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ENGEL ELEMENTS IN GROUPS OF AUTOMORPHISMS OF ROOTED TREES

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I hereby declare that this dissertation is all my own work, except as indicated in the text:

Date: ______________________. Signature: ______________________.

Approval of the thesis:

Engel elements in groups of automorphisms of rooted trees

submitted by Marialaura Noce in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the University of Salerno and University of the Basque Country.

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Camins, somnis i promeses.

...ὅπερ ἔδει δεῖξαι. (Euclide)

A mamma...

non metterò mai in ordine!

Abstract

Groups of automorphisms of d-adic rooted trees ($\operatorname{Aut}\mathcal{T}$ for short) have been studied for years as an important source of groups with interesting properties. For example, many of their subgroups constitute a counterexample to the General Burnside Problem. The question of whether every Engel group is locally nilpotent is the analogue of the General Burnside Problem in the realm of Engel groups.

In this thesis we present recent results about Engel conditions in groups and we study Engel elements in some important families of subgroups of $\operatorname{Aut}\mathcal{T}$.

First, we complete the description, given by Bartholdi [6], of Engel elements in the Grigorchuk group [43]. Then we consider the families of fractal and (weakly) branch groups. About the former, we prove that fractal groups with torsion-free abelianization have no non-trivial Engel elements [17]. On the other hand, with a completely different approach, we show that all non-torsion branch groups have no non-trivial Engel elements, and we also prove some results concerning left Engel elements when we add the condition of being torsion. We conclude by proving that all weakly branch groups have no non-trivial bounded left Engel elements, and, under some additional conditions, that the same happens for right Engel elements [20]. For all these families (fractal and (weakly) branch groups), we present several applications to specific groups, for instance we consider the Basilica group, the Brunner-Sidki-Vieira group, the (multi-)GGS-groups, the Hanoi Tower group, the group of finitary automorphisms, and the whole $\operatorname{Aut}\mathcal{T}$. Also, we study a Lie algebra introduced in [51] and from this we construct the first example of a group which contains a left 3-Engel element whose normal closure is not nilpotent [44].

Sommario. I gruppi di automorfismi $\operatorname{Aut}\mathcal{T}$ che agiscono su un albero regolare e con radice rivestono particolare interesse nell'ambito della teoria dei gruppi. In particolare, molti sottogruppi di $\operatorname{Aut}\mathcal{T}$ rappresentano un controesempio al Problema Generale di Burnside. Stabilire se ogni gruppo di Engel è localmente nilpotente è l'analogo del Problema Generale di Burnside nella teoria di Engel.

La tesi qui presentata è incentrata sullo studio delle condizioni di Engel in alcuni sottogruppi di $\operatorname{Aut}\mathcal{T}$.

In primis, si completa la descrizione, ottenuta da Bartholdi [6], degli elementi di Engel nel gruppo di Grigorchuk [43]. Poi, si descrivono gli elementi di Engel nelle famiglie di gruppi frattali e gruppi branch. Riguardo i primi, si dimostra che tutti i gruppi frattali con abelianizzazione libera da torsione, non hanno elementi di Engel [17]. Inoltre, con un approccio completamente diverso, si dimostra che tutti i gruppi non periodici branch non hanno elementi di Engel, e si fornisce una descrizione degli elementi di Engel a sinistra quando il gruppo branch in questione è di torsione. Per concludere, si dimostra che tutti i gruppi weakly branch non hanno elementi limitati di Engel e, con alcune ipotesi aggiuntive, che non ci sono nemmeno elementi di Engel a destra [20]. Per tutte queste famiglie (gruppi frattali e gruppi (weakly) branch), vengono presentate varie applicazioni a gruppi noti, ad esempio il gruppo di Basilica, il gruppo di Brunner-Sidki-Vieira, i gruppi (multi-)GGS, il gruppo di Hanoi, il gruppo di automorfismi finitari e anche l'intero $\operatorname{Aut}\mathcal{T}$.

Si presenta, inoltre, un argomento basato sulla costruzione di una algebra di Lie introdotta in [51] dalla quale si costruisce il primo esempio di gruppo che contiene un elemento 3-Engel a sinistra la cui chiusura normale non è nilpotente [44].

Laburpena. Zuhaitz d-adiko errotuen automorfismoen taldeak sakonki aztertuak izan dira azken urteetan, propietate interesgarriak dituzten taldeen iturri aberatsa baitira. Esate baterako, horien azpitalde asko Burnsideren Problema Orokorraren kontra adibideak dira. Engelen taldeen alorrean, Burnsideren Problema Orokorraren analogoa Engelen taldeak lokalki nilpotenteak diren edo ez aztertzea da. Tesi honetan, Engelen baldintzaren inguruan izan diren azken emaitzak azalduko ditugu eta Engelen elementuak aztertuko ditugu zuhaitz errotuen automorfismoen taldeen familia berezi batzuetan: talde fraktaletan eta talde adarkatuetan esaterako. Lehenik, Bartholdik [6]-en hasi zuen Grigorchuken taldearen Engelen elementuen deskribapena amaituko dugu [43]. Ondoren, talde fraktalen eta talde (ahulki) adarkatuen taldeak kontsideratuko ditugu. Talde fraktalei dagokienez, bihurdura gabeko abelianizazioa badute Engelen elementu ez-tribialik ez dutela frogatuko dugu [17]. Bestalde, ikuspuntu guztiz berri batekin, bihurdura taldeak ez diren talde adarkatuek Engelen elementu ez-tribialik ez dutela ere ikusiko dugu eta, gainera, ezkerreko Engel elementuen inguruan emaitza ezberdinak frogatuko ditugu talde adarkatua bihurdura taldea denean. Honela, talde ahulki adarkatuek ezkerreko Engelen elementu bornatu ez-tribialik ez dutela frogatuko dugu eta, baldintza batzuk gehituz, eskuineko Engelen elementuekin gauza bera gertatzen dela ikusiko dugu [20]. Emaitza guzti hauek, talde fraktalen eta talde (ahulki) adarkatuen familian aurki daitezkeen talde berezi batzuetan aplikatuko ditugu, besteak beste, Basilica taldean, (multi-)GGS-taldeetan, Hanoi Dorrearen taldean, automorfismo finitarioen taldean eta $\operatorname{Aut}\mathcal{T}$ talde osoan. Azkenik, [51]-en definitutako Lie-ren aljebra bat aztertuko dugu eta, honela, eskuineko 3-Engelen elementuekin gertatzen ez den bezala, itxitura normal ez-nilpotentea duen ezkerreko 3-Engelen elementu baten adibidea eraikiko dugu [44].

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Chapter 1

Introduction

Subgroups of the group of automorphisms of a d-adic rooted tree ($\operatorname{Aut}\mathcal{T}_d$ for short) have been studied for years as an important source of groups with interesting properties. For instance, the Grigorchuk group is the first example of a group of intermediate word growth, and amenable but not elementary amenable. Together with the Grigorchuk group, other subgroups of $\operatorname{Aut}\mathcal{T}_d$ like the Gupta-Sidki p-groups and many groups in the family of the so-called Grigorchuk-Gupta-Sidki groups (known simply as *GGS-groups*) are shown to be a counterexample to the General Burnside Problem. The question of whether every Engel group is locally nilpotent is the analogue of the General Burnside Problem in the realm of Engel groups. In the following we will refer to this problem as the "Burnside Engel Problem". For some classes of groups, the answer is positive. Nevertheless, there exist examples of infinite Engel p-groups (in the family of the so-called Golod-Shafarevich groups) which are finitely generated but not locally nilpotent. Notice that Golod-Shafarevich groups provide also a negative answer to the General Burnside Problem. It seems natural to ask whether subgroups of $\operatorname{Aut}\mathcal{T}_d$ provide a negative answer also to the Burnside Engel Problem. In [6] Bartholdi proved that the Grigorchuk group and the Gupta-Sidki 3-group are not Engel. Hence, so far, Golod-Shafarevich groups are the only counterexample to the Burnside Engel Problem.

This thesis is mostly devoted to the study of Engel elements in certain classes of subgroups of $\operatorname{Aut}\mathcal{T}_d$, and more generally of $\operatorname{Aut}\mathcal{T}$, where $\mathcal{T}$ is a spherically homogeneous rooted tree.

We recall that an element x of a group G is said to be *left Engel* if for any $g \in G$ there exists a positive integer n depending on both g and x such that $[g,_n x] = [[g,_{n-1} x], x] = 1$. Similarly, x is *right Engel* if the variable g appears on the right. The sets of all left and right Engel elements of G are denoted by $\mathrm{L}(G)$ and $\mathrm{R}(G)$, respectively. If $\mathrm{L}(G) = G$ (or, equivalently, $\mathrm{R}(G) = G$) we say that G is an Engel group. Moreover, an element of G is said to be an n-Engel element if the choice of n is independent of g. In this case we denote the respective sets $\mathrm{L}_n(G)$ and $\mathrm{R}_n(G)$. Also we write $\overline{\mathrm{L}}(G) = \cup_{n\in\mathbb{N}} \mathrm{L}_n(G)$ and $\overline{\mathrm{R}}(G) = \cup_{n\in\mathbb{N}} \mathrm{R}_n(G)$ the sets of bounded left and right Engel elements, respectively.

A big area of interest in Engel theory is the study of the subsets $\mathrm{L}(G)$, $\mathrm{R}(G)$, $\overline{\mathrm{L}}(G)$ and $\overline{\mathrm{R}}(G)$. Indeed, in general, it is not known whether these are subgroups or not. In 2006, Bludov constructed a group based on the Grigorchuk group that is the first example of a group in which the set of left Engel elements is not a subgroup (see [9]). In 2016, Bartholdi [6] refined this example by proving that the only left Engel elements in the Grigorchuk group itself are the involutions. However the problem for $\mathrm{R}(G)$, $\overline{\mathrm{L}}(G)$, and $\overline{\mathrm{R}}(G)$ is still open. It is also noticeable that there exists an automaton group where determining if an element is left Engel is undecidable [22].

The topic of this thesis stems in two directions. On the one hand we have investigated Engel groups and Engel elements, and on the other hand we have studied groups of automorphisms of rooted trees. More precisely, the thesis is organized as follows.

In Chapter 2, we give the definitions of Engel groups and Engel elements and we survey some relevant results and open problems in Engel theory. In particular, we present the famous Burnside problems and we underline their connection with Engel groups. We finish this chapter by giving the definition of almost Engel groups and showing that a group that is almost Engel and residually nilpotent is an Engel group (Section 2.3).

In Chapter 3, we analyze a problem concerning left 3-Engel elements. It was proved by Newell [42] that any right 3-Engel element a of a group G belongs to the Hirsch Plotkin radical $\mathrm{HP}(G)$ (the subgroup generated by all normal locally nilpotent subgroups of G). Actually, Newell proved the stronger result that the normal closure of a is nilpotent of class at most 3. The natural question of whether the analogous holds for left 3-Engel elements arises.

We prove that this is not the case, by providing the first example of a group with a left 3-Engel element whose normal closure is not nilpotent. More precisely, we prove the following.

Theorem A. *There exists a locally finite* 2*-group* G *with a left* 3*-Engel element* a *such that* $\langle a\rangle^G$ *is not nilpotent.*

In fact, this theorem is a corollary of deeper results concerning the construction of a Lie algebra over a field of characteristic 2. Indeed, one of the purposes of this chapter is to further investigate the properties of a Lie algebra introduced in [52]. This algebra provides an example of a non-solvable Engel 3-Lie algebra of characteristic 2 containing an element generating a non-nilpotent ideal. In the same paper, Traustason proved that every Engel 3-Lie algebra of odd characteristic is soluble [52]. This is an example of how different the cases of odd and even characteristic can behave in the study of Engel conditions in Lie algebras.
All these results have given rise to the paper [44], published in *Journal of Pure and Applied Algebra* written jointly with G. Tracey and G. Traustason.

Chapter 4 is devoted to introduce groups acting on spherically homogeneous rooted trees. We present some alternative approaches to these groups, and also study some subgroups of $\operatorname{Aut}\mathcal{T}$ which play an important role in this context, namely stabilizers and rigid stabilizers of different levels of $\mathcal{T}$. We define self-similar groups, fractal groups and (weakly) branch groups, and we prove some results connecting these groups. We finish by introducing the notion of just infinite groups and we prove a criterion for a regular branch group to be just infinite.

In Chapter 5 we focus on specific groups of automorphisms of rooted trees (including the Grigorchuk group, the multi-GGS groups, the Basilica group, the Brunner-Sidki-Vieira group, and the group of finitary automorphisms) and some of their remarkable properties. Also, we provide an alternative proof of the fact that the Brunner-Sidki-Vieira group has torsion-free abelianization [49].

In Chapter 6 we present the state of the art of Engel theory in $\operatorname{Aut}\mathcal{T}$. In particular, in Chapter 6.1 we reconstruct the example of Bludov which is the first example of a group where the set of left Engel elements is not a subgroup. As pointed out at the beginning of this introduction, this example was never published and it is based on the first Grigorchuk group. Ten years later, Bartholdi showed [6] that the Grigorchuk group is not Engel, and also that all its left Engel elements are the involutions. In Chapter 6.2 we complete the description of Engel elements in the first Grigorchuk group.

Theorem B. *Let $\mathfrak{G}$ be the first Grigorchuk group. Then* $\overline{\mathrm{L}}(\mathfrak{G}) = \mathrm{R}(\mathfrak{G}) = \overline{\mathrm{R}}(\mathfrak{G}) = \{1\}$.

This result has led to the paper [43] jointly written with A. Tortora and published in *International Journal of Group Theory*.

There are two important subclasses of groups of automorphisms of rooted trees: fractal groups and branch groups (both defined in Chapter 4). Chapters 7 and 8 are devoted to study Engel elements in these two classes of groups. We remark that since $\mathrm{R}(G)^{-1}, \overline{\mathrm{L}}(G), \overline{\mathrm{R}}(G)^{-1} \subseteq \mathrm{L}(G)$ for any group G [33], by proving that $\mathrm{L}(G) = 1$ one automatically obtains that each one of these sets is also trivial.

In Chapter 7 we generalize the notions of commutators and left Engel elements to the context of group actions. We then prove that a certain class of fractal groups has no non-trivial left Engel elements.

Our main results are stated below and are collected in [17], a paper jointly written with G.A. Fernández-Alcober and A. Garreta and published in Monatshefte für Mathematik.

We denote by Γ a standard Sylow pro-p subgroup of $\operatorname{Aut}\mathcal{T}_p$, for p a prime. Our first theorem reads as follows.

Theorem C. *Let $G \leq \Gamma$ be a fractal group such that $|G' : \mathrm{st}_G(1)'| = \infty$. Then $\mathrm{L}(G) = 1$.*

As shown in Section 7.4, the conditions $|G' : \mathrm{st}_G(1)'| = \infty$ and being fractal cannot be omitted in the theorem above. The following result is a consequence of Theorem C.

Theorem D. *Let $G \leq \Gamma$ be a fractal group with torsion-free abelianization. Then $\mathrm{L}(G) = 1$.*

A natural application to some well-known fractal groups follows.

Corollary. *The Basilica group and the Brunner-Sidki-Vieira group have no non-trivial left Engel elements.*

The case of the GGS-group $\mathcal{G}$ with constant defining vector is more difficult since the group has finite abelianization. However, we reduce the study of Engel elements to a subgroup $K \leq \mathcal{G}$ which meets the requirements of Theorem D. After proving that $\mathrm{L}(\mathcal{G}) \subseteq \mathrm{L}(K)$, we obtain the following.

Theorem E. *The GGS-group with constant defining vector has no non-trivial left Engel elements.*

In Chapter 8 we study Engel elements in (weakly) branch groups. The results presented in this chapter are collected in [20], a paper submitted and written jointly with G.A. Fernández-Alcober and G. Tracey.

The main tool to our approach to Engel problems in weakly branch groups is the reduction of the action of an automorphism f from the whole tree to one or several "reduced trees" determined by some special orbits of the set of vertices of $\mathcal{T}$. Then we describe some properties of orbits of automorphisms of $\mathcal{T}$, and we prove several key results regarding Engel elements in wreath products.

This machinery allows us to study Engel elements in weakly branch groups. Our main theorems read as follows.

Theorem F. *Let G be a weakly branch group. Then:*

(i) $\overline{\mathrm{L}}(G) = 1$.

(ii) If $\mathrm{L}(G)$ *contains non-trivial elements of finite order then:*

- *All such elements have p-power order for some prime p.*
- $\mathrm{rst}_G(n)$ *is a p-group for some $n \geq 1$.*

Moreover, if we add the stronger condition that the group is branch, we obtain the following.

Theorem G. *Let G be a branch group. Then:*

(i) If G is not periodic, then $\mathrm{L}(G) = 1$.

(ii) If G is periodic, then $\mathrm{L}(G)$ *consists of p-elements for some prime p.*

(iii) If $\mathrm{L}(G) \neq 1$, *then G is virtually a p-group for the same prime as in (ii).*

Compare this last theorem with the situation in the Grigorchuk group. In that case, $\mathrm{L}(\mathfrak{G})$ consists of all elements of order 2 in $\mathfrak{G}$, and $\mathfrak{G}$ is a 2-group. Theorem F and G can be applied to several groups, as the two corollaries below show.

Corollary 1. Let $\mathcal{T}$ be a spherically homogeneous rooted tree. Then the following hold:

(i) If G is an infinitely iterated wreath product of finite transitive permutation groups of degree at least 2, then $\mathrm{L}(G) = 1$. This applies in particular to the whole group of automorphisms of $\mathcal{T}$, and also to its Sylow pro-p subgroups if $\mathcal{T}$ is a p-adic tree, where p is a prime.

(ii) If $\mathcal{F}$ is the group of finitary automorphisms of $\mathcal{T}$, and there are infinitely many levels in which the number of descendants is greater than 2, then $\mathrm{L}(\mathcal{F}) = 1$. If $\mathcal{T}$ is a p-adic tree and $\mathcal{F}_p$ is the group of p-finitary automorphisms of $\mathcal{T}$, then $\overline{\mathrm{L}}(\mathcal{F}_p) = 1$.

(iii) If $\mathcal{H}$ is the Hanoi Tower group, then $\mathrm{L}(\mathcal{H}) = 1$.

(iv) If G is a multi-GGS groups, then $\mathrm{R}(G) = 1$. If G is furthermore non-torsion, then $\mathrm{L}(G) = 1$.

Corollary 2. Let p be a prime and let $\sigma \in \mathrm{Sym}(p)$ be a fixed p-cycle. Then the subgroup $\mathcal{F}_p$ of $\operatorname{Aut} \mathcal{T}_p$ formed by the finitary automorphisms all of whose labels lie in $\langle \sigma \rangle$ (in other words, the intersection of $\mathcal{F}$ with the standard Sylow pro-p subgroup of $\operatorname{Aut} \mathcal{T}_p$ corresponding to σ) satisfies that:

(i) $\mathrm{L}(\mathcal{F}_p) = \mathcal{F}_p$.

(ii) $\overline{\mathrm{L}}(\mathcal{F}_p) = 1$.

In Chapter 8 we also prove some results concerning right Engel elements in branch groups.

Theorem H. *Let G be a weakly branch group. If $\mathrm{rst}_G(n)$ is not Engel for any n, then $\mathrm{R}(G) = 1$.*

Theorem H can be applied to show that GGS-groups have no non-trivial right Engel elements:

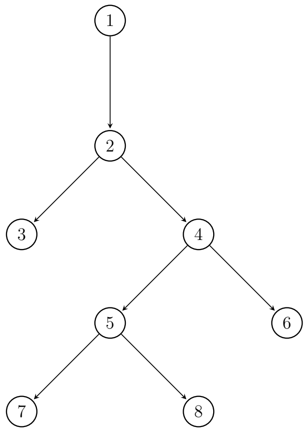
Corollary 3. Let G be a GGS-group. Then $\mathrm{R}(G) = 1$.

We conclude this introduction by remarking that our results suggest that being Engel is a condition too strong for (infinite) finitely generated fractal/branch groups. Also, as pointed out before, Golod-Shafarevich groups are the only known counterexample to the Engel Burnside Problem and they cannot be branch (see [16, Proposition 8.11]). This motivates the following question.

Question. *Can a finitely generated branch/fractal group be Engel?*

Note that, without finite generation, the group $\mathcal{F}_p$ shows that the answer is positive (see Corollary 2). Also, observe that weakly branch groups cannot satisfy a law (see [3, Corollary 1.4]) and so cannot be n-Engel for a fixed n. Thus we are asking whether finite generation makes it impossible for them to be Engel as well.

Here we give a suggestion to the reader about a possible approach to read this thesis.



A *non-fractal tree* showing logical dependence among chapters.

Part I

Engel conditions in groups

Chapter 2

Engel elements in groups

In this chapter, we give basic definitions of Engel groups and Engel elements and we survey some results and open problems in Engel theory. In particular, we present the Burnside problems and we underline their connections with Engel groups.
Finally, in Section 2.3 we provide the definition of almost Engel group and we show that a group that is almost Engel and residually nilpotent is an Engel group.

2.1 Burnside problems

The origin of the general Burnside problem is the famous paper [11] where in 1902 Burnside posed his attention to a "still undecided point" on torsion groups. Recall that a group G is *torsion* (or *periodic*) if all its elements have finite order, that is, for any $g \in G$ there exists an integer $n \geq 1$ such that $g^n = 1$. If the orders of the elements are also bounded, the group G is said to have *finite exponent* and the least common multiple of the orders is the *exponent* of G. On the other hand, a group G is *torsion-free* if the only element of G of finite order is the identity.

In his 1902 paper [11], Burnside posed the following question, that is known as the *General Burnside Problem* (GBP for short).

1. Is a finitely generated periodic group necessarily finite?

In 1911 Schur proved that every finitely generated periodic subgroup of the general linear group of degree $n \geq 1$ over the complex field is finite. No more progress was made until 1964, when Golod provided a counterexample. More precisely, for any prime p and any integer $r \geq 2$, Golod showed that there exists an infinite p-group $G = \langle x_1, \dots, x_r \rangle$ such that every $(r-1)$-generator subgroup of G is finite [15]. Later much simpler examples have been found. For instance, in 1980 Grigorchuk constructed an infinite 3-generator 2-group, which will be discussed in Section 5.1.

Since Question 1 seems quite difficult, the GBP can be restated adding the stronger hypothesis that the group is of finite exponent. Thus we have the following.

2. Is a finitely generated periodic group of finite exponent necessarily finite?

Question 2 is known simply as the *Burnside Problem* and is the same as asking whether the free Burnside groups are finite. In the following we recall the definition of free Burnside groups.

Definition 2.1. Let F_m be a free group of rank m. The free Burnside group $B(m, n)$ is the group F_m/F_m^n, where F_m^n is the group generated by all the n-th powers of elements of F_m.

Note that $B(m, n)$ is the free group in which the identity $x^n = 1$ holds and for this reason it is the biggest group generated by m elements of exponent n.

It is easy to prove that for any m the 2-group $B(m, 2)$ is elementary abelian (i.e. an abelian group in which every non-trivial element has order 2) and so finite. Burnside proved that also $B(m, 3)$ is finite for any m. Levi and van der Waerden in 1993 proved that if $m \geq 3$, then the Burnside group $B(m, 3)$ is finite and nilpotent of class 3. One can also prove that since $B(m, 3)$ is a group of exponent 3, then it is a 2-Engel group (see Definition 2.9 in the next section). In 1940, Sanov proved that also the group $B(m, 4)$ is finite for every m. For $n = 5$ the problem is still open. Although for some small values of n the Burnside groups are finite, in 1968, Novikov and Adian proved that, in general, Burnside groups need not be finite. Indeed they showed that if $m \geq 2$, and n is odd and greater than 4381, then $B(n, m)$ is infinite. This bound was improved later by Adian who showed that n can be chosen odd and greater than or equal to 665. Also, in 1979 Ol'shanskii constructed infinite (simple) 2-generator groups all of whose proper non-trivial subgroups are of prime order p [1]. They are called Tarski monster groups (for further information see [45]).

These results suggest that any counterexample to Questions 1 and 2 will be difficult. In 1930-40s the topic was resurrected by the suggestion of a third variant, which thanks to Magnus is known as the *Restricted Burnside Problem* [39].

3. Are there only finitely many finite groups generated by m elements of exponent n?

The Restricted Burnside Problem is the only "Burnside question" with positive answer. It was solved by Zelmanov in 1991 ([56] for groups of exponent 2 and [55] for groups of odd exponent), who was awarded the Fields medal.

2.2 Background on Engel theory

In this section, we give definitions of Engel groups and Engel elements and we state some results that can be found, for example, in [47].

2.2.1 First definitions and properties

Let G be a group and let $x_1, x_2, \ldots$ be elements of G. We define the commutator of weight $n \geq 1$ recursively by the rule

$$[x_1, \ldots, x_n] = \begin{cases} x_1 & \text{if } n = 1 \\ [[x_1, \ldots, x_{n-1}], x_n] & \text{if } n > 1, \end{cases}$$

where $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2 = x_1^{-1}x_1^{x_2}$.

If $x = x_1$ and $y = x_2 = \cdots = x_{n+1}$, we use the following shorthand notation

$$[x, {}_n y] = [x, y, \overset{n}{\ldots}, y].$$

In the following we collect some basic properties of commutators.

Lemma 2.2. *Let G be a group and let $x, y, z \in G$. Then:*

(i) $[y, x] = [x, y]^{-1}$.

(ii) $[x^{-1}, y] = [y, x]^{x^{-1}}$ *and* $[x, y^{-1}] = [y, x]^{y^{-1}}$.

(iii) $[xy, z] = [x, z][x, z, y][y, z]$ *and* $[x, yz] = [x, z][x, y][x, y, z]$.

(iv) $[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1$ *(Hall-Witt's identity).*

If we add the condition that the group G is metabelian (i.e. its derived subgroup G' is abelian), we can prove the following.

Lemma 2.3. *Let G be a metabelian group. Then for any $x, y, z \in G$, $u \in G'$ and for any $n \geq 1$, we have:*

(i) $[u, x, y] = [u, y, x]$.

(ii) $[x, y, z][y, z, x][z, x, y] = 1$ *(Hall-Witt's identity).*

(iii) $[x^{-1}, {}_n y] = [x, {}_n y]^{-x^{-1}}$.

(iv) $[xy, {}_n z] = [x, {}_n z][x, {}_n z, y][y, {}_n z]$.

Definition 2.4. Let G be a group and $g \in G$. We say that g is a *right Engel element* if for any $x \in G$ there exists $n = n(g, x) \geq 1$ such that $[g, {}_n x] = 1$. If n is a positive integer, an element $g \in G$ is a *right n-Engel element* of G if $[g,_n x] = 1$ for all $x \in G$. An element $g \in G$ is called a *bounded right Engel element* if it is right n-Engel for some n.
Similarly, g is a *left Engel element* if for any $x \in G$ there exists $n = n(g, x) \geq 1$ such that $[x, {}_n g] = 1$. Again, if the choice of n is independent from x, then g is a *left n-Engel element*, and an element $g \in G$ is called a bounded left Engel element if it is left n-Engel for some n.

We denote by $\mathrm{R}(G), \mathrm{R}_n(G)$ and $\overline{\mathrm{R}}(G)$ ($\mathrm{L}(G), \mathrm{L}_n(G)$ and $\overline{\mathrm{L}}(G)$) the set of right (left) Engel elements, right (left) n-Engel elements, bounded right (left) Engel elements of G.

The following gives the interaction between (bounded) right and (bounded) left Engel elements.

Proposition 2.5. *In any group G, the inverse of a right Engel element is a left Engel element and the inverse of a right n-Engel element is a left $(n+1)$-Engel element. In other words:*

(i) $\mathrm{R}(G)^{-1} \subseteq \mathrm{L}(G)$.

(ii) $\mathrm{R}_n(G)^{-1} \subseteq \mathrm{L}_{n+1}(G)$.

It is still an open question whether every right Engel element is a left Engel element. However, it is easy to see that the converse does not hold in general. For example, let Sym(3) be the symmetric group of degree 3 and consider the element $g = (1\ 2\ 3)$. Since $\langle g \rangle$ is normal in Sym(3), we have $[x, g] \in \langle g \rangle$ for any $x \in G$. Hence, $[x, g, g] = 1$ and g is a left 2-Engel element. On the other hand, if $x = (1\ 2)$, then $[g, x] = g$. It follows by induction that $[g, {}_n x] = g$ for any $n \geq 1$. Thus, g is not a right Engel element.

It is a long-standing problem, raised by Plotkin, whether the sets $\mathrm{R}(G)$ and $\mathrm{L}(G)$ ($\overline{\mathrm{R}}(G)$ and $\overline{\mathrm{L}}(G)$, respectively) of a given group G are subgroups (see Problem 16.15 and Problem 16.16 in the Kourovka notebook [36]).

There are some results for left k-Engel elements. For instance, consider the standard wreath product $H = C_2 \wr K$ of a group of order 2 with an elementary abelian group of order 4. The group H is generated by left

2-Engel elements but $H \neq \mathrm{L}_2(H)$ because of the following proposition. We denote $\langle x\rangle^G$ the normal closure of x in G.

Proposition 2.6. [2, Proposition 3.25] *Let G be a group. We have $\mathrm{L}_2(G) = \{x \in G \mid \langle x\rangle^G$ is abelian$\}$.*

In general, for any integer $n \geq 2$, there exists a group G for which $a, b \in \mathrm{L}_n(G)$ but $ab \notin \mathrm{L}_n(G)$. More precisely, we have the following.

Proposition 2.7. [2, Proposition 3.26] *Let $k \geq 1$ and let G be a group of exponent 2^k. Take the standard wreath product $H = G \wr (\langle x\rangle \times \langle y\rangle)$, where $\langle x\rangle$ and $\langle y\rangle$ are cyclic groups of order 2. Then $\mathrm{L}_{k+1}(H)$ is not a subgroup.*

In 2006, during a conference at the University of Debrecen, Bludov announced that there exists a group in which the set of left Engel elements is not a subgroup [9]. This example, which has never been published anywhere, will be discussed in Chapter 6.1. The question for right Engel elements is still unsolved.
We conclude this section by stating and proving a key result for this thesis.

Proposition 2.8. *Let G be group and let $g \in G$ be such that $g^2 = 1$. Then for any $x \in G$ and any $n \geq 1$:*

$$[x,{}_n g] = [x, g]^{(-2)^{n-1}}.$$

In particular, every involution in any 2-group is a left Engel element.

Proof. We proceed by induction on n, the case $n = 1$ being trivial. Suppose then that $n > 1$. Since $[x,{}_{n+1} g] = [[x,{}_n g], g] = [x,{}_n g]^{-1}[x,{}_n g]^g$, by induction hypothesis we have

$$[x,_n g]^{-1}[x,_n g]^g = [x,g]^{-(-2)^{n-1}}([x,g]^g)^{(-2)^{n-1}}.$$

Also, using (ii) of Lemma 2.2 and the fact that $g^2 = 1$, we have

$$[x,g]^g = [x,g]^{g^{-1}} = [g^{-1},x] = [g,x] = [x,g]^{-1}.$$

It follows that

$$[x,_{n+1} g] = [x,g]^{-(-2)^{n-1}}[x,g]^{-(-2)^{n-1}} = [x,g]^{(-2)^n}.$$

This completes the proof. □

2.2.2 An Engel version of the general Burnside problem

Let G be a group, and recall that G is locally nilpotent if each finitely generated subgroup of G is nilpotent.

Definition 2.9. We say that G is an *Engel group* if $G = \mathrm{R}(G)$ or, equivalently, $G = \mathrm{L}(G)$. Furthermore, G is *n-Engel* if there exists $n \geq 1$ such that $[x,_n y] = 1$ for all $x, y \in G$.

It is easy to see that every locally nilpotent group is Engel. Indeed, for all $x, y \in G$, the subgroup $\langle x, y \rangle$ is nilpotent and therefore $[x,_n y] = 1$ for some $n = n(x,y) \geq 1$. The question whether every finitely generated Engel group is nilpotent is the analogue of the general Burnside problem for Engel groups. Also in this case, the infinite p-group $G = \langle x_1 \ldots, x_r \rangle$ constructed by Golod is a counterexample. In fact, with $r = 3$, every 2-generator subgroup of G is nilpotent and so G is an Engel group. However, if G is nilpotent, then it is finite: a contradiction. So far, in the periodic case, Golod's group is the only known example of a finitely generated Engel group that is not nilpotent. On the other hand, the general Burnside problem for Engel groups has a positive answer for some families of groups such as: finite groups (Zorn [57]), groups that satisfy the maximal condition for subgroups (Baer [5]), solvable groups (Gruenberg [31]), and Hausdorff compact topological groups (Medvedev [40]). Nevertheless the main open question in the realm of Engel groups is whether every n-Engel group is locally nilpotent.

2.3 Almost Engel groups

Almost Engel groups were introduced by Khukhro and Shumyatsky in 2016, [37]. They are defined as follows.

Definition 2.10. A group G is *almost Engel* if for every $g \in G$ there is a finite set $\mathcal{E}(g)$ such that for every $x \in G$ there exists a positive integer $n(g, x)$ for which $[x,_n g] \in \mathcal{E}(g)$ for all $n \geq n(g, x)$.

For example in the case of an Engel group G, we have $\mathcal{E}(g) = \{1\}$.

Lemma 2.11. [37, Lemma 2.1] *If G is almost Engel, then for any $g \in G$, there exists a unique minimal finite set $\mathcal{E}(g)$ that consists precisely of all elements z such that $z = [z, g, \overset{n}{\ldots}, g]$, for some n.*

To prove the following, first recall what is a residually nilpotent group.

Definition 2.12. Let G be a group, and $\mathfrak{X}$ a property of groups (for instance being finite, being nilpotent, etc.). We say that G is residually $\mathfrak{X}$ if for any $1 \neq g \in G$, there exists N normal in G such that:

1. $g \notin N$.

2. G/N is in the class $\mathfrak{X}$.

In other words, G is residually $\mathfrak{X}$ if and only if

$$\bigcap\{N \trianglelefteq G \mid G/N \in \mathfrak{X}\} = 1.$$

Theorem 2.13. *Let G be a residually nilpotent almost Engel group. Then G is Engel.*

Proof. We want to show that for every $g \in G$, then $\mathcal{E}(g) = \{1\}$ (by taking $\mathcal{E}(g)$ minimal). Suppose towards contradiction that $\mathcal{E}(g) \neq \{1\}$ and consider $1 \neq z \in \mathcal{E}(g)$. Then by Lemma 2.11 there exists n such that $z = [z,_n g]$. Since G is residually nilpotent, consider a normal subgroup N of G such that $z \notin N$ and G/N is nilpotent. Suppose that the nilpotency class of G/N is k. Then $z = [z,_{nk} g] \in \gamma_{k+1}(G) \leq N$, a contradiction. $\square$

Chapter 3

On left 3-Engel elements in groups

In this chapter we present a paper carried out in collaboration with G. Traustason and G. Tracey, both from the University of Bath, and published in *Journal of Pure and Applied Algebra* [44].

3.1 Preliminaries

In what follows $\mathrm{HP}(G)$ denotes the Hirsch-Plotkin radical of a group G, that is the subgroup generated by the union of the normal locally nilpotent subgroups of G.

Remark 3.1. Notice that any element of the Hirsch-Plotkin radical $\mathrm{HP}(G)$ of a group G is a left Engel element. Indeed, take $g \in G$ and $x \in \mathrm{HP}(G)$. Then $[g, x] \in \mathrm{HP}(G)$ and the subgroup generated by x and

$[g, x]$ is contained in $\mathrm{HP}(G)$ and so it is nilpotent. Hence $[g,_n x] = 1$ for some $n \geq 1$, and $\mathrm{HP}(G) \subseteq \mathrm{L}(G)$.

The converse is known to be true for some classes of groups, including finite groups (more generally groups satisfying the maximal condition on subgroups) and solvable groups [5, 32]. However, the converse is not true in general because of Golod's examples. Concerning bounded left Engel elements, one can readily see that a left 2-Engel element is always in the Hirsch-Plotkin radical. On the other hand, it is is still an open question if the same is true for left 3-Engel elements. Recently it was proved that any left 3-Engel element of odd order is contained in $\mathrm{HP}(G)$. From [50] one also knows that in order to generalize this to left 3-Engel elements of any finite order it suffices to deal with elements of order 2.

One can also see (for example in [34, 35]) that for sufficiently large n we do not have in general that a left n-Engel element is contained in the Hirsch-Plotkin radical. Using the fact that groups of exponent 4 are locally finite [48], one can also see that if all left 4-Engel elements of a group G of exponent 8 are in $\mathrm{HP}(G)$ then G is locally finite.

In [42] Newell proved that if a is a right 3-Engel element in G then $a \in \mathrm{HP}(G)$ and in fact he proved the stronger result that $\langle a \rangle^G$ (i.e. the normal closure of a in G) is nilpotent of class at most 3. The natural question arises whether the analogous result holds for left 3-Engel elements. In this chapter we show that this is not the case by giving an example of a locally finite 2-group with a left 3-Engel element a such that $\langle a \rangle^G$ is not nilpotent.

3.2 Left 3-Engel element whose normal closure is not nilpotent

3.2.1 The construction of a Lie algebra

Our construction will be based on an example of a Lie algebra given in [51]. Let $\mathbb{F}$ be the field of order 2 and consider a 4-dimensional vector space $V = \mathbb{F}x + \mathbb{F}u + \mathbb{F}v + \mathbb{F}w$ where

$$u \cdot v = u,\ v \cdot w = w,\ w \cdot u = v,\ u \cdot x = 0, v \cdot x = 0, w \cdot x = u.$$

Also, $v \cdot u = -u = u$, $w \cdot v = -w = w$, and $u \cdot w = -v = v$, and the product of a basis element with itself is 0.

We then extend the product linearly on V. One can check that V is a Lie algebra with a trivial center and where $W = \mathbb{F}u + \mathbb{F}v + \mathbb{F}w$ is a simple ideal (see [51]). Let $E = \langle \mathrm{ad}(x), \mathrm{ad}(u), \mathrm{ad}(v), \mathrm{ad}(w) \rangle \leq \mathrm{End}(V)$ be the associative enveloping algebra of V. Recall that for $l \in V$, the adjoint map $\mathrm{ad}(l) : V \to V$ is defined by $m\,\mathrm{ad}(l) = (m, l)$, for $m \in V$. Let

$$\begin{aligned}
&e_1 = \mathrm{ad}(w), && e_2 = \mathrm{ad}(w)^2, && e_3 = \mathrm{ad}(w)^3,\\
&e_4 = \mathrm{ad}(v), && e_5 = \mathrm{ad}(v)\,\mathrm{ad}(w), && e_6 = \mathrm{ad}(v)\,\mathrm{ad}(w)^2,\\
&e_7 = \mathrm{ad}(u), && e_8 = \mathrm{ad}(u)\,\mathrm{ad}(w), && e_9 = \mathrm{ad}(u)\,\mathrm{ad}(w)^2,\\
&e_{10} = \mathrm{ad}(x), && e_{11} = \mathrm{ad}(x)\,\mathrm{ad}(w), && e_{12} = \mathrm{ad}(x)\,\mathrm{ad}(w)^2.
\end{aligned}$$

Lemma 3.2. *The associative enveloping algebra E is 12-dimensional with basis $e_1, \ldots, e_{12}$.*

Proof. We first show that E is spanned by products of the form

$$\mathrm{ad}(x)^{\epsilon} \cdot \mathrm{ad}(u)^{r} \cdot \mathrm{ad}(v)^{s} \cdot \mathrm{ad}(w)^{t}$$

where ϵ, r, s, t are non-negative integers. To see this we need to show that any product $\mathrm{ad}(y_1) \cdots \mathrm{ad}(y_m)$, with $y_1, \ldots, y_m \in \{x, u, v, w\}$ can be written as a linear combination of elements of the required form. We use induction on m. This is obvious when $m = 1$. Now suppose $m \geq 2$ and that the statement is true for all shorter products (since the product of two basis elements is either 0 or another basis element). Suppose there are ϵ entries of x, r entries of u, s entries of v and t entries of w in the product $\mathrm{ad}(y_1) \cdots \mathrm{ad}(y_m)$. Using the fact that $\mathrm{ad}(y_i)\,\mathrm{ad}(y_j) = \mathrm{ad}(y_j)\,\mathrm{ad}(y_i) + \mathrm{ad}(y_i y_j)$, we see that modulo shorter products we have

$$\mathrm{ad}(y_1) \cdots \mathrm{ad}(y_m) = \mathrm{ad}(x)^{\epsilon}\,\mathrm{ad}(u)^{r}\,\mathrm{ad}(v)^{s}\,\mathrm{ad}(w)^{t}.$$

Hence the statement is true for products of length m. This finishes the inductive proof of our claim.

From the fact that

$$\mathrm{ad}(x)^2 = \mathrm{ad}(x)\,\mathrm{ad}(u) = \mathrm{ad}(u)\,\mathrm{ad}(x) = \mathrm{ad}(u)\,\mathrm{ad}(x) = 0,$$

and $\mathrm{ad}(v)^2 = \mathrm{ad}(v)$, $\mathrm{ad}(u)^2 = \mathrm{ad}(x)\,\mathrm{ad}(v)$ and $\mathrm{ad}(u)\,\mathrm{ad}(v) = \mathrm{ad}(u) + \mathrm{ad}(x)\,\mathrm{ad}(w)$, we can assume that $0 \leq \epsilon, r, s \leq 1$ and that if $\mathrm{ad}(u)$ is included then we can assume that neither $\mathrm{ad}(x)$ nor $\mathrm{ad}(v)$ is included. This together with $\mathrm{ad}(x) = \mathrm{ad}(x)\,\mathrm{ad}(v)$ and $\mathrm{ad}(w)^4 = \mathrm{ad}(v)\,\mathrm{ad}(w)^3 = \mathrm{ad}(u)\,\mathrm{ad}(w)^3 =$

$\mathrm{ad}(x)\,\mathrm{ad}(w)^3 = 0$ shows that E is generated by $e_1, \ldots, e_{12}$. It remains to see that these elements are linearly independent. Suppose $\alpha_1 e_1 + \cdots + \alpha_{12} e_{12} = 0$ for some $\alpha_1, \ldots, \alpha_{12} \in \mathbb{F}$. Then

$$0 = x(\alpha_1 e_1 + \ldots + \alpha_{12} e_{12}) = \alpha_1 u + \alpha_2 v + \alpha_3 w$$

gives that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Then

$$0 = u(\alpha_4 e_4 + \ldots + \alpha_{12} e_{12}) = \alpha_4 u + \alpha_5 v + \alpha_6 w$$

implies that $\alpha_4 = \alpha_5 = \alpha_6 = 0$. Likewise

$$0 = v(\alpha_7 e_7 + \ldots + \alpha_{12} e_{12}) = \alpha_7 u + \alpha_8 v + \alpha_9 w$$

giving $\alpha_7 = \alpha_8 = \alpha_9 = 0$. Finally

$$0 = w(\alpha_{10} e_{10} + \alpha_{12} e_{11} + \alpha_{12} e_{12}) = \alpha_{10} u + \alpha_{11} v + \alpha_{12} w$$

and thus $\alpha_{10} = \alpha_{11} = \alpha_{12} = 0$. $\square$

We use this example to construct a certain locally nilpotent Lie algebra over $\mathbb{F}$ of countably infinite dimension. For ease of notation it will be useful to introduce the following modified union of subsets of $\mathbb{N}$. We let

$$A \sqcup B = \begin{cases} A \cup B & (\text{if } A \cap B = \emptyset) \\ \emptyset & (\text{otherwise}) \end{cases}.$$

For each non-empty subset A of $\mathbb{N}$, we let W_A be a copy of the vector space W, where $W = \mathbb{F}u + \mathbb{F}v + \mathbb{F}w$. That is $W_A = \{z_A : z \in W\}$ with addition $z_A + t_A = (z+t)_A$. We then take the direct sum of these

$$W^* = \bigoplus_{\emptyset \neq A \subseteq \mathbb{N}} W_A$$

that we turn into a Lie algebra with multiplication $z_A \cdot t_B = (zt)_{A \sqcup B}$ when $z_A \in W_A$ and $t_B \in W_B$ that is then extended linearly on W^*. The interpretation here is that $z_\emptyset = 0$. Finally we extend this to a semidirect product with $\mathbb{F}x$

$$V^* = W^* \oplus \mathbb{F}x$$

induced from the action $z_A \cdot x = (zx)_A$.

Notice that V^* has basis $\{x\} \cup \{u_A, v_A, w_A : \emptyset \neq A \subseteq \mathbb{N}\}$ and that

$$u_A \cdot u_B = v_A \cdot v_B = w_A \cdot w_B = 0,$$
$$u_A \cdot x = 0,\ v_A \cdot x = 0,\ w_A \cdot x = u_A$$

and

$$u_A \cdot v_B = u_{A \sqcup B},\ v_A \cdot w_B = w_{A \sqcup B},\ w_A \cdot u_B = v_{A \sqcup B}.$$

Notice that any finitely generated subalgebra of V^* is contained in some $S = \langle x, u_{A_1}, \ldots, u_{A_r}, v_{B_1}, \ldots, v_{B_s}, w_{C_1}, \ldots, w_{C_t} \rangle$. From the fact that $zxx = 0$ for all $z \in V^*$ it follows that S is nilpotent of class at most $2(r+s+t)$. Hence V^* is locally nilpotent. The next aim is to find a group $G \leq \mathrm{GL}(V^*)$ containing $1 + \mathrm{ad}(x)$ where $1 + \mathrm{ad}(x)$ is a left 3-Engel element in G but where $\langle 1+\mathrm{ad}(x)\rangle^G$ is not nilpotent. The next lemma is a preparation for this.

Lemma 3.3. *The adjoint linear operator* $\mathrm{ad}(x)$ *on* V^* *satisfies:*

(i) $\mathrm{ad}(x)^2 = 0$.

(ii) $\mathrm{ad}(x)\,\mathrm{ad}(y)\,\mathrm{ad}(x) = 0$ *for all* $y \in V^*$.

Proof. (i) Follows from the fact that $x \cdot x = u_A \cdot x = v_A \cdot x = 0$ and $(w_A \cdot x) \cdot x = u_A \cdot x = 0$.
(ii) Follows from $w_A \cdot x \cdot u_B = u_A \cdot u_B = 0$, $w_A \cdot x \cdot v_B \cdot x = u_A \cdot v_B \cdot x = u_{A\sqcup B} \cdot x = 0$ and $w_A \cdot x \cdot w_B \cdot x = u_A \cdot w_B \cdot x = v_{A\sqcup B} \cdot x = 0$. □

Let y be any of the generators x, u_A, v_A, w_A. As $\mathrm{ad}(y)^2 = 0$ it follows that

$$(1+\mathrm{ad}(y))^2 = 1 + 2\,\mathrm{ad}(y) + \mathrm{ad}(y)^2 = 1.$$

Thus $1+\mathrm{ad}(y)$ is an involution in $\mathrm{GL}(V^*)$.

3.2.2 Our counterexample

Notice that for any $A, B \subseteq \mathbb{N}$, the pairs

$$(\mathrm{ad}(u_A), \mathrm{ad}(u_B)), (\mathrm{ad}(v_A), \mathrm{ad}(v_B)), (\mathrm{ad}(w_A), \mathrm{ad}(w_B))$$

consist of elements that commute. Thus the subgroups

$$\begin{aligned}
\mathcal{U} &= \langle 1+\mathrm{ad}(u_A) :\ A \subseteq \mathbb{N}\rangle, \\
\mathcal{V} &= \langle 1+\mathrm{ad}(v_A) :\ A \subseteq \mathbb{N}\rangle, \\
\mathcal{W} &= \langle 1+\mathrm{ad}(w_A) :\ A \subseteq \mathbb{N}\rangle,
\end{aligned}$$

are elementary abelian of countably infinite rank. We will be working with the group $G = \langle 1+\mathrm{ad}(x), \mathcal{U}, \mathcal{V}, \mathcal{W}\rangle$.

Lemma 3.4. *The following commutator relations hold in G:*

(i) $[1+\mathrm{ad}(u_A), 1+\mathrm{ad}(v_B)] = 1+\mathrm{ad}(u_{A\sqcup B})$.

(ii) $[1+\mathrm{ad}(v_A), 1+\mathrm{ad}(w_B)] = 1+\mathrm{ad}(w_{A\sqcup B})$.

(iii) $[1+\mathrm{ad}(w_A), 1+\mathrm{ad}(u_B)] = 1+\mathrm{ad}(v_{A\sqcup B})$.

(iv) $[1+\mathrm{ad}(u_A), 1+\mathrm{ad}(x)] = 1$.

(v) $[1+\mathrm{ad}(v_A), 1+\mathrm{ad}(x)] = 1$.

(vi) $[1+\mathrm{ad}(w_A), 1+\mathrm{ad}(x)] = 1+\mathrm{ad}(u_A)$.

Proof. (i) We have

$$\begin{aligned}
&[1+\mathrm{ad}(u_A), 1+\mathrm{ad}(v_B)] = \\
&\quad = (1+\mathrm{ad}(u_A))\cdot(1+\mathrm{ad}(v_B))\cdot(1+\mathrm{ad}(u_A))\cdot(1+\mathrm{ad}(v_B)) \\
&\quad = 1+\mathrm{ad}(u_A)\,\mathrm{ad}(v_B)+\mathrm{ad}(v_B)\,\mathrm{ad}(u_A) \\
&\quad = 1+\mathrm{ad}(u_Av_B) \\
&\quad = 1+\mathrm{ad}(u_{A\sqcup B}).
\end{aligned}$$

(ii) and (iii) are proved similarly. For (vi) we have

$$\begin{aligned}
[1+&\operatorname{ad}(w_A), 1+\operatorname{ad}(x)] \\
&= (1+\operatorname{ad}(w_A))\cdot(1+\operatorname{ad}(x))\cdot(1+\operatorname{ad}(w_A))\cdot(1+\operatorname{ad}(x)) \\
&= 1+\operatorname{ad}(w_A)\operatorname{ad}(x)+\operatorname{ad}(x)\operatorname{ad}(w_A)+\operatorname{ad}(x)\operatorname{ad}(w_A)\operatorname{ad}(x) \\
&= 1+\operatorname{ad}(w_A\cdot x) \\
&= 1+\operatorname{ad}(u_A).
\end{aligned}$$

Here in the 2nd last equality, we have used Lemma 3.3. Parts (iv) and (v) are proved similarly. $\square$

Remark 3.5. Notice that as V^* is locally nilpotent, it follows from Lemma 3.4 that G is locally nilpotent. The next proposition clarifies further the structure of G.

Proposition 3.6. *We have $G = \langle 1+\operatorname{ad}(x)\rangle \mathcal{U}\mathcal{V}\mathcal{W}$. Furthermore every element $g \in G$ has a unique expression $g = (1+\operatorname{ad}(x))^\epsilon rst$ with $\epsilon \in \{0,1\}$, $r \in \mathcal{U}$, $s \in \mathcal{V}$ and $t \in \mathcal{W}$.*

Proof. We first deal with the existence of such a decomposition. Suppose

$$g = l_0(1+\operatorname{ad}(x))l_1\cdots(1+\operatorname{ad}(x))l_n$$

where $l_0,\ldots,l_n$ are products of elements of the form $1+\operatorname{ad}(u_A), 1+\operatorname{ad}(v_A)$ and $1+\operatorname{ad}(w_A)$. From Lemma 3.4 we know that $(1+\operatorname{ad}(w_A))(1+\operatorname{ad}(x)) = (1+\operatorname{ad}(x))(1+\operatorname{ad}(w_A))(1+\operatorname{ad}(u_A))$ and $1+\operatorname{ad}(x)$ commutes with all products of the form $1+\operatorname{ad}(u_A)$ and $1+\operatorname{ad}(v_A)$. We can thus collect the

$(1+\mathrm{ad}(x))$'s to the left, starting with the leftmost occurrence. This may introduce more elements of the form $(1+\mathrm{ad}(u_A))$ but no new $1+\mathrm{ad}(x)$. We thus see that

$$g = (1+\mathrm{ad}(x))^n g_1 \cdots g_m$$

where each g_i is of the form $1+\mathrm{ad}(u_A)$, $1+\mathrm{ad}(v_A)$ or $1+\mathrm{ad}(w_A)$. This reduces our problem to the case when $g \in \langle \mathcal{U}, \mathcal{V}, \mathcal{W} \rangle$. Suppose

$$g = h_0(1+\mathrm{ad}(u_{A_1}))h_1 \cdots (1+\mathrm{ad}(u_{A_n}))h_n$$

where $h_0, \ldots, h_n$ are products of elements of the form $1+\mathrm{ad}(v_A)$ and $1+\mathrm{ad}(w_A)$. Suppose that the elements occurring in these products are $1+\mathrm{ad}(v_{A_{n+1}}), \ldots, 1+\mathrm{ad}(v_{A_{n+l}}), 1+\mathrm{ad}(w_{A_{n+l+1}}), \ldots, 1+\mathrm{ad}(w_{A_m})$. Using Lemma 3.4 we know that $(1+\mathrm{ad}(v_B))(1+\mathrm{ad}(u_A)) = (1+\mathrm{ad}(u_A))(1+\mathrm{ad}(v_B))(1+\mathrm{ad}(u_{A\sqcup B}))$ and that $(1+\mathrm{ad}(w_B))(1+\mathrm{ad}(u_A)) = (1+\mathrm{ad}(u_A))(1+\mathrm{ad}(w_B))(1+\mathrm{ad}(v_{A\sqcup B}))$. We can thus collect $1+\mathrm{ad}(u_{A_1}), \ldots, 1+\mathrm{ad}(u_{A_n})$ to the left. In doing so we may introduce new terms of the form $1+\mathrm{ad}(u_A)$, with A of the form $A_{i_1} \sqcup \cdots \sqcup A_{i_s}$, and $s \geq 2$. This shows that

$$g = (1+\mathrm{ad}(u_{A_1})) \cdots (1+\mathrm{ad}(u_{A_n}))g_1 \cdots g_m$$

where each g_j is of the form $1+\mathrm{ad}(v_B), 1+\mathrm{ad}(w_B)$ or $1+\mathrm{ad}(u_A)$, and A is a modified union of at least 2 sets from $\{A_1, \ldots, A_m\}$. We can repeat this procedure, collecting all the new $(1+\mathrm{ad}(u_A))$s. In doing so, we possibly introduce some new such elements but these will then be with an A that is a modified union of at least 3 sets from $\{A_1, \ldots, A_m\}$. Continuing like this the procedure will end after at most m steps as every modified union of $m+1$ sets from $\{A_1, \ldots, A_m\}$ will be trivial. We have thus seen that $g = rh$ with $r \in \mathcal{U}$ and $h \in \langle \mathcal{V}, \mathcal{W} \rangle$. We are now only left with the situation when $g \in \langle \mathcal{V}, \mathcal{W} \rangle$. Suppose

$$g = l_0(1 + \mathrm{ad}(v_{A_1}))l_1 \cdots (1 + \mathrm{ad}(v_{A_n}))l_n$$

where $l_0, l_1, \ldots, l_n$ are of the form $1 + \mathrm{ad}(w_A)$. As $(1 + \mathrm{ad}(w_B))(1 + \mathrm{ad}(v_A)) = (1 + \mathrm{ad}(v_A))(1 + \mathrm{ad}(w_B))(1 + \mathrm{ad}(w_{A \sqcup B}))$, we can now collect $1 + \mathrm{ad}(v_{A_1}), \ldots, 1 + \mathrm{ad}(v_{A_n})$ to the left and in doing so, all the new terms introduced will be of the form $1 + \mathrm{ad}(w_A)$. Thus $g = st$ with $s \in \mathcal{V}$ and $t \in \mathcal{W}$. This completes the existence part. We now want to show that such a decomposition is unique. Suppose

$$\begin{gathered}
(1 + \eta\ \mathrm{ad}(x))(1 + \alpha_1\ \mathrm{ad}(u_{A_1})) \cdots (1 + \alpha_r\ \mathrm{ad}(u_{A_r})) \\
(1 + \gamma_1\ \mathrm{ad}(v_{B_1})) \cdots (1 + \gamma_s\ \mathrm{ad}(v_{B_s}))(1 + \epsilon_1\ \mathrm{ad}(w_{C_1})) \cdots (1 + \epsilon_t\ \mathrm{ad}(w_{C_t})) \\
= \\
(1 + \tau\ \mathrm{ad}(x))(1 + \beta_1\ \mathrm{ad}(u_{A_1})) \cdots (1 + \beta_r\ \mathrm{ad}(u_{A_r})) \\
(1 + \delta_1\ \mathrm{ad}(v_{B_1})) \cdots (1 + \delta_s\ \mathrm{ad}(v_{B_s}))(1 + \nu_1\ \mathrm{ad}(w_{C_1})) \cdots (1 + \nu_t\ \mathrm{ad}(w_{C_t})),
\end{gathered}$$

where all these coefficients can be either 0 or 1.

Applying both sides to $w_{\mathbb{N}}$ we get

$$w_{\mathbb{N}} + \eta\, u_{\mathbb{N}} = w_{\mathbb{N}} + \tau\, u_{\mathbb{N}}$$

from which we get $\eta = \tau$. Applying both sides to x we get

$$(x + \epsilon_1 u_{C_1} + \cdots + \epsilon_r u_{C_t})(\epsilon_1\epsilon_2 v_{C_1 \sqcup C_2} + \cdots + \epsilon_{t-1}\epsilon_t v_{C_{t-1}\sqcup C_t})$$
$$(\epsilon_1\epsilon_2\epsilon_3 w_{C_1\sqcup C_2\sqcup C_3} + \cdots + \epsilon_{t-2}\epsilon_{t-1}\epsilon_t w_{C_{t-2}\sqcup C_{t-1}\sqcup C_t})$$
$$=$$
$$(x + \nu_1 u_{C_1} + \cdots + \nu_r u_{C_t})(\nu_1\nu_2 v_{C_1 \sqcup C_2} + \cdots + \nu_{t-1}\nu_t v_{C_{t-1}\sqcup C_t})$$
$$(\nu_1\nu_2\nu_3 w_{C_1\sqcup C_2\sqcup C_3} + \cdots + \nu_{t-2}\nu_{t-1}\nu_t w_{C_{t-2}\sqcup C_{t-1}\sqcup C_t})$$

from which we see that $\epsilon_1 = \nu_1, \ldots, \epsilon_t = \nu_t$. Thus

$$(1 + \alpha_1 \text{ ad}(u_{A_1}))\cdots(1 + \alpha_r \text{ ad}(u_{A_r}))(1 + \gamma_1 \text{ ad}(v_{B_1}))\cdots(1 + \gamma_s \text{ ad}(v_{B_s}))$$
$$= (1 + \beta_1 \text{ ad}(u_{A_1}))\cdots(1 + \beta_r \text{ ad}(u_{A_r}))(1 + \delta_1 \text{ ad}(v_{B_1})\cdots(1 + \delta_s \text{ ad}(v_{B_s})).$$

We can assume that $A_j \not\subseteq A_i$ and $B_j \not\subseteq B_i$ when $i < j$. Applying both sides to $u_{\mathbb{N}\setminus B_1}$ gives

$$u_{\mathbb{N}\setminus B_1} + \gamma_1 u_{\mathbb{N}} = u_{\mathbb{N}\setminus B_1} + \delta_1 u_{\mathbb{N}}$$

from which we see that $\gamma_1 = \delta_1$. Cancelling on both sides by $1+\gamma_1 \text{ ad}(v_{B_1})$ and then applying both sides to $u_{\mathbb{N}\setminus B_2}$ likewise gives $\gamma_2 = \delta_2$. Continuing in this manner gives $\gamma_1 = \delta_1, \ldots, \gamma_s = \delta_s$. We then have

$$(1+\alpha_1 \text{ ad}(u_{A_1}))\cdots(1+\alpha_r \text{ ad}(u_{A_r})) = (1+\beta_1 \text{ ad}(u_{A_1}))\cdots(1+\beta_r \text{ ad}(u_{A_r})).$$

A similar argument as before, applying both sides to $v_{\mathbb{N}\setminus A_1}, v_{\mathbb{B}\setminus A_2}, \ldots$ gives likewise $\alpha_1 = \beta_1, \ldots, \alpha_r = \beta r$. This finishes the proof. □

We are now ready to prove the main result of this chapter.

Theorem 3.7. *The element* $1 + \mathrm{ad}(x)$ *is a left* 3*-Engel element in* G*. However* $\langle 1 + \mathrm{ad}(x)\rangle^G$ *is not nilpotent.*

Proof. Let $g = h(1 + \mathrm{ad}(w_{A_1})) \cdots (1 + \mathrm{ad}(w_{A_n}))$ be an arbitrary element in G where $h \in \langle 1 + \mathrm{ad}(x)\rangle\mathcal{U}\mathcal{V}$. We want to show that g is left 3-Engel that is the same as showing the following

$$[(1 + \mathrm{ad}(x))^g,_2 1 + \mathrm{ad}(x)] = 1.$$

Notice first that if $y \in V$ then

$$\begin{aligned}(1 + \mathrm{ad}(y))^{1+\mathrm{ad}(w_A)} &= (1 + \mathrm{ad}(w_A))(1 + \mathrm{ad}(y))(1 + \mathrm{ad}(w_A)) \\ &= 1 + \mathrm{ad}(y) + \mathrm{ad}(yw_A).\end{aligned}$$

Notice that

$$(1 + \mathrm{ad}(x))^g = (1 + \mathrm{ad}(x))^{(1+\mathrm{ad}(w_{A_1}))\cdots(1+\mathrm{ad}(w_{A_n}))}$$

and a straightforward induction shows that

$$(1 + \mathrm{ad}(x))^g = 1 + \mathrm{ad}(y)$$

where

$$y = x + \sum_{1\le i\le n} u_{A_i} + \sum_{1\le i<j\le n} v_{A_i\sqcup A_j} + \sum_{1\le i<j<k\le n} w_{A_i\sqcup A_j\sqcup A_k}.$$

Since $\mathrm{ad}(x)\,\mathrm{ad}(y)\,\mathrm{ad}(x) = 0$ (see Lemma 3.3), the commutator of $(1 + \mathrm{ad}(x))^g$ with $1 + \mathrm{ad}(x)$ is

$$\begin{aligned}&(1+\mathrm{ad}(y))(1+\mathrm{ad}(x))(1+\mathrm{ad}(y))(1+\mathrm{ad}(x))\\ &= 1+\mathrm{ad}(y)\,\mathrm{ad}(x)+\mathrm{ad}(x)\,\mathrm{ad}(y)+\mathrm{ad}(y)\,\mathrm{ad}(x)\,\mathrm{ad}(y).\end{aligned}$$

Then

$$\begin{aligned}&[(1+\mathrm{ad}(x))^g,_2 1+\mathrm{ad}(x)] = ((1+\mathrm{ad}(y))(1+\mathrm{ad}(x))^4\\ &= ((1+\mathrm{ad}(y)\,\mathrm{ad}(x)+\mathrm{ad}(x)\,\mathrm{ad}(y)+\mathrm{ad}(y)\,\mathrm{ad}(x)\,\mathrm{ad}(y))^2 = 1\end{aligned}$$

using again the fact that $\mathrm{ad}(x)\,\mathrm{ad}(y)\,\mathrm{ad}(x) = 0$.

Then the normal closure of $1 + \mathrm{ad}(x)$ in G is though not nilpotent as for $A_i = \{i\}$ we have

$$\begin{aligned}&[1+\mathrm{ad}(w_{A_1}), 1+\mathrm{ad}(x), 1+\mathrm{ad}(w_{A_2}), 1+\mathrm{ad}(w_{A_3}), \ldots\\ &\qquad \ldots, 1+\mathrm{ad}(x), 1+\mathrm{ad}(w_{A_{2n}}), 1+\mathrm{ad}(w_{A_{2n+1}})] = 1+\mathrm{ad}(w_A),\end{aligned}$$

where $A = A_1 \sqcup \ldots \sqcup A_{2n+1} = \{1, 2, \ldots, 2n+1\}$. □

Our next aim is to show however that if we take any r conjugates $(1 + \mathrm{ad}(x))^{g_1}, \ldots, (1 + \mathrm{ad}(x))^{g_r}$ of $1 + \mathrm{ad}(x)$ in G, they generate a nilpotent subgroup of r-bounded class that grows linearly with r.

We first work in a more general setting. For each $e \in E$ and $\emptyset \neq A \subseteq \mathbb{N}$, let $e(A) \in \mathrm{End}(V^*)$ where

$$u_B e(A) = (ue)_{B \sqcup A}.$$

Then let $E^* = \langle \mathrm{ad}(x), e(A) : e \in E \text{ and } \emptyset \neq A \subseteq \mathbb{N} \rangle$. As V^* is locally nilpotent, one sees readily that the elements of E^* are nilpotent and thus $1 + E^*$ is a subgroup of $\mathrm{End}(V^*)$. We are going to see that $1 + E^*$ is of finite exponent.

Remark 3.8. Notice that $\mathrm{ad}(u_{\mathbb{N}}) = \mathrm{ad}(v_{\mathbb{N}}) = 0$.

Lemma 3.9. *The elements* $\mathrm{ad}(w_{\mathbb{N}})$ *and* $\{e_i(A) : 1 \leq i \leq 12, \emptyset \neq A \subset \mathbb{N}\} \cup \{e_1(\mathbb{N}), e_2(\mathbb{N}), e_3(\mathbb{N})\}$ *form a basis for* E^*.

Proof. One sees that these elements span E^* as a vector space in a similar way as in the proof of Lemma 3.2. We then show that these elements are linearly independent. Suppose

$$\epsilon \, \mathrm{ad}(x) + \sum_{i=1}^{12} \sum_{A} \epsilon_A^i e_i(A) = 0,$$

where only finitely many of the coefficients ϵ, ϵ_A^i are non-zero. Denote the left hand side by T. Then

$$0 = xT = \sum_A \epsilon_A^1 u_A + \sum_A \epsilon_A^2 v_A + \sum_A \epsilon_A^3 w_A$$

implying that $\epsilon_A^i = 0$ for all A and $i = 1, 2, 3$. Then

$$0 = u_{\mathbb{N} \setminus A} T = \sum_{B \subseteq A} \epsilon_B^4 u_{(\mathbb{N} \setminus A) \sqcup B} + \sum_{B \subseteq A} \epsilon_B^5 v_{(\mathbb{N} \setminus A) \sqcup B} + \sum_{B \subseteq A} \epsilon_B^6 w_{(\mathbb{N} \setminus A) \sqcup B}.$$

In particular $\epsilon_A^i = 0$ for all $A \neq \mathbb{N}$ and $i = 4, 5, 6$. We continue in a similar way. Next

$$0 = v_{\mathbb{N}\setminus A}T = \sum_{B\subseteq A} \epsilon_B^7 u_{(\mathbb{N}\setminus A)\sqcup B} + \sum_{B\subseteq A} \epsilon_B^8 v_{(\mathbb{N}\setminus A)\sqcup B} + \sum_{B\subseteq A} \epsilon_B^9 w_{(\mathbb{N}\setminus A)\sqcup B}$$

that shows that $\epsilon_A^i = 0$ for all A and $i = 7, 8, 9$. Finally

$$0 = w_{\mathbb{N}}T = \epsilon u_{\mathbb{N}}$$

giving $\epsilon = 0$ and

$$0 = w_{\mathbb{N}\setminus A}T = \sum_{B\subseteq A} \epsilon_B^{10} u_{(\mathbb{N}\setminus A)\sqcup B} + \sum_{B\subseteq A} \epsilon_B^{11} v_{(\mathbb{N}\setminus A)\sqcup B} + \sum_{B\subseteq A} \epsilon_B^{12} w_{(\mathbb{N}\setminus A)\sqcup B}$$

and $\epsilon_A^i = 0$ for all A and $i = 10, 11, 12$. This finishes the proof. $\square$

Corollary 3.10. *We have* $(1 + E^*)^{32} = 1$.

Proof. Let $\bar{E}$ be the subalgebra of E^* generated by all $e_i(A)$ where $1 \leq i \leq 12$ and $\emptyset \neq A \subseteq \mathbb{N}$. Let $f = \mathrm{ad}(x) + e \in E$ where $e \in \bar{E}$. Then $f^2 = \mathrm{ad}(x)^2 + e^2 + (e\,\mathrm{ad}(x) + \mathrm{ad}(x)e) = e^2 + (e\,\mathrm{ad}(x) - \mathrm{ad}(x)e)$. Since $\bar{E}$ is an ideal in the Lie algebra E^*, it is straightforward to see that $e\,\mathrm{ad}(x) - \mathrm{ad}(x)e \in \bar{E}$ and thus $f^2 \in \bar{E}$. It thus suffices to show that $\bar{E}^{16} = 0$, as then it will follow that $(E^*)^{32} = 0$ and therefore $(1 + e)^{32} = 1 + e^{32} = 1$ for all $e \in E^*$. Let $e = y_1 + \cdots + y_m$ be any element in $\bar{E}$ where $y_1, \ldots, y_m$ belong to the basis $\{e_i(A) : 1 \leq i \leq 12, \emptyset \neq A \subset \mathbb{N}\} \cup \{e_1(\mathbb{N}), e_2(\mathbb{N}), e_3(\mathbb{N})\}$ for $\bar{E}$ given in Lemma 3.9. As any product with a repeated term is 0 we see that e^{16}

is a sum of terms of the form

$$\sum_{\sigma \in S_{16}} f_{\sigma(1)} \cdots f_{\sigma(16)} \tag{3.1}$$

with $f_1, \ldots, f_{16} \in \{y_1, \ldots, y_m\}$. As $16 > 12$ some two of $f_1, \ldots, f_{16}$ must be of the same type. Without loss of generality we can assume that these are $f_{15} = e_i(A)$ and $f_{16} = e_i(B)$. Notice that the sum 3.1 splits naturally into $16!/2$ sums of pairs

$$\begin{aligned}
&\sum_{\sigma \in S_{16}} f_{\sigma(1)} \cdots f_{\sigma(16)} \\
&= \sum_{\sigma \in S_{14}} (e_i(A)e_i(B) + e_i(B)e_i(A)) f_{\sigma(1)} \cdots f_{\sigma(14)} + \cdots \\
&+ \sum_{\sigma \in S_{14}} f_{\sigma(1)} \cdots f_{\sigma(14)} (e_i(A)e_i(B) + e_i(B)e_i(A)),
\end{aligned}$$

one for each of the $\binom{16}{2}$ positions of the pair $(e_i(A), e_i(B))$ within the product. But for each such choice of positions the two elements in the pair have the same value and as the characteristic is 2, the sum of each pair is 0. Thus the sum in (3.1) is zero and we have shown that $\bar{E}^{16} = 0$. □

Proposition 3.11. *Any r-generator subgroup of $1 + E^*$ is nilpotent of r-bounded class.*

Proof. From Corollary 3.10 we know that $1 + E^*$ is of bounded exponent. The result thus follows from Zelmanov's solution to the Restricted Burnside Problem. □

Despite the fact that the normal closure of $1 + \mathrm{ad}(x)$ in G is not nilpotent, it turns out that the nilpotency class of the subgroup generated by any k conjugates grows linearly with respect to k. In order to see this we first introduce some more notation. Let $A_1, A_2, \ldots, A_k$ be any k subsets of $\mathbb{N}$. For each k-tuple $(i1, i2, \ldots, ir)$ of non-negative integers and each $e \in E$ we let

$$e^{(i1,\ldots,ir)} = \sum_{\substack{B_1 \subseteq A_1 \\ |B_1| = i1}} \cdots \sum_{\substack{B_k \subseteq A_k \\ |B_k| = ik}} e(B_1 \sqcup B_2 \sqcup \cdots \sqcup B_k).$$

Notice that

$$e^{(i1,\ldots,ir)} f^{(j1,\ldots,jr)} = \binom{i1+j1}{i1} \cdots \binom{ir+jr}{ir} (ef)^{(i1+j1,\ldots,ir+jr)}.$$

Now notice that $\binom{3+i}{3}$ is even for $i = 1, 2, 3$ and the same is true for $\binom{2+2}{2}$ and $\binom{1+1}{1}$. However $\binom{2+1}{2}$ is odd. From this it follows that

$$Q = \langle \mathrm{ad}(x), e^{(i1,\ldots,ir)} : e \in E,\, 0 \leq i1, \ldots, ir \leq 3,\; i1 + \cdots + ir \geq 1 \rangle$$

is a subalgebra of E^*.

Also, a non-zero product in Q can have at most $2r$ elements of the form $e^{(i1,\ldots,ir)}$ and as $\mathrm{ad}(x)^2 = 0$ we could then have at most $1 + 2r$ occurrences of $\mathrm{ad}(x)$ in a non-zero product. Thus $Q^{4r+2} = 0$. With these remarks in mind, we prove the last result of this chapter.

Proposition 3.12. *Let $(1+\mathrm{ad}(x))^{g_1}, \ldots, (1+\mathrm{ad}(x))^{g_r}$ be any r conjugates of $1+\mathrm{ad}(x)$ in G. Then the group generated by these conjugates is nilpotent of class at most $4r+2$.*

Proof. Take some r conjugates of $(1 + \mathrm{ad}(x))$ in G. Recall that each conjugate is of the from $(1 + \mathrm{ad}(x))^{(1+\mathrm{ad}(w_{C_1}))\cdots(1+\mathrm{ad}(w_{C_j}))}$. For ease of notation we will assume that each C_k is a singleton set. The following argument also works for the more general case. Let

$$A_1 = \{1, \ldots, k_1\},\ A_2 = \{k_1 + 1, \ldots, k_2\}, \ldots, A_r = \{k_{r-1} + 1, \ldots, k_r\}.$$

Then we have seen (see the proof of Theorem 3.7) that

$$\begin{aligned}
&(1 + \mathrm{ad}(x))^{(1+\mathrm{ad}(w_1))\cdots(1+\mathrm{ad}(w_{k_1}))} \\
&= 1 + \mathrm{ad}(x) + e_7^{(1,0,\ldots,0)} + e_4^{(2,0,\ldots,0)} + e_1^{(3,0,\ldots,0)} \\
&\vdots \\
&(1 + \mathrm{ad}(x))^{(1+\mathrm{ad}(w_{k_{r-1}+1}))\cdots(1+\mathrm{ad}(w_{k_r}))} \\
&= 1 + \mathrm{ad}(x) + e_7^{(0,\ldots,0,1)} + e_4^{(0,\ldots,0,2)} + e_1^{(0,\ldots,0,3)}.
\end{aligned}$$

In other words the r conjugates are all in $1+Q$. Hence if H is the subgroup of $\mathrm{GL}(V^*)$ generated by the r conjugates then

$$\gamma_{4r+2}(H) \leq \gamma_{4r+2}(1 + Q) \leq 1 + Q^{4r+2} = 1.$$

This completes the proof. $\square$

Part II

Groups of automorphisms of rooted trees

Chapter 4

Preliminaries

All our wisdom is stored in the trees.

Santosh Kalwar

In this chapter we define groups of automorphisms of spherically homogeneous rooted trees (in the following, $\operatorname{Aut}\mathcal{T}$ for short). We exhibit some important families of subgroups of $\operatorname{Aut}\mathcal{T}$ such as branch groups, (strongly) fractal groups, and just infinite groups.

4.1 Spherically homogeneous rooted trees

A tree is a connected graph with no cycles. A *rooted* tree is a tree with a designated vertex called the root. Let $\overline{m} = \{m_i\}_{i=1}^{N}$, be a sequence of natural numbers $m_i \geq 2$, where N can be either a natural number or infinity. The tree is called *homogeneous* because at every level there is the same number of descendants (but this could be different at different levels). In other words, all the vertices at distance n from the root have the same number m_{n+1} of immediate descendants. We will write $\mathcal{T}_{\overline{m}}$ to denote the *spherically homogeneous rooted tree* corresponding to the sequence $\overline{m}$.

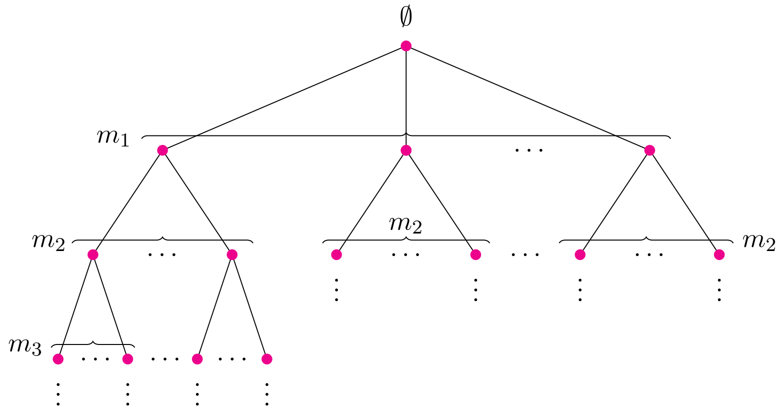


Figure 4.1: A spherically homogeneous rooted tree corresponding to the sequence $m_1, m_2, m_3, \ldots$

Let $\overline{m}$ be an infinite constant sequence $\overline{m} = m, m, \ldots$, with $m \geq 2$, then we say that $\mathcal{T}_{\overline{m}}$ is a *regular rooted tree* of degree m or a m-adic tree. For every positive integer m, there is a unique regular rooted tree of degree m (up to isomorphism). We will call it simply $\mathcal{T}_m$ and it looks as in Figure 4.2.

Let $X = \{x_1, x_2, \ldots, x_m\}$. A word of length n in X is an expression of the form $x_{i_1} x_{i_2} \ldots x_{i_n}$, where $i_1, i_2, \ldots, i_n \in \{1, \ldots, m\}$. Let X^n be the set of all words of length n and $X^* = \bigcup_{n \in \mathbb{N}} X^n$ the set of all word of finite lengths over the alphabet X. It is customary to denote a vertex of the tree as a word in X^*, where X^n represents all the vertices of the n-th level of $\mathcal{T}_m$. The free monoid generated by X is X^*, where we can multiply together

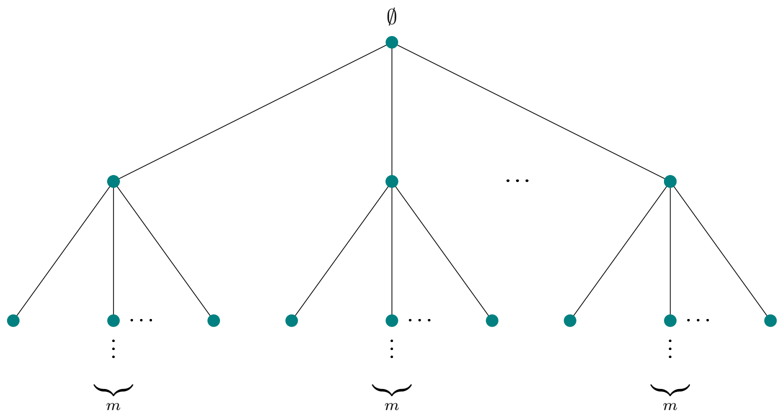


Figure 4.2: The m-adic tree $\mathcal{T}_m$

elements by juxtaposition. Indeed, if we consider $u \in X^n$ and $v \in X^m$, then $uv \in X^{n+m}$. Moreover, chosen a vertex u, the subtree of $\mathcal{T}_m$ hanging from u constitutes the set uX^* and it is a copy of $\mathcal{T}_m$. If we deal with the regular rooted tree, it is customary to choose the set $X = \{1, 2, \dots, m\}$.

More generally, let $\mathcal{T}_{\overline{m}}$ be a spherically homogeneous rooted tree, and let $\overline{X} = X_1, X_2, \dots$ be a sequence of alphabets such that $|X_i| = m_i$. A word of length n over $\overline{X}$ is an expression of the form $x_1 x_2 \dots x_n$ where $x_i \in X_i$ for all i. We denote with $\emptyset$ the empty word (the word of length 0), and the set of all words over $\overline{X}$ is $\overline{X}^*$.

For ease of notation and unless it is strictly necessary, we write $\mathcal{T}$ to denote a generic spherically homogeneous tree $\mathcal{T}_{\overline{m}}$. Furthermore, we usually consider as a generic sequence, a sequence of the form $\overline{m} = m, m_2, \dots$ (we omit the subindex in the first component).

4.2 The group $\operatorname{Aut}\mathcal{T}$ and some of its subgroups

If we consider bijective maps of the set X^* which preserve the root and incidence, i.e. automorphisms of $\mathcal{T}$, the set of all of these maps is a group with respect to the ordinary composition between functions, where $(fg)(u) = g(f(u))$. We call this set $\operatorname{Aut}\mathcal{T}$. For example we have the identity map or we can permute rigidly the subtrees hanging from the vertices $1, 2, \ldots, m$ using a permutation $\alpha \in \operatorname{Sym}(m)$. This is called the *rooted automorphism* corresponding to α. Suppose that $X_i = \{1, 2, \ldots, m_i\}$. Let g be in $\operatorname{Aut}\mathcal{T}$, and let u be a vertex in the level $i-1$. If g sends u to v then

$$\begin{aligned} u1 &\longmapsto v\alpha(1) \\ u2 &\longmapsto v\alpha(2) \\ &\vdots \\ u(m_i) &\longmapsto v\alpha(m_i), \end{aligned}$$

where $\alpha \in \operatorname{Sym}(m_i)$. Furthermore α is called the *label* of g at the vertex u and it is denoted by $g_{(u)}$. The *portrait* of g is the set of all labels of g. The portrait of an element is a way of describing automorphisms of $\operatorname{Aut}\mathcal{T}$. Indeed, given $g \in \operatorname{Aut}\mathcal{T}$, if its portrait is known, then we can calculate the image under g of any vertex of $\mathcal{T}$. More precisely,

$$g(i_1 i_2 \ldots i_n) = g_{(\emptyset)}(i_1) g_{(i_1)}(i_2) \ldots g_{(i_1 \ldots i_{n-1})}(i_n). \tag{4.1}$$

Conversely, given any portrait on the tree with elements in $\mathrm{Sym}(m_i)$ for every i, by formula (4.1) we can define an automorphism of $\mathcal{T}$.

For every $n \in \mathbb{N}$, we write $\mathcal{L}_n$ for the set of all vertices on the n-th level of $\mathcal{T}$. If u is a vertex of $\mathcal{T}$, the *vertex stabilizer* of u is denoted $\mathrm{st}(u)$ and it is defined as the subgroup of $\mathrm{Aut}\,\mathcal{T}$ consisting of all those automorphisms of $\mathrm{Aut}\,\mathcal{T}$ that fix the vertex u. The *n-th level stabilizer* is the normal finite index subgroup of $\mathrm{Aut}\,\mathcal{T}$ consisting of the automorphisms of $\mathcal{T}$ that fix all the vertices on the level n, that is

$$\mathrm{st}(n) = \{f \in \mathrm{Aut}\,\mathcal{T} \mid f(u) = u \ \forall u \in \mathcal{L}_n\}.$$

In particular, $\mathrm{st}(n) = \bigcap_{u \in \mathcal{L}_n} \mathrm{st}(u)$. Also, if $H \leq \mathrm{Aut}\,\mathcal{T}$, we define $\mathrm{st}_H(n) = H \cap \mathrm{st}(n)$.

Using the portrait one can see that if $f \in \mathrm{Aut}\,\mathcal{T}$, it is easy to establish when $f \in \mathrm{st}(n)$ because up to level n, all the labels of f must be 1.

Note that there is a chain of subgroups of $\mathrm{Aut}\,\mathcal{T}$

$$\mathrm{Aut}\,\mathcal{T} \supseteq \mathrm{st}(1) \supseteq \mathrm{st}(2) \supseteq \dots \supseteq \mathrm{st}(n) \supseteq \dots$$

where $\bigcap_{n \in \mathbb{N}} \mathrm{st}(n) = 1$. Hence $\mathrm{Aut}\,\mathcal{T}$ is a residually finite group (i.e. a group in which the intersection of all its normal subgroups of finite index is trivial).

In the following, we show that $\mathrm{Aut}\,\mathcal{T}$ is a profinite group. To this end, we first need to recall the definition of inverse system and inverse limit and to fix some notation.

Definition 4.1. An *inverse system* of groups over a directed set $\mathcal{S}$ is a family of groups $\{G_s\}_{s\in\mathcal{S}}$ with homomorphisms $\pi_{s_1,s_2} : G_{s_1} \longrightarrow G_{s_2}$ (for all $s_1 \geq s_2$) such that:

1. For any $s_1, s_2, s_3 \in \mathcal{S}$, $\pi_{s_2,s_3} \circ \pi_{s_1,s_2} = \pi_{s_1,s_3}$.

2. The map $\pi_{s,s}$ is the identity map of G_s for any $s \in \mathcal{S}$.

Definition 4.2. The *inverse limit* with respect to an inverse system

$$\Big(\{G_s\}_{s\in\mathcal{S}}, \{\pi_{s,s'} \mid s, s' \in \mathcal{S}\}\Big)$$

is the subgroup of the Cartesian product $\prod_{s\in\mathcal{S}} G_s$ of the tuples (g_s) such that when $s_1 \geq s_2$, we have $\pi_{s_1,s_2}(g_{s_1}) = (g_{s_2})$. We denote such group G by:

$$G = \varprojlim G_s = \varprojlim \{G_s\}_{s\in\mathcal{S}}.$$

A group G is said to be *profinite* if G is an inverse limit of an inverse system of finite groups. Additionally, if each G_s has order a power of p for some fixed prime p, then G is a *pro-p* group.

Let $\mathcal{T}_{[n]}$ be the tree with all finitely many vertices up to level n, that is $\mathcal{T}[n] = \mathcal{T}_{(m_1,\dots,m_n)}$. From the definition of n-th level stabilizer, it follows that $\operatorname{Aut}\mathcal{T}_{[n]} \cong \operatorname{Aut}\mathcal{T}/\operatorname{st}(n)$. Also, for any $n, m \in \mathbb{N}$ with $m \geq n$ we have the projections $\pi_{m,n} : \operatorname{Aut}\mathcal{T}_{[m]} \longrightarrow \operatorname{Aut}\mathcal{T}_{[n]}$ that form an inverse system. One can prove that the full group of automorphisms is the inverse limit of $\operatorname{Aut}\mathcal{T}_{[n]}$ and so is a profinite group. Thus we have

$$\operatorname{Aut}\mathcal{T} \cong \varprojlim_{n\to\infty} \operatorname{Aut}\mathcal{T}_{[n]}.$$

We conclude this section by defining another important subgroup of $\operatorname{Aut}\mathcal{T}$:

$$H_n = \{g \in \operatorname{Aut}\mathcal{T} \mid g_{(u)} = 1 \ \forall u \in \mathcal{L}_{\geq n}\},$$

where $\mathcal{L}_{\geq n}$ is the set of all vertices in a level $k \geq n$.

One can check that $\operatorname{Aut}\mathcal{T} = H_n \ltimes \operatorname{st}(n)$. Since $H_1 = \operatorname{Sym}(m)$, then in particular we have $\operatorname{Aut}\mathcal{T} = \operatorname{Sym}(m) \ltimes \operatorname{st}(1)$. Also, all the automorphisms in H_1 are rooted automorphisms of $\mathcal{T}$.

4.3 Other facts about $\operatorname{Aut}\mathcal{T}$

In the following, we denote by $\mathcal{T}_u$ the subtree hanging from the vertex u of the tree. Since for any two vertices u, u' of the same level $\mathcal{L}_n$ we have $\mathcal{T}_u \cong \mathcal{T}_{u'}$, we denote by $\mathcal{T}_{\langle n\rangle}$ any tree isomorphic to a subtree with root at a generic vertex of level n.

Let $f \in \operatorname{Aut}\mathcal{T}$ and suppose that $f(u) = u$, where $u \in X^*$. The *section* of f at u is the restriction of f to the subtree hanging from u, which is identified with $\mathcal{T}_{\langle n\rangle}$. It is denoted by f_u. Then f_u can be described as $f(uv) = uf_u(v)$.

More generally, without requiring the condition $f(u) = u$, the section of f at u is defined by $f(uv) = f(u)f_u(v)$. Below we collect some useful formulas for sections (these are also satisfied for labels).

Lemma 4.3. *Let $f, g \in \operatorname{Aut}\mathcal{T}$ and let u be a vertex of $\mathcal{T}$. Then the following hold:*

(i) $(fg)_u = f_u g_{f(u)}$.

(ii) $(f^{-1})_u = (f_{f^{-1}(u)})^{-1}$.

(iii) $(f^g)_u = (g_{g^{-1}(u)})^{-1} f_{g^{-1}(u)} g_{f(g^{-1}(u))}$.

Proof. We prove (i). Let v be a vertex of $\mathcal{T}$. We have

$$\begin{aligned}(fg)(uv) &= g(f(uv)) = g(f(u)f_u(v)) = g(f(u))g_{f(u)}(f_u(v)) \\ &= (fg)(u)(f_u g_{f(u)})(v).\end{aligned}$$

Then we obtain that $(fg)_u(v) = f_u g_{f(u)}(v)$ for every vertex of $\mathcal{T}$ and $(fg)_u = f_u g_{f(u)}$. Items (ii) and (iii) easily follow by using same arguments of (i). $\square$

Using sections, one can define the following isomorphism that will be of fundamental importance throughout the thesis. Let $n \in \mathbb{N}$, we define ψ_n as

$$\begin{aligned}\psi_n : \mathrm{st}(n) &\longrightarrow \operatorname{Aut} \mathcal{T}_{\langle n\rangle} \times \overset{m_1\cdots m_n}{\cdots} \times \operatorname{Aut} \mathcal{T}_{\langle n\rangle} \\ g &\longmapsto (g_u)_{u\in\mathcal{L}_n},\end{aligned}$$

for every $g \in \mathrm{st}(n)$.

Observe that by giving the image of $\psi_n(g)$, one can use ψ_n to define an element $g \in \mathrm{st}(n)$. We usually write ψ instead of ψ_1.

If the tree is regular of degree m, the situation is much easier. Indeed we have

$$\mathrm{st}(n) \cong \operatorname{Aut} \mathcal{T}_m \times \overset{m^n}{\cdots} \times \operatorname{Aut} \mathcal{T}_m.$$

This implies that $\operatorname{Aut} \mathcal{T}_m$ contains direct products $\operatorname{Aut} \mathcal{T}_m \times \overset{m^n}{\cdots} \times \operatorname{Aut} \mathcal{T}_m$, a fact that is not true for a general group (if we consider $\mathbb{Z}$, there does not exist a subgroup H of $\mathbb{Z}$ such that $H \cong \mathbb{Z} \times \mathbb{Z}$).

Notice that the "semidirect" structure of $\operatorname{Aut} \mathcal{T}_m$ and the fact that $\operatorname{st}(n) \cong \operatorname{Aut} \mathcal{T}_m \times \overset{m^n}{\cdots} \times \operatorname{Aut} \mathcal{T}_m$ imply that $\operatorname{Aut} \mathcal{T}_m$ can also be seen as an iterated permutational wreath product

$$\begin{aligned} \operatorname{Aut} \mathcal{T}_m &\cong \operatorname{Sym}(m) \ltimes \operatorname{st}(1) \cong \operatorname{Aut} \mathcal{T}_m \wr \operatorname{Sym}(m) \\ &\cong ((\operatorname{Sym}(m) \wr \dots) \wr \operatorname{Sym}(m)) \wr \operatorname{Sym}(m). \end{aligned}$$

From this it follows that an element $g \in \operatorname{Aut} \mathcal{T}_m$ can be written in the form $g = h\sigma$, where $h = (h_1, \dots, h_m) \in \operatorname{st}(1)$, and $\sigma \in \operatorname{Sym}(m)$.

Definition 4.4. Let $\mathcal{T}_p$ be the p-adic tree, for p a prime. If $\sigma = (1 \dots p)$, we define a standard *Sylow pro-p subgroup* of $\operatorname{Aut} \mathcal{T}_p$ as the subgroup $\Gamma \leq \operatorname{Aut} \mathcal{T}_p$ that is mapped isomorphically to

$$((\langle \sigma \rangle \wr \dots) \wr \langle \sigma \rangle) \wr \langle \sigma \rangle = ((C_p \wr \dots) \wr C_p) \wr C_p.$$

In this case, every element $g \in \Gamma$ can be written in the form $g = h\sigma^t$, for some $t \in \mathbb{Z}$ and $h \in \operatorname{st}(1)$ such that $\psi(h) \in \Gamma \times \overset{p}{\cdots} \times \Gamma$.

4.4 Self-similar and fractal groups

In this section, we provide some definitions and basic properties of self-similar and fractal groups. Before giving the definition of these groups, we need to recall what a spherically transitive group is.

Definition 4.5. A group $G \leq \operatorname{Aut} \mathcal{T}$ is *spherically transitive* (or *level* *transitive*) if it acts transitively on each level of the tree. Also, a spherically transitive group cannot be finite as the following proposition shows.

Proposition 4.6. *Let G be a spherically transitive group. Then G is infinite.*

Proof. If we fix a vertex $u_1 \in \mathcal{L}_n$, then for every vertex $u_i \in \mathcal{L}_n$, there exists $g_i \in G$ such that $g_i(u_1) = u_i$. Clearly, $g_i \neq g_j$ for any $i \neq j$. Then, since $\{g_1, \ldots, g_{|\mathcal{L}_n|}\} \subseteq G$, we have $|G| \geq |\mathcal{L}_n| = m_1 \cdots m_n$. Since this holds for every $n \in \mathbb{N}$, the only possibility is that G is infinite. $\square$

Definition 4.7. A group $G \leq \operatorname{Aut} \mathcal{T}$ is *self-similar* if $\psi(h) \in G \times \overset{m}{\cdots} \times G$ for all $g = h\sigma \in G$, with $h \in \operatorname{st}_G(1)$ and $\sigma \in \operatorname{Sym}(m)$.

Let $\psi_u : \operatorname{st}_G(u) \to \operatorname{Aut} \mathcal{T}$ be the homomorphism which sends each $g \in \operatorname{st}_G(u)$ to the section g_u. Neither this map nor the map ψ_n defined before need be surjective, as in the case of $\operatorname{Aut} \mathcal{T}$. Hence it is useful to give some definitions and considerations related to this problem.

Definition 4.8. Let $G \leq \operatorname{Aut} \mathcal{T}$ be a self-similar group. We say that:

- G is *fractal* (or *self-replicating*) if for all $u \in \mathcal{T}$, $\psi_u(\operatorname{st}_G(u)) = G$.
- G is *strongly fractal* if $\pi_i(\psi(\operatorname{st}_G(1))) = G$ for all $i = 1, \ldots, m$, where π_i is the projection onto the i-th component of $G \times \overset{m}{\ldots} \times G$. Equivalently, one can say that G is strongly fractal if for all $g \in G$, there exist $g_2, \ldots, g_m \in G$ such that $(g, g_2, \ldots, g_m) \in G$ (and similarly for all components other than the first).

Recall that Γ denotes the standard Sylow pro-p subgroup of $\operatorname{Aut} \mathcal{T}_p$. If

$G \leq \Gamma$, then G is fractal if and only if it is strongly fractal, by [53, Lemma 2.5]. We now prove a useful lemma for fractal groups.

Lemma 4.9. *Let G be a fractal group. Then G is spherically transitive if and only if G acts transitively on the first level of the tree $\mathcal{L}_1$.*

Proof. One implication is trivial. Suppose that G acts transitively on the first level of the tree. We want to prove that for any u and v in $\mathcal{L}_n$ there exists $g \in G$ such that $g(u) = v$. We argue by induction on n. The case $n = 1$ is trivially satisfied. Thus, suppose that G acts transitively on the level $\mathcal{L}_{n-1}$ and let us prove that G acts transitively also on $\mathcal{L}_n$. Take $u, v \in \mathcal{L}_n$, and write $u = xw$ and $v = x'w'$ where $x, x' \in \mathcal{L}_1$ and $w, w' \in \mathcal{L}_{n-1}$. By induction, there exists $h \in G$ such that $h(x) = x'$. If we consider an arbitrary $g \in G$ and we evaluate $hg(u)$, we have

$$hg(u) = hg(xw) = hg(x)(hg)_x(w) = g(h(x))g_{h(x)}(h_x(w)).$$

Again by induction hypothesis, since if $w \in \mathcal{L}_{n-1}$, then $h_x(w) \in \mathcal{L}_{n-1}$, there exists $f \in G$ such that $f(h_x(w)) = w'$. Moreover, the group G is fractal, that is $\psi_{h(x)}(\mathrm{st}_G(h(x))) = G$. Then, there exists $g \in \mathrm{st}_G(h(x))$ such that $g_{h(x)} = f$. Then, finally, we obtain

$$g(h(x))g_{h(x)}(h_x(w)) = h(x)f(h_x(w)) = x'w' = v,$$

as required. $\square$

4.5 Branch groups

In this section we define (weakly) branch groups and we show some important properties of these groups.

Branch groups, which were first defined by Grigorchuk [8] at the Groups at St. Andrews conference in Bath in 1997, are generalizations of the famous p-groups constructed by Grigorchuk himself, and Gupta and Sidki (see Section 5 for a detailed account on these groups). Despite their relatively recent introduction, branch groups have appeared in the literature in the past, without being explicitly defined. For instance, the class of branch groups contains one of the three classes of groups in John Wilson's famous characterization of just infinite groups [54]. This is one of the primary motivations for their study. Another important motivation comes from the remarkable properties that examples of these groups, such as the Grigorchuk and Gupta-Sidki groups mentioned above, possess. Among others, we mention the fact that (weakly) branch groups do not satisfy any law, a result proved by Abért [3].

Let G be a subgroup of $\operatorname{Aut}\mathcal{T}$. In the following, for simplicity we denote by ψ_n the map

$$\psi_n : \mathrm{st}_G(n) \longrightarrow \psi_n(\mathrm{st}_G(n)),$$

where $\psi_n(\mathrm{st}_G(n))$ need not be a direct product. Let $n \geq 0$ and $u \in \mathcal{L}_n$. The *rigid stabilizer* of the vertex u is the subgroup of G that consists of all those automorphisms that fix all vertices not having u as a prefix, i.e.

$$\mathrm{rst}_G(u) = \{g \in \mathrm{st}_G(n) \mid \ g_v = 1 \ \forall \ v \in \mathcal{L}_n \text{ where } u \neq v\}.$$

If u is a vertex of $\mathcal{L}_n$, then $\psi_n(\mathrm{rst}_G(u))$ has all coordinates equal to 1 except at the position u. As a consequence, if G is self-similar, for some subgroup R_u of $\mathrm{Aut}\,\mathcal{T}$, we have

$$\{1\} \times \cdots \times \{1\} \times R_u \times \{1\} \times \cdots \times \{1\} \subseteq \psi_n(\mathrm{st}_G(n)).$$

The *rigid stabilizer* of the n-th level is

$$\mathrm{rst}_G(n) = \langle \mathrm{rst}_G(u) \mid u \in L_n \rangle.$$

Clearly, $\mathrm{rst}_G(n) \subseteq \mathrm{st}_G(n)$ and we have the following decomposition:

$$\mathrm{rst}_G(n) = \prod_{u \in \mathcal{L}_n} \mathrm{rst}_G(u).$$

Obviously, if G is the whole $\mathrm{Aut}\,\mathcal{T}$ then the rigid stabilizer coincides with the n-th level stabilizer. However, this is not usually the case for arbitrary subgroups of $\mathrm{Aut}\,\mathcal{T}$.

Remark 4.10. Note that by Lemma 4.3, one can readily prove that

$$\mathrm{rst}_G(v)^g = \mathrm{rst}_G(g(v))$$

for every vertex of $\mathcal{T}$ and $g \in G$. Thus if G is spherically transitive, then each level rigid stabilizer $\mathrm{rst}_G(n)$ is a direct product of isomorphic subgroups for all $n \in \mathbb{N}$.

The subgroup $\mathrm{rst}_G(n)$ is highly important in the study of branch groups. Indeed, informally speaking, the subgroup $\psi_n(\mathrm{rst}_G(n))$ is the largest subgroup of $\psi_n(\mathrm{st}_G(n))$ which is a "geometric" direct product. We recall that if H_i are normal subgroups of G_i for $i = 1, \dots, n$, then $H_1 \times \cdots \times H_n$ is a normal subgroup of $G_1 \times \cdots \times G_n$. We call a subgroup of $G_1 \times \cdots \times G_n$ of the form $H_1 \times \cdots \times H_n$ is a *subproduct* of $G_1 \times \cdots \times G_n$.

Proposition 4.11. *Let $H \leq \mathrm{st}_G(n)$ and $\psi_n(H)$ be a subproduct of* $\mathrm{im}(\psi_n)$*. Then $H \leq \mathrm{rst}_G(n)$.*

Proof. Let H_u be such that

$$\{1\} \times \cdots \times H_u \times \cdots \times \{1\} \subseteq \psi_n(H) = \prod_{u \in \mathcal{L}_n} H_u.$$

Moreover, we can consider $J_u \subseteq \mathrm{st}_G(n)$ where

$$\psi_n(J_u) = \{1\} \times \cdots \times H_u \times \cdots \times \{1\},$$

then we have $\psi_n(J_u) \subseteq \psi_n(H)$. Now we have $\psi_n(H) = \prod_{u \in \mathcal{L}_n} H_u = \psi_n\left(\prod_{u \in \mathcal{L}_n} J_u\right)$ and since ψ_n is injective, $H = \prod_{u \in \mathcal{L}_n} J_u$. Moreover, if we consider $h \in J_u$, since $h_v = 1$ for any $v \in \mathcal{L}_n$ different from u, we have $J_u \subseteq \mathrm{rst}_G(u)$. Finally, $H = \prod_{u \in \mathcal{L}_n} J_u \subseteq \mathrm{rst}_G(n)$. This completes the proof. □

Definitions and basic properties

In this section we provide basic definitions and properties of branch groups.

Definition 4.12. Let $G \leq \mathrm{Aut}\,\mathcal{T}$ be a spherically transitive group.

- We say that G is a *branch group* if for all $n \geq 1$, the index of the rigid n-th level stabilizer in G is finite. In other words, for all $n \geq 1$,

$$|G : \mathrm{rst}_G(n)| < \infty.$$

Notice that this is the same as asking that $|\,\mathrm{st}_G(n) : \mathrm{rst}_G(n)| < \infty$.

- If G is self-similar, we say that G is a *weakly regular branch group* (respectively, regular branch group) if there exists a non-trivial subgroup K of $\mathrm{st}_G(1)$ (of finite index) such that $K \times \cdots \times K \subseteq \psi(K)$. If we want to emphasize the subgroup K, we say that G *is weakly regular branch over* K (regular branch group over K).

- Note that level stabilizers cannot be trivial. Indeed, they are subgroups of finite index of G and G is infinite by Proposition 4.6. However, this is not the case of rigid level stabilizers, since they might be trivial. We say that G is a *weakly branch group* if all of its rigid vertex stabilizers are non-trivial for every vertex of the tree.

We remark that since a branch group is spherically transitive and hence infinite (see Proposition 4.6), then any branch group is a weakly branch group. From now on, we will always assume that G is a subgroup of $\mathrm{Aut}\,\mathcal{T}$ acting spherically transitive on $\mathcal{T}$.
As the following result shows, one could have defined weakly branch groups by requiring that all its rigid stabilizer are infinite.

Proposition 4.13. *Let G be a weakly branch group. Then all its rigid vertex stabilizers of G are infinite.*

Proof. Since all rigid vertex stabilizers of G are non-trivial, consider a vertex u of the tree and $1 \neq g \in \mathrm{rst}_G(u)$. Let v be a vertex of $\mathcal{T}$ where g has non-trivial label, and consider a descendant of v, say w. Now again $\mathrm{rst}_G(w)$ must be non-trivial, and also $g \notin \mathrm{rst}_G(w)$ because it has non-trivial label at $v \notin \mathcal{T}_w$. So we can find a non-trivial automorphism $h \in \mathrm{rst}_G(u)$ different from g. We can continue in this way and we get infinitely many different elements inside $\mathrm{rst}_G(u)$. Hence $\mathrm{rst}_G(u)$ cannot be finite. □

We also remark that, in general, it is possible for subgroups of $\mathrm{Aut}\,\mathcal{T}$ to have all trivial rigid stabilizers. As the proposition below shows, an example of this case is the *adding machine*. The adding machine is the group $H = \langle x \rangle$ acting on the binary tree, where $x = (1, x)\sigma$ and σ is the rooted automorphism corresponding to the cycle $(1\ 2)$.

Proposition 4.14. *Let H be the adding machine. Then all its rigid stabilizers are trivial.*

Proof. Note first that $x^{2n} = (x^n, x^n)$, and $x^{2n+1} = (x^n, x^n)\sigma$. As a consequence, the portrait of x^{2n} is as follows: it has trivial label in $\mathcal{L}_{<n}$ and the permutation σ at every vertex in $\mathcal{L}_n$. Also, the stabilizer of the first level is $\langle x^2 \rangle$, and more generally $\mathrm{st}_H(n) = \langle x^{2n} \rangle$. Thus since no power of x belongs to $\mathrm{rst}_H(n)$, all rigid stabilizers of H are trivial. This concludes the proof. □

We conclude this section providing an important property of regular branch groups.

Lemma 4.15. *Let $K \leq G$. If G is regular branch over K, then G is branch.*

Proof. Write $K_n = \psi_n^{-1}(K \times \overset{m^n}{\cdots} \times K)$ and so $\psi_n(K_n) \leq K \times \overset{m^n}{\cdots} \times K$. First we show that also the converse inclusion is true in the latter. Note that

$$\psi_n(K_n) = (K \times \overset{m^n}{\cdots} \times K) \cap \psi_n(\mathrm{st}_G(n)) = K \times \overset{m^n}{\cdots} \times K.$$

In this case, we have

$$K \times \overset{m^n}{\cdots} \times K = \psi_n(K_n) \leq \psi_n(\mathrm{st}_G(n)) \leq G \times \overset{m^n}{\cdots} \times G,$$

where the last inclusion holds because G is self-similar. By Proposition 4.11, we have $K_n \leq \mathrm{rst}_G(n)$, and since ψ is injective, we obtain the following inequalities

$$\begin{aligned} |\,\mathrm{st}_G(n) : \mathrm{rst}_G(n)| &\leq |\,\mathrm{st}_G(n) : K_n| = |\psi_n(\mathrm{st}_G(n)) : \psi_n(K_n)| \\ &\leq |G \times \overset{m^n}{\cdots} \times G : K \times \overset{m^n}{\cdots} \times K| = |G : K|^{m^n} < \infty. \end{aligned}$$

Since from the expression above we have $|\,\mathrm{st}_G(n) : \mathrm{rst}_G(n)| < \infty$, then G is branch. This completes the proof. $\square$

4.6 Just infinite groups

In this section we define just infinite groups and we observe how the fact that the group is branch gives us some additional properties of the group. For further references one can see Chapter 4 of [14].

Definition 4.16. Let H be a group and $\mathfrak{X}$ a property of groups. We say that H is *just* $\mathfrak{X}$ if H has the property $\mathfrak{X}$ and every proper quotient of

H does not have the property $\mathfrak{X}$. In particular, a group G is *just infinite* if G is infinite and if all of its proper quotients are finite.

The following is a criterion for a regular branch group to be just infinite. We denote with K' the derived subgroup of K and we use the notation of K_n introduced in Lemma 4.15. Also, we for a group P, and a subgroup $R \leq P$, the (normal) core of R is the largest normal subgroup of P that is contained in R.

Theorem 4.17. [30, Theorem 4] *If G is regular branch over K, then G is just infinite if and only if $|K : K'| < \infty$. Furthermore if $|K : K'| < \infty$ then for every non-trivial normal subgroup N of G, there exits $n \in \mathbb{N}$ such that $K_n' \subseteq N$.*

Proof. First note that K' is of infinite index in K, and so also its core is of infinite index. Also, since weakly branch groups do not satisfy any law [3], they cannot have abelian subgroups of finite index. Indeed, if there exists an abelian subgroup of index m in G, then G satisfies the law $[x^m, y^m]$ for any $x, y \in G$. Hence, K' must be non-trivial. This implies that G is not just infinite.
Conversely, let $|K : K'| < \infty$. Let $g \in G$ such that $g \neq 1$ and write $H = \langle g \rangle^G$. We will prove that H contains K_n', for some n. It is enough to prove that G is just infinite. Indeed, in this case, we obtain

$$|G : H| \leq |G : K_n'| \leq |G : K| \cdot |K : K_n'| < \infty,$$

for some n, where the last inequality holds because we are assuming that G is regular branch over K and that $|K : K'| < \infty$. Now, suppose that g

is in $\mathrm{st}_G(n) \setminus \mathrm{st}_G(n+1)$, and write $\psi_n(g) = (*, \ldots, *, g_u, *, \ldots, *)$, where the vertex u has length n and is such that $g_u \notin \mathrm{st}_G(1)$. Then we can write $g_u = ha$, with $h \in \mathrm{st}_G(1)$ and $a \neq 1$ sending x to y, where x and y are different vertices of $\mathcal{L}_1$. If we consider an arbitrary element $\xi \in K$, we can associate $f \in \mathrm{st}_G(n+1)$ defined as follows:

$$\psi_n(f) = (1, \ldots, 1, f_u, 1, \ldots, 1) \text{ and } \psi(f_u) = (1, \ldots, 1, \xi, 1, \ldots, 1),$$

with ξ in position x. Notice that this choice is possible since G is regular branch over K and then $K_{n+1} = \psi_{n+1}^{-1}(K \times \overset{m^{n+1}}{\cdots} \times K) \subseteq \mathrm{st}_G(n+1)$, as shown in the proof of Lemma 4.15. The commutator $[g, f]$ belongs to H because H is a normal subgroup. Also, $[g, f]$ belongs to $\mathrm{st}_G(n+1)$, and we have

$$\begin{aligned} \psi_n([g, f]) &= (1, \ldots, 1, [g_u, f_u], 1, \ldots, 1) \\ &= (1, \ldots, 1, a^{-1}h^{-1}f_u^{-1}haf_u, 1, \ldots, 1). \end{aligned}$$

Then, in the $(n+1)$-st level, we have

$$\psi_{n+1}([g, f]) = (1, \ldots, 1, \xi^{-h_x}, 1, \ldots, 1, \xi, 1, \ldots, 1),$$

where ξ^{-h_x} is in position y and ξ in position x. Consider another element $\eta \in K$ and define $l \in \mathrm{st}_G(n+1)$ by the following expression

$$\psi_{n+1}(l) = (1, \ldots, 1, \eta, 1, \ldots, 1).$$

Then, $[[g,f],l] = (1,\ldots,1,[\xi,\eta],1,\ldots,1)$. Since ξ and η are arbitrary elements of K, we have

$$\{1\} \times \cdots \times \{1\} \times K' \times \{1\} \times \cdots \times \{1\} \leq \psi_{n+1}(\mathrm{st}_H(n+1)).$$

As a consequence

$$K' \times \overset{m^{n+1}}{\cdots} \times K' \leq \psi_{n+1}(\mathrm{st}_H(n+1))$$

because H is normal in G and the action of G is transitive. Also,

$$K' \times \overset{m^{n+1}}{\cdots} \times K' = (K \times \overset{m^{n+1}}{\cdots} \times K)' = \psi_{n+1}(K_{n+1})' = \psi_{n+1}(K'_{n+1}).$$

Hence we obtain $K'_{n+1} \subseteq \mathrm{st}_H(n+1)$. This concludes the proof. $\square$

As a consequence, a finitely generated torsion regular branch group is just infinite. Also, the study of branch groups is motivated by the following theorem, whose proof can be found in [30].

Theorem 4.18 (Proposition 3, [30])**.** *If G is a finitely generated infinite group, then G can be mapped onto a just infinite group.*

Hence, every finitely generated infinite group has a just infinite quotient. However, there exist infinitely generated groups that do not have just infinite quotients, for instance the additive group of rational numbers $\mathbb{Q}$.

Chapter 5

Some important groups of automorphisms

In this chapter we provide some important examples of groups of automorphisms acting on a regular rooted tree. In particular, we define the (first) Grigorchuk group, the (multi-)GGS-groups, the Basilica group, the lamplighter group, the group of finitary automorphisms, the Hanoi Tower group, and the Brunner-Sidki-Vieira group. About the latter, in Section 5.5.1, we give a different proof of the fact that it has torsion-free abelianization.

5.1 The Grigorchuk group

The (first) Grigorchuk group $\mathfrak{G}$, introduced by Grigorchuk [24] in 1980 is a subgroup of $\operatorname{Aut} \mathcal{T}_2$. The Grigorchuk group possesses a lot of interesting properties. For instance, it was the first group shown to be of intermediate growth [25], it is a counterexample to the General Burnside Problem and it is amenable but not elementary amenable [29]. For more information about growth of groups and on amenability see [41] and [46], respectively. The group is generated by $a = (1\ 2)$ that is the swap at the root, and other three elements $b, c, d \in \mathrm{st}(1)$ defined recursively as follows

$$\psi(b) = (a, c), \quad \psi(c) = (a, d), \quad \psi(d) = (1, b),$$

whose portrait is represented below:

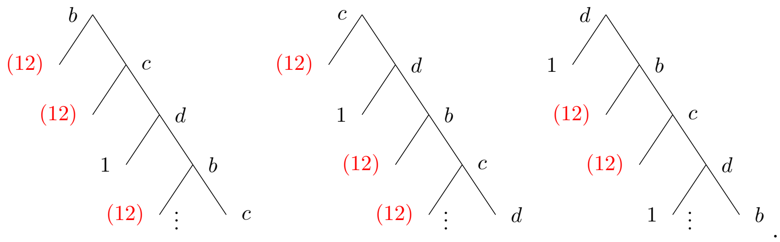


Figure 5.1: Portrait of the elements b, c and d of the Grigorchuk group

For further information and for the proof of the following theorem, one can see [13, Chapter 7].

Here we collect some properties of $\mathfrak{G}$ that we will use later in this thesis.

Theorem 5.1. *The group $\mathfrak{G}$ satisfies the following properties:*

(i) $\mathfrak{G}$ *is strongly fractal.*

(ii) $\mathfrak{G}$ *is regular branch over* $K = \langle [a, b] \rangle^{\mathfrak{G}}$.

(iii) $\mathfrak{G}$ *is a 2-group.*

(iv) $\mathfrak{G}$ *is just infinite.*

5.2 The GGS-groups

Let p be an odd prime, and $\mathbf{e} = (e_1, \dots, e_{p-1})$ a vector where $e_i \in \{1, \dots, p\}$ and such that not all the e_i's are 0. The GGS-group $\mathcal{G}_{\mathbf{e}}$, named after Grigorchuk, Gupta, and Sidki, is the group generated by the two automorphisms $a, b \in \operatorname{Aut} \mathcal{T}_p$, where a is the rooted automorphism corresponding to the cycle $(1 \dots p)$, and $b \in \operatorname{st}(1)$ is

$$\psi(b) = (a^{e_1}, \dots, a^{e_{p-1}}, b).$$

For example the famous Gupta-Sidki p-group is a GGS-group whose defining vector $\mathbf{e}$ is $\mathbf{e} = (1, -1, 0, \dots, 0)$.

The group $\mathcal{G}_{\mathbf{e}}$ is strongly fractal for any vector $\mathbf{e}$, but other properties depend on the choice of $\mathbf{e}$ (see [19]). For instance, $\mathcal{G}_{\mathbf{e}}$ is just infinite and branch if and only if $\mathbf{e}$ is not constant. If the defining vector is constant the situation is much different. For the rest of the section we consider $\mathcal{G}_{\mathbf{e}}$ in the case when $\mathbf{e}$ is constant, i.e. $e_1 = \dots = e_{p-1} = n$ for some non-zero n. We may assume that $n = 1$ since proportional non-zero vectors define the same GGS-group, and for ease of notation we let $\mathcal{G} = \mathcal{G}_{\mathbf{e}}$. We collect here some results from [19] which will be used later in Chapter 8. Before doing this, we need to fix some notation. Write $y_0 = ba^{-1}$ and $y_i = y_0^{a^i}$ for every $i = 1, \dots, p$, and let K be the normal closure of y_0 in $\mathcal{G}$.

Lemma 5.2. [19, Lemma 4.2] *Let $\mathcal{G}$ and K be as before. Then the following hold:*

(i) $|\mathcal{G} : \mathcal{G}'| = p^2$.

(ii) $|\mathcal{G} : K| = p$.

(iii) $K = \langle y_0, \dots, y_p \rangle$.

(iv) $\mathcal{G}$ *is weakly regular branch over its derived subgroup* K'.

We remark that (i) of the Lemma above is satisfied by all GGS-groups.

5.3 Multi-GGS groups

Given an odd prime p and a non-trivial subspace $\mathbf{E}$ of $\mathbb{F}_p^{p-1}$, we define the multi-GGS group $G_{\mathbf{E}}$ (again, GGS stands for Grigorchuk, Gupta, and Sidki) as the following subgroup of $\operatorname{Aut}\mathcal{T}_p$. The group $G_{\mathbf{E}}$ is generated by the rooted automorphism a of order p corresponding to the p-cycle $(1\ 2\ \dots\ p)$, and by the elementary abelian p-subgroup B consisting of all automorphisms $b_{\mathbf{e}}$, with

$$\mathbf{e} = (e_1, \dots, e_{p-1}) \in \mathbf{E},$$

defined recursively via

$$b_{\mathbf{e}} = (a^{e_1}, \dots, a^{e_{p-1}}, b_{\mathbf{e}}). \tag{5.1}$$

If $\dim \mathbf{E} = 1$ then $G_{\mathbf{E}}$ is simply a GGS group. Multi-GGS groups are usually presented by giving a basis $(\mathbf{e}_1, \dots, \mathbf{e}_r)$ of $\mathbf{E}$ and defining $b_i \in B$ from $\mathbf{e}_i$ as in (5.1) for each $i = 1, \dots, r$, so that $G_{\mathbf{E}} = \langle a, b_1, \dots, b_r \rangle$. We refer the reader to [4] for general facts about multi-GGS groups. Multi-GGS groups are infinite and provide a wealth of examples giving a negative answer to the General Burnside Problem. In general, a multi-GGS group is periodic if and only if $\mathbf{E}$ is contained in the hyperplane of $\mathbb{F}_p^{p-1}$ given by the equation $e_1 + \cdots + e_{p-1} = 0$ [4, Theorem 3.2]. On the other hand, multi-GGS groups are known to be branch unless $\mathbf{E} = \langle(1, \ldots, 1)\rangle$ consists of constant vectors, in which case it is weakly branch [4, Proposition 3.7].

5.4 The Basilica group

The Basilica group $\mathcal{B}$ was introduced by Grigorchuk and Zuk in 2002 [28]. The name *Basilica* comes from the Schreier graph of $\mathcal{B}$ (Figure 5.2 below), since it has the shape of the basilica of San Marco in Venice together with its reflection in the water, see [12].

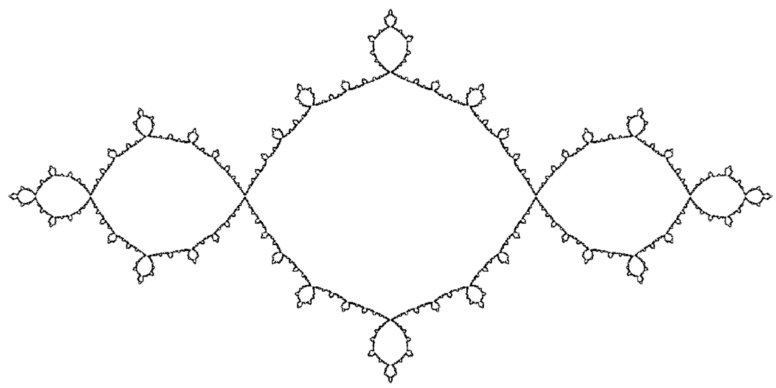


Figure 5.2: The Schreier graph of the Basilica group $\mathcal{B}$

The Basilica group is the subgroup of $\operatorname{Aut} \mathcal{T}_2$ generated by the two automorphisms a and b defined as

$$a = (1, b) \quad \text{and} \quad b = (1, a)\sigma,$$

where σ is the rooted automorphism of $\mathcal{T}_2$ that corresponds to the permutation $(1\ 2)$. Here, we abuse notation by writing $a = (1, b)$ instead of $\psi(a) = (1, b)$, and similarly for the element $(1, a)$. The portrait of these two automorphisms is represented below.

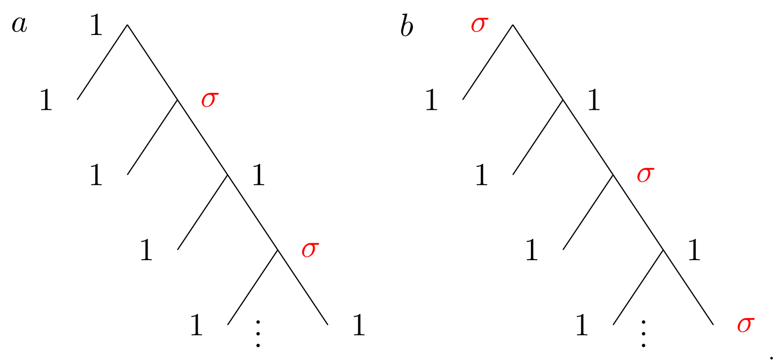


Figure 5.3: The portrait of the generators of the Basilica group $\mathcal{B}$

Here we collect some properties of $\mathcal{B}$ that can be found in [28]. We recall that for a group G, its abelianization is the quotient of the group by its commutator subgroup G'.

Theorem 5.3. *Let $\mathcal{B}$ be the Basilica group. Then:*

(i) $\mathcal{B}$ is strongly fractal.

(ii) $\mathcal{B}$ is weakly regular branch over its derived subgroup $\mathcal{B}'$.

(iii) $\mathcal{B}$ is torsion-free.

(iv) $\mathcal{B}$ *is just non-solvable.*

(v) $\mathcal{B}$ *has torsion-free abelianization.*

5.5 The Brunner-Sidki-Vieira group

The *Brunner-Sidki-Vieira group* $\mathcal{S}$ is the group of automorphisms of $\mathcal{T}_2$ generated by

$$c = (1, c^{-1})\sigma \quad \text{and} \quad d = (1, d)\sigma,$$

where σ is the rooted automorphism of $\mathcal{T}_2$ corresponding to the cycle (1 2). Again, we have omitted the map ψ in the definition of c and d.
In the following we present some relevant properties of $\mathcal{S}$ that can be found in [10] and also in [49].

Theorem 5.4. *Let* $\mathcal{S}$ *be the Brunner-Sidki-Vieira group. Then:*

(i) $\mathcal{S}$ *is strongly fractal.*

(ii) $\mathcal{S}$ *is weakly regular branch over its derived subgroup* $\mathcal{S}'$*.*

(iii) $\mathcal{S}$ *is torsion-free.*

(iv) $\mathcal{S}$ *is just non-solvable.*

(v) $\mathcal{S}$ *has torsion-free abelianization.*

In the subsection below, we present a different proof of the fact that $\mathcal{S}$ has torsion-free abelianization.

5.5.1 Abelianization of the Brunner-Sidki-Vieira group

Throughout the section, we let $\mathcal{M}$ be the free monoid $\mathcal{M}(c,d)$ generated by c and d. First observe that any $g \in \mathcal{S}$ can be written uniquely as $g = h\sigma^{\varepsilon}$, for some $h \in \mathrm{st}(1)$ and some $\varepsilon = 0, 1$. The element h may not belong to $\mathcal{S}$, but $\psi(h) = (h_1, h_2)$ satisfies $h_1, h_2 \in \mathcal{S}$. We call h_1 and h_2 the *components of the stable part of* h. Let $w \in \mathcal{M}$, and let w_1 and w_2 be the components of the stable part of $\pi(w)$, where π is the natural projection of the monoid to the group. There are natural words u_1 and u_2 such that $\pi(u_1) = w_1$ and $\pi(u_2) = w_2$. These can be obtained as follows: if $|w| = 1$ and $w = c$, then $u_1 = 1$ and $u_2 = c^{-1}$. If $w = d$, then $u_1 = 1$ and $u_2 = d$. Inductively, if $w = w'c$ for some subword w' and if $\pi(w') \in \mathrm{st}_{\mathcal{S}}(1)$, then $u_1 = w'_1$ and $u_2 = w'_2 c^{-1}$. If $\pi(w') \notin \mathrm{st}_{\mathcal{S}}(1)$, then $u_1 = w'_1 c^{-1}$ and $u_2 = w'_2$ (this comes from the fact that, in this case, $\pi(w')$ ends with σ, and from the computation $\psi(\sigma c \sigma) = (c^{-1}, 1)$). The remaining cases are treated similarly.
We will denote u_1 and u_2 by $\rho_1(w)$ and $\rho_2(w)$. Notice that then $\pi \circ \rho_1(w) = w_1$ and $\pi \circ \rho_2(w) = w_2$. We also let $n_c(w)$ and $n_d(w)$ be the exponent sum of c and d in w, respectively.

Lemma 5.5. *The following hold for any $w \in \mathcal{M}$:*

(i) $|w| = |\rho_1(w)| + |\rho_2(w)|$.

(ii) $n_c(w) = n_c(\rho_1(w)) + n_c(\rho_2(w)), \quad n_d(w) = n_d(\rho_1(w)) + n_d(\rho_2(w))$.

Proof. The results follow immediately by induction on the length of w. □

Lemma 5.6. *Suppose that $|w| \geq 2$. Then $|(\rho_i \circ \rho_j)(w)| < |w|$ for all $i = 1, 2$, $j = 1, 2$.*

Proof. The case $|w| = 2$ is immediate by direct computations. Now assume that $|w| \geq 3$, so that $w = uv$ for some subwords u and v, with $|v| = 2$. One has that $\rho_1(w) = \rho_1(u)\rho_1(v)$, $\rho_2(w) = \rho_2(u)\rho_2(v)$ or $\rho_1(w) = \rho_1(u)\rho_2(v)$, $\rho_2(w) = \rho_2(u)\rho_1(v)$, depending on whether $\pi(u)$ and $\pi(v)$ belong to $\mathrm{st}_S(1)$ or not. More generally, for both $j = 1, 2$, $\rho_j(w) = \rho_j(u)\rho_{j-\varepsilon}(v)$, where $\varepsilon \in \{0, 1\}$, and $\rho_0(v)$ is defined to be $\rho_2(v)$. Now, for $i = 1, 2$ and $j = 1, 2$, we have

$$|\left(\rho_i \circ \rho_j\right)(w)| = |\rho_i(\rho_j(w))| = |\rho_i(\rho_j(u)\rho_{\varepsilon-j}(v))|. \tag{5.2}$$

Fix j, and let $u' = \rho_j(u)$, $v' = \rho_{j-\varepsilon}(v)$, $w' = u'v'$. Repeating the arguments above, $\rho_1(w') = \rho_1(u')\rho_{\varepsilon'}(v')$ and $\rho_2(w') = \rho_2(u')\rho_{1-\varepsilon'}(v')$ for some $\varepsilon' = 0, 1$. Plugging these into (5.2), we have

$$\begin{aligned}
|\left(\rho_i \circ \rho_j\right)(w)| &= |\rho_i(\rho_j(u)\rho_{j-\varepsilon}(v))| = |\rho_i(w')| \\
&= |\rho_i(\rho_j(u))\rho_{i-\varepsilon'}(\rho_{j-\varepsilon}(v))| \\
&= |\left(\rho_i \circ \rho_j\right)(u)| + |\left(\rho_{i-\varepsilon'} \circ \rho_{j-\varepsilon}\right)(v)| \\
&< |\left(\rho_i \circ \rho_j\right)(u)| + |v| \qquad (5.3) \\
&\leq |\rho_j(u)| + |v| \leq |u| + |v| = |w| \qquad (5.4)
\end{aligned}$$

where (5.3) has been obtained using that the lemma is true for words v of length 2, and (5.4) is a consequence of (i) of Lemma 5.5 applied twice. □

Lemma 5.7. *Suppose $\pi(w) = 1$, i.e. suppose that w represents the identity element of $\mathcal{S}$. Then $n_c(w) = n_d(w) = 0$.*

Proof. We proceed by induction on the number of letters $|w|$ of w. If $|w| \leq 2$ the statement is clear. Suppose the lemma holds when $|w| \geq 2$. By taking

appropriate conjugates, we can assume that $w = uc^{\varepsilon}d$, for some subword u and some $\varepsilon = -1, 1$. Suppose $\varepsilon = 1$. Since $c^{\varepsilon}d \in \mathrm{st}_{\mathcal{S}}(1)$, then $u \in \mathrm{st}_{\mathcal{S}}(1)$ where $\psi(c^{\varepsilon}d) = (d, c^{-1})$, and $\rho_1(w) = \rho_1(u)d$, $\rho_2(w) = \rho_2(u)c^{-1}$.
Since $\pi(w) = 1$, then $\pi(\rho_1(w)) = \pi(\rho_2(w)) = 1$. By (i) of Lemma 5.5, $|\rho_1(u)d| + |\rho_2(u)c^{-1}| = |w|$. Now, if $|\rho_1(u)d| < |w|$ and $|\rho_2(u)c^{-1}| < |w|$, by induction,

$$
\begin{aligned}
n_c(\rho_1(u)d) &= n_d(\rho_1(u)d) = 0, \\
n_c(\rho_2(u)c^{-1}) &= n_d(\rho_2(u)c^{-1}) = 0. \qquad (5.5)
\end{aligned}
$$

Using (ii) of Lemma 5.5, and (5.5), we have

$$
\begin{aligned}
n_c(w) &= n_c(\rho_1(u)d) + n_c(\rho_2(u)c^{-1}) = 0, \\
n_d(w) &= n_d(\rho_1(u)d) + n_d(\rho_2(u)c^{-1}) = 0,
\end{aligned}
$$

as required. Now suppose that $|\rho_2(u)c^{-1}| = 0$ (the other case follows similarly). Then $\rho_2(u)c^{-1} = 1$ and $\psi(\pi(w)) = (\pi(\rho_1(u)d), 1)$, with $|w| = |\rho_1(u)d| \geq 2$. Since we must have $\pi(\rho_1(u)d) = 1$, we can repeat the same arguments used so far with $\rho_1(u)d$. By Lemma 5.6, it is not possible that $\rho_1(\rho_1(u)d)$ or $\rho_2(\rho_1(u)d)$ is trivial (equivalently, both words must have length strictly smaller than $|\rho_1(u)d| = |w|$). Hence, by induction,

$$
n_c(\rho_1(u)d) = n_d(\rho_2(u)d) = 0,
$$

and since $w = \rho_1(u)d$, we have $n_c(w) = n_c(\rho_1(u)d) = 0$, and $n_d(w) = n_d(\rho_1(u)) = 0$. ☐

Before proving the main result of this subsection, we recall that given a free abelian group G, its rank is the smallest cardinality of a generating set for G.

Theorem 5.8. *The abelianization of $\mathcal{S}$ is the free abelian group of rank 2 generated by $c\mathcal{S}'$ and $d\mathcal{S}'$.*

Proof. Clearly, $\mathcal{S}/\mathcal{S}'$ is generated by $c\mathcal{S}'$ and $d\mathcal{S}'$. To prove the lemma, it suffices to show that if $c^i d^j \mathcal{S}' = \mathcal{S}'$ then $i = j = 0$. Indeed, suppose $c^i d^j \mathcal{S}' = \mathcal{S}'$ holds. There must exist a word $u \in \mathcal{M}(c,d)$ that represents an element from $\mathcal{S}'$, and $c^i d^j = u$ in $\mathcal{S}$. Since we are taking elements in $\mathcal{S}'$, i.e. commutator words, we have $n_c(u) = n_d(u) = 0$, and, consequently, $n_c(u^{-1}) = n_d(u^{-1}) = 0$. Since $\pi(c^i d^j u^{-1}) = 1$, Lemma 5.7 implies that $n_c(c^i d^j u^{-1}) = 0$ and $n_d(c^i d^j u^{-1}) = 0$. But then

$$n_c(c^i d^j u^{-1}) = n_c(c^i d^j) + n_c(u^{-1}) = i + 0 = 0$$

and, similarly, $n_d(c^i d^j u^{-1}) = j = 0$. This completes the proof. □

5.6 The lamplighter group

The lamplighter group $\mathcal{L}$ is the wreath product $C_2 \wr C_\infty$. It is given by the presentation

$$\mathcal{L} = \langle a, t \mid a^2, [a^{t^n}, a^{t^m}] \text{ for } m, n \in \mathbb{Z} \rangle.$$

The name *lamplighter* comes from viewing the group as acting on a doubly infinite sequence of street lamps $\ldots, l_{-2}, l_{-1}, l_0, l_1, l_2, \ldots$. Each of them can be `on` or `off`, and a lamplighter standing at some lamp l_k. The generator t increments k, so that the lamplighter moves to the next lamp (t^{-1} decrements k), while the generator a means that the state of lamp l_k is changed (from off to on or from on to off).

The lamplighter group $\mathcal{L}$ is a metabelian self-similar group of exponential growth [27]. It can be seen also as a group acting on a binary tree defined by $a = (a, a\sigma)$ and the rooted automorphism σ corresponding to the cycle $(1\ 2)$.

5.7 The Hanoi Tower group

The Hanoi Tower game is a mathematical puzzle which consists of disks of different sizes and n pegs. The goal is to move the entire stack to another peg, following some rules:

- One disk can be moved at a time.
- Each move consists of taking the upper disk from one of the stacks and placing it on top of another or on an empty peg.
- No disk may be placed on top of a smaller disk.

Let the number of pegs be 3. A word in $X = \{1, 2, 3\}$ is a configuration of the disks and the length of the word is the number of disks. For example the configuration 231123, being of length 6, is a configuration of the game with 6 disks. Also, it means that the first smaller disk is on the second peg (231123), that the second smaller is on the third (231123), and so on. Then the goal is to send $11.\overset{n}{\dots}.1$ to $33.\overset{n}{\dots}.3$. Thus, a configuration (i.e. a sequence of n digits from $\{1, 2, 3\}$) can be seen as a vertex on the n-th level in a rooted ternary tree, and each move takes one vertex on the n-th level of the tree to another vertex on the n-th level. Hence each move in the Hanoi Tower game can be thought of as an automorphism of the rooted ternary tree.

We define the move a as follows:

- Search for the first time a 1 or 2 appears in the configuration and switch it with 2 or 1.
- Apply the identity in the other digits of the configuration.

For example $a(21322) = 11322$. This means that a, reading the configuration from the smaller disk to the biggest, does the only movement allowed to do between pegs 1 and 2. In a similar way one can define the remaining two moves b and c, the swap of 1 with 3 and 2 with 3, respectively.

The Hanoi Tower group $\mathcal{H}$ is the subgroup of $\operatorname{Aut}\mathcal{T}_3$ generated by the three automorphisms a, b and c given by the following recursive formulas (we abuse notation by omitting the map ψ):

$$a = (1, 1, a)(1\ 2),$$

$$b = (1, b, 1)(1\ 3),$$

$$c = (c, 1, 1)(2\ 3).$$

In [23] it has been proved that $\mathcal{H}$ is weakly regular branch over its derived subgroup $\mathcal{H}'$.

5.8 The group of finitary automorphisms

An automorphism of $\mathcal{T}$ is called *finitary* if it has finitely many non-trivial labels in its portrait. Finitary automorphisms form a locally finite subgroup $\mathcal{F}$ of $\operatorname{Aut}\mathcal{T}$. If $\mathcal{T}$ is a p-adic tree for a prime p and we fix a p-cycle σ in $\mathrm{Sym}(p)$, the group $\mathcal{F}_p$ of finitary automorphisms whose labels are all powers of σ constitute a subgroup of $\mathcal{F}$. We call this the *group of p-finitary automorphisms* of $\mathcal{T}$ (we give no reference to σ, since different choices of the p-cycle give rise to isomorphic groups). Observe that $\mathcal{F}_p$ is locally a finite p-group.

Chapter 6

Engel elements in $\mathrm{Aut}\,\mathcal{T}$: state of the art

In this chapter, we motivate the search of Engel elements in subgroups of $\mathrm{Aut}\,\mathcal{T}$. Indeed, the example of Bludov (explained below in Section 6.1), based on the Grigorchuk group $\mathfrak{G}$, is the first example of a group in which the set of left Engel elements is not a subgroup. Bartholdi in [6] refined this example by proving that the only left Engel elements of the Grigorchuk group $\mathfrak{G}$ are the involutions (i.e. elements of order 2). In particular, this shows that the Grigorchuk group is not Engel.
In Section 6.2 we complete the study of the Engel sets in the first Grigorchuk group. These results have given rise to a paper produced with A. Tortora and published in *International Journal of Group Theory* [43].
In Section 6.3 we give an alternative proof (suggested by Bartholdi in [7]) that the Grigorchuk group is not Engel.

6.1 The example of Bludov

In 2006 Bludov announced the possibility to construct a group based on the Grigorchuk group in which the set of left Engel elements is not a subgroup [9]. This was the first example of a group with this property. Let $\mathfrak{G}$ be the first Grigorchuk group, and let $K = \langle g \rangle$ be a cyclic group of order 4. By taking $G = \mathfrak{G} \wr K$, we will show that there exists an element $g \in G$ that is not a left Engel element of G. This leads to the following.

Lemma 6.1. *Let* $g \in K$*, and* $h = (1, ab, ca, d) \in \mathfrak{G} \times \mathfrak{G} \times \mathfrak{G} \times \mathfrak{G}$*. We have* $[h,_n g] \neq 1$ *for any* $n \geq 1$*.*

Proof. If we calculate $[h,_n g]$ starting from $n = 1$, we have

$$\begin{aligned}
h_1 &= [h, g] = h^{-1}h^g = (1, ba, ac, d)(d, 1, ab, ca)\\
&= (d, ba, c^a b, dca) = (d, ba, c^a b, ba);\\
h_2 &= [h_1, g] = (d, ab, bc^a, ab)(ba, d, ba, c^a b)\\
&= (ca, ac, bc^a ba, b^a cab);\\
h_3 &= [h_2, g] = (ac, ca, abc^a b, bacb^a)(b^a cab, ca, ac, bc^a ba)\\
&= (acb^a cab, cc^a, abc^a bac, bacb^a bc^a ba)\\
&= (c^a bc^a b, cc^a, b^a cb^a c, bc^a bb^a cb^a).
\end{aligned}$$

Since h_3 is in the first level stabilizer, we can consider $\psi(h_{3,i}) = (h_{3,a_i}, h_{3,b_i})$ for all $i = 1, \ldots, 4$ and set

$$h_4 = (h_{3,a_1}, h_{3,a_2}, h_{3,a_3}, h_{3,a_4}).$$

Then, we have

$$\begin{aligned}
\psi(h_{3,1}) &= \psi(c^a b c^a b) = (d,a)(a,c)(d,a)(a,c) = (dd^a, c^a c);\\
\psi(h_{3,2}) &= \psi(cc^a) = (a,d)(d,a) = (ad, da);\\
\psi(h_{3,3}) &= \psi(b^a c b^a c) = (c,a)(a,d)(c,a)(a,d) = (cc^a, d^a d);\\
\psi(h_{3,4}) &= \psi(bc^a bb^a cb^a) = (a,c)(d,a)(a,c)(c,a)(a,d)(c,a)\\
&= (d^a cac, cacd^a).
\end{aligned}$$

Thus, we set $h_4 = (dd^a, ad, cc^a, d^a cac)$ and we compute $h_5 = [h_4, g]$. We have

$$\begin{aligned}
h_5 &= [h_4, g] = (d^a dd^a cac, dd^a da, c^a cad, cc^a dc^a ca);\\
h_6 &= [h_5, g]\\
&= (cc^a dd^a dc^a cd^a cc^a, d^a dd^a dd^a dc^a c, dc^a cd^a dd^a, c^a cd^a cc^a cc^a d).
\end{aligned}$$

Arguing as before, we have

$$\begin{aligned}
\psi(h_{6,1}) &= \psi(cc^a dd^a dc^a cd^a cc^a) = (b^a bad, da);\\
\psi(h_{6,2}) &= \psi(d^a dd^a dd^a dc^a c) = (ca, bad);\\
\psi(h_{6,3}) &= \psi(dc^a cd^a dd^a) = (da, bac);\\
\psi(h_{6,4}) &= \psi(c^a cd^a cc^a cc^a d) = (db^a dad, dab).
\end{aligned}$$

As for h_4, we set $h_7 = (b^a bad, ca, da, db^a dad)$ and compute

$$h_8 = [h_7, g] = (db^a bd^a bd^a d, c^a bb^a d, b^a, dd^a b)$$

and

$$\psi(h_{8,1}) = \psi(db^a bd^a bd^a d) = (cb^a b, bab);$$

$$\psi(h_{8,2}) = \psi(c^a bb^a d) = (dac, c^a b);$$

$$\psi(h_{8,3}) = \psi(b^a) = (c, a);$$

$$\psi(h_{8,4}) = \psi(dd^a b) = (ba, bc).$$

We set $h_9 = (cb^a b, dac, c, ba)$ and

$$h_{10} = [h_9, g] = (bb^a da, cb^a bab, bac, ad);$$

$$h_{11} = [h_{10}, g] = (d^a bb^a d, bb^a bd^a bd^a, cd^a bb^a b, db^a c).$$

Then

$$\psi(h_{11,1}) = \psi(d^a bb^a d) = (bac, cab);$$

$$\psi(h_{11,2}) = \psi(bb^a bd^a bd^a) = (c^a bab, ca);$$

$$\psi(h_{11,3}) = \psi(cd^a bb^a b) = (b^a ca, bac);$$

$$\psi(h_{11,4}) = \psi(db^a c) = (ca, bad).$$

Again with $h_{12} = (bac, c^a bab, b^a ca, ca)$, we have

$$h_{13} = [h_{12}, g] = (cd^a, bb^a cb^a c, c^a db^a b, c^a bc^a).$$

Then

$$\psi(h_{13,1}) = \psi(cd^a) = (ab, d);$$
$$\psi(h_{13,2}) = \psi(bb^a cb^a c) = (c^a ca, cd^a d);$$
$$\psi(h_{13,3}) = \psi(c^a db^a b) = (ba, b^a c);$$
$$\psi(h_{13,4}) = \psi(c^a bc^a) = (dad, c^a).$$

Now we set $h_{14} = (h_{13,b_1}, h_{13,b_2}, h_{13,b_3}, h_{13,b_4}) = (d, cd^a d, b^a c, c^a)$ and we apply again ψ. We have

$$\psi(h_{13,b_1}) = \psi(d) = (1, b);$$
$$\psi(h_{13,b_2}) = \psi(cd^a d) = (ab, c);$$
$$\psi(h_{13,b_3}) = \psi(b^a c) = (ca, ad);$$
$$\psi(h_{13,b_4}) = \psi(c^a) = (d, a).$$

Finally, setting $h_{15} = (1, ab, ca, d)$, we find again h. So we fall in a loop and consequently $[h,_n g] \neq 1$ for all $n \geq 1$. Thus, g is not an Engel element of G. □

Now we are able to prove the main theorem of this section.

Theorem 6.2. *Let $D_8 = \langle x, y \mid x^2 = y^2 = (xy)^4 = 1\rangle$, and let $G = \mathfrak{G} \wr D_8$, where the action of D_8 on $\mathfrak{G}$ is seen as the action on the set of vertices of a square. Then* $\mathrm{L}(G)$ *is not a subgroup.*

Proof. Suppose D_8 is generated by x and y. Define $g = xy$, and $h = (1, ab, ca, d) \in \mathfrak{G} \times \mathfrak{G} \times \mathfrak{G} \times \mathfrak{G}$, as before. We have $[h,_n g] \neq 1$ for any $n \geq 1$

by Lemma 6.1. Since every involution in a 2-group is a left Engel element, by Proposition 2.8, we deduce that x, $y \in \mathrm{L}(G)$. However $xy \notin \mathrm{L}(G)$. Hence $\mathrm{L}(G)$ is not a subgroup. $\square$

According to this theorem, there exists a 2-group generated by involutions with an element of order four which is not left Engel. This suggests the following question.

Question 6.3 (Bludov)**.** *Assuming that G is not a 2-group, is $\mathrm{L}(G)$ a subgroup of G?*

Since the Grigorchuk group $\mathfrak{G}$ is a 2-group generated by involutions, one might wonder whether $\mathfrak{G}$ is an Engel group but the answer is negative, as shown by Bartholdi.

Theorem 6.4. [6, Theorem 1] *Let $\mathfrak{G}$ be the first Grigorchuk group. Then*

$$\mathrm{L}(\mathfrak{G}) = \{g \in \mathfrak{G} \mid g^2 = 1\}.$$

In particular, $\mathfrak{G}$ is not an Engel group.

Notice that in the next Chapter 6.3 we propose an alternative proof without using the result of Bartholdi about the fact that the Grigorchuk group is not an Engel group. The proof is based on an idea given by Bartholdi in [7].

The following natural question now arises: *are $\overline{\mathrm{L}}(\mathfrak{G}), \mathrm{R}(\mathfrak{G})$ and $\overline{\mathrm{R}}(\mathfrak{G})$ subgroups of $\mathfrak{G}$?*

6.2 Engel elements in the first Grigorchuk group

In this section we present a paper written with A. Tortora and published in *International Journal of Group Theory* [43]. Let $\mathfrak{G}$ be the first Grigorchuk group. According to a result of Bartholdi, the only left Engel elements of $\mathfrak{G}$ are the involutions. This implies that the set of left Engel elements of $\mathfrak{G}$ is not a subgroup. The natural question arises whether this is also the case for the sets of bounded left Engel elements, right Engel elements and bounded right Engel elements of $\mathfrak{G}$. Motivated by this, we prove that these three subsets of $\mathfrak{G}$ coincide with the identity subgroup.

Let us first see what consequences can be derived if one of these subsets is a subgroup of $\mathfrak{G}$.

Recall that $\mathfrak{G}$ is just-infinite. As a consequence, if $\overline{\mathrm{L}}(\mathfrak{G})$ were a nontrivial subgroup of $\mathfrak{G}$, then $\overline{\mathrm{L}}(\mathfrak{G})$ would be finitely generated and, by Theorem 6.4, also abelian. Hence $\overline{\mathrm{L}}(\mathfrak{G})$ would be finite and then trivial as otherwise $\mathfrak{G}$ would be an extension of a finite group by a finite group giving the contradiction that $\mathfrak{G}$ is finite. Thus the only possibility for $\overline{\mathrm{L}}(\mathfrak{G})$ to be a subgroup is to be trivial. Notice also that the same holds for $\mathrm{R}(\mathfrak{G})$ and $\overline{\mathrm{R}}(\mathfrak{G})$.

For the proof of our main theorem of this section, we require two lemmas concerning commutators between specific elements of $\mathfrak{G}$.

Lemma 6.5. *Let $x = ag$ be an involution in $\mathfrak{G}$ where $g \in \mathrm{st}_{\mathfrak{G}}(1)$ and $\psi(g) = (g_1, g_2)$. Let $y \in \mathrm{st}_{\mathfrak{G}}(1)$ where $\psi(y) = (k, 1)$. Then for every $m \geq 1$ we have*

$$\psi([y,_m x]) = (k^{(-1)^m 2^{m-1}}, (k^{g_2})^{(-2)^{m-1}}).$$

Proof. Since x is an involution we have $[y,_m x] = [y,x]^{(-2)^{m-1}}$ for every $m \geq 1$ (see Proposition 2.6). Thus

$$\begin{aligned}
\psi([y,_m x]) = \psi([y,x])^{(-2)^{m-1}} &= \psi(y^{-1}y^{ag})^{(-2)^{m-1}} \\
&= \Big(\psi(y^{-1})\psi(y^a)^{\psi(g)}\Big)^{(-2)^{m-1}} \\
&= (k^{-1}, k^{g_2})^{(-2)^{m-1}} = (k^{(-1)^m 2^{m-1}}, (k^{g_2})^{(-2)^{m-1}}),
\end{aligned}$$

as desired. □

Lemma 6.6. *Let $x = ag$ where $g \in \mathrm{st}_{\mathfrak{G}}(1)$ and $\psi(g) = (g_1, g_2)$. Let $y \in \mathrm{st}_{\mathfrak{G}}(1)$ with $\psi(y) = (y_1, y_2)$. Then for every $m \geq 1$ we have*

$$\psi([x,_{m+1} y]) = ([(y_2^{-1})^{g_1},_m y_1]^{y_1}, [(y_1^{-1})^{g_2},_m y_2]^{y_2}).$$

Proof. Of course, $[x,_n y] \in \mathrm{st}_{\mathfrak{G}}(1)$ for every $n \geq 1$. Thus

$$\begin{aligned}
\psi([x,y]) &= \psi((y^{-1})^x y) = \psi((y^{-1})^a)^{\psi(g)}\psi(y) \\
&= ((y_2^{-1})^{g_1}, (y_1^{-1})^{g_2})(y_1, y_2) = ((y_2^{-1})^{g_1} y_1, (y_1^{-1})^{g_2} y_2).
\end{aligned}$$

It follows that

$$\begin{aligned}
\psi([x,y,y]) &= [\psi([x,y]), \psi(y)] \\
&= [((y_2^{-1})^{g_1} y_1, (y_1^{-1})^{g_2} y_2), (y_1, y_2)] \\
&= ([(y_2^{-1})^{g_1} y_1, y_1], [(y_1^{-1})^{g_2} y_2, y_2]) \\
&= ([(y_2^{-1})^{g_1}, y_1]^{y_1}, [(y_1^{-1})^{g_2}, y_2]^{y_2}).
\end{aligned}$$

This proves the result when $m = 1$. Let $m > 1$. Then, by induction, we conclude that

$$\begin{aligned}\psi([x,_{m+1} y]) &= [\psi([x,_m y]), \psi(y)] \\ &= [([(y_2^{-1})^{g_1},_{m-1} y_1]^{y_1}, [(y_1^{-1})^{g_2},_{m-1} y_2]^{y_2}), (y_1, y_2)] \\ &= ([(y_2^{-1})^{g_1},_m y_1]^{y_1}, [(y_1^{-1})^{g_2},_m y_2]^{y_2}),\end{aligned}$$

as required. □

We are now ready to prove our main theorem, which is stated below.

Theorem 6.7. *Let $\mathfrak{G}$ be the first Grigorchuk group. Then*

$$\overline{\mathrm{L}}(\mathfrak{G}) = \mathrm{R}(\mathfrak{G}) = \overline{\mathrm{R}}(\mathfrak{G}) = \{1\}.$$

Proof. Let x be a nontrivial element of $\mathfrak{G}$ where x is either in $\overline{\mathrm{L}}(\mathfrak{G})$ or $\mathrm{R}(\mathfrak{G})$. First, notice that we may assume $x \notin \mathrm{st}_{\mathfrak{G}}(1)$. In fact, if $x \in \mathrm{st}_{\mathfrak{G}}(n) \setminus \mathrm{st}_{\mathfrak{G}}(n+1)$ then

$$\psi_n(x) = (x_1, \ldots, x_{2^n})$$

where all the x_i's are Engel elements (since $\mathfrak{G}$ is strongly fractal) of the same kind as x and one of the x_i's does not belong to $\mathrm{st}_{\mathfrak{G}}(1)$. Hence $x = ag$, for some $g \in \mathrm{st}_{\mathfrak{G}}(1)$ with $\psi(g) = (g_1, g_2)$. We distinguish two cases: $x \in \overline{\mathrm{L}}(\mathfrak{G})$ and $x \in \mathrm{R}(\mathfrak{G})$.
Assume $x \in \overline{\mathrm{L}}(\mathfrak{G})$. Then for some m, $[y,_m x] = 1$ for every $y \in \mathfrak{G}$. Also $x^2 = 1$, by Theorem 6.4. Since K is not of finite exponent, we can take

$k \in K$ of order $> 2^{m-1}$. On the other hand $\psi(K) \geq K \times K$, so there exists $y \in K \leq \mathrm{st}_{\mathfrak{G}}(1)$ such that $(k, 1) = \psi(y)$. Thus, by Lemma 6.5, we have

$$(1,1) = \psi(1) = \psi([y,_m x]) = \left(k^{(-1)^m 2^{m-1}}, (k^{g_2})^{(-2)^{m-1}}\right).$$

It follows that $k^{2^{m-1}} = 1$, a contradiction. This proves that $\overline{\mathrm{L}}(\mathfrak{G}) = \{1\}$.

Assume $x \in \mathrm{R}(\mathfrak{G})$. Since K is not abelian, it cannot be an Engel group by Theorem 6.4. Thus $[h,_m y_1] \neq 1$ for some $h, y_1 \in K$ and for every $m \geq 1$. Put $y_2 = [y_1, h]^{g_1^{-1}}$. Obviously, $y_2 \in K$ and $(y_2^{-1})^{g_1} = [h, y_1]$. Now $\mathfrak{G}$ is regular branch over K, so there exists $y \in K \subseteq \mathrm{st}_{\mathfrak{G}}(1)$ such that $\psi(y) = (y_1, y_2)$. Furthermore, there is $m = m(x, y) \geq 1$ such that $[x,_m y] = 1$. Applying Lemma 6.6, we get

$$\begin{aligned}(1,1) &= \psi(1) = \psi([x,_{m+1} y]) \\ &= ([(y_2^{-1})^{g_1},_m y_1]^{y_1}, [(y_1^{-1})^{g_2},_m y_2]^{y_2}) \\ &= ([h,_{m+1} y_1]^{y_1}, [(y_1^{-1})^{g_2},_m y_2]^{y_2}).\end{aligned}$$

This implies that $[h,_{m+1} y_1] = 1$, which is a contradiction. Therefore $\mathrm{R}(\mathfrak{G}) = \overline{\mathrm{R}}(\mathfrak{G}) = \{1\}$, and the proof is complete. $\square$

6.3 The Grigorchuk group is not Engel

In this section, we give an alternative proof of the fact that the Grigorchuk group is not an Engel group. This proof is based on a final remark in Bartholdi's paper [7].

We are going to consider a subgroup H of $\mathfrak{G}$, and an epimorphism, say ρ, from H to $G = \mathfrak{G} \wr D_8$. Notice that, as before, D_8 is the dihedral group of 8 elements and it can be seen as a permutation group over the vertices of a square (i.e. $D_8 = \langle (1\ 2\ 3\ 4), (2\ 4) \rangle$). As we have already seen in Section 6.1, the group G is not Engel. Our goal here is to find a suitable $H \leq \mathfrak{G}$ and ρ such that

$$\frac{H}{\ker \rho} \cong G.$$

If this is the case, $\mathfrak{G}$ will contain a subgroup H that is not Engel so $\mathfrak{G}$ itself is not an Engel group.

Now, we construct H and ρ. Before doing this, we need some preliminary results. Consider four vertices $\{x_1, x_2, x_3, x_4\}$ of the third level of the binary tree, where $x_1 = 111, x_2 = 112, x_3 = 211, x_4 = 212$.

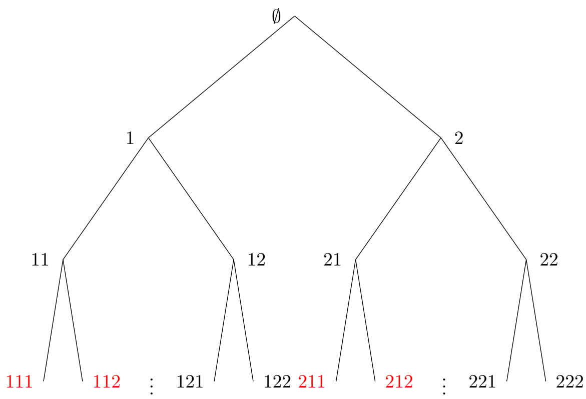


Figure 6.1: The vertices x_1, x_2, x_3 and x_4

Recall that the group $\mathfrak{G}$ is regular branch over the subgroup

$$K = \langle [a, b], [b, d^a], [b^a, d] \rangle \leq \mathfrak{G}.$$

Then $\psi^{-1}(K \times \overset{8}{\ldots} \times K) \leq \mathrm{rst}_{\mathfrak{G}}(3) \leq \mathfrak{G}$.

In the following we set

$$L = \psi^{-1}(K \times K \times \{1\} \times \{1\} \times K \times K \times \{1\} \times \{1\}) \leq K \leq \mathfrak{G}.$$

Also, we write the elements of $\langle a, d\rangle$ as cycles acting on x_1, x_2, x_3 and x_4. We have

$$a = (x_1\ x_3)(x_2\ x_4), d = (x_3\ x_4), ad = (x_1\ x_4\ x_2\ x_3).$$

Proposition 6.8. *We have*

$$\langle L, a, d\rangle = L \rtimes \langle a, d\rangle \cong K \wr D_8,$$

where D_8 acts naturally on four letters.

Proof. First, we prove that $L \cap \langle a, d\rangle = \{1\}$. Obviously, $\langle a, d\rangle \not\subseteq \mathrm{st}_{\mathfrak{G}}(1)$. If we take $J = \langle (ad)^2, d\rangle = \{1, (ad)^2, d, (ad)^2 d\}$, we have $J \leq \mathrm{st}_{\mathfrak{G}}(1)$ but still J is not contained in K. Indeed $\psi((ad)^2) = (b, b)$, $\psi(d) = (1, b)$, and $\psi(d^a) = (b, 1)$. Hence the intersection of L with $\langle a, d\rangle$ must be trivial, because $L \subseteq \mathrm{st}(3)$ but $(ad)^2, d$ and $(ad)^2 d$ are not in $\mathrm{st}(3)$.

Also, L is normal in $\langle L, a, d\rangle$. Indeed if we consider $l \in L$, such that $\psi(l) = (k_1, k_2, 1, 1, k_3, k_4, 1, 1)$, then we have

$$\psi(l^d) = (k_1, k_2, 1, 1, k_4, k_3, 1, 1) \in L$$

and also $\psi(l^a) = (k_3, k_4, 1, 1, k_2, k_1, 1, 1) \in L$. This concludes the proof.

□

For ease of notation we write $K_0 = \mathrm{st}_K(u)$, where u is the vertex 1111. As $K_0 \subseteq K$, we take $L_0 = \psi^{-1}(K_0 \times K_0 \times \{1\} \times \{1\} \times K_0 \times K_0 \times \{1\} \times \{1\})$. Then we have, as before,

$$\langle L_0, a, d\rangle = L_0 \rtimes \langle a, d\rangle \cong K_0 \wr D_8.$$

Lemma 6.9. *The group K_0 has a quotient isomorphic to $\mathfrak{G}$.*

Proof. Take $u = 1111$. Let ψ_u be the map sending an element f of K_0 to its section f_u. We prove that ψ_u is onto $\mathfrak{G}$. Take $[a,b]^4, [b,d^a]^4, [a,b]^{(ac)^2} \in K_0$. By easy computations, one can see that $\psi_u([a,b]^4) = a$, $\psi_u([b,d^a]^4) = b$ and $\psi_u([a,b]^{(ac)^2}) = c$. Then

$$\mathfrak{G} \cong \frac{K_0}{\ker \psi_u},$$

and the result follows. □

In the following we write a general lemma regarding wreath products whose proof is straightforward.

Lemma 6.10. *Let A, B be arbitrary groups, and consider a wreath product $A \wr B$, corresponding to some action of B on a set X. Consider the following maps:*

$$\begin{array}{ccc} A & \wr & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle Id_B} \\ A^* & \wr & B \end{array}$$

where α is a homomorphism sending A to the group A^*, and Id_B denotes the identity map. For $(a_1, \dots, a_n)b \in A \wr B$, define

$$\rho((a_1, \dots, a_n)b) = \alpha(a_1, \dots, a_n)b.$$

Then ρ is a homomorphism.

Now we are ready to prove the main theorem of this section.

Theorem 6.11. *The Grigorchuk group is not an Engel group.*

Proof. Suppose towards contradiction that $\mathfrak{G}$ is an Engel group. Then also its subgroup $\langle L, a, d\rangle \cong K \wr D_8$ is Engel. The latter contains $K_0 \wr D_8$. Let η be the map sending K_0 to $\mathfrak{G}$ via ψ_u (defined in Lemma 6.9), and D_8 to D_8 via the identity map. By the previous Lemma 6.10, η is a epimorphism and so $G = \mathfrak{G} \wr D_8$ is also an Engel group. This contradicts the fact that G is not Engel as showed in Section 6.1. □

6.3.1 Branch groups and almost Engel groups

In Theorem 2.3 we proved that any residually nilpotent almost Engel group is necessarily an Engel group.
Since the Grigorchuk group is not Engel, we conclude this chapter by pointing out that it cannot be almost Engel either. Moreover this is the same situation for all those branch groups lying in the Sylow pro-p subgroup Γ that are not Engel (see Chapter 7 for more groups in this family) since they are residually finite p-groups and hence residually nilpotent. Hence, we have the following theorem whose proof is straightforward.

Theorem 6.12. *Let $G \leq \Gamma$ be a branch group. If G is non-Engel, then G cannot be almost Engel.*

Part III

Engel elements in some subgroups of $\mathrm{Aut}\,\mathcal{T}$

Chapter 7

Engel elements in some fractal groups

In this chapter we present a paper [17] produced in collaboration with G.A. Fernández-Alcober and A. Garreta and published in *Monatshefte für Mathematik*.

We will consider group actions by using a more general notion of commutator. This is achieved by defining $[h, g] = h^{-1}h^g$, where h^g is the action of g on h, and $h \in H$, $g \in G$. One then denotes by $\mathrm{L}(G \curvearrowright H)$ the set of all $g \in G$ such that for all $h \in H$, we have $[h, g, \overset{n}{\ldots}, g] = 1$ for some n. With this in mind, we prove some properties concering Engel actions and then we apply fractal groups. In particular, we consider a prime p and a subgroup G of a Sylow pro-p subgroup of the group of automorphisms of the p-adic tree. We prove that if G is fractal and $|G' : \mathrm{st}_G(1)'| = \infty$, then the set $\mathrm{L}(G)$ of left Engel elements of G is trivial. This result applies to fractal non-abelian groups with torsion-free abelianization, for example the Basilica group, the Brunner-Sidki-Vieira group, and also to the GGS-group with constant defining vector. We further provide two examples showing that neither of the requirements $|G' : \mathrm{st}_G(1)'| = \infty$ and being fractal can be dropped.

7.1 Engel elements in strongly fractal groups

We start by studying the set of left Engel elements of strongly fractal groups $G \le \operatorname{Aut} \mathcal{T}_d$. We will use the following identity

$$\psi\left([h,_n g]\right) = \left([h_1,_n g_1], \ldots, [h_d,_n g_d]\right), \tag{7.1}$$

which holds for any two elements $h, g \in \mathrm{st}(1)$. Here the h_i's and g_i's are the components of $\psi(g)$ and $\psi(h)$.

Lemma 7.1. *Let $G \le \operatorname{Aut} \mathcal{T}_d$ be a strongly fractal group. Then*

$$\mathrm{L}(G) \cap \mathrm{st}_G(1) = \left\{ h \in \mathrm{st}_G(1) \mid \psi(h) \in \mathrm{L}(G) \times \overset{d}{\cdots} \times \mathrm{L}(G) \right\}.$$

Proof. Let $g \in \mathrm{L}(G) \cap \mathrm{st}_G(1)$, and write $\psi(g) = (g_1, \ldots, g_d)$. We show that each g_i is left Engel in G. Assume for contradiction that there exists $h \in G$ be such that $[h,_n g_i] \neq 1$ for all n. Since G is strongly fractal, there exists an element $s \in \mathrm{st}_G(1)$ such that $\psi(s) = (h_1, \ldots, h_{i-1}, h, h_{i+1}, \ldots, h_d)$, for some $h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_d \in G$. Then, by (7.1), we obtain that

$\psi([s,_n g])$ has components

$$([h_1,_n g_1], \ldots, [h_{i-1},_n g_{i-1}], [h,_n g_i], [h_{i+1},_n g_{i+1}], \ldots, [h_d,_n g_d]) \, .$$

This element is non-trivial because $[h,_n g_i] \neq 1$ for all n, and so $[s,_n g] \neq 1$ for all n, contradicting the fact that $g \in \mathrm{L}(G)$.
To prove the reverse inclusion, let $h \in \mathrm{st}_G(1)$ be such that all components of $\psi(h)$ are left Engel, and let g be any element of G. Then, since $\mathrm{st}_G(1)$ is normal in G, we have $[g,h] \in \mathrm{st}_G(1)$. It follows now from the equality (7.1) that $[g,_n h] = 1$ for some n, as required. □

The following constitutes a key observation.

Proposition 7.2. *Let S be a subset of* $\mathrm{Aut}\,\mathcal{T}_d$. *Suppose that* $S \subseteq \mathrm{st}(1)$ *and that* $\psi(S) \subseteq S \times \overset{d}{\ldots} \times S$. *Then* $S = 1$.

Proof. Suppose that $S \neq 1$. Then there exists a maximum n such that $S \subseteq \mathrm{st}(n)$, but $S \not\subseteq \mathrm{st}(n+1)$. Let $s \in S$. In particular, $s \in \mathrm{st}(1)$, and the components $s_1, \ldots, s_d$ of $\psi(s)$ belong to S by hypothesis. Since $S \subseteq \mathrm{st}(n)$, each s_i stabilizes the first n levels of $\mathcal{T}_d$, and hence $s \in \mathrm{st}(n+1)$. This occurs with all $s \in S$, yielding the contradictory inclusion $S \subseteq \mathrm{st}(n+1)$. Therefore $S = 1$. □

The following is an immediate consequence of Lemma 7.1 and Proposition 7.2.

Corollary 7.3. *Let $G \leq$* $\mathrm{Aut}\,\mathcal{T}_d$ *be a strongly fractal group such that* $\mathrm{L}(G) \subseteq \mathrm{st}_G(1)$. *Then* $\mathrm{L}(G) = 1$.

7.2 Engel elements, group actions, and fractal groups

Here, we generalize the notion of commutator and of left Engel element to the context of group actions $G \curvearrowright H$. We will use exponential notation to refer to the action of a group G on another group H, so that $h \cdot g = h^g$ for all $g \in G, h \in H$.

Given an action of groups $G \curvearrowright H$, one may define commutators by letting $[h, g] = h^{-1}h^g$, for $g \in G$ and $h \in H$. Then, $[h,_n g]$ is defined similarly as before. Of course, if H is a normal subgroup of G and the action $G \curvearrowright H$ is the standard conjugation, then $[h, g]$ is the usual commutator $h^{-1}g^{-1}hg$.

An element $g \in G$ is called *left Engel with respect to* $G \curvearrowright H$ if for all $h \in H$ there exists $n \geq 1$ such that $[h,_n g] = 1$. The set of left Engel elements of G with respect to $G \curvearrowright H$ will be denoted $\mathrm{L}(G \curvearrowright H)$. If $\mathrm{L}(G \curvearrowright H) = G$, then $G \curvearrowright H$ is called an *Engel action*. Given $S \subseteq H$, and $T \subseteq G$, we denote $[S, T] = \langle [s, t] \mid s \in S, t \in T \rangle$.

We start by proving a key fact regarding how periodic groups can act on finitely generated abelian groups.

Proposition 7.4. *Let $G \curvearrowright A$ be an Engel action of a finite group G on a finitely generated abelian group A. Then $[A, G]$ is finite. As a consequence, if A is free abelian, then the action $G \curvearrowright A$ is trivial.*

Proof. Let ℓ be the order of G, and take two elements $a \in A$, and $g \in G$. We claim that if $[a,_n g] = 1$ for some n, then $[a, g]^{\ell^{n-1}} = 1$. We argue by induction on n, the case $n = 1$ being obvious. Then let $n > 1$. If we denote $s = [a,_{n-2} g]$, we have $[s, g, g] = 1$. One can then prove by induction that

$[s, g^k] = [s, g]^k$ for all $k \in \mathbb{Z}$. Then $[a,_{n-1} g]^\ell = [s, g]^\ell = [s, g^\ell] = 1$. Write $K = \langle [a,_{n-1} g]^h \mid h \in G \rangle$, $\overline{A} = A/K$, and consider the Engel action $G \curvearrowright \overline{A}$, which is still an Engel action of a finite group on a finitely generated abelian group. Then, $[\bar{a},_{n-1} g] = \overline{[a,_{n-1} g]} = \overline{1}$ and, by induction, $[\bar{a}, g]^{\ell^{n-2}} = \overline{1}$. Thus, $[a, g]^{\ell^{n-2}} \in K$, and so $[a, g]^{\ell^{n-1}} = 1$ because K is abelian generated by elements of order dividing ℓ. This completes the proof of the claim.

We have proved that each element of the generator set of $[A, G]$ has finite order. Since $[A, G]$ is finitely generated and abelian, we conclude that $[A, G]$ is finite. The last part of the lemma immediately follows. □

Now we can proceed to the proof of the main theorem of this chapter.

Theorem 7.5. *Let $G \leq \Gamma$ be a fractal group such that $|G' : \mathrm{st}_G(1)'| = \infty$. Then* $\mathrm{L}(G) = 1$.

Proof. Write S for $\mathrm{st}_G(1)$. We claim that $\mathrm{L}(G) \subseteq S$, from which $\mathrm{L}(G) = 1$ follows by using Corollary 7.3 and the fact that subgroups of Γ are strongly fractal if and only if they are fractal (Lemma 2.5 of [53]). By way of contradiction, we assume that $\mathrm{L}(G) \nsubseteq S$. Since $G \leq \Gamma$, then $|G : S| \leq |\Gamma : \mathrm{st}_\Gamma(1)| = p$. Note that $|G : S| \neq 1$ because otherwise G would be trivial. Then $|G : S| = p$ and also the factor group G/S' is solvable. By a result of Gruenberg (Theorems 2 and 4 of [32]), $\mathrm{L}(G/S')$ is a subgroup of G/S'. Since S/S' is an abelian normal subgroup of G/S', we have $S/S' \subseteq \mathrm{L}(G/S')$ and then either $\mathrm{L}(G/S') = S/S'$ or G/S'. In the former case, we have $\mathrm{L}(G) \subseteq S$, which is a contradiction. In the latter, G/S' is an Engel group. Since S/S' is abelian, the action of G on S by conjugation induces an action of G/S on S/S', by which we have $[sS', gS] = [s, g]S'$

for all $s \in S$ and $g \in G$. Thus this action is Engel. By Proposition 7.4, the subgroup $[G/S, S/S']$ is finite, i.e. $[G,S]/S'$ is finite. Now since G/S is cyclic, it follows that $G' = [G,S]$ and we conclude that G'/S' is finite. This contradiction completes the proof of the theorem. $\square$

7.3 Applications to specific fractal groups

In this section we apply Theorem 7.5 to a family of fractal groups.

Since $\mathrm{R}(G)^{-1}, \overline{\mathrm{L}}(G), \overline{\mathrm{R}}(G)^{-1} \subseteq \mathrm{L}(G)$ for any group G, by proving that $\mathrm{L}(G) = 1$ one automatically obtains that each one of these sets is also trivial. We will omit this observation in the statement of the subsequent results.

Theorem 7.6. *Let $G \leq \Gamma$ be a non-abelian fractal group with torsion-free abelianization. Then* $\mathrm{L}(G) = 1$.

Proof. Denote $S = \mathrm{st}_G(1)$. By Theorem 7.5, it suffices to prove that $|G' : S'| = \infty$. Suppose towards contradiction that this is not the case. Let π_i be the projection of $G \times \overset{p}{\ldots} \times G$ onto its i-th component. Notice the chain of inclusions $S' \leq G' \leq S \leq G$. Since G is strongly fractal, $\pi_i(\psi(S')) = G'$ for all $i = 1, \ldots, p$. By assumption $|G' : S'| < \infty$, hence $|\psi(G') : \psi(S')| < \infty$ and $|\pi_i(\psi(G')) : \pi_i(\psi(S'))| < \infty$, for all $i = 1, \ldots, p$. Thus, $\pi_i(\psi(G'))/G'$ is a finite subgroup of G/G'. Since G/G' is torsion-free, it follows that $\pi_i(\psi(G')) = G'$ for all i. This implies that $\psi(G') \leq G' \times \overset{p}{\ldots} \times G'$. In this case, by Proposition 7.2, we have $G' = 1$, which is a contradiction. $\square$

This theorem can be directly applied to a variety of groups that are weakly regular branch groups over their derived subgroups, for example the Basilica group and the Brunner-Sidki-Vieira group. Note that these groups are fractal and have torsion free abelianization (see Sections 5.4 and 5.5, respectively). We have the following.

Theorem 7.7. *The Basilica group and the Brunner-Sidki-Vieira group have no non-trivial left Engel elements.*

7.3.1 The GGS-group with constant defining vector

Throughout this section we let $\mathcal{G}$ denote the GGS-group corresponding to the vector $\mathbf{e} = (1, \dots, 1)$.

The case of this group is different because it has finite abelianization thus we cannot apply directly Theorem 7.6. Hence, we reduce the study of $\mathrm{L}(\mathcal{G})$ to a subgroup K that meets the requirements of Theorem 7.6 and we show first that $\mathrm{L}(\mathcal{G}) \subseteq \mathrm{L}(K)$ and then we prove that $\mathrm{L}(K) = 1$.

According to Lemma 4.2 of [19], $\mathcal{G}$ has a normal subgroup K of index p such that $\mathcal{G}$ is weakly regular branch over K'. Moreover, if $^{-}$ denotes the projection $\mathcal{G} \to \mathcal{G}/K'$, then by Proposition 3.4 of [18], we have $\overline{\mathcal{G}} = \overline{K} \ltimes \langle \overline{a} \rangle$, with $\overline{K} \cong C_\infty \times \overset{p-1}{\cdots} \times C_\infty$, and $\langle \overline{a} \rangle \cong C_p$.

Lemma 7.8. *We have* $\mathrm{L}(\overline{\mathcal{G}}) = \overline{K}$*. As a consequence,* $\mathrm{L}(\mathcal{G}) \subseteq \mathrm{L}(K)$.

Proof. Since $\overline{\mathcal{G}}$ is solvable, $\mathrm{L}(\overline{\mathcal{G}})$ is a subgroup of $\overline{\mathcal{G}}$, and since $\overline{K}$ is normal abelian, we have $\overline{K} \leq \mathrm{L}(\overline{\mathcal{G}})$. Thus either $\mathrm{L}(\overline{\mathcal{G}}) = \overline{K}$ or $\overline{\mathcal{G}}$. In the latter case, the action of $\langle \overline{a} \rangle$ on $\overline{K}$ is Engel and, since $\overline{K}$ is free abelian, this action is trivial, by Proposition 7.4. Hence $\overline{\mathcal{G}}$ is abelian and $\mathcal{G}' = K'$, which is a contradiction, since $|\mathcal{G} : \mathcal{G}'| = p^2$ is finite by Theorem 2.1 of [19]. It follows that $\mathrm{L}(\overline{\mathcal{G}}) = \overline{K}$ and, since $\overline{\mathrm{L}(\mathcal{G})} \subseteq \mathrm{L}(\overline{\mathcal{G}})$, we get $\mathrm{L}(\mathcal{G}) \subseteq \mathrm{L}(K)$. $\square$

We now proceed to prove that $\mathrm{L}(K) = 1$. Unfortunately, K is not even self-similar. This can be fixed by appropriately conjugating K in $\operatorname{Aut} \mathcal{T}_p$.

Lemma 7.9 ([21]). *Let $h \in \mathrm{st}(1)$ be such that*

$$\psi(h) = (ah, a^2 h, \ldots, a^{p-1} h, h).$$

Then K^h is strongly fractal.

Proof. As shown in Lemma 4.2 of [19], K is generated by $y_0 = ba^{-1}$, and $y_i = y_0^{a^i}$ $(i = 1, \ldots, p-1)$. Define

$$\begin{aligned} z_1 &= (z_1, 1, \ldots, 1)a^{-1}, \\ z_2 &= (1, z_2, 1, \ldots, 1)a^{-1}, \\ &\vdots \\ z_p &= (1, \ldots, 1, z_p)a^{-1}. \end{aligned}$$

By making computations, one may check that $y_i^h = z_i$ for all i (reading the subindices modulo p). Hence, K^h is generated by the z_i. It is now clear that K^h is self-similar, and strong-fractalness of K^h is a consequence of the identity $z_i^p = (z_i, \ldots, z_i)$, which holds for all i. □

Theorem 7.10. *The GGS-group with constant defining vector has no non-trivial Engel elements.*

Proof. By Lemma 7.8, it suffices to show that $\mathrm{L}(K) = 1$. Let h be the element of Lemma 7.9. Since $K \cong K^h$, one has that $\mathrm{L}(K) = 1$ if and only if $\mathrm{L}(K^h) = 1$. To prove the latter we will use Theorem 7.6. Lemma 7.9

states that K^h is fractal, and clearly K^h is a subgroup of the Sylow pro-p subgroup Γ. On the other hand, $(K^h)/(K^h)' \cong K/K'$ is torsion-free. Hence, by Theorem 7.6, we have $\mathrm{L}(K^h) = 1$, and we conclude that $\mathrm{L}(\mathcal{G}) = 1$. $\square$

7.4 The lamplighter group and the adding machine: examples

Theorem 7.5 states that $\mathrm{L}(G) = 1$ for any fractal group G such that $G \leq \Gamma \leq \operatorname{Aut} \mathcal{T}_p$ and $|G' : \mathrm{st}_G(1)'| = \infty$. In this part we show that if $|G : G'| = \infty$ and $|G' : \mathrm{st}_G(1)'| < \infty$ this is no longer true. We also prove that the condition of being fractal cannot be dropped.

Recall that the lamplighter group $\mathcal{L}$ is the metabelian group $C_2 \wr C_\infty$. It is well known that $\mathcal{L}$ can also be seen as the group of automorphisms of the binary tree $\mathcal{T}_2$ generated by $a = (a, a\sigma)$ and the rooted automorphism σ corresponding to the cycle (1 2).

Proposition 7.11. *The lamplighter group $\mathcal{L}$ satisfies the following properties: $\mathcal{L} \leq \Gamma = \operatorname{Aut} \mathcal{T}_2$, $|\mathcal{L} : \mathcal{L}'| = \infty$, $\mathcal{L}$ is fractal, $|\mathcal{L}' : \mathrm{st}_{\mathcal{L}}(1)'|$ is finite, and $\mathrm{L}(\mathcal{L}) \neq 1$.*

Proof. The first property is trivially satisfied. To see that the group $\mathcal{L}$ is fractal, notice that the first components of a and a^σ generate $\mathcal{L}$, since $a = (a, a\sigma)$ and $a^\sigma = (a\sigma, a)$. The same holds for the second components. It is also well known that the abelianization of $\mathcal{L}$ is isomorphic to $C_2 \times C_\infty$ with $\langle \sigma\mathcal{L}' \rangle \cong C_2$ and $\langle a\mathcal{L}' \rangle \cong C_\infty$.

We now prove that the index of $\mathrm{st}_{\mathcal{L}}(1)'$ in $\mathcal{L}'$ is finite. Let us write $S = \mathrm{st}_{\mathcal{L}}(1)$. Notice that $S' \subseteq \mathcal{L}' \subseteq S \subseteq \mathcal{L}$. One can compute that $S = \langle a, a^\sigma \rangle$. Now, letting $c = [a, \sigma]$,

$$S/S' = \langle aS', a^\sigma S' \rangle = \langle aS', acS' \rangle = \langle aS', cS' \rangle. \tag{7.2}$$

We claim that $\mathcal{L}'/S' = \langle cS' \rangle$. Indeed, let $y \in \mathcal{L}'/S'$. In particular, $y \in S/S'$, and by (7.2), $y = a^n c^m S'$, for some m and n. Then, $a^n S' \in \mathcal{L}'/S'$ and thus $n = 0$, because $\langle a\mathcal{L}' \rangle \cong C_\infty$. It follows that $\mathcal{L}'/S' \leq \langle cS' \rangle$, and so $\mathcal{L}'/S' = \langle cS' \rangle$. Notice that c has order 2 because $c = [a, \sigma] = a^{-1}a^\sigma = (\sigma, \sigma)$. Then, $|\mathcal{L}' : S'| \leq 2$ (in fact, $|\mathcal{L}' : S'| = 2$ by Proposition 7.2).
Finally, since the base group of $\mathcal{L}$ is abelian and normal, it is contained in $\mathrm{L}(\mathcal{L})$ and $\mathrm{L}(\mathcal{L}) \neq 1$. $\square$

We next show that the requirement of being fractal is necessary in Theorem 7.5. Let H be the subgroup of $\mathrm{Aut}\,\mathcal{T}_2$ generated by σ and x, where σ is again the rooted automorphism corresponding to the cycle (1 2), and $x = (1, x)\sigma$ is the so-called adding machine.

Proposition 7.12. *The group $H \leq \Gamma = \mathrm{Aut}\,\mathcal{T}_2$ is not fractal, $|H' : \mathrm{st}_H(1)'| = \infty$, and $\mathrm{L}(H) = \mathrm{st}_H(1)$.*

Proof. Define $b = x\sigma$. Note that $b = (1, x) \in \mathrm{st}(1)$ and $x^2 = (x, x)$. Then both elements x and b have infinite order. By easy computations, one can see that $\mathrm{st}_H(1) = \langle b, b^\sigma \rangle$ and $\mathrm{st}_H(1) \cong C_\infty \times C_\infty$. In particular, $\mathrm{st}_H(1)' = 1$. Moreover $H' = \langle [\sigma, b] \rangle^H = \langle (x^{-1}, x) \rangle^H = \{(x^{\pm n}, x^{\mp n}) \mid n \in \mathbb{Z}\}$, and it follows that $|H' : \mathrm{st}_H(1)'| = \infty$. Note also that H is not fractal because we can never obtain $\sigma \in H$ in a component of an element of $\mathrm{st}_H(1)$ since

$b = (1, x)$ and $b^\sigma = (x, 1)$.

We now prove that $\mathrm{L}(H) = \mathrm{st}_H(1)$. One inclusion is obvious, since $\mathrm{st}_H(1)$ is abelian and normal in H. Since H is solvable, it follows that $\mathrm{L}(H) = \mathrm{st}_H(1)$ or H. In the latter case, the action of $H/\mathrm{st}_H(1)$ on $\mathrm{st}_H(1)$ is Engel and, by Proposition 7.4, this action must be trivial. This implies that $H' = [H, \mathrm{st}_H(1)] = 1$, which is a contradiction. $\square$

Chapter 8

Engel elements in weakly branch groups

Here we present a recent submitted paper carried out in collaboration with G. A. Fernández-Alcober and G. Tracey.

In this chapter, we study properties of Engel elements in weakly branch groups, lying in the group of automorphisms of a spherically homogeneous rooted tree. More precisely, we prove that the set of bounded left Engel elements is always trivial in weakly branch groups. In the case of branch groups, the existence of non-trivial left Engel elements implies that these are all p-elements and that the group is virtually a p-group (and so periodic) for some prime p. We also show that the set of right Engel elements of a weakly branch group is trivial under a relatively mild condition. Also, we apply these results to well-known families of weakly branch groups, like the multi-GGS groups.

8.1 Orbits of automorphisms of $\mathcal{T}$

The main tool in our approach to Engel problems in weakly branch groups is the reduction of the action of an automorphism f from the whole tree to one or several "reduced trees" determined by some special orbits of f on $V(\mathcal{T})$. Hence we start by describing some properties of orbits of automorphisms of $\mathcal{T}$.

Definition 8.1. If $f \in \operatorname{Aut}\mathcal{T}$ and $v \in V(\mathcal{T})$, the *f-orbit* of v is the orbit of v under the action of $\langle f \rangle$ on $V(\mathcal{T})$, i.e. the set $\{f^i(v) \mid i \in \mathbb{Z}\}$. The f-orbit is *trivial* if it consists of only one vertex, that is, if $f(v) = v$.

In the statement of the following lemma, we consider the least common multiple of an unbounded family of positive integers to be infinity.

Lemma 8.2. *Let $f \in \operatorname{Aut}\mathcal{T}$ and, for every vertex $v \in V(\mathcal{T})$, let $\mathcal{O}_v$ be the f-orbit of v. Then the following hold:*

(i) If w is a descendant of v, then $|\mathcal{O}_v|$ divides $|\mathcal{O}_w|$.

(ii) $|f| = \operatorname{lcm}(|\mathcal{O}_v| \mid v \in V(\mathcal{T}))$.

(iii) If $|f|$ is finite then there exists a finite subset V of $V(\mathcal{T})$ satisfying that $|f| = \operatorname{lcm}(|\mathcal{O}_v| \mid v \in V)$ and that, whenever w is a descendant of a vertex $v \in V$, we have $|\mathcal{O}_w| = |\mathcal{O}_v|$. Also if f is non-trivial then all the orbits $\mathcal{O}_v$ with $v \in V$ are non-trivial. Furthermore, V can be chosen to lie in $\mathcal{L}_n$ for some n.

Proof. (i) This is obvious by the orbit-stabilizer Theorem, since $\operatorname{st}(w) \subseteq \operatorname{st}(v)$.

(ii) Set $H = \langle f \rangle$. Then $|\mathcal{O}_v| = |H/\operatorname{st}_H(v)|$ for all $v \in V(\mathcal{T})$. The natural map φ from H to the cartesian product of finite groups $\prod_{v \in V(\mathcal{T})} H/\operatorname{st}_H(v)$ is injective, since the intersection of all vertex stabilizers is trivial. Consequently

$$\begin{aligned}|f| = |\varphi(f)| &= \operatorname{lcm}(|f \operatorname{st}_H(v)| \mid v \in V(T)) \\ &= \operatorname{lcm}(|H/\operatorname{st}_H(v)| \mid v \in V(T)),\end{aligned}$$

which proves the result.

(iii) Let $L = \{|\mathcal{O}_v| \mid v \in V(\mathcal{T})\}$. If $|f|$ is finite then, by (ii), it can be achieved as the least common multiple of a finite subset of L. Let k be the minimum cardinality of such a subset and let

$$\mathcal{S} = \{S \subseteq L \mid |S| = k \text{ and } \operatorname{lcm}(S) = |f|\}.$$

Observe that $\mathcal{S}$ is a finite set.

We introduce a relation $\leq_{\mathrm{d}}$ in $\mathcal{S}$ by letting $S \leq_{\mathrm{d}} T$ if there exists a bijection $\alpha : S \to T$ such that $s \mid \alpha(s)$ for all $s \in S$. By (i), this models the situation when we pass from the orbits of a set of vertices to the orbits of a set of descendants of those vertices. We claim that $\leq_{\mathrm{d}}$ is an order relation in $\mathcal{S}$. Obviously, only antisymmetry needs to be checked. Assume that $\alpha : S \to T$ and $\beta : T \to S$ are such that $s \mid \alpha(s)$ and $t \mid \beta(t)$ for all $s \in S$ and $t \in T$. Then s divides $\beta(\alpha(s))$ and, if they are not equal, we get $\operatorname{lcm}(S \smallsetminus \{s\}) = |f|$. This is contrary to the minimality condition imposed on k. Thus $\beta(\alpha(s)) = s$ and, since $s \mid \alpha(s)$ and $\alpha(s) \mid \beta(\alpha(s))$, we obtain that $\alpha(s) = s$ for all $s \in S$. We conclude that $S = T$, which proves antisymmetry of $\leq_{\mathrm{d}}$.

Now choose S in $\mathcal{S}$ that is maximal with respect to the order $\leq_{\mathrm{d}}$, and let $V = \{v_1, \ldots, v_k\} \subseteq V(\mathcal{T})$ be such that $S = \{|\mathcal{O}_{v_1}|, \ldots, |\mathcal{O}_{v_k}|\}$. Consider an arbitrary set of vertices $W = \{w_1, \ldots, w_k\}$, where each w_i is a descendant of v_i, and let $T = \{|\mathcal{O}_{w_1}|, \ldots, |\mathcal{O}_{w_k}|\}$. Then $S \leq_{\mathrm{d}} T$ and, by the maximality of S, we have $S = T$. This implies that $|\mathcal{O}_{w_i}| = |\mathcal{O}_{v_i}|$ for all $i = 1, \ldots, k$. Observe also that the minimality of k implies that, if f is non-trivial, no orbit $\mathcal{O}_v$ with $v \in V$ is of length 1. Hence V satisfies the properties stated in (iii).

Finally, observe that also the set W satisfies the required properties. Thus by considering, for a suitable n, a subset of $\mathcal{L}_n$ consisting of one descendant of each vertex in V, we may assume that $V \subseteq \mathcal{L}_n$. □

Vertices and orbits as in part (iii) of the previous lemma will play a fundamental role in the rest of the chapter, and it is convenient to introduce some terminology.

Definition 8.3. Let $f \in \operatorname{Aut} \mathcal{T}$ and let $\mathcal{O}$ be an f-orbit. We say that $\mathcal{O}$ is *totally splitting* if for every descendant w of a vertex $v \in \mathcal{O}$, the length of the f-orbit of w is equal to $|\mathcal{O}|$.

Equivalently, an f-orbit $\mathcal{O}$ is totally splitting when the set of descendants of the vertices in $\mathcal{O}$ at every level of the tree splits into the maximum possible number of f-orbits.

Definition 8.4. Let $f \in \operatorname{Aut} \mathcal{T}$ be an automorphism of finite order. If V is a finite set of vertices satisfying the conditions in (iii) of Lemma 8.2, all of them lying on the same level of $\mathcal{T}$, we say that V is a *fundamental* *system of vertices* for f.

Next we give a sufficient condition for two automorphisms of $\mathcal{T}$ to generate a wreath product.

Lemma 8.5. *Let $f \in \operatorname{Aut}\mathcal{T}$ be an automorphism of finite order m, and assume that the f-orbit of a vertex $v \in V(\mathcal{T})$ has length m. Then for every $g \in \operatorname{rst}(v)$, the subgroup $\langle g, f\rangle$ of $\operatorname{Aut}\mathcal{T}$ is isomorphic to the regular wreath product $\langle g\rangle \wr \langle f\rangle$.*

Proof. Let $\mathcal{O}$ be the f-orbit of v. Since $|\mathcal{O}| = |f|$, we have $\langle f\rangle \cap \operatorname{st}(v) = 1$. As a consequence, if v lies at level n of the tree, also $\langle f\rangle \cap \operatorname{st}(n) = 1$ and

$$\langle g, f\rangle = \langle f\rangle \langle g, g^f, \dots, g^{f^{m-1}}\rangle = \langle f\rangle \ltimes \langle g, g^f, \dots, g^{f^{m-1}}\rangle, \tag{8.1}$$

since $g \in \operatorname{rst}(v)$ implies that $\langle g, g^f, \dots, g^{f^{m-1}}\rangle \subseteq \operatorname{st}(n)$.
Now set $v_i = f^i(v)$ for all $i \in \mathbb{Z}$, so that $\mathcal{O} = \{v_0, v_1, \dots, v_{m-1}\}$. Since $g \in \operatorname{rst}_G(v)$, from Remark 4.10, we get $g^{f^i} \in \operatorname{rst}_G(v_i)$ for all $i = 0, \dots, m-1$, and then

$$\langle g^{f^i}\rangle \cap \langle g, g^f, \dots, g^{f^{i-1}}\rangle \subseteq \operatorname{rst}_G(v_i) \cap \operatorname{rst}_G(\{v_1, \dots, v_{i-1}\}) = 1.$$

Also $[g^{f^i}, g^{f^j}] = 1$ for every $i, j \in \{0, \dots, m-1\}$. It follows that

$$\langle g, g^f, \dots, g^{f^{m-1}}\rangle = \langle g\rangle \times \langle g^f\rangle \times \cdots \times \langle g^{f^{m-1}}\rangle,$$

and since $g^{f^m} = g$, we conclude from (8.1) that $\langle g, f\rangle \cong \langle g\rangle \wr \langle f\rangle$. □

The result in Lemma 8.5 raises the question of whether an automorphism

$f \in \operatorname{Aut}\mathcal{T}$ of finite order m must have a regular orbit on $V(\mathcal{T})$, i.e. an orbit of length m. This is clearly the case if m is a prime power, by (ii) of Lemma 8.2, but it usually fails otherwise. Indeed, one can consider for example a rooted automorphism corresponding to a permutation whose order is strictly bigger than the lengths of its disjoint cycles. However, as we see in Lemma 8.8 below, it is always possible to derive a collection of automorphisms f_i from f, acting not on $\mathcal{T}$ but on some other rooted trees $\mathcal{R}_i$ obtained from $\mathcal{T}$, and having the property that every f_i has a regular orbit on $V(\mathcal{R}_i)$. These automorphisms f_i will allow us to study Engel conditions regarding f by using Lemma 8.5.

As we will see, Lemma 8.8 is essentially a reformulation of (iii) of Lemma 8.2. Before proceeding we need to introduce the concept of reduced tree. Note that reduced trees are somehow related to the trees obtained by deletion of layers defined by Grigorchuk and Wilson in [26].

Definition 8.6. Let V be a subset of vertices of $\mathcal{T}$, all lying on the same level n. We define the *reduced tree* of $\mathcal{T}$ at V, denoted by $\mathcal{R}(V)$, as the rooted tree consisting of the subtrees $\mathcal{T}_v$ for $v \in V$, all connected to a common root. In other words, the set of vertices of $\mathcal{R}(V)$ is

$$\{\emptyset\} \cup \{vw \mid v \in V,\ w \in \mathcal{T}_{s^{n+1}(\overline{d})}\},$$

where as before s denotes the shift operator on sequences.

For example, in the following figure, we consider the rooted automorphism f of the ternary tree $\mathcal{T}_3$ corresponding to the permutation (1 2 3) and we show in red the reduced tree at the orbit of the vertex 13:

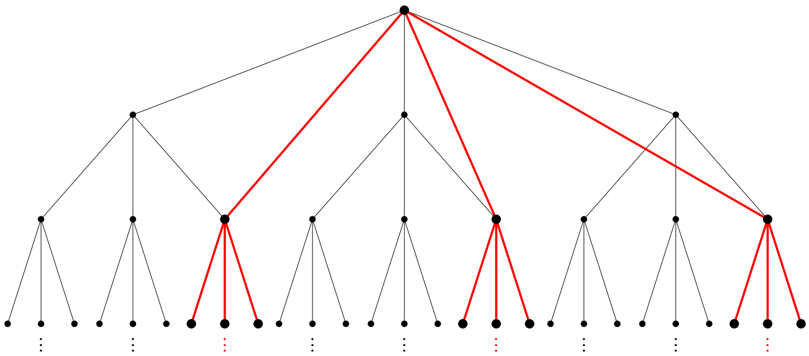


Figure 8.1: An f-orbit and its corresponding reduced tree

Every $f \in \operatorname{Aut} \mathcal{T}$ such that $f(V) = V$ induces by restriction an automorphism $f_V \in \operatorname{Aut} \mathcal{R}(V)$. Clearly, the map $\Phi_V : f \longmapsto f_V$ is a homomorphism of groups. The effect of Φ_V is to focus on the action of f only on the subtrees $\mathcal{T}_v$ with $v \in V$, so to speak. We will use reduced trees mainly in the case where V is an orbit of f.

Remark 8.7. If v is a vertex of the reduced tree $\mathcal{R}(V)$ and $f \in \operatorname{Aut} \mathcal{T}$ is such that $f(V) = V$, then the f_V-orbit of v coincides with the f-orbit of v as a vertex in $V(\mathcal{T})$. In particular, if $\mathcal{O}$ is a totally splitting f-orbit and we consider the induced automorphism $x = \Phi_{\mathcal{O}}(f)$ of $\mathcal{R}(\mathcal{O})$, then (ii) of Lemma 8.2 implies that $|x| = |\mathcal{O}|$. In other words, $\mathcal{O}$ is a regular orbit of x in $\mathcal{R}(\mathcal{O})$.

Given a subgroup G of $\operatorname{Aut} \mathcal{T}$, we write G_V for the image of the setwise stabilizer of V in G under the homomorphism Φ_V. In other words,

$$G_V = \{f_V \mid f \in G \text{ and } f(V) = V\}.$$

Then G_V is a subgroup of $\operatorname{Aut}\mathcal{R}(V)$, and for every vertex $v \in V$ we have $\Phi_V(\operatorname{rst}_G(v)) \subseteq \operatorname{rst}_{G_V}(v)$ (the inclusion can be proper, since there can be automorphisms in G whose action is trivial on $\mathcal{T}_w$ for every $w \neq v$ with $w \in V$, but non-trivial for some $w \notin V$).

On the other hand, if $f \in G$ stabilizes the set V and $x = \Phi_V(f)$ is the induced automorphism of $\mathcal{R}(V)$, then $f \in \mathrm{L}(G)$ or $f \in \overline{\mathrm{L}}(G)$ imply that $x \in \mathrm{L}(G)$ or $x \in \overline{\mathrm{L}}(G)$, respectively. In particular, by choosing V to be an f-orbit, this will allow us to transfer the analysis of a given Engel element in a subgroup of $\operatorname{Aut}\mathcal{T}$ to a more restricted situation where, for example, the Engel element acts transitively on the first level of the tree.

Actually the most convenient strategy is to reduce the tree to non-trivial totally splitting f-orbits, since the induced automorphisms will then have regular orbits. More precisely, we will rely on the following lemma, which is basically a rephrasing of part of Lemma 8.2 in the language of reduced trees.

Lemma 8.8. *Let $f \in \operatorname{Aut}\mathcal{T}$ be an automorphism of finite order $m > 1$ and let $\{v_1, \ldots, v_k\}$ be a fundamental system of vertices for f. For every $i = 1, \ldots, k$, let $\mathcal{O}_i$ be the f-orbit of v_i, set $\mathcal{R}_i = \mathcal{R}(\mathcal{O}_i)$, and let f_i be the automorphism of $\mathcal{R}_i$ induced by f. Then the following hold:*

(i) $\operatorname{lcm}(|\mathcal{O}_1|, \ldots, |\mathcal{O}_k|) = m$.

(ii) $\mathcal{O}_i$ is a non-trivial totally splitting f-orbit for every $i = 1, \ldots, k$.

(iii) $|f_i| = |\mathcal{O}_i|$ *for every* $i = 1, \ldots, k$.

Proof. The first two items follow from (iii) of Lemma 8.2, and (iii) from Remark 8.7. □

8.2 Some properties of Engel elements in wreath products

In this section we prove several results regarding Engel elements in wreath products. These will provide the basis for the proof of the main theorem, which will be addressed in Sections 8.3 and 8.4.

We start by studying left Engel elements lying outside the base group of a regular wreath product of two cyclic groups. To this purpose, we rely on the paper [38] by Liebeck.

Before proving the next lemma, we need a preliminary definition and a remark.

Definition 8.9. We say that the *Engel degree* of x on g is the smallest n such that $[g,_n x] = 1$.

Remark 8.10. For completeness of the next lemma, we recall that Baer's Theorem states that if a subgroup K satisfies the maximal condition on subgroups, then $K = \mathrm{HP}(K)$ [5]. In particular, $\mathrm{L}(K)$ coincides with the Fitting subgroup if K is finite.

We recall that a group K, we denote by $F(K)$ its Fitting subgroup (the subgroup generated by all normal nilpotent subgroups of K).

Lemma 8.11. *Let $X = \langle x \rangle$ and $Y = \langle y \rangle$ be two non-trivial cyclic groups, where X is finite, and let $W = Y \wr X$ be the corresponding regular wreath product. If $x \in \mathrm{L}(W)$ then X and Y are finite p-groups for some prime p. Furthermore, the Engel degree of x on $g = (y, 1, \dots, 1)$ is equal to*

$$|x| + \frac{1}{p}(\log_p |y| - 1)(p-1)|x|.$$

Proof. Let m be the order of x, and let p be an arbitrary prime divisor of m. Also, write d for the Engel degree of x on $g = (y, 1, \dots, 1)$.

First of all, suppose that Y is finite. Then W is finite and, as a consequence, x lies in the Fitting subgroup $F(W)$, by Baer's Theorem. We claim that Y is then a p-group. To this purpose, assume that $|Y|$ is divisible by a prime $q \neq p$, and let $Z = \langle z \rangle \neq 1$ be the subgroup of Y of order q. Consider the direct product Z^X inside the base group of W. Since Z^X is abelian and normal in W, it lies in $F(W)$. Now Z^X is a q-group and $x_p = x^{m/p}$ is a p-element, and both lie in the nilpotent group $F(W)$. It follows that x_p centralizes Z^X, which is clearly a contradiction, since x_p does not commute with $(z, 1, \dots, 1)$. This proves the claim, and since this property holds for every prime divisor of m, it also follows that X is a p-group. Observe that, since both X and Y are finite p-groups, the proof of Theorem 5.1 of [38] yields that

$$d = m + \frac{1}{p}(\log_p |y| - 1)(p-1)m \tag{8.2}$$

in this case.

Now we prove that it is impossible for Y to be infinite. To see this, argue by way of contradiction and observe that the wreath product $W_n = (Y/Y^{p^n}) \wr X$ can be seen as a factor group of W for all $n \in \mathbb{N}$. By applying (8.2) to W_n, we get

$$d \geq m + \frac{1}{p}(n-1)(p-1)m$$

for every n, which is impossible. □

Now we digress from Engel elements for a moment, but still working with wreath products of cyclic groups, in order to prove that rigid stabilizers of weakly branch groups are not only infinite, but have infinite exponent (Proposition 8.13 below).

Lemma 8.12. *Let $X = \langle x \rangle$ and $Y = \langle y \rangle$ be two finite cyclic groups, where Y is non-trivial, and let $W = Y \wr X$ be the corresponding regular wreath product. If $g = (y, 1, \ldots, 1)$ then $|xg| > |x|$.*

Proof. Set $m = |x|$ and let $n \in \{1, \ldots, m\}$ be arbitrary. Then

$$(xg)^n = x^n\, \overbrace{(\underbrace{y, \ldots, y}_{n}, 1, \ldots, 1)}^{m}.$$

In particular, $(xg)^n \neq 1$ for $1 \leq n \leq m$, and consequently $|xg| > m$, as desired. □

Proposition 8.13. *Let G be a weakly branch group. Then the exponent of $\mathrm{rst}_G(n)$ is not finite for every $n \in \mathbb{N}$.*

Proof. By way of contradiction, assume that $\mathrm{rst}_G(n)$ has finite exponent. Thus $\mathrm{rst}_G(n)$ is periodic and there is a bound for the orders of its elements.

For every $k \geq n$, let π_k be the (finite) set of prime divisors of the orders of the elements of $\mathrm{rst}_G(k)$. Then $\{\pi_k\}_{k\geq n}$ is a decreasing sequence of non-empty finite sets and consequently their intersection is also non-empty. Let p be a prime in $\cap_{k\geq n} \pi_k$.

Consider a p-element $f \in \mathrm{rst}_G(n)$ of maximum order, say m. Since the order of f is the least common multiple of the orders of the components of $\psi_n(f)$, we may assume without loss of generality that $f \in \mathrm{rst}_G(u)$ for some vertex u of the nth level. By (ii) of Lemma 8.2, there is a vertex v in the tree $\mathcal{T}$ such that the f-orbit of v has length m. Of course, v must be a descendant of u. Now let g be a non-trivial p-element in $\mathrm{rst}_G(v)$ and set $H = \langle g, f\rangle$. By Lemma 8.5, we have $H \cong \langle g\rangle \wr \langle f\rangle$. In particular, H is a finite p-group. On the other hand, by Lemma 8.12, H contains an element of order greater than m. This is a contradiction with the choice of m, since $H \subseteq \mathrm{rst}_G(n)$. □

Now we continue with our analysis of Engel elements in some wreath products. Before proceeding, we introduce some further notation. If G is a group and $S \subseteq G$, we write $\mathrm{L}_G(S)$ to denote the set of all $x \in G$ that are left Engel elements on every element of S, that is, such that for all $s \in S$ there exists $n = n(s, x)$ such that $[s,_n x] = 1$. We define the set $\overline{\mathrm{L}}_G(S)$ in the obvious way, and if $x \in \overline{\mathrm{L}}_G(S)$ then the Engel degree of x on S is the maximum of the Engel degrees of x on the elements of S.

Lemma 8.14. *Let $W = Y \wr X$ be a regular wreath product of two non-trivial groups, where X is finite cyclic of order n, and let $\pi : W \to X$ be the natural projection. Assume that $D = D_1 \times \cdots \times D_n \neq 1$ is a subgroup of the base group of W, and that $w \in W$ is such that $\pi(w)$ is a generator* *of X. Then the following hold:*

(i) If $w \in \overline{\mathrm{L}}_W(D)$ has Engel degree d on D then $d \geq n$.

(ii) If $w \in \mathrm{L}_W(D)$ then $C_D(w)$ is periodic.

Proof. Write $w = (y_1, \dots, y_n)x$, where $y_i \in Y$ and x generates X. We may assume that x permutes the components of the base group according to the cycle $(1 \dots n)$.

(i) Without loss of generality, we may assume that $D_1 \neq 1$. Choose a non-trivial element $g = (y, 1, \dots, 1) \in D$ and let $1 \leq i \leq n-1$. One can easily check by induction on i that

$$[g,_i w] = (y^{(-1)^i}, \dots, y^{y_1 \cdots y_i}, 1, \dots, 1),$$

where the last non-trivial component is in position $i+1$. It follows that $[g,_{n-1} w] \neq 1$ and $d \geq n$.

(ii) By contradiction, assume that $h = (z_1, \dots, z_n) \in C_D(w)$ is of infinite order. For notational convenience, set $z_0 = z_n$ and $y_0 = y_n$. Then from the condition $h = h^w$ we get $z_i = z_{i-1}^{y_{i-1}}$ for all $i = 1, \dots, n$. Hence all components of h are conjugate and they are all of infinite order.

Now let $g = (z_1, 1, \dots, 1) \in D$. For every $k \geq 0$, let us write $[g,_k w] = (z_{k,1}, \dots, z_{k,n})$ and, as before, set $z_{k,0} = z_{k,n}$. We claim that the following hold for every $k \geq 0$:

(a) $z_{k,i} \in \langle z_i \rangle$ for every $i = 1, \dots, n$.

(b) If we write $z_{k,i} = z_i^{m_{k,i}}$, then there exists $i \in \{1, \dots, n\}$ such that $m_{k,i} \neq m_{k,i-1}$.

We argue by induction on k. The result is obvious for $k = 0$, so assume $k \geq 1$ and that the claim is true for values less than k. Since $[g,_k w] = [g,_{k-1} w]^{-1}[g,_{k-1} w]^w$, it follows that

$$z_{k,i} = z_{k-1,i}^{-1}\, z_{k-1,i-1}^{y_{i-1}} = z_i^{-m_{k-1,i}}\, (z_{i-1}^{y_{i-1}})^{m_{k-1,i-1}} = z_i^{m_{k-1,i-1} - m_{k-1,i}}$$

for all $i = 1, \ldots, n$. This proves (a) and, if (b) does not hold, then

$$\begin{aligned} m_{k-1,1} - m_{k-1,2} &= m_{k-1,2} - m_{k-1,3} \\ &= \cdots = m_{k-1,n-1} - m_{k-1,n} = m_{k-1,n} - m_{k-1,1}. \end{aligned}$$

Now the sum of the $n-1$ first terms in this chain of equalities is the same as $n-1$ times the last one, i.e.

$$m_{k-1,1} - m_{k-1,n} = (n-1)(m_{k-1,n} - m_{k-1,1}).$$

From this, it readily follows that

$$m_{k-1,1} = m_{k-1,2} = m_{k-1,3} = \cdots = m_{k-1,n},$$

which is contrary to the induction hypothesis.

Finally, observe that (b) above implies that $m_{k,i}$ and $m_{k,i-1}$ cannot both be zero. Since z_i and z_{i-1} are of infinite order, we conclude that $[g,_k w] \neq 1$ for all $k \geq 1$ and consequently $w \notin \mathrm{L}_W(D)$. This contradiction completes the proof. $\square$

8.3 Left Engel elements in weakly branch groups

At this point, we can start combining all the machinery developed in Section 8.2 in order to prove our main results. In this section we consider left Engel elements. The following is an expanded version of our main theorem.

Theorem 8.15. *Let G be a subgroup of $\operatorname{Aut}\mathcal{T}$ in which all rigid vertex stabilizers are non-trivial. Then:*

(i) If f is a non-trivial left Engel element of finite order, and $\mathcal{O}$ is a non-trivial totally splitting f-orbit, then for some prime number p the length of $\mathcal{O}$ is a p-power and $\operatorname{rst}_G(\mathcal{O})$ is a p-subgroup.

If G is furthermore weakly branch, then:

(ii) If the set of finite order elements of $\mathrm{L}(G)$ is non-trivial then it is a p-set for some prime p, and $\operatorname{rst}_G(n)$ is a p-group for some $n \geq 1$.

(iii) $\overline{\mathrm{L}}(G) = 1$.

Proof. (i) Denote the reduced tree $\mathcal{R}(\mathcal{O})$ by $\mathcal{R}$, and set $x = \Phi_{\mathcal{O}}(f)$ and $H = G_{\mathcal{O}}$. We observe that $|x| = |\mathcal{O}|$ by Remark 8.7. Consider now a vertex v in $\mathcal{O}$ and an arbitrary element $g \in \operatorname{rst}_G(v)$, and set $y = \psi_v(g)$ (here v is considered as a vertex in $\mathcal{T}$). Then $h = \Phi_{\mathcal{O}}(g)$ lies in $\operatorname{rst}_H(v)$ and $\psi_v(h) = y$ (here v is considered as a vertex in $\mathcal{R}$). By Lemma 8.5, we have $\langle h, x\rangle \cong \langle h\rangle \wr \langle x\rangle$. Since $x \in \mathrm{L}(H)$, Lemma 8.11 implies that both $|y|$ and $|x|$ are p-powers for some prime p. Thus $|g|$ and $|\mathcal{O}|$ are p-powers.

Since $g \in \mathrm{rst}_G(v)$ was arbitrary and f acts transitively on $\mathcal{O}$, we conclude that $\mathrm{rst}_G(\mathcal{O})$ is a p-group.

(ii) Let again $f \in \mathrm{L}(G)$ be a non-trivial element of finite order. By applying Lemma 8.8 to f, we obtain non-trivial totally splitting f-orbits $\mathcal{O}_1, \ldots, \mathcal{O}_k$, all lying on the same level n of the tree, such that $|f| = \mathrm{lcm}(|\mathcal{O}_1|, \ldots, |\mathcal{O}_k|)$. Let us fix $i \in \{1, \ldots, k\}$. By (i), there exists a prime p (in principle, depending on i) such that $|\mathcal{O}_i|$ is a p-power and $\mathrm{rst}_G(\mathcal{O}_i)$ is a p-group. Since G acts now level transitively on $\mathcal{T}$, all rigid vertex stabilizers are isomorphic by Remark 4.10 in Chapter 4. It follows that p is the same for all i and consequently $\mathrm{rst}_G(n)$ is a p-group. Also the length of all orbits $\mathcal{O}_1, \ldots, \mathcal{O}_k$ is a power of p and f is a p-element.

(iii) By contradiction, assume that $f \in \overline{\mathrm{L}}(G)$, $f \neq 1$. Let d be the Engel degree of f.

Assume first that f is of finite order. Let $\mathcal{O}$ be a non-trivial totally splitting f-orbit. Define x and y as in the proof of (i), and recall that these are p-elements. By Lemma 8.11,

$$d \geq |x| + \frac{1}{p}(\log_p |y| - 1)(p-1)|x|.$$

On the other hand, since the exponent of $\mathrm{rst}_G(n)$ is not finite by Proposition 8.13, the order of y is unbounded. This is a contradiction.

Assume now that the order of f is infinite. By Lemma 8.2, there exists an f-orbit $\mathcal{O}$ of length $\ell > d$. Let once again $\mathcal{R}$ be the reduced tree $\mathcal{R}(\mathcal{O})$, and set $h = \Phi_{\mathcal{O}}(f)$ and $H = G_{\mathcal{O}}$. Then $h \in S\langle x \rangle$, where S is the first level stabilizer in $\mathrm{Aut}\,\mathcal{R}$ (i.e. the stabilizer of $\mathcal{O}$) and x is a rooted automorphism corresponding to a cycle of length ℓ. Observe that $S\langle x\rangle$ is isomorphic to a regular wreath product $W = Y \wr X$, where Y is the stabilizer in $\mathcal{R}$ of a vertex in $\mathcal{O}$ and $X = \langle x\rangle$ is cyclic of order ℓ. Under this isomorphism, h corresponds to an element w with $\pi(w) = x$. Also h lies in $\overline{\mathrm{L}}_H(D)$ with Engel degree at most d, where $D = \Phi_{\mathcal{O}}(\mathrm{rst}_G(\mathcal{O}))$ corresponds to a non-trivial direct product inside the base group of W. Now, by applying (i) of Lemma 8.14, we get $d \geq \ell$, which is a contradiction. This completes the proof of (iii). □

Now we proceed to prove the main theorem of this section.

Theorem 8.16. *Let G be a branch group. If $\mathrm{L}(G) \neq 1$ then G is periodic and there exists a prime p such that:*

(i) $\mathrm{L}(G)$ consists of p-elements.

(ii) G is virtually a p-group.

Proof. It suffices to show that $\mathrm{L}(G)$ does not contain any elements of infinite order. Indeed, since $\mathrm{L}(G) \neq 1$, the theorem then follows immediately from (ii) of Theorem 8.15, by taking into account that $|G : \mathrm{rst}_G(n)|$ is always finite if G is a branch group.

Let us assume then that $f \in \mathrm{L}(G)$ is of infinite order. Consider an f-orbit $\mathcal{O}$ in $V(\mathcal{T})$ of length $\ell \geq 2$, and let n be the level of $\mathcal{T}$ containing $\mathcal{O}$. Set $\mathcal{R} = \mathcal{R}(\mathcal{O})$, $h = \Phi_{\mathcal{O}}(f)$ and $H = G_{\mathcal{O}}$. Then for every vertex $v \neq \emptyset$ of $\mathcal{R}$ we have $\Phi_{\mathcal{O}}(\mathrm{rst}_G(v)) \subseteq \mathrm{rst}_H(v)$, and consequently all rigid vertex stabilizers of H are non-trivial. Also $h \in \mathrm{L}(H)$.

If h has finite order, then by (i) of Theorem 8.15, the rigid stabilizer in H of some vertex $v \neq \emptyset$ of $\mathcal{R}$ is periodic. Consequently $\mathrm{rst}_G(v)$ is periodic, and by level transitivity of G, also $\mathrm{rst}_G(n)$ is periodic. Since $|G : \mathrm{rst}_G(n)|$ is finite, it follows that G itself is periodic, which is a contradiction.

Assume now that the order of h is infinite. As in the proof of (iii) of Theorem 8.15, h lies in $S\langle x\rangle$, where S is the first level stabilizer of $\mathrm{Aut}\,\mathcal{R}$, and x is a rooted automorphism corresponding to a cycle of length ℓ. We can identify $S\langle x\rangle$ with the regular wreath product $W = Y \wr X$, where $X = \langle x\rangle$ is cyclic of order ℓ and h maps onto x. Then $h \in \mathrm{L}_W(D)$, where

$$D = \Phi_{\mathcal{O}}(\mathrm{rst}_G(\mathcal{O})) = \Phi_{\mathcal{O}}(\mathrm{rst}_G(n))$$

corresponds to a non-trivial direct product inside the base group of W. By (ii) of Lemma 8.14, $C_D(h)$ is periodic. However, since G is branch we have $f^k \in \mathrm{rst}_G(n)$ for some $k \geq 1$ and then $h^k = \Phi_{\mathcal{O}}(f^k) \in D$. It follows that $h^k \in C_D(h)$ is an element of infinite order, which is a contradiction. □

Now we can apply Theorems 8.15 and 8.16 to some distinguished subgroups of $\mathrm{Aut}\,\mathcal{T}$. All the groups below have been introduced in Chapter 5.

Corollary 8.17. *In all the following groups, the only left Engel element is the identity:*

(i) Every infinitely iterated wreath product of finite transitive permutation groups of degree at least 2*. In particular,* $\mathrm{Aut}\,\mathcal{T}$ *and* Γ_p*, for* p *a prime.*

(ii) The group $\mathcal{F}$ *of all finitary automorphisms of* $\mathcal{T}$*, provided that the sequence* $\overline{d}$ *defining* $\mathcal{T}$ *contains infinitely many terms greater than* 2*.*

(iii) All non-periodic multi-GGS groups $G_{\mathbf{E}}$*, i.e. those with at least one*

vector $\mathbf{e} \in \mathbf{E}$ *having non-zero sum in* $\mathbb{F}_p$.

(iv) The Hanoi Tower group $\mathcal{H}$.

Proof. (i) For every $n \in \mathbb{N}$, let K_n be a finite transitive permutation group of degree $d_n \geq 2$, and let W be the iterated wreath product of all these groups. Let $\mathcal{T}$ be the spherically homogeneous rooted tree corresponding to the sequence $\overline{d} = \{d_n\}_{n\in\mathbb{N}}$. Then W is isomorphic to the subgroup K of $\operatorname{Aut}\mathcal{T}$ consisting of all automorphisms whose labels at level n are elements of K_{n+1}. Observe that K is a branch group, since every K_n is transitive and obviously $\operatorname{rst}_K(n) = \operatorname{st}_K(n)$ in this case.

According to Theorem 8.16, we only need to construct an element of infinite order in K to conclude that $\mathrm{L}(W) = 1$. To this purpose, we choose an infinite sequence $\{k_n\}_{n\in\mathbb{N}}$ of non-trivial permutations $k_n \in K_n$, and an infinite sequence $\{v_n\}_{n\in\mathbb{N}\cup\{0\}}$ of vertices, where $v_n \in \mathcal{L}_n$ and $k_n(v_n) \neq v_n$. Also, let $\mathcal{O}_n$ denote the orbit of v_n under $\langle k_n\rangle$ and set $\ell_n = |\mathcal{O}_n|$.

Now we define f to be the automorphism of $\mathcal{T}$ having label k_{n+1} at vertex v_n for all $n \in \mathbb{N}\cup\{0\}$. We claim that the length of the f-orbit of v_n is $\ell_1 \ldots \ell_n$ for all $n \in \mathbb{N}$. Since $\ell_i \geq 2$ for every i, we conclude that f is of infinite order by using (ii) of Lemma 8.2.

We prove the claim by induction on n. The result is obvious for $n = 1$, since f behaves as k_1 on the first level of $\mathcal{T}$. Then f^{ℓ_1} fixes all vertices in the orbit $\mathcal{O}_1$, and a simple calculation shows that on all those vertices the section of f^{ℓ_1} coincides with the section of f at v_1, let us call it g. Since v_n lies at level $n-1$ for g, by induction the length of the g-orbit of v_n is $\ell_2 \ldots \ell_n$. From this one can readily see that the f-orbit of v_1 has length $\ell_1 \ldots \ell_n$, as desired.

(ii) Obviously, $\mathcal{F}$ is spherically transitive and $\mathrm{rst}_{\mathcal{F}}(n) = \mathrm{st}_{\mathcal{F}}(n)$ for all $n \in \mathbb{N}$. Thus $\mathcal{F}$ is a branch group. In this case, all elements of $\mathcal{F}$ are of finite order, but we still get $\mathrm{L}(\mathcal{F}) = 1$ from Theorem 8.16, because there is no prime p for which $\mathcal{F}$ is virtually a p-group. Indeed, assume for a contradiction that N is a normal p-subgroup of $\mathcal{F}$ of finite index m. Under this assumption, if H is a q-subgroup of $\mathcal{F}$ for a prime $q \neq p$, the order of H cannot exceed m. However, as we see in the next paragraph, the condition on the sequence $\overline{d}$ implies that $\mathcal{F}$ has 2-subgroups and 3-subgroups of arbitrarily high order, and we get a contradiction.
Consider the following subset of $\mathbb{N}$:

$$S = \{n \in \mathbb{N} \mid d_n \geq 3\}.$$

By hypothesis, S is infinite. For every $n \in S$, let H_n be the subgroup of $\mathcal{F}$ consisting of all automorphisms with labels lying in $\langle(1\ 2)\rangle$ for all vertices in $\mathcal{L}_n$ and trivial labels elsewhere. Then the order of H_n is $2^{\,d_1 \cdots d_n}$, which tends to infinity as $n \to \infty$. We can define similarly a subgroup J_n of order $3^{d_1 \cdots d_n}$ for every $n \in S$, by using the 3-cycle $(1\ 2\ 3)$. Thus we get 2-subgroups and 3-subgroups of $\mathcal{F}$ of arbitrarily high order, as desired.
(iii) If $\mathbf{E} = \langle(1, \ldots, 1)\rangle$ then $\mathrm{L}(G_{\mathbf{E}}) = 1$ by [17, Theorem 7]. Otherwise $G_{\mathbf{E}}$ is a branch group, and the result follows immediately from Theorem 8.16 and from the characterisation of periodic multi-GGS groups.
(iv) The Hanoi Tower group is known to be a branch group [23, Theorem 5.1]. Let us see that the element $ab = (b, 1, a)(1\ 2\ 3)$ is of infinite order. Assume, for a contradiction, that $|ab| = k$ is finite. Observe that $k = 3\ell$

for some ℓ, since ab has order 3 modulo the first level stabilizer. But then

$$(ab)^{3\ell} = ((ba)^{\ell}, (ab)^{\ell}, (ab)^{\ell})$$

implies that $(ab)^{\ell} = 1$, which is a contradiction. $\square$

Corollary 8.18. *Let p be a prime and let $\sigma \in \mathrm{Sym}(p)$ be a fixed p-cycle. Then the subgroup $\mathcal{F}_p$ of $\mathrm{Aut}\,\mathcal{T}_p$ formed by the finitary automorphisms all of whose labels lie in $\langle\sigma\rangle$ (in other words, the intersection of $\mathcal{F}$ with the standard Sylow pro-p subgroup of $\mathrm{Aut}\,\mathcal{T}_p$ corresponding to σ) satisfies that:*

(i) $\mathrm{L}(\mathcal{F}_p) = \mathcal{F}_p$.

(ii) $\overline{\mathrm{L}}(\mathcal{F}_p) = 1$.

Proof. Since $\mathcal{F}_p$ is locally a finite p-group, (i) is clear. On the other hand, since $\mathcal{F}_p$ is spherically transitive and $\mathrm{rst}_{\mathcal{F}_p}(n) = \mathrm{st}_{\mathcal{F}_p}(n)$ for all n, (ii) follows directly from Theorem 8.15. $\square$

8.4 Right Engel elements in weakly branch groups

In this section, we prove our main theorem regarding right Engel elements in weakly branch groups, and then we apply it to show that $\mathrm{R}(G) = 1$ whenever G is a GGS-group. Before proceeding, we need a straightforward lemma.

Lemma 8.19. *Suppose that $\mathcal{T}$ has d vertices in the first level, and consider $x, y \in \mathrm{Aut}\,\mathcal{T}$ such that:*

1. *$y = az$, where a is the rooted automorphism corresponding to the cycle $(1\ 2\ \dots\ d)$ and $z \in \mathrm{st}(1)$ is given by $\psi(z) = (z_1, \dots, z_d)$.*

2. *$x \in \mathrm{st}(1)$ is given by $\psi(x) = (x_1, \dots, x_d)$.*

Then, for all $k \geq 2$, we have

$$\psi([y,_k x]) = ([(x_d^{-1})^{z_1},_{k-1} x_1]^{x_1}, \dots, [(x_{d-1}^{-1})^{z_d},_{k-1} x_d]^{x_d}).$$

Proof. We have

$$\begin{aligned}\psi([y,x]) &= \psi((x^{-1})^y x) = \psi((x^{-1})^a)^{\psi(z)}\psi(x) \\ &= ((x_d^{-1})^{z_1}x_1, (x_1^{-1})^{z_2}x_2, \dots, (x_{d-1}^{-1})^{z_d}x_d).\end{aligned}$$

Now the result follows immediately by observing that taking subsequent commutators with x is performed componentwise. □

Theorem 8.20. *Let G be a weakly branch group. If $\mathrm{rst}_G(n)$ is not an Engel group for all $n \in \mathbb{N}$, then $\mathrm{R}(G) = 1$.*

Proof. Let $f \in G$, $f \neq 1$, and assume by way of contradiction that $f \in \mathrm{R}(G)$. Choose a non-trivial f-orbit $\mathcal{O} = \{v_1, \dots, v_d\}$, and assume that f permutes cyclically the vertices v_i. Let $\mathcal{R} = \mathcal{R}_{\mathcal{O}}$, $H = G_{\mathcal{O}}$ and $y = \Phi_{\mathcal{O}}(f) \in \mathrm{R}(H)$. Then we can write $y = az$, where a is rooted in $\mathcal{R}$ corresponding to the cycle $(1\ 2\ \dots\ d)$ and z is in the first level stabilizer. Write $\psi(z) = (z_1, \dots, z_d)$.

Let n be the level of $\mathcal{T}$ where $\mathcal{O}$ lies. Since $\Phi_{\mathcal{O}}(\mathrm{rst}_G(n)) \subseteq \mathrm{rst}_H(1)$ and $\mathrm{rst}_G(n)$ is not Engel by hypothesis, it follows that $\mathrm{rst}_H(1)$ is not an Engel group. If L is the first component of the direct product $\psi(\mathrm{rst}_H(1))$ then L is not an Engel group either, and we can choose $a, b \in L$ such that $[b,_k a] \neq 1$ for all $k \geq 1$. Now consider $r_1, r_2 \in \mathrm{rst}_H(1)$ such that

$$\psi(r_1) = (a, 1, \dots, 1) \qquad \text{and} \qquad \psi(r_2) = (b, 1, \dots, 1),$$

and define $x = r_1 (r_2^{-1})^{y^{-1}}$, so that

$$\psi(x) = (a, 1, \dots, 1, (b^{-1})^{z_1^{-1}}).$$

By applying the formula in Lemma 8.19, we get

$$\psi([y,_k x]) = ([b,_{k-1} a]^a, *, \dots, *)$$

and consequently $[y,_k x] \neq 1$ for all $k \geq 2$. This is a contradiction, since $y \in \mathrm{R}(H)$ and $x \in H$. □

Theorem 8.20 can be applied to show that GGS-groups have no non-trivial right Engel elements. We first need to prove the weaker result that they are not Engel groups.

Lemma 8.21. *Let G be a GGS-group. Then G is not an Engel group.*

Proof. We show that there is a power of b that is not a left Engel element of G. Let $\mathbf{e}$ be the defining vector of b. Consider any index $i \in \{1, \dots, p-1\}$ such that $e_{p-i} \neq 0$ in $\mathbb{F}_p$, and choose $\lambda \in \mathbb{F}_p^\times$ such that $\lambda e_{p-i} = -i$. Then we have

$$\psi((b^{-\lambda})^{a^i}) = (*, \dots, *, a^i), \tag{8.3}$$

where we use $*$ to denote unspecified elements of G.

Since $(b^{-\lambda})^{a^i} = [a^i, b^\lambda]b^{-\lambda}$, it follows that, for every $k \geq 2$,

$$\begin{aligned}
[(b^{-\lambda})^{a^i}, b^\lambda, \overset{k-1}{\ldots}, b^\lambda] &= [[a^i, b^\lambda]b^{-\lambda}, b^\lambda, \overset{k-1}{\ldots}, b^\lambda] \\
&= [[a^i, b^\lambda, b^\lambda]^{b^{-\lambda}}, b^\lambda, \overset{k-2}{\ldots}, b^\lambda] \\
&= [a^i, b^\lambda, \overset{k}{\ldots}, b^\lambda]^{b^{-\lambda}}.
\end{aligned}$$

By using (8.3), it follows that

$$\begin{aligned}
\psi([a^i, b^\lambda, \overset{k}{\ldots}, b^\lambda]^{b^{-\lambda}}) &= \psi([(b^{-\lambda})^{a^i}, b^\lambda, \overset{k-1}{\ldots}, b^\lambda]) \\
&= (*, \ldots, *, [a^i, b^\lambda, \overset{k-1}{\ldots}, b^\lambda]).
\end{aligned} \tag{8.4}$$

Now if b^λ is a left Engel element of G, choose the minimum $k \geq 1$ such that $[a^i, b^\lambda, \overset{k}{\ldots}, b^\lambda] = 1$. Since a^i and b^λ do not commute, we have $k \geq 2$ and so $[a^i, b^\lambda, \overset{k-1}{\ldots}, b^\lambda] \neq 1$. According to (8.4), this is a contradiction. $\square$

Corollary 8.22. *Let G be a GGS-group. Then* $\mathrm{R}(G) = 1$.

Proof. If the defining vector $\mathbf{e}$ is constant, then $\mathrm{L}(G) = 1$ by [17, Theorem 7], and consequently also $\mathrm{R}(G) = 1$. Thus in the remainder we assume that $\mathbf{e}$ is not constant. By [19, Lemma 3.2 and Lemma 3.4], we know that G is regular branch over K, where $K = \gamma_3(G)$ if $\mathbf{e}$ is symmetric and $K = G'$ otherwise. Since $\mathrm{rst}_G(n)$ contains a copy of $K \times \overset{p^n}{\cdots} \times K$ for every $n \in \mathbb{N}$, if we prove that K is not Engel then Theorem 8.20 applies to conclude that $\mathrm{R}(G) = 1$.

In order to show that K is not Engel, we are going to find a vertex v of the first level of the tree such that $\psi_v(K) = G$. Since G is not Engel by

Lemma 8.21, it follows that K is not Engel either, as desired.

We consider separately the cases when $\mathbf{e}$ is symmetric and non-symmetric. Assume first that $\mathbf{e}$ is non-symmetric, so that $K = G'$. We have

$$\psi([b,a]) = (a^{-e_1}b, a^{e_1-e_2}, a^{e_2-e_3}, \dots, a^{e_{p-2}-e_{p-1}}, b^{-1}a^{e_{p-1}}).$$

Since $\mathbf{e}$ is not constant, there exists $i \in \{1, \dots, p-2\}$ such that $e_i \neq e_{i+1}$ in $\mathbb{F}_p$. If v is the vertex $i+1$ on the first level of the tree, then

$$\psi_v([b,a]) = a^{e_i - e_{i+1}} \qquad \text{and} \qquad \psi_v([b,a]^{a^i}) = a^{-e_1}b.$$

Since the subgroup $\langle a^{e_i-e_{i+1}}, a^{-e_1}b\rangle$ coincides with G, we get the desired equality $\psi_v(G') = G$.

Now let $\mathbf{e}$ be symmetric, i.e. such that $e_i = e_{p-i}$ for all $i = 1, \dots, p-1$. Since $\mathbf{e}$ is not constant, this implies that $p \geq 5$. We have

$$\begin{aligned}\psi([b,a,a]) &= \psi([b,a]^{-1})\psi([b,a]^a)\\ &= (b^{-1}a^{e_1}b^{-1}a^{e_{p-1}}, a^{-2e_1+e_2}b, a^{e_1-2e_2+e_3}, \dots,\\ &\qquad a^{e_{p-3}-2e_{p-2}+e_{p-1}}, a^{-e_{p-1}}ba^{e_{p-2}-e_{p-1}}).\end{aligned}$$

If $e_i - 2e_{i+1} + e_{i+2} \neq 0$ for some $i \in \{1, \dots, p-3\}$, we have a non-trivial power of a in one of the components of $\psi([b,a,a])$ and we can argue as above to prove that $\psi_v(\gamma_3(G)) = G$ for a vertex v in the first level. On the other hand, if $e_i - 2e_{i+1} + e_{i+2} = 0$ for all $i = 1, \dots, p-3$, then

$$\begin{aligned} e_3 &= 2e_2 - e_1, \\ e_4 &= 2e_3 - e_2 = 3e_2 - 2e_1, \\ e_5 &= 2e_4 - e_3 = 4e_2 - 3e_1, \\ &\vdots \\ e_{p-1} &= 2e_{p-2} - e_{p-3} = (p-2)e_2 - (p-3)e_1. \end{aligned}$$

Since $e_{p-1} = e_1$, the last equation implies that $e_1 = e_2$, and then using all other equations, we get that all components e_i are equal to e_1. Thus the vector $\mathbf{e}$ is constant, which is a contradiction. $\square$

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