^*$ and $x_1 \in T_2(x_1^*)$ having property $\widetilde{d}(x_n, x_{n+1}) \preceq M^n \widetilde{d}(x_0, x_1)$, where $M = [I - q(B + C)]^{-1} q(A + B + C)$, for $q \in (1, Q)$, where Q is defined by Lemma 3.

If we consider that the sequence $(x_n)_{n \in \mathbb{N}} \in X$ converges to x_2^* we have $x_2^* \in T_2(x_2^*)$. Moreover, for each $n, p \in \mathbb{N}$ we have $\widetilde{d}(x_n, x_{n+p}) \preceq M^n (I - M)^{-1} \widetilde{d}(x_0, x_1)$.

Letting $p \to \infty$ we get that $\widetilde{d}(x_n, x_2^*) \preceq I(I - M)^{-1} \widetilde{d}(x_0, x_1)$.

Choosing $n = 0$ we get that $\widetilde{d}(x_0, x_2^*) \preceq I(I - M)^{-1} \widetilde{d}(x_0, x_1)$ and using above notations we get the conclusion: $\widetilde{d}(x_1^*, x_2^*) \preceq (I - M)^{-1} \eta I_{m \times 1}$. □

4. Conclusions

Our main purpose in this paper is to establish new generalizations of a contractive condition of Hardy–Rogers type in metric spaces of Perov's sense. Using a complete new approach in the definition of Hardy–Rogers type contractive conditions for multivalued mappings in generalized metric spaces of Perov's type, we obtained results which generalize, extend, complement and enrich several recent ones in the existing literature. The Ulam–Hyers stability, the well-posedness of the fixed point problem and the data dependence of the fixed point set, are also studied.

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References

1. Perov, A.I. On Cauchy problem for a system of ordinary differential equations, (in Russian). *Priblizhen. Metody Reshen. Differ. Uravn.* **1964**, *2*, 115–134.
2. Abbas, M.; Nazir, T.; Rakočević, V. Common Fixed Points Results of Multivalued Perov type Contractions on Cone Metric Spaces with a Directed Graph. *Bull. Belg. Math. Soc. Simon Stevin* **2018**, *25*, 1–24. [CrossRef]
3. Huang, L.-G.; Zhang, X. Cone metric spaces and fixed point theorems of contractive mappings. *J. Math. Anal. Appl.* **2007**, *332*, 1468–1476. [CrossRef]
4. Zabreĭko, P.P. K-metric and K-normed linear spaces: Survey. *Collect. Math.* **1997**, *48*, 825–859.
5. Janković, S.; Kadelburg, Z.; Radenović, S. On cone metric spaces: A survey. *Nonlinear Anal.* **2011**, *74*, 2591–2601. [CrossRef]
6. Kadelburg, Z.; Radenović, S.; Rakočević, V. A note on equivalence of some metric and cone metric fixed point results. *Appl. Math. Lett.* **2011**, *24*, 370–374. [CrossRef]
7. Aleksić, S.; Kadelburg, Z.; Mitrović, Z.D.; Radenović, S. A new survey: Cone metric spaces. *J. Int. Math. Virtual Inst.* **2019**, *9*, 93–121.
8. Jachimski, J.; Klima, J. Around Perov's fixed point theorem for mappings on generalized metric spaces. *Fixed Point Theory* **2016**, *17*, 367–380.

9. Rus, I.A. The theory of a metrical fixed point theorem: Theoretical and applicative relevances. *Fixed Point Theory* **2008**, *9*, 541–559.
10. Hardy, G.E.; Rogers, A.D. A generalisation of fixed point theorem of Reich. *Canad. Math. Bull.* **1973**, *16*, 201–208. [CrossRef]
11. Rus, I.A. Remarks on Ulam stability of the operatorial equations. *Fixed Point Theory* **2009**, *10*, 305–320.
12. Petruşel, A.; Rus, I.A.; Sântămărian, A. Data dependence of the fixed point set of multivalued weakly Picard operators. *Nonlinear Anal.* **2003**, *52*, 1947–1959.
13. Petruşel, A. Multivalued weakly Picard operators and applications. *Sci. Math. Japon.* **2004**, *59*, 169–202.
14. Brzdek, J.; Popa, D.; Xu, B. The Hyers-Ulam stabililty of nonlinear recurrences. *J. Math. Anal. Appl.* **2007**, *335*, 443–449. [CrossRef]
15. Brzdek, J.; Popa, D.; Xu, B. Hyers-Ulam stabililty for linear equations of higher orders. *Acta Math. Hung.* **2008**, *120*, 1–8. [CrossRef]
16. Hyers, D.H. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222–224. [CrossRef] [PubMed]
17. Hyers, D.H.; Isac, G.; Rassias, T.M. Stability of Functional Equations in Several Variables. *Proc. Am. Math. Soc.* **1998**, *126*, 425–430. [CrossRef]
18. Jung, S.-M.; Lee, K.-S. Hyers-Ulam stability of first order linear partial differrential equations with constant coefficients. *Math. Ineq. Appl.* **2007**, *10*, 261–266.
19. Petru, T.P.; Petruşel, A.; Yao, J.-C. Ulam–Hyers stability for operatorial equations and inclusions via nonself operators. *Taiwanese J. Math.* **2011**, *15*, 2195–2212. [CrossRef]
20. Popa, D. Hyers-Ulam stabililty of the linear recurrence with constant coefficients. *Adv. Differ. Equ.* **2005**, *2*, 101–107.