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Generalized Geometry for Relativistic Theories of Gravity

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Abstract

The TGR is constructed over a Pseudo-Riemaniann manifolod, where the torsion is assumed to vanish. However, there is not any physical argument behind it and when torsion is allowed, the spacetime in which the theory is constructed generalizes, from a V_4 to a U_4 spacetime, which carries several interesting consequences to be analyzed in this work. The most important one is the idea that the spin angular momentum acts, along with mass, as a source of gravitational interaction. In the first part of this work the TGR is formulated from an action principle. Later, the torsionless restriction is removed and the same procedure leads to slightly different field equations, in which spin is involved. Finally, requiring to curved spacetimes to posses symmetry under local Poincaré transformations, the same U_4 spacetime is deduced. Its implications are discussed in the end.

Laburpena

Erlatibitate orokorraren teoria barietate Sasi-Riemanndarrean eraikitzen da, non definizioz, tortsioa nulua den. Honek, ordea, ez du zertan hala izan, ez baitago tortsioa nulutzat hartzeko arrazoi fisikorik eta tortsioa agertzen denean, V_4 espazio-denboratik U_4 espazio-denborara igarotzen da, non lan honetan aztertzen diren hainbat ezberdinatasun nabari antzeman daitezkeen. Garrantzitsuena, beharbada, spin momentu angeluarrak grabitazio iturri bezala jokatzen duela da, masak bezalaxe. Lan honen lehen zatian espazio-denbora Sasi-Riemanndarrean akzio printzio batetatik abiatuz, Einsteinen teoria garatzen da. Ondoren, prozesu berbera jarraitzen da U_4 espazio-denboran, spinari lotutako eremu ekuazio berriak lortuz. Azkenik, kurbaduradun espazio-denborari Poincaré taldearekiko simetria lokala derrigortuz, U_4 espazio-denbora topatzen da. Honek eduki ditzakeen ondorio fisikoak eztabaidatzen dira amaieran.

Resumen

La Teoría de la Relatividad General se construye en una variedad Pseudo-Riemanniana, donde la torsión es nula. Sin embargo, no hay ningún argumento físico que impida la aparición de la torsión. Esto generaliza el espacio-tiempo V_4, convirtiendolo en U_4, donde aparecen ciertas características que se analizan en este trabajo, de las cuales destaca el hecho de que el espín, junto con la masa, actúa como fuente de gravitación. En la primera parte del trabajo se deduce la teoría de Einstein desde un principio de acción y después, se repite el proceso pero esta vez con torsión no nula, del cual se obtienen las ecuaciones de campo relacionadas al espín. Finalmente se identifica el espacio-tiempo U_4 cuando se requiere que el espacio-tiempo curvo sea invariante bajo el grupo de transformaciones locales de Poincaré.

Contents

1. Fundamentals of differential geometry and manifolds with torsion and non-metricity

Most of modern theories of gravity are based on the concept of curvature of the spacetime and for a good understanding of it, I consider of great importance a proper mathematical treatment of many aspects of differential geometry. That is why this first chapter is focused on the definition of some mathematical tools, specially on those that are not usually covered in Geneal Relativiy textbooks, and later bring all the physical intuition. This first chapter is developed for a general n-dimensional manifolds and later sections deal with four dimensional spacetimes of different types.

1.1 Basics and notation

At any point P described by $P = x^\mu$ in some coordinate system $\{x^\mu\}$ in a n-manifold M, it is possible to define a basis formed by the tangent vectors $e_\mu \equiv \partial/\partial x^\mu = \partial_\mu$, called coordinate basis, which expand the tangent vector space at P, denoted by $T_P M$. Correspondingly, it is denoted $T^*_P M$ to the dual vector space formed by all mappings $T_P M \to \mathbb{R}$, called the cotangent space, and usually chosen the dual basis vectors $e^\mu \equiv dx^\mu$ so that the inner product gives

$$\langle e_\mu, e^\nu \rangle = \frac{\partial x^\nu}{\partial x^\mu} = \delta^\nu_\mu \, . \tag{1.1}$$

Let the manifold be endowed with a metric tensor field of rank $(0, 2)$, $g : T_P M \times T_P M \to \mathbb{R}$, which provides the possibility to make measurements on the manifold and it is given by

$$g = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu \, . \tag{1.2}$$

Here $g_{\mu\nu}(x)$ are called the covariant components of the metric tensor field, which come naturally defined by the product of coordinate basis vectors, $g_{\mu\nu}(x) = \langle e_\mu, e_\nu \rangle|_x$. Hence, for two arbitrary tangent vectors[1] $V = V^\mu e_\mu$, $U = U^\mu e_\mu \in T_P M$:

$$\begin{aligned} g(V, U) &= g(V^\mu e_\mu, U^\nu e_\nu) \\ &= V^\mu U^\nu g(e_\mu, e_\nu) \\ &= V^\mu U^\nu g_{\alpha\beta}(P) \langle e_\mu, e^\alpha \rangle \langle e_\nu, e^\beta \rangle \\ &= V^\mu U^\nu g_{\alpha\beta}(P) \delta^\alpha_\mu \delta^\beta_\nu \\ &= V^\mu U^\nu g_{\mu\nu}(P) \, . \end{aligned} \tag{1.3}$$

[1] The Einstein sum convention is used during the whole work.

Similarly the contravariant components of the metric tensor field can be written like

$$g^{\mu\nu}(P) = \langle e^\mu, e^\nu \rangle|_P \quad ,$$

and conclude the important relation between contravariant and covariant components of the metric tensor:

$$g^{\mu\lambda} g_{\lambda\nu} = g_{\nu\lambda} g^{\lambda\mu} = \delta^\mu_\nu \ . \tag{1.4}$$

The metric tensor components can be used to move between contravariant and covariant expressions of tensors, for example: $T^{\alpha\beta} = T^\alpha{}_\lambda g^{\lambda\beta} = T_{\nu\lambda} g^{\lambda\beta} g^{\nu\alpha}$.

NOTE: in this last expression and here on, I will not explicitly write where the tensor field components are evaluated and it has to be understood that they are fields on the manifold.

The metric $g_{\mu\nu}$ is symmetric by definition, and consequently, its eigenvalues real. If the metric is positive-definite it is called a *Riemannian metric* and the manifold itself is a *Riemannian manifold*. Similarly, if the metric is nondegenerate it is called a *Pseudo-Riemannian metric* and the manifold itself a *Pseudo-Riemannian manifold*, which is a slight generalization and will be denoted V_n. A vector field over a manifold M is defined as a map that smoothly maps a tangent vector to every point in M. In this work I will refer to the set of all vector fields over M by $\mathfrak{X}(M)$, to the set of all scalar fields over M by $\mathfrak{F}(M)$ and to the set of tensor fields of rank (q,r) over M by $\mathfrak{T}^q_r(M)$.

1.2 Affine connections

A connection is a mathematical construct that specifies how the information is transported along curves in a manifold. A general manifold may be endowed with a connection, which is in principle, an independent geometric structure that characterizes M. Specifically, if the connection gives information about how tangent spaces are related, it is called an affine connection, which is an essential object to develop differential calculus on smooth manifolds. Let $X, Y, Z \in \mathfrak{X}(M)$ and $f \in \mathfrak{F}(M)$. Formally, an affine connection maps $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$, denoted by $\nabla_X Y$, in a way that the following properties are satisfied:

$$\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z \tag{1.5}$$
$$\nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z \tag{1.6}$$
$$\nabla_{(fX)} Y = f \nabla_X Y \tag{1.7}$$
$$\nabla_X(fY) = X[f]Y + f\nabla_X Y \tag{1.8}$$

If the connection is applied on two coordinate basis vector fields, the resulting vector field written in the coordinate basis would be

$$\nabla_{e_\mu} e_\nu = \Gamma^\lambda{}_{\mu\nu} e_\lambda, \tag{1.9}$$

where the n^3 coefficients $\Gamma^\lambda{}_{\mu\nu}$ are called the connection coefficients (and often just connection). For simpler notation, it is usually written just $\nabla_\mu e_\nu$ instead of $\nabla_{e_\mu} e_\nu$. Now, for two arbitrary vector fields $X = X^\mu e_\mu$ and $Y = Y^\mu e_\mu$ and making use of the properties above:

$$\begin{aligned}\nabla_X Y &= \nabla_{(X^\mu e_\mu)}(Y^\mu e_\mu) \\ &= X^\mu \left(\partial_\mu Y^\lambda + Y^\nu \Gamma^\lambda{}_{\mu\nu}\right) e_\lambda \,.\end{aligned}$$

If X was taken just a coordinate basis vector field, the vector components of the resulting vector field, are $(1,1)$ tensor field components

$$\nabla_\mu Y^\lambda = \partial_\mu Y^\lambda + Y^\nu \Gamma^\lambda{}_{\mu\nu} \,. \tag{1.10}$$

This is the definition of the usual covariant derivative, and represents the change of the λ component of the vector field Y along the e_μ direction. A more intuitive and less mathematical derivation of this can be found in most General Relativity books, like in [9, 17].

For a curve $c(t) \to M$, where $t \in \mathbb{R}$, and the vector field $X \equiv d/dt = (dx^\mu/dt)(\partial/\partial x^\mu)$ defined along the curve, it is said that the vector field Y is parallel transported along it if it does not change its direction, i.e.

$$\nabla_X Y = 0 \,, \tag{1.11}$$

which can be written like

$$\frac{dY^\mu}{dt} + \Gamma^\mu{}_{\nu\lambda} \frac{dx^\nu}{dt} Y^\lambda = 0 \,. \tag{1.12}$$

Parallelism in two and three dimesional Euclidean spaces seems a totally natural and intuitive notion. However, in manifolds with curvature there is not a natural way of defining "parallelism" but the one that the connection itself uniquely provides. If the parallel transported vector field Y is tangent to the curve, i.e. $Y^\mu = dx^\mu/dt$, the vector is being parallel transported along the straightest curve possible and the curve is said to be a geodesic, which satisfies the following equation:

$$\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\lambda} \frac{dx^\nu}{dt} \frac{dx^\lambda}{dt} = 0 \,. \tag{1.13}$$

When vectors are parallel transported along closed curves, they generally change direction when they return to the starting point. Geodesics, however, conserve the direction of the vector by definition. Geodesics in V_n happen to be the shortest curves as well as the straightest ones, though this is not true for more general manifolds, as it is analyzed in chapter 3.

One interesting feature of a connection is whether or not it conserves the length of a vector upon parallel transport. This would mean, that

$$\nabla_\mu \left[g(X,X)\right] = 0\,. \tag{1.14}$$

Using equation 1.3:

$$\nabla_\mu \left[g(X,X)\right] = X^\nu X^\lambda \nabla_\mu g_{\nu\lambda} = 0 \Longrightarrow \nabla_\mu g_{\nu\lambda} = 0\,. \tag{1.15}$$

A connection that satisfies equation 1.15 is called a *metric connection* and the tensor

$$Q_{\lambda\mu\nu} \equiv \nabla_\lambda g_{\mu\nu} \tag{1.16}$$

is named the *non-metricity tensor*. V_n manifolds are characterized by having a connection for which the non-metricity tensor vanishes.

1.3 Torsion and Curvature

Curvature is a core concept in the understanding of General Relativity, which is very well studied in most textbooks. That is why I am not going to focus on explaining the physical aspects and analyzing examples about curvature. However, I would like to remark some important properties and consequences about it. It is remarkable that when a vector is parallel transported from one point to another along two distinct curves, the direction of the vector is not the same in general, and the difference between them is proportional to the curvature tensor [18]. A vector parallel transported along an infinitesimal loop does also have a change in direction proportional to the curvature tensor, which can be derived by integrating the equation of parallelism along a closed loop [9]. Hence, curvature can be understood as the defect of the manifold when parallel transporting vectors, and by defect I mean the inherent imperfection that makes it distinct from an Euclidean space.

The concept of torsion is rarely studied in General Relativity textbooks, so it is worth analyzing its physical sense properly here. Say a point $P \in M$ has the coordinates $\{x^\mu\}$. Let $u = \epsilon^\mu e_\mu$ and $v = \delta^\mu e_\mu$ be two infinitesimal vectors in $T_P M$. Since they are infinitesimal, it can be considered that each of them points a new point in M, $Q = \{x^\mu + \epsilon^\mu\}$ and $S = \{x^\mu + \delta^\mu\}$. See figure 1.1. The parallel transported u from P to S is

$$\tilde{u} = \left(\epsilon^\mu - \Gamma^\mu{}_{\lambda\nu}\epsilon^\nu\delta^\lambda\right) e_\mu, \tag{1.17}$$

which is easily obtained from equation (10). Similarly, parallel transport of v from P to Q is

$$\tilde{v} = \left(\delta^\mu - \Gamma^\mu{}_{\nu\lambda}\epsilon^\nu\delta^\lambda\right) e_\mu. \tag{1.18}$$

The components of the vectors from the origin to the new points defined by $\tilde{u}$ and $\tilde{v}$ are

$$r_1^\mu \equiv v + \tilde{u} = \delta^\mu + \epsilon^\mu - \Gamma^\mu{}_{\lambda\nu}\epsilon^\nu\delta^\lambda \tag{1.19}$$

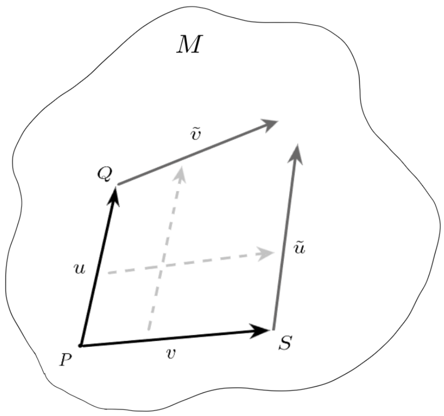

Figure 1.1: the parallelogram one would expect to be formed in euclidean space is open when torsion is non-vanishing.

$$r_2^\mu \equiv u + \tilde{v} = \epsilon^\mu + \delta^\mu - \Gamma^\mu{}_{\nu\lambda}\epsilon^\nu\delta^\lambda \tag{1.20}$$

and their difference comes given by

$$r_1^\mu - r_2^\mu = \left(\Gamma^\mu{}_{\nu\lambda} - \Gamma^\mu{}_{\lambda\nu}\right)\epsilon^\nu\delta^\lambda = T^\mu{}_{\nu\lambda}\epsilon^\nu\delta^\lambda \;, \tag{1.21}$$

where the torsion tensor $T^\lambda{}_{\mu\nu}$, is the antisymmetric part of the connection[2]. So generally r_1 and r_2 are not the same, and this four vectors do not form a parallelogram, as one would expect in Euclidean space. This happens only in the case where torsion components vanish. Hence, torsion can be understood as the twist of vectors (and frames) under parallel transport, as an inherent property of the connection of the manifold. To illustrate this, consider $\mathbb{R}^3$ and let $\{X, Y, Z\}$ be the cartesian coordinate vector fields. Choose a connection such that

$$\begin{aligned} \nabla_X Y &= -Z & \qquad \nabla_Y X &= Z \\ \nabla_X Z &= Y & \qquad \nabla_Z X &= -Y \\ \nabla_Y Z &= -X & \qquad \nabla_Z Y &= X \;. \end{aligned}$$

From equation 1.9 one can see that the only non-zero connection components are:

$$\begin{aligned} \Gamma^Z{}_{XY} &= -1 & \qquad \Gamma^Z{}_{YX} &= 1 \\ \Gamma^X{}_{YZ} &= -1 & \qquad \Gamma^X{}_{ZY} &= 1 \\ \Gamma^Y{}_{ZX} &= -1 & \qquad \Gamma^Y{}_{XZ} &= 1 \;. \end{aligned}$$

It is clear that this connection has a non-vanishing torsion, since it is not symmetric. From the parallel transport equation 1.12 the rate of change of the vector components is

[2] I will call torsion to the more convenient tensor $S^\lambda{}_{\mu\nu} \equiv \Gamma^\lambda{}_{[\mu\nu]} = T^\lambda{}_{\mu\nu}/2$ though

$$\frac{dV^a}{dt} = -\Gamma^a{}_{bc} V^b \frac{dx^c}{dt} \ ,$$

which can be written for each component using the connection coefficients above like

$$\begin{aligned}
\frac{dX}{dt} &= \left(Z\frac{dy}{dt} - Y\frac{dz}{dt} \right) \\
\frac{dY}{dt} &= \left(X\frac{dz}{dt} - Z\frac{dx}{dt} \right) \\
\frac{dZ}{dt} &= \left(Y\frac{dx}{dt} - X\frac{dy}{dt} \right) .
\end{aligned}$$

Now, take as the initial vector just a coordinate basis vector, $V_0 = Z$ for example, and parallel transport it along the y direction with velocity v. Along this path we know that

$$\frac{dx}{dt} = 0 \qquad \frac{dz}{dt} = 0 \qquad \frac{dy}{dt} = v \ .$$

Hence, the equations of motion are

$$\frac{dX}{dt} = Zv \qquad \frac{dY}{dt} = 0 \qquad \frac{dZ}{dt} = -Xv \ .$$

The X component decreases or increases proportional to the velocity v depending on the sign of Z and the opposite happens to the Z component. It clearly rotates with respect to the y direction. A body undergoing parallel transport along straight lines in $\mathbb{R}^3$ endowed with this connection, spins around the direction of motion with angular velocity proportional to its speed. The inertial frame spins like Rugby ball, see figure 1.2

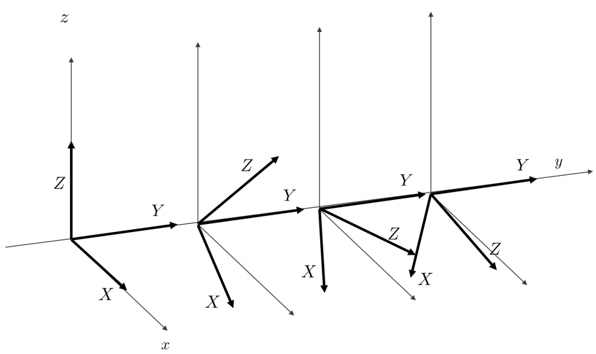

Figure 1.2: A cartesian frame being parallel transported along a straight line in $\mathbb{R}^3$ which rotates due to a connection with torsion.

Formally, though, *torsion* and *curvature* (operators, not tensors) are two multilinear mappings $\mathcal{T} : \mathfrak{X}(M) \otimes \mathfrak{X}(M) \to \mathfrak{X}(M)$ and $\mathcal{R} : \mathfrak{X}(M) \otimes \mathfrak{X}(M) \otimes \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined in the following way:

$$\mathcal{T}(X,Y) \equiv \nabla_X Y - \nabla_Y X - [X,Y] \tag{1.22}$$

$$\mathcal{R}(X,Y)Z \equiv \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{1.23}$$

for $X, Y, Z \in \mathfrak{X}(M)$. Since they map vectors, applied to a covector it will map on $\mathbb{R}$, and that is how the torsion tensor and curvature tensor are defined:

$$\mathcal{T} = T^{\lambda}{}_{\mu\nu} e_\lambda \otimes e^\mu \otimes e^\nu \tag{1.24}$$

$$\mathcal{R} = R^{\lambda}{}_{\mu\nu\kappa} e_\lambda \otimes e^\mu \otimes e^\nu \otimes e^\kappa \ , \tag{1.25}$$

where the tensor components are defined as said before:

$$\begin{aligned} T^{\lambda}{}_{\mu\nu} &= \langle e^\lambda, \mathcal{T}(e_\mu, e_\nu)\rangle \\ &= \langle e^\lambda, \nabla_\mu e_\nu - \nabla_\nu e_\mu \rangle \\ &= \langle e^\lambda, (\Gamma^{\gamma}{}_{\mu\nu} - \Gamma^{\gamma}{}_{\nu\mu})\, e_\gamma \rangle \\ &= (\Gamma^{\gamma}{}_{\mu\nu} - \Gamma^{\gamma}{}_{\nu\mu})\, \delta^{\lambda}_{\gamma} \\ &= \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu} \\ &= 2S^{\lambda}{}_{\mu\nu} \ . \end{aligned} \tag{1.26}$$

Same can be done with the curvature torsion components [18]

$$\begin{aligned} R^{\lambda}{}_{\mu\nu\gamma} &= \langle e^\lambda, \mathcal{R}(e_\mu, e_\nu, e_\gamma)\rangle \\ &= \partial_\nu \Gamma^{\lambda}{}_{\gamma\mu} - \partial_\gamma \Gamma^{\lambda}{}_{\nu\mu} + \Gamma^{\alpha}{}_{\gamma\mu}\Gamma^{\lambda}{}_{\nu\alpha} - \Gamma^{\alpha}{}_{\nu\mu}\Gamma^{\lambda}{}_{\gamma\alpha} \ . \end{aligned} \tag{1.27}$$

From the equations 1.22 and 1.23 the following relations are deduced:

$$S^{\lambda}{}_{\mu\nu} = -S^{\lambda}{}_{\nu\mu} \tag{1.28}$$

$$R^{\lambda}{}_{\mu\nu\gamma} = -R^{\lambda}{}_{\mu\gamma\nu} \ . \tag{1.29}$$

By contracting the curvature tensor the *Ricci tensor* is obtained:

$$R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}{}_{\nu\mu} - \partial_\nu \Gamma^{\lambda}{}_{\lambda\mu} + \Gamma^{\gamma}{}_{\nu\mu}\Gamma^{\lambda}{}_{\lambda\gamma} - \Gamma^{\gamma}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\nu\gamma} \tag{1.30}$$

And further contraction gives the *scalar curvature*:

$$R \equiv g^{\mu\nu} R_{\mu\nu} \ . \tag{1.31}$$

1.4 Levi-Civita connection

In a general n-manifold endowed with a metric and an affine connection, L_n, the most general expression for the connection components is the following one [7]:

$$\Gamma^{\lambda}{}_{\mu\nu} = \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} + K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu} \,. \tag{1.32}$$

The first term are the *Christoffel* symbols of the second kind (which are not components of some tensor field) defined in terms of partial derivatives of the metric:

$$\left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} = \frac{1}{2} g^{\lambda\alpha} \left(\partial_{\mu} g_{\alpha\nu} + \partial_{\nu} g_{\mu\alpha} - \partial_{\alpha} g_{\mu\nu} \right) \,. \tag{1.33}$$

The second term is called the *contorsion tensor* and can be expressed in terms of the torsion tensor $S^{\lambda}{}_{\mu\nu}$ like

$$K^{\lambda}{}_{\mu\nu} \equiv S^{\lambda}{}_{\mu\nu} - S_{\mu\nu}{}^{\lambda} - S_{\nu\mu}{}^{\lambda} \,. \tag{1.34}$$

The third term is defined in terms of the non-metricity tensor in the following way:

$$L^{\lambda}{}_{\mu\nu} \equiv \frac{1}{2}(Q^{\lambda}{}_{\mu\nu} - Q_{\mu\nu}{}^{\lambda} - Q_{\nu\mu}{}^{\lambda}) \,. \tag{1.35}$$

Therefore, the whole geometry of the L_n manifold is completely defined by three independent tensors: $g_{\mu\nu}$, $S^{\lambda}{}_{\mu\nu}$ and $Q^{\lambda}{}_{\mu\nu}$. In a (Pseudo-)Riemaniann manifold, however, $Q_{\lambda\mu\nu} = 0$ and $S^{\lambda}{}_{\mu\nu} = 0$. Thus, the connection coefficients are just the Christoffel symbols, as it can be seen from equation 1.32 and the metric tensor field is sufficient to describe the whole manifold. This connection is called the Levi-Civita connection, which is the connection used in Einstein's Theory of General Relativity.

Fundamental Theorem of (pseudo-)Riemannian manifolds: Given a (pseudo-)Riemannian n-manifold V_n, it exists a unique Levi-Civita connection. In other words, there is a well defined[3] connection such that

$$S^{\lambda}{}_{\mu\nu} \equiv \frac{1}{2}(\Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu}) = 0 \tag{1.36}$$

$$Q_{\lambda\mu\nu} \equiv \nabla_{\lambda} g_{\mu\nu} = 0 \tag{1.37}$$

are satisfied. The proof is pretty straightforward and can be found in [18], chapter 7.

[3]By well defined I mean that it's existance and uniqueness is proven.

2. General Relativity

From all the interactions observed in nature, gravity remains as the most enigmatic one and yet hard to fully comprehend it in a fundamental level. In spite of all the attempts done by modern theoretical physicists trying to describe it in the same framework with electromagnetic, weak and strong interactions, we still do not have a successful theory. However, gravitational force is the oldest one being studied and theorised, and still nowadays, most of the celestial mechanical problems are solved using methods based in Newtonian mechanics. The great success of Newton's theory in order to describe the motion of objects with mass and it's accordance with experimental observations, settled and solidified by the end of XVII his equations of motion and the idea of a rigid space and absolute time, which seemed logic from every-day experience. It was not until the end of the XIX. century that some shortcomings of the theory started to be seen by the scientific community and the assumptions were no longer satisfactory.

For instance, as Newton's theory states, any particle with mass in a gravitational field receives the same acceleration no matter it's mass. This means that the mass, feature of the particle that interacts with the field, does not actually characterize the motion of it. This is suspicious and it is not the case with other interactions. For example, the motion of a charged particle in an electric field is characterized by it's charge. The larger the charge, the greater the acceleration. This was an apparently subtle but annoying thing about the gravitational interaction, which probably stole hours and hours of thinking from physicists of that time and gave a hint that a new gravitational theory was needed. This phenomenon happens because of the "coincidence" or equivalence of the gravitational and inertial mass and would later be known as the Weak Equivalence Principle (WEP).

Another visible weakness of Newtonian theory of gravity was that unlike with Maxwell equations, Newton's law of gravitation is not invariant under Lorentz group transformations. Therefore, whether the whole classical theory of Electromagnetism and Special theory of relativity was incorrect, or Newton's theory would not satisfy the Principle of Relativity and needed to be modified.

From 1907 to 1915 Einstein developed the Theory of General Relativity which would replace the Newtonian theory for gravity and generalize the Theory of Special Relativity. Although it was not completely accepted by the scientific community at first, due to the mathematical complexity and non intuitive notion of space and time, it can not be discussed nowadays that it is not only one of the most beautiful and sophisticated approaches when interpreting nature, but also a great success. A long list of experiments have been made since the theory was published to test it's validity and even in the ones carried out the last decades, with the huge advance observational astronomy and technology has made, the theory remains satisfactory at large scale.

2.1 The Theory of General Relativity

Einstein built his theory on two essential physical principles:

(1) Principle of general covariance: the physics described from any coordinate system must be the same. Consequently, all physical laws must be invariant under coordinate transformations. This is why the theory is developed in tensorial formalism.

(2) Principle of equivalence: it always exists a coordinate system for which the effect of gravity cancels and the physics of Special Relativity is recovered. In other words, a change of coordinate system can always be made so that the metric becomes locally flat and the spacetime locally a Minkowski spacetime.

In Einsteins's General Relativity space and time are part of the same structure called spacetime. Mathematically it is a Lorentzian 4-manifold, which is a Pseudo-Riemannian manifold with signature $(1,3)$. Therefore, the connection is the Levi-Civita connection, which implies that

$$\Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu} \tag{2.1}$$

$$Q_{\lambda\mu\nu} = \nabla_{\lambda} g_{\mu\nu} = 0 \, , \tag{2.2}$$

as seen before, and therefore the connection coefficients are simply defined in terms of the metric by the Christoffel symbols:

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} \left(\partial_{\mu} g_{\alpha\nu} + \partial_{\nu} g_{\mu\alpha} - \partial_{\alpha} g_{\mu\nu} \right) . \tag{2.3}$$

The choice of the Levi-Civita connection is convenient in order to describe the physics in spacetime. First of all , as mentioned in chapter 1, because the existence and uniqueness is mathematically proven. Besides, many tensor identities simplify and it is much easier to mathematically deal with them. For example, Ricci tensor is symmetric and Bianchi identities become much simpler. Moreover, the Equivalence Principle satisfies directly from [9]:

let a point $O \in V_4$ be written in components x^{μ}_{O} in some arbitrary coordinate system $\{x^{\mu}\}$. Let me define a new primed coordinate system in the following form:

$$x^{\mu'} = x^{\mu} - x^{\mu}_{O} + \frac{1}{2} (\Gamma^{\mu}{}_{\nu\lambda})_{O} (x^{\nu} - x^{\nu}_{O})(x^{\lambda} - x^{\lambda}_{O}). \tag{2.4}$$

Check that the origin is at O. Differentiation with respect to x^{α} gives

$$
\begin{aligned}
X^{\mu'}_{\alpha} &= \delta^{\mu}_{\alpha} + \frac{1}{2}(\Gamma^{\mu}{}_{\nu\lambda})_O\delta^{\nu}_{\alpha}(x^{\lambda} - x^{\lambda}_O) + \frac{1}{2}(\Gamma^{\mu}{}_{\nu\lambda})_O(x^{\nu} - x^{\nu}_O)\delta^{\lambda}_{\alpha} \\
&= \delta^{\mu}_{\alpha} + \frac{1}{2}(\Gamma^{\mu}{}_{\alpha\lambda})_O(x^{\lambda} - x^{\lambda}_O) + \frac{1}{2}(\Gamma^{\mu}{}_{\nu\alpha})_O(x^{\nu} - x^{\nu}_O),
\end{aligned}
\tag{2.5}
$$

where $X^{\mu'}_{\alpha} \equiv \partial x^{\mu'}/\partial x^{\alpha}$ are the components of the Jacobi matrix. Since the connection is the Levi-Civita connection it is symmetric. Hence,

$$X^{\mu'}_{\alpha} = \delta^{\mu}_{\alpha} + (\Gamma\mu_{\nu\alpha})_O(x^{\nu} - x^{\nu}_O). \tag{2.6}$$

And evaluated at point O:

$$(X^{\mu'}_{\alpha})_O = \delta^{\mu}_{\alpha}\,. \tag{2.7}$$

Differentiating 2.6 again it gives:

$$X^{\mu'}_{\alpha\beta} \equiv \frac{\partial x^{\mu'}}{\partial x^{\alpha}\partial x^{\beta}} = (\Gamma^{\mu}{}_{\nu\alpha})_O\delta^{\nu}_{\beta} = (\Gamma^{\mu}{}_{\beta\alpha})_O\,. \tag{2.8}$$

Making a change of coordinate from the unprimed to the primed coordinate system the connection leaves (remember it is not a tensor and transforms as the following expression)

$$\Gamma^{\mu'}{}_{\nu'\lambda'} = \Gamma^{\alpha}{}_{\beta\gamma}X^{\mu'}_{\alpha}X^{\beta}_{\nu'}X^{\gamma}_{\lambda'} - X^{\alpha}_{\nu'}X^{\beta}_{\lambda'}X^{\mu'}_{\alpha\beta}. \tag{2.9}$$

And evaluated at the origin O:

$$
\begin{aligned}
(\Gamma^{\mu'}{}_{\nu'\lambda'})_O &= (\Gamma^{\alpha}{}_{\beta\gamma})_O\delta^{\mu}_{\alpha}\delta^{\beta}_{\nu}\delta^{\gamma}_{\lambda} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\lambda}(\Gamma^{\mu}{}_{\alpha\beta})_O \\
&= (\Gamma^{\mu}{}_{\nu\lambda})_O - (\Gamma^{\mu}{}_{\nu\lambda})_O \\
&= 0
\end{aligned}
\tag{2.10}
$$

In conclusion, if the Levi-Civita connection is chosen, it exists a coordinate system in which the connection coefficients vanish locally in the neighbourhood of O and hence, there is no curvature. Another Lorentz transformation can be made to reduce the metric to $diag\{-1, 1, 1, 1\}$. Minkowski spacetime is recovered locally, and Special Relativity physics rule. Therefore, for a Levi-Civita connection Equivalence principle is always satisfied.

2.2 Einstein field equations

Einstein field equations (EFE) are the ultimate mathematical result triggered by Einstein's brilliant physical intuition and persistence, being without any doubt one of the biggest successes in joining mathematical elegance and physical sense. It is a tensor equation of symmetric 4x4 tensors, which describes the interaction between mass-energy distribution and the geometry of the spacetime:

$$G^{\mu\nu} = \kappa T^{\mu\nu}\,, \tag{2.11}$$

where κ is some proportionality factor, $G^{\mu\nu}$ are the components of some tensor describing the geometry of the spacetime and $T^{\mu\nu}$ are the energy-momentum tensor components which describe the energy and mass distribution. From conservation of energy, the divergence of $T^{\mu\nu}$ must vanish, i.e

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{2.12}$$

It is also sensible to think that in the Newtonian limit the Poisson equation for gravitational potential should be recovered:

$$\nabla^2\Phi = 4\pi G\rho\,. \tag{2.13}$$

It can be shown pretty easily (chapter 2.7 from [9]), that for a nearly Cartesian coordinate system, g_{00} reduces to $1 + 2\Phi c^2$, so we can expect that in any general coordinate system the metric tensor components $g_{\mu\nu}$ directly represent the gravitational potential. Therefore, in order to recover the equation 2.13 in the Newtonian limit, the tensor components $G^{\mu\nu}$ should contain second order derivatives of $g^{\mu\nu}$. After many trials and exhausting work, by the end of 1915, Einstein found the form of $G^{\mu\nu}$ that satisfies all these requirements, what is now known as Einstein tensor:

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\,. \tag{2.14}$$

The proportionality factor κ is easily found by requiring to recover equation 2.13 in the Newtonian limit, finally giving the system of 10 second order nonlinear coupled partial differential equations called Einstein field equations:

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu}\,. \tag{2.15}$$

Later, in 1917, Einstein would modify the equations adding the Cosmological term, giving birth to the field of Cosmology and leaving the EFE like

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \Lambda g^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu}\,. \tag{2.16}$$

2.2.1 Einstein-Hilbert action

Like in any field theory, the field equations can be obtained from a variation of an action in the hamiltonian formalism. In order to obtain the EFE, the lagrangian density must contain a term describing the geometry of the spacetime, $\mathscr{L}_g$, and another term describing the matter-energy distribution, $\mathscr{L}_m$. Requiring that the lagrangian density must be a scalar and contain second derivatives of the metric, the simplest guess for $\mathscr{L}_g$ is the only natural contraction of the curvature tensor, the scalar curvature: $\mathscr{L}_g \propto R$. The proportionality factor is obtained from the Newtonian limit, and the result is the Einstein-Hilbert action[1]:

[1] $\Omega_M = \sqrt{-g}d^nx$ is the invariant volume element of a n-manifold endowed with a metric $g_{\mu\nu}$, where $g = det(g_{\mu\nu})$.

$$S_{EH} = \frac{1}{2\kappa}\int R\sqrt{-g}d^4x \ , \tag{2.17}$$

where $\kappa = 8\pi G/c^4$. On the other side, the lagrangian density that describes the matter-energy content might vary a lot, but in general it depends on some matter field[2] $\phi(x)$, its derivatives and the metric. Hence, the matter-energy content can be described in general by

$$S_M = \int \mathscr{L}_m(\phi, \partial\phi, g)\sqrt{-g}d^4x \ , \tag{2.18}$$

and the total action leaves:

$$S[g, \partial g, \phi, \partial\phi] = \frac{1}{2\kappa}\int R\sqrt{-g}d^4x + \int \mathscr{L}_m(\phi, \partial\phi, g)\sqrt{-g}d^4x, \tag{2.19}$$

The action has 10 degrees of freedom from the metric $g_{\mu\nu}$ and one (or more) from the matter field $\phi(x)$. However, variation with respect to the field gives nothing interesting, just the matter equation $\delta\mathscr{L}_m/\delta\phi = 0$. Now, for simplicity, let me compute the variation with respect to $g^{\mu\nu}$ separately in S_{EH} and S_m:

$$\begin{aligned}
\delta S_{EH} &= \frac{1}{2\kappa}\int \delta\Big(R\sqrt{-g}\Big)d^4x \\
&= \frac{1}{2\kappa}\int \Big[\delta R\sqrt{-g} + R\delta\Big(\sqrt{-g}\Big)\Big]d^4x \\
&= \frac{1}{2\kappa}\int \left[\delta(R_{\mu\nu}g^{\mu\nu})\sqrt{-g} - \frac{1}{2}R\frac{\delta g}{\sqrt{-g}}\right]d^4x \\
&= \frac{1}{2\kappa}\int \sqrt{-g}\left[\delta R_{\mu\nu}g^{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\delta g^{\mu\nu}\right]d^4x
\end{aligned}$$

In this last step I have used $\delta g = \delta det(g^{\mu\nu}) = gg_{\mu\nu}\delta g^{\mu\nu}$. Now, we can make use of the Palatini identity[3] to write $\delta R_{\mu\nu}$:

$$\begin{aligned}
\delta S_{EH} &= \frac{1}{2\kappa}\int \sqrt{-g}\left\{\left[\nabla_\lambda(\delta\Gamma^\lambda{}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\lambda{}_{\lambda\mu})\right]g^{\mu\nu} + \left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]\delta g^{\mu\nu}\right\}d^4x \\
&= \frac{1}{2\kappa}\int \sqrt{-g}\left\{\left[\nabla_\lambda(g^{\mu\nu}\delta\Gamma^\lambda{}_{\nu\mu}) - \nabla_\nu(g^{\mu\nu}\delta\Gamma^\lambda{}_{\lambda\mu})\right] + \left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]\delta g^{\mu\nu}\right\}d^4x \\
&= \frac{1}{2\kappa}\int \sqrt{-g}\left\{\nabla_\gamma\left[g^{\mu\nu}\delta\Gamma^\gamma{}_{\nu\mu} - g^{\mu\gamma}\delta\Gamma^\nu{}_{\nu\mu}\right] + \left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]\delta g^{\mu\nu}\right\}d^4x
\end{aligned}$$

The first part is a divergence term called the Gibbons-Hawking-York [21, 10] boundary term, which vanishes when integrated in a closed spacetime where $\delta g_{\mu\nu} =$

[2]Matter fields are in general tensor fields, although they can also have some internal degrees of freedom, such as spinor fields. An example of a scalar field would be the real scalar field $\mathscr{L}_m = -(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2)/2$, an example of a $(1,0)-$tensor field would be the Maxwell field $\mathscr{L}_m = -F_{\mu\nu}F^{\mu\nu}/4$, where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, where the field is a four-potential $A(x)$ and an example of a spinor field would be the Dirac field $\mathscr{L}_m = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$. By now the matter filed $\phi(x)$ is assumed to be scalar, since the procedure is exactly the same in this case for any field type, and later in section 3.2 intrinsic degrees of freedom will be considered.

[3]$\delta R_{\mu\nu} = \nabla_\gamma(\delta\Gamma^\gamma{}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\gamma{}_{\gamma\mu})$

0 at the boundary by the Stokes-Cartan theorem, and the second term is just the Einstein tensor. Note that the first step made here is of crucial importance: the metric $g^{\mu\nu}$ comes inside the covariant derivative, because the connection is Levi-Civita and consequently $\nabla_\lambda g^{\mu\nu} = 0$. Coming back to the action of the matter fields, variation with respect to the metric gives, by definition, the Hilbert energy-momentum tensor:

$$\delta S_m = \int \frac{\delta \mathscr{L}_m}{\delta g^{\mu\nu}} \delta g^{\mu\nu} \sqrt{-g} d^4x = \frac{-1}{2} \int T_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g} d^4x \; . \tag{2.20}$$

Putting it all together:

$$\delta S = \frac{1}{2} \int \delta g^{\mu\nu} \sqrt{-g} \left[\frac{1}{\kappa} \Big(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \Big) - T_{\mu\nu} \right] d^4x = 0 \; , \tag{2.21}$$

which gives the EFE of equation 2.15. The modified equations with the cosmological term are derived the same way, but with a constant term added to the Ricci tensor in the Einstein-Hilbert action: $R \to R + \Lambda$.

2.2.2 Palatini formalism

In order to study different or modified theories of gravity, one usually seeks to take out constrains in the derivation of the field equations. In the section before, I have worked directly with the Levi-Civita connection, without a convincing argument of choice, just for convenience, letting the metric be the unique geometrical structure needed to completely define the spacetime. However, one could simply not assume metric-compatible and/or torsionless conditions, leading to a spacetime where the metric and the connection are independent. This formalism has many advantages and does not make any assumption about the geometry of spacetime. Hence, in the following derivation the non-metricity and torsionless conditions 1.37 and 1.36 are no longer assumed. When applying now variation principle to the Einstein-Hilbert action, the variation with respect to the connection has to be considered as well, since it is not defined by the metric. By now, I will assume the matter field is scalar and does not couple to the connection, but only to the metric. Then, the action is

$$S[g, \Gamma, \partial\Gamma, \phi, \partial\phi] = \frac{1}{2\kappa} \int g^{\mu\nu} R_{\mu\nu}(\Gamma) \sqrt{-g} d^4x + \int \mathscr{L}_m(\phi, \partial\phi, g) \sqrt{-g} d^4x \; . \tag{2.22}$$

The big difference here is that the Ricci tensor does not depend on ∂g terms anymore, but on Γ and $\partial\Gamma$ terms, as it can bee seen from the definition 1.30. Therefore, the action in principle depends on the 64 components of $\Gamma^\lambda{}_{\mu\nu}$, the 10 components of $g_{\mu\nu}$ and the scalar matter field $\phi(x^\mu)$. Variation with respect to $g^{\mu\nu}$ leads to the same field equations as seen before, since the $g^{\mu\nu}\delta R_{\mu\nu}$ term vanished anyways. Variation with respect to the matter field leads to the same matter equation as before as well. However, variation with respect to $\Gamma^\lambda{}_{\mu\nu}$ has to be made carefully:

-With the new generalized connection coefficients $\Gamma^{\lambda}{}_{\mu\nu}$, despite the definition of the Ricci tensor does not change, it does change the palatini identity. The modified Palatini identity has an extra term related to the torsion:

$$\delta R_{\mu\nu} = \nabla_\lambda(\delta\Gamma^{\lambda}{}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\lambda}{}_{\lambda\mu}) + 2S^{\gamma}{}_{\nu\lambda}\delta\Gamma^{\lambda}{}_{\mu\gamma} \; . \tag{2.23}$$

- Now $\nabla_\lambda g^{\mu\nu} \neq 0$, and it appears an extra term when writing $g^{\mu\nu}$ inside the covariant derivative, due to the chain rule:

$$g^{\mu\nu}\nabla_\gamma(\delta\Gamma^{\lambda}{}_{\mu\nu}) = \nabla_\gamma(g^{\mu\nu}\delta\Gamma^{\lambda}{}_{\mu\nu}) - (\nabla_\lambda g^{\mu\nu})\delta\Gamma^{\lambda}{}_{\mu\nu} \; . \tag{2.24}$$

Therefore,

$$\begin{aligned}
\delta S &= \frac{1}{2\kappa}\int g^{\mu\nu}\delta R_{\mu\nu}(\Gamma)\sqrt{-g}d^4x \\
&= \frac{1}{2\kappa}\int g^{\mu\nu}\left[\nabla_\lambda(\delta\Gamma^{\lambda}{}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\lambda}{}_{\lambda\mu}) + 2S^{\gamma}{}_{\nu\lambda}\delta\Gamma^{\lambda}{}_{\mu\gamma}\right]\sqrt{-g}d^4x \\
&= \frac{1}{2\kappa}\int \left\{\nabla_\lambda\left[g^{\mu\nu}(\delta\Gamma^{\lambda}{}_{\nu\mu})\right] - \nabla_\nu\left[g^{\mu\nu}(\delta\Gamma^{\lambda}{}_{\lambda\mu})\right] + (\nabla_\nu g^{\mu\nu})\delta\Gamma^{\lambda}{}_{\lambda\mu} - (\nabla_\lambda g^{\mu\nu})\delta\Gamma^{\lambda}{}_{\nu\mu}\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. + 2g^{\mu\nu}S^{\gamma}{}_{\nu\lambda}\delta\Gamma^{\lambda}{}_{\mu\gamma}\right\}\sqrt{-g}d^4x \\
&= \frac{1}{2\kappa}\int \nabla_\lambda\left[g^{\mu\nu}(\delta\Gamma^{\lambda}{}_{\nu\mu}) - g^{\mu\lambda}(\delta\Gamma^{\gamma}{}_{\gamma\mu})\right]\sqrt{-g}d^4x \\
&\quad + \frac{1}{2\kappa}\int \left[(\nabla_\nu g^{\mu\nu})\delta\Gamma^{\lambda}{}_{\lambda\mu} - (\nabla_\lambda g^{\mu\nu})\delta\Gamma^{\lambda}{}_{\nu\mu} + 2g^{\mu\nu}S^{\gamma}{}_{\nu\lambda}\delta\Gamma^{\lambda}{}_{\mu\gamma}\right]\sqrt{-g}d^4x.
\end{aligned}$$

The first term is the same divergence term as before, so it vanishes as well. Therefore, only the second integral is left and by the least action principle:

$$\int \left[(\nabla_\nu g^{\mu\nu})\delta\Gamma^{\lambda}{}_{\lambda\mu} - (\nabla_\lambda g^{\mu\nu})\delta\Gamma^{\lambda}{}_{\nu\mu} + 2g^{\mu\nu}S^{\gamma}{}_{\nu\lambda}\delta\Gamma^{\lambda}{}_{\mu\gamma}\right] d^4x = 0 \; . \tag{2.25}$$

Note that if now I assume the connection to metric compatible, the first two terms are zero and hence, it implies that it is also torsionless. The other way works as well. If the connection is torsionless, the last term vanishes and it implies that the first two do so, i.e. it is also metric compatible. The conclusion is that following the Palatini formalism for a completely independent and unknown connection Γ, the Levi-Civita connection comes up in a natural way, and standard General Relativity is recovered. This derivation has been made considering the connection to be completely independent. However, there is a very important relation between Γ, g, S and Q from 1.32 which will be used in the next chapter.

3. U_4 geometry and gravitation in U_4 spacetime

Einstein's General Relativity works perfectly fine in a macroscopic level and although it is true that it is not satisfactory within the same framework as other interactions, one could think it does not really matter, since the gravitational interaction at microscopic level is really weak in comparison. Maybe there is no need to understand it in a fundamental level and put efforts trying to take General Relativity to further generalization, at least until experimental disagreement, in order to have a good understanding of both macroscopic and microscopic physics separately. Nevertheless, many open questions in high energy physics, cosmology and particle physics could be answered by a more solid theory of gravity. In my opinion, and being less pragmatical and holist, the search of beauty and the understanding of nature as a whole is much more satisfactory rather than having different theories for different scales.

Relativistic quantum field theory manages to construct a very good description of electromagnetic, strong and weak interactions. This description is carried out in a flat Minkowski spacetime though, while gravity is the curvature of the spacetime itself. The field theory description of the three interactions break, however, when curvature is added into the spacetime. There is no problem when the spacetime is asymptotically flat, but when the curvature is not negligible at all like in the vicinity of massive objects or in the early universe high density plasma, the theory is unsatisfactory. A description of gravitational interaction without curvature would be much easier to include it in the framework of relativistic quantum mechanics. There are, in fact, some theories that try to explain gravity as a result of the torsion of the spacetime, instead of the result curvature, called teleparallelism theories although they are unsatisfactory so far. Einstein himself was the first with this attempt [8].

In this chapter my intention is to review different geometric alternatives for possible spacetimes and analyze in detail the Einstein-Cartan spacetime U_4. I intend to focus on analyzing how does a U_4 spacetime behave in comparison with V_4 and deriving the gravitational field equations the same way it was previously done. This time the matter field is going to be taken a general tensor field $\Phi(x)$, with its degrees of freedom. This is important, since it implies that that covariant derivative carries the torsion degrees of freedom.

3.1 Riemann-Cartan spacetime

Instead of directly considering U_4 spacetime, let me start from the most general mathematical manifold and see how this reduces to different kind of spacetimes. It is remarkable that U_4 spacetime is the most general one obtained by physical restriction.

The most general four-dimensional spacetime one can assume from a mathematical point of view, considering that it is continuum, is just M_4, where any point is described by one time coordinate and three spacial coordinates. In order to compute differential calculus on this spacetime, there must be a defined way to transport vectors and tensors along curves in the spacetime. In other words, the manifold M_4 must be endowed with some affine connection ∇ connecting the tangent spaces of different points. In order to do physics, and what I mean by this is the possibility to make measurements, it is essential to define a metric tensor field as well. A manifold satisfying these conditions is called a linearly connected manifold endowed with a metric: (L_4, g). In such manifold, the non-metricity tensor defined in 1.16 expresses how does the metric tensor field vary along parallel transport, and the torsion tensor field expresses the closure failure when constructing parallelograms by parallel transport. Q determines how vectors change in length and S determines they spin.

Following the principle of general covariance, it looks reasonable to set the metric to be covariantly constant, i.e. $Q_{\mu\nu\lambda} = 0$. This guarantees that measurements of spacetime (distances, angles, ...) are preserved under parallel transport and that physical laws expressed by tensors and covariant derivatives remain invariant under coordinate transformations. Such spacetime is called a *Riemann-Cartan spacetime*, U_4. In this spacetime the metric and the torsion are the unique independent geometric structures, and they provide all the geometric information about the manifold.

Setting more restrictions on a U_4 spacetime like torsionless condition leads to the usual Pseudo-Riemannian spacetime V_4. One could also restrict the spacetime to be flat instead, vanishing curvature everywhere as a restriction and leave torsion to be non-zero. Such spacetime is called *Weitzenbock spacetime*, and theories of teleparallel gravity mentioned before attempt to construct a precise description of gravity in such a spacetime. Of course when both curvature and torsion vanish, one just obtains a Minkowski spacetime. Figure 3.1 shows a scheme of these spacetimes, based on the shorter scheme given in [11].

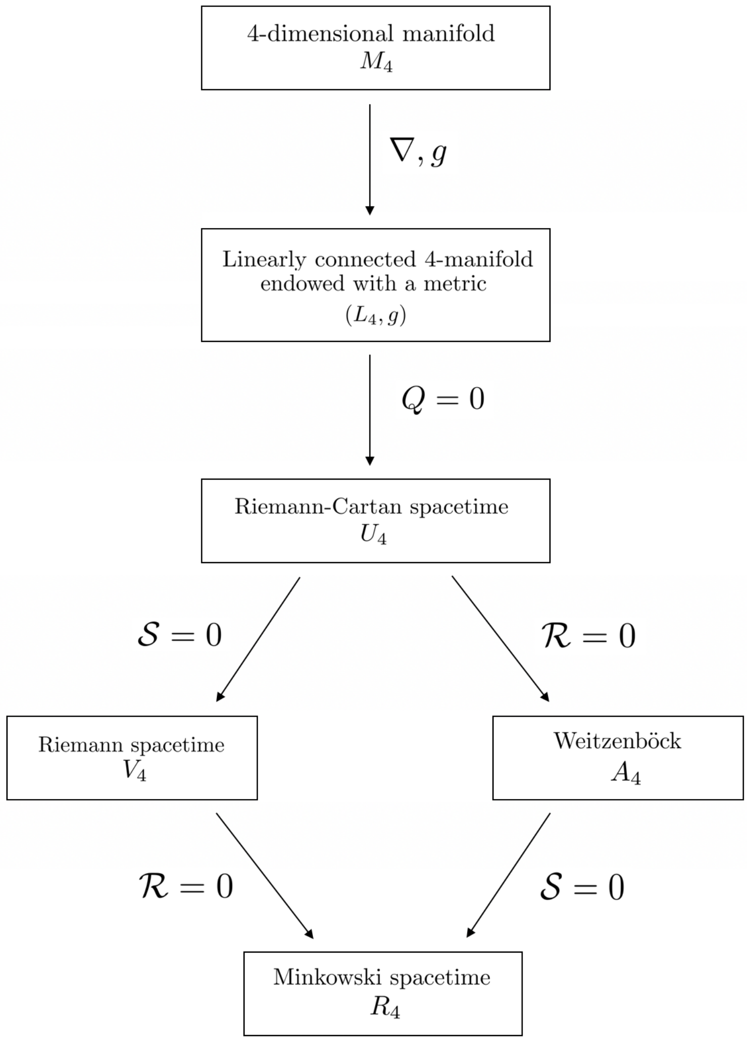

Figure 3.1: Different manifolds that can potentially describe the spacetime and their respective restriction.

Despite the classical Theory of General Relativity is developed in a Pseudo-Riemannian spacetime, there is not physical argument to vanish torsion. Hence, nothing stops one from considering a U_4 spacetime. The main difference between a U_4 and a V_4 spacetime is that while in U_4 $\Gamma^{\lambda}{}_{\mu\nu}$ and $g_{\mu\nu}$ are independent, they are not in V_4, and $\Gamma^{\lambda}{}_{\mu\nu}$ can by expressed in terms of $g_{\mu\nu}$ by the Christoffel symbols of the second kind. In U_4 spacetimes the contorsion tensor appears when relating these two obejects. Based on the general equation 1.32, for U_4 spacetimes one has

$$\Gamma^{\lambda}{}_{\mu\nu} = \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} + K^{\lambda}{}_{\mu\nu} \, . \tag{3.1}$$

3.1.1 Geodesics in U_4

One important consequence of the generalized U_4 spacetime comes when one tries to define geodesics. Recall that in V_4, geodesics were defined as "shortest" and the "straightest" curves between two points, though the geodesic equation was derived from the condition that a vector tangent to the curve was parallel to itself, without making any assumption about the connection nor the spacetime. Therefore, the straightest curves in U_4, named *autoparallels*, are define the same way:

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt} = 0 \,. \tag{3.2}$$

However, it is important to keep in mind that the $\Gamma^\lambda{}_{\mu\nu}$ coefficients are no more the symmetric Christoffel symbols, but the connection coefficients of U_4. It is easy to see by separating the symmetric and antisymmetric parts,

$$\Gamma^\mu{}_{\nu\lambda} = \frac{1}{2}\Big(\Gamma^\mu{}_{(\nu\lambda)} + \Gamma^\mu{}_{[\nu\lambda]}\Big) \,, \tag{3.3}$$

that the antisymmetric part of the connection does not take part in the equation 3.2. However, the symmetric part of the connection depends on the symmetric part of the contorsion tensor, which is generally non-zero, as it can be checked from equation 1.34.

On the other side, one can look for the "shortest" curve between two points. This is computed by applying the action principle on the distance functional:

$$\delta S = 0 \,, \qquad \text{where} \qquad S = \int ds \,. \tag{3.4}$$

Knowing that $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, it is much easier to insert $\delta(ds^2)$, related to $\delta(ds)$ by

$$\delta(ds^2) = 2ds\delta(ds) \rightarrow \delta(ds) = \frac{\delta(ds^2)}{2ds} \,. \tag{3.5}$$

Hence, variation of the distance action is

$$\delta S = \int \delta(ds) = \int \frac{\delta(ds^2)}{2ds} = \frac{1}{2}\int \delta\Big(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{ds}\Big)d\tau = 0 \,, \tag{3.6}$$

where τ is just parameterizing the curve from some initial point P to some final point Q. Getting rid of the 1/2 factor outside the integral and using the chain rule:

$$\delta S = \int \Big(\delta g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{ds} + g_{\mu\nu}\frac{d(\delta x^\mu)}{d\tau}\frac{dx^\nu}{ds} + g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d(\delta x^\nu)}{ds}\Big)d\tau = 0 \,. \tag{3.7}$$

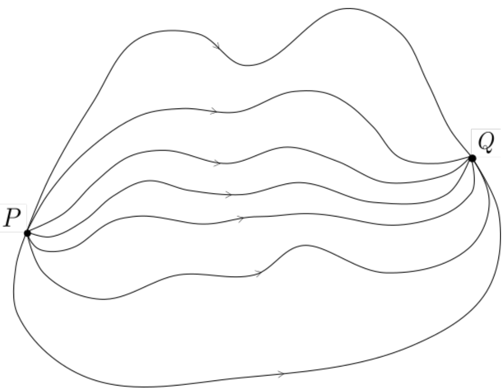

Figure 3.2: Some possible paths from P to Q to ilustrate the least action principle.

Variation on the components of the metric can be expressed by

$$\delta g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\delta x^\lambda = \partial_\lambda g_{\mu\nu}\delta x^\lambda \tag{3.8}$$

and the last two terms of the integral are simply the same by rearranging the indices. Hence, it gives

$$\delta S = \int \Big(\partial_\lambda g_{\mu\nu}\delta x^\lambda \frac{dx^\mu}{d\tau}\frac{dx^\nu}{ds} + 2g_{\mu\nu}\frac{d(\delta x^\mu)}{d\tau}\frac{dx^\nu}{ds}\Big)d\tau\ . \tag{3.9}$$

Now, the second term can be integrated by parts,

$$\int \Big(2g_{\mu\nu}\frac{dx^\nu}{ds}\Big)\Big(d(\delta x^\mu)\Big) = \Big[2g_{\mu\nu}\frac{dx^\nu}{ds}\delta x^\mu\Big]_P^Q - \int 2\frac{d}{ds}\Big(g_{\mu\nu}\frac{dx^\nu}{ds}\Big)\delta x^\mu ds\ . \tag{3.10}$$

The first term is zero since the variation of the coordinates in the boudaries is obviusly zero (see picture 3.2). Using the chain rule with the term left and replacing it in the equation 3.9 one gets

$$\begin{aligned}\delta S &= \int \Big(\partial_\lambda g_{\mu\nu}\delta x^\lambda \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - 2\delta x^\mu \partial_\lambda g_{\mu\nu}\frac{dx^\lambda}{ds}\frac{dx^\nu}{ds} - 2\delta x^\mu g_{\mu\nu}\frac{d^2x^\nu}{ds^2}\Big)ds \\ &= \int \Big(\partial_\lambda g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - 2\partial_\mu g_{\lambda\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - 2g_{\lambda\nu}\frac{d^2x^\nu}{ds^2}\Big)\delta x^\lambda ds \\ &= \int \Big(\partial_\lambda g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - \partial_\mu g_{\lambda\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - \partial_\nu g_{\lambda\mu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - 2g_{\lambda\nu}\frac{d^2x^\nu}{ds^2}\Big)\delta x^\lambda ds \\ &= 0\ .\end{aligned} \tag{3.11}$$

Therefore it can be see that the curve that minimizes the distance between two points is given by:

$$2g_{\lambda\nu}\frac{d^2x^\nu}{ds^2} + \Big(\partial_\lambda g_{\mu\nu} - \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\lambda\mu}\Big)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0\ , \tag{3.12}$$

or contracting with $g^{\gamma\nu}$ and dividing by two:

$$\begin{aligned}
&\frac{d^2x^\gamma}{ds^2} + \frac{1}{2} g^{\gamma\nu}\Big(\partial_\lambda g_{\mu\nu} - \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\lambda\mu}\Big)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \\
&= \frac{d^2x^\gamma}{ds^2} + \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \\
&= 0\,.
\end{aligned} \tag{3.13}$$

If now one compares the autoparallelism equation 3.2 and the extremal equation 3.13, there is a clear difference between the equations that characterizes the "straightest" curve and the "shortest" curve between two points. The extremals or shortest paths are the same in U_4 and in V_4, but the definition of straightness changes when generalizing to U_4, due to the intrinsic twist of the spacetime coming from torsion.

3.2 Field equations

Variation of the Einstein-Hilbert action is quite different now in a U_4 spacetime, since the connection (and hence covariant derivatives) depend on torsion degrees of freedom. The big difference between the analysis of section 2.2.2 and the following one is that before, the matter fields would not couple to the connection, but now they do, by the covariant derivative on the field. The covariant derivative of a scalar field is just a partial derivative, and does not depend on the connection. However, covariant derivatives of tensor fields and spinor fields do depend on the connection (see equation A.13) and hence, there is a big difference between V_4 and U_4. Therefore, let me now consider an arbitrary tensor field[1] $\Phi(x)$ in the lagrangian density of matter fields, which implies new degrees of freedom due to torsion,

$$\mathscr{L}_m(\Phi, \nabla\Phi, g) = \mathscr{L}_m(\Phi, \partial\Phi, \omega, g) \to \mathscr{L}_m(\Phi, \partial\Phi, g, \partial g, S)\,, \tag{3.14}$$

and hence, new variations of the action. This lagrangian density depends on ψ, the ten components of $g_{\mu\nu}$ and the 24 components of $S^\lambda{}_{\mu\nu}$. Variation with respect to different components lead to different tensors:

$$T^{\mu\nu} = 2\frac{\delta \mathscr{L}_m}{\delta g_{\mu\nu}} \qquad \text{Hilbert energy-momentum tensor} \tag{3.15}$$

$$s_\lambda{}^{\mu\nu} = \frac{\delta \mathscr{L}_m}{\delta S^\lambda{}_{\mu\nu}} \qquad \text{spin-energy potential} \tag{3.16}$$

$$\sigma_\lambda{}^{\mu\nu} = \frac{\delta \mathscr{L}_m}{\delta K^\lambda{}_{\mu\nu}} \qquad \text{spin angular momentum tensor} \tag{3.17}$$

$$\Theta^{\mu\nu} = T^{\mu\nu} - \tilde{\nabla}_\lambda s^{\mu\nu\lambda} \qquad \text{canonical stress-energy tensor}\,, \tag{3.18}$$

[1]The following procedure could be easily done for spinor fields as well[12] using non-cordinate frames (see appendix A)

where in the last equation is used the modified divergence $\tilde{\nabla}_\mu = \nabla_\mu + 2S^\nu{}_{\mu\nu}$. When trying to determine $\mathscr{L}_g$, in comparison with the Palatini procedure carried out before, we know that the connection depends on the derivatives of the metric and the torsion. Thus, the independent ingredients that define the geometry and might participate in the construction of $\mathscr{L}_g$ are g and S:

$$\mathscr{L}_g(g, \Gamma) \to \mathscr{L}_g(g, \partial g, S) \; . \tag{3.19}$$

Without making any assumptions by now about $\mathscr{L}_g$, variation of the total action with respect to the different independent variables goes as follows. From the total action

$$S = \int \mathscr{L}_g(g, \partial g, S)\sqrt{-g}d^4x + \int \mathscr{L}_m(\Phi, \partial\Phi, g, \partial g, S)\sqrt{-g}d^4x \; . \tag{3.20}$$

Variation with respect to matter fields gives the matter equation:

$$\frac{\delta \mathscr{L}_m}{\delta \Phi} = 0 \; . \tag{3.21}$$

Variation with respect to the metric:

$$-\frac{\delta \mathscr{L}_g}{\delta g_{\mu\nu}} = \kappa T^{\mu\nu} \; , \tag{3.22}$$

which can be written, by using equation 3.18, more conveniently for latter purposes like

$$-\frac{\delta \mathscr{L}_g}{\delta g_{\mu\nu}} - \frac{g^{\lambda\mu}}{2}\tilde{\nabla}_\gamma\Big(\frac{\delta \mathscr{L}_g}{\delta S^\lambda{}_{\nu\gamma}}\Big) = \kappa \Theta^{\mu\nu} \; . \tag{3.23}$$

Variation with respect to the torsion:

$$-\frac{\delta \mathscr{L}_g}{\delta S^\lambda{}_{\mu\nu}} = 2\kappa s_\lambda{}^{\nu\mu} \; , \tag{3.24}$$

which can also be written like

$$-\Big(\frac{g^{\gamma[\mu}}{2}\Big)\frac{\delta \mathscr{L}_g}{\delta S^\gamma{}_{\nu]\lambda}} = \kappa \sigma^{\mu\nu\lambda} \tag{3.25}$$

by using equation 3.18 and the the identity $s^{\lambda\mu\nu} = -\sigma^{\lambda\mu\nu} + \sigma^{\mu\nu\lambda} - \sigma^{\nu\lambda\mu}$ easily derived from the definition of torsion. It is time to set the explicit form of $\mathscr{L}_g$ now, and as it has been explained in section 2.2.1 for V_4 spacetimes, the most logical choice for U_4 spacetimes is the curvature scalar density $\mathscr{L}_g = R$ as well. Variations with respect to its independent variables are obtained from [13]:

$$\frac{\delta \mathscr{L}_g}{\delta g_{\mu\nu}} = -G^{\mu\nu} + \tilde{\nabla}_\lambda\Big(\tilde{S}^{\lambda\mu\nu} - \tilde{S}^{\mu\nu\lambda} - \tilde{S}^{\nu\lambda\mu}\Big) \tag{3.26}$$

$$\frac{\delta \mathscr{L}_g}{\delta S^\lambda{}_{\mu\nu}} = -2\Big(\tilde{S}^\nu{}_\lambda{}^\mu - \tilde{S}_\lambda{}^{\mu\nu} - \tilde{S}^{\mu\nu}{}_\lambda\Big) \; , \tag{3.27}$$

where $\tilde{S}^{\lambda}{}_{\mu\nu} = S^{\lambda}{}_{\mu\nu} + 2g^{\lambda}_{[\mu}S^{\gamma}{}_{\nu]\gamma}$ is the modified torsion tensor. Now inserting these variations into equation 3.23:

$$G^{\mu\nu} - \tilde{\nabla}_{\lambda}\Big(\tilde{S}^{\lambda\mu\nu} - \tilde{S}^{\mu\nu\lambda} - \tilde{S}^{\nu\lambda\mu}\Big) + g^{\lambda\mu}\tilde{\nabla}_{\gamma}\Big(\tilde{S}^{\gamma}{}_{\lambda}{}^{\nu} - \tilde{S}_{\lambda}{}^{\nu\gamma} - \tilde{S}^{\nu\gamma}{}_{\lambda}\Big) = \kappa\Theta^{\mu\nu}$$

$$G^{\mu\nu} = \kappa\Theta^{\mu\nu}\ . \tag{3.28}$$

And for equation 3.25:

$$-\frac{1}{4}\Big(g^{\gamma\mu}\frac{\delta\mathscr{L}_g}{\delta S^{\gamma}{}_{\nu\lambda}} - g^{\gamma\nu}\frac{\delta\mathscr{L}_g}{\delta S^{\gamma}{}_{\mu\lambda}}\Big) = \kappa\sigma^{\mu\nu\lambda}$$
$$\frac{1}{2}\Big[g^{\gamma\mu}\Big(\tilde{S}^{\lambda}{}_{\gamma}{}^{\nu} - \tilde{S}_{\gamma}{}^{\nu\lambda} - \tilde{S}^{\nu\lambda}{}_{\gamma}\Big) - g^{\gamma\nu}\Big(\tilde{S}^{\lambda}{}_{\gamma}{}^{\mu} - \tilde{S}_{\gamma}{}^{\mu\lambda} - \tilde{S}^{\mu\lambda}{}_{\gamma}\Big)\Big] = \kappa\sigma^{\mu\nu\lambda}$$
$$\frac{1}{2}\Big(\tilde{S}^{\lambda\mu\nu} - \tilde{S}^{\mu\nu\lambda} - \tilde{S}^{\nu\lambda\mu} - \tilde{S}^{\lambda\nu\mu} + \tilde{S}^{\nu\mu\lambda} + \tilde{S}^{\mu\lambda\nu}\Big) = \kappa\sigma^{\mu\nu\lambda}$$

Now check from the definition of the modified torsion tensor that it is antisymmetric on its last two indices. Hence, it leaves

$$\tilde{S}^{\mu\nu\lambda} = \kappa\sigma^{\mu\nu\lambda}\ . \tag{3.29}$$

Equations 3.28 and 3.29 are the ultimate field equations in a U_4 spacetime. First equation is known as the modified Einstein's field equation and check that since the Ricci tensor (and the Einstein tensor) is not symmetric in U_4, neither is the canonical stress-energy tensor. Although it might seem weird to have a non-symmetric stress-energy tensor, it is not the first time this happens when torsion gets involved. The same phenomenon happens in the continuum theory of crystal dislocations [16]. The second one is the Cartan spin equation, in which the spin tensor 3.17 is in many cases related to the matter spin distribution. The origin of this tensor is analyzed in detail in the next chapter.

3.3 Spin and gravitation

By assuming a U_4 spacetime, an interesting result is suggested from the new field equations: spin angular momentum contributes, along with matter, in the dynamics of the geometry of spacetime, which is directly related to torsion[2]. Therefore, spin angular momentum would be a source of gravity as well. This is the ultimate result of the so called Einstein-Cartan-Sciama-Kibble (ECSK) theory.

When describing fundamental physics at microscopic levels, elementary particles are classified by irreducible unitary representations, where mass-energy conservation is associated to translation symmetries and angular momentum (including

[2] In [3] S. Capozziello and C. Stonaiolo make a deep analysis by decomposing the torsion tensor into three irreducible tensors and classifying different type of torsion tensors. Many of them, not all though, are related to spin.

spin) is associated with rotation symmetries. Knowing that spin and mass are somewhat intrinsic and elementary features of matter, the description of gravity in which they both play a similar role as sources of gravity, should not sound so weird.

After careful observation, one would notice that Cartan spin equation is not a differential equation, but an algebraic equation. This means that torsion does not propagate in vacuum and it simply vanishes outside matter. Moreover, it can be substituted in the modified Einstein equation and by separating the Riemaniann part, it can be written with the symmetric Einstein tensor in the left side and an effective symmetric energy momentum tensor in the right hand side, like in [20]:

$$G^{\mu\nu}_{\text{Riemann}} = \kappa T^{\mu\nu}_{\text{effective}} \,. \tag{3.30}$$

The effective energy-momentum tensor is composed by the symmetric Hilbert energy-momentum tensor and a large term cuadratic in the spin tensor:

$$T^{\mu\nu}_{\text{effective}} = T^{\mu\nu} + \kappa O^{\mu\nu}(\sigma^2) \,. \tag{3.31}$$

Check from the equation 3.18 that the symmetric tensor $T^{\mu\nu}$ can be written like in terms of the spin tensor like

$$T^{\mu\nu} = \Theta^{\mu\nu} + \tilde{\nabla}_\lambda(\sigma^{\nu\lambda\mu} - \sigma^{\mu\nu\lambda} - \sigma^{\lambda\mu\nu}) \,. \tag{3.32}$$

4. Poincaré group symmetries, conservation laws and sources of gravity

Many years after Cartan proposed the consideration of a U_4 spacetime and derived the new field equations which included spin angular momentum[1], by the end of the 50s, D. Sciama and T. Kibble worked on gauge approaches to gravitation and both independently reached the same modified field equation from a local gauge theory for the Poincaré group. The necessity of a U_4 structure comes totally natural when demanding the spacetime to possess a locally gauge symmetry for rotations and translations.

The Poincaré group is the group of isometries[2] of the Minkoski spacetime of special relativity, which generalizes the Lorentz group of transformations (boosts and rotations) taking into account translations as well. From the principle of equivalence, Minkowski spacetime (and consequently special relativity) should be locally recovered from the curved spacetime where the theory is constructed, and therefore, it is spected to have a locally gauged symmetry for the Poincar é group. However, a (pseudo-)Riemaniann spacetime does not have a locally gauged Lorentz symmetry, and consequently, conservation laws can not be expressed for rotational and boost symmetries, and neither can spinor representations be described in such curved spacetimes.

The aim of this section is not to derive the whole gauge theory formalism in order to recover the equations before. For a rigorous development of a local gauge theory for the Poincaré group, check chapter IV of [13], where requiring a pseudo-Riemaniann spacetime to have a locally gauged Poincaré symmetry, U_4 geometry comes up naturally. Here, I just aim to analyze the local conservation laws that arise from different symmetries of the Minkowski spacetime separately, and use the coupling carried out in [13] to generalize it to curved spacetime and hopefully, give a better understanding of where the spin tensor of equation 3.29 comes from.

[1] It is interesting to mention that when Cartan derived the new field equations in 1922 which led him to include intrinsic angular momentum, the spin had not been discovered yet.

[2] An isometry is a metric preserving diffeomorphism (transformation).

4.1 Symmetry transformations and Noether currents

Say that a lagrangian density does not explicitly depend on the spacetime itself and, thus, the action depends only on some scalar field and its derivative

$$S = \int \mathscr{L}(\phi, \partial\phi)\sqrt{-g}d^4x \,. \tag{4.1}$$

Suppose there is a set of transformations that transform the field and the lagrangian density like $\phi(x) \to \phi(x)+\delta_s\phi(x)$ and $\mathscr{L}(\phi,\partial\phi) \to \mathscr{L}(\phi,\partial\phi)+\delta_s\mathscr{L}(\phi,\partial\phi)$ respectively, where

$$\delta_s\phi(x) = \epsilon_s\Psi_s(x,\phi,\partial\phi) \quad \text{and} \quad \delta_s\mathscr{L}(\phi,\partial\phi) = \epsilon_s\partial_\mu\Lambda^\mu_s(x,\phi,\partial\phi) \,, \tag{4.2}$$

for some functions Ψ and Λ^μ. The subscript s is to denote the different transformations in the transformation set and the ϵ_s are infinitesimal quantities. Then, such transformations are called symmetry transformations because they leave the action invariant and, by Noether's theorem, there exist some Noether currents [3]

$$j^\mu_s = \Lambda^\mu_s - \frac{\partial\mathscr{L}}{\partial\partial_\mu\phi}\Psi_s \tag{4.3}$$

with vanishing four-divergence, i.e.

$$\partial_\mu j^\mu_s = 0 \,. \tag{4.4}$$

This is a local conservation law, since it comes from infinitesimal transformations. However, assuming that the field $\phi(x)$ vanishes at spatial infinity, one can easily obtain the Noether charges and the global conservation laws by integrating over all the space:

$$Q_s(t) = \int j^0_s(\vec{x},t)d^3x \tag{4.5}$$

$$\frac{dQ}{dt} = \int \partial_0 j^0_s(\vec{x},t)d^3x = 0 \,. \tag{4.6}$$

4.2 Global Poincaré transformations in the Minkowski spacetime

In order to analyze the local symmetries of curved spacetimes, the first step is to analyze the global symmetry of the flat Minkowski spacetime and then, make the transition to curved spacetimes. It is important to focus on the flat spacetime first and analyze matter fields with internal degrees of freedom like spinors.

[3] the notation $\frac{\partial}{\partial\partial_\mu\phi}$ stands for $\frac{\partial}{\partial(\partial\phi/\partial x^\mu)}$.

4.2.1 Translation symmetry and Canonical energy-momentum tensor

For the principle of general covariance to be satisfied, the Lagrangian density of matter fields must be invariant under translations. In other words, translations must be symmetry transformations. To examine this, take an infinitesimal translation

$$x^\mu \to x^\mu + \epsilon^\mu \, . \tag{4.7}$$

See that translation of coordinates 4.7 is equivalent to evaluate the field displaced in the opposite direction:

$$\phi(x^\mu) \to \phi(x^\mu - \epsilon^\mu) \sim \phi(x^\mu) - \epsilon^\mu \partial_\mu \phi \, , \tag{4.8}$$

from which one can identify the variation of the field $\delta\phi = -\epsilon^\mu \partial_\mu \phi$. Variation of the lagrangian density is rather simple as well:

$$\begin{gathered} \mathscr{L}(\phi, \partial\phi) \to \mathscr{L}(\phi', \partial\phi') = \mathscr{L}(\phi, \partial\phi) + \delta\mathscr{L}(\phi, \partial\phi) \\ \delta\mathscr{L} = \frac{\partial \mathscr{L}}{\partial \phi} \delta\phi + \frac{\partial \mathscr{L}}{\partial \partial_\mu \phi} \partial_\mu \delta\phi = -\Big[\epsilon^\mu \frac{\partial \mathscr{L}}{\partial \phi} \partial_\mu \phi + \epsilon^\nu \frac{\partial \mathscr{L}}{\partial \partial_\mu \phi} \partial_\mu \partial_\nu \phi \Big] \, . \end{gathered} \tag{4.9}$$

Now using the Euler-Lagrange equation in the first term and the chain rule:

$$\delta\mathscr{L} = -\epsilon^\nu \partial_\mu \Big(\frac{\partial \mathscr{L}}{\partial \partial_\mu \phi} \cdot \partial_\nu \phi \Big) = -\epsilon^\nu \partial_\nu \mathscr{L} \, . \tag{4.10}$$

Therefore, translation is a symmetry transformation and its associated Noether currents (the subindex ν represents each direction in spacetime) are constructed like in equation 4.3:

$$\Theta^\mu{}_\nu = \frac{\partial \mathscr{L}}{\partial \partial_\mu \phi} \partial_\nu \phi - \delta^\mu_\nu \mathscr{L} \, , \tag{4.11}$$

which, of course, satisfies

$$\partial_\mu \Theta^\mu{}_\nu = 0 \, . \tag{4.12}$$

The Noether charges associated with it are the components of the four-momentum,

$$P^\mu = \int \Theta^{0\mu} d^3x \, , \tag{4.13}$$

and thus, the global conservation laws are just the well known total energy and linear momentum conservation laws. $\Theta^\mu{}_\nu$ transforms like a tensor, and it is usually referred as the *canonical energy-momentum tensor*. It coincides with the energy-momentum tensor defined in 3.18. Symmetry upon translations in space are related to the linear momentum conservation, and symmetry upon translations in time to the energy conservation.

4.2.2 Lorentz transformations

An infinitesimal Lorentz transformation can be approximated in Minkowski cartesian coordinates like

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu \sim x^\mu + \Omega^\mu{}_\nu x^\nu \ , \tag{4.14}$$

where $\Omega_{\mu\nu} = \Omega_{[\mu\nu]}$ are constant parameters. By a similar procedure as before, one can identify that scalar fields and the lagrangian density transforms like

$$\delta\phi = -\partial_\mu\phi \, \Omega^\mu{}_\nu x^\nu \tag{4.15}$$

$$\delta\mathscr{L} = -\partial_\mu(\mathscr{L} x^\nu)\Omega^\mu{}_\nu \tag{4.16}$$

and construct the following Noether currents [2]:

$$L^{\mu\nu\gamma} = x^\mu \Theta^{\nu\gamma} - x^\nu \Theta^{\mu\gamma} \ . \tag{4.17}$$

The associated Noether Charge is the angular momentum

$$L^{\mu\nu} = \int L^{\mu\nu 0} d^3x \tag{4.18}$$

and it is conserved if and only if the canonical energy-momentum tensor is symmetric:

$$\partial_\gamma L^{\mu\nu\gamma} = \partial_\gamma (x^\mu \Theta^{\nu\gamma} - x^\nu \Theta^{\mu\gamma}) = \Theta^{\nu\mu} - \Theta^{\mu\nu} = 0 \ . \tag{4.19}$$

Canonical energy-momentum is indeed symmetric only as long as the field $\phi(x)$ is scalar. In this case, the translation transformations plus the Lorentz group of transformations (rotations and boosts) are symmetries, and the theory is invariant under the whole Poincaré group.

4.2.3 Spinor fields

Instead of assuming that the matter field is scalar, let me now work with spinor fields. If the matter field is a spinor field $\psi(x)$, which has intrinsic spin degrees of freedom, the canonical energy-momentum tensor is no longer symmetric and hence, angular momentum is not conserved. Thus, there must be some extra term in the Noether current 4.17 associated with the intrinsic degrees of freedom of the field.

In order to express spinor transformations, let me introduce some orthonormal frame fields (see appendix A):

$$e_a = e_a{}^\mu \partial_\mu \qquad \theta^a = e^a{}_\mu dx^\mu \qquad g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab} \ . \tag{4.20}$$

In flat Minkowski spacetime the orthonormal frames coincide with the cartesian frames, so the frame fields become just

$$e_a{}^\mu = \delta_a{}^\mu \qquad e^a{}_\mu = \delta^a{}_\mu \ . \tag{4.21}$$

An infinitesimal lorentz transformation like 4.14 transforms the spinor field like

$$\psi(x) \rightarrow \psi(x) + [\Omega^{ab}\gamma_{ab} - \Omega_a{}^c \delta^a{}_\mu x^\mu \partial_c]\psi(x) \ , \tag{4.22}$$

which leads to the same Noether current plus a new term:

$$J^{ab\mu} = L^{ab\mu} + \sigma^{ab\mu} = x^a \Theta^{b\mu} - x^b \Theta^{a\mu} + \sigma^{ab\mu} \ . \tag{4.23}$$

The new term is known as the *spin current* $\sigma^{\mu ab}$ and by the procedure done before it can be written like

$$\sigma^\mu{}_{ab} = -\frac{\partial \mathscr{L}}{\partial \partial_\mu \psi}\gamma_{ab}\psi \ . \tag{4.24}$$

The divergence of the new Noether current $J^{ab\mu}$ vanishes even if the canonical energy-momentum tensor is not symmetric:

$$\partial_\gamma J^{ab\gamma} = 0 \ . \tag{4.25}$$

Note that this requirement implies a very close relation between the spin current and the antisymmetric part of the canonical energy-momentum tensor:

$$\partial_\mu \sigma^{ab\mu} = \Theta^{ba} - \Theta^{ab} \ , \tag{4.26}$$

which is generally non-zero, except for spin zero scalar fields. The conserved Noether charge will, of course, have a new term now

$$J^{ab} = \int (L^{ab0} + \sigma^{ab0}) d^3x = L^{ab} + \sigma^{ab} \ . \tag{4.27}$$

The first term is the orbital angular momentum defined before and the second is the spin angular momentum, as one might recognize from elementary quantum mechanics. The total angular momentum is always conserved, no matter how the canonical energy-momentum tensor is constructed. Taking into account that spin degrees of freedom break the symmetry of the canonical energy-momentum tensor, one would like to construct a symmetric energy-momentum tensor which still preserves the properties of the old canonical tensor. Indeed, this construction exists [1] and it is called the Belinfante-Rosenfeld tensor, which appears naturally when one works with a Lagrangian density of spinor fields.

4.3 Local Poincaré transformation

So far, having introduced the new spin term in the current, everything works fine in the Minkowski spacetime. Conservation laws of energy, linear momentum and total angular momentum have been deduced from the symmetry of the Poincaré group. It is time to start curving the spacetime, and letting gravity appear.

4.3.1 Coupling and the Belinfante-Rosenfeld tensor

The coupling process here is not that simple though. For a detailed development with physical arguments check IV. B section of [13], where U_4 geometry is recognized from the coupling. In the end, the coupling is basically given by

$$\delta^a{}_\mu \to e^a{}_\mu \tag{4.28}$$

$$\partial_\mu \to \nabla_\mu = \partial_\mu + \omega_\mu{}^{ab}\gamma_{ab} \ , \tag{4.29}$$

where $\omega_\mu{}^{ab}$ happens to be identical to the spin connection of the U_4 spacetime. Therefore, the lagrangian density of matter field couples like

$$\mathscr{L}_m(\psi, \partial\psi) \to \mathscr{L}_m(g, \psi, \nabla\psi) = \mathscr{L}_m(e, \psi, \partial\psi, \omega, \gamma) \ . \tag{4.30}$$

In the case of scalar matter fields, the covariant derivative is just the partial derivative and it is easy to see variation with respect to the frame field gives the symmetric Hilbert energy-momentum tensor 2.20. Torsion does not appear in this case and the V_4 analysis is completely equivalent. However, when the matter field is a spinor field, the spin connection comes up in the lagrangian density. Variation with respect to the frame field, keeping the spin connection constant, gives the canonical energy-momentum tensor:

$$\delta S = \int \left(\frac{\delta \mathscr{L}_m}{\delta e_a{}^\mu}\right)_\omega \delta e_a{}^\mu \sqrt{-g} d^4x = \int \Big(\Theta_{bc}\eta^{ba} e^c{}_\mu\Big)\delta e_a{}^\mu \sqrt{-g} d^4x \ , \tag{4.31}$$

Which in principle, is not symmetric. Now, since the connection is that of U_4, check that from the equation A.16 variation with respect to the spin connection is just the same as the variation with respect to the contorsion tensor, which was defined to be the spin tensor 3.17:

$$\delta S = \int \left(\frac{\delta \mathscr{L}_m}{\delta \omega_\mu{}^{ab}}\right)_e \delta\omega_\mu{}^{ab} \sqrt{-g} d^4x = -\int \sigma^\mu{}_{ab}\delta\omega_\mu{}^{ab}\sqrt{-g} d^4x \ . \tag{4.32}$$

Here is where the spin torsion comes up. The coupling requires a U_4 structure for the spacetime when matter contains spin. Now if the torsion vanishes, the variation of the spin connection can be written in terms of the variation of the frame field[1], and the variation of the whole action reads like

$$\begin{aligned} \delta S &= \int \left[\left(\frac{\delta \mathscr{L}_m}{\delta e_a{}^\mu}\right)_\omega \delta e_a{}^\mu + \left(\frac{\delta \mathscr{L}_m}{\delta \omega_\mu{}^{ab}}\right)_e \delta\omega_\mu{}^{ab}\right] \sqrt{-g} d^4x \\ &= \int \Big[\Theta_{ab} + \nabla_c(\sigma^c{}_{ba} - \sigma_{ba}{}^c - \sigma_{ab}{}^c)\Big]\eta^{ad} e^b{}_\mu \delta e^\mu{}_d \ . \end{aligned} \tag{4.33}$$

From this last expression, the symmetric Belinfante-Rosenfeld tensor

$$\tilde{T}_{ab} = \Theta_{ab} + \nabla_c(\sigma^c{}_{ba} - \sigma_{ba}{}^c - \sigma_{ab}{}^c) \tag{4.34}$$

is obtained. It is remarkable that the exact same symmetrized energy-momentum tensor (plus cuadratic terms) was obtained when the algebraic Cartan spin equation was used in the modified Einstein equation to obtain a unique set of symmetric equations 3.32 by removing torsion. This may be very convenient because it is symmetric and turns out that the total angular momentum is derived the same way as in 4.17 with the canonical energy-momentum tensor, i.e.

$$J^{\mu\nu} = \int (x^{\mu}\tilde{T}^{\nu 0} - x^{\nu}\tilde{T}^{\mu 0}) d^3x \ . \tag{4.35}$$

U_4 spacetime, unlike V_4, has a locally gauged Poincaré symmetry.

5. Conclusions and implications

It is clear that U_4 spacetime is a totally plausible spacetime in order to define nature and physics on it. First, it does not rely on any mathematical assumption about the geometry and only restrictions come from physical intuition: the principle of general covariance and the principle of equivalence. U_4 spacetime is the most general spacetime that satisfies these physical requirements. Secondly, the U_4 structure has a locally gauged Poincaré symmetry and consequently conservation laws for rotations and lorentzian boosts can be expressed, as well as describe spinors in curved spacetime, where gravitational interaction is considerable.

The existence of torsion in the spacetime has an important conceptual implication: straightest curves are not the shortest ones. The definition of straightness takes into account the internal twist that objects suffer when are parallel transported, for which torsion is responsible. The concept of parallelism was not intuitive in V_4 already (and what I mean by this is that there is not a natural way of defining parallel transport), and it becomes even less intuitive when torsion is added.

The nature of gravitational interaction has a relevant change in U_4, since spin angular momentum acts as a source. This, in principle, is a very drastic change with respect to Einstein General Relativity and would suppose a big difference between the TGR and ECSK theory. Nevertheless, since the Cartan spin equation is algebraic, torsion does not propagate and the EFE would be recovered in vacuum, where all of the TGR experimental tests have been carried out, and hence the ECSK theory would be experimentally valid. By analyzing equation 3.31, it can be seen that when second order terms in spin are neglected, the effective symmetric energy-momentum tensor becomes just the Belinfante-Rosenfeld tensor, and the TGR and ECSK theories become equivalent. Moreover, at macroscopic level the averaged spin of matter is close to zero in most cases and therefore the ECSK theory would not contribute anything new. An interesting phenomenon at macroscopic level would be the formation of gravitational waves due to alternating big spin distributions over time. Neutron stars, due to their strong magnetic fields, would be a potential candidate in which this phenomenon could happen. However, theoretical estimations in [14] predict that the gravitational spin forces are 10^{36} times smaller than magnetic forces, and hence, effects due to spin would be impossible to detect.

Only at microscopic levels, where significant spin distributions are taken into account, could deviations between TGR and ECSK theory be observed. From equation 3.31 it is seen that the relative order between mass and spin is

$$m \sim \kappa s^2 \,. \tag{5.1}$$

Considering that all spins are aligned, the spin distribution and matter distribution are

$$\rho_{\text{spin}} = n\frac{\hbar}{2} \qquad \text{and} \qquad \rho_{\text{mass}} = nm \tag{5.2}$$

respectively. Therefore, for the spin density to equal the mass density, the critical particle density is given by:

$$\bar{n} = \frac{4m}{\kappa\hbar^2} \sim 6.43255 \cdot 10^{85} m \quad (\text{kg}^{-1}\text{m}^{-3}) \,. \tag{5.3}$$

This gives a critical density of $\bar{n}_{electron} \sim 5.86 \cdot 10^{55}\,\text{m}^{-3}$ for electrons and $\bar{n}_{neutron} \sim 1.08 \cdot 10^{59}\,\text{m}^{-3}$ for neutrons. This numbers are only reachable at cases of study like final stage of gravitational collapse, quantum gravitational phenomena and physics of really early universe (planck epoch) in cosmological models with singularities. This estimations make ECSK theory absolutely indistinguishable from the TGR experimentally, and there is little hope that observations on the deviation between theories can be made any soon. However, the difference at such huge densities is very notorius and it implies a fundamental difference on the fabric of the spacetime, what make it theoretically really interesting.

As said before, many open questions in cosmology, high energy physics and particle physics might be faced and solved by a better understanding of gravity at fundamental level, though it is so difficult to directly measure it experimentally. Anyways, it is a beautiful theoretical work developed by Cartan on the foundations and definitions of U_4 geometry [4, 5, 6], Sciama and Kibble on their gauge approaches to gravity [19, 15], Trautmann on his modern and elegnat description of the theory in terms of differential forms [20] and many others that help future generations to look at gravity with different eyes.

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A. Non-coordinate frames

Previously, in chapter 2, it has been shown that for a Pseudo-Riemannian manifold V_4, one can always find a basis for which the manifold becomes locally flat. Although the Minkowski spacetime is obtained in a neighbourhood of a given point, it can not generally be chosen a frame which recovers $g_{\mu\nu} \to \eta_{\mu\nu}$ for the whole (or at least a "big" region of) the spacetime. However, different frames can be chosen in each point that make them locally flat. These local frames are called *tetrads*:

$$e_a = \{e_0, e_1, e_2, e_3\} \tag{A.1}$$

which are usually chosen to be orthonormal

$$e_a \cdot e_b = \eta_{ab} \ . \tag{A.2}$$

Say in the tangent space of a point $P \in V_4$ it is defined the tetrad $\{e_a\}$. These basis vectors can be written in terms of the global coordinate frame $\{e_\mu\} \equiv \{\partial_\mu\}$,

$$e_a = {e_a}^\mu e_\mu \ , \tag{A.3}$$

where $\{{e_a}^\mu\} \in GL(4, \mathbb{R})$ is called the *vierbein* or *frame field* and it is a 4 by 4 matrix that relates the coordinate frame with the orthonormal tetrad frame in each point of the spacetime. Similarly, the dual basis (tetrad coframe) is obtained from the coordinate covectors basis $\{e^\mu\} \equiv \{dx^\mu\}$:

$$\theta^a = {e^a}_\mu e^\mu \ . \tag{A.4}$$

The following relations are easily derived:

$$\begin{aligned} {e^a}_\mu {e_b}^\mu &= \delta^a_b & \text{(A.5)} \\ {e^a}_\mu {e_a}^\nu &= \delta^\nu_\mu & \text{(A.6)} \\ \eta_{ab} &= {e_a}^\mu {e_b}^\nu g_{\mu\nu} & \text{(A.7)} \\ g_{\mu\nu} &= {e^a}_\mu {e^b}_\nu \eta_{ab} \ . & \text{(A.8)} \end{aligned}$$

Note that the latin index refer to tetrad (or lorentzian) index and greek index refer to coordinate index. The vierbein can be seen as a 16 component matrix field which allows to work with locally Minskowski spacetime in any desired point. The price one has to pay are those extra 6 degrees of freedom compared to the 10 that the metric has, which come from the tetrad lorentz gauge transformations, 3 from tetrad rotations and 3 from tetrad lorentz boosts. A physical analogy would be that there were observers in the inertial frame of every point of spacetime giving information about how it transforms with the global coordinate frame. In contrast with the coordinate frame, the tetrad frame is in general anhalonomic, which means that the basis vector fields do not commute. For a tetrad defined by the equation A.3, the commutator of basis vectors is:

$$[e_a, e_b] = \mathcal{C}_{ab}{}^c e_c \qquad \text{where} \qquad \mathcal{C}_{ab}{}^c = 2e^c{}_\nu e_{[a}{}^\mu e_\mu e_{b]}{}^\nu \tag{A.9}$$

One can define covariant derivatives in the local orthonormal frame, the same way it has been done in the coordinate frame. The only difference is that the partial derivatives are replaced by tetrads $\partial_\mu \to \partial_a$, which do not commute, and the connection coefficients are replace by the *tetrad connection coefficients* or *spin connection coefficients* $\Gamma^\lambda{}_{\mu\nu} \to \omega^a{}_{bc}$. Therefore, the tetrad covariant derivative of an arbitrary tetrad tensor is simply

$$\nabla_a T^b{}_c = \partial_a T^b{}_c + \omega^b{}_{da} T^d{}_c - \omega^d{}_{ac} T^b{}_d \ . \tag{A.10}$$

If the tetrad metric is constant, for example if it is taken to be orthonormal like in A.2, the spin connection is antisymmetric in $\omega^a{}_{bc} = \omega^a{}_{[bc]}$. Assuming that the connection is a metric connection, it is easy to see that the condition

$$\nabla_a e_b{}^\mu = 0 \tag{A.11}$$

must be satisfied. Hence, one can identify the spin connection coefficients that generate covariant derivatives in the lorentz frame defined by A.10. This would require the spin connection coefficients to be

$$\omega_\mu{}^{ab} = e^a{}_\nu \Gamma^\nu{}_{\gamma\mu} e^{\gamma b} - e^{\nu b} \partial_\mu e^a{}_\nu \ . \tag{A.12}$$

Now that the spin connection has been defined, covariant derivative of spinor fields can be written explicitly. It's derivation is obtained by demanding to $\nabla_a \psi$ to transform like a product of a spacetime vector and a spinor under local lorentz transformations. It goes like this:

$$\nabla_a \psi = \partial_a \psi + \omega_{\mu\nu a} \gamma^{\mu\nu} \psi \ , \tag{A.13}$$

where $\gamma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$ is the commutator of two gamma matrices, which forms a representation of the generators of the lorentz group.

Torsion and curvature can be again defined in the tetrad frame. The commutator of tetrad covariant derivatives applied on a scalar field ϕ is related to the torsion tensor:

$$\nabla_{[a} \nabla_{b]} \phi = T^c{}_{ab} \partial_c \phi \ , \tag{A.14}$$

and applied on a vector V_a is related to the curvature tensor:

$$\nabla_{[a} \nabla_{b]} V_c = R_{abcd} V^d \ . \tag{A.15}$$

In U_4 the connection coefficients are given by equation 3.1, or written in the orthonormal frame

$$\omega_{abc} = -\mathcal{C}_{abc} + \mathcal{C}_{bca} - \mathcal{C}_{cab} - K_{abc} \ , \tag{A.16}$$

from [13]. Recall that the tetrad commutators $\mathcal{C}_{abc}$ can be written in terms of $e, \partial e$ frame fields or $g, \partial g$.