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Applying Fixed Point Techniques to Stability Problems in Intuitionistic Fuzzy Banach Spaces

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Abstract: In this paper we investigate Hyers-Ulam-Rassias stability of certain nonlinear functional equations. Considerations of such stabilities in different branches of mathematics have been very extensive. Again the fuzzy concepts along with their several extensions have appeared in almost all branches of mathematics. Here we work on intuitionistic fuzzy real Banach spaces, which is obtained by combining together the concepts of fuzzy Banach spaces with intuitionistic fuzzy sets. We establish that pexiderized quadratic functional equations defined on such spaces are stable in the sense of Hyers-Ulam-Rassias stability. We adopt a fixed point approach to the problem. Precisely, we use a generxalized contraction mapping principle. The result is illustrated with an example.

Keywords: Hyers-Ulam stability; pexider type functional equation; intuitionistic fuzzy normed spaces; alternative fixed point theorem

1. Introduction

In this paper, we derive Hyers-Ulam-Rassias stability results for certain functional equations in the context of intuitionistic fuzzy Banach spaces (IFBS). The problem of stability that we study here was for the first time mathematically formulated by Ulam [1]. It was partly solved and further generalized by Hyers [2] and Rassias [3]. Today we know such stability problems as the problems of the Hyers-Ulam-Rassias (H-U-R) stability. It has many extended forms and has been studied in several domains of mathematics including differential equations [4], functional equations [5], isometries [6], etc. Our interest is in the study of such stabilities for certain functional equations. H-U-R stability for functional equations on linear spaces has been discussed in quite a large number of papers, some of which are noted in [7–14].

The fuzzy concept was mathematically introduced by Zadeh [15] in 1965. Over the following years it was adopted in almost all the domains of mathematics including linear algebra and functional analysis. The idea of a fuzzy set has many extensions of itself. One such extension is the concept of intuitionistic fuzzy set introduced by Atanassov [16]. Here we have an additional degree of membership, which is sometimes referred to as the degree of non-belongingness.

In this paper we consider the intuitionistic fuzzy linear spaces as defined by S. Shakeri [17]. It is a generalization of the definition of fuzzy normed linear space given by Mirmostafaee [18]. Stability of functional equations on the above-mentioned space have been considered in works like [19–21].

Precisely in this paper we consider the H-U-R stability problem for pexiderized quadratic functional equations. These equations are generalized quadratic functional equations and appeared in the literature in works like [22–24]. Amongst several approaches to H-U-R stability problems we adopt the fixed point approach where the stability is established through an application of a fixed point theorem obtained in complete generalized metric spaces [25].

2. Mathematical Background

The following is the definition of a pexiderized quadratic functional equation.
A mapping $f : R \to R$ is said to be a quadratic form if $f(x) = cx^2$ for all $x, c \in R$.

Let X and Y be a real vector space and a Banach space, respectively, and corresponding to a mapping $f : X \to Y$, consider the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1}$$

Any solution of Equation (1) is termed as quadratic mapping. Particularly, if $X = Y = R$, the quadratic form $f(x) = cx^2$ is a solution of (1). The form

$$f\,(x\,+\,y)\,+\,f\,(x\,-\,y)\,=\,2\,g\,(x)\,+\,2\,h\,(y) \tag{2}$$

is known as pexiderized quadratic functional equation [26,27], which is an extension of the above definition of quadratic functional equation.

Definition 1 ([28,29]). *Consider the set L^* and the order relation $\leq_{L^*}$ defined by*

$$Ł^{*} \,=\, \{\,(\alpha_1\,,\,\alpha_2)\,:\,(\alpha_1\,,\,\alpha_2)\,\in\,[\,0\,,\,1\,]^2 \text{ and } \alpha_1\,+\,\alpha_2\,\leq\,1\,\},$$

$$(\alpha_1\,,\,\alpha_2)\,\leq_{L^*}\,(\beta_1\,,\,\beta_2)\,\Leftrightarrow\,\alpha_1\,\leq\,\beta_1\,,\,\alpha_2\,\geq\,\beta_2\,,\,\forall\,(\alpha_1\,,\,\alpha_2)\,,\,(\beta_1\,,\,\beta_2)\,\in\,L^*\,.$$

Then $(L^*\,,\,\leq_{L^*})$ is a complete lattice.
The elements $0_{L^*} \,=\, (0\,,\,1)$ and $1_{L^*} \,=\, (1\,,\,0)$ are its units.

Definition 2 ([16]). *An intuitionistic fuzzy set A of E where E is a nonempty set, is $A = \{\,(x\,,\,\mu_A\,(x)\,,\,\nu_A\,(x))\,:\,x\,\in\,E\,\}$, in which case the functions $\mu_A\,:\,E\,\to\,[\,0\,,\,1\,]$ and $\nu_A\,:\,E\,\to\,[\,0\,,\,1\,]$ are the degree of membership and the degree of non-membership respectively for every $x\,\in\,E$ satisfying $0\,\leq\,\mu_A\,(x)\,+\,\nu_A\,(x)\,\leq\,1$.*
For our notational purposes we denote an intuitionistic fuzzy set on X by any function $A_{\mu,\nu}\,=\,X\,\to\,L^*$ given by $A_{\mu,\nu}\,(x)\,=\,(\mu_A\,(x)\,,\,\nu_A\,(x))$ with $\mu_A,\,\nu_A\,:\,X\,\to\,[0,1]$ satisfying $0\,\leq\,\mu_A\,(x)\,+\,\nu_A\,(x)\,\leq\,1$.

Definition 3 ([30]). *A triangular norm (t-norm) on L^* is a mapping $\Gamma\,:\,(L^*)^2\,\to\,L^*$ satisfying the following conditions:*

(a) $(\forall\,\alpha\,\in\,L^*)\,(\Gamma\,(\alpha\,,\,1_{L^*})\,=\,\alpha)$ *(boundary condition),*
(b) $(\forall\,(\alpha\,,\,\beta)\,\in\,(L^*)^2)\,(\Gamma\,(\alpha\,,\,\beta)\,=\,\Gamma\,(\beta\,,\,\alpha))$ *(commutativity),*
(c) $(\forall\,(\alpha\,,\,\beta\,,\,\gamma)\,\in\,(L^*)^3)\,(\Gamma\,(\alpha\,,\,\Gamma\,(\beta\,,\,\gamma))\,=\,\Gamma\,(\Gamma\,(\alpha\,,\,\beta)\,,\,\gamma))$ *(associativity),*
(d) $(\forall\,(\alpha\,,\,\alpha'\,,\,\beta\,,\,\beta')\,\in\,(L^*)^4)\,(\alpha\,\leq_{L^*}\,\alpha'$ *and* $\beta\,\leq_{L^*}\,\beta'\,\Rightarrow\,\Gamma\,(\alpha\,,\,\beta)\,\leq_{L^*}\,\Gamma\,(\alpha'\,,\,\beta'))$ *(monotonicity).*

If Γ is continuous then Γ is called a continuous t-norm.

Definition 4 ([30]). *A triangular conorm (t-conorm) on L^* is a mapping $S\,:\,(L^*)^2\,\to\,L^*$ satisfying the following conditions:*

(a) $(\forall\,\alpha\,\in\,L^*)\,(S\,(\alpha\,,\,0_{L^*})\,=\,\alpha)$ *(boundary condition),*

(b) $(\forall (\alpha, \beta) \in (L^*)^2)\ (S(\alpha, \beta) = S(\beta, \alpha))$ *(commutativity)*,
(c) $(\forall (\alpha, \beta, \gamma) \in (L^*)^3)\ (S(\alpha, S(\beta, \gamma)) = S(S(\alpha, \beta), \gamma))$ *(associativity)*,
(d) $(\forall (\alpha, \alpha', \beta, \beta') \in (L^*)^4)\ (\alpha \leq_{L^*} \alpha'$ *and* $\beta \leq_{L^*} \beta' \Rightarrow S(\alpha, \beta) \leq_{L^*} S(\alpha', \beta'))$ *(monotonicity)*.

Example 1. *Let*

$$M(\alpha, \beta) = (\min\{\alpha_1, \beta_1\}, \max\{\alpha_2, \beta_2\})$$

for all $\alpha = (\alpha_1, \alpha_2),\ \beta = (\beta_1, \beta_2) \in L^*$.
Then $M(\alpha, \beta)$ *is a continuous t-norm.*

Definition 5 ([30])**.** *A continuous t-norm* Γ *on* L^* *is said to be continuous t-representable if we can find a continuous t-norm* $*$ *and a continuous t-conorm* $\diamond$ *on* $[0, 1]$ *such that for all* $x = (\alpha_1, \alpha_2),\ y = (\beta_1, \beta_2) \in L^*,\ \Gamma(x, y) = (\alpha_1 * \beta_1, \alpha_2 \diamond \beta_2)$
We now define the iterated sequence Γ^n *recursively by* $\Gamma^1 = \Gamma$ *and*

$$\Gamma^n(x^{(1)}, x^{(2)}, \cdots, x^{(n+1)}) = \Gamma(\Gamma^{(n-1)}(x^{(1)}, x^{(2)} \cdots, x^{(n)}), x^{(n+1)}),$$

$\forall\, n \geq 2,\ x^{(i)} \in L^*$.

Intuitionistic fuzzy normed linear space was defined by Saadati [31]. Shakeri [17] has stated this definition in a more compact form. We state the definition in the form used by Shakeri [17]

Definition 6 ([17])**.** *We call the triple* $(X, P_{\mu,\nu}, \Gamma)$ *an intuitionistic fuzzy normed space (briefly IFN-space) if X is a vector space,* Γ *is a continuous t-norm and* $P_{\mu,\nu}$ *is a mapping* $X \times (0, \infty) \to L^*$ *which is an intuitionistic fuzzy set satisfying the following conditions:*
for all $x, y \in X$ *and* $t, s > 0$,

(i) $P_{\mu,\nu}(x, 0) = 0_{L^*}$;
(ii) $P_{\mu,\nu}(x, t) = 1_{L^*}$ *if and only if* $x = 0$;
(iii) $P_{\mu,\nu}(\alpha x, t) = P_{\mu,\nu}\left(x, \frac{t}{|\alpha|}\right)$ *for all* $\alpha \neq 0$;
(iv) $P_{\mu,\nu}(x + y, t + s) \geq_{L^*} \Gamma(P_{\mu,\nu}(x, t), P_{\mu,\nu}(y, s))$.

It can be noted that $P_{\mu,\nu}$ *has the form* $P_{\mu,\nu}(x, t) = (\mu_x(t), \nu_x(t)) = (\mu(x, t), \nu(x, t))$ *such that* $0 \leq \mu_x(t) + \nu_x(t) \leq 1$ *for all* $x \in X$ *and* $t > 0$*. Then with* μ *and* ν *the above definition reduces to the more explicit form used in [31].*

Definition 7 ([17])**.** (1) *The sequence* $\{x_n\}$ *is said to be convergent to a point* $x \in X$ *if*

$$P_{\mu,\nu}(x_n - x, s) \to 1_{L^*} \text{ as } n \to \infty \text{ for every } s > 0.$$

(2) *A sequence* $\{x_n\}$ *in an IFN-space* $(X, P_{\mu,\nu}, M)$ *is said to be a Cauchy sequence if given any* $\varepsilon > 0$ *and* $s > 0$*, we can find* $n_0 \in N$ *such that*

$$P_{\mu,\nu}(x_n - x_m, s) >_{L^*} (1 - \varepsilon, \varepsilon),\ \forall n, m \geq n_0$$

(3) *An IFN-space* $(X, P_{\mu,\nu}, M)$ *is said to be complete if every Cauchy sequence in* $(X, P_{\mu,\nu}, M)$ *is convergent in* $(X, P_{\mu,\nu}, M)$*. A complete intuitionistic fuzzy normed space is called an intuitionistic fuzzy Banach space.*

We require the following fixed point result to establish our result of stability in this paper.

Definition 8 ([25])**.** *Let* X *be a nonempty set. A function* $d : X \times X \to [0, \infty]$ *is called a generalized metric on* X *if* d *satisfies*

(i) $d\,(p\,,\,q)\;=\;0$ *if and only if* $p\;=\;q\,;$
(ii) $d\,(p\,,\,q)\;=\;d\,(q\,,\,p)$ *for all* $p\,,\,q\;\in\;X\,;$
(iii) $d\,(p\,,\,q)\;\leq\;d\,(p\,,\,r)\,+\,d\,(r\,,\,q)$ *for all* $p\,,\,q\,,\,r\;\in\;X$. *Then* (X,d) *is called a generalized metric space.*

Theorem 1 ([12,23,32]). *Let* $(\,X\,,\,d)$ *be a complete generalized metric space and let* $J\,:\,X\,\rightarrow\,X$ *be a strictly contractive mapping with Lipschitz constant* $0\,<\,L\,<\,1$ *, that is ,*

$$d(J\,p\,,\,J\,q)\;\leq\;L\,d\,(p\,,\,q),$$

for all $p\,,\,q\;\in\;X$ *. Then for each* $p\;\in\;X$ *, either*

$$d\,(J^{\,k}\,p\,,\,J^{\,k+1}\,p)\;=\;\infty,\;\forall\;k\;\geq\;0$$

or,

$$d\,(J^{\,k}\,p\,,\,J^{\,k+1}\,p)\;<\;\infty\quad\forall\;k\;\geq\;k_o$$

for some non-negative integers k_0. *Moreover, if the second alternative holds then*

(1) *the sequence* $\{\,J^{\,k}\,p\,\}$ *converges to a fixed point* $q^{\star}$ *of* $J\,;$
(2) $q^{\star}$ *is the unique fixed point of* J *in the set*

$$Y\;=\;\{\,q\;\in\;X\,:\,d\,(J^{\,k_0}\,p\,,\,q)\;<\;\infty\,\};$$

(3) $d(q\,,\,q^{\star})\;\leq\;(\frac{1}{1-L})\,d\,(q\,,\,J\,q)$ *for all* $q\;\in\;Y\,.$

3. The Hyers-Ulam-Rassias Stability Result

Throughout the result of the paper, X is considered to be a normed linear space, $(Y\,,\,P_{\mu,\nu}\,,\,M)$ an IF-real Banach space, $(Z\,,\,P\,'_{\mu,\nu}\,,\,M)$ an IFN-space and M is continuous $t-$ norm defined in Example 2, also consider

$$\begin{aligned} &M_1\,(x\,,\,t)\\ &=M^2\left\{P\,'_{\mu,\nu}\left(\phi\,(x\,,\,x)\,,\,\frac{t}{3}\right),\,P\,'_{\mu,\nu}\left(\phi\,(x\,,\,0)\,,\,\frac{t}{3}\right),\right.\\ &\qquad\qquad\left.P\,'_{\mu,\nu}\left(\phi\,(0\,,\,x)\,,\,\frac{t}{3}\right)\right\} \end{aligned}\tag{3}$$

where $\phi\,:\,X\times\,X\,\rightarrow\,Z$.

Lemma 1. *Let* $(Z\,,\,P'_{\mu,\nu},\,M)$ *be an IFN-space. Let* $\phi\;:\;X\,\times\,X\;\rightarrow\;Z$ *be a mapping and further let* $E\;=\;\{\,g\,|\,g\,:\,X\,\rightarrow\,Y\,\}$. *Let* $d:E\times E\rightarrow[0,\infty]$ *be defined by*

$$\begin{aligned} &d\,(g\,,\,h)\\ &\qquad:=\;\inf\{\,k\;\in\;R^{+}\;:\;P_{\mu,\nu}(g(x)\;-\;h(x)\,,\,kt)\;\geq_{L^*}\;M_1\,(x\,,\,t)\;\text{for all}\,x\;\in\;X\,,\,t\;>\;0\} \end{aligned}$$

and $g\,,\,h\;\in\;E.$
Then $(E\,,d)$ *is a complete generalized metric space.*

Proof. Let $f\,,\,g\,,\,h\;\in\;E$ and $d\,(f\,,g)\;=\;k_1\;<\;\infty\,,\,d\,(g\,,h)\;=\;k_2\;<\;\infty\,.$

Then $P_{\mu,\nu}(f(x)\;-\;g(x)\,,\,k_1\,t)\;\geq_{L^*}\;M_1\,(x\,,\,t)$ and $P_{\mu,\nu}(g(x)\;-\;h(x)\,,\,k_2\,t)\;\geq_{L^*}\;M_1\,(x\,,\,t)$

Therefore $P_{\mu,\nu}(f(x)\;-\;h(x)\,,\,(k_1\,+\,k_2)\,t)$

$\geq_{L^*}\;M\,\left(P_{\mu,\nu}(f(x)\;-\;g(x)\,,\,k_1\,t)\,,\,P_{\mu,\nu}(g(x)\;-\;h(x)\,,\,k_2\,t)\,\right)$ (by property iv of Definition 6)

$\geq_{L^*}$ $M\,(M_1\,(x\,,\,t)\,,\,M_1\,(x\,,\,t)\,)$ (by the monotonicity property) $=\ _{L^*}\,M_1\,(x\,,\,t)$ (Idempotent property)s for all $x\,\in\,X,\,t\,>\,0.$

Hence $d\,(f\,,\,h)\,\leq\,k_1\,+\,k_2$ so that $d\,(f,\,h)\,\leq\,d\,(f\,,\,g)\,+\,d\,(g\,,\,h)$ which is the triangle inequality. The other axioms are obvious, and hence, $(E\,,\,d)$ is a generalized metric space. Now we prove that $(E\,,\,d)$ is complete.

Let $\{\,g_n\}$ be a Cauchy sequence in $(E\,,\,d)$. Now for each fixed $x\in\,X$ and for every $t>0$ and $\epsilon\,>\,0$ there exists $\lambda>0$ such that $M_1\,(x\,,\,\frac{t}{\lambda})\,>\,1-\varepsilon$. Since $\{\,g_n\}$ is a Cauchy sequence in $(E,\,d)$ corresponding to $\lambda\,>\,0,$ there exists $n_0\,\in\,N$ such that $d(g_n,\,g_m)\,<\,\lambda$ for all $m\,,\,n\geq\,n_0\,.$

Since $g_n,\,g_m\,\in E$ so we find,

$$d\,(g_n\,,\,g_m)\,=\,\inf\{\,k\,\in\,R^+\,:\,P_{\mu,\nu}\,(g_n\,(x)\,-\,g_m\,(x)\,,\,kt)\,\geq\,M_1\,(x\,,\,t)\,\}$$

That is,

$$d\,(g_n\,,\,g_m)\,=\,\inf\{\,k\,\in\,R^+\,:\,P_{\mu,\nu}\,(g_n\,(x)\,-\,g_m\,(x)\,,\,t)\,\geq\,M_1\,(x\,,\,\frac{t}{k})\,\}$$

then there exists $k_3\,\in\,[0\,,\,\infty)$ such that $d\,(g_n\,,\,g_m)\,\leq\,k_3\,<\,\lambda$ for all $m\,,\,n\geq\,n_0$ and $P_{\mu,\nu}\,(g_n\,(x)\,-\,g_m\,(x)\,,\,t)\,\geq\,M_1\,(x\,,\,\frac{t}{k_3})\geq\,M_1\,(x\,,\,\dfrac{t}{\lambda})\,>\,1-\varepsilon,$ as $P_{\mu,\nu}\,(x\,,\,t)$ is non-decreasing with respect to t for all $m\,,\,n\geq\,n_0.$

Thus, for fixed $x\,\in\,X,\,\{\,g_n\,(x)\,\}$ is a Cauchy sequence in Y. Again since Y is Banach space, every $x\in\,X,$ there exists $g(x)\in\,Y$ such that $g_n\,(x)\,\to g\,(x)$ as $n\to\infty$. Then the mapping $g\,:\,X\,\to\,Y$ is such that $g_n\,(x)\,\to g\,(x)$ as $n\to\infty$ for all $x\,\in\,X.$

Again, $\{\,g_n\,\}$ is a Cauchy sequence in $(E,\,d)$ therefore for $\epsilon\,>\,0,\,t\,>\,0$ there exists $n_0\,\in\,N$ such that $d\,(g_n,\,g_m)\,<\,\epsilon\quad\forall\,m,\,n\,\geq\,n_0$ and hence there exists $k'\,\in\,[0,\,\infty)$ such that $d\,(g_n,\,g_m)\,\leq\,k'\,<\,\epsilon\,\,\forall\,m,\,n\,\geq\,n_0$

$$P_{\mu,\nu}\,(g_n\,(x)\,-\,g_m\,(x)\,,\,t)\,\geq\,M_1\,(x\,,\,\frac{t}{k'})\geq\,M_1\,(x\,,\,\frac{t}{\epsilon}).$$

That is

$$P_{\mu,\nu}\,(g_m\,(x)\,-\,g_n\,(x)\,,\,\epsilon\,t)\,\geq\,M_1\,(x\,,\,t),\,\,\forall\,n\,,\,m\,\geq\,n_0$$

Now let $\epsilon\,,\,\delta\,>\,0$ be given and $m,\,n\,>\,n_0\,,\,t\,>\,0,$ then

$$\begin{gathered}
P_{\mu,\nu}\,(g_n\,(x)\,-\,g\,(x)\,,\,(\epsilon\,+\,\delta)\,t)\\
\geq_{L^*}\,M\,\{P_{\mu,\nu}\,(g_n\,(x)\,-\,g_m\,(x)\,,\,\epsilon\,t)\,,\,P_{\mu,\nu}\,(g_m\,(x)\,-\,g\,(x)\,,\,\delta\,t)\}\\
\geq_{L^*}\,M\,\{\,M_1\,(x\,,\,t)\,,\,P_{\mu,\nu}\,(g_m\,(x)\,-\,g\,(x)\,,\,\delta\,t)\}\\
\geq_{L^*}\,M\,\{\,M_1\,(x\,,\,t)\,,\,1_{L^*}\}\,[\,bytakinglimitas\,m\,\to\,\infty\,]\,=_{L^*}\,M_1\,(x\,,\,t))
\end{gathered}$$

that is, $d\,(g_n,\,g)\,\leq\,\epsilon\,+\,\delta$ for all $x\,\in\,X$ and $m,\,n\,\geq\,n_0\,.$

Taking $\delta\to 0$ we have a mapping $g\,:\,X\,\to\,Y$ such that

$$g\,(x)\,=\,P_{\mu,\nu}\,-\lim_{n\to\infty}\,g_n\,(x)\,\in\,E\,.$$

Therefore, $(E\,,\,d)$ is a complete generalized metric space. $\square$

For our purpose, we denote

$$Let\,\,D\,f\,(x\,,\,y)\,=\,f\,(x\,+\,y)\,+\,f\,(x\,-\,y)\,-\,2\,g\,(x)\,-\,2\,h\,(y) \tag{4}$$

Theorem 2. *Let X be a linear space, $(Z, P'_{\mu,\nu}, M)$ be an IFN-space, $\phi : X \times X \to Z$ be such that*

$$P'_{\mu,\nu}(\phi(2x, 2x), t) \geq_{L^*} P'_{\mu,\nu}(\alpha\phi(x, x), t) \tag{5}$$

for some real α with $0 < \alpha < 2$, $(\forall x \in X, t > 0)$ and

$$\lim_{n\to\infty} P'_{\mu,\nu}(\phi(2^n x, 2^n x), 2^n t) = 1_{L^*}$$

where $x \in X$ and $t > 0$. Further let $(Y, P_{\mu,\nu}, M)$ be a complete IFN-space. If $f, g, h : X \to Y$ are odd mappings such that

$$P_{\mu,\nu}(Df(x, y), t) \geq_{L^*} P'_{\mu,\nu}(\phi(x, y), t) \tag{6}$$

$(\forall x \in X, t > 0)$, where $Df(x,y)$ is given by Equation (4). Then there exists a unique additive mapping $A : X \to Y$ define by $A(x) := \lim_{n\to\infty}\left(\frac{f(2^n x)}{2^n}\right)$ for all $x \in X$ satisfying

$$P_{\mu,\nu}(f(x) - A(x), t) \geq_{L^*} M_1(x, t(2-\alpha)) \tag{7}$$

and

$$P_{\mu,\nu}(A(x) - g(x) - h(x), t) \geq_{L^*} M_1\left(x, \frac{t \times 3(2-\alpha)}{5-\alpha}\right). \tag{8}$$

Proof. Interchanging the role of x and y in Equation (6) we get

$$P_{\mu,\nu}(f(x+y) - f(x-y) - 2g(y) - 2h(x), t)$$

$$\geq_{L^*} P'_{\mu,\nu}(\phi(y, x), t) \tag{9}$$

Also from Equation (6) and using Equation (9) we get

$$P_{\mu,\nu}(2f(x+y) - 2g(x) - 2h(y) - 2g(y) - 2h(x), 2t)$$

$$\geq_{L^*} M\{P'_{\mu,\nu}(\phi(x, y), t), P'_{\mu,\nu}(\phi(y, x), t)\}$$

that is,

$$P_{\mu,\nu}(f(x+y) - g(x) - h(y) - g(y) - h(x), t)$$

$$\geq_{L^*} M\{P'_{\mu,\nu}(\phi(x, y), t), P'_{\mu,\nu}(\phi(y, x), t)\} \tag{10}$$

Now putting $y = 0$ in Equation (10) we have

$$P_{\mu,\nu}(f(x) - g(x) - h(x), t)$$

$$\geq_{L^*} M\{P'_{\mu,\nu}(\phi(x, 0), t), P'_{\mu,\nu}(\phi(0, x), t)\} \tag{11}$$

Replacing y by x in Equation (11) we get

$$P_{\mu,\nu}(f(y) - g(y) - h(y), t)$$

$$\geq_{L^*} M\{P'_{\mu,\nu}(\phi(y, 0), t), P'_{\mu,\nu}(\phi(0, y), t)\} \tag{12}$$

Hence using Equations (10)–(12) we get

$$P_{\mu,\nu}(f(x+y) - f(x) - f(y), 3t)$$

$$\geq_{L^*} M^5\{P'_{\mu,\nu}(\phi(x, y), t), P'_{\mu,\nu}(\phi(y, x), t)$$

$$P'_{\mu,\nu}\,(\phi\,(x\,,0)\,,t\,,)\;P'_{\mu,\nu}\,(\phi\,(0\,,x)\,,t\,)$$
$$P'_{\mu,\nu}\,(\phi\,(y\,,0)\,,t\,)\;,P'_{\mu,\nu}\,(\phi\,(0\,,y)\,,t\,)$$

Therefore

$$\begin{aligned}&P_{\mu,\nu}(f\,(x+y)-f\,(x)-f\,(y)\,,t)\\ \geq_{L^*}\;& M^5\left\{P'_{\mu,\nu}\left(\phi\,(x\,,y)\,,\frac{t}{3}\right),P'_{\mu,\nu}\left(\phi\,(y\,,x)\,,\frac{t}{3}\right),\right.\\ &P'_{\mu,\nu}\left(\phi\,(x\,,0)\,,\frac{t}{3}\right),P'_{\mu,\nu}\left(\phi\,(0\,,x)\,,\frac{t}{3}\right),\\ &\left.P'_{\mu,\nu}\left(\phi\,(y\,,0)\,,\frac{t}{3}\right),P'_{\mu,\nu}\left(\phi\,(0\,,y)\,,\frac{t}{3}\right)\right\}\end{aligned}\tag{13}$$

Also we put $y = x$ in Equation (13)

$$\begin{aligned}&P_{\mu,\nu}(f\,(2x)-2f\,(x)\,,t)\\ \geq_{L^*}\;& M^5\left\{P'_{\mu,\nu}\left(\phi\,(x\,,x)\,,\frac{t}{3}\right),P'_{\mu,\nu}\left(\phi\,(x\,,x)\,,\frac{t}{3}\right),\right.\\ &P'_{\mu,\nu}\left(\phi\,(x\,,0)\,,\frac{t}{3}\right),P'_{\mu,\nu}\left(\phi\,(0\,,x)\,,\frac{t}{3}\right),\\ &\left.P'_{\mu,\nu}\left(\phi\,(x\,,0)\,,\frac{t}{3}\right),P'_{\mu,\nu}\left(\phi\,(0\,,x)\,,\frac{t}{3}\right)\right\}\\ =\;& M^2\left\{P'_{\mu,\nu}\left(\phi\,(x\,,x)\,,\frac{t}{3}\right),P'_{\mu,\nu}\left(\phi\,(x\,,0)\,,\frac{t}{3}\right),\right.\\ &\left.P'_{\mu,\nu}\left(\phi\,(0\,,x)\,,\frac{t}{3}\right)\right\}\\ =\;& M_1\,(x\,,t)\end{aligned}$$

that is,

$$P_{\mu,\nu}(f\,(2x)-2f\,(x)\,,t)\geq_{L^*} M_1\,(x\,,t)\tag{14}$$

Now define a mapping $J : E \to E$ by $J\,g\,(x) = \frac{1}{2}\,g\,(2x)$ for all $g \in E$ and $x \in X$, where (E, d) is a complete generalized metric space as in Lemma 1. We now prove that J is a strictly contractive mapping of E with the Lipschitz constant $\frac{\alpha}{2}$.

Let $g\,,h \in E$ and $\epsilon > 0$. Then there exists $k' \in R^+$ satisfying $P_{\mu,\nu}\,(g\,(x) - h\,(x)\,,k'\,t) \geq_{L^*} M_1\,(x\,,t)$ such that $d(g, h) \leq k' < d(g, h) + \epsilon$ for any $\epsilon > 0$.

Then $\inf\left\{k \in R^+ : P_{\mu,\nu}(g(x) - h(x)\,,kt) \geq_{L^*} M_1(x\,,t)\right\} \leq k' < d(g, h) + \epsilon$ that is, $\inf\left\{k \in R^+ : P_{\mu,\nu}(\frac{g(2x)}{2} - \frac{h\,(2x)}{2}, \frac{kt}{2}) \geq_{L^*} M_1(2x\,,t)\right\} < d\,(g\,,h) + \epsilon$ that is, $\inf\left\{k \in R^+ : P_{\mu,\nu}(Jg(x) - Jh(x), \frac{kt}{2}) \geq_{L^*} M_1\,(2x, t)\right\} < d\,(g\,,h) + \epsilon$ that is, $\inf\left\{k \in R^+ : P_{\mu,\nu}(Jg\,(x) - Jh\,(x), \frac{k\alpha t}{2}) \geq_{L^*} M_1(x\,,t)\right\} < d(g\,,h) + \epsilon$ as $M_1\,(2^n\,x, t) = M_1\,(x, \frac{t}{\alpha^n})$ or, $d\,\{\frac{2}{\alpha}\,(J\,g\,,J\,h)\} < d\,(g\,,h) + \epsilon$ or, $d\,\{(J\,g\,,J\,h)\} < \frac{\alpha}{2}\,\{d\,(g\,,h) + \epsilon\}$. Taking $\epsilon \to 0$ we get $d\,\{(J\,g\,,J\,h)\} < \frac{\alpha}{2}\,\{d\,(g\,,h)\,\}$. Therefore, J is strictly contractive mapping with Lipschitz constant $\frac{\alpha}{2}$.

Also from Equation (14) $d\,(f\,,J\,f) \leq \frac{1}{2}$ and $d\,(J\,f\,,J^2\,f) \leq \frac{\alpha}{2}\,d\,(f\,,J\,f) < \infty$. Again, replacing x by $2^n\,x$ in Equation (14) we get $P_{\mu,\nu}\,(f\,(2^{n+1}x) - 2\,f\,(2^n\,x)\,,t) \geq_{L^*} M_1(2^n\,x\,,t)$
or, $P_{\mu,\nu}\,(\frac{f\,(2^{n+1}x)}{2^{n+1}} - \frac{f\,(2^n\,x)}{2^n}\,,\frac{t}{2^{n+1}}) \geq_{L^*} M_1\,(2^n\,x\,,t)$

$$\geq_{L^*} M_1\,(x\,,\frac{t}{\alpha^n})$$

or, $P_{\mu,\nu}\left(J^{n+1}f(x) - J^{n}f(x), t\frac{(\frac{\alpha}{2})^{n}}{2}\right) \geq_{L^*} M_1(x,t)$

Hence, $d(J^{n+1}f, J^{n}f) \leq \frac{1}{2}(\frac{\alpha}{2})^{n} < \infty$ has Lipschitz constant $\frac{\alpha}{2} < 1$ for $n \geq n_0 = 1$.

Therefore, by Theorem 1 there exists a mapping $A : X \to Y$ such that the following holds:

1. A is a fixed point of J for which $A(2x) = 2A(x)$ for all $x \in X$.

Further, A is a unique fixed point of J in the set $E_1 = \{g \in E : d(J^{n_0}f, g) = d(Jf, g) < \infty\}$. Therefore, $d(Jf, A) < \infty$.

Also from Equation (14) $d(Jf, f) \leq \frac{1}{2} < \infty$. Thus $f \in E_1$. Now, $d(f, A) \leq d(f, Jf) + d(Jf, A) < \infty$. Thus, there exists $k \in (0, \infty)$ satisfying

$$P_{\mu,\nu}(f(x) - A(x), kt) \geq_{L^*} M_1(x,t)$$

for all $x \in X, t > 0$;

2. $d(J^{n}f, A)$
$= \inf\{k \in R^{+} : P_{\mu,\nu}(J^{n}f(x) - A(x), kt) \geq_{L^*} M_1(x,t)\}$
$= \inf\{k \in R^{+} : P_{\mu,\nu}(f(2^{n}x) - A(2^{n}x), 2^{n}kt) \geq_{L^*} M_1(x, (\frac{2}{\alpha})^{n}t)\}$

Therefore, $d(J^{n}f, A) \leq (\frac{\alpha}{2})^{n} \to 0$ as $n \to \infty$. This implies the equality

$$A(x) = \lim_{n\to\infty} J^{n}f(x) = \lim_{n\to\infty}\frac{f(2^{n}x)}{2^{n}} \tag{15}$$

for all $x \in X$.

3. $d(f, A) \leqslant \frac{1}{1-L}d(f, Jf)$ with $f \in E_1$ which implies the inequality

$$d(f, A) \leq \frac{1}{1-\frac{\alpha}{2}} \times \frac{1}{2} = \frac{1}{2-\alpha}$$

then it follows that

$$P_{\mu,\nu}(A(x) - f(x), \frac{1}{2-\alpha}t) \geq_{L^*} M_1(x,t)$$

It implies that

$$P_{\mu,\nu}(A(x) - f(x), t) \geq_{L^*} M_1(x, (2-\alpha)t) \tag{16}$$

for all $x \in X; t > 0$.

Replacing x and y by $2^{n}x$ and $2^{n}y$ in Equation (13) we have

$$\begin{aligned}
&P_{\mu,\nu}\left(\frac{f(2^{n}(x+y))}{2^{n}} - \frac{f(2^{n}x)}{2^{n}} - \frac{f(2^{n}y)}{2^{n}}, t\right)\\
&\geq_{L^*} M^{5}\left\{P'_{\mu,\nu}\left(\phi(2^{n}x, 2^{n}y), \frac{2^{n}t}{3}\right), P'_{\mu,\nu}\left(\phi(2^{n}y, 2^{n}x), \frac{2^{n}t}{3}\right),\right.\\
&\qquad P'_{\mu,\nu}\left(\phi(2^{n}x, 0), \frac{2^{n}t}{3}\right), P'_{\mu,\nu}\left(\phi(0, 2^{n}x), \frac{2^{n}t}{3}\right),\\
&\qquad\qquad \left.P'_{\mu,\nu}\left(\phi(2^{n}y, 0), \frac{2^{n}t}{3}\right), P'_{\mu,\nu}\left(\phi(0, 2^{n}y), \frac{2^{n}t}{3}\right)\right\}
\end{aligned} \tag{17}$$

Taking the limit as $n \to \infty$ in Equation (17) and using

$$\lim_{n\to\infty} P'_{\mu,\nu}(\phi(2^{n}x, 2^{n}y), 2^{n}t) = 1_{L^*}$$

we have

$$P_{\mu,\nu}(A(x+y) - A(x) - A(y), t) = 1_{L^*}$$

$$A(x+y) = A(x) + A(y) \tag{18}$$

that is, A is additive.

Also from Equation (11) we have

$$P_{\mu,\nu}(A(x) - g(x) - h(x), t\frac{5-\alpha}{3})$$
$$= P_{\mu,\nu}(A(x) - f(x) + f(x) - g(x) - h(x), t + \frac{2-\alpha}{3}t)$$
$$\geq_{L^*} M\left(P_{\mu,\nu}(A(x) - f(x), t), P_{\mu,\nu}\left(f(x) - g(x) - h(x), \frac{2-\alpha}{3}t\right)\right)$$

$$\geq_{L^*} M\left(M_1(x, (2-\alpha)t), M\left(P'_{\mu,\nu}\left(\phi(x,0), \frac{2-\alpha}{3}t\right), P'_{\mu,\nu}\left(\phi(0,x), \frac{2-\alpha}{3}t\right)\right)\right)$$
$$\geq_{L^*} M\left(M_1(x, (2-\alpha)t), M_1(x, (2-\alpha)t)\right)$$
$$\geq_{L^*} M_1(x, (2-\alpha)t)$$

Therefore,

$$P_{\mu,\nu}(A(x) - g(x) - h(x), t) \geq_{L^*} M_1\left(x, \frac{t \times 3(2-\alpha)}{5-\alpha}\right).$$

Again, A is the unique fixed point of J with the following property that there exists $u \in (0, \infty)$ such that

$$P_{\mu,\nu}(f(x) - A(x), ut) \geq_{L^*} M_1(x, t)$$

for all $x \in X$ and $t > 0$ [23]. This establishes the uniqueness of A. This completes the proof of the theorem. □

Theorem 3 ([23]). *Let X be a linear space and $(Z, P'_{\mu,\nu}, M)$ be an IFN-space. Let $\phi : X \times X \to Z$ be such that*

$$P'_{\mu,\nu}(\phi(2x, 2x), t) \geq_{L^*} P'_{\mu,\nu}(\alpha\phi(x,x), t) \tag{19}$$

for some real α with $0 < \alpha < 4$, $(\forall x \in X, t > 0)$ and

$$\lim_{n\to\infty} P'_{\mu,\nu}(\phi(2^n x, 2^n x), 4^n t) = 1_{L^*}$$

for all $x, y \in X$ and $t > 0$. Let $(Y, P_{\mu,\nu}, M)$ be a complete IFN-space. If $f, g, h : X \to Y$ are even mappings with $f(0) = g(0) = h(0) = 0$ such that

$$P_{\mu,\nu}(Df(x,y), t) \geq_{L^*} P'_{\mu,\nu}(\phi(x,y), t) \tag{20}$$

$(\forall x \in X, t > 0)$, *where* D *is given by Equation (4). Then there exists a unique quadratic mapping* $Q : X \to Y$ *defined by* $Q(x) := \lim_{n\to\infty} \left(\frac{f(2^n x)}{4^n}\right)$ *for all* $x \in X$ *satisfying*

$$P_{\mu,\nu}(f(x) - Q(x), t) \geq_{L^*} M_1(x, t(4-\alpha)) \tag{21}$$

and

$$P_{\mu,\nu}(Q(x) - g(x), t) \geq_{L^*} M_1\left(x, \frac{t \times 6(4-\alpha)}{10-\alpha}\right). \tag{22}$$

also

$$P_{\mu,\nu}(Q(x) - h(x), t) \geq_{L^*} M_1\left(x, \frac{t \times 6(4-\alpha)}{10-\alpha}\right).$$

Proof. Putting $y = x$ in Equation (20)

$$P_{\mu,\nu}(f(2x) - 2g(x) - 2h(x), t) \geq_{L^*} P'_{\mu,\nu}(\phi(x,x), t) \tag{23}$$

Also putting $x = 0$ in Equation (20)

$$P_{\mu,\nu}(2f(y) - 2h(y), t) \geq_{L^*} P'_{\mu,\nu}(\phi(0,y), t) \tag{24}$$

Again putting $y = 0$ in Equation (20)

$$P_{\mu,\nu}(2f(x) - 2g(x), t) \geq_{L^*} P'_{\mu,\nu}(\phi(x,0), t) \tag{25}$$

Now using Equations (20), (24), (25)

$$\begin{aligned} & P_{\mu,\nu}\{f(x+y) + f(x-y) - 2f(x) - 2f(y), 3t\} \\ = {} & P_{\mu,\nu}\{f(x+y) + f(x-y) - 2g(x) - 2h(y) - \\ & \{2f(y) - 2h(y)\} - \{2f(x) - 2g(x)\}, 3t\} \\ \geq_{L^*} {} & M^2\{P'_{\mu,\nu}(\phi(x,y), t), P'_{\mu,\nu}(\phi(0,y), t), P'_{\mu,\nu}(\phi(x,0), t)\} \end{aligned}$$

Therefore

$$\begin{aligned} & P_{\mu,\nu}\{f(x+y) + f(x-y) - 2f(x) - 2f(y), t\} \\ \geq_{L^*} {} & M^2\left\{P'_{\mu,\nu}\left(\phi(x,y), \frac{t}{3}\right), P'_{\mu,\nu}\left(\phi(0,y), \frac{t}{3}\right), \right. \\ & \left. P'_{\mu,\nu}\left(\phi(x,0), \frac{t}{3}\right)\right\} \end{aligned} \tag{26}$$

Now putting $y = x$ in Equation (26) we get

$$\begin{aligned} & P_{\mu,\nu}(f(2x) - 4f(x), t) \\ \geq_{L^*} {} & M^2\left(P'_{\mu,\nu}\left(\phi(x,x), \frac{t}{3}\right), P'_{\mu,\nu}\left(\phi(0,x), \frac{t}{3}\right), \right. \\ & \left. P'_{\mu,\nu}\left(\phi(x,0), \frac{t}{3}\right)\right) \\ = {} & M_1(x, t) \end{aligned}$$

Thus,

$$P_{\mu,\nu}(f(2x) - 4f(x), t) \geq_{L^*} M_1(x, t)$$

Similar to before [23], we consider the set $E := \{g : X \to Y\}$ and introduce a complete generalized metric on E. Again, define a mapping $J : E \to E$ by $Jg(x) = \frac{1}{4}g(2x)$ for all $g \in E$ and $x \in X$. And in a similar way as before we prove that J is strictly contractive mapping with Lipschitz constant $\frac{\alpha}{4}$ and $d(f, Jf) \leq \frac{1}{4}$.

Therefore by Theorem 1 there exists a mapping $Q : X \to Y$ such that the followings hold:

1. Q is a fixed point of J, that is, $Q(2x) = 4Q(x)$ for all $x \in X$.

The mapping Q is a unique fixed point of J in the set $E_1 = \{g \in E : d(J^{n_0}f, g) = d(Jf, g) < \infty\}$ and there exists $k \in (0, \infty)$ satisfying

$$P_{\mu,\nu}(f(x) - Q(x), kt) \geq_{L^*} M_1(x, t)$$

for all $x \in X$, $t > 0$;

2. $d(J^n f, Q) \leq (\frac{\alpha}{4})^n \to 0$ as $n \to \infty$. This implies the equality

$$Q(x) = \lim_{n\to\infty} J^n f(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$$

3. $d(f, Q) \leqslant \frac{1}{1-L} d(f, Jf)$ with $f \in E_1$, which implies the inequality

$$d(f, Q) \leq \frac{1}{1 - \frac{\alpha}{4}} \times \frac{1}{4} = \frac{1}{4 - \alpha}$$

then it follows that

$$P_{\mu,\nu}(Q(x) - f(x), \frac{1}{4-\alpha} t) \geq_{L^*} M_1(x, t)$$

It implies that

$$P_{\mu,\nu}(Q(x) - f(x), t) \geq_{L^*} M_1(x, (4-\alpha)t)$$

for all $x \in X; t > 0$.

Replacing x and y by $2^n x$ and $2^n y$ in Equation(26) we have

$$P_{\mu,\nu}\left\{\frac{f(2^n(x+y))}{4^n} + \frac{f(2^n(x-y))}{4^n} - \frac{2f(2^n x)}{4^n} - \frac{2f(2^n y)}{4^n}, t\right\}$$
$$\geq_{L^*} M^2\left\{P'_{\mu,\nu}\left(\phi(2^n x, 2^n y), \frac{4^n t}{3}\right), P'_{\mu,\nu}\left(\phi(0, 2^n y), \frac{4^n t}{3}\right),\right.$$
$$\left. P'_{\mu,\nu}\left(\phi(2^n x, 0), \frac{4^n t}{3}\right)\right\}$$

Taking limit as $n \to \infty$ we get

$$P_{\mu,\nu}(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y), t) = 1_{L^*}$$

that is, $Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$ that is, Q is quadratic.

Also from Equation (25) we have

$$P_{\mu,\nu}(Q(x) - g(x), \frac{10-\alpha}{6} t)$$

$$= P_{\mu,\nu}(Q(x) - f(x) + f(x) - g(x), t + \frac{(4-\alpha)}{6} t)$$

$$\geq_{L^*} M\left(P_{\mu,\nu}(Q(x) - f(x), t), P_{\mu,\nu}\left(f(x) - g(x), \frac{(4-\alpha)}{2.3} t\right)\right)$$

$$\geq_{L^*} M\left(M_1(x, (4-\alpha)t), M\left(P'_{\mu,\nu}\left(\phi(x,0), \frac{(4-\alpha)}{3} t\right)\right)\right)$$

$$\geq_{L^*} M\left(M_1(x, (4-\alpha)t), M_1(x, (4-\alpha)t)\right)$$

$$\geq_{L^*} M_1(x, (4-\alpha)t)$$

Therefore,

$$P_{\mu,\nu}(Q(x) - g(x), t) \geq_{L^*} M_1\left(x, \frac{t \times 6(4-\alpha)}{10-\alpha}\right).$$

Similarly,

$$P_{\mu,\nu}(Q(x) - h(x), t) \geq_{L^*} M_1\left(x, \frac{t \times 6(4-\alpha)}{10-\alpha}\right).$$

□

Corollary 1. *Let* $p < 1$ *be a non-negative real number and* X *be norm linear space with norm* $\|.\|$, $(Z, P'_{\mu,\nu}, M)$ *be an IFN-space,* $(Y, P_{\mu,\nu}, M)$ *be a complete IFN-space and* $z_0 \in Z$. *If* $f, g, h : X \to Y$ *are odd mappings such that*

$$P_{\mu,\nu}(f(x+y) + f(x-y) - 2g(x) - 2h(y), t)$$
$$\geq_{L*} P'_{\mu,\nu}(z_0(\|x\|^p + \|y\|^p), t)$$
$$(x, y \in X, t > 0, z_0 \in Z)$$

then there exists a unique additive mapping $A : X \to Y$ *such that*

$$P_{\mu,\nu}(f(x) - A(x), t) \geq_{L^*} P'_{\mu,\nu}\left(z_0\|x\|^p, \frac{t}{6}(2-2^p)\right)$$

and $P_{\mu,\nu}(A(x) - g(x) - h(x), t) \geq_{L^*} P'_{\mu,\nu}\left(z_0\|x\|^p, \frac{(2-2^p)}{10-2^{p+1}} t\right)$
for all $x \in X$ *and* $t > 0, z_0 \in Z$.

Proof. Define $\phi(x, y) = z_0(\|x\|^p + \|y\|^p)$, then the corollary is proved exactly as Theorem 2 with $\alpha = 2^p$. □

Corollary 2. *Let* $p < 2$ *be a non-negative real number and* X *be norm linear space with norm* $\|.\|$, $(Z, P'_{\mu,\nu}, M)$ *be an IFN-space,* $(Y, P_{\mu,\nu}, M)$ *be a complete IFN-space and* $z_0 \in Z$. *If* $f, g, h : X \to Y$ *are even mappings such that*

$$P_{\mu,\nu}(f(x+y) + f(x-y) - 2g(x) - 2h(y), t)$$
$$\geq_{L*} P'_{\mu,\nu}(z_0(\|x\|^p + \|y\|^p), t)$$
$$(x, y \in X, t > 0, z_0 \in Z)$$

then there exists a unique quadratic mapping $Q : X \to Y$ *such that*

$$P_{\mu,\nu}(f(x) - Q(x), t) \geq_{L^*} P'_{\mu,\nu}\left(z_0\|x\|^p, \frac{t}{6}(4-2^p)\right)$$

and $P_{\mu,\nu}(Q(x) - g(x), t) \geq_{L^*} P'_{\mu,\nu}\left(z_0\|x\|^p, \frac{(4-2^p)}{10-2^p} t\right)$ *for all* $x \in X$ *and* $t > 0, z_0 \in Z$.

Proof. Define $\phi\,(x\,,\,y)\;=\;z_0\;(\|x\|^p\;+\;\|y\|^p\;)$. Then the corollary is proved exactly as Theorem 3 with $\alpha\;=\;2^{\,p}$ $\square$

Example 2. *Let* $(X\,,\,\|.\|)$ *be a Banach algebra and let Z be a normed linear space, M a continuous t-norm as defined in Example 1. Then* $(X\,,\,P_{\mu,\nu}\,,\,M)$ *is a complete IFN-space in which* $P_{\mu,\nu}(x,t) = (\mu_x(t), \nu_x(t))$.
Define $f\,,g\,,h\,:\,X\,\to\,X$, *by* $f(x)\;=\;x^{\,2}\;+\;A\,\|x\|x_0$, $g(x)\;=\;x^{\,2}\;+\;B\,\|x\|x_0$, $h(y)\;=\;y^{\,2}\;+\;C\,\|y\|x_0$, $||x_0|| = 1$ *in X and* A, B, C *are positive real numbers.*
Then $\|f\,(x\,+\,y)\,+\,f\,(x\,-\,y)\,-\,2\,g\,(x)\,-\,2\,h\,(y)\,\|\;\le\;2\,(A\,+\,B)\|x\|\,+\,2\,(A\,+\,C)\|y\|$
for all $x\,,\,y\;\in\;X$.
Let $\phi : X \times X \to Z$ *be defined as*
$\phi\,(x\,,\,y)\;=\;2\,(A\,+\,B)\|x\|\,z_0\,+\,2\,(A\,+\,C)\|y\|\,z_0$ *for all* $x\,,\,y\;\in\;X$ *and* z_0 *be a unit vector in Z.*
Thus, $P_{\mu,\nu}\,(f\,(x\,+\,y)\,+\,f\,(x\,-\,y)\,-\,2\,g\,(x)\,-\,2\,h\,(y)\,,\,t)$

$$\ge_{L^*}\;P'_{\mu,\nu}\;(2\,(A\,+\,B)\|x\|\,z_0\,+\,2\,(A\,+\,C)\|y\|\,z_0\,,\,t\,)$$

$$=_{L^*}\;P'_{\mu,\nu}(\phi(x,y),\,t)$$

for all $x\,,\,y\;\in\;X$ *and* $t\;>\;0$.
Then $P_{\mu,\nu}\,(\phi\,(2\,x\,,\,2\,y)\,,\,t)\;\ge\;P'_{\mu,\nu}\,(2\,\phi\,(x\,,\,y)\,,\,t)$ *for all* $x\,,\,y\;\in\;X$ *and* $t\;>\;0$. *Hence, all the conditions of Theorem 3 are valid for* $\alpha = 2 < 4$.
Therefore, f *can be approximated by a mapping* $Q\;:\;X\;\to\;X$ *such that*

$$\begin{aligned}&P_{\mu,\nu}\,(f(x)\;-\;Q\,(x),\,t)\\&\ge_{L^*}\;M_1\,(x,\,2t)\\&\quad=\;P'_{\mu,\nu}\left(\|x\|\,z_0\,,\,\frac{t}{6\min\{(A\,+\,B)\,,\,(2\,A\,+\,B\,+\,C),\,(A\,+\,C)\}}\right)\end{aligned}$$

for all $x\,,\,y\;\in\;X$ *and* $t\;>\;0$.

4. Conclusions

Our consideration in this paper is a pexiderized quadratic functional equation, which is an extension of the quadratic functional equation. It may be possible to extend the cubic and higher order functional equations on similar lines. In our proof of the main theorem, we have made extensive use of the characteristics of intuitionistic fuzzy Banach spaces. As a future problem, we can think of the problem of Hyers-Ulam-Rassias stability for more general forms of functional equations in intuitionistic fuzzy linear spaces.

5. Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

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