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Proximally Compatible Mappings and Common Best Proximity Points

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Abstract: The purpose of this paper is to introduce and analyze a new idea of proximally compatible mappings and we extend some results of Jungck via proximally compatible mappings. Furthermore, we obtain common best proximity point theorems for proximally compatible mappings through two different ways of construction of sequences. In addition, we provide an example to support our main result.

Keywords: commuting mappings; common fixed point; (ϵ, δ)-contractions; compatible mappings; common best proximity point

1. Introduction

A study of best proximity point theory is a useful tool for providing optimal approximate solutions when a mapping does not have a fixed point. In other words, optimization problems can be converted to the problem of finding best proximity points. Hence, the existence of best proximity points develops the theory of optimization.

Interestingly, these best proximity point theorems also serve as a natural generalization of fixed point theorems and a best proximity point becomes a fixed point if the mapping under consideration is a self-mapping.

In [1], Jungck introduced the notion of compatible mappings and derived results on common fixed points for the compatible mappings. Sessa [2] defined the term weakly commuting pairs and obtained fixed point theorems. The following theorem via commuting mappings was studied in [3].

Theorem 1. *Let (X, D) be complete metric space. Then, a continuous function $\Lambda : X \to X$ has a fixed point if and only if there exists $s \in (0,1)$ and a function $\Gamma : X \to X$ which commutes with Λ $(\Gamma\Lambda = \Lambda\Gamma)$ and satisfies: $\Gamma(X) \subset \Lambda(X)$ and $D(\Gamma(\eta), \Gamma(\omega)) \leq sD(\Lambda(\eta), \Lambda(\omega))$ for $\eta, \omega \in X$.*

One can note that the above theorem is a generalization of the Banach contraction theorem. Das et al. [4] generalized the result of Jungck [3] and proved existence of common fixed point for mappings which need not be continuous. In [5], Chang generalized and unified many fixed point theorems in complete metric spaces. Later, Conserva [6] proved three existence of common fixed point theorems for commuting mappings on a metric space which generalize the various fixed point results. In 1998, Jungck and Rhoades [7] initiated the concept of weakly compatible mappings and proved that the class of weakly compatible mappings contains the class of compatible mappings. Furthermore, Chugh and Kumar [8] proved theorems on existence of a common fixed point for weakly compatible mappings.

In the sequel, Basha et al. [9] gave existence of common best proximity points for pairs of non-self mappings in metric spaces. Aydi et al. [10] established the existence result of common best proximity point for generalized $\alpha - \psi-$proximal contractive pair of non-self mappings. In [11], Mongkolkeha et al. proved existence of common best proximity point for a pair of proximity commuting mappings in a complete metric space. On the other hand, Cvetković et al. [12] showed existence of common fixed point for four mappings in cone metric spaces. Parvaneh Lo'lo' et al. [13] proved a result which gives sufficient condition to exist a common best proximity point for four different mappings in metric-type spaces. One can get some ideas on results of common best proximity point for several kinds of non-self mappings which are available in [14–18]

In this research paper, we provide the concept of proximally compatible mappings and we give common best proximity point theorems for proximally compatible non-self mappings. First, we prove some basic results from Jungck [1], which are analogous of self mappings. Using these results, we give enough conditions that make sure the existence of a common best proximity point.

2. Preliminaries

Here we start with some notions:
Let M, N be two subsets of a metric space (X, D).

$$\begin{aligned}
&dist(M,N) = D(M,N) = inf\{D(\eta,\omega) : \eta \in M, \omega \in N\};\\
&D(\eta,N) = inf\{D(\eta,\omega) : \omega \in N\};\\
&M_0 = \{\eta \in M : D(\eta,\omega') = dist(M,N) \text{ for some } \omega' \in N\};\\
&N_0 = \{\omega \in N : D(\eta',\omega) = dist(M,N) \text{ for some } \eta' \in M\}.
\end{aligned}$$

Definition 1 ([13])**.** *An element $\eta \in M$ is said to be a common best proximity point of the nonself-mappings $\Lambda_1, \Lambda_2, ..., \Lambda_n : M \to N$ if it satisfies*

$$D(\eta, \Lambda_1\eta) = D(\eta, \Lambda_2\eta) = ... = D(\eta, \Lambda_n\eta) = D(M, N).$$

Definition 2 ([13])**.** *Mappings $\Lambda : M \to N$ and $\Gamma : M \to N$ are said to be commute proximally if they satisfy*

$$[D(v, \Lambda\eta) = D(\nu, \Gamma\eta) = D(M, N)] \quad implies \quad \Lambda\nu = \Gamma v,$$

for some $v, \nu, \eta \in M$.

Definition 3. *Let Λ, Γ be two non self-mappings $\Lambda, \Gamma : M \to N$. A point $\eta \in M$ is said to be coincidence point if $\Lambda(\eta) = \Gamma(\eta)$.*

Definition 4. *A pair of mappings Λ and Γ is called weakly commuting proximally pair if they commute proximally at coincidence points.*

Definition 5 ([13])**.** *If $M_0 \neq \emptyset$ then the pair (M, N) is said to have the P-property if for any $\eta_1, \eta_2 \in M_0$ and $\omega_1, \omega_2 \in N_0$*

$$\begin{cases} D(\eta_1, \omega_1) = D(M, N) \\ D(\eta_2, \omega_2) = D(M, N) \end{cases} \quad implies \quad D(\eta_1, \eta_2) = D(\omega_1, \omega_2).$$

Definition 6. *A function $\chi : [0, \infty) \to [0, \infty)$ is said to be an altering distance function if it satisfies,*

(i) χ *is non-decreasing and continuous,*
(ii) $\chi(t) = 0$ *iff* $t = 0$.

In [2], Khan et al. extended the fixed point theorems for contractive type mappings using altering distance function.

Let M and N be subsets of a metric space (X, D). Let $\Gamma : M \to N$ be a continuous and nondecreasing mapping such that

$$\chi(D(\Gamma(\eta), \Gamma(\omega))) \leq \chi(D(\eta, \omega)) - \varphi(D(\eta, \omega)) \text{ for } \eta, \omega \in M,$$

where χ, φ are altering distance functions.

3. Proximally Compatible Mappings

Now we extend the definition of compatible mappings (Definition 2.1 of [1]) for the case of non-self mappings.

Definition 7. *Let M and N be two subsets of a metric space (X, D). Two non-self mappings Λ and Γ from M to N are proximally compatible if for any sequences $\{\eta_n\}$, $\{\upsilon_n\}$ and $\{\nu_n\}$ in M*

$$\begin{cases} D(\upsilon_n, \Lambda\eta_n) = D(M, N) \\ D(\nu_n, \Gamma\eta_n) = D(M, N) \end{cases} \quad \textit{implies} \quad \lim_{n\to\infty} D(\Lambda\nu_n, \Gamma\upsilon_n) = 0,$$

whenever $\lim_{n\to\infty} \upsilon_n = \lim_{n\to\infty} \nu_n = t$.

Example 1. *Let $X = \mathbb{R}^2$ and $D(\eta, \omega) = \sqrt{(\eta_1 - \omega_1)^2 + (\eta_2 - \omega_2)^2}$, where $\eta = (\eta_1, \eta_2), \omega = (\omega_1, \omega_2)$. In addition, we consider $M = \{(0, \eta) : 0 \leq \eta \leq 1\}, N = \{(1, \omega) : 0 \leq \omega \leq 1\}$, then $M_0 = M, N_0 = N$. Define the functions $\Lambda, \Gamma : M \to N$ by $\Lambda(0, \eta) = (1, \eta^3)$ and $\Gamma(0, \eta) = (1, 2\eta^3)$. Now if*

$$\begin{cases} D((0, \eta_n^3), (1, \eta_n^3)) = 1 \\ D((0, 2\eta_n^3), (1, 2\eta_n^3)) = 1 \end{cases}$$

then, we have $D(\Lambda(0, 2\eta_n^3), \Gamma(0, \eta_n^3)) = D((1, 2^3\eta_n^9), (1, 2\eta_n^9)) = 6\eta_n^9$ and $D((0, \eta_n^3), (0, 2\eta_n^3)) = \eta_n^3$. Since $\eta_n^3 \to 0$ as $n \to \infty$, $6\eta_n^9 \to 0$. So Λ and Γ are proximally compatible.

Proposition 1. *Let M and N be two subsets of a metric space (X, D). Let $\Lambda, \Gamma : M \to N$ be continuous and let $B = \{a \in M : D(a, \Lambda a) = D(a, \Gamma a) = D(M, N)\}$. Assume that the pair (M, N) satisfies P-property. Then Λ and Γ are proximally compatible if any one of the following holds:*

(1) If $\upsilon_n, \nu_n \to t \in M$ as $n \to \infty$, then $t \in B$,
(2) $D(\upsilon_n, \nu_n) \to 0$ as $n \to \infty$ implies $D(\upsilon_n, B) \to 0$ as $n \to \infty$,
(3) B is compact and $D(\upsilon_n, \nu_n) \to 0$ as $n \to \infty$ imply $D(\eta_n, B) \to 0$ as $n \to \infty$,

where $\upsilon_n, \nu_n, \eta_n$ are same as in Definition 7.

Proof. We assume $\lim_{n\to\infty} \upsilon_n = \lim_{n\to\infty} \nu_n = t$ for some $t \in M$.
If (1) holds, since $t \in B$, $D(t, \Lambda t) = D(t, \Gamma t) = D(M, N)$. By P-property, $D(\Lambda t, \Gamma t) = 0$. Since Λ, Γ are continuous, which gives

$$\begin{cases} \lim_{n\to\infty} \Lambda\nu_n = \Lambda t \\ \lim_{n\to\infty} \Gamma\upsilon_n = \Gamma t \end{cases} \quad \text{implies} \quad D(\Lambda\nu_n, \Gamma\upsilon_n) \to D(\Lambda t, \Gamma t) = 0,$$

result follows. If (2) holds, and by noting that B is closed, since $D(\upsilon_n, B) \to 0$ as $n \to \infty$, then there exists a sequence $\{b_n\}$ in B such that $D(\upsilon_n, b_n) \to 0$ as $n \to \infty$. This implies that $b_n \to t$ as $n \to \infty$, and then $t \in B$. So the result follows from (1). If (3) holds, since B is compact, there is a subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$ such that $\{\eta_{n_k}\}$ converges to η^* and $\eta^* \in B$, that is, $D(\eta^*, \Lambda\eta^*) = D(\eta^*, \Gamma\eta^*) = D(M, N)$. The continuity of Λ implies that $\Lambda\eta_{n_k} \to \Lambda\eta^*$. From Definition 7, in particular, we have $D(\upsilon_{n_k}, \Lambda\eta_{n_k}) = D(M, N)$, and as $k \to \infty$, we obtain $D(t, \Lambda\eta^*) = D(M, N)$, and by P-property, we get $t = \eta^*$. Therefore Λ, Γ are proximally compatible by (1). □

Proposition 2. *Let M and N be two subsets of a metric space (X, D). Let $\Lambda, \Gamma : M \to N$, be proximally compatible and the pair (M, N) satisfy P- property.*

(1) If $\Lambda t = \Gamma t$, with

$$\begin{cases} D(\upsilon, \Lambda t) = D(M, N) \\ D(\nu, \Gamma t) = D(M, N), \end{cases}$$

then $\Lambda\nu = \Gamma\upsilon$.

(2) Suppose that $\lim_{n\to\infty} \upsilon_n = \lim_{n\to\infty} \nu_n = t$ for some t in M, where υ_n, ν_n are same as in Definition 7.

  - *(a) If Λ is continuous at t, then $\lim_{n\to\infty} \Gamma\upsilon_n = \Lambda t$.*
  - *(b) If Λ, Γ are continuous at t, with*

$$\begin{cases} D(\upsilon, \Lambda t) = D(M, N) \\ D(\nu, \Gamma t) = D(M, N), \end{cases}$$

  then $\upsilon = \nu$ and $\Lambda\nu = \Gamma\upsilon$.

Proof. For (1), suppose that $\Lambda t = \Gamma t$ and

$$\begin{cases} D(\upsilon, \Lambda t) = D(M, N) \\ D(\nu, \Gamma t) = D(M, N). \end{cases}$$

By P-property, we have $D(\upsilon, \nu) = D(\Lambda t, \Gamma t) = 0$, and this implies that $\upsilon = \nu$. Now, assume $\upsilon_n\ upsilon, \nu_n = \nu, \eta_n = t$, for all $n \in \mathbb{N}$. So

$$\begin{cases} D(\upsilon_n, \Lambda\eta_n) = D(M, N) \\ D(\nu_n, \Gamma\eta_n) = D(M, N), \end{cases} \quad \text{implies} \quad \lim_{n\to\infty} D(\Lambda\nu_n, \Gamma\upsilon_n) = 0$$

by proximally compatible. Then $D(\Lambda\nu, \Gamma\upsilon) = 0$, proving (1). Now we prove $2(a)$, since $\lim_{n\to\infty} \nu_n = t$, $\lim_{n\to\infty} \Lambda\nu_n = \Lambda t$ by continuity of Λ. Now

$$D(\Gamma\upsilon_n, \Lambda t) \le D(\Gamma\upsilon_n, \Lambda\nu_n) + D(\Lambda\nu_n, \Lambda t).$$

Since Λ, Γ are proximally compatible, $D(\Gamma\upsilon_n, \Lambda t) \to 0$.

For $2(b)$, $\Gamma\upsilon_n \to \Lambda t$ by $2(a)$ and by continuity $\Gamma\upsilon_n \to \Gamma t$. Thus $\Lambda t = \Gamma t$. By the P-property we have $\upsilon = \nu$. In addition, also $\Lambda\nu = \Gamma\upsilon$, by (1). □

4. Common Best Proximity Points for $(\epsilon, \delta, \chi, \varphi)$-Contractions

Motivated by Definition 3.1 in [1], we define the following.

Definition 8. *Let (X, D) be a metric space. Let M and N be two subsets of X. A pair of nonself mappings Λ and Γ from M to N are $(\epsilon, \delta, \chi, \varphi) - F, G-$ contractions relative to mappings $F, G : M \to N$ if and only if $\Lambda(M) \subset G(M), \Gamma(M) \subset F(M)$, and there exists a mapping $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\delta(\epsilon) > \epsilon$ for all $\epsilon > 0$ and for $\eta, \omega \in M$:*

1. $\epsilon \leq \chi\big(D(F\eta, G\omega)\big) - \varphi\big(D(F\eta, G\omega)\big) < \delta(\epsilon)$ *implies* $\chi\big(D(\Lambda\eta, \Gamma\omega)\big) < \epsilon$, *and*
2. $\Lambda\eta = \Gamma\omega$ *whenever* $F\eta = G\omega$,

where χ, φ *are altering distance functions.*

Note that if Λ and Γ are $(\epsilon, \delta, \chi, \varphi) - F, G-$ contractions, then $\chi\big(D(\Lambda\eta, \Gamma\omega)\big) \leq \chi\big(D(F\eta, G\omega)\big) - \varphi\big(D(F\eta, G\omega)\big)$ for all $\eta, \omega \in M$.
Let M and N be subsets of a metric space (X, D). Let Λ, Γ, F, and G be nonself mappings from M to N such that $\Lambda(M) \subset G(M)$ and $\Gamma(M) \subset F(M)$ for $\eta_0 \in M$. Any sequence $\{\omega_n\}$ is constructed by $\omega_{2n-1} = G\eta_{2n-1} = \Lambda\eta_{2n-2}$ and $\omega_{2n} = F\eta_{2n} = \Gamma\eta_{2n-1}$ for $n \in \mathbb{N}$ - called an F, G - iteration of η_0 under Λ and Γ.

Lemma 1. *Let M and N be two subsets of a metric space (X, D). Let F and G be nonself mappings from M to N and let (Λ, Γ) be $(\epsilon, \delta, \chi, \varphi) - F, G-$ contraction. If $\eta_0 \in M$ and $\{\omega_n\}$ is an F, G-iteration of η_0 under Λ and Γ, then*

(i) *for each $\epsilon > 0$, $\epsilon \leq \chi\big(D(\omega_p, \omega_q)\big) - \varphi\big(D(\omega_p, \omega_q)\big) < \delta(\epsilon)$ implies $\chi\big(D(\omega_{p+1}, \omega_{q+1})\big) < \epsilon$, where p, q are opposite parity,*
(ii) *$D(\omega_n, \omega_{n+1}) \to 0$ as $n \to \infty$, and*
(iii) *$\{\omega_n\}$ is a Cauchy sequence.*

Proof. To prove (i), let $\epsilon > 0$. Since Λ and Γ are $(\epsilon, \delta, \chi, \varphi) - F, G$-contractions,

$$\epsilon \leq \chi\big(D(F\eta, G\omega)\big) - \varphi\big(D(F\eta, G\omega)\big) < \delta(\epsilon) \text{ implies } \chi\big(D(\Lambda\eta, \Gamma\omega)\big) < \epsilon. \tag{1}$$

Now suppose that $\epsilon \leq \chi\big(D(\omega_p, \omega_q)\big) - \varphi\big(D(\omega_p, \omega_q)\big) < \delta(\epsilon)$, where $p = 2n$ and $q = 2m - 1$. We have

$$\chi\big(D(\omega_{p+1}, \omega_{q+1})\big) = \chi\big(D(\omega_{2n+1}, \omega_{2m})\big) = \chi\big(D(\Lambda\eta_{2n}, \Gamma\eta_{2m-1})\big)$$

and

$$\begin{aligned} \chi\big(D(\omega_p, \omega_q)\big) - \varphi\big(D(\omega_p, \omega_q)\big) &= \chi\big(D(\omega_{2n}, \omega_{2m-1})\big) - \varphi\big(D(\omega_{2n}, \omega_{2m-1})\big) \\ &= \chi\big(D(F\eta_{2n}, G\eta_{2m-1})\big) - \varphi\big(D(F\eta_{2n}, G\eta_{2m-1})\big). \end{aligned}$$

By (1), we have,

$$\epsilon \leq \chi\big(D(F\eta_{2n}, G\eta_{2m-1})\big) - \varphi\big(D(F\eta_{2n}, G\eta_{2m-1})\big) < \delta(\epsilon),$$

which gives $\chi\big(D(\Lambda\eta_{2n}, \Gamma\eta_{2m-1})\big) = \chi\big(D(\omega_{p+1}, \omega_{q+1})\big) < \epsilon$.
For (ii), we know from the hypothesis $\chi\big(D(\Lambda\eta, \Gamma\omega)\big) \leq \chi\big(D(F\eta, G\omega)\big) - \varphi\big(D(F\eta, G\omega)\big)$ for all $\eta, \omega \in M$. Suppose n is even, say, $n = 2m$,

$$\begin{aligned} \chi\big(D(\omega_n, \omega_{n+1})\big) &= \chi\big(D(\omega_{2m}, \omega_{2m+1})\big) \\ &= \chi\big(D(\Gamma\eta_{2m-1}, \Lambda\eta_{2m})\big) \\ &\leq \chi\big(D(F\eta_{2m}, G\eta_{2m-1})\big) - \varphi\big(D(F\eta_{2m}, G\eta_{2m-1})\big) \\ &\leq \chi\big(D(F\eta_{2m}, G\eta_{2m-1})\big) \\ &= \chi\big(D(\omega_{2m}, \omega_{2m-1})\big) = \chi\big(D(\omega_n, \omega_{n-1})\big). \end{aligned}$$

Similarly, one can prove that $\chi\big(D(\omega_n, \omega_{n+1})\big) \leq \chi\big(D(\omega_{n-1}, \omega_n)\big)$ if $n = 2m + 1$. Then the sequence $\{\chi\big(D(\omega_n, \omega_{n+1})\big)\}$ is nonincreasing which shows $D(\omega_n, \omega_{n+1}) \leq D(\omega_n, \omega_{n-1})$ for all n. Hence, the sequence $\{D(\omega_n, \omega_{n+1})\}$ is bounded and nonincreasing. Thus, there exists $r \geq 0$ such that

$$\lim_{n \to \infty} D(\omega_n, \omega_{n+1}) = r. \tag{2}$$

Suppose $r > 0$. Let us assume n is odd, that is, $n = 2m - 1$. Then the inequality

$$\begin{aligned}
\chi\big(D(\omega_n, \omega_{n+1})\big) &= \chi\big(D(\omega_{2m-1}, \omega_{2m})\big) \\
&= \chi\big(D(\Lambda\eta_{2m-2}, \Gamma\eta_{2m-1})\big) \\
&\leq \chi\big(D(F\eta_{2m-2}, G\eta_{2m-1})\big) - \varphi\big(D(F\eta_{2m-2}, G\eta_{2m-1})\big) \\
&\leq \chi\big(D(F\eta_{2m-2}, G\eta_{2m-1})\big) \\
&= \chi\big(D(\omega_{2m-2}, \omega_{2m-1})\big) \\
&= \chi\big(D(\omega_{n-1}, \omega_n)\big)
\end{aligned}$$

which implies that

$$\lim_{m\to\infty} \varphi\big(D(F\eta_{2m-2}, G\eta_{2m-1})\big) = \lim_{m\to\infty} \varphi\big(D(\omega_{2m-2}, \omega_{2m-1})\big) = \lim_{n\to\infty} \varphi\big(D(\omega_{n+1}, \omega_n)\big) = 0. \tag{3}$$

However, as $0 < r \leq D(\omega_{n+1}, \omega_n)$ and φ is nondecreasing function,

$$0 < \varphi(r) \leq \varphi\big(D(\omega_{n+1}, \omega_n)\big),$$

and this implies $\lim_{n\to\infty} \varphi\big(D(\omega_{n+1}, \omega_n)\big) \geq \varphi(r) > 0$ which contradicts to (3). Similarly one can easily varify that for the case of n is even. Then we obtain,

$$\lim_{n\to\infty} D(\omega_n, \omega_{n+1}) = 0. \tag{4}$$

To show (iii), suppose $\{\omega_{2n}\}$ is not a Cauchy sequence. Then we can choose an $\epsilon > 0$ such that for any integer l, there exist $m(l)$ and $n(l)$ with $m(l) > n(l) \geq l$ such that

$$D(\omega_{2m(l)}, \omega_{2n(l)}) > \epsilon. \tag{5}$$

For each $2l$, let $2m(l)$ be the smallest integer exceeding $2n(l)$ satisfying both (5) and the next inequality

$$D(\omega_{2n(l)}, \omega_{2m(l)-2}) \leq \epsilon. \tag{6}$$

Then for each $2l$, we have

$$\begin{aligned}
\epsilon &\leq D(\omega_{2n(l)}, \omega_{2m(l)}) \\
&\leq D(\omega_{2n(l)}, \omega_{2m(l)-2}) + D(\omega_{2m(l)-2}, \omega_{2m(l)-1}) + D(\omega_{2m(l)-1}, \omega_{2m(l)}).
\end{aligned}$$

Using (6), we obtain

$$\epsilon \leq D(\omega_{2n(l)}, \omega_{2m(l)}) \leq \epsilon + D(\omega_{2m(l)-2}, \omega_{2m(l)-1}) + D(\omega_{2m(l)-1}, \omega_{2m(l)}).$$

From part (ii) and by Sandwich lemma, we get

$$D(\omega_{2n(l)}, \omega_{2m(l)}) \to \epsilon \text{ as } l \to \infty. \tag{7}$$

Again from part (ii) and (7), the inequality

$$D(\omega_{2m(l)}, \omega_{2n(l)}) \leq D(\omega_{2m(l)}, \omega_{2m(l)+1}) + D(\omega_{2m(l)+1}, \omega_{2n(l)})$$

as $l \to \infty$, gives that

$$\epsilon \leq \lim_{l\to\infty} D(\omega_{2m(l)+1}, \omega_{2n(l)}).$$

Now again we have the inequality

$$D(\omega_{2m(l)+1}, \omega_{2n(l)}) \leq D(\omega_{2m(l)+1}, \omega_{2m(l)}) + D(\omega_{2m(l)}, \omega_{2n(l)})$$

as $l \to \infty$, we obtain

$$\lim_{l\to\infty} D(\omega_{2m(l)+1}, \omega_{2n(l)}) \leq \epsilon.$$

Hence $\lim_{l\to\infty} D(\omega_{2m(l)+1}, \omega_{2n(l)}) = \epsilon$. In the same way, one can obtain

$$\lim_{l\to\infty} D(\omega_{2m(l)}, \omega_{2n(l)-1}) = \epsilon.$$

Therefore, we have

$$\begin{aligned}
\chi\big(D(\omega_{2n(l)}, \omega_{2m(l)+1})\big) &= \chi\big(D(\Gamma\eta_{2n(l)-1}, \Lambda\eta_{2m(l)})\big) \\
&\leq \chi\big(D(F\eta_{2m(l)}, G\eta_{2n(l)-1})\big) - \varphi\big(D(F\eta_{2m(l)}, G\eta_{2n(l)-1})\big) \\
&= \chi\big(D(\omega_{2m(l)}, \omega_{2n(l)-1})\big) - \varphi\big(D(\omega_{2m(l)}, \omega_{2n(l)-1})\big).
\end{aligned}$$

Letting $l \to \infty$, we get

$$\chi(\epsilon) \leq \chi(\epsilon) - \varphi(\epsilon),$$

which implies a contradiction, since $\epsilon > 0$. Thus, $\{\omega_{2n}\}$ is a Cauchy sequence in N and so $\{\omega_n\}$. □

Lemma 2. *Let M and N be two subsets of a metric space (X, D). Let $F, G : M \to N$ be nonself mappings. Let Λ and Γ be $(\epsilon, \delta, \chi, \varphi) - F, G$-contractions such that the pairs $\{\Lambda, F\}$ and $\{\Gamma, G\}$ are proximally compatible. Assume (M, N) satisfies P-property and $\Lambda(M_0) \subset N_0$. If there exists $z \in M_0$ such that $\Lambda z = Fz$ and $\Gamma z = Gz$, then Λ, Γ, F and G have unique common best proximity points.*

Proof. By the definition of $(\epsilon, \delta, \chi, \varphi) - F, G$-contractions implies

$$\begin{aligned}
\chi\big(D(\Lambda\eta, \Gamma\omega)\big) &\leq \chi\big(D(F\eta, G\omega)\big) - \varphi\big(D(F\eta, G\omega)\big) \text{ if } F\eta \neq G\omega \\
&< \chi\big(D(F\eta, G\omega)\big).
\end{aligned}$$

Suppose $z \in M_0$ such that $\Lambda z = Fz$ and $\Gamma z = Gz$. Thus if $Fz \neq Gz$, then

$$\begin{aligned}
\chi\big(D(\Lambda z, \Gamma z)\big) &\leq \chi\big(D(Fz, Gz)\big) - \varphi\big(D(Fz, Gz)\big) \\
&< \chi\big(D(Fz, Gz)\big),
\end{aligned}$$

which is contradiction. Then $\Lambda z = \Gamma z = Fz = Gz$.

Since $\Lambda(M_0) \subset N_0$, there exists $\eta \in M_0$ such that $D(\eta, \Lambda z) = D(\eta, \Gamma z) = D(\eta, Fz) = D(\eta, Gz) = D(M, N)$.

As $\{\Lambda, F\}$ and $\{\Gamma, G\}$ proximally compatible, implies that $\Lambda\eta = \Gamma\eta = F\eta = G\eta$. Since $\Lambda(M_0) \subset N_0$, there exists $\omega \in M_0$ such that $D(\omega, \Lambda\eta) = D(\omega, \Gamma\eta) = D(\omega, F\eta) = D(\omega, G\eta) = D(M, N)$. Since the pair (M, N) has the P-property

$$\begin{aligned}
\chi\big(D(\eta, \omega)\big) &= \chi\big(D(\Lambda z, \Gamma\eta)\big) \\
&\leq \chi\big(D(Fz, G\eta)\big) - \varphi\big(D(Fz, G\eta)\big) \\
&= \chi\big(D(\eta, \omega)\big) - \varphi\big(D(\eta, \omega)\big), \\
\varphi\big(D(\eta, \omega)\big) &\leq 0.
\end{aligned}$$

These imply that $\eta = \omega$. Therefore $D(\eta, \Lambda\eta) = D(\eta, \Gamma\eta) = D(\eta, F\eta) = D(\eta, G\eta) = D(M, N)$.

To prove the uniqueness, suppose that w is another common best proximity point of the mappings Λ, Γ, F and G, so that, $D(w, \Lambda w) = D(w, \Gamma w) = D(w, Fw) = D(w, Gw) = D(M, N)$. As the pair (M, N) has the P-property

$$
\begin{aligned}
\chi(D(\eta, w)) &= \chi(D(\Lambda\eta, \Gamma w)) \\
&\leq \chi(D(F\eta, Gw)) - \varphi(D(F\eta, Gw)) \\
&= \chi(D(\eta, w)) - \varphi(D(\eta, w)), \\
\varphi(D(\eta, w)) &\leq 0,
\end{aligned}
$$

which imply $\eta = w$. This completes the proof. □

Now we prove the existence of common best proximity point for four mappings.

Theorem 2. *Let M and N be two subsets of a complete metric space (X, D). Let F and G be mappings from M to N and let Λ and Γ be $(\epsilon, \delta, \chi, \varphi) - F, G$- contractions such that the pairs (Λ, F) and (Γ, G) are proximally compatible and assume $\Lambda(M_0) \subseteq G(M_0) \subseteq N_0, \Gamma(M_0) \subseteq F(M_0) \subseteq N_0$ with $G(M_0), F(M_0)$ and N_0 are closed. Then Λ, Γ, F and G have unique common best proximity point.*

Proof. Let η_0 in M_0. Since $\Lambda(M_0) \subset G(M_0)$, there exists η_1 in M_0 such that $\Lambda(\eta_0) = G(\eta_1)$. Similarly, a point $\eta_2 \in M_0$ can be chosen such that $\Gamma(\eta_1) = F(\eta_2)$. Continuing in this way, we obtain a sequence $\{\omega_n\} \subset N_0$ such that

$$\omega_{2n} = \Lambda(\eta_{2n}) = G(\eta_{2n+1}) \text{ and } \omega_{2n+1} = \Gamma(\eta_{2n+1}) = F(\eta_{2n+2}), n = 0, 1, 2, 3, ... \tag{8}$$

By Lemma 1, $\{\omega_n\}$ is a Cauchy sequence in N_0. Since N_0 is complete, there is a point $z \in N_0$ such that $\lim_{n\to\infty} \omega_n = z$. Therefore $\lim_{n\to\infty} \Lambda\eta_{2n} = \lim_{n\to\infty} G\eta_{2n+1} = z$ and $\lim_{n\to\infty} \Gamma\eta_{2n+1} = \lim_{n\to\infty} F\eta_{2n+2} = z$. Then

$$\lim_{n\to\infty} \Lambda\eta_{2n} = \lim_{n\to\infty} G\eta_{2n+1} = \lim_{n\to\infty} \Gamma\eta_{2n+1} = \lim_{n\to\infty} F\eta_{2n+2} = z.$$

Since $F(M_0)$ is closed, $z \in F(M_0)$. Then there exists a point $v \in M_0$ such that $Fv = z$. Then,

$$\chi(D(\Lambda v, \Gamma\eta_{2n+1})) \leq \chi(D(Fv, G\eta_{2n+1})) - \varphi(D(Fv, G\eta_{2n+1})).$$

As $n \to \infty$,

$$\chi(D(\Lambda v, z)) \leq 0. \tag{9}$$

Therefore $\Lambda v = z = Fv$.
Since $G(M_0)$ is closed, $z \in G(M_0)$. Then there exists a point $\nu \in M_0$ such that $G\nu = z$. Then,

$$
\begin{aligned}
\chi(D(z, \Gamma\nu)) &= \chi(D(\Lambda v, \Gamma\nu)) \\
&\leq \chi(D(Fv, G\nu)) - \varphi(D(Fv, G\nu)) \\
&\leq \chi(D(Fv, G\nu)) \\
&= \chi(D(z, z)) = 0.
\end{aligned}
$$

Therefore $\Gamma\nu = z = G\nu$. Thus $\Lambda v = Fv = \Gamma\nu = G\nu \in N_0$.

Then there exists $\eta \in M_0$ such that $D(\eta, \Lambda v) = D(\eta, \Gamma\nu) = D(\eta, Fv) = D(\eta, G\nu) = D(M, N)$. Since the pair (Λ, F) and (Γ, G) are proximally compatible, $\Lambda\eta = F\eta$ and $\Gamma\eta = G\eta$ and the theorem follows from Lemma 2. □

Through the following example we illustrate our result.

Example 2. *Let $X = \mathbb{R}^2$, $D(\eta,\omega) = \sqrt{(\eta_1-\omega_1)^2+(\eta_2-\omega_2)^2}$ where $\eta = (\eta_1,\eta_2)$, $\omega = (\omega_1,\omega_2)$ and let $M = \{(0,\omega) : \omega \in [1,\infty)\}, N = \{(1,\omega) : \omega \in [1,\infty)\}$. Then clearly, $M_0 = M$, $N_0 = N$. The functions $\Lambda,\Gamma,F,G : M \to N$ are defined by $\Lambda(0,\omega) = (1,\omega^2)$, $\Gamma(0,\omega) = (1,\omega)$, $F(0,\omega) = (1,\frac{\omega^4+1}{2})$, $G(0,\omega) = (1,\frac{\omega^2+1}{2})$. Here (M,N) satisfies P-property with $D(M,N) = 1$ and $F(M_0) = N_0, G(M_0) = N_0$. Now we claim that Λ and F are proximally compatible. Indeed, let $\{(0,\eta_n)\} \subseteq M_0$, we have*

$$\begin{cases} D((0,\eta_n^2),(1,\eta_n^2)) = 1 \\ D((0,\frac{\eta_n^4+1}{2})),(1,\frac{\eta_n^4+1}{2})) = 1 \end{cases}$$

whenever $(0,\eta_n^2) \to (0,t)$ as $n \to \infty$ and $(0,\frac{\eta_n^4+1}{2}) \to (0,t)$ as $n \to \infty$, which implies that $t = 1$. Now

$$D\Big(\Lambda\Big(0,\frac{\eta_n^4+1}{2}\Big),F(0,\eta_n^2)\Big) = D\Big(\Big(1,\Big(\frac{\eta_n^4+1}{2}\Big)^2\Big),\Big(1,\frac{\eta_n^8+1}{2}\Big)\Big) = (\eta_n^4-1)^2/4.$$

As $n \to \infty$, since $\eta_n^2 \to 1$, we get $(\eta_n^4-1)^2/4 \to 0$. This proves $\{\Lambda,F\}$ is proximally compatible. Similarly, one can easily verify that the pair $\{\Gamma,G\}$ is also proximally compatible. Now suppose $\chi(t) = 2t$, $\varphi(t) = t$, and if $\epsilon \le \chi\big(D(F(0,\eta),G(0,\omega)\big) - \varphi\big(D(F(0,\eta),G(0,\omega)\big) = D(F(0,\eta),G(0,\omega)) = (\eta^4-\omega^2)/2 < \delta(\epsilon)$ then, because of $\eta,\omega \ge 1$, we get $\eta^2+\omega \ge 1+\sqrt{2\epsilon+1}$. In addition, also, since $(\eta^4-\omega^2)/2 < \delta(\epsilon)$, we obtain that $\chi\big(D(\Lambda(0,\eta),\Gamma(0,\omega))\big) = (\eta^2-\omega) < 4\delta(\epsilon)(1+\sqrt{2\epsilon+1})^{-1} = \epsilon$, if $\delta(\epsilon) = \epsilon(1+\sqrt{2\epsilon+1})/4$. Therefore by Theorem 2, there exists a common best proximity point $(0,1) \in M_0$.

We give another method to find best proximity point by changing the construction of sequence.

Let M and N be subsets of a metric space (X,D). Let Λ,Γ,F, and G be nonself mappings from M to N such that $\Lambda(M_0) \subset G(M_0)$ and $\Gamma(M_0) \subset F(M_0)$. Fix η_0 in M_0, since $\Lambda(M_0) \subset G(M_0)$, there exists an element η_1 in M_0 such that $\Lambda\eta_0 = G\eta_1$. Similarly, a point $\eta_2 \in M_0$ can be chosen such that $\Gamma\eta_1 = F\eta_2$. By continuing, we get a sequence $\{\eta_n\}$ in M_0 such that $\Lambda\eta_{2n} = G\eta_{2n+1}$ and $\Gamma\eta_{2n+1} = F\eta_{2n+2}$, for $n = 0,1,2,3,\ldots$

Suppose $\Lambda(M_0) \subset N_0$ and $\Gamma(M_0) \subset N_0$, there exists $\{v_n\}$ in M_0 such that

$$D\big(v_{2n},\Lambda\eta_{2n}\big) = D(M,N) \text{ and } D\big(v_{2n+1},\Gamma\eta_{2n+1}\big) = D(M,N). \tag{10}$$

Therefore

$$D\big(v_{2n},\Lambda\eta_{2n}\big) = D\big(v_{2n},G\eta_{2n+1}\big) = D\big(v_{2n+1},\Gamma\eta_{2n+1}\big) = D\big(v_{2n+1},F\eta_{2n+2}\big) = D(M,N).$$

Lemma 3. *Let Λ and Γ be $(\epsilon,\delta,\chi,\varphi)-F,G$-contractions and assume that the pair (M,N) has the P-property. Then the sequence $\{v_n\}$ defined by (10) above is Cauchy in M_0.*

Proof. Let $D_n = D(v_n,v_{n+1}), n = 0,1,2,\ldots$. Now, we prove $D_n \le D_{n-1}$ for $n = 1,2,3,\ldots$. By the P-property, we have

$$\begin{aligned} \chi(D(v_{2n},v_{2n+1})) &= \chi\big(D(\Lambda\eta_{2n},\Gamma\eta_{2n+1})\big) \\ &\le \chi\big(D(F\eta_{2n},G\eta_{2n+1})\big) - \varphi\big(D(F\eta_{2n},G\eta_{2n+1})\big) \\ &\le \chi\big(D(F\eta_{2n},G\eta_{2n+1})\big) \\ &= \chi\big(D(v_{2n-1},v_{2n})\big), \\ \chi(D_{2n}) &\le \chi(D_{2n-1}). \end{aligned}$$

These imply that $D_{2n} \leq D_{2n-1}$. Similarly,

$$\begin{aligned}
\chi\big(D(v_{2n+1}, v_{2n+2})\big) &= \chi\big(D(\Lambda\eta_{2n+2}, \Gamma\eta_{2n+1})\big) \\
&\leq \chi\big(D(F\eta_{2n+2}, G\eta_{2n+1})\big) - \varphi\big(D(F\eta_{2n+2}, G\eta_{2n+1})\big) \\
&\leq \chi\big(D(F\eta_{2n+2}, G\eta_{2n+1})\big) \\
&= \chi\big(D(v_{2n+1}, v_{2n})\big), \\
\chi(D_{2n+1}) &\leq \chi(D_{2n}).
\end{aligned}$$

These imply that $D_{2n+1} \leq D_{2n}$. Therefore, we have $D_n \leq D_{n-1}$.
Therefore, the sequence $\{D(v_n, v_{n+1})\}$ is bounded and non-increasing. Then there exists $r \geq 0$ such that

$$\lim_{n\to\infty} D(v_n, v_{n+1}) = r. \tag{11}$$

Suppose that $\lim_{n\to\infty} D(v_n, v_{n+1}) = r > 0$. Let us assume n is odd, that is, $n = 2m - 1$. Again by the P-property and using Λ and Γ are $(\epsilon, \delta, \chi, \varphi) - F, G$-contractions, we obtain

$$\begin{aligned}
\chi\big(D(v_n, v_{n+1})\big) &= \chi\big(D(v_{2m-1}, v_{2m})\big) \\
&= \chi\big(D(\Gamma\eta_{2m-1}, \Lambda\eta_{2m})\big) \\
&\leq \chi\big(D(F\eta_{2m}, G\eta_{2m-1})\big) - \varphi\big(D(F\eta_{2m}, G\eta_{2m-1})\big) \\
&\leq \chi\big(D(F\eta_{2m}, G\eta_{2m-1})\big) \\
&= \chi\big(D(v_{2m-1}, v_{2m-2})\big) \\
&= \chi\big(D(v_n, v_{n+1})\big).
\end{aligned}$$

Now using (11) and continuity of χ in the above inequality, we can obtain

$$\lim_{m\to\infty} \varphi\big(D(F\eta_{2m}, G\eta_{2m-1})\big) = \lim_{m\to\infty} \varphi\big(D(v_{2m-1}, v_{2m-2})\big) = \lim_{n\to\infty} \varphi\big(D(v_n, v_{n+1})\big) = 0. \tag{12}$$

However, as $0 < r \leq D(v_n, v_{n+1})$ and φ is nondecreasing function,

$$0 < \varphi(r) \leq \varphi\big(D(v_n, v_{n+1})\big),$$

so $\lim_{n\to\infty} \varphi\big(D(v_n, v_{n+1})\big) \geq \varphi(r) > 0$ which contradicts to (12). Similarly one can easily varify that for the case of n is even. Hence,

$$\lim_{n\to\infty} D(v_n, v_{n+1}) = 0. \tag{13}$$

Suppose that $\{v_{2n}\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ and for any even integer $2l$ for which we can find subsequences $\{v_{2m(l)}\}$ and $\{v_{2n(l)}\}$ of $\{v_{2n}\}$ such that $2n(l)$ is smallest index for which

$$2n(l) > 2m(l) \geq 2l, \ D(v_{2m(l)}, v_{2n(l)}) \geq \epsilon.$$

This means that

$$D(v_{2m(l)}, v_{2n(l)-1}) < \epsilon. \tag{14}$$

Then, we obtain

$$\begin{aligned}
\epsilon &\leq D(v_{2m(l)}, v_{2n(l)}) \\
&\leq D(v_{2m(l)}, v_{2n(l)-1}) + D(v_{2n(l)-1}, v_{2n(l)}) \\
&< \epsilon + D(v_{2n(l)-1}, v_{2n(l)}).
\end{aligned}$$

Letting $l \to \infty$ and using (13) we conclude that

$$\lim_{l\to\infty} D(v_{2m(l)}, v_{2n(l)}) = \epsilon. \tag{15}$$

Again from (13) and (15), the inequality

$$D(v_{2m(l)}, v_{2n(l)}) \le D(v_{2m(l)}, v_{2m(l)-1}) + D(v_{2m(l)-1}, v_{2n(l)}),$$

as $l \to \infty$, gives that

$$\epsilon \le \lim_{l\to\infty} D(v_{2m(l)-1}, v_{2n(l)}). \tag{16}$$

Now again we have the inequality

$$D(v_{2m(l)-1}, v_{2n(l)}) \le D(v_{2m(l)-1}, v_{2m(l)}) + D(v_{2m(l)}, v_{2n(l)}),$$

as $l \to \infty$, we obtain

$$\lim_{l\to\infty} D(v_{2m(l)-1}, v_{2n(l)}) \le \epsilon. \tag{17}$$

Then from (16) and (17), we have

$$\lim_{l\to\infty} D(v_{2m(l)-1}, v_{2n(l)}) = \epsilon. \tag{18}$$

Now we prove $\lim_{l\to\infty} D(v_{2n(l)-1}, v_{2m(l)-2}) = \epsilon$. By (13) and (18), we have

$$D(v_{2n(l)-1}, v_{2m(l)-2}) \le D(v_{2n(l)-1}, v_{2n(l)}) + D(v_{2n(l)}, v_{2m(l)-1}) + D(v_{2m(l)-1}, v_{2m(l)-2}),$$

as $l \to \infty$, gives that

$$\lim_{l\to\infty} D(v_{2n(l)-1}, v_{2m(l)-2}) \le \epsilon.$$

By triangle inequality

$$\begin{aligned} D(v_{2n(l)}, v_{2m(l)}) \le\ & D(v_{2n(l)}, v_{2n(l)-1}) + D(v_{2n(l)-1}, v_{2m(l)-2}) + D(v_{2m(l)-2}, v_{2m(l)-1}) \\ & + D(v_{2m(l)-1}, v_{2m(l)}). \end{aligned}$$

Now using (13), (15) and taking limit on both side of the above inequality, we get

$$\epsilon \le \lim_{l\to\infty} D(v_{2n(l)-1}, v_{2m(l)-2}).$$

Therefore

$$\lim_{l\to\infty} D(v_{2n(l)-1}, v_{2m(l)-2}) = \epsilon. \tag{19}$$

Using (18) and (19), we have

$$\begin{aligned} \chi(D(v_{2m(l)-1}, v_{2n(l)})) &= \chi(D(\Lambda\eta_{2n(l)}, \Gamma\eta_{2m(l)-1})) \\ &\le \chi(D(F\eta_{2n(l)}, G\eta_{2m(l)-1})) - \varphi(D(F\eta_{2n(l)}, G\eta_{2m(l)-1})) \\ &= \chi(D(v_{2n(l)-1}, v_{2m(l)-2})) - \varphi(D(v_{2n(l)-1}, v_{2m(l)-2})). \end{aligned}$$

Letting $l \to \infty$, we get

$$\chi(\epsilon) \le \chi(\epsilon) - \varphi(\epsilon),$$

which implies a contradiction, since $\epsilon > 0$. Thus, $\{v_{2n}\}$ is a Cauchy sequence in M_0 and so $\{v_n\}$. □

Theorem 3. *Let M and N be two subsets of a complete metric space (X, D). Assume the pair (M, N) satisfies P-property. Let F and G be mappings from M to N and let Λ and Γ be $(\epsilon, \delta, \chi, \varphi) - F, G$- contractions such that the pairs (Λ, F) and (Γ, G) are proximally compatible and assume $\Lambda(M_0) \subset G(M_0), \Gamma(M_0) \subset F(M_0)$ and $\Lambda(M_0) \subset N_0, \Gamma(M_0) \subset N_0$ with M_0 is closed. If Λ, Γ, F and G are continuous on M then Λ, Γ, F and G have unique common best proximity point.*

Proof. By Lemma 3, the sequence $\{v_n\}$ is Cauchy and since M_0 is closed, there exists $v \in M_0$ such that $\{v_n\}$ converges to v.

Since the pair $\{\Lambda, F\}$ is proximally compatible, by Definition 7, $D(\Lambda v_{2n+1}, Fv_{2n}) \to 0$. However, since Λ and F are continuous, $D(\Lambda v_{2n+1}, Fv_{2n}) \to D(\Lambda v, Fv)$. Therefore, $\Lambda v = Fv$.
Similarly, the pair $\{\Gamma, G\}$ is proximally compatible, by Definition 7, $D(\Gamma v_{2n}, Gv_{2n+1}) \to 0$. Also, the continuity of Γ and G implies that $D(\Gamma v_{2n}, Gv_{2n+1}) \to D(\Gamma v, Gv)$. Therefore, $\Gamma v = Gv$. Further the theorem follows from Lemma 2. □

5. Conclusions

The fixed point theorems help to provide sufficient conditions to ensure the existence of solution for many nonlinear problems. On the other word, the fixed point theorems give the solution of equations of the form $Tx = x$, where T is self mapping. Suppose the mapping T is non-self, there is no guarantee for solution. In this situation, the best proximity point theorems provide approximate solution to the nonlinear problems. In the literature, there are many articles deal the existence of best proximity point for various kind of non-self mappings. The more general version of best proximity point theorems which involve more than one non-self mappings known as common best proximity point theorems. There are many research works that provide the existence of common best proximity points. In the sequel, we want to find existence of common best proximity point for a large class of non-self mappings. Therefore, in this paper, we give a new idea of proximally compatible mappings with an interesting example and using this class of mappings, we extend some results of Jungck. Furthermore, we introduce the concept $(\epsilon, \delta, \chi, \varphi)$ contractions, this class of mappings contains the class of (ϵ, δ) contractions in [1]. In addition, using this class, we provide common best proximity point theorems for proximally compatible mappings. Finally, we provide an example to support our main result.

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