Facultad de Economía y Empresa

Dpto. Análisis Económico

eman ta zabal zazu

Universidad del País Vasco
Euskal Herriko Unibertsitatea

Three Essays on Game Theory and Social Choice

Oihane Gallo[1] [2]

Ph.D Thesis

Advisors: Elena Inarra and Jorge Alcalde-Unzu

Bilbao, 2020

[1]Dpto. Análisis Económico, Facultad de Economía y Empresa, Universidad del País Vasco (UPV/EHU), Av. Lehendakari Aguirre 83, 48015 Bilbao (SPAIN), , email: oihane.gallo@ehu.es, personal page: https://sites.google.com/site/oihanegallofernandez/.

[2]This research was supported by the Basque Government through scholarship PRE_2019_2_0116.

Acknowledgements

After five years of hard work it is time to look back and realize how many people have supported me, not only when there were successes, but also when forces faltered.

This adventure was not in my plans when I started my Master's degree and if it had not been for one of my advisors, Elena Iñarra of the University of the Basque Country, I would never have embarked on it. I would like to thank Elena for choosing me, guiding me, and supporting me through this long process. The development of this dissertation would not have been possible without her patience and the time she has devoted to me.

It was in my first workshop, in Vigo, where I met a person who would play a relevant role in my thesis. He is my advisor Jorge Alcalde-Unzu of the Public University of Navarre. He introduced me to a new field and even though the process has not been easy, he has always encouraged me to improve my reasoning and resolution skills. I would like to thank Jorge for his support, the patience and the time he has devoted to me.

One of the chapters of this thesis is written in collaboration with Bettina Klaus of the University of Lausanne. Bettina hosted me at the University of Lausanne for three months and she was always available to help and spend time with me during my stay. I had the pleasure to take part in the reading group she organized and come up with an idea to explore and work in together. I would like to thank her for the effort and the time she has devoted to the project. I also want to thank her for the great opportunity she has offered me by hiring me as a postdoc student at the University of Lausanne. I am excited for the new challenge I will face during the next four years.

Apart from the visit to Lausanne, I also benefited from two additional ones: the first one was at Universitat Autònoma de Barcelona hosted by Jordi Massó. I have to thank him for his support during the stay, his proposals and ideas to improve my thesis. I also want to thank the CODE for funding me both my visit to Barcelona and several workshops.

A second stay made me roll the puddle to Rochester. I spent four months at the

University of Rochester hosted by William Thomson. He was always ready to help and spend time with me during my stay. I would like to thank him for his supportive attitude, our productive meetings and the time he devoted to me. Attending one of his courses as well as participating in his reading groups was a very beneficial and interesting experience for my academic development. I also have to thank him all his effort to teach me how a good paper should be written. Although I still have a long way to get it I always have his advice in mind.

I am also grateful to the members of the departments of Foundations of the Economic Analysis I and Applied Economics IV for giving me the opportunity to start my career as a researcher. Special thanks go out to Ana Isabel Saracho and Marta Escapa for facilitating my work in the department and Nagore Iriberri, Iñigo Hernández-Arenaz, Jaromir Kovarick and Federico Grafe for their insightful comments that greatly contributed to enhancing the quality of my work. Very special gratitude also goes out to the Basque Government for providing the funding for this dissertation, to Project IT1367-19 of the Basque Government and Project ECO2015-67519P of the Spanish Ministry of Education. Many thanks to Karmele Agirrezabala and Ester Vidal for their administrative support. I want to specially thank Idoia Aguirre, Iñigo Hernández-Arenaz and Peio Zuazo-Garin for their essential technical support to make possible that you are reading this thesis now.

This thesis has also given me the opportunity to meet a wonderful group of students who have become almost part of my family. First, I thank all my fellow colleagues at the University of the Basque Country for all the lunches and coffee breaks that have been my savior in hard times. I specially thank Idoia Aguirre and Jone Ascorbebeitia for being a vital moral support during this long process. I also want to thank Ata Atay and Paco Robles for making my visit to Barcelona and several conferences more pleasant. I am also very grateful to my friends of Rochester, Alex and Paco, and all my friends at Lausanne (I cannot mention all of them because I would forget some for sure) for their company and support during my visit there.

I cannot conclude this section without thanking my parents, Elena and Jose Manuel, and my sister, Ziortza, for their infinite patience, their support, and their constant and unconditional love. My friends also deserve a special mentions; I am greatly indebted to them for being part of my everyday life enjoying with me the good moments and supporting me in the bad ones. This accomplishment would not have been possible without all of them.

Contents

Introduction

The thesis is structured in two different parts that encompass aspects of two fields of Economics. The first part focuses on Game Theory and the second part on Social Choice Theory.

Game Theory is the study of mathematical models of strategic interaction among rational decision-makers. Modern Game Theory began with the publication of the ground-breaking book “Theory of Games and Economic Behaviour,” by Von Neumann and Morgenstern (1945). All subsequent work in Game Theory is strongly influenced by this book, which defines the basis of what today is known as “classic Game Theory". The methods of this discipline are currently applied successfully to a large number of fields such as economics, biology, sociology and political science.

In Game Theory two different approaches are distinguished: on the one hand, there are non-cooperative or competitive games, in which each player has strategies and looks for her maximum benefit without making any binding agreement between players. On the other hand, there are cooperative games in which players can make binding agreements. The thesis follows the latter approach; *i.e.*, players can cooperate by forming coalitions in order to obtain benefits. These games focus on predicting what coalitions will form and how benefits will be distributed among the players.

Cooperative games with transferable utility (TU games) consist of a set of agents and a characteristic function that allocates a worth to each possible coalition. A solution is a function that associates each game with a non-empty set of payoff vectors. Based on different notions of fairness, different solution concepts have been proposed. One of the most important solution concepts is the core (Gillies, 1953), which assigns payoff vectors to each game such that no coalition can simultaneously provide a higher payoff to each of its members. The core of a game may be empty (see Bondareva, 1963; Shapley, 1965). If the core is non-empty, it does not necessarily contain a unique payoff vector. Two relevant solutions that assign a unique payoff vector to each game are the Shapley value (Shapley, 1953) and the nucleolus (Schmeidler, 1969).

TU games have been related to rationing problems. O'Neill (1982) provides a simple mathematical model, which has been extensively analyzed, to explain a wide variety of economic problems such as bankruptcy, partnerships, and taxation, among others. This model, known as the (conflicting) claims problem, studies situations in which there are a group of agents with individual claims and a resource (endowment) to be allocated which is insufficient to meet all agents' claims. In a rationing problem, the question to be addressed is how the resource should be divided among the agents. A division rule is a function that associates each rationing problem with a division proposal so that each agent receives a nonnegative amount that does not exceed her claim. O'Neill shows that a rationing problem can be rationalized in terms of transferable utilities as a TU game, in which the worth of each coalition is calculated as the maximum worth between zero and the difference between the endowment and the sum of the claims of the agents out of that coalition. Thus, in some cases, a solution for a rationing problem coincides with an explicit solution of TU games. For instance, the recursive completion method (also known as the random arrival rule) coincides with the Shapley value and the Contested Garment Division rule (also known as the Babylonian Talmud rule) coincides with the nucleolus.

A frequent assumption in TU games is that the grand coalition will form. Thus, many game-theoretic solutions have been defined only with reference to that coalition. However, there are situations in which acting together can be costly or unfeasible so the grand coalition fails to form. Hence, a coalition structure or partition is defined as a collection of pairwise disjoint coalitions whose union is equal to the set of agents. Aumann and Drèze (1974) introduce cooperative games with coalition structures to define solutions also with reference to an arbitrary partition.

Later, Dreze and Greenberg (1980) consider cooperative games with coalition structures as a natural framework for studying of situations in which agents have preferences over the coalitions to which they may belong. In these games, formally known as hedonic games or coalition formation problems, each agent only cares about the identity of the agents in her coalition. Coalition formation problems encompass a wide array of models studied in the literature. Depending on which coalitions are feasible, matching problems are a well-known subclass of coalition formation problems. Examples of such problems are the marriage problem, the roommate problem, and the many-to-one and many-to-many matching problems such as hospitals/doctors or students/schools problems. There are also economic environments in which coalitions produce outputs to be divided among their members according to a pre-specified sharing rule. In such situations, the sharing rule naturally induces a coalition formation game where each agent ranks the coalitions to which she may belong according to the payoffs obtainable in each of them.

In some economic environments, the formation of coalitions may lead to a conflict to decide what coalition to form. Such conflicts are well described by Demange (1994), who explains the existence of two opposing fundamental forces: on the one hand, the increasing returns to scale, which incentivizes agents to cooperate and therefore to form large groups, and on the other hand, the heterogeneity of agents, which pushes them towards forming small groups. As a result of these forces, two main questions arise: (i) What coalitions will be formed? and (ii) how will agents share the profits resulting from the coalitions formed? Observe that these two questions are related: agents' payoffs depend on what coalitions form, while the coalitions formed depend on the payoffs available to each agent in each possible coalition.

In answering the question of what partitions will form, the most appealing notion for these games is core stability. A partition is (core) stable if there is no coalition whose members strictly prefer that coalition to the one to which they belong in the partition. Since coalition formation problems may have an empty core, many papers restrict the domain of these games in order to guarantee the existence of stability (for more details, see Banerjee et al., 2001; Bogomolnaia et al., 2002; Iehlé, 2007). Observe that, unlike Aumann and Drèze (1974), in coalition formation problems, a stable partition is obtained endogenously as the outcome of the problem.[1]

Social Choice Theory studies the different ways of aggregating agents' opinions or preferences in order to take a collective decision. It concerns the aggregation of individual inputs (e.g., votes, preferences, welfare) into collective outputs (*e.g.*, collective decisions, preferences, welfare). Some of the main questions are: How can a group of individuals choose a winning outcome (*e.g.*, policy, electoral candidate) from a given set of alternatives? Which are the properties of different voting mechanisms? How can a collective (*e.g.*, electorate, legislature, or committee) arrive at collective preference on the basis of its members' individual preferences? The influence of Social Choice Theory extends across economics, political science, philosophy, mathematics, and recently computer science.

At the heart of Social Choice Theory is the analysis of preference aggregation, understood as the aggregation of several individuals' preference rankings of two or more alternatives into a single, collective preference ranking (or choice) over these alternatives. Arrow (1963) introduces a general approach to the study of preference aggregation. He considers a class of possible aggregation methods, which he calls social welfare functions, and asks which of them satisfy certain axioms or desiderata. He proves that when there are three or more alternatives, no function can convert the

[1] Hart and Kurz (1983) also study cooperative games with endogenous coalition structures for the case of NTU games.

preference rankings of the agents into a social ranking meeting a specified set of criteria: unrestricted domain (*i.e.*, all rankings across the alternatives are feasible for each agent), non-dictatorship (*i.e.*, the social ranking need not be always equal to the ranking of a particular agent), Pareto efficiency (*i.e.*, if all agents agree on a particular binary comparison, then the social ranking should rank that pair according to that unanimous opinion), and independence of irrelevant alternatives (*i.e.*, the social ranking across any two alternatives should depend only on the individual preferences of that pair). This result, known as Arrow's impossibility theorem, prompted much work and many debates in Social Choice Theory and welfare economics.

Some of the work done in Social Choice Theory since then has focused on the vulnerability of social choice rules to strategic voting. A basic result in this context is given by the Gibbard-Satterthwaite theorem (Gibbard et al., 1973; Satterthwaite, 1975), which states an impossibility regarding the selection of a winning alternative as the social outcome (instead of a social ranking). The theorem states that there is no rule with a range other than two that satisfies the properties of unrestricted domain, non-dictatorship, and strategy-proofness (*i.e.*, no agent would obtain a better outcome by lying about her preference ranking). This result is the basis of a growing literature in strategy-proofness. One main conclusion of that theorem is that in order to construct non-dictatorial social choice rules that induce truth-telling, it is necessary either to restrict the range of the rules to two alternatives or to restrict the domain of admissible preferences of each agent. Since rules with range of two alternatives are not Pareto efficient on the universal preference domain, much of the research on strategy-proofness has been focused on the existence of domain restrictions which allow for non-trivial strategy-proof social choice rules, and the characterization of such rules, when possible.

Many domain restrictions have been studied for the analysis of strategy-proofness, *e.g.*, domains where preferences are single-peaked (Black, 1948*a*,*b*) or single-dipped (Barberà et al., 2012). Both these domains arise naturally when alternatives can be located on the real line such as in the location of public facilities. Single-peaked preferences are commonly used when proximity to the public facility is desirable and single-dipped preferences when such proximity is not desirable.

A preference is single-peaked if (i) there is a single most preferred alternative (the *peak*); and (ii) for each alternative x situated between the peak and another alternative y, x is preferred to y. The domain of single-peaked preferences was discussed first by Black (1948*a*,*b*), who shows that the median voter rule, which selects the median of the declared peaks, is strategy-proof and selects the Condorcet winner. Moulin (1980) and Barberà and Jackson (1994) characterize the set of all strategy-proof rules on this domain: Generalized median voter rules. In contrast, a preference is single-dipped whenever (i) there is a single worst alternative (the *dip*); and (ii) for each alternative x located between another alternative y and the dip, y is preferred to x. Barberà et al. (2012) and Manjunath (2014) show that in this domain all non-dictatorial strategy-proof rules have a range of two alternatives. They then show that this domain cannot escape from the classical impossibility. However, these authors also fortunately prove that there are rules for the single-dipped preference domain with range two that are strategy-proof, non-dictatorial, and Pareto efficient. They also provide a characterization of such rules.

This thesis consists of three chapters, where questions raised by previous literature are analyzed from a theoretical point of view. Specifically, the thesis proposes a new model for dealing with coalition formation games that emerge under rationing situations in which stability and the structure of the stable partitions are analyzed (Chapters 1 and 2) and analyzes the strategy-proof social choice rules for locating public facilities in a particular preference domain(Chapter 3). Despite the close interconnection in their approaches, especially between the first two chapters, all the chapters are intended to stand as self-contained papers. The main results are briefly described below.

Chapter 1: Rationing rules and stable partitions.

Chapter 1 studies the formation of coalitions under rationing situations. In several economic and political environments such as the provision of public goods and the formation of clubs or labor unions, agents act by forming coalitions in order to get a joint output. These outputs are not usually sufficient to meet all the claims and they are divided according to a pre-specified rule. When several coalitions can be formed, agents establish a preference ranking over coalitions according to their individual payoffs obtained by the rule applied. An example used in the chapter is that of a call for funding research projects. Researchers form teams to submit a project in order to receive funding to develop the project. The money that each team may receive depends on the competence of the group and the quality of the project. Moreover, it usually happens that the money to be assigned is insufficient to meet all researchers' claims. Observe that claims here may be related to the CVs of the researchers or to the amount of money that each researcher considers necessary to carry out her part of the project. Since each researcher looks for the highest possible amount of money, the payoffs obtained by the rule applied to distribute the money among the researchers induces them to rank the teams in which they may participate. Chapter 1 (Subsection 1.2.1) introduces a generalized claims problem to deal with coalition formation problems in bankruptcy situations bringing together two different branches of the literature hitherto analyzed separately: claims problems and coalition formation problems. We also introduce a new class of coalition formation problems, regular coalition formation problems, and show that this class guarantees the non-emptiness of the core (Proposition 1.3.1). The main result (Theorem 1.3.8) provides a characterization of all rationing rules that guarantee the existence of (core) stable partitions. We show that these rules satisfy two appealing properties (along with continuity): Resource monotonicity and consistency. Finally, we analyze a well-known class of rules that satisfy the properties required: Parametric rules (Stovall, 2014; Young, 1987). The main feature of these rules is that the individual payoff of each agent can be obtained by a function that depends only on her claim and a parameter, which is common to all agents. Significant parametric rules are the Proportional Rule, the Constrained Equal Awards (CEA) Rule, the Constrained Equal Losses (CEL) Rule, and the Talmud Rule. We provide an alternative proof of the main result of the chapter for the class of parametric rules in Proposition 1.3.9.

Chapter 2: Stable partitions for proportional generalized claims problems.

Chapter 2 continues the analysis of the model introduced in Chapter 1. The characterization provided in Chapter 1 is an existence result of stability but say nothing about the structure of the stable partitions. This second chapter focuses on analyzing how agents organize themselves into coalitions to form stable partitions and seeks to answer the questions of what size stable partitions have will have and which agents will sort themselves together. The results in Chapter 1 hold for any vector of claims and any endowments, which gives full flexibility and makes complex the study of the structure of stable partitions a complex task. We therefore introduce a non-singleton proportional generalized claims problem as a generalized claims problem where endowments are restricted in such a way that singleton coalitions get a zero endowment and the remaining endowments are a fixed proportion of the sum of their members' claims. Note that by giving a zero endowment to singletons, the model encourages cooperation between agents, which usually happens in many economic situations such as research team funding. With this model in hand, we characterize all stable partitions when the rule applied is continuous, strict resource monotonic, and consistent. More specifically, we find that in each stable partition there is at most one singleton and coalitions of a size larger than two are feasible only if each agent in those coalitions receives a proportional payoff (Theorem 2.3.2). Moreover, when the monotonicity condition is weakened, not all stable partitions are characterized but the existence of a pairwise stable partition; *i.e.*, a stable partition formed by coalitions with a size of at most two is guaranteed (Theorem 2.3.3). Clearly, the way in which agents sort themselves into those pairwise stable partitions may differ depending on the rule applied. To analyze the particular structure of such pairwise stable partitions, we examine two egalitarian parametric rules: The CEA and CEL rules. We propose two algorithms, one for each rule, for determining a pairwise stable partition. More specifically, the CEA algorithm gives rise to a pairwise stable partition by sequentially pairing off either the two highest claim agents or the highest with the lowest claim agent (Theorem 2.3.4). For CEL, the algorithm sequentially pairs off the two lowest claim agents (Theorem 2.3.6).

Chapter 3: Strategy-proofness on a mixed domain of single-peaked and single-dipped preferences.

Chapter 3 analyzes the problem of locating a public facility taking into account individuals' preferences. In these real-life situations, a social planner (the government of a city or a region) must decide where a public good (such as a school or a hospital) or a public bad (such as a prison or a nuclear power plant) is located. In addition to technical constraints and monetary limitations, public officials may also be interested in considering citizens' preferences when taking the final decision. This could be interpreted as a political strategy or simply a desire to maximize their citizens' welfare. When the public facility to be considered is a public good such as a school, a natural restricted domain is the single-peaked preference domain, while if a public bad has to be located, then the single-dipped preference domain arises naturally. This chapter considers the location of a public facility that does not cause a unanimous opinion among agents such as a dog park or a dance club. A mixed domain that includes both single-peaked and single-dipped preferences may seem a natural restricted domain for such cases. However, Berga and Serizawa (2000) and Achuthankutty and Roy (2018) show that when all single-peaked and all single-dipped preferences are admissible for each agent, the Gibbard-Satterthwaite impossibility holds. Therefore, more restrictions on the domain are needed. Restricted mixed domains have been studied by Thomson (2008) and Alcalde-Unzu and Vorsatz (2018), among others. In Chapter 3, we construct a new domain in which the social planner knows the type of preference of each agent (single-peaked or single-dipped), but is totally uncertain as to the structure of those preferences and, in particular, about the location of the peak or the dip. We characterize all strategy-proof rules on this domain and show that they are all also group strategy-proof on that domain (Theorem 3.4.8). We also analyze which rules of the characterized family satisfy Pareto efficiency (Proposition 3.5.1).

Bibliography

Achuthankutty, G. and Roy, S. (2018), 'Dictatorship on top-circular domains', *Theory and Decision* **85**(3-4), 479–493.

Alcalde-Unzu, J. and Vorsatz, M. (2018), 'Strategy-proof location of public facilities', *Games and Economic Behavior* **112**, 21–48.

Arrow, K. (1963), 'Social choice and individual values. new york 1951b', *ArrowSocial Choice and Individual Values1951* .

Aumann, R. J. and Drèze, J. H. (1974), 'Cooperative games with coalition structures', *International Journal of game theory* **3**(4), 217–237.

Banerjee, S., Konishi, H. and Sönmez, T. (2001), 'Core in a simple coalition formation game', *Social Choice and Welfare* **18**(1), 135–153.

Barberà, S., Berga, D. and Moreno, B. (2012), 'Domains, ranges and strategy-proofness: the case of single-dipped preferences', *Social Choice and Welfare* **39**(2-3), 335–352.

Barberà, S. and Jackson, M. (1994), 'A characterization of strategy-proof social choice functions for economies with pure public goods', *Social Choice and Welfare* **11**(3), 241–252.

Berga, D. and Serizawa, S. (2000), 'Maximal domain for strategy-proof rules with one public good', *Journal of Economic Theory* **90**(1), 39–61.

Black, D. (1948*a*), 'The decisions of a committee using a special majority', *Econometrica: Journal of the Econometric Society* pp. 245–261.

Black, D. (1948*b*), 'On the rationale of group decision-making', *Journal of political economy* **56**(1), 23–34.

Bogomolnaia, A., Jackson, M. O. et al. (2002), 'The stability of hedonic coalition structures', *Games and Economic Behavior* **38**(2), 201–230.

Bondareva, O. N. (1963), 'Some applications of linear programming methods to the theory of cooperative games', *Problemy kibernetiki* **10**, 119–139.

Demange, G. (1994), 'Intermediate preferences and stable coalition structures', *Journal of Mathematical Economics* **23**(1), 45–58.

Dreze, J. H. and Greenberg, J. (1980), 'Hedonic coalitions: Optimality and stability', *Econometrica: Journal of the Econometric Society* pp. 987–1003.

Gibbard, A. et al. (1973), 'Manipulation of voting schemes: a general result', *Econometrica* **41**(4), 587–601.

Gillies, D. (1953), 'Some theorems on n-person games (ph. d. thesis)', *Princeton, New Jersey* .

Hart, S. and Kurz, M. (1983), 'Endogenous formation of coalitions', *Econometrica: Journal of the econometric society* pp. 1047–1064.

Iehlé, V. (2007), 'The core-partition of a hedonic game', *Mathematical Social Sciences* **54**(2), 176–185.

Manjunath, V. (2014), 'Efficient and strategy-proof social choice when preferences are single-dipped', *International Journal of Game Theory* **43**(3), 579–597.

Moulin, H. (1980), 'On strategy-proofness and single peakedness', *Public Choice* **35**(4), 437–455.

O'Neill, B. (1982), 'A problem of rights arbitration from the talmud', *Mathematical social sciences* **2**(4), 345–371.

Satterthwaite, M. A. (1975), 'Strategy-proofness and arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions', *Journal of economic theory* **10**(2), 187–217.

Schmeidler, D. (1969), 'The nucleolus of a characteristic function game', *SIAM Journal on applied mathematics* **17**(6), 1163–1170.

Shapley, L. S. (1953), 'A value for n-person games', *Contributions to the Theory of Games* **2**(28), 307–317.

Shapley, L. S. (1965), On balanced sets and cores, Technical report, RAND CORP SANTA MONICA CALIF.

Stovall, J. E. (2014), 'Asymmetric parametric division rules', *Games and Economic Behavior* **84**, 87–110.

Thomson, W. (2008), Where should your daughter go to college? an axiomatic analysis, Technical report, Tech. rep., Mimeo University of Rochester.

Von Neumann, J. and Morgenstern, O. (1945), 'Theory of games and economic behavior', *Bull. Amer. Math. Soc* **51**(7), 498–504.

Young, H. P. (1987), 'On dividing an amount according to individual claims or liabilities', *Mathematics of Operations Research* **12**(3), 398–414.

Chapter 1

Rationing rules and stable partitions

Abstract

This chapter introduces a new model of coalition formation. It assumes that agents have claims over the outputs that they could produce by forming coalitions. Each output is insufficient to meet the claims of its members and is rationed by a rule whose proposals of division induce each agent to rank the coalitions to which she belongs. As a result, a coalition formation problem arises. Assuming continuous rules, we show that resource monotonic and consistent rules are the only rules that always induce coalition formation problems that admit stable partitions.

1.1 Introduction

Agents such as individuals, firms and institutions seek to form alliances with the aim of achieving profits to be divided according to their aspirations,[1] which are often impossible to satisfy. When there are many profitable coalitions and agents have conflicts over in which coalition take part, the rule used to distribute profits will specify which coalitions are likely to form.

Consider, for instance, a community of households that has to install a public facility which provides a certain benefit to each of them. Suppose that each subgroup of households can equip itself with that facility as long as it is efficient (joint benefit is greater than the provision cost) to do so. Each coalitional cost is divided by a rule among its members according to individual benefits. Each household will seek to minimize its payment and will rank the coalitions in which it takes part accordingly. Consequently, which coalitions of households build the facility will depend on the

[1] Aspirations are depicted by a one-dimensional quantifiable factor such as claim, demand, effort, etc. depending on the context.

rule used. Examples of such situations arise in the provision of local public goods and of club goods, the formation of jurisdictions and of research teams and others. As a second scenario, consider a call for funding research projects. Researchers form teams to apply for funding knowing that the outcome depends on their composition and the quality of the project. Typically, the money assigned to a funded project falls short of meeting the researchers' aspirations[2] and participation in a single team is a prerequisite of the call. Clearly, each researcher aims for the highest possible share of the funding so the payoffs proposed by the rule, which takes the agents' aspirations as input, in the distribution of funding play a key role in how researchers rank the teams in which they may participate. Regardless of whether agents maximize payoffs or minimize payments, these two examples are formally identical. In this chapter we mainly use the second interpretation.

We deal with problems with the following ingredients: *(i)* a set of agents with their claims, which are commonly known; *(ii)* for each subgroup of agents, called coalition, an endowment which is insufficient to meet the claims of its members and (*iii*) a rule that for each coalition induces a payoffs' vector. Consequently, each agent establishes a preference relation over coalitions. Each agent's preference relation is based on her payoffs in the coalitions to which she belongs but it does not depend on the identity of the other members of the coalition. The structure of those preferences is decisive in the formation of the coalitions. Whether or not the resulting preferences generate a stable partition of coalitions is an essential issue. Thus, analyzing which class of rules always induces stable partitions is relevant not only in providing a better understanding of coalition formation but also from a normative point of view when choosing division rules.

The formal literature on claims problems began with O'Neill (1982). In a claims problem the endowment is insufficient to meet all agents' claims. A rule proposes a division such that every agent receives a non-negative payoff which does not exceed her claim. Two appealing properties of rules are resource monotonicity and consistency. Resource monotonicity is the requirement that when the endowment increases, each agent should receive at least as much as initially. The idea behind consistency is the following: Consider a problem and the distribution given by the rule for it. Imagine that some agents take their payoffs and leave. At that point, reassess the situation of the remaining agents, that is, consider the problem of dividing among them what remains of the endowment. A consistent rule should assign to the remaining agents the same payoffs as initially.[3]

[2]In this example aspirations are linked to expertise and although researchers may tend to overestimate their own, objective measures such as CVs bind them.

[3]Consistency is a property with a fertile history in the literature on social choice and cooperative game theory. This principle adapted to diverse areas differs in the precise definition of the reduced

The literature on coalition formation problems, initiated by Dreze and Greenberg (1980), is based on the idea that each agent's preference relation over coalitions depends on the identities of their members. Informally, a partition (of agents into coalitions) is stable if there is no coalition in which each agent is better off. Coalition formation problems may not have stable partitions and identifying classes of coalition formation problems that have stable partitions has been a central issue in this literature.[4]

Our approach to coalition formation problems bridges these two branches of the literature. Specifically, the question addressed in this chapter is which rules, given any list of claims problems, that is, any generalized claims problem, induce coalition formation problems that have stable partitions. In answering this question we introduce a new class of coalition formation problems, called regular coalition formation problems. This class includes the problems that satisfy the "common ranking property" (Farrell and Scotchmer, 1988) and is contained in the class of stable coalition formation problems that satisfy the "top coalition property" (Banerjee et al., 2001) (Proposition 1.3.1).

Then, we characterize the class of rules that always induce coalition formation problems that have stable partitions. Our main result is that only resource monotonic and consistent rules (among continuous rules) guarantee the existence of coalition formation problems that have stable partitions (Theorem 1.3.8). Thus, non-consistent rules commonly analyzed in claims problems such as the random arrival rule (O'Neill, 1982) fail to guarantee stability. However, a host of continuous rules satisfy resource monotonicity and consistency. The most important ones belong to the class of (symmetric and asymmetric) parametric rules (see Stovall, 2014; Young, 1987).[5] In Proposition 1.3.9, we give a simple proof of the fact that parametric rules always induce coalition formation problems that have stable partitions. However, they are not the only rules that generate coalition formation problems that have stable partitions. There are continuous non-parametric rules that also do so. This is the grounds for justifying a characterization of rules beyond the class of parametric rules stated in Theorem 1.3.8.

problems (see, for instance Thomson and Lensberg, 1989). For an extensive review of consistency see Moulin (2004) and Thomson (2015).

[4]See for instance Banerjee et al. (2001), Bogomolnaia et al. (2002) and Iehlé (2007).

[5]Proportional, constrained equal awards, constrained equal losses, the Talmud rule (Auman and Maschler, 1985), the reverse Talmud rule (Schummer et al., 2001) and the dictatorial rule with priority are parametric rules.

Related literature

Our work is inspired by Pycia (2012), who deals with a unified coalition formation model that includes many-to-one matching problems with externalities. He introduces the "pairwise alignment property", which requires that any two agents order any two coalitions to which they both belong in the same way. Then he shows that in the setting of bargaining problems this property guarantees stability. By contrast, we restrict ourselves to coalition formation problems and we weaken the property of pairwise alignment by requiring only that any two agents who share two coalitions do not order them in opposite ways. This property is not sufficient to guarantee stability and other requirements have to be met to achieve stability. We impose absence of "rings" (cyclicity among coalitions).[6] These two conditions define the class of "regular" coalition formation problems.

As an application, Pycia enriches the coalition formation model by assuming that coalitions produce outputs to be divided among their members according to their bargaining power. Then, he characterizes the bargaining rules that induce coalition formation problems that have stable partitions. Unlike Pycia, we do not specify a utility function for each agent, but rather a claim; we assume that each coalitional output is insufficient to accommodate the claims of all its members. Then we impose rules that satisfy continuity, resource monotonicity, and consistency. Pycia has already established a link between consistency and stability (see footnote 5 in his paper). Exploiting this idea, we find that consistency together with resource monotonicity and continuity characterize the class of rules guaranteeing the existence of coalition formation problems that have stable partitions. The intuition why this coupling is fruitful is the following: The exit (or entry) of some agents from (to) a coalition, regardless of whether or not it is accompanied by changes in output, does not affect the payoffs of the remaining agents in opposite directions. This generates weak pairwise alignment preferences without rings.

Two other articles bear some relation with the current chapter. Alcalde and Revilla (2004) explore the existence of stable research teams, when each agent's preference depends on the identity of the members of the team with which she collaborates. The assumption that agents' preferences satisfy the "tops responsiveness" condition[7] guarantees the existence of stable research teams. The formation of research teams facing a call for funding has been used to motivate our coalition formation

[6] There are different definitions of cyclicity among coalitions: For instance, Chung (2000) defines it for roommate problems with weak preferences while Inal (2015) does it for coalition formation problems with strict preferences. To avoid confusion, as in computer science, we use the word rings to describe cyclicity among coalitions.

[7] This assumption is based on the idea of how each researcher thinks different colleagues can complement her abilities.

problem. However, unlike these authors, we do not impose restrictions on agents' preferences other than those induced by the rule. Barberà et al. (2015) consider coalitions in which individuals are endowed with productivity parameters whose sum gives an output. Members of each coalition decide by a majority vote between meritocratic and egalitarian divisions of the output so that one coalition may choose meritocracy while another chooses egalitarianism. Like them, we endow each individual with a claim, but we assume that the output is insufficient to satisfy all claims and that its division among agents is dictated by a single rule. These two articles, like ours, analyze the stability of coalition formation problems.

The rest of the chapter is organized as follows: Section 1.2 contains the preliminaries on claims problems and on coalition formation problems used to define our coalition formation problem. It also contains a result on coalition formation problems. The characterization and other results are presented in Section 1.3. Section 1.4 concludes.

1.2 The model

This section presents the preliminaries of two models that have been extensively analyzed in the literature: claims problems and coalition formation problems. It also presents a new model defined by combining some ingredients of the two literatures.

First, we introduce some notation. Let $\mathbb{N}$ be the set of potential agents and $\mathcal{N}$ the set of all non-empty finite subsets of $\mathbb{N}$. Given $N \in \mathcal{N}$ and $x, y \in \mathbb{R}^N$, we write that $x \geqq y$ if for each $i \in C,\ x_i \geq y_i$.

1.2.1 Claims problems

Consider a group of agents who have claims on a certain endowment, this endowment being insufficient to satisfy all the claims. A primary example is bankruptcy, where agents are the creditors of a firm and the endowment is its liquidation value, but here, we have in mind a more general interpretation of the data (see introduction).

Formally, let $N \in \mathcal{N}$. Let c_i be agent i's claim and $c = (c_i)_{i \in N}$ the **claims vector**. Let $E \in \mathbb{R}_+$ be the **endowment**. A **claims problem with agent set *S*** is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum_{i \in N} c_i \geq E$. Let $\boldsymbol{\mathcal{C}^N}$ denote the class of such problems and $\boldsymbol{\mathcal{C}} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{C}^N$.

An **allocation for (c,E)** $\in \mathcal{C}^N$ is a vector $x = (x_i)_{i\in N} \in \mathbb{R}^N_+$ that satisfies the non-negativity and claim boundedness conditions $0 \leqq x \leqq c$, and the efficiency condition $\sum_{i\in N} x_i = E$.

A **rule** is a function F defined on $\mathcal{C}$ that associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$ an allocation for (c, E). Let $\boldsymbol{\mathcal{F}}$ denote the set of rules.

A rule is **continuous** if small changes in the data of the problem do not lead to large changes in the chosen allocation.[8] Hereafter, we restrict ourselves to continuous rules.

We now introduce our main axioms. Let $F \in \mathcal{F}$ be a generic rule:

Consider a claims problem and the allocation given by the rule for it. We require that if the endowment increases, each agent should receive at least as much as initially.

Resource monotonicity: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $E' > E$, if $\sum_{i\in N} c_i \geq E'$, then $F(c, E') \geqq F(c, E)$.

Consider a claims problem and the allocation given by the rule for it. Imagine that some agents leave with their payoffs. At that point, reassess the situation of the remaining agents, that is, consider the problem of dividing among them what remains of the endowment. We require that the rule should assign the same payoffs to each of them as initially.

Consistency: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $S \subset N$, and letting $x \equiv F(c, E)$, then $x_S = F((c_i)_{i\in S}, E - \sum_{i\in N\setminus S} x_i)$ or equivalently $x_S = F((c_i)_{i\in S}, \sum_{i\in S} x_i)$.

Next, we generalize the notion of a claims problem. Consider a set of agents, each of them with a claim. This time, for each subgroup of agents there is an endowment to distribute among its members. The endowment is insufficient to satisfy their claims.

Formally, given $N \in \mathcal{N}$, a **generalized claims problem with agent set N** is a pair $(c, E) = ((c_i)_{i\in N}, (E_S)_{S\subseteq N}) \in \mathbb{R}^N_+ \times \mathbb{R}^{2^{|N|-1}}$, such that for each $S \subseteq N$, $\sum_{i\in S} c_i \geq E_S$. Let $\boldsymbol{\mathcal{G}^N}$ denote the class of such problems and $\boldsymbol{\mathcal{G}} \equiv \bigcup_{N\in\mathcal{N}} \mathcal{G}^N$.

[8]A rule F is continuous if for each $(c, E) \in \mathcal{C}^N$ and for each sequence of problems $\{(c^k, E^k)\}$ of elements of $\mathcal{C}^N$, if (c^k, E^k) converges to (c, E) then the solution $F(c^k, E^k)$ converges to $F(c, E)$.

An **allocation configuration for (*c*,*E*)** $\in \mathcal{G}^N$ is a list $(x_S)_{S\subseteq N}$ such that for each $S \subseteq N$, x_S is an allocation for the claims problem derived from (c, E) for agent set S, (c_S, E_S).[9]

A **generalized rule** is a function F defined on $\mathcal{G}$ that associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{G}^N$ an allocation configuration for (c, E).

1.2.2 Coalition formation problems

Consider a society where each agent can rank the coalitions that she may belong to. Some well-known examples of such problems are matching problems, in particular, marriage and roommate problems.

Formally, let $N \in \mathcal{N}$. For each agent $i \in N$, $\succsim_i$ is a complete and transitive preference relation over the subsets of N containing i. Given $S, S' \subseteq N$ such that $i \in S \cap S'$, $S \succsim_i S'$ means that agent i finds coalition S at least as desirable as coalition S'. Let $\mathcal{R}_i$ the set of such preference relations for agent i and $\mathcal{R}^N \equiv \Pi_{i\in N}\mathcal{R}_i$. A **coalition formation problem with agent set *N*** consists of a list of preference relations, one for each $i \in N$, $\succsim = (\succsim_i)_{i\in N} \in \mathcal{R}^N$. Let $\boldsymbol{\mathcal{D}^N}$ be the class of such problems with agent set N and $\boldsymbol{\mathcal{D}}\mathcal{D} \equiv \bigcup_{N\in\mathcal{N}} \mathcal{D}^N$.

A partition is a set of disjoint coalitions of N whose union is N and whose pairwise intersections are empty. Formally, a **partition** of a finite set of agents $N = \{1, \ldots, n\}$ is a list $\pi = \{S_1, \ldots, S_m\}$, ($m \leq n$ is a positive integer) such that *(i)* for each $l = 1, \ldots, m$, $S_l \neq \emptyset$, *(ii)* $\bigcup_{l=1}^m S_l = N$, and *(iii)* for each pair $l, l' \in \{1, \ldots, m\}$ with $l \neq l'$, $S_l \cap S_{l'} = \emptyset$. Let $\boldsymbol{\Pi(N)}$ denote the set of all partitions of N. For each $\pi \in \Pi(N)$ and each $i \in N$, let $\boldsymbol{\pi(i)}$ denote the unique coalition in π which contains agent i.

The main goal of the literature on coalition formation problems is to identify the properties of the preferences of individual agents and of preference profiles that guarantee the existence of a partition from which no agent wants to deviate. Formally, let $\succsim \in \mathcal{D}^N$. A partition $\pi \in \Pi(N)$ is **stable** for $\succsim$ if there is no coalition $T \subseteq N$ such that for each $i \in T, T \succ_i \pi(i)$. Let $\boldsymbol{St(\succsim)}$ denote the set of all stable partitions of $\succsim$.

We introduce some properties of preference profiles that express a commonality of preferences among the set of agents and that are sufficient for stability.

[9]The notion of allocation (payoff) configuration was introduced by **?** in his characterization of the Harsanyi NTU solution.

The first property is “pairwise alignment". A preference profile is “pairwise aligned" if any two agents rank coalitions that contain both of them in the same way (Pycia, 2012).[10] We introduce a weaker version of this property which says that if one agent ranks coalitions S and S' in one way, the other cannot rank them in the opposite way. This weaker version will be a necessary property for us: A coalition formation problem $\succsim \in \mathcal{D}$ satisfies the **weak pairwise alignment** property if for each pair S, S' and each pair $i, j \in S \cap S'$, $S \succ_i S' \Rightarrow S \succsim_j S'$. Note that this definition allows agent i to be indifferent between the two coalitions while agent j prefers one to the other. Let $\mathcal{D}_{WPA}$ denote the class of all coalition formation problems that satisfy the weak pairwise alignment property.

A second property that is the “top coalition property" (Banerjee et al., 2001). This says that for each non-empty subset S of agents, there is a coalition $S' \subseteq S$ such that all members of S' prefer S' to any other coalition that consists of some (or all) members of S. Formally, let $S \subseteq N$. A non-empty subset $S' \subseteq S$ is a **top coalition of *S*** if for each $i \in S'$ and each $T \subseteq S$ with $i \in T$, we have $S' \succsim_i T$. A coalition formation problem satisfies the **top coalition property** if each non-empty set of agents $S \subseteq N$ has a top coalition.[11]

A **ring** for $\succsim \in \mathcal{D}$ is an ordered list of coalitions $(S_1, ..., S_m)$, $m > 2$, such that for each $l = 1, \ldots, m$ (subscripts modulo l) and each $i \in A_l = S_l \cap S_{l+1}$, $S_{l+1} \succsim_i S_l$, and for at least one $i \in A_l$, $S_{l+1} \succ_i S_l$.

That is, there is at least one agent in the intersection of any two consecutive coalitions with a strict preference between both coalitions while his intersection-mates can be indifferent between them. Thus, an agent is unable to move from one coalition to the next unless her intersection-mates allow her to do so.[12]

Finally, the lack of rings is also a sufficient condition for stability but it is not a necessary condition, that is, a coalition formation problem with rings may have stable partitions (see Example 2). It depends on the “position" of the components of the ring in the problem under consideration (Bloch and Diamantoudi, 2011).

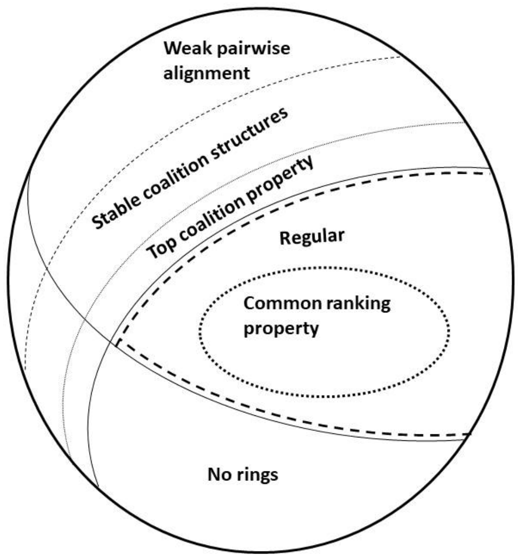
[10]A preference profile $\succsim$ is pairwise aligned if for each $i, j \in S \cap S'$, $S \succsim_i S' \iff S \succsim_j S'$. This definition implies that if agent i is indifferent between two coalitions so is agent j.

[11]This property is a generalization of the “common ranking property" (Farrell and Scotchmer, 1988), which requires the existence of a linear ordering over all the coalitions such that each agent ranks the coalitions she belongs to according to this ordering.

[12]Inal (2015) defines cyclicity by requiring that only one agent at the intersection of two consecutive coalitions prefers the former over the latter. Pycia (2012) only requires a weak preference of a single agent in the intersection of any two consecutive coalitions with at least one strict preference. In both definitions other members who belong to two consecutive coalitions can oppose the transition from one coalition to the next.

Hereafter a coalition formation problem that satisfies the weak pairwise alignment property and has no rings is called **regular**.

The relations among all these properties are illustrated in the following graph.



1.2.3 Coalition formation problems induced by a generalized claims problem and a rule

Given a generalized claims problem and a rule, each agent, by calculating her payoff in each coalition, can form preferences over coalitions giving rise to a coalition formation problem. Examples of this situation can be the formation of jurisdictions and research groups. More examples can be found in the introduction.

Formally, given $N \in \mathcal{N}$, $(c, E) \in \mathcal{G}^N$, and $F \in \mathcal{F}$, the **coalition formation problem with agent set *N* induced by ((*c*,*E*),*F*)** $\in \mathcal{G}^N \times \mathcal{F}$ consists of the list of preferences $\succsim^{((c,E),F)} = (\succsim_i)_{i \in N}^{((c,E),F)}$ defined as follows: for each $i \in N$, and each pair $S, S' \subseteq N$ such that $i \in S \cap S'$, $S \succsim_i^{((c,E),F)} S'$ if and only if $F_i(c_S, E_S) \geq F_i(c_{S'}, E_{S'})$.

Our aim is to identify properties of F that guarantee the existence of stable partitions for coalition formation problems induced by pairs $((c, E), F) \in \mathcal{G}^N \times \mathcal{F}$.

1.3 Results

In this section we characterize the rules that, given any generalized claims problem, induce coalition formation problems that have stable partitions. The characterization is based on two properties that a coalition formation problem has to satisfy, the weak pairwise alignment property and the lack of rings.

The interest of our first proposition stems from the fact that it identifies a specific class of coalition formation problems. This class includes the class of problems that satisfy the common ranking property (the proof is straightforward) and is contained in the class of problems that satisfy the top coalition property.[13]

Proposition 1.3.1. *If a coalition formation problem is regular, then it satisfies the top coalition property.*

Proof: Let $N \in \mathcal{N}$ and $\succsim \in \mathcal{D}^N$ be a regular coalition formation problem. Let $S \subseteq N$ and $s \equiv |S|$. For each $i \in N$, let $Ch_i(S)$ be the set of most preferred coalitions for agent i, that is, $Ch_i(S) \equiv \{S' \subseteq S : i \in S'$ and for each $T \subseteq S',\ S' \succsim_i T\}$.[14] We show that there is at least one coalition $S' \subseteq S$ such that for each $i \in S',\ S' \in Ch_i(S)$, and therefore S' is a top coalition of S.

Assume by contradiction that there is no such S'. Then for each $i \in S$, $\{i\} \notin Ch_i(S)$ and, for each $i \in S$ and each $S' \in Ch_i(S)$, there are $j \in S'$, $j \neq i$ and $S'' \in Ch_j(S)$ such that $S'' \succ_j S'$. We define an iterative process with at most $s-1$ steps to find a contradiction. We show that either the weak pairwise alignment property is violated or there is a ring at some step. Therefore, a contradiction is reached and the process stops.

Take any agent of S, let us name her agent 1.

Step 1 : Let $S_1 \in Ch_1(S)$. Then there are an agent, let us name her agent 2, and a coalition in $Ch_2(S)$, let us name it S_2, such that $S_2 \succ_2 S_1$. If $1 \in S_2$ and $S_1 \succ_1 S_2$ then the weak pairwise alignment property is violated. Otherwise, go to Step 2.

...

[13] There is no inclusive relation between the class of regular coalition formation problems and the class of coalition formation problems that satisfy the (weak) top-choice property (see supplementary material, Karakaya, 2011) or with the class of coalition formation problems induced by separable preferences (Burani and Zwicker, 2003). The notions of *k*-acyclicity (Inal, 2015) and ordinal balance (Bogomolnaia et al., 2002) are only defined for strict preferences.

[14] Note that if a coalition formation problem shows indifference between coalitions any agent may have several preferred sets.

Step $k \leq s-1$: Let $S_k \in Ch_k(S)$. Then there are an agent, let us name her agent $k+1$, and a coalition in $Ch_{k+1}(S)$, let us name it S_{k+1}, such that $S_{k+1} \succ_{k+1} S_k$. Two cases are distinguished:

(i) For some $i \in \{1,\ldots,k-1\}$, $S_{k+1} = S_i$, .

- If $S_{k+1} = S_{k-1}$, then, by Step $k-1$, $S_k \succ_k S_{k-1}$ and the weak pairwise alignment property is violated.
- If for some $i \in \{1,\ldots,k-2\}$, $S_{k+1} = S_i$, then $S_i \succ_{k+1} S_k$. By the $k-i$ previous steps, for each $j = i,\ldots,k$, $S_{j+1} \succ_{j+1} S_j$. Then $\{S_i, S_{i+1},\ldots,S_k\}$ is a ring.[15]

(ii) For each $i = 1,\ldots,k-1$, $S_{k+1} \neq S_i$.

- If $S_k \succ_k S_{k+1}$ then the weak pairwise alignment property is violated.
- If for some $i \in \{1,\ldots,k-1\}$, $S_i \succ_i S_{k+1}$, and given that, by the $k-i$ previous steps, for each $j = i+1,\ldots,k+1$, $S_{j+1} \succ_{j+1} S_j$, then $\{S_{k+1}, S_i,\ldots,S_k\}$ is a ring.

Otherwise, if $k < s-1$, go to Step $k+1$.

Note that if $k = s-1$, given that $S_s \neq \{s\}$ and there is $j \in S$ such that $S_j \succ_j S_s$, then either for some $i \in \{1,\ldots,s-2\}$, $S_s = S_i$, and a contradiction is reached in (i), or for each $i = 1,\ldots,s-2$, $S_s \neq S_i$, and a contradiction is reached in (ii). □

In contrast to Proposition 1.3.1, a coalition formation problem that satisfies the top coalition property may not satisfy the weak pairwise alignment property or may have rings, as the following examples illustrate.

Example 1.3.2. [16] *Let $N = \{1,2,3\}$ and $\succsim \in \mathcal{D}^N$ be as follows:*

$$\succsim_1: \{1\} \succ_1 \{12\} \succ_1 \{13\} \succ_1 \{123\},$$
$$\succsim_2: \{123\} \succ_2 \{12\} \succ_2 \{23\} \succ_2 \{2\},$$
$$\succsim_3: \{123\} \succ_3 \{23\} \succ_3 \{13\} \succ_3 \{3\}.$$

This problem $\succsim$ satisfies the top coalition property but not the weak pairwise alignment property: Agent 1 prefers $\{12\}$ to $\{123\}$ while agent 2 orders these coalitions in the opposite way.

[15] Note that any other agent in the intersection of two consecutive coalitions of the ring should order them in the same way or be indifferent, otherwise the weak pairwise alignment property is violated.

[16] This is Example 3.5 in Bloch and Diamantoudi (2011).

Example 1.3.3. *Let* $N = \{1,2,3\}$ *and* $\succsim \in \mathcal{D}^N$ *be as follows:*

$$\succsim_1: \{123\} \succ_1 \{12\} \succ_1 \{13\} \succ_1 \{1\},$$
$$\succsim_2: \{123\} \succ_2 \{23\} \succ_2 \{12\} \succ_2 \{2\},$$
$$\succsim_3: \{123\} \succ_3 \{13\} \succ_3 \{23\} \succ_3 \{3\}.$$

This problem $\succsim$ *satisfies the top coalition property and has ring* $\{\{13\},\{12\},\{23\}\}$.

The top coalition property guarantees the existence of stable partitions and if preferences are strict, there is a unique such partition (Banerjee et al., 2001). If indifference is allowed, there may be more than one stable partition.

Let $N \in \mathcal{N}$, $(c,E) \in \mathcal{G}^N$, and $F \in \mathcal{F}$. Now, we show that $\succsim^{((c,E),F)}$ satisfies both the weak pairwise alignment property (Lemma 1.3.4) and has no rings (Lemma 1.3.7) only if F is resource monotonic and consistent. These two lemmas and Proposition 1.3.1 prove our characterization (Theorem 1.3.8).

Lemma 1.3.4. *Let* $F \in \mathcal{F}$. *For each* $N \in \mathcal{N}$ *and each* $(c,E) \in \mathcal{G}^N$, $\succsim^{((c,E),F)} \in \mathcal{D}_{WPA}$ *if and only if* F *is resource monotonic and consistent.*

Proof: *(a)* We prove that for each $N \in \mathcal{N}$, if $F \in \mathcal{F}$ violates either resource monotonicity or consistency, there is $(c,E) \in \mathcal{G}^N$ such that $\succsim^{((c,E),F)} \notin \mathcal{D}_{WPA}$.

(i) Assume that F is not resource monotonic. Then there are $S \in \mathcal{N}$, $(c,E) \in \mathcal{C}^S$, $E' > E$ with $\sum_{i \in S} c_i \geq E'$, and $i \in S$, such that $F_i(c,E') < F_i(c,E)$. Then there is $j \in S$ such that $F_j(c,E') > F_j(c,E)$. Let $x \equiv F(c,E)$ and $x' \equiv F(c,E')$.

Case 1: $|S| > 2$. Let $y \equiv F(c_{\{i,j\}}, x_i + x_j)$. Two cases are distinguished:

Subcase 1.1: $(y_i, y_j) = (x_i, x_j)$. Then $y_i > x'_i$ and $y_j < x'_j$. Let $(\bar{c}, \bar{E}) \in \mathcal{G}^S$ be such that $\bar{c} = c$, $\bar{E}_S = E'$, and $\bar{E}_{\{i,j\}} = x_i + x_j$. Then agents i and j order coalitions S and $\{i,j\}$ in opposite ways. Hence, $\succsim^{((\bar{c},\bar{E}),F)} \notin \mathcal{D}_{WPA}$.

Subcase 1.2: $(y_i, y_j) \neq (x_i, x_j)$. Then $y_i > x_i$ and $y_j < x_j$. Let $(\bar{c}, \bar{E}) \in \mathcal{G}^S$ be such $\bar{c} = c$, $\bar{E}_S = E$, and $\bar{E}_{\{i,j\}} = x_i + x_j$. Then agents i and j order coalitions S and $\{i,j\}$ in opposite ways. Hence, $\succsim^{((\bar{c},\bar{E}),F)} \notin \mathcal{D}_{WPA}$.

Case 2: $|S| = 2$. W.l.o.g. let $S = \{i,j\}$. Let $N \supset S$ and $(c', x_i + x_j) \in \mathcal{C}^N$ be such that for each $k \in S$, $c'_k = c_k$ and each $k \neq i,j$, $c'_k = 0$. Let $y \equiv F(c', x_i + x_j)$. Note that for each $k \neq i,j$, $y_k = 0$. Let $(\bar{c}, \bar{E}) \in \mathcal{G}^N$. Reasoning as in Subcases 1.1. and 1.2., we obtain that agents i and j order coalitions S and N in opposite ways. Hence, $\succsim^{((\bar{c},\bar{E}),F)} \notin \mathcal{D}_{WPA}$.

(ii) Assume that F is not consistent. Let $S \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^S$. Let $F \in \mathcal{F}$ and $x \equiv F(c, E)$. Since F is not consistent, there are $S' \subset S$ and $(c_{S'}, \sum_{i \in S'} x_i) \in \mathcal{C}^{S'}$ such that there are at least two agents in S', say agents i and j, such that $F_i(c_{S'}, \sum_{k \in S'} x_k) > x_i$ and $F_j(c_{S'}, \sum_{k \in S'} x_k) < x_j$. Let $(\bar{c}, \bar{E}) \in \mathcal{G}^S$ be such that $\bar{c} = c$, $\bar{E}_S = E$ and $\bar{E}_{S'} = \sum_{k \in S'} x_k$. Then $S' \succ_i S$ and $S \succ_j S'$. Hence, $\succsim^{((\bar{c}, \bar{E}), F)} \notin \mathcal{D}_{WPA}$.

(b) Conversely, we prove that if F is resource monotonic and consistent, then for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{G}^N$, $\succsim^{((c,E),F)} \in \mathcal{D}_{WPA}$.

Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{G}^N$. Let $S, S' \subseteq N$ and $j, k \in S \cap S'$. Let $x \equiv F(c_S, E_S)$ and $x' \equiv F(c_{S'}, E_{S'})$. By consistency,

$$x_{\{j,k\}} = F(c_{\{j,k\}}, (x_j + x_k)) \text{ and } x'_{\{j,k\}} = F(c_{\{j,k\}}, (x'_j + x'_k)).$$

By resource monotonicity, one of the following two cases holds:

Case 1: $x_j + x_k \geq x'_j + x'_k$.

Then $x_{\{j,k\}} = F(c_{\{j,k\}}, (x_j + x_k)) \geqq F(c_{\{j,k\}}, (x'_j + x'_k)) = x'_{\{j,k\}}$.

Since $x_{\{j,k\}} = F_{\{j,k\}}(c_S, E_S)$ and $x'_{\{j,k\}} = F_{\{j,k\}}(c_{S'}, E_{S'})$,

$$F_{\{j,k\}}(c_S, E_S) \geqq F_{\{j,k\}}(c_{S'}, E_{S'}).$$

Therefore, agents j, k do not rank S and S' in opposite ways.

Case 2: $x_j + x_k \leq x'_j + x'_k$.

Proceeding as in Case 1, agents j, k do not rank S and S' in opposite ways.

Thus, either $F_{\{j,k\}}(c_S, E_S)) \geqq F_{\{j,k\}}(c_{S'}, E_{S'})$ or $F_{\{j,k\}}(c_S, E_S) \leqq F_{\{j,k\}}(c_{S'}, E_{S'})$, *i.e.* agents j, k do not rank S and S' in opposite ways. As this argument holds for each pair $S, S' \subseteq N$ and each pair $j, k \in S \cap S'$, $\succsim^{((c,E),F)} \in \mathcal{D}_{WPA}$. □

The next example shows that there are coalition formation problems satisfying the weak pairwise alignment property that do not admit stable partitions.

Example 1.3.5. *Let $N = \{1, 2, 3\}$ and $\succsim \in \mathcal{D}^N$ be as follows:*

$$\succsim_1: \{12\} \succ_1 \{13\} \succ_1 \{123\} \succ_1 \{1\},$$
$$\succsim_2: \{23\} \succ_2 \{12\} \succ_2 \{123\} \succ_2 \{2\},$$
$$\succsim_3: \{13\} \succ_3 \{23\} \succ_3 \{123\} \succ_3 \{3\}.$$

This problem $\succsim$ satisfies the weak pairwise alignment property. However, due to the existence of ring $\{\{13\}, \{12\}, \{23\}\}$, there is no stable partition.

The following lemma shows that for each $N \in \mathcal{N}$ and each $((c, E), F) \in \mathcal{G}^N$, if F is resource monotonic and consistent, then weak pairwise alignment and rings do not coexist in $\succsim^{((c,E),F)}$.

First, we introduce a lemma that says that the payoffs given by a consistent rule in a claims problem can be obtained in an extended claims problem in which one or more agents with positive claims are added.

Lemma 1.3.6. *Let $S \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^S$. Let $F \in \mathcal{F}$ be a consistent rule and $x \equiv F(c, E)$. Let $S' \supset S$. Then there is $\bar{E}$ such that letting $(c', \bar{E}) \in \mathcal{C}^{S'}$ such that for each $i \notin S$, $c'_i > 0$ and for each $i \in S$, $c'_i = c_i$, then for each $i \in S$, $F_i(c', \bar{E}) = x_i$.*

Proof: Let $S \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^S$. Let F be a consistent rule and $x \equiv F(c, E)$. Let $S' \supset S$. Let $(c', E') \in \mathcal{C}^{S'}$ be such that for each $i \in S$, $c'_i = c_i$ and for each $i \notin S$, $c_i > 0$. Let $\alpha(E') = \sum_{i \in S} F_i(c', E')$. Since F is continuous, α is a continuous function defined on $[0, \sum_{i \in S'} c_i]$ with $\alpha(0) = 0$ and $\alpha(\sum_{i \in S'} c_i) = \sum_{i \in S} c_i$. By continuity of α, there is $\bar{E} \in [0, \sum_{i \in S'} c_i]$ such that $\alpha(\bar{E}) = \sum_{i \in S} x_i$. Let $y \equiv F(c', \bar{E})$. By construction, $\sum_{i \in S} y_i = \sum_{i \in S} x_i$ and by consistency, for each $i \in S$, $F_i(c', \bar{E}) = x_i$. □

Lemma 1.3.7. *Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{G}^N$ and $F \in \mathcal{F}$. If F is resource monotonic and consistent, then $\succsim^{((c,E),F)}$ has no rings.*

Proof: Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{G}^N$. Let $F \in \mathcal{F}$ be a resource monotonic and consistent rule. Assume by contradiction that $\succsim^{((c,E),F)}$ has a ring $(S_1, \ldots, S_l)$, $l > 2$.

By definition of a ring, for each $k = 1, \ldots, l$ (subscripts modulo k), there is at least one agent, say agent $j_{k+1} \in S_{k+1} \cap S_k$, such that $S_{k+1} \succ_{j_{k+1}} S_k$. By transitivity of preferences, it cannot be that $j_1 = \cdots = j_l$. Let $S^* \equiv \bigcup_{k=1}^{l} S_k$. We claim that S^* is not a component of $(S_1, \ldots, S_l)$. Assume by contradiction that for some $k \in \{1, \ldots, l\}$, $S^* = S_k$, say $S^* = S_1$. Then $S_2 \succ_{j_2} S^*$ and $S^* \succ_{j_1} S_l$. Since $S_3 \succ_{j_3} S_2$ and, by weak pairwise alignment, $S_2 \succsim_{j_3} S^*$, then $S_3 \succ_{j_3} S^*$. Likewise, for each $k > 3$, $S_k \succ_{j_k} S^*$. Now, if $j_l = j_1$, then $S^* \succ_{j_1} S_l$ and $S_l \succ_{j_1} S^*$ and therefore, agent j_1's preference is not transitive. Hence, $\succsim^{((c,E),F)}$ is not a well-defined coalition formation problem. If $j_l \neq j_1$, then $S^* \succ_{j_1} S_l$ and $S_l \succ_{j_l} S^*$. Therefore, the weak pairwise alignment property is not satisfied and, by Lemma 1, F is not both *resource monotonic* and *consistent*.

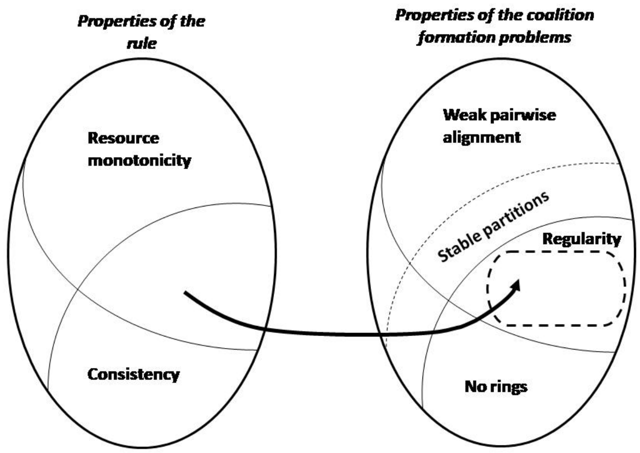
Now, let $(c', E') \in \mathcal{G}^N$ be such that for each $i \in N$, $c'_i = c_i$, for each $S \neq S^*$, $E' = E$ and let E'_{S^*} be derived from Lemma 2 as follows: let $x_{S_1} \equiv F(c_{S_1}, E_{S_1})$; since $S_1 \subset S^*$, there is E'_{S^*} such that for each $i \in S_1$, $F_i(c_{S^*}, E'_{S^*}) = x_i$.[17] Then by construction, the agents in S_1 receive the same payoffs in S_1 as in S^*, and therefore they are indifferent between these two coalitions. Hence, $\succsim^{((c',E'),F)}$ has a ring of which S^* is a component, which is impossible from the previous argument. Therefore, if F is resource monotonic and consistent, $\succsim^{((c,E),F)}$ has no rings. □

[17]Note that if for each $i \in S^* \backslash S_1$, $c_i = 0$, then $E'_{S^*} = E_{S_1}$.

Finally, Lemmas 1.3.4 and 1.3.7 jointly with Proposition 1.3.1 prove the characterization result:

Theorem 1.3.8. *Let $F \in \mathcal{F}$. For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{G}^N$, the set of stable partitions of the coalition formation problem induced by $((c, E), F)$ is non-empty if and only if F is resource monotonic and consistent.*

These results are illustrated in the following picture:



1.3.1 Parametric rules induce stability

For claims problems many division rules are resource monotonic and consistent. The most important ones are the so-called "parametric" rules. For a rule in this class, there is a function of two variables such that for each problem, there is parameter λ such that each agent's payoff is the value given by this function where the first argument is his claim and the second one is λ, which is the same for all agents. This parameter is chosen so that the sum of agents' payoffs is equal to the endowment. Young (1987) characterizes parametric rules on the basis of symmetry,[18] continuity and bilateral consistency.[19] Recently, there has been growing interest in studying parametric rules (see, for instance, Erlanson and Flores-Szwagrzak, 2015; Kaminski, 2006; Stovall, 2014). Stovall characterizes the family of symmetric and asymmetric parametric rules on the basis of continuity, resource monotonicity, bilateral consistency and two additional axioms, "N-continuity" and "intrapersonal consistency".

[18] Two agents with equal claims should receive equal payoffs.

[19] Bilateral consistency requires consistency only when $|M| = 2$.

Therefore, this family of rules always induces coalition formation problems that have stable partitions. A simple proof of this result is given below.

A parametric rule is defined as follows:

Let f be a collection of functions $\{f_i\}_{i\in N}$,[20] where each $f_i : \mathbb{R}_+ \times [a,b] \longrightarrow \mathbb{R}_+$ is continuous and weakly increasing in λ, $\lambda \in [a,b]$, $-\infty \leq a < b \leq \infty$ and for each $i \in N$ and $d_i \in \mathbb{R}_+$, $f_i(c_i, a) = 0$ and $f_i(c_i, b) = c_i$. Hence, for each f a rule F for claims problem $(c, E) \in \mathcal{C}^N$ is defined as follows.
For each $i \in N$

$$F_i(c, E) = f_i(c_i, \lambda) \text{ where } \lambda \text{ is chosen so that } \sum\nolimits_{i\in N} f_i(c_i, \lambda) = E.$$

Thus, f is said to be a **parametric representation of F**.

The proportional, constrained equal awards, constrained equal losses, and the Talmud and reverse Talmud rules are symmetric parametric rules while the sequential priority rule associated with $\prec$ is an asymmetric parametric rule.[21]

Proposition 1.3.9. *A parametric rule induces coalition formation problems that have at least one stable partition.*

Proof: (i) First, we show that, given any generalized claims problem, parametric rules induce coalition formation problems that satisfy the weak pairwise alignment property.

Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{G}^N$ and F be a parametric rule. Let $\succsim^{((c,E),F)}$ be such that the weak pairwise alignment property is violated. Then there are $S, S' \subseteq N$ and $i, j \in S \cap S'$ such that $S \succ_i S'$ and $S' \succ_j S$. Let $x \equiv F(c_S, E_S)$ and $y \equiv F(c_{S'}, E_{S'})$. By definition of F, for each $S \subseteq N$ and each $i \in S$, there exist a collection of functions f and a parameter λ such that $x_i = f_i(c_i, \lambda)$. For the sake of convenience, we denote the value of λ for coalition S by $\lambda(S)$. Let $x_i = f_i(c_i, \lambda(S))$ and $y_i = f_i(c_i, \lambda(S'))$ the allocations given by F to agent i and similarly for agent j. As $S \succ_i S'$, then $x_i > y_i$ and therefore $\lambda(S) > \lambda(S')$. On the other hand, as $S' \succ_j S$, then $y_j > x_j$ and therefore $\lambda(S') > \lambda(S)$, and a contradiction is reached.

(ii) Second, we show that, given any generalized claims problem, parametric rules induce coalition formation problems with no rings.

Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{G}^N$, and F be a parametric rule. Let $\succsim^{((c,E),F)}$ be such that it contains ring $\{S_1, \ldots, S_m\}, m > 2$. Let $\{A_1, \ldots, A_l\}$ be the sets of agents such that for each $l = 1, \ldots, m$ (subscripts modulo l), $A_l = S_l \cap S_{l+1}$.

[20] When the rule is symmetric, f_i is the same for all agents.

[21] Moulin (2000) characterizes a class of asymmetric rules using consistency and other properties, "upper and lower composition".

Let $x(S_l) \equiv F(c_{S_l}, E_{S_l})$. By definition of F, for each $l = 1, \ldots, m$ and each agent $i \in S_l$ there are a collection of functions f and a parameter λ such that $x_i = f_i(c_i, \lambda)$. For the sake of convenience, we denote the payoff of agent i in coalition S_l by $x_i(S_l)$ and the value of λ associated to coalition S_l by $\lambda(S_l)$.

By definition of a ring, for each $l = 1, \ldots, m$ and each $i \in A_l$, $S_{l+1} \succsim_i S_l$ and for at least one $j \in A_l$, $S_{l+1} \succ_j S_l$. Observe that no other agent in the intersection opposes the agent j's strict preference of S_{l+1} over S_l. Therefore,

$$S_{l+1} \succsim_{A_l} S_l \Longrightarrow S_{l+1} \succ_j S_j \Longrightarrow x_j(S_{l+1}) > x_j(S_l) \Longrightarrow$$

$$f_j(c_j, \lambda(S_{l+1})) > f_j(c_j, \lambda(S_l)) \Longrightarrow \lambda(S_{l+1}) > \lambda(S_l)$$

Since this holds for each S_l, $l = 1, \ldots, m$ (subscripts modulo l), a contradiction is reached.

Finally, considering (i) and (ii) together with Proposition 1.3.1, it can be stated that parametric rules induce coalition formation problems that have at least one stable partition. □

Parametric rules, however, are not the only rules covered by our results. There are continuous non-parametric rules that induce stability in coalition formation problems, such as the following example borrowed from Stovall (2014) shows.

Example 1.3.10. *For $i \neq 1$, let $f_i(c_i, \lambda) = \lambda c_i$ be i's parametric function on $[0, 1]$. For $i = 1$, f_1 is not a function, but a correspondence on $[0, 1]$ defined by:*

$$f_1(c_1, \lambda) = \begin{cases} 0 & \text{for} \quad \lambda < \frac{c_1}{1+c_1} \\ [0, c_1] & \text{for} \quad \lambda = \frac{c_1}{1+c_1} \\ c_1 & \text{for} \quad \lambda > \frac{c_1}{1+c_1} \end{cases} \tag{1.1}$$

Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{G}^N$. Observe that for each $S \subseteq N$, there is a unique λ such that $E_S \in \sum_{i \in S} f_i(c_i, \lambda)$. So define a rule F as follows:
For $i \neq 1$,

$$F_i(c_S, E_S) = f_i(c_i, \lambda) = \lambda c_i$$

and for $i = 1$,

$$F_1(c_S, E_S) = E_S - \sum_{i \in S \setminus \{1\}} F_i(c_S, E_S),$$

where λ is chosen so that $E_S \in \sum_{i \in S} f_i(c_i, \lambda)$.

Stovall (2014) shows that F has no parametric representation but nevertheless satisfies continuity, consistency, and resource monotonicity. This is a simplified exposition of Stovall's example in our context that justifies the characterization of rules beyond the class of parametric rules. Such a characterization is given in Theorem 1.3.8.

1.4 Concluding remarks

In this chapter we introduce a coalition formation problem induced by a generalized claims problem and a rule that links the literatures on claims problems and coalition formation problems. We start with a group of agents with claims such that each subgroup produces an output that is insufficient to meet agents' claims. Each coalitional output is rationed among its members by a rule, which takes the agent's claims over the outputs as input. Thus, every agent orders the coalitions that she can join according to the payoffs proposed by the rule. The orderings define a coalition formation problem. It turns out that only resource monotonic and consistent rules (among continuous rules) induce regular coalition formation problems, which are proven to have stable partitions.

In our approach claims and outputs are assumed to be exogenous and independent of each other. Other assumptions about claims and outputs may also be realistic: Each agent's claim may well depend on the identity of the members of the coalition she can join, outputs may be a function of the size of the coalition or may be contingent on how the remaining agents are organized.[22] These and related considerations offer potential for future research.

Bibliography

Alcalde, J. and Revilla, P. (2004), 'Researching with whom? stability and manipulation', *Journal of Mathematical Economics* **40**(8), 869–887.

Auman, R. and Maschler, M. (1985), 'Game theoretic analysis of a bankruptcy problem from the talmud', *Journal of Economic Theory* **36**(2), 195–213.

Banerjee, S., Konishi, H. and Sönmez, T. (2001), 'Core in a simple coalition formation game', *Social Choice and Welfare* **18**(1), 135–153.

Barberà, S., Beviá, C. and Ponsatí, C. (2015), 'Meritocracy, egalitarianism and the stability of majoritarian organizations', *Games and Economic Behavior* **91**, 237–257.

Bloch, F. and Diamantoudi, E. (2011), 'Noncooperative formation of coalitions in hedonic games', *International Journal of Game Theory* **40**(2), 263–280.

Bogomolnaia, A., Jackson, M. O. et al. (2002), 'The stability of hedonic coalition structures', *Games and Economic Behavior* **38**(2), 201–230.

[22]Some of these modifications may generate externalities across coalitions and call for the analysis of partition function form games (see Thrall and Lucas, 1963).

Burani, N. and Zwicker, W. S. (2003), 'Coalition formation games with separable preferences', *Mathematical Social Sciences* **45**(1), 27–52.

Chung, K.-S. (2000), 'On the existence of stable roommate matchings', *Games and economic behavior* **33**(2), 206–230.

Dreze, J. H. and Greenberg, J. (1980), 'Hedonic coalitions: Optimality and stability', *Econometrica: Journal of the Econometric Society* pp. 987–1003.

Erlanson, A. and Flores-Szwagrzak, K. (2015), 'Strategy-proof assignment of multiple resources', *Journal of Economic Theory* **159**, 137–162.

Farrell, J. and Scotchmer, S. (1988), 'Partnerships', *The Quarterly Journal of Economics* **103**(2), 279–297.

Iehlé, V. (2007), 'The core-partition of a hedonic game', *Mathematical Social Sciences* **54**(2), 176–185.

Inal, H. (2015), 'Core of coalition formation games and fixed-point methods', *Social Choice and Welfare* **45**(4), 745–763.

Kaminski, M. M. (2006), 'Parametric rationing methods', *Games and Economic Behavior* **54**(1), 115–133.

Karakaya, M. (2011), 'Hedonic coalition formation games: A new stability notion', *Mathematical Social Sciences* **61**(3), 157–165.

Moulin, H. (2000), 'Priority rules and other asymmetric rationing methods', *Econometrica* **68**(3), 643–684.

Moulin, H. (2004), *Fair division and collective welfare*, MIT press.

O'Neill, B. (1982), 'A problem of rights arbitration from the talmud', *Mathematical social sciences* **2**(4), 345–371.

Pycia, M. (2012), 'Stability and preference alignment in matching and coalition formation', *Econometrica* **80**(1), 323–362.

Schummer, J. P., Chun, Y. and Thomson, W. (2001), 'Constrained egalitarianism: A new solution for claims problems', *Seoul Journal of Economics* **14**, 269–298.

Stovall, J. E. (2014), 'Asymmetric parametric division rules', *Games and Economic Behavior* **84**, 87–110.

Thomson, W. (2015), 'Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: an update', *Mathematical Social Sciences* **74**, 41–59.

Thomson, W. and Lensberg, T. (1989), 'Axiomatic theory of bargaining with a variable population'.

Thrall, R. M. and Lucas, W. F. (1963), 'N-person games in partition function form', *Naval Research Logistics Quarterly* **10**(1), 281–298.

Young, H. P. (1987), 'On dividing an amount according to individual claims or liabilities', *Mathematics of Operations Research* **12**(3), 398–414.

Chapter 2

Stable partitions for proportional generalized claims problems

Abstract

We consider a set of agents, *e.g.*, a group of researchers, who have claims on an endowment, *e.g.*, a research budget from a national science foundation. The research budget is not large enough to cover all claims. Agents can form coalitions and coalitional funding is proportional to the sum of the claims of its members, except for singleton coalitions which do not receive any funding. We analyze the structure of stable partitions when coalition members use well-behaved rules to allocate coalitional endowments, *e.g.*, the well-known constrained equal awards rule (CEA) or the constrained equal losses rule (CEL).

For continuous, (strictly) resource monotonic, and consistent rules, stable partitions with (mostly) pairwise coalitions emerge. For CEA and CEL we provide algorithms to construct such a pairwise stable partition. While for CEL the resulting stable pairwise partition is assortative and sequentially matches two lowest claim agents, for CEA the resulting stable pairwise partition is obtained by either assortatively matching two highest claim agents or by matching a lowest claim with a highest claim agent.

2.1 Introduction

The formation of coalitions is a pervasive aspect of social, economic, or political environments. Agents form coalitions in very different situations in order to achieve some joint benefits. Cooperation between agents is sometimes hampered by the existence of two opposing fundamental forces: on the one hand, the increasing returns to scale, which incentivizes agents to cooperate and, therefore, to form large coalitions and, on the other hand, the heterogeneity of agents, which causes instability and pushes towards the formation of only small coalitions.

Chapter 1 introduces *generalized claims problems* to deal with coalition formation in a bankruptcy framework. A generalized claims problem consists of a group of agents, each of them with a claim and a set of "coalitional endowments", one for each possible coalition, which are not sufficient to meet the claims of their members. Coalitional endowments are divided among their members according to a prespecified rule, which thus is a decisive element of the coalition formation process. Their main result states that, given a generalized claims problem, only the continuous rules that are resource monotonic, and consistent induce coalition formation problems with stable partitions. In this chapter, we study the structure of stable partitions under different rules satisfying those properties to answer two types of questions: What coalition sizes can emerge? & Who are the coalition partners?

The model proposed in Chapter 1 does not impose any restriction on coalitional endowments and consequently answering the above questions is not really possible in their general model. In contrast, we consider *non-singleton proportional generalized claims problems* where singleton coalitions have zero endowments and all remaining coalitional endowments are a fixed proportion of the sum of their members' claims. Proportionality is justified in many situations such as the funding of research projects where the budgets are often divided proportionally to funding needs or according to other funding criteria such as project quality.[1] Moreover, in many situations, institutions are interested to spark cooperation and hence, discourage singleton coalitions.

Non-singleton proportional generalized claims problems are a subclass of the class of generalized claims problems studied in Chapter 1 and hence its results hold. Given a non-singleton proportional generalized claims problem, we characterize all stable partitions when the rule applied satisfies continuity, strict resource monotonicity and consistency. We show that at most one singleton coalition belongs to each stable partition and that for each coalition in the stable partition with size larger than two, each agent of the coalition receives a proportional payoff (Theorem 2.3.2). Furthermore, considering resource monotonicity instead of its strict version, even though we do not characterize all stable partitions, we show that a stable partition formed by pairwise coalitions, *i.e.*, coalitions of size two, with the exception of at most one singleton coalition if the set of agents is odd, exists (Theorem 2.3.3).

With the result of Theorem 2.3.3 as the departure point, we analyze how agents sort themselves into pairwise coalitions under some parametric rules (see Stovall, 2014; Young, 1987). Parametric rules are well-studied in the literature because the

[1]Other examples can be found in a bankruptcy situation, where assets have to be allocated among creditors according to their claims or, in a legislature, where seats are distributed among the parties according to their vote totals.

payoff of each agent is given by a function that depends only on the claim of the agent and a parameter that is common to all agents. We focus on two well-known parametric rules that represent two egalitarian principles: the constrained equal awards rule (CEA) and the constrained equal losses rule (CEL). On the one hand, CEA divides the endowment as equally as possible subject to no agent receiving more than her claim (*e.g.*, rationing toilet paper when shortage occurs). On the other hand, CEL divides the losses as equally as possible subject to no agent receiving a negative amount (*e.g.*, equal sacrifice taxation when utility is measured linearly[2]).

We propose two algorithms, one for each rule, to find a pairwise stable partition. The *CEA algorithm* sequentially pairs off either two highest claim agents or a highest with a lowest claim agent (Theorem 2.3.4). Examples of the first type of cooperation are found in social environments where agents tend to join other agents with similar characteristics. In contrast, the second type of cooperation may be interpreted as a transfer of knowledge between agents as happens, for instance, between apprentices and advisors. While the CEA algorithm produces stable partitions that can contain *assortative* as well as *extremal pairwise coalitions*, the *CEL algorithm* is purely assortative and sequentially pairs off lowest claim agents (Theorem 2.3.6).

There is a large number of papers that pay attention to the structure of coalition formation. Becker (1973) and Greenberg and Weber (1986) introduce the notion of assortative coalitions.[3] Observe that in both our algorithms assortative coalitions (in terms of claims) may form. We discuss some papers in which similar results concerning assortative stable coalitions are obtained in our conclusion (Section 2.4).

The rest of the chapter is organized as follows. In Section 2.2 we introduce all ingredients needed to define the class of generalized claims problems and our subclass of non-singleton proportional generalized claims problems. We also introduce the class of parametric claims rules (including CEA and CEL) and their key properties (continuity, (strict) resource monotonicity, and consistency). Section 2.3 contains the results we have discussed above with subsections dedicated to CEA (Subsection 2.3.1) and CEL (Subsection 2.3.2). We conclude in Section 2.4.

2.2 The model

Consider a **coalition of agents**, *e.g.*, a group of researchers, who have claims on an **endowment**, *e.g.*, a research budget from a national science foundation. The re-

[2]The idea of the equal sacrifice principle in taxation is that all tax payers end up sacrificing equally, according to some cardinal utility function. Young (1988) provides a characterization of the family of equal-sacrifice rules based on a few compelling principles and, more recently, Chambers and Moreno-Ternero (2017) generalize the previous family.

[3]Assortativeness is based on an ordering of agents according to a specific variable such as claims, productivity, or location. Alternative terminology includes that of consecutive coalitions.

searchers' **claims** could be related to the past performance or productivity of researchers or be an estimate of the research costs or the expected research outputs. The research budget is not large enough to cover all claims. Now assume that the national science foundation prefers to subsidize research projects or teams instead of individual researchers and that researchers will later need to allocate the research funding within the research teams. On the other hand, anticipating this method of allocating funding, researchers might prefer to be members of certain research teams over others. This situation was analyzed in Chapter 1. Before fully specifying this class of problems, we present the preliminaries of the two classical type of problems it is based on: **claims problems** and **coalition formation problems**.

First, we introduce some notation. Let $\mathbb{N}$ be the set of potential agents and $\mathcal{N}$ the set of all non-empty finite subsets or **coalitions** of $\mathbb{N}$. Given $N \in \mathcal{N}$ and $x, y \in \mathbb{R}^N$, if for each $i \in N$, $x_i > y_i$, then $x \gg y$ and if for each $i \in N$, $x_i \geq y_i$, then $x \geqq y$. Furthermore, for each $x \in \mathbb{R}^N$ and each $S \subseteq N$, $x_S := (x_j)_{j \in S}$ denotes the **restriction of x to coalition S**.

Generalized claims problems

Consider a coalition of agents who have claims on a certain endowment, this endowment being insufficient to satisfy all the claims. A primary example is bankruptcy, where agents are the creditors of a firm and the endowment is its liquidation value; however, we have a more general interpretation of the data in mind.

Formally, let $N \in \mathcal{N}$. For $i \in N$, let c_i be agent i's claim and $c = (c_j)_{j \in N}$ the **claims vector**. Let $E \in \mathbb{R}_+$ be the **endowment**. A **claims problem with coalition N** is a pair $(c, E) \in \mathbb{R}^N_{++} \times \mathbb{R}_+$ such that $\sum_{j \in N} c_j \geq E$. Let $\boldsymbol{\mathcal{C}^N}$ denote the class of such problems and $\boldsymbol{\mathcal{C}} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{C}^N$.

An **allocation for (c, E)** $\in \mathcal{C}^N$ is a (payoff) vector $x = (x_i)_{i \in N} \in \mathbb{R}^N_+$ that satisfies the non-negativity and claim boundedness conditions $0 \leqq x \leqq c$, and the efficiency condition $\sum_{j \in N} x_j = E$. A **rule** is a function F defined on $\mathcal{C}$ that associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$ an allocation for (c, E). Let $\boldsymbol{\mathcal{F}}$ denote the set of rules.

A rule is **continuous** if small changes in the data of the problem do not lead to large changes in the chosen allocation.

Continuity. For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each sequence of problems $\{(c^k, E^k)\}$ of elements of $\mathcal{C}^N$, if (c^k, E^k) converges to (c, E) then $F(c^k, E^k)$ converges to $F(c, E)$.

Consider a claims problem and the allocation given by the rule for it. We require that if the endowment increases, each agent should receive at least as much as (more than, respectively) initially.

Resource monotonicity. For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $E' > E$, if $\sum_{j \in N} c_j \geq E'$, then $F(c, E') \geqq F(c, E)$.

Strict resource monotonicity. For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $E' > E$, if $\sum_{j \in N} c_j \geq E'$, then $F(c, E') \gg F(c, E)$.

Consider a claims problem and the allocation given by the rule for it. Imagine that some agents leave with their payoffs. At that point, reassess the situation of the remaining agents, that is, consider the problem of dividing what remains of the endowment among them. We require that the rule should assign the same payoffs to each of them as initially.

Consistency. For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, each $S \subsetneq N$, and letting $x \equiv F(c, E)$, we have $x_S = F(c_S, E - \sum_{j \in N \setminus S} x_j)$ or, equivalently, $x_S = F(c_S, \sum_{j \in S} x_j)$.

For claims problems many rules are continuous, resource monotonic, and consistent. The most important ones are the so-called "parametric" rules. For a rule in this class, there is a function of two variables such that for each problem, each agent's payoff is the value of this function when the first argument is her claim and the second one is parameter λ, which is the same for all agents. This parameter is chosen so that the sum of agents' payoffs is equal to the endowment. Young (1987) characterizes parametric rules on the basis of symmetry,[4] continuity, and bilateral consistency.[5] Stovall (2014) characterizes the family of possibly asymmetric parametric rules on the basis of continuity, resource monotonicity, bilateral consistency, and two additional axioms, "N-continuity" and "intrapersonal consistency."

A **parametric rule** is defined as follows: Let f be a collection of functions $\{f_i\}_{i \in N}$,[6] where each $f_i : \mathbb{R}_+ \times [a, b] \longrightarrow \mathbb{R}_+$ is continuous and weakly increasing in λ, $\lambda \in [a, b]$, $-\infty \leq a < b \leq \infty$ and for each $i \in N$ and $c_i \in \mathbb{R}_+$, $f_i(c_i, a) = 0$ and $f_i(c_i, b) = c_i$. Hence, for each f a rule F for claims problem $(c, E) \in \mathcal{C}^N$ is defined as follows. For each $i \in N$,

$$F_i(c, E) = f_i(c_i, \lambda) \text{ where } \lambda \text{ is chosen so that } \sum_{j \in N} f_j(c_j, \lambda) = E.$$

Then, f is said to be a **parametric representation of rule F**.

The proportional, constrained equal awards, constrained equal losses, the Talmud, reverse Talmud, and Piniles rules are symmetric parametric rules while the sequential priority rule associated with a strict priority $\succ$ on agents is an asymmetric parametric rule. We define the first three rules and refer to Thomson (2003, 2015, 2019) for the definition of the other rules mentioned above.

[4] Two agents with equal claims should receive equal payoffs.
[5] Bilateral consistency requires consistency only when $|S| = 2$.
[6] When the parametric rule is symmetric, f_i is the same for all agents.

The most commonly used rule in practice makes awards proportional to claims.

Proportional rule, $\boldsymbol{P}$. For each $(c, E) \in \mathcal{C}^N$ and each $i \in N$, $P_i(c, E) = \lambda c_i$, where λ is chosen so that $\sum_{j \in N} \lambda c_j = E$.

Our next rule divides endowment as equally as possible subject to no agent receiving more than her claim.

Constrained equal awards rule, $\boldsymbol{CEA}$. For each $(c, E) \in \mathcal{C}^N$ and each $i \in N$, $CEA_i(c, E) = \min\{c_i, \lambda\}$, where λ is chosen so that $\sum_{j \in N} \min\{c_j, \lambda\} = E$.

An alternative to the constrained equal awards rule is obtained by focusing on the losses agents incur (what they do not receive), as opposed to what they receive, and to assign losses to all agents as equally as possible subject to no one receiving a negative amount.

Constrained equal losses rule, $\boldsymbol{CEL}$. For each $(c, E) \in \mathcal{C}^N$ and each $i \in N$, $CEL_i(c, E) = \max\{0, c_i - \lambda\}$, where λ is chosen so that $\sum_{j \in N} \max\{0, c_j - \lambda\} = E$.

Next, we generalize the notion of a claims problem. Consider $(c, E) \in \mathcal{C}^N$. Then, each coalition of agents $S \subseteq N$ has the **reduced claims vector $\boldsymbol{c_S}$** $= (c_i)_{i \in S}$. Next, assume that for each coalition $S \subseteq N$ there is a **coalitional endowment $\boldsymbol{E_S}$** such that $(c_S, E_S) \in \mathcal{C}^S$ and $E_N = E$. Formally, given $N \in \mathcal{N}$, a **generalized claims problem with coalition $\boldsymbol{N}$** is a pair $(c, (E_S)_{S \subseteq N}) \in \mathbb{R}^N_{++} \times \mathbb{R}^{2^{|N|}-1}_{+}$, such that for each coalition $S \subseteq N$, $(c_S, E_S) \in \mathcal{C}^S$. Let $\boldsymbol{\mathcal{G}^N}$ denote the class of such problems and $\boldsymbol{\mathcal{G}} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{G}^N$.

More specifically, we first study the subclass of **proportional generalized claims problems** where given $N \in \mathcal{N}$, $(c, (E_S)_{S \subseteq N}) \in \mathcal{G}^N$, and $\alpha := \frac{E}{c^N} \in (0, 1)$, for each coalition $S \subseteq N$, the coalitional endowment E_S is proportional to the **coalitional claim $\boldsymbol{c^S}$** $:= \sum_{j \in S} c_j$, *i.e.*, $E_S = \alpha c^S$ (where $\alpha := \frac{E}{c^N} \in (0, 1)$). Let $\boldsymbol{\mathcal{P}^N}$ denote the class of such problems and $\boldsymbol{\mathcal{P}} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{P}^N$. Since for each coalition $S \subseteq N$, all coalitional endowments E_S are completely determined by c and E, we will simplify notation and denote a proportional generalized claims problem $(c, (E_S)_{S \subseteq N}) \in \mathcal{P}^N$ by (c, E).

An **allocation configuration for $\boldsymbol{(c, E)}$** $\in \mathcal{P}^N$ is a list $(x_S)_{S \subseteq N}$ such that for each $S \subseteq N$, x_S is an allocation for the claims problem derived from (c, E) for coalition S, (c_S, E_S).[7] Any rule $F \in \mathcal{F}$ can be extended to a **generalized rule** defined on $\mathcal{P}$ by associating with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{P}^N$ an allocation configuration $F(c, E) = (F(c_S, E_S))_{S \subseteq N}$. Since it should not lead to any confusion we use $\boldsymbol{\mathcal{F}}$ to also denote the set of generalized rules.

[7] The notion of allocation (payoff) configuration was introduced by Hart (1985) in his characterization of the Harsanyi NTU solution.

Coalition formation problems

Consider a society where each agent can rank the coalitions that she may belong to. Some well-known examples of such problems are matching problems, in particular, marriage and roommate problems.

Formally, let $N \in \mathcal{N}$. For each agent $i \in N$, $\succsim_i$ is a complete and transitive preference relation over the coalitions of N containing i. Given $S, S' \subseteq N$ such that $i \in S \cap S'$, $S \succsim_i S'$ means that agent i finds coalition S at least as desirable as coalition S'. Let $\mathcal{R}_i$ be the set of such preference relations for agent i and $\mathcal{R}^N \equiv \Pi_{i \in N} \mathcal{R}_i$. A **coalition formation problem with agent set $\boldsymbol{N}$** consists of a list of preference relations, one for each $i \in N$, $\boldsymbol{\succsim} = (\succsim_j)_{j \in N} \in \mathcal{R}^N$. Let $\boldsymbol{\mathcal{D}^N}$ be the class of such problems and $\boldsymbol{\mathcal{D}} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{D}^N$.

A partition of a set of agents $N \in \mathcal{N}$ is a set of disjoint coalitions of N whose union is N and whose pairwise intersections are empty. Formally, a **partition** of N is a list $\pi = \{S_1, \ldots, S_m\}$, ($m \leq |N|$ is a positive integer) such that (i) for each $l = 1, \ldots, m$, $S_l \neq \emptyset$, (ii) $\bigcup_{l=1}^{m} S_l = N$, and (iii) for each pair $l, l' \in \{1, \ldots, m\}$ with $l \neq l'$, $S_l \cap S_{l'} = \emptyset$. Let $\boldsymbol{\Pi(N)}$ denote the set of all partitions of N. For each $\pi \in \Pi(N)$ and each $i \in N$, let $\boldsymbol{\pi(i)}$ denote the unique coalition in π which contains agent i. We refer to the partition $\boldsymbol{\pi^0}$ at which each agent forms a singleton coalition, *i.e.*, for each $i \in N$, $\pi^0(i) = \{i\}$, as the **singleton partition**.

An important question for coalition formation problems is the existence of partitions from which no agent wants to deviate. Let $\succsim \in \mathcal{D}^N$ and consider a partition $\pi \in \Pi(N)$. Then, coalition $T \subseteq N$ **blocks** π if for each agent $i \in T$, $T \succ_i \pi(i)$. A partition $\pi \in \Pi(N)$ is **stable** for $\succsim$ if it is not blocked by any coalition $T \subseteq N$. Let $\boldsymbol{St(\succsim)}$ denote the set of all stable partitions of $\succsim$. We refer to the coalitions that are part of a stable partition as **stable coalitions**.

Coalition formation problems induced by a generalized claims problem and a rule

Given a generalized claims problem and a rule, each agent, by calculating her payoff in each coalition, can form preferences over coalitions, giving rise to a coalition formation problem. Examples of this situation can be the formation of jurisdictions and research groups (for further examples, see Chapter 1).

Formally, given $N \in \mathcal{N}$, $(c, (E_S)_{S \subseteq N}) \in \mathcal{G}^N$, and $F \in \mathcal{F}$, the **coalition formation problem induced by $\boldsymbol{((c, (E_S)_{S \subseteq N}), F)}$** consists of the list of preferences $\succsim^{((c,E),F)} = (\succsim_j)_{j \in N}^{((c,E),F)}$ defined as follows: for each $i \in N$, and each pair $S, S' \subseteq N$ such that $i \in S \cap S'$, $S \succsim_i^{((c,E),F)} S'$ if and only if $F_i(c_S, E_S) \geq F_i(c_{S'}, E_{S'})$.

Proposition 1.3.9 in Chapter 1 shows that, given any generalized claims problem, parametric rules always induce coalition formation problems that have stable partitions. More generally, they show that a coalition formation problem induced by any generalized claims problem and any rule has a stable partition if and only if the rule F is continuous, resource monotonic, and consistent (Theorem 1.3.8 in Chapter 1).

2.3 Stable partitions for (non-singleton) proportional generalized claims problems

We restrict attention to the class of proportional generalized claims problems $\mathcal{P}$. Thus, consider $N \in \mathcal{N}$, $(c, E) \in \mathcal{P}^N$, and $F \in \mathcal{F}$. Even without any further assumptions on rule F, the coalition formation problem induced by $((c, E), F)$ has a stable partition. Due to the assumption that the generalized claims problems we consider are proportional, the singleton-partition is always stable and all stable partitions are payoff equivalent to the singleton partition.

Proposition 2.3.1. *Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{P}^N$, and $F \in \mathcal{F}$. Then, for the coalition formation problem with agent set N induced by $((c, E), F)$, the singleton-partition π^0 is stable and each stable partition π induces the proportional allocation configuration where each agent $i \in N$ receives αc_i.*

Proof: Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{P}^N$, and $F \in \mathcal{F}$. Consider the coalition formation problem with agent set N induced by $((c, E), F)$.

First, assume by contradiction that there exists a stable partition π that does not induce the proportional allocation configuration. Hence, there exists some agent $i \in N$ who receives an under-proportional payoff $F_i(c_{\pi(i)}, E_{\pi(i)}) < \alpha c_i$. However, since $(c, E) \in \mathcal{P}^N$, agent i by forming the singleton coalition $\{i\}$ obtains $\alpha c_i > F_i(c_{\pi(i)}, E_{\pi(i)})$ and blocks π; contradicting the stability of π. Thus, each stable partition π induces the proportional allocation configuration.

Second, consider the singleton partition π^0 at which each agent $i \in N$ obtains αc_i. Since $(c, E) \in \mathcal{P}^N$, for each $S \subseteq N$, $E_S = \alpha c^S$. Thus, no coalition $S \subseteq N$ can achieve payoffs larger than αc_i for its members $i \in S$ and block π^0. Hence, π^0 is stable. □

Proposition 2.3.1 illustrates that for proportional generalized claims problems essentially only the "trivial" singleton partition is stable. Going back to our motivating example, the allocation of research funding to research teams, Proposition 2.3.1 suggests that if the main principle of research funding allocation from the funding agency to teams is proportionality, then, since essentially only individual proportional funding is stable, the formation of larger research collaborations is unlikely. However, many scientific funding schemes are aimed at the promotion of cooperation of researchers from different countries or disciplines and require research teams of at least size two; see, for instance the international programs of the Swiss National

Sciences Foundation (http://www.snf.ch/en/funding/directaccess/international/), and the European Union funded COST (European Cooperation in Science and Technology) actions that focus on research and innovation networks (https://www.cost.eu/).

We take this as the departure point to modify the class of proportional generalized claims problems in order to disincentivizing singleton coalitions by assuming that singleton coalitions are not funded. Hence, the second subclass of generalized claims problems that we consider is the following adjustment of the class of proportional generalized claims problems where, given $N \in \mathcal{N}$, $(c,E) \in \mathcal{C}^N$, and $\alpha := \frac{E}{c^N} \in (0,1)$), for each $S \subseteq N$ such that $|S| \geq 2$, $E_S = \alpha c^S$ and for each $S \subseteq N$ such that $|S| = 1$, $E_S = 0$. Let $\widetilde{\boldsymbol{\mathcal{P}}}^{\boldsymbol{N}}$ be the class of such problems. We refer to this subclass of generalized claims problems as **non-singleton proportional generalized claims problems** and denote it by $\widetilde{\boldsymbol{\mathcal{P}}} \equiv \bigcup_{N \in \mathcal{N}} \widetilde{\mathcal{P}}^N$. We again simplify notation and denote a non-singleton proportional generalized claims problem $(c, (E_S)_{S \subseteq N}) \in \widetilde{\mathcal{P}}^N$ by (c, E).

Next, we analyze the structure of coalition formation problems that are induced by non-singleton proportional generalized claims problems and describe the structure of stable partitions if the underlying rule is continuous, strictly resource monotonic, and consistent. By Chapter 1 the set of stable partitions is nonempty. Furthermore, for each stable partition, there is at most one singleton coalition and each other coalition either allocates proportional payoffs or is of size at most two. Hence, in comparison to the benchmark result of proportional stable sharing that is equivalent to the singleton-partition, non-proportional cooperation can be sustained in a stable way in pairwise research teams.

Theorem 2.3.2. *Let $N \in \mathcal{N}$ and $(c,E) \in \widetilde{\mathcal{P}}^N$. Consider $F \in \mathcal{F}$ satisfying continuity, strict resource monotonicity, and consistency. Then, for the coalition formation problem with agent set N induced by $((c,E),F)$, the set of stable partitions is nonempty and each stable partition π is such that*

(i) there is at most one singleton coalition and

ii) if $S \in \pi$ such that $|S| > 2$, then for all $i \in S$, $F_i(c_S, E_S) = \alpha c_i$.

Proof: Let N, (c,E), and F as specified in the theorem. First, by Chapter 1, each coalition formation problem with agent set N induced by $((c,E),F)$ has at least one stable partition.

(i) Let $\pi \in St(\succsim^{((c,E),F)})$ and assume, by contradiction, that there exist $i,j \in N$, $i \neq j$, such that $\pi(i) = \{i\}$ and $\pi(j) = \{j\}$. For the trivial claims problem $(c_{\{i,j\}}, 0) \in \mathcal{C}^{\{i,j\}}$, we have $F_i(c_{\{i,j\}}, 0) = F_j(c_{\{i,j\}}, 0) = 0$. Then, since $E_{\{i,j\}} > 0$, by strict resource monotonicity, $F_i(c_{\{i,j\}}, E_{\{i,j\}}), F_j(c_{\{i,j\}}, E_{\{i,j\}}) > 0$. Given that $(c,E) \in \widetilde{\mathcal{P}}^N$, we have

$E_{\{i\}} = E_{\{j\}} = 0$. Then, $F_i(c_{\{i\}}, 0) = F_j(c_{\{j\}}, 0) = 0$. Thus, coalition $\{i, j\}$ blocks π, which is a contradiction.
(ii) Assume that $\pi \in St(\succsim^{((c,E),F)})$ is such that $S \in \pi$, $|S| > 2$, and for some $i \in S = \pi(i)$, $F_i(c_S, E_S) \neq \alpha c_i$. Without loss of generality, assume that agent i receives an over-proportional payoff $F_i(c_S, E_S) > \alpha c_i$. By consistency, for each $j \in S \setminus \{i\}$,

$$F_j\left(c_{S\setminus\{i\}}, \sum\nolimits_{k\in S\setminus\{i\}} F_k(c_S, E_S)\right) = F_j(c_S, E_S).$$

Furthermore, $E_{S\setminus\{i\}} = \alpha c^{S\setminus\{i\}} > \sum_{k\in S\setminus\{i\}} F_k(c_S, E_S)$ and the agents in coalition $S\setminus\{i\}$ have a larger endowment to share among themselves if they get rid of agent i. Then, by strict resource monotonicity, for each $j \in S \setminus \{i\}$,

$$F_j\left(c_{S\setminus\{i\}}, E_{S\setminus\{i\}}\right) > F_j\left(c_{S\setminus\{i\}}, \sum\nolimits_{k\in S\setminus\{i\}} F_k(c_S, E_S)\right).$$

Hence, for each $j \in S \setminus \{i\}$,

$$F_j\left(c_{S\setminus\{i\}}, E_{S\setminus\{i\}}\right) > F_j(c_S, E_S)$$

and coalition $S \setminus \{i\}$ blocks π, which is a contradiction. □

We now weaken strict resource monotonicity to resource monotonicity and show that among all possible stable partitions that can exist in a coalition formation problem induced by a non-singleton proportional generalized claims problem, there is always one stable partition that is composed of pairwise research teams (with at most one singleton coalition if the number of agents is odd).

Theorem 2.3.3. *Let $N \in \mathcal{N}$ and $(c, E) \in \widetilde{\mathcal{P}}^N$. Consider $F \in \mathcal{F}$ satisfying continuity, resource monotonicity, and consistency. Then, for the coalition formation problem with agent set N induced by $((c, E), F)$, there exists a stable partition π such that*

(i) if $|N|$ is even, then for each $i \in N$, $|\pi(i)| = 2$ and

(ii) if $|N|$ is odd, then there exists an agent $j \in N$, such that $p(j) = \{j\}$ and for each $i \in N \setminus \{j\}$, $|\pi(i)| = 2$.

We prove Theorem 2.3.3 in Appendix 2.4.

Given the result of Theorem 2.3.3, from now on we focus on coalitions of size two. Note that Theorem 2.3.3 does not give information about how agents sort themselves into those stable pairwise coalitions. In fact, the stable partitions may differ depending on how endowments are divided among agents. Next, we will study the specific structure of stable partitions under two egalitarian parametric rules: the constrained equal awards (CEA) and the constrained equal losses (CEL) rule. Egalitarianism is a natural principle applied in many economic environments and hence, studying CEA and CEL in our context seems a natural first step to understand stable partitions.

2.3.1 Stable coalitions under the constrained equal awards rule

We first analyze how agents organize themselves into stable pairwise coalitions when endowments are distributed under the constrained equal awards rule, CEA. Recall that this rule divides endowments as equally as possible subject to no agent receiving more than her claim. Hence, given a claims vector, under CEA, those agents with lower claims receive over-proportional payoffs (more so the lower the claims are) while agents with higher claims receive under-proportional payoffs (more so the higher the claims are). Furthermore, the higher an agent's claim, the higher her contribution towards the endowment of any coalition she is part of. So intuitively, in order to form a stable pairwise coalition, one could suspect that an agent with a high claim will pair up with another high claim agent. Indeed, high claim agents play a special role in our construction of stable pairwise coalitions.

Let $N = \{1, \ldots n\}$ and assume that $c_1 \leq c_2 \leq \ldots \leq c_n$. First, consider a highest claim agent, say agent n. As explained above, agent n in a coalition with agent $n-1$ would be a contender to be part of a stable partition with the following possible justification: agent n needs to team up with some agent to obtain a positive payoff and agent $n-1$ provides the highest possible contribution to coalition $\{n-1, n\}$ while requiring a smaller transfer from the proportional payoff compared to other agents. This reasoning is correct, unless a smallest claim agent, say agent 1, has such a small claim that conceding this small claim to agent 1 is a smaller loss from the proportional payoff for agent n than the transfer from the proportional payoff of n to agent $n-1$. We capture this intuition in an algorithm that determines a stable partition by sequentially pairing off either two highest claim agents or a highest with a lowest claim agent.

CEA Algorithm.

Input: $N \in \mathcal{N}$ and $(c, E) \in \widetilde{\mathcal{P}}^N$.

Step 1. Let $N_1 := N$, $|N_1| > 2$, and

$$\delta_1 := \frac{2c_1}{2c_1 - c_{n-1} + c_n}.$$

We distinguish two cases:

(i) If $\alpha \leq \delta_1$, then set $S_1 := \{n-1, n\}$.

(ii) If $\alpha > \delta_1$, then set $S_1 := \{1, n\}$.

Set $N_2 := N \setminus S_1$. If $|N_2| \leq 2$, then set $S_2 := N_2$, define $\pi := \{S_1, S_2\}$, and stop. Otherwise, go to Step 2.

Step k ($k > 1$). Recall from Step $k-1$ that $N_k := N \setminus \left(\cup_{j=1}^{k-1} S_j\right)$ and $|N_k| > 2$. We relabel the agents in N_k such that $N_k = \{1', \ldots, n'\}$ and $c_{1'} \leq \ldots \leq c_{n'}$. Let

$$\delta_k := \frac{2c_{1'}}{2c_{1'} - c_{(n-1)'} + c_{n'}}.$$

We distinguish two cases:

(i) If $\alpha \leq \delta_k$, then set $S_k := \{(n-1)', n'\}$.

(ii) If $\alpha > \delta_k$, then set $S_k := \{1', n'\}$.

Set $N_{k+1} := N \setminus \cup_{j=1}^{k} S_j$. If $|N_{k+1}| \leq 2$, then set $S_{k+1} := N_{k+1}$, define $\pi := \{S_1, \ldots, S_{k+1}\}$, and stop. Otherwise, go to Step $k+1$.

Output: A partition $\pi = \{S_1, \ldots, S_l\}$ for the coalition formation problem with agent set N induced by $((c, E), CEA)$. If $|N|$ is even, then partition π is constructed in $\frac{n-2}{2}$ steps. If $|N|$ is odd, then partition π is constructed in $\frac{n-1}{2}$ steps.

The following result states that the partition obtained by the CEA algorithm is stable.

Theorem 2.3.4. *Let $N \in \mathcal{N}$ and $(c, E) \in \widetilde{\mathcal{P}}^N$. Consider the coalition formation problem with agent set N induced by $((c, E), CEA)$. Then, the partition obtained by the CEA algorithm is stable.*

We prove Theorem 2.3.4 in Appendix 2.4 but we explain the key intuition of the proof here. Observe that in each step of the CEA algorithm either two agents with the highest claims are matched or an agent with the highest claim is matched with an agent with the lowest claim. We first show (Lemma 2.4.3) that each agent will always prefer to be matched to an agent with a higher claim, the higher the better. That means that a highest claim agent, say agent n is a most desirable pairwise coalition partner for all other agents. Hence, by matching agent n with her most desirable pairwise coalition partner, a stable pairwise coalition is formed (in fact, a pairwise *top coalition* is formed, see Appendix 2.4).

Second (Lemma 2.4.4), we show that when agent n is in a pairwise coalition where the endowment is split equally, she prefers a pairwise coalition with a higher-claim agent. This happens when parameter α is lower than $\frac{2c_1}{c_1+c_n}$. Third (Lemma 2.4.6), we show that when agent n is in a pairwise coalition where the other agent receives his claim, she prefers a pairwise coalition with a lower-claim agent. This happens when parameter α higher than $\frac{2c_{n-1}}{c_{n-1}+c_n}$.

Lemmas 2.4.4 and 2.4.6 now imply that agents 1 and $n-1$ are potential "stable partners" for agent n. We need to determine the value for parameter α that makes agent n's partner be $n-1$ or 1. We show that $\frac{2c_1}{2c_1-c_{n-1}+c_n}$ is the value for α that makes agent n form coalition with agent $n-1$ or agent 1 (Lemma 2.4.8). Note that that value coincides with the one specified in Step 1 of the CEA algorithm to trigger either Case (i) with stable coalition $\{n-1, n\}$ or Case (ii) with stable coalition $\{1, n\}$.

Finally, Lemmas 2.4.4, 2.4.6, and 2.4.8 are used to show that in each step of the CEA algorithm a stable pairwise coalition is chosen. In particular, it follows that for values of α low enough to trigger Case (i) in each step of the CEA algorithm, a stable partition is formed by assortative pairwise coalitions (we call this an *assortative stable partition*) starting from an agent with the highest claim. This implies that if n is odd, then the singleton coalition will be formed by an agent with the lowest claim. Similarly, for values of α high enough to trigger case (ii) in each step of the CEA algorithm, a stable partition is formed by pairwise coalitions in which a highest and a lowest claim agent are matched in each step (we call this an *extremal stable partition*). So, depending on α, the CEA algorithm constructs a stable partition that is either assortative, extremal, or a mix of both type of pairwise coalitions.

Assortative matching of high types has been observed in other contexts, for instance, the neoclassical marriage model by Becker (1973). In our example of research team formations, assortative matching of high types can indeed be observed in practice but it can also be observed in other situations such as the formation of pairs of students for class projects or other social environments. However, we also observe extremal research team formations, for example in mentor-mentee relationships such as between a PhD student and her advisor.

Finally, the stable partitions obtained by the CEA algorithm are not unique (even beyond tie-breaking between cases (i) and (ii) in the algorithm). The following example demonstrates that stable partitions with larger coalition sizes are possible.

Example 2.3.5. Let $N = \{1,2,3\}$, $c = (1,2,3)$, $E = 5.4$, and $(c,E) \in \widetilde{\mathcal{P}}^N$ (hence, $\alpha = \frac{9}{10}$). Let $F = CEA$. Hence,

Coalition	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
Endowment	0	0	0	2.7	3.6	4.5	5.4
Allocation	(0)	(0)	(0)	$(1,1.7)$	$(1,2.6)$	$(2,2.5)$	$(1,2,2.4)$

The coalition formation problem induced, $\succsim^{((c,E),CEA)}$, is

$$\succsim_1^{((c,E),CEA)}: \{12\} \sim_1 \{13\} \sim_1 \{123\} \succ_1 \{1\},$$
$$\succsim_2^{((c,E),CEA)}: \{23\} \sim_2 \{123\} \succ_2 \{12\} \succ_2 \{2\},$$
$$\succsim_3^{((c,E),CEA)}: \{13\} \succ_3 \{23\} \succ_3 \{123\} \succ_3 \{3\}.$$

The stable partition obtained by our algorithm in Theorem 2.3.4 is $\{\{13\},\{2\}\}$. However, it can be easily verified that partitions $\{\{1\},\{23\}\}$ and $\{\{123\}\}$ are also stable.

2.3.2 Stable coalitions under the constrained equal losses rule

We next analyze how agents organize themselves into pairwise coalitions in a stable partition when endowments are distributed according to the constrained equal losses rule, CEL. Recall that this rule allocates losses as equally as possible subject to no agent receiving a negative amount. Hence, given a claims vector, under CEL, agents with lower claims receive under-proportional payoffs (more so the lower the claims are) while agents with higher claims receive over-proportional payoffs (more so the higher the claims are). Furthermore, the lower an agent's claim, the lower his contribution towards the loss of any coalition she is part of is. So intuitively, in order to form a stable pairwise coalition, one could suspect that an agent with a low claim will pair up with another low claim agent. So in contrast to the CEA, when the CEL is used, low claim agents play a special role in our construction of stable pairwise coalitions.

Let $N = \{1, \ldots n\}$ and assume that $c_1 \leq c_2 \leq \ldots \leq c_n$. First, consider a lowest claim agent, say agent 1. As explained above, agent 1 in a coalition with agent 2 would be a contender to be part of a stable partition with the following possible justification: agent 1 needs to team up with some agent to obtain a positive payoff and agent 2 provides the lowest possible loss to coalition $\{1,2\}$ while requiring a smaller transfer from the proportional payoff compared to other agents. We capture this intuition in an algorithm that determines a stable partition by sequentially pairing off lowest claim agents.

CEL Algorithm.

Input: $N \in \mathcal{N}$ and $(c, E) \in \widetilde{\mathcal{P}}^N$.

Step 1. Let $N_1 := N$, $|N_1| > 2$. Set $S_1 := \{1,2\}$ and $N_2 := N \setminus S_1$. If $|N_2| \leq 2$, then set $S_2 := N_2$, define $\pi := \{S_1, S_2\}$, and stop. Otherwise, go to Step 2.

Step k ($k > 1$). Recall from Step $k-1$ that $N_k := N \setminus \left(\cup_{j=1}^{k-1} S_j\right)$ and $|N_k| > 2$. Set $S_k := \{2k-1, 2k\}$ and $N_{k+1} := N \setminus \cup_{j=1}^{k} S_j$. If $|N_{k+1}| \leq 2$, then set $S_{k+1} := N_{k+1}$, define $\pi := \{S_1, \ldots, S_{k+1}\}$, and stop. Otherwise, go to Step $k+1$.

Output: A partition $\pi = \{S_1, \ldots, S_l\}$ for the coalition formation problem with agent set N induced by $((c, E), CEL)$. If $|N|$ is even, then partition π is constructed in $\frac{n-2}{2}$ steps. If $|N|$ is odd, then partition π is constructed in $\frac{n-1}{2}$ steps.

The following result states that the partition obtained by the CEA algorithm is stable.

Theorem 2.3.6. *Let $N \in \mathcal{N}$ and $(c, E) \in \widetilde{\mathcal{P}}^N$. Consider the coalition formation problem with agent set N induced by $((c, E), CEL)$. Then, the partition obtained by the CEL algorithm is stable.*

We prove Theorem 2.3.6 in Appendix 2.4 but we explain the key intuition of the proof here. Consider $i, j \in N$ such that $c_i < c_j$. Then, for coalition $\{i, j\}$, $E_{\{i,j\}} = \alpha(c_i + c_j)$ and the associated loss equals $(1 - \alpha)(c_i + c_j)$. Hence, the loss decreases if either agent i or j switches to a pairwise coalition with a lower-claim agent, and it increases if they switch to a higher-claim agent. Since losses are split as equally as possible (taking zero as lower bound), sequentially matching the pairs of lowest claim agents will lead to a stable partition.

Observe that in each step of the CEL algorithm two agents with the lowest claims are paired and we obtain an assortative stable partition starting from the agent with a lowest claim. In particular, this implies that if n is odd, then the singleton coalition will be formed by an agent with the highest claim (this contrasts the assortative case of the CEA algorithm where an agent with the lowest claim would form the singleton coalition).

Finally, the partition obtained by the CEL algorithm is not unique. The following example demonstrates that stable partitions with larger coalition sizes are possible.

Example 2.3.7. Let $N = \{1, 2, 3\}$, $c = (1, 3, 11)$, $E = 7.5$, and $(c, E) \in \widetilde{\mathcal{P}}^N$ (hence, $\alpha = \frac{1}{2}$). Let $F = CEL$. Hence,

Coalition	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
Endowment	0	0	0	2	6	7	7.5
Allocation	(0)	(0)	(0)	(0, 2)	(0, 6)	(0, 7)	(0, 0, 7.5)

The coalition formation problem induced, $\succsim^{((c,E),CEL)}$ is

$$\succsim_1^{((c,E),CEL)}: \{12\} \sim_1 \{13\} \sim_1 \{123\} \succ_1 \{1\},$$
$$\succsim_2^{((c,E),CEL)}: \{12\} \succ_2 \{123\} \succ_2 \{23\} \succ_2 \{2\},$$
$$\succsim_3^{((c,E),CEL)}: \{123\} \succ_3 \{23\} \succ_3 \{13\} \succ_3 \{3\}.$$

It can be checked that any partition of N is stable.

2.4 Concluding remarks

In this chapter, we continue the analysis of the model introduced in Chapter 1. Chapter 1 focuses on the existence of stable partitions but it does not analyze their exact structure. To make more precise predictions about the possible size and composition of stable partitions, we restrict attention to what we call non-singleton proportional generalized claims problem where singleton coalitions receive no endowments and all remaining coalitional endowments are a fixed proportion of the sum of the claims of coalition members. Let us briefly summarize our results.

We first characterize all stable partitions when the rule applied to allocate coalitional endowments satisfies continuity, strict resource monotonicity, and consistency. For the weaker notion of resource monotonicity, we demonstrate the existence of a pairwise stable partition with at most one singleton coalition if the set of agents is odd. Furthermore, we provide two algorithms to construct stable pairwise coalitions for CEA and CEL, respectively. For CEA, the obtained stable partition assortatively pairs off either highest claim agents (assortative coalition) or a highest and a lowest claim agent (extremal coalition). For CEL, an assortative stable partition is obtained by sequentially pairing off lowest claim agents.

Observe that our results are based on the assumption of proportional coalitional endowments. Future research could consider another principle of assigning coalitional endowments than proportionality. More generally, a two-step model in which first the total endowment is split between coalitions (by one claims rule) and second, within each coalition the coalitional endowment is split between its members (by another rule, possibly the same than the first one) could be considered ((for a related two-step model in a bankruptcy framework see, for instance, Izquierdo and Timoner, 2019).[8] Therefore, coalition formation will depend both on the rule that divides the total endowment among the different coalitions and on the rule that is used to distribute the coalitional endowments among its members. Observe that our model can be straightforwardly extended to a two-step procedure in which the rule used in the first step is the proportional rule for any coalition of size larger than one and the constant zero rule for singleton coalitions, and the rule applied in then second step satisfies continuity, (strict) resource monotonicity, and consistency (or, for some of our results, equals CEA or CEL).

Finally, as already mentioned in the introduction, there are many papers dealing with assortative stable coalitions. We briefly discuss three of them.

Barberà et al. (2015) consider a model in which each agent is endowed with a productivity level and agents can cooperate to perform certain tasks. Each coalition generates an output equal to the sum of its members' productivities. The authors then analyze the formation of coalitions when all agents in a society vote between meritocracy and egalitarianism. They find societies where assortative and non-assortative partitions (in terms of productivity) arise.

[8] Two-step procedures have been also analyzed, among others, by Lorenzo-Freire et al. (2010) and Bergantiños et al. (2010) for multi-issue allocation problems. Moreno-Ternero (2011) studies a coalition procedure (two or more steps) for bankruptcy situations.

In a bargaining framework, Pycia (2012) presents a model in which each agent has a utility function and, for each possible coalition of agents, there is an output to be distributed among its members. He analyzes coalition formation games induced by different bargaining rules and shows that when agents are endowed with productivity levels and "when shares are divided by a stability-inducing sharing rule, agents sort themselves into coalitions in a predictably assortative way". Pycia (2012) deals with many-to-one problems and his notion of assortativeness implies that the most productive agents join the most productive firms.

Finally, Bogomolnaia et al. (2008) study societies where agents are located in an interval and form jurisdictions to consume public projects, which are located in the same interval. Agents share their costs equally and they divide transportation costs to the location of the public project based its distance to each agent. They analyze both core and Nash stable partitions with a focus on assortative and non-assortative (in terms of location) stable jurisdiction structures.

Appendix

Proof of Theorem 2.3.3

We first introduce some lemmas that will be used to prove Theorem 2.3.3. By the first lemma we show that, given a non-singleton proportional generalized claims problem and a consistent rule, if in a coalition all agents receive proportional payoffs, then all agents in any subcoalition (except singleton subcoalitions) receive proportional payoffs as well.

Lemma 2.4.1. *Let $N \in \mathcal{N}$, $(c, E) \in \widetilde{\mathcal{P}}^N$, and $F \in \mathcal{F}$ satisfying consistency. If $S \subseteq N$ is such that for each $i \in S$, $F_i(c_S, E_S) = \alpha c_i$, then for each $S' \subsetneq S$ with $|S'| > 1$ and each $j \in S'$, $F_j(c_{S'}, E_{S'}) = \alpha c_j$.*

Proof: Let N, (c, E), and F as specified in the lemma. Consider $S \subseteq N$ such that for each $i \in S$, $F_i(c_S, E_S) = \alpha c_i$ and consider $S' \subsetneq S$ with $|S'| > 1$ and problem $(c_{S'}, E_{S'}) \in C^{S'}$. By consistency, for each $j \in S'$, $F_j\left(c_{S'}, \sum_{k \in S'} F_k(c_S, E_S)\right) = F_j(c_S, E_S)$ and hence $F_j\left(c_{S'}, \sum_{k \in S'} F_k(c_S, E_S)\right) = \alpha c_j$. Since $(c, E) \in \widetilde{\mathcal{P}}^N$ we have that $E_{S'} = \alpha c^{S'} = \sum_{k \in S'} F_k(c_S, E_S)$. Thus, for each $j \in S'$, $F_j\left(c_{S'}, E_{S'}\right) = \alpha c_j$. □

We next show that, given a non-singleton proportional generalized claims problem and a resource monotonic and consistent rule, if some agent in a coalition of size larger than two does not receive a proportional payoff, then there exists a subcoalition that is strictly preferred by at least one member of the subcoalition.

Lemma 2.4.2. *Let $N \in \mathcal{N}$, $(c, E) \in \widetilde{\mathcal{P}}^N$, and $F \in \mathcal{F}$ satisfying resource monotonicity and consistency. If $S \subseteq N$, $|S| > 2$, is such that for some $i \in S$, $F_i(c_S, E_S) \neq \alpha c_i$, then there exists a subcoalition $S' \subsetneq S$ such that for some agent $l \in S'$, $S' \succ_l^{((c,E),F)} S$.*

Proof: Let N, (c, E), and F as specified in the lemma. Consider $S \subseteq N$, $|S| > 2$, such that for some $i \in S$, $F_i(c_S, E_S) \neq \alpha c_i$. By consistency, for each $j \in S \setminus \{i\}$,

$$F_j\left(c_{S\setminus\{i\}}, \sum_{k\in S\setminus\{i\}} F_k(c_S, E_S)\right) = F_j(c_S, E_S). \tag{2.1}$$

Since $E_S = \alpha c^S$ and $F_i(c_S, E_S) \neq \alpha c_i$, without loss of generality we can assume that agent i receives an over-proportional share, *i.e.*, $F_i(c_S, E_S) > \alpha c_i$. Therefore, subcoalition $S \setminus \{i\}$ can achieve a larger joint endowment without agent i compared to what they jointly receive at (c_S, E_S), *i.e.*,

$$E_{S\setminus\{i\}} = \alpha c^{S\setminus\{i\}} > \sum_{k\in S\setminus\{i\}} F_k(c_S, E_S). \tag{2.2}$$

Hence, the endowment at problem $(c_{S\setminus\{i\}}, E_{S\setminus\{i\}})$ is larger than the endowment at problem $\left(c_{S\setminus\{i\}}, \sum_{k\in S\setminus\{i\}} F_k(c_S, E_S)\right)$ and by resource monotonicity, for each agent $j \in S \setminus \{i\}$,

$$F_j(c_{S\setminus\{i\}}, E_{S\setminus\{i\}}) \geq F_j\left(c_{S\setminus\{i\}}, \sum_{k\in S\setminus\{i\}} F_k(c_S, E_S)\right) \stackrel{(2.1)}{=} F_j(c_S, E_S),$$

and, by strict inequality (2.2), for some agent $l \in S \setminus \{i\}$,

$$F_l(c_{S\setminus\{i\}}, E_{S\setminus\{i\}}) > F_l\left(c_{S\setminus\{i\}}, \sum_{k\in S\setminus\{i\}} F_k(c_S, E_S)\right) \stackrel{(2.1)}{=} F_l(c_S, E_S).$$

Hence, for subcoalition $S' = S \setminus \{i\}$, there exists an agent $l \in S'$ such that $S' \succ_l^{((c,E),F)} S$. □

Now we introduce a property for coalition formation problems that plays an important role in the proof of Theorem 2.3.3. Let $\succsim$ be a coalition formation problem with agent set N. Then, for each coalition S of agents, there exists a coalition $S' \subseteq S$ such that all members of S' are at least as well off at S' than at any subcoalition of S. Formally, let $S \subseteq N$. Then, a coalition $S' \subseteq S$ is a **top coalition of S** if for each $i \in S'$ and each $T \subseteq S$ with $i \in T$, we have $S' \succsim_i T$. A coalition formation problem satisfies the **top coalition property** if each non-empty set of agents $S \subseteq N$ has a top coalition. This property (Banerjee et al. (2001)) is sufficient to guarantee stability.

We are now ready to prove Theorem 2.3.3.

Proof: Let $N \in \mathcal{N}$, $(c, E) \in \widetilde{\mathcal{P}}^N$, and $F \in \mathcal{F}$ satisfying continuity, resource monotonicity, and consistency. By Theorem 1.3.8 of Chapter 1, the coalition formation problem with coalition N induced by $((c, E), F)$ has a nonempty stable set and by Lemmas 1.3.4, 1.3.7 and Proposition 1.3.1 in Chapter 1, $\succsim^{((c,E),F)}$ satisfies the top coalition property. If $|N| \leq 2$, we have nothing further to prove. Hence, assume that $|N| > 2$.

First, we iteratively construct a stable partition $\pi \in St(\succsim^{((c,E),F)})$ with coalition sizes of at most two.

Step 1. Let $N_1 := N$, $|N_1| > 2$. There exists a top coalition $S_1 \subseteq N_1$ of N_1. Without loss of generality, assume that S_1 is a top coalition of minimal size. We prove that $|S_1| \leq 2$. Assume, by contradiction, that $|S_1| > 2$. We distinguish two cases:

(i) For each $i \in S_1$, $F_i(c_{S_1}, E_{S_1}) = \alpha c_i$. Then, by Lemma 2.4.1, for each $S \subsetneq S_1$ with $|S| > 1$ and each $j \in S$, $F_j(c_S, E_S) = \alpha c_j$. Hence, any subcoalition $S \subsetneq S_1$ such that $|S| = 2$ is a top coalition of N_1 as well, contradicting our assumption that top coalition S_1 was of minimal size.

(ii) For some $i \in S_1$, $F_i(c_{S_1}, E_{S_1}) \neq \alpha c_i$. Then, by Lemma 2.4.2, there exists a subcoalition $S \subsetneq S_1$ and an agent $j \in S$ such that $S \succ_j^{((c,E),F)} S_1$, which contradicts that S_1 is a top coalition of N_1.

Note that agents in S_1 can never be strictly better off in any other coalition $S \subseteq N_1 = N$. Hence, if S_1 is part of a stable partition, no agent in S_1 can block it.

Set $N_2 := N \setminus S_1$. If $|N_2| \leq 2$, then set $S_2 := N_2$, define $\pi := \{S_1, S_2\}$, and stop. Otherwise, go to Step 2.

Step *k* (*k* > 1). Recall from Step $k-1$ that $N_k := N \setminus \left(\cup_{i=1}^{k-1} S_i\right)$ and $|N_k| > 2$. There exists a top coalition $S_k \subseteq N_k$ of N_k. Without loss of generality, assume that S_k is a top coalition of minimal size. We prove that $|S_k| \leq 2$. Assume, by contradiction, that $|S_k| > 2$. We distinguish two cases:

(i) For each $i \in S_k$, $F_i(c_{S_k}, E_{S_k}) = \alpha c_i$. Then, by Lemma 2.4.1, for each $S \subsetneq S_k$ with $|S| > 1$ and each $j \in S$, $F_j(c_S, E_S) = \alpha c_j$. Hence, any subcoalition $S \subsetneq S_k$ such that $|S| = 2$ is a top coalition of N_k as well, contradicting our assumption that top coalition S_k was of minimal size.

(ii) For some $i \in S_k$, $F_i(c_{S_k}, E_{S_k}) \neq \alpha c_i$. Then, by Lemma 2.4.2, there exists a subcoalition $S \subsetneq S_k$ and an agent $j \in S$ such that $S \succ_j^{((c,E),F)} S_k$, which contradicts that S_k is a top coalition of N_k.

Note that agents in S_k can never be strictly better off in any other coalition $S \subseteq N_k$. In addition, it follows from previous steps that for each $j \in \{1, \dots, k\}$, agents in S_j can never be strictly better off in any other coalition $S \subseteq N_j$. Hence, if $S_1, \dots, S_k$ are part of a stable partition, no agent in $\cup_{i=1}^k S_i$ can block it.

Set $N_{k+1} := N \setminus \left(\cup_{i=1}^k S_i\right)$. If $|N_{k+1}| \leq 2$, then set $S_{k+1} := N_{k+1}$, define $\pi := \{S_1, \dots, S_{k+1}\}$, and stop. Otherwise, go to Step $k+1$.

After at most $|N|-2$ steps, we have constructed a stable partition $\pi = \{S_1, \dots, S_l\}$ of coalitions with size at most two.

Finally, we prove that by merging any two singleton coalitions of π, we obtain another stable partition. Assume that for two distinct agents $i, j \in N$, $\{i\}, \{j\} \in \pi$ and denote the partition obtained from π by replacing sets $\{i\}$ and $\{j\}$ with $\{i, j\}$ by $\tilde{\pi}$. Note, that since $F_i(c_{\{i\}}, E_{\{i\}}) = F_j(c_{\{j\}}, E_{\{j\}}) = 0$, we have that $F_i(c_{\{i,j\}}, E_{\{i,j\}}) \geq F_i(c_{\{i\}}, E_{\{i\}})$ and $F_j(c_{\{i,j\}}, E_{\{i,j\}}) \geq F_j(c_{\{j\}}, E_{\{j\}})$. Hence, since at partition $\tilde{\pi}$ payoffs only changed for agents i and j, any blocking coalition for partition $\tilde{\pi}$ would also be a blocking coalition for π (contradicting $\pi \in St(\succsim^{((c,E),F)})$) and thus, $\tilde{\pi} \in St(\succsim^{((c,E),F)})$. □

Proof of Theorem 2.3.4

Recall that $N = \{1, \dots n\}$ and $c_1 \leq c_2 \leq \dots \leq c_n$. For each $S \subseteq N$, we denote the CEA parameter associated with (c_S, E_S) by λ_S, *i.e.*, for each $i \in S$, $CEA_i(c_S, E_S) = \min\{c_i, \lambda_S\}$, where λ_S is chosen so that $\sum_{j \in S} \min\{c_j, \lambda_S\} = E_S$.

We first introduce some lemmas that will be used to prove Theorem 2.3.4. By the first lemma, each agent $i \in N \setminus \{n\}$ weakly prefers to form a pairwise coalition with a highest claim agent, say agent n, instead of with any other agent.

Lemma 2.4.3. *Let $N \in \mathcal{N}$, $(c, E) \in \widetilde{\mathcal{P}}^N$, and $\succsim^{((c,E),CEA)}$ be the coalition formation problem with agent set N induced by $((c, E), CEA)$. Then, for each $i \in N \setminus \{n\}$ and each $j \in N \setminus \{i, n\}$,*

$$\{i, n\} \succsim_i^{((c,E),CEA)} \{i, j\}.$$

Proof: Let N, (c, E), and $\succsim^{((c,E),CEA)}$ as specified in the lemma and let $i \in N \setminus \{n\}$ and $j \in N \setminus \{i, n\}$. Given that $(c, E) \in \widetilde{\mathcal{P}}^N$, we have $E_{\{i,n\}} = \alpha(c_i + c_n)$ and $E_{\{j,n\}} = \alpha(c_j + c_n)$. We prove $\{i, n\} \succsim_i^{((c,E),CEA)} \{i, j\}$ by showing that

$$CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) \geq CEA_i(c_{\{i,j\}}, E_{\{i,j\}}).$$

If $CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) = c_i$, then the above inequality holds automatically. Hence, assume that $CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) = \lambda_{\{i,n\}} < c_i$. Since $c_i \leq c_n$, this implies $CEA_n(c_{\{i,n\}}, E_{\{i,n\}}) = \lambda_{\{i,n\}} \leq c_n$. Thus, $CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) = \frac{\alpha(c_i+c_n)}{2}$. We distinguish two cases:

Case 1. $CEA_j(c_{\{i,j\}}, E_{\{i,j\}}) = c_j$. Hence, $CEA_i(c_{\{i,j\}}, E_{\{i,j\}}) = \alpha(c_i + c_j) - c_j$. Then,

$$CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) \geq CEA_i(c_{\{i,j\}}, E_{\{i,j\}})$$

$$\begin{aligned} \Leftrightarrow \quad & \frac{\alpha(c_i + c_n)}{2} \geq \alpha(c_i + c_j) - c_j \\ \Leftrightarrow \quad & \alpha c_i + \alpha c_n \geq 2\alpha c_i + 2(\alpha - 1)c_j \\ \Leftrightarrow \quad & \underbrace{\alpha}_{>0}(\underbrace{c_n - c_i}_{\geq 0}) \geq 2(\underbrace{\alpha - 1}_{<0})\underbrace{c_j}_{\geq 0}. \end{aligned}$$

Case 2. $CEA_j(c_{\{i,j\}}, E_{\{i,j\}}) = \lambda_{\{i,j\}} = \frac{\alpha(c_i+c_j)}{2}$. Thus,

$$CEA_i(c_{\{i,j\}}, E_{\{i,j\}}) = \frac{\alpha(c_i + c_j)}{2} \leq \frac{\alpha(c_i + c_n)}{2} = CEA_i(c_{\{i,n\}}, E_{\{i,n\}}).$$

□

We will now focus on agent n and discover with which agents she wants to form a pairwise coalition.

Consider $i, j \in N \setminus \{n\}$ such that $i < j < n$ ($c_i \leq c_j \leq c_n$). By our next lemma, if at coalition $\{i, n\}$ both agents receive the same payoff, then agent n weakly prefers coalition $\{j, n\}$.

Lemma 2.4.4. *Let $N \in \mathcal{N}$, $(c, E) \in \widetilde{\mathcal{P}}^N$, and $\succsim^{((c,E),CEA)}$ be the coalition formation problem with agent set N induced by $((c, E), CEA)$. Let $i \in N \setminus \{n\}$ such that*

$$CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) = \lambda_{\{i,n\}}.$$

Then,

$$\alpha \leq \frac{2c_i}{(c_i + c_n)}$$

and for each $j \in N \setminus \{i, n\}$ such that $i < j < n$,

$$CEA_n(c_{\{j,n\}}, E_{\{j,n\}}) \geq CEA_n(c_{\{i,n\}}, E_{\{i,n\}}).$$

Proof: Let N, (c, E), $\succsim^{((c,E),CEA)}$, and $i, j \in N \setminus \{n\}$ as specified in the lemma. Given that $(c, E) \in \widetilde{\mathcal{P}}^N$, we have $E_{\{i,n\}} = \alpha(c_i + c_n)$ and $E_{\{j,n\}} = \alpha(c_j + c_n)$. Since $CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) = \lambda_{\{i,n\}} \leq c_i$, $c_i \leq c_n$ implies $CEA_n(c_{\{i,n\}}, E_{\{i,n\}}) = \lambda_{\{i,n\}} \leq c_n$. Thus, $CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) = \frac{\alpha(c_i+c_n)}{2} \leq c_i$. Thus,

$$\alpha \leq \frac{2c_i}{c_i + c_n} \text{ and } \alpha c_n \leq (2 - \alpha)c_i.$$

Since $(2 - \alpha) > 0$ and $c_i \leq c_j$, this implies

$$\alpha c_n \leq (2 - \alpha)c_j$$

$$\Leftrightarrow \quad \alpha c_n + \alpha c_j \leq 2c_j$$
$$\Leftrightarrow \quad \frac{\alpha(c_j + c_n)}{2} \leq c_j.$$

Hence, $\lambda_{\{j,n\}} = \frac{\alpha(c_j+c_n)}{2} \leq c_j \leq c_n$ and $CEA_n(c_{\{j,n\}}, E_{\{j,n\}}) = \frac{\alpha(c_j+c_n)}{2} \geq \frac{\alpha(c_i+c_n)}{2} = CEA_n(c_{\{i,n\}}, E_{\{i,n\}})$. Thus,

$$CEA_n(c_{\{j,n\}}, E_{\{j,n\}}) \geq CEA_n(c_{\{i,n\}}, E_{\{i,n\}}).$$

□

Lemma 2.4.4 implies the following corollary.

Corollary 2.4.5. *Let $N \in \mathcal{N}$, $(c, E) \in \widetilde{\mathcal{P}}^N$, and $\succsim^{((c,E),CEA)}$ be the coalition formation problem with agent set N induced by $((c, E), CEA)$. If $\alpha \leq \frac{2c_1}{(c_1+c_n)} \equiv \beta_1$, then for each $i \in N \setminus \{n-1, n\}$, $CEA_n(c_{\{n-1,n\}}, E_{\{n-1,n\}}) \geq CEA_n(c_{\{i,n\}}, E_{\{i,n\}})$.*

Consider again $i, j \in N \setminus \{n\}$ such that $i < j < n$ ($c_i \leq c_j \leq c_n$). By our next lemma, if at coalition $\{j, n\}$ agent j receives her claim, then agent n weakly prefers coalition $\{i, n\}$.

Lemma 2.4.6. *Let $N \in \mathcal{N}$, $(c, E) \in \widetilde{\mathcal{P}}^N$, and $\succsim^{((c,E),CEA)}$ be the coalition formation problem with agent set N induced by $((c, E), CEA)$. Let $j \in N \setminus \{n\}$ such that*

$$CEA_j(c_{\{j,n\}}, E_{\{j,n\}}) = c_j.$$

Then,

$$\alpha \geq \frac{2c_j}{(c_j + c_n)} \tag{2.3}$$

and for each $i \in N \setminus \{i, n\}$ such that $i < j < n$,

$$CEA_n(c_{\{i,n\}}, E_{\{i,n\}}) \geq CEA_n(c_{\{j,n\}}, E_{\{j,n\}}). \tag{2.4}$$

Proof: Let N, (c, E), $\succsim^{((c,E),CEA)}$, and $i, j \in N \setminus \{n\}$ as specified in the lemma. Given that $(c, E) \in \widetilde{\mathcal{P}}^N$, we have $E_{\{i,n\}} = \alpha(c_i + c_n)$ and $E_{\{j,n\}} = \alpha(c_j + c_n)$. Since $CEA_j(c_{\{j,n\}}, E_{\{j,n\}}) = c_j \leq \lambda_{\{j,n\}}$, we have that $c_j \leq \frac{\alpha(c_j+c_n)}{2}$ and

$$\alpha \geq \frac{2c_j}{c_j + c_n} \text{ and } \alpha c_n \geq (2 - \alpha)c_j.$$

Since $(2 - \alpha) > 0$ and $c_i \leq c_j$, this implies

$$\alpha c_n \geq (2 - \alpha)c_i$$

$$\Leftrightarrow \quad \alpha c_n + \alpha c_i \geq 2c_i$$

$$\Leftrightarrow \quad \frac{\alpha(c_i + c_n)}{2} \geq c_i.$$

This implies $CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) = c_i$ and $CEA_n(c_{\{i,n\}}, E_{\{i,n\}}) = \alpha(c_i + c_n) - c_i = \alpha c_n + (\underbrace{\alpha - 1}_{<0}) \underbrace{c_i}_{\leq c_j} \geq \alpha c_n + (\alpha - 1)c_j = \alpha(c_j + c_n) - c_j = CEA_n(c_{\{j,n\}}, E_{\{j,n\}})$. Thus,

$$CEA_n(c_{\{i,n\}}, E_{\{i,n\}}) \geq CEA_n(c_{\{j,n\}}, E_{\{j,n\}}).$$

□

Lemma 2.4.6 implies the following corollary.

Corollary 2.4.7. *Let $N \in \mathcal{N}$, $(c, E) \in \widetilde{\mathcal{P}}^N$, and $\succsim^{((c,E),CEA)}$ be the coalition formation problem with agent set N induced by $((c, E), CEA)$. If $\alpha \geq \frac{2c_{n-1}}{c_{n-1}+c_n} \equiv \gamma_1$, then for each $i \in N \setminus \{1, n\}$, $CEA_n(c_{\{1,n\}}, E_{\{1,n\}}) \geq CEA_n(c_{\{i,n\}}, E_{\{i,n\}})$.*

Corollaries 2.4.5 and 2.4.7 now imply that agents 1 and $n-1$ are potential "stable partners" for agent n. When α is very low ($\alpha \leq \frac{2c_1}{c_1+c_n} \equiv \beta_1$), then $\{n-1, n\}$ is a stable pairwise coalition, and when α is very large ($\alpha > \frac{2c_{n-1}}{c_{n-1}+c_n} \equiv \gamma_1$), then $\{1, n\}$ is a stable pairwise coalition. Thus, we next need to determine a threshold value δ_1 for parameter α when $\beta_1 < \alpha \leq \gamma_1$ to see when agent n's partner of choice is $n-1$ (for $\alpha \leq \delta_1$) and when it is 1 (for $\alpha > \delta_1$).

We next show that $\delta_1 \equiv \frac{2c_1}{2c_1 - c_{n-1} + c_n}$, the value specified in Step 1 of the CEA algorithm to trigger either Case (i) with stable coalition $\{n-1, n\}$ or Case (ii) with stable coalition $\{1, n\}$.

Lemma 2.4.8. *Let $N \in \mathcal{N}$, $(c, E) \in \widetilde{\mathcal{P}}^N$, and $\succsim^{((c,E),CEA)}$ be the coalition formation problem with agent set N induced by $((c, E), CEA)$. Assume that $\beta_1 \leq \alpha \leq \gamma_1$. Then, for $\delta_1 \equiv \frac{2c_1}{2c_1 - c_{n-1} + c_n}$ we have*

(i) *If $\alpha \leq \delta_1$, then $CEA_n(c_{\{n-1,n\}}, E_{\{n-1,n\}}) \geq CEA_n(c_{\{1,n\}}, E_{\{1,n\}})$.*

(ii) *If $\alpha \geq \delta_1$, then $CEA_n(c_{\{1,n\}}, E_{\{1,n\}}) \geq CEA_n(c_{\{n-1,n\}}, E_{\{n-1,n\}})$.*

Proof: Let N, (c, E), and $\succsim^{((c,E),CEA)}$ as specified in the lemma. Given that $(c, E) \in \widetilde{\mathcal{P}}^N$, we have $E_{\{1,n\}} = \alpha(c_1 + c_n)$ and $E_{\{n-1,n\}} = \alpha(c_{n-1} + c_n)$. Assume that $\beta_1 \leq \alpha \leq \gamma_1$, which implies

$$(2 - \alpha)c_1 \leq \alpha c_n \leq (2 - \alpha)c_{n-1}.$$

This together with Lemmas 2.4.4 and 2.4.6 imply that $CEA_1(c_{\{1,n\}}, E_{\{1,n\}}) = c_1$ and $CEA_{n-1}(c_{\{n-1,n\}}, E_{\{n-1,n\}}) = \frac{\alpha(c_{n-1}+c_n)}{2}$. Hence, $CEA_n(c_{\{1,n\}}, E_{\{1,n\}}) = \alpha(c_1 + c_n) - c_1$ and $CEA_n(c_{\{n-1,n\}}, E_{\{n-1,n\}}) = \frac{\alpha(c_{n-1}+c_n)}{2}$. Now, we consider

$$CEA_n(c_{\{1,n\}}, E_{\{1,n\}}) = \alpha(c_1 + c_n) - c_1 \gtrless \frac{\alpha(c_{n-1} + c_n)}{2} = CEA_n(c_{\{n-1,n\}}, E_{\{n-1,n\}})$$

$$\Leftrightarrow \quad \alpha(2c_1 + 2c_n - c_{n-1} - c_n) \gtrless 2c_1$$
$$\Leftrightarrow \quad \alpha \gtrless \frac{2c_1}{(2c_1 - c_{n-1} + c_n)} = \delta_1.$$

It now follows that

(i) If $\alpha \leq \delta_1$, then $CEA_n(c_{\{n-1,n\}}, E_{\{n-1,n\}}) \geq CEA_n(c_{\{1,n\}}, E_{\{1,n\}})$.

(ii) If $\alpha \geq \delta_1$, then $CEA_n(c_{\{1,n\}}, E_{\{1,n\}}) \geq CEA_n(c_{\{n-1,n\}}, E_{\{n-1,n\}})$.

□

Finally, we show that parameters β_1, γ_1, and δ_1 as defined in Corollaries 2.4.5 and 2.4.7 and Lemma 2.4.8 satisfy

$$\beta_1 \leq \delta_1 \leq \gamma_1. \tag{2.5}$$

$$\beta_1 \leq \delta_1$$
$$\Leftrightarrow \quad \frac{2c_1}{(c_1 + c_n)} \leq \frac{2c_1}{(2c_1 - c_{n-1} + c_n)}$$
$$\Leftrightarrow \quad \frac{2c_1}{(c_1 + c_n)} \leq \frac{2c_1}{(c_1 + c_n) - (\underbrace{c_{n-1} - c_1}_{\geq 0})}$$

and

$$\delta_1 \leq \gamma_1$$
$$\frac{2c_1}{(2c_1 - c_{n-1} + c_n)} \leq \frac{2c_{n-1}}{(c_{n-1} + c_n)}$$
$$\Leftrightarrow \quad 2c_1(c_{n-1} + c_n) \leq 2c_{n-1}(2c_1 - c_{n-1} + c_n)$$
$$\Leftrightarrow \quad c_1c_{n-1} + c_1c_n \leq 2c_1c_{n-1} - c_{n-1}c_{n-1} + c_{n-1}c_n$$
$$\Leftrightarrow \quad 0 \leq c_1c_{n-1} - c_1c_n - c_{n-1}c_{n-1} + c_{n-1}c_n$$
$$\Leftrightarrow \quad 0 \leq c_{n-1}(c_n - c_{n-1}) - c_1(c_n - c_{n-1})$$
$$\Leftrightarrow \quad 0 \leq (\underbrace{c_{n-1} - c_1}_{\geq 0})(\underbrace{c_n - c_{n-1}}_{\geq 0}).$$

We are now ready to prove Theorem 2.3.4.

Proof: Let $N \in \mathcal{N}$ and $(c, E) \in \widetilde{\mathcal{P}}^N$. Consider the coalition formation problem with agent set N induced by $((c, E), CEA)$.

If $|N| \leq 2$, then $\pi = \{N\}$ is stable and we have nothing further to prove. Hence, assume that $|N| > 2$. We show that in each step of the CEA algorithm, a top coalition is chosen. This implies that the resulting partition π is stable.

Step 1. Recall that $N_1 := N$, $|N_1| > 2$, and according to Cases (i) or (ii) in Step 1 of the CEA algorithm, , $S_1 \in \{\{n-1, n\}, \{1, n\}\}$. We show that in either case, S_1 is a top coalition.

Case (i). $\alpha \leq \delta_1$. Note that since $(c, E) \in \widetilde{\mathcal{P}}^N$ we have that for each $i \in \{n-1, n\}$,

$$\{n-1, n\} \succsim_i^{((c,E),CEA)} \{i\}.$$

Next, let $i \in \{n-1, n\}$ and $j \in N \setminus \{n-1, n\}$. We prove $\{n-1, n\} \succsim_i^{((c,E),CEA)} \{i, j\}$ by showing that

$$CEA_i(c_{\{n-1,n\}}, E_{\{n-1,n\}}) \geq CEA_i(c_{\{i,j\}}, E_{\{i,j\}}). \tag{2.6}$$

For $i = n - 1$, inequality (2.6) follows with Lemma 2.4.3. For $i = n$, inequality (2.6) follows with Corollary 2.4.5 and Lemma 2.4.8 (i).

Case (ii). $\alpha > \delta_1$. Note that since $(c, E) \in \widetilde{\mathcal{P}}^N$ we have that for each $i \in \{1, n\}$,

$$\{1, n\} \succsim_i^{((c,E),CEA)} \{i\}.$$

Next, let $i \in \{1, n\}$ and $j \in N \setminus \{1, n\}$. We prove $\{1, n\} \succsim_i^{((c,E),CEA)} \{i, j\}$ by showing that

$$CEA_i(c_{\{1,n\}}, E_{\{1,n\}}) \geq CEA_i(c_{\{i,j\}}, E_{\{i,j\}}). \tag{2.7}$$

For $i = 1$, inequality (2.7) follows with Lemma 2.4.3. For $i = n$, inequality (2.6) follows with Corollary 2.4.7 and Lemma 2.4.8 (ii).

We have shown that S_1 is a top coalition of N.

Step *k* (*k* > 1). Recall from Step $k-1$ that $N_k := N \setminus \left(\cup_{j=1}^{k-1} S_j\right)$ and $|N_k| > 2$. Furthermore, agents in N_k are relabelled such that $N_k = \{1', \ldots, n'\}$, $c_{1'} \leq \ldots \leq c_{n'}$, and $\delta_k = \frac{2c_{1'}}{2c_{1'} - c_{(n-1)'} + c_{n'}}$. Then, according to Cases (i) or (ii) in Step k of the CEA algorithm, $S_k \in \{\{n-1, n\}, \{1, n\}\}$. Hence, by a similar reasoning than in Step 1 (with agents $1'$, $(n-1)'$, and n' in the roles of agents 1, $n-1$, and n respectively) it follows that in both cases S_k is a top coalition of N_k.

We have proven that the CEA algorithm in each step assigns a pairwise top coalition. Therefore, by the same arguments as in the proof of Theorem 2.3.3, the resulting partition π is stable. □

Proof of Theorem 2.3.6

Recall that $N = \{1, \ldots n\}$ and $c_1 \leq c_2 \leq \ldots \leq c_n$. Furthermore, for each $S \subseteq N$, we denote the CEL parameter associated with (c_S, E_S) by λ_S, *i.e.*, for each $i \in S$,

$CEL_i(c_S, E_S) = \max\{0, c_i - \lambda_S\}$, where λ_S is chosen so that $\sum_{j \in S} \max\{0, c_j - \lambda_S\} = E_S$.

Proof: Let $N \in \mathcal{N}$ and $(c, E) \in \widetilde{\mathcal{P}}^N$. Consider the coalition formation problem with agent set N induced by $((c, E), CEL)$.

If $|N| \leq 2$, then $\pi = \{N\}$ is stable and we have nothing further to prove. Hence, assume that $|N| > 2$. We show that in each step of the CEL algorithm, a top coalition is chosen. This implies that the resulting partition π is stable.

Step 1. Recall that $N_1 := N$, $|N_1| > 2$, and $S_1 := \{1, 2\}$. We show that S_1 is a top coalition. Note that since $(c, E) \in \widetilde{\mathcal{P}}^N$ we have that for each $i \in \{1, 2\}$,

$$\{1, 2\} \succsim_i^{((c,E),CEL)} \{i\}.$$

Next, let $i \in \{1, 2\}$ and $j \in N \setminus \{1, 2\}$. We prove $\{1, 2\} \succsim_i^{((c,E),CEL)} \{i, j\}$ by showing that

$$CEL_i(c_{\{1,2\}}, E_{\{1,2\}}) \geq CEL_i(c_{\{i,j\}}, E_{\{i,j\}}) \text{ or, equivalently, } \lambda_{\{1,2\}} \leq \lambda_{\{i,j\}}. \quad (2.8)$$

Note that $i < j$ and hence $c_1 \leq c_i \leq c_j$.

If $CEL_i(c_{\{i,j\}}, E_{\{i,j\}}) = 0$, then inequality (2.8) holds automatically. Hence, assume that $CEL_i(c_{\{i,j\}}, E_{\{i,j\}}) = c_i - \lambda_{\{i,j\}} > 0$. Since $c_i \leq c_j$, this implies $CEL_j(c_{\{i,j\}}, E_{\{i,j\}}) = c_j - \lambda_{\{i,j\}} > 0$ and $E_{\{i,j\}} = c_i + c_j - 2\lambda_{\{i,j\}}$. Given that $(c, E) \in \widetilde{\mathcal{P}}^N$, we also have $E_{\{i,j\}} = \alpha(c_i + c_j)$ and a corresponding loss $(1 - \alpha)(c_i + c_j)$. Thus,

$$\lambda_{\{i,j\}} = \frac{(1 - \alpha)(c_i + c_j)}{2} < c_i \leq c_j. \quad (2.9)$$

Next, let $\{i, k\} = \{1, 2\}$. Note that $c_k \leq c_j$, together with inequality (2.9), implies

$$\frac{(1 - \alpha)(c_1 + c_2)}{2} = \frac{(1 - \alpha)(c_i + c_k)}{2} \leq \lambda_{\{i,j\}} < c_i.$$

Given that $(c, E) \in \widetilde{\mathcal{P}}^N$, we have $E_{\{1,2\}} = \alpha(c_1 + c_2)$ with corresponding loss $(1 - \alpha)(c_1 + c_2)$.

If $\frac{(1-\alpha)(c_1+c_2)}{2} \leq c_1$, then $\lambda_{\{1,2\}} = \frac{(1-\alpha)(c_1+c_2)}{2} \leq \lambda_{\{i,j\}}$ and inequality (2.8) holds.

Hence, assume that $\frac{(1-\alpha)(c_1+c_2)}{2} > c_1$. Then, $CEL_1(c_{\{1,2\}}, E_{\{1,2\}}) = 0$ and $CEL_2(c_{\{1,2\}}, E_{\{1,2\}}) = c_2 - \lambda_{\{1,2\}} > 0$. Hence, $i = 2$ and $E_{\{1,2\}} = c_2 - \lambda_{\{1,2\}}$. Thus, $\lambda_{\{1,2\}} = (1 - \alpha)c_2 - \alpha c_1$. Then, inequality (2.9) together with $c_2 \leq c_j$, implies

$$\lambda_{\{2,j\}} = \frac{(1 - \alpha)(c_2 + c_j)}{2} \geq \frac{(1 - \alpha)2c_2}{2} = (1 - \alpha)c_2 > (1 - \alpha)c_2 - \alpha c_1 = \lambda_{\{1,2\}}$$

and inequality (2.8) holds.

We have shown that S_1 is a top coalition of N.

Step *k* (*k* $>$ 1). Recall from Step $k-1$ that $N_k := N \setminus \left(\cup_{j=1}^{k-1} S_j\right)$ and $|N_k| > 2$. Set $S_k := \{2k-1, 2k\}$. Note that agents $2k-1$ and $2k$ are lowest claim agents in N_k. Hence, by a similar reasoning than in Step 1 (with agents $2k-1$ and $2k$ in the roles of agents 1 and 2 respectively) it follows that S_k is a top coalition of N_k.

We have proven that the CEL algorithm in each step assigns a pairwise top coalition. Therefore, by the same arguments as in the proof of Theorem 2.3.3, the resulting partition π is stable. ☐

Bibliography

Banerjee, S., Konishi, H. and Sönmez, T. (2001), 'Core in a simple coalition formation game', *Social Choice and Welfare* **18**(1), 135–153.

Barberà, S., Beviá, C. and Ponsatí, C. (2015), 'Meritocracy, egalitarianism and the stability of majoritarian organizations', *Games and Economic Behavior* **91**, 237–257.

Becker, G. (1973), 'A theory of marriage: Part i', *Journal of Political Economy* pp. 813–846.

Bergantiños, G., Lorenzo, L. and Lorenzo-Freire, S. (2010), 'A characterization of the proportional rule in multi-issue allocation situations', *Operations Research Letters* **38**(1), 17–19.

Bogomolnaia, A., Le Breton, M., Savvateev, A. and Weber, S. (2008), 'Stability of jurisdiction structures under the equal share and median rules', *Economic Theory* **34**(3), 525–543.

Chambers, C. P. and Moreno-Ternero, J. D. (2017), 'Taxation and poverty', *Social Choice and Welfare* **48**(1), 153–175.

Greenberg, J. and Weber, S. (1986), 'Strong tiebout equilibrium under restricted preferences domain', *Journal of Economic Theory* **38**(1), 101–117.

Hart, S. (1985), 'An axiomatization of harsanyi's nontransferable utility solution', *Econometrica: Journal of the Econometric Society* pp. 1295–1313.

Izquierdo, J. M. and Timoner, P. (2019), 'Decentralized rationing problems', *Economics Letters* **175**, 88–91.

Lorenzo-Freire, S., Casas-Méndez, B. and Hendrickx, R. (2010), 'The two-stage constrained equal awards and losses rules for multi-issue allocation situations', *Top* **18**(2), 465–480.

Moreno-Ternero, J. D. (2011), 'A coalitional procedure leading to a family of bankruptcy rules', *Operations research letters* **39**(1), 1–3.

Pycia, M. (2012), 'Stability and preference alignment in matching and coalition formation', *Econometrica* **80**(1), 323–362.

Stovall, J. E. (2014), 'Asymmetric parametric division rules', *Games and Economic Behavior* **84**, 87–110.

Thomson, W. (2003), 'Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey', *Mathematical Social Sciences* **45**, 249–297.

Thomson, W. (2015), 'Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: an update', *Mathematical Social Sciences* **74**, 41–59.

Thomson, W. (2019), *How to Divide when There Isn't Enough: From Aristotle, the Talmud, and Maimonides to the Axiomatics of Resource Allocation*, Vol. 62, Cambridge University Press.

Young, H. P. (1987), 'On dividing an amount according to individual claims or liabilities', *Mathematics of Operations Research* **12**(3), 398–414.

Young, H. P. (1988), 'Distributive justice in taxation', *Journal of Economic Theory* **44**(2), 321–335.

Chapter 3

Strategy-proofness on a mixed domain of single-peaked and single-dipped preferences

Abstract

We analyze the problem of locating a public facility in a mixed domain of single-peaked and single-dipped preferences where the type of preference (single-peaked or single-dipped) of each agent is known, but there is no information about the position of her peak or dip and the rest of the preference. In this framework, we characterize all strategy-proof social choice rules and show that they all are also group strategy-proof. We find that each of these rules can be decomposed into two steps: in the first step, agents with single-peaked preferences are asked about their peaks and then, at most two alternatives are preselected; in the second step, agents with single-dipped preferences are asked about their dips to complete the decision between the preselected alternatives. We also study which strategy-proof rules satisfy Pareto efficiency.

3.1 Introduction

Governments are continually improving their cities by constructing new public facilities such as schools, hospitals or parks. When deciding the location of these facilities, public officials must consider technical constraints (*e.g.*, not all locations may be feasible) and monetary limitations (*e.g.*, construction costs of facilities could differ from one location to another), but they may also be interested in considering the preferences of the population affected by the decision. However, since preferences are private information and agents are strategic, the rule applied may not always induce agents to reveal their preferences. This chapter seeks to construct social choice rules that incentivize people always to reveal their preferences truthfully, a property known as strategy-proofness.

The literature on strategy-proofness is based on Gibbard-Satterthwaite's theorem (Gibbard et al., 1973; Satterthwaite, 1975), which states that dictatorial rules are the only strategy-proof rules on the universal preference domain with more than two alternatives in their range. Therefore, to construct non-dictatorial social choice rules that induce truth-telling, one has to restrict either the range of the rules to two alternatives or the domain of admissible preferences. Since rules with a range of two are not Pareto efficient on the universal preference domain, the literature has focused on restricting the domain of preferences, which makes a good fir for some real-life situations such as the location of public facilities.

This chapter proposes a new preference domain for locating a public facility in any countable subset of the real line. In this domain, the type of preference of each agent (single-peaked or single-dipped) is known but the location of the peak or dip and the rest of the preference are private information. In this setting, the set of admissible preferences for an agent with single-peaked (single-dipped) preferences is equal to the set of all single-peaked (single-dipped) preferences.

Our main result is a characterization of all strategy-proof rules on the aforementioned preference domain. In particular, we find that all strategy-proof rules can be described in two steps. In the first step, each agent with single-peaked preferences is asked about her best alternative in the range of the rule (her "peak") and, as a result, one or two alternatives are preselected. If only one alternative is preselected, this is the final outcome. Otherwise, we find that any pair of alternatives that arise as the outcome of the first step is formed by contiguous alternatives in the range of the rule. In the second step, each agent with single-dipped preferences is asked about her worst alternative in the range of the rule (her "dip"). Finally, taking into account the information about the "peaks" and "dips", one of the two preselected alternatives is chosen. We also study which of the strategy-proof rules are Pareto efficient. We find that any strategy-proof and Pareto efficient rule has either a range of two, coinciding with the "extreme points" of the set of feasible alternatives, or a range equal to the set of feasible alternatives.

Related literature

There is a considerable body of literature on strategy-proofness that studies domains that escape from the Gibbard-Satterthwaite impossibility (see Barberà (2011) for a survey). For the case of locating public facilities, the literature has analyzed in depth the domain of single-peaked preferences and the domain of single-dipped preferences.

When the facility is a public good, *e.g.*, a school or a hospital, the domain of single-peaked preferences is appealing. A preference is said to be single-peaked if (a) there is a single most preferred alternative (the peak); and (b) for each alternative x situated between the peak and another alternative y, the preference declares that x is preferred to y. The single-peaked preference domain was first discussed by Black (1948*a*,*b*), who shows that the median voter rule, which selects the median of the declared peaks, is strategy-proof and selects the Condorcet winner. Later, Moulin (1980) and Barberà and Jackson (1994) characterize all strategy-proof rules on this domain: The *generalized median voter rules*. By contrast, if the facility to be located is a prison or a cemetery, *i.e.*, a facility that is considered a public bad, the single-dipped preference domain emerges naturally. A preference is single-dipped whenever (a) there is a single worst alternative (the dip); and (b) for each alternative x located between another alternative y and the dip, the preference declares that y is preferred to x. Barberà et al. (2012) and Manjunath (2014) show that for this domain all strategy-proof rules have a range of two.

However, there are examples of public facilities which do not give rise to unanimous opinions for which preferences may be either single-peaked or single-dipped in the society, *e.g.*, dog parks, soccer stadiums or dance clubs. For example, dog owners may have single-peaked preferences on the location of a dog park, but individuals who do not like dogs will probably have single-dipped preferences. Consequently, a mixed domain that includes both single-peaked and single-dipped preferences is needed to deal with such situations. When the set of admissible preferences of each agent coincides with the set of all single-peaked and all single-dipped preferences, Berga and Serizawa (2000) and Achuthankutty and Roy (2018) show that the Gibbard-Satterthwaite impossibility applies. Hence, the domain of preferences needs to be constrained even further. Thomson (2008) studies a restricted domain with two agents. In that domain, the social planner knows the type of preferences of each agent: one agent has single-peaked preferences while the other has single-dipped preferences. Moreover, the peak and the dip of these preferences are located at the same point and this location is public information. The rest of the preference of each agent in that domain is private information. Later, Feigenbaum and Sethuraman (2015) consider a restricted domain in which the type of the preference of each agent (single-peaked or single-dipped) is known and preferences are cardinally determined by the distance between each location and the peak/dip, which is the only issue in the preference of each agent that is private information. More recently, Alcalde-Unzu and Vorsatz (2018) analyze a model closer to the one presented in this chapter. In their model, the peak or the dip of each agent, which corresponds to the location of the individual in the real line, is public information but the social planner is uncertain whether a particular agent has single-peaked or single-dipped preferences and what the rest of the preference of each agent is. Unlike them, we consider a domain in which the type of preference of each agent (single-peaked or single-dipped) is public information, but the social planner has no information about the location of the peak or the dip and the rest of the preference of each agent. Observe that in this new domain, in contrast to the domain of Alcalde-Unzu and Vorsatz (2018), agents could misrepresent their preferences by lying about the location of the peak or dip, but not about the type of preference (*i.e.*, an agent with single-peaked preferences cannot declare that she has single-dipped preferences, and vice versa).

The strategy-proof rules on our domain are somewhat similar to the strategy-proof rules on the single-peaked and single-dipped preference domains. Observe that in the first step of our rules, only the agents with single-peaked preferences are asked about their peaks. Note that the outcome of this first step may contain a single alternative or a pair of contiguous alternatives. We show that if we define a set formed by single alternatives and all pairs formed by contiguous alternatives under a particular order, the procedure used in this first step to choose an element of that set consists of applying a generalized median voter function on that set. Moreover, in the second step of our rules, a binary decision problem is faced and we find that the choice between the two locations is made in the same way as in the strategy-proof rules of the single-dipped preference domain.

The rest of the chapter is organized as follows. Section 3.2 presents the model. Section 3.3 introduces the general structure that any strategy-proof rule must have. Section 3.4 completes the characterization of all strategy-proof rules.Section 3.5 analyzes which strategy-proof rules also satisfy Pareto efficiency. Section 3.6 concludes. All proofs are relegated to the Appendix.

3.2 The model

Consider a social planner who wants to locate a public facility at a point in a countable set $X \subset \mathbb{R}$ of feasible locations. There is a finite group of agents N, divided into two groups: A, with cardinality a, and D, with cardinality $n-a$. Let R_i be the **weak preference relation of agent $\boldsymbol{i} \in \boldsymbol{N}$ on $\boldsymbol{X}$**. Formally, R_i is a complete, transitive, and antisymmetric binary relation. P_i denotes the strict preference relation induced by R_i. We assume that the agents of A (respectively, D) have single-peaked (respectively, single-dipped) preferences on X. That is, if $i \in A$, R_i is a **single-peaked preference with a peak $\boldsymbol{p(R_i)}$**, which means that for all $x, y \in X$ such that $p(R_i) \geq x > y$ or $p(R_i) \leq x < y$, it follows that $x \, P_i \, y$. Similarly, if $i \in D$, R_i is a **single-dipped preference with a dip $\boldsymbol{d(R_i)}$**, which means that for all $x, y \in X$ such that $d(R_i) \geq x > y$ or $d(R_i) \leq x < y$, it follows that $y \, P_i \, x$. We denote the preference domain of agent i by $\mathcal{R}_i$. If $i \in A$ ($i \in D$), then $\mathcal{R}_i$ corresponds with the set of all single-peaked preferences (the set of all single-dipped preferences).

A **preference profile** is a set of preferences $R \equiv (R_i)_{i \in N}$ such that $R_i \in \mathcal{R}_i$ for each $i \in N$. For each profile R and each set $S \subset N$, we denote by R_S and R_{-S} the subprofiles of R restricted to the set of agents S and $N \setminus S$, respectively.[1] In particular, the preferences of the agents with single-peaked (single-dipped) preferences in R is denoted by R_A (R_D). We also write for any $S \subseteq N$, $\mathcal{R}^S = \times_{i \in S} \mathcal{R}_i$ and, for the sake of simplicity, $\mathcal{R} = \mathcal{R}^N$.

Given any $R \in \mathcal{R}$, for any $S \subseteq A$, we denote the vector of peaks of the agents of S by $p(R_S)$. Similarly, $d(R_S)$ denotes the vector of dips of the agents of S for any $S \subseteq D$ at profile R. With a slight abuse of notation, we write $p(R)$ and $d(R)$ instead of $p(R_A)$ and $d(R_D)$, respectively.

The solution concept is a **social choice rule** $f : \mathcal{R} \rightarrow X$ that selects for each preference profile $R \in \mathcal{R}$ a feasible location $f(R) \in X$. We denote the range of f by r_f, *i.e.*, $r_f = \{x \in X : \exists R \in \mathcal{R} \text{ such that } f(R) = x\}$. Given any $R \in \mathcal{R}$, we define $\omega(p(R))$ as the set of alternatives in the range of f that appear as the outcome of f for the profiles with vector of peaks $p(R)$, *i.e.*, $\omega(p(R)) = \{x \in r_f : \exists R' \in \mathcal{R} \text{ such that } p(R') = p(R) \text{ and } f(R') = x\}$.

We focus on rules that incentivize the truthful representation of preferences. A rule f is said to be manipulable by group $S \subseteq N$ if there is a preference profile $R \in \mathcal{R}$ in which each agent in S benefits from a simultaneous misrepresentation of preferences. Formally, a social choice rule f is manipulable by group $S \subseteq N$ if there is a preference profile $R \in \mathcal{R}$ and a subprofile $R'_S \in \mathcal{R}^S$ such that $f(R'_S, R_{-S})\, P_i\, f(R)$ for each $i \in S$. Thus, a rule f is **group strategy-proof (GSP)** if it is not manipulable by any group $S \subseteq N$. Similarly, f is said to be **strategy-proof (SP)** if it is not manipulable by any group $S \subseteq N$ such that $|S| = 1$. A rule f is **Pareto efficient (PE)** if for each $R \in \mathcal{R}$, there is no $x \in X$ such that $x\, P_i\, f(R)$ for each $i \in N$. Finally, a rule f with $|r_f| \geq 2$ is **dictatorial** if there is an agent $i \in N$ (the *dictator*) such that $f(R)\, R_i\, x$ for all $R \in \mathcal{R}$ and all $x \in r_f$.

3.3 General structure of strategy-proof rules

This section derives the common structure that each SP rule will have. First, it is important to note that all rules f with $|r_f| = 1$ are SP. Therefore, we focus throughout the chapter only on rules f with $|r_f| \geq 2$. The first result establishes an equivalence between SP and GSP on our domain.

Proposition 3.3.1. *The social choice rule f is SP if and only if it is GSP.*

In the following proposition, we show that if the preference of any agent changes but she continues to rank all alternatives in r_f in the same way as before, then the

[1]With a slight abuse of notation, we write R_{-i} instead of $R_{-\{i\}}$.

alternative chosen should be the same. That is, the rule f should be independent of preferences over alternatives that are not in the range of f.

Proposition 3.3.2. *Let f be SP. For each $R, R' \in \mathcal{R}$ such that $x\, P_i\, y \Leftrightarrow x\, P'_i\, y$ for all $x, y \in r_f$ and all $i \in N$, then $f(R) = f(R')$.*

Given Proposition 3.3.2, it is useful to define for each $i \in A$ and each $R_i \in \mathcal{R}_i$ the **peak over the alternatives in the range of *f*** as $p_f(R_i) = \{x \in r_f \,:\, x\, P_i\, y$ for all $y \in r_f \setminus \{x\}\}$. Thus, $p_f(R)$ is an element of r_f^A. Similarly, for each $i \in D$ and each $R_i \in \mathcal{R}_i$, we define the **dip over the alternatives in the range of *f*** as $d_f(R_i) = \{x \in r_f \,:\, y\, P_i\, x$ for all $y \in r_f \setminus \{x\}\}$.

We now discuss the implications that SP (and, thus, GSP) imposes on the rules. The following result, which is essential to understand the structure of any SP rule, establishes that if a vector of peaks $p_f(R)$ is set, there are at most two alternatives in the range of f.

Proposition 3.3.3. *Let f be SP. Then, for each $R \in \mathcal{R}$, $|\omega(p_f(R))| \leq 2$.*

Proposition 3.3.3 implies that any SP rule can be decomposed in two steps. In the first step, agents with single-peaked preferences have to declare only their peaks and, depending on those peaks, a set of at most two alternatives is preselected. If one alternative is preselected, that alternative is finally implemented. If two alternatives are preselected, the alternative finally selected must be determined in the second step of the procedure. This idea is summarized in the following proposition. We first introduce some notation: for each $R \in \mathcal{R}$, denote $\min \omega(p_f(R))$ by $\underline{\omega}(p_f(R))$ and $\max \omega(p_f(R))$ by $\overline{\omega}(p_f(R))$.

Proposition 3.3.4. *If f is SP, there is a function $\omega : r_f^A \to r_f^2$ and a set of binary decision functions $\{g_{\{x,y\}} : \mathcal{R} \to \{l, r\}\}_{\{x,y\} \in r_f^2}$ such that for each $R \in \mathcal{R}$,*

$$f(R) = \begin{cases} \underline{\omega}(p_f(R)) & \text{if } g_{\omega(p_f(R))}(R) = l, \\ \overline{\omega}(p_f(R)) & \text{if } g_{\omega(p_f(R))}(R) = r. \end{cases}$$

Proposition 3.3.4 explains the structure of any SP rule. Each SP rule f depends on a set of functions. The first of them, ω, determines the set of alternatives that are preselected when the agents of A have declared their peaks. That is, $\omega(p_f(R))$ gives the alternatives that can be selected when the vector of peaks is $p_f(R)$. By Proposition 3.3.3, we know that this set of preselected alternatives includes at most two alternatives. If only one alternative x is preselected, we denote that the outcome of ω is $\{x, x\}$. Thus, the outcome of ω is always in r_f^2. To choose between the preselected alternatives, $\underline{\omega}(p_f(R))$ and $\overline{\omega}(p_f(R))$, the SP rules apply a binary decision function $g_{\omega(p_f(R))} : \mathcal{R} \to \{l, r\}$ in such a way that if the outcome of $g_\omega(p_f(R))$ at profile R is l, then $\underline{\omega}(p_f(R))$ is chosen by f. Otherwise, $\overline{\omega}(p_f(R))$ is selected by f. For example, if $\omega(p_f(R)) = \{x, y\}$, with $x < y$, and $g_{\{x,y\}}(R) = r$, then $f(R) = y$. Note that, when only one alternative x has been preselected in the first step, that is, $\omega(p_f(R)) = \{x, x\}$, it always holds that $f(R) = x$ because $\underline{\omega}(p_f(R)) = \overline{\omega}(p_f(R)) = x$.

Observe that so far we have not imposed any conditions on ω and $\{g_{\{x,y\}}\}_{\{x,y\}\in r_f^2}$. For that reason, although all SP rules can be decomposed in the way described before, there are rules that follow the structure of Proposition 3.3.4 that are not SP. Therefore, there is a need to study how SP restricts each function.

3.4 Characterization of strategy-proof rules

In this section, we characterize all rules that are SP on our preference domain. We start by providing the necessary conditions upon Proposition 3.3.4 that each of the two steps has to satisfy.

3.4.1 Conditions on the first step

We study what additional conditions are required in the first step to guarantee SP, *i.e.*, what conditions have to be satisfied by the function ω. In particular, we analyze the structure of the single alternatives and pairs of alternatives that can appear as the outcome of the first step.

The first result states that if the outcome of ω for a vector of peaks contains two alternatives, then there is no other alternative between them in the range of f. We say for any $\vec{p_f} \in r_f^A$ that $|\omega(\vec{p_f})| = 1$ when $\omega(\vec{p_f}) = \{x, x\}$ for some $x \in r_f$ and that $|\omega(\vec{p_f})| = 2$ when $\omega(\vec{p_f}) = \{x, y\}$ for some $x, y \in r_f$ with $x \neq y$.

Proposition 3.4.1. *Let f be SP. If $|\omega(\vec{p}_f)| = 2$ for some $\vec{p}_f \in r_f^A$, then $r_f \cap (\underline{\omega}(\vec{p}_f), \overline{\omega}(\vec{p}_f)) = \emptyset$.*

Given Proposition 3.4.1, it is useful to define V_f as the set that contains all ordered pairs that can be formed with contiguous alternatives of r_f, *i.e.*, $V_f = \{(x, y) \in r_f^2 : x < y \text{ and } (x, y) \cap r_f = \emptyset\}$. If the outcome of ω is a single alternative, then ω takes a value in r_f. Otherwise, if the outcome of ω is a pair of contiguous alternatives, then ω takes a value in V_f. Therefore, the range of ω is a (not necessarily strict) subset of $V_f \cup r_f$.[2] To simplify notation, from now on we write the outcome of $\omega(\vec{p_f})$ by x when it is $\{x, x\}$ and by (x, y) when it is $\{x, y\}$, with $x < y$.

[2]There is a slight abuse of notation in this sentence because the range of ω consists of non-ordered pairs, while V_f is defined as ordered pairs. However, since any pair can be ordered, we use both notations without distinction throughout the chapter.

At this point, it must be remembered that r_f is countable (because X is countable), so there may be no $\min r_f$ and $\max r_f$. However, all the following definitions and results can be applied for any possible r_f independently of whether or not the set has a maximum and/or a minimum.[3]

The second result states that for each interior alternative in the range of f, there is a vector of peaks such that the outcome of ω is uniquely that alternative.[4]

Proposition 3.4.2. *Let f be SP. For each $x \in r_f \setminus \{\min r_f, \max r_f\}$, there is $\vec{p}_f \in r_f^A$ such that $\omega(\vec{p}_f) = x$.*

Observe that if $\min r_f$ or $\max r_f$ exists, Proposition 3.4.2 gives the flexibility to include them or not in the range of ω. Therefore, the range of ω always includes $V_f \cup r_f \setminus \{\min r_f, \max r_f\}$ and may or may not include the alternatives $\min r_f$ and $\max r_f$.

Observe that $r_f \subseteq X \subseteq \mathbb{R}$. Then, there is a natural order $<$ over the set r_f. This order can be extended to $V_f \cup r_f$ with an order $<^*$ in the following way:

(i) $x <^* y \Leftrightarrow x < y$,

(ii) for each $(x, y) \in V_f$ with $x \neq y$, $x <^* (x, y) <^* y$.

That is, $<^*$ ranks all single alternatives in the range in the same way as $<$ and inserts each pair of contiguous alternatives in the middle of them.[5] Figure 3.1 illustrates how V_f and $<^*$ are constructed from r_f.

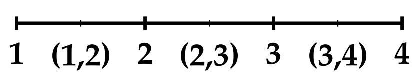


Figure 3.1: Order $<^*$ over $V_f \cup r_f$ with $r_f = \{1, 2, 3, 4\}$.

We now analyze the structure of ω in order to ensure SP. Observe that in this first step there is a set of agents with single-peaked preferences and a set $V_f \cup r_f$ of elements with an order $<^*$. It is well-known in the literature (see Barberà, 2011) that the SP rules on a domain in which all agents have single-peaked preferences over an ordered set are generalized median voter rules. However, the existing results cannot

[3] In those results in which the statement makes reference to the elements $\min r_f$ and/or $\max r_f$, if the particular r_f under analysis does not have that element, then the reference to that element should be taken as eliminated on reading the statement.

[4] With a slight abuse of notation, when $\omega(p(R)) = \{x, x\}$, we write $\omega(p(R)) = \{x\}$ or simply $\omega(p(R)) = x$.

[5] Observe that an order $>^*$ can be defined in a similar way. We also make use of the weak orders $\leq^*$ and $\geq^*$ constructed from $<^*$ and $>^*$ in the standard way, respectively.

be applied directly because even though the preferences of the agents of A are single-peaked over X, they may not be single-peaked over $V_f \cup r_f$. To see why, consider an agent whose peak is at $x \in X$. Thus, it seems that this agent will strictly prefer (x, y) over y as the outcome of the function ω. However, this is not always the case because the final decision will depend on the entire profile of preferences: if, given the preferences of the other agents, y is going to be selected in the second step in both cases, then the agent is indifferent between (x, y) and y. Nevertheless, it turns out that the unique ω functions compatible with an SP rule on our domain are also generalized median voter functions.

The first important issue that deserves to be mentioned about the class we are going to define is that it is defined not necessarily on $V_f \cup r_f$, but on a set T_f such that $[V_f \cup (r_f \setminus \{\min r_f, \max r_f\})] \subseteq T_f \subseteq V_f \cup r_f$. To clarify this point, consider the rules f with $r_f = \{1, 2, 3, 4\}$. Then, since alternatives $1 = \min r_f$ and $4 = \max r_f$ may or may not belong to T_f, there are four possible sets T_f for these rules: (i) $T_f = V_f \cup r_f$, that is, $T_f = \{1, (1, 2), 2, (2, 3), 3, (3, 4), 4\}$; (ii) $T_f = V_f \cup (r_f \setminus \{1\})$, that is, $T_f = \{(1, 2), 2, (2, 3), 3, (3, 4), 4\}$; (iii) $T_f = V_f \cup (r_f \setminus \{4\})$, that is, $T_f = \{1, (1, 2), 2, (2, 3), 3, (3, 4)\}$; and (iv) $T_f = V_f \cup (r_f \setminus \{1, 4\})$, that is, $T_f = \{(1, 2), 2, (2, 3), 3, (3, 4)\}$.

The second main point in defining a generalized median voter function on a set (in this case, on T_f) is the concept of a left coalition system. We first introduce the formal definition of it.

Definition 3.4.3. *Consider a set T_f such that $[V_f \cup (r_f \setminus \{\min r_f\}, \{\max r_f\})] \subseteq T_f \subseteq V_f \cup r_f$. A **left coalition system on T_f** is a correspondence $\mathcal{L} : T_f \to 2^A$ assigning to each $\alpha \in T_f$ a collection of coalitions $\mathcal{L}(\alpha)$ such that:*

(i) *if $C \in \mathcal{L}(\alpha)$ and $C \subset C'$, then $C' \in \mathcal{L}(\alpha)$,*

(ii) *if $\alpha <^* \beta$ and $C \in \mathcal{L}(\alpha)$, then $C \in \mathcal{L}(\beta)$, and*

(iii) *if r_f has a maximum and $\max r_f \notin T_f$, then $\emptyset \in \mathcal{L}(\max T_f) \setminus \mathcal{L}(\alpha)$ for each $\alpha \in T_f \setminus \{\max T_f\}$.*

(iv) *if r_f does not have a maximum, then $\emptyset \notin \mathcal{L}(\alpha)$ for each $\alpha \in T_f$.*

A left coalition system on T_f includes a set of coalitions $\mathcal{L}(\alpha)$ for each element $\alpha \in T_f$. The coalitions in $\mathcal{L}(\alpha)$ can be interpreted as the "support" or "winning coalitions" needed to implement an alternative to the left of or equal (with the order $\leq^*$) to α. Condition (i) implies that if a coalition is winning at α, all its supercoalitions are also winning at α. Condition (ii) implies that if a coalition is winning at α, it is also winning at any $\beta >^* \alpha$. Finally, conditions (iii) and (iv) guarantee, as shown below, that any alternative in the range of f appears as the outcome of f.

A generalized median voter function can thus be defined using the notion of a left coalition system.

Definition 3.4.4. *Given a left coalition system $\mathcal{L}$ on T_f, its* ***associated generalized median voter function*** $\boldsymbol{\omega}$ *is defined so that for each $\vec{p}_f \in r_f^A$ and each $R \in \mathcal{R}$ such that $p_f(R) = \vec{p}_f$,*

$$\omega(\vec{p}_f) = \alpha \Leftrightarrow \{i \in A : p_f(R_i) \leq^* \alpha\} \in \mathcal{L}(\alpha) \text{ and}$$

$$\{i \in A : p_f(R_i) \leq^* \beta\} \notin \mathcal{L}(\beta) \text{ for all } \beta \in T_f \text{ such that } \beta <^* \alpha.$$

Once a set T_f and a left coalition system $\mathcal{L}$ on T_f are obtained, the associated generalized median voter function ω chooses the first element $\alpha \in T_f$, starting from the left, such that the set of agents whose peaks are to the left of or equal to α belongs to $\mathcal{L}(\alpha)$ under the order $\leq^*$.

We are now ready to explain the relevance of conditions (iii) and (iv) in Definition 3.4.3. Note first that once the empty set belongs to $\mathcal{L}(\alpha)$ for some $\alpha \in T_f$, no alternative to the right of α appears as the outcome of ω for any profile. To see why, observe that if $\emptyset \in \mathcal{L}(\alpha)$ for some $\alpha \in T_f$, then $\mathcal{L}(\alpha) = 2^A$ by condition (i) in Definition 3.4.3. Therefore, for any $R \in \mathcal{R}$, $\{i \in A : p_f(R_i) \leq^* \alpha\} \in \mathcal{L}(\alpha)$ and thus, $\omega(p_f(R)) \leq^* \alpha$. So if the empty set belongs to an element of T_f, then T_f has a maximum, which requires that r_f should have a maximum too. Thus, condition (iv) in Definition 3.4.3 means that when r_f does not have a maximum, the empty set does not belong to $\mathcal{L}(\alpha)$ for any $\alpha \in T_f$. To understand condition (iii), consider, for instance, a rule f with $r_f = \{1, 2, 3, 4\}$ and $T_f = V_f \cup (r_f \setminus \{4\})$. We first show that the empty set must belong to $\mathcal{L}(\alpha)$ for some $\alpha \in T_f$. To see this, consider a profile $R \in \mathcal{R}$ such that $p_f(R_i) = 4$ for each $i \in A$. If $\emptyset \notin \mathcal{L}(\alpha)$ for each $\alpha \in T_f$, then $\omega(p_f(R)) >^* (3, 4) = \max T_f$, which is not possible. Second, we show that the empty set belongs only to $\mathcal{L}(\max T_f) = \mathcal{L}(3, 4)$. To see this, assume that for some $\alpha \neq (3, 4)$, $\emptyset \in \mathcal{L}(\alpha)$. Then, $\mathcal{L}(\alpha) = 2^A$ by condition (i) in Definition 3.4.3. Therefore, for any $R \in \mathcal{R}$, $\omega(p_f(R)) \leq^* \alpha$ and $(3, 4)$ will never appear as the outcome of ω. Hence, alternative 4 will never be the outcome of f, which is not possible given that $4 \in r_f$.

Finally, observe that Proposition 3.4.2 states that all interior alternatives of r_f must be in the range of ω but there is no obligation to include either the single alternatives $\min r_f$ and $\max r_f$ or each of the pairs of V_f. All this flexibility is included in Definition 3.4.4. On the one hand, as shown, the set T_f may or may not include $\min r_f$ and $\max r_f$. On the other hand, even though all pairs $(x, y) \in V_f$ also belong to T_f, some of them may not belong to the range of ω. In particular, this happens when $\mathcal{L}(x) = \mathcal{L}(x, y)$. To see why, consider, for instance, that $\mathcal{L}(2) = \mathcal{L}(2, 3)$. Thus, for any $R \in \mathcal{R}$, $\{i \in A : p_f(R_i) \leq^* 2\} = \{i \in A : p_f(R_i) \leq^* (2, 3)\}$ given that $p_f(R_i) \in r_f$. Therefore, if $\{i \in A : p_f(R_i) \leq^* (2, 3)\} \in \mathcal{L}(2, 3)$, then $\{i \in A : p_f(R_i) \leq^* 2)\} \in \mathcal{L}(2)$ and then, $\omega(p_f(R)) \neq (2, 3)$. Thus, the pair $(2, 3)$ will never appear as the outcome of ω.

The result below shows that the function ω has to be a generalized median voter function to guarantee SP.

Proposition 3.4.5. *Let f be SP. The function ω is a generalized median voter function on a set T_f, with $[V_f \cup (r_f \setminus \{\min r_f, \max r_f\})] \subseteq T_f \subseteq V_f \cup r_f$.*

3.4.2 Conditions on the second step

We now introduce the additional conditions that SP imposes on the second step; *i.e.*, the conditions on the functions $\{g_{\{x,y\}} : \mathcal{R} \to \{l, r\}\}_{\{x,y\} \in r_f^2}$. Note first that, given the analysis of the previous subsection, only the pairs of T_f may appear as the outcome of ω. Additionally, observe that if the outcome of ω is a singleton, then the definition of the second step is irrelevant. Therefore, the only functions that need to be analyzed are $\{g_{(x,y)}\}_{(x,y) \in V_f}$. We introduce the following notation: for each $(x, y) \in V_f$ and each $R \in \mathcal{R}$, denote the set of agents that prefer x to y at R by $L_{(x,y)}(R) = \{i \in N : x\, P_i\, y\}$.

We define a particular class of binary decision functions called voting by collections of left-decisive sets. Any of these functions $g_{(x,y)}$ can be defined by specifying a set of coalitions $W(g_{(x,y)}) \subseteq 2^N$ (called left-decisive sets) such that $g_{(x,y)}$ chooses l if the set of agents that prefer x to y, $L_{(x,y)}(R)$, is a superset of any coalition that belongs to $W(g_{(x,y)})$, and r otherwise. Before the formal definition is introduced, some concepts need to be defined. A coalition $S \subseteq N$ is **minimal in a set of coalitions** $V \subseteq 2^N$ if $S \in V$ and for each $S' \subset S, S' \notin V$. Then, we say that a set of coalitions $V \subseteq 2^N$ is **minimal** if all its coalitions are minimal in that set.

Definition 3.4.6. *Given a generalized median voter function ω defined on a set T_f, with $[V_f \cup (r_f \setminus \{\min r_f, \max r_f\})] \subseteq T_f \subseteq V_f \cup r_f$, and any $(x, y) \in V_f$, the binary decision function $g_{(x,y)}$ is called a* ***voting by collections of left-decisive sets*** *if there is a minimal set of coalitions $W(g_{(x,y)}) \subseteq 2^N$ such that for each $R \in \mathcal{R}$, with $\omega(p_f(R)) = (x, y)$,*

$$g_{(x,y)}(R) = \begin{cases} l & \text{if } C \subseteq L_{(x,y)}(R) \text{ for some } C \in W(g_{(x,y)}) \\ r & \text{otherwise,} \end{cases}$$

and the following conditions are satisfied:

(i) For each $C \in W(g_{(x,y)})$, $C \cap D \neq \emptyset$.

(ii) For each minimal coalition B of $\mathcal{L}(x, y) \setminus \mathcal{L}(x)$, there is $C \in W(g_{(x,y)})$ such that $C \cap A = B$.

Observe that two additional conditions are required in order to complete the description of the minimal set of coalitions $W(g_{(x,y)})$. These conditions guarantee that, given any $\vec{p}_f \in r_f^A$ such that $\omega(\vec{p}_f) = (x, y)$, both alternatives x and y appear as the outcome of f for some profiles with a vector of peaks $\vec{p}_f$. To see why, observe that if for some $\vec{p}_f \in r_f^A$ we have $\omega(\vec{p}_f) = (x, y)$, then the set of agents of A that prefer x to

y, say B, is in $\mathcal{L}(x,y) \setminus \mathcal{L}(x)$. Thus, condition (ii) requires there to be a coalition of $W(g_{(x,y)})$ such that its agents with single-peaked preferences are exactly those agents of B. Additionally, note that given any $\vec{p}_f \in r_f^A$ such $\omega(\vec{p}_f) = (x,y)$, the preference between x and y of all agents of A is known because there is no alternative in the range between x and y: if for some agent $i \in A$, $(\vec{p}_f)_i \leq x$, then agent i prefers x to y. Otherwise, if $(\vec{p}_f)_i \geq y$, then agent i prefers y to x. Thus, if the preferences of the agents with single-dipped preferences are not considered, the outcome of f will be either always x or always y for all profiles with a vector of peaks $\vec{p}_f$, which is not possible because $\omega(\vec{p}_f) = (x,y)$. Therefore, condition (i) requires there to be at least one agent with single-dipped preferences in each coalition of $W(g_{(x,y)})$.

Proposition 3.4.7. *Let f be SP and ω be its associated generalized median voter function on a set T_f, with $[V_f \cup (r_f \setminus \{\min r_f, \max r_f\})] \subseteq T_f \subseteq V_f \cup r_f$. Then, the family of binary decision functions $\{g_{(x,y)} : \mathcal{R} \to \{l,r\}\}_{(x,y)\in V_f}$ is such that for each $(x,y) \in V_f$, $g_{(x,y)}$ is a voting by collections of left-decisive sets.*

Observe that no conditions are imposed on the relationship between functions $g_{(x,y)}$ and $g_{(z,w)}$, so it is possible to use different votings by collections of left-decisive sets for each pair of V_f.

3.4.3 The characterization

Some necessary conditions have been defined on the two steps into which a social choice rule can be decomposed to guarantee SP. The main theorem states that all these conditions are also sufficient to obtain SP (and, by Proposition 3.3.1, GSP).

Theorem 3.4.8. *The following statements are equivalent:*

(i) f is SP.

(ii) f is GSP.

(iii) There is a generalized median voter function ω on a set T_f, with $[V_f \cup (r_f \setminus \{\min r_f, \max r_f\})] \subseteq T_f \subseteq V_f \cup r_f$, and a set of voting by collections of left-decisive sets $\{g_{(x,y)} : \mathcal{R} \to \{l,r\}\}_{(x,y)\in V_f}$ such that for each $R \in \mathcal{R}$,

$$f(R) = \begin{cases} \omega(p_f(R)) & \text{if } \omega(p_f(R)) \in r_f, \\ \underline{\omega}(p_f(R)) & \text{if } \omega(p_f(R)) \in V_f \text{ and } g_{\omega(p_f(R))}(R) = l, \\ \overline{\omega}(p_f(R)) & \text{if } \omega(p_f(R)) \in V_f \text{ and } g_{\omega(p_f(R))}(R) = r. \end{cases}$$

A relevant characteristic of the rules characterized in Theorem 3.4.8 is how little information is required about agents' preferences: any of these rules only need to know the peak of the preference of each agent $i \in A$ among the alternatives in the range of the rule, $p_f(R_i)$, and the dip of the preference of each agent $j \in D$ among the alternatives in the range of the rule, $d_f(R_j)$. To see why, observe on the one hand that the function ω only requires information about the location of all peaks, $p_f(R)$. On the other hand, the functions $g_{(x,y)}$ only need information about agents' preferences between alternatives x and y. Since there is no alternative of r_f between x and y, the whole vector of peaks and dips among the alternatives of r_f provide the required information: if for an agent $k \in N$, $p_f(R_k) \leq x$ or $d_f(R_k) \geq y$, agent k prefers x to y, while in the remaining cases agent k prefers y to x.

It is also important to mention that the social planner can design any of the rules characterized in Theorem 3.4.8 as follows. First, the social planner decides a range $r_f \subseteq X$. Once r_f is known, V_f is straightforwardly defined. Second, the social planner define a set T_f by deciding whether $\min r_f$ and/or $\max r_f$ belong to T_f or not. Once T_f is defined, the social planner decides on a left coalition system $\mathcal{L}$ on T_f that meets the conditions in Definition 3.4.3. Thus, the function ω associated with $\mathcal{L}$ is known to be a generalized median voter function. Since the outcome of ω may belong to V_f, the social planner finally has to specify for each $(x,y) \in V_f$, a set of minimal left-decisive sets $W(g_{(x,y)})$ satisfying the conditions in Definition 3.4.6.

The following examples clarify how some of the rules in the family characterized in Theorem 3.4.8 are described with the four components explained in the previous paragraph: r_f, T_f, $\mathcal{L}(\alpha)$ for each $\alpha \in T_f$, and $W(g_{(x,y)})$ for each $(x,y) \in V_f$. The first two examples correspond to cases $D = \emptyset$ and $A = \emptyset$ respectively.

Example 3.4.9. It has been established by Moulin (1980) and Barberà and Jackson (1994) that generalized median voter rules on a set $S \subseteq X$ are the only SP rules on the single-peaked preference domain. Thus, to describe those rules in terms of the family characterized in Theorem 3.4.8, the outcome of ω must take a value of r_f for each profile. These rules correspond to $r_f = S$, $T_f = V_f \cup r_f$, a left coalition system $\mathcal{L}$ such that $\mathcal{L}(x,y) = \mathcal{L}(x)$ for each $(x,y) \in V_f$, and any sets $\{W(g_{(x,y)})\}_{(x,y) \in V_f}$ because the second step is never required. □

Example 3.4.10. Manjunath (2014) shows that if all agents have single-dipped preferences, the only SP rules on that domain have range two and that the choice between the two alternatives is made with a voting by collections of left-decisive sets. These rules can be expressed in terms of the family characterized in Theorem 3.4.8 by making the outcome of the first step always the same pair of alternatives. These rules correspond to any range r_f such that $|r_f| = 2$, $T_f = \{(\min r_f, \max r_f)\}$, a set $W(g_{(\min r_f, \max r_f)})$ such that $C \subseteq D$ for each $C \in W(g_{(\min r_f, \max r_f)})$, and any left coalition system $\mathcal{L}$ because since T_f only includes the element $(\min r_f, \max r_f)$, the first step is irrelevant. Since the outcome of ω is $(\min r_f, \max r_f)$ for any vector of peaks, the choice will be always made in the second step of the procedure according to a voting by collection of left-decisive sets, $g_{(\min r_f, \max r_f)}$. □

Dictatorial rules are SP on the universal preference domain, so they are also SP on all restricted preference domains such as the domain studied in this chapter. The following two examples show how they can be described in terms of the family characterized in Theorem 3.4.8.[6]

Example 3.4.11. Consider the dictatorial rule over a set $S \subseteq X$ in which the dictator is an agent with single-peaked preferences, $i \in A$. This rule corresponds to $r_f = S$, $T_f = V_f \cup r_f$, a left coalition system $\mathcal{L}$ such that for each $\alpha \in T_f$, $[C \in \mathcal{L}(\alpha) \Leftrightarrow i \in C]$, and any sets $\{W(g_{x,y})\}_{(x,y)\in V_f}$. Since for each $(x, y) \in V_f$, $\mathcal{L}(x, y) = \mathcal{L}(x)$, the outcome of ω will be always an alternative of r_f. Moreover, given that a coalition is decisive in ω if and only if i belongs to it, the outcome of ω for any profile R will be $p_f(R_i)$, the best alternative in the range according to i's preference. Observe that the second step is irrelevant, so $\{W(g_{x,y})\}_{(x,y)\in V_f}$ does not need to be specified. □

Example 3.4.12. Consider the dictatorial rule in which the dictator is an agent with single-dipped preferences, $i \in D$. This rule corresponds to any range r_f such that $|r_f| = 2$, $T_f = \{(\min r_f, \max r_f)\}$, $W(g_{(\min r_f, \max r_f)}) = \{\{i\}\}$, and any left coalition system $\mathcal{L}$. Since $W(g_{(\min r_f, \max r_f)})$ only contains coalition $\{i\}$, the outcome of $g_{(\min r_f, \max r_f)}$ for any profile R will depend only on $d_f(R_i)$. That is, f selects $\min r_f$ if and only if agent i prefers $\min r_f$ to $\max r_f$. Hence, agent i always prefer the alternative selected by the rule to the other alternative in the range. □

Apart from the rules set out above, there are many other SP rules on our domain that take into account the preferences of all agents. We describe one of them in the following example.

Example 3.4.13. Suppose that $X = \mathbb{R}$, $A = \{i_1, i_2, i_3\}$ and $D = \{j_1, j_2, j_3\}$. Consider any rule with $r_f = \{1, 2, 3, 4\}$. Then, $V_f = \{(1, 2), (2, 3), (3, 4)\}$, so $\{(1, 2), 2, (2, 3), 3, (3, 4)\} \subseteq T_f \subseteq \{1, (1, 2), 2, (2, 3), 3, (3, 4), 4\}$. Consider in particular the rule f such that:

- $T_f = V_f \cup r_f$.
- $\mathcal{L}(1) = \mathcal{L}(1, 2) = \mathcal{L}(2) = \{S \subseteq A : |S| \geq 2\}$ and $\mathcal{L}(2, 3) = \mathcal{L}(3) = \mathcal{L}(3, 4) = \mathcal{L}(4) = \{S \subseteq A : |S| \geq 1\}$.
- $W(g_{(1,2)}) = \{S \subseteq N : |S| = 3 \text{ and } |S \cap A| = 2\}$ and $W(g_{(2,3)}) = W(g_{(3,4)}) = \{S \subseteq N : |S| = 3 \text{ and } |S \cap A| = 1\}$.

This rule is SP because it belongs to the family characterized in Theorem 3.4.8. The outcome of this function in some profiles can be seen below:

[6] Observe that the families of rules defined in Examples 3.4.9 and 3.4.10 already include the dictatorial rules where the dictator is a single-peaked or a single-dipped agent, respectively.

- Consider a profile $R \in \mathcal{R}$ such that $p_f(R) = p_f(R_{i_1}, R_{i_2}, R_{i_3}) = (1, 2, 4)$ and $d_f(R) = d_f(R_{j_1}, R_{j_2}, R_{j_3}) = (1, 3, 3)$. Thus, $\omega(p_f(R)) = 2$ because $\{i \in A : p_f(R_i) \leq^* 2\} = \{i_1, i_2\} \in \mathcal{L}(2)$ and $\{i \in A : p_f(R_i) \leq^* (1, 2)\} = \{i_1\} \notin \mathcal{L}(1, 2)$. Since $2 \in r_f$, then $f(R) = 2$.

- Consider a profile $R' \in \mathcal{R}$ such that $p_f(R') = (1, 3, 4)$ and $d_f(R') = (1, 3, 3)$. Thus, $\omega(p_f(R')) = (2, 3)$ because $\{i \in A : p_f(R'_i) \leq^* (2, 3)\} = \{i_1\} \in \mathcal{L}(2, 3)$ and $\{i \in A : p_f(R'_i) \leq^* 2\} = \{i_1\} \notin \mathcal{L}(2)$. Since $(2, 3) \in V_f$, it is necessary to analyze $g_{(2,3)}$. Observe that $L_{(2,3)}(R') = \{i_1, j_2, j_3\}$. Given that $|\{i_1, j_2, j_3\}| = 3$ and $|\{i_1, j_2, j_3\} \cap A| = 1$, it follows that $\{i_1, j_2, j_3\} \in W(g_{(2,3)})$. Thus, $f(R') = 2$.

- Consider a profile $R'' \in \mathcal{R}$ such that $p_f(R'') = (1, 3, 4)$ and $d_f(R'') = (1, 2, 3)$. Thus, $\omega(p_f(R'')) = (2, 3)$ because $\{i \in A : p_f(R''_i) \leq^* (2, 3)\} = \{i_1\} \in \mathcal{L}(2, 3)$ and $\{i \in A : p_f(R''_i) \leq^* 2\} = \{i_1\} \notin \mathcal{L}(2)$. Observe that $L_{(2,3)}(R'') = \{i_1, j_3\}$. Given that $|L_{(2,3)}(R'')| = 2$, there is no $C \in W(g_{(2,3)})$ such that $C \subseteq L_{(2,3)}(R'')$. Thus, $f(R'') = 3$.

Observe that this rule corresponds to the following procedure: First, the function ω is the median of the peaks of the three agents in A and four phantom voters situated at $-\infty$, 1, $(2, 3)$ and $+\infty$. Second, if a pair of alternatives is preselected by ω, a majority rule is applied (selecting the left element of the pair if there is a tie). □

Finally, observe that the family characterized in Theorem 3.4.8 has similarities with the results provided in the literature for the domains of only single-peaked preferences or only single-dipped preferences. On the one hand, in the first step of the rules characterized in Theorem 3.4.8, a generalized median voter function is applied as in the single-peaked preference domain. However, on the single-peaked preference domain the generalized median voter function is defined on r_f, while here it is defined on a set T_f. On the other hand, the second step of the rules characterized in Theorem 3.4.8 faces a binary decision problem as it occurs with the SP rules on the single-dipped preference domain and in both cases a voting by collections of left-decisive sets has to be applied.

3.5 Strategy-proof and Pareto efficient rules

The family characterized in Theorem 3.4.8 is quite large, so it may be useful to reduce it by imposing additional axioms. In this section, we analyze the consequences of adding PE. The results are summarized in the following proposition.

Proposition 3.5.1. *The following statements hold:*

(i) *All SP rules f such that $r_f = X$ are PE.*

(ii) *All SP rules f such that $r_f \notin \{X, \{\min X, \max X\}\}$ are not PE.*

(iii) *A SP rule f such that $r_f = \{\min X, \max X\} \neq X$ is PE if and only if $A = \emptyset$ or $T_f = \{(\min X, \max X)\}$.*

It can be seen that PE imposes a strong restriction on the range of f. All of those SP rules whose range is equal to the set of feasible alternatives but only some of the SP rules whose range is equal to the minimum and maximum points of the set of feasible alternatives are PE. Let us explain the intuition behind this result. Note first that if a SP rule f has a range equal to X but is not PE, then there is an alternative in X that is unanimously preferred to the one chosen by f. Since that alternative is in the range of f, the rule is not GSP and, by Proposition 3.3.1, nor is SP. Observe that this argument only explains part (i) in Proposition 3.5.1. Now we focus on the remaining cases.

First, it is easy to check that $\min X$ and $\max X$ must be in the range of f to guarantee PE. To see why, assume, for instance, that $\min X$ is not in the range of f and consider a preference profile in which all the peaks over X are in $\min X$ and all the dips over X are in $\max X$. Then, $\min X$ Pareto dominates any alternative in the range of f, so f is not PE.

Second, if $\min X$ and $\max X$ are not the only elements in the range of f, then there is an interior alternative in r_f. Moreover, since the range of f does not coincide with X, there is another interior alternative in $X \setminus r_f$. Let $x \in X \setminus r_f$ and $y \in r_f$, with y being the closest alternative to x. Consider a preference profile where all the peaks over X are located in x and all the peaks and dips over r_f are in y. It can be checked that any SP rule chooses y in that profile by Theorem 3.4.8. However, x Pareto dominates y, which contradicts PE. Therefore, if a SP rule f, with $r_f \neq X$, is PE, then $r_f = \{\min X, \max X\}$.

Finally, if the range of f is equal to $\{\min X, \max X\}$, then more restrictions are needed to guarantee PE. These restrictions are provided in part (iii) of Proposition 3.5.1. Observe that these conditions require the set D to be non-empty. Moreover, by the role of the agents of D in $W(g_{(\min X, \max X)})$, $T_f = \{(\min X, \max X)\}$ implies that if f chooses $\min X$ (respectively, $\max X$) it is because there is an agent of D such that $\min X$ (respectively, $\max X$) is preferred to any other alternative of X, which guarantees PE.

3.6 Concluding remarks

We analyze the problem of locating a public facility that generates different opinions among agents: Some agents consider it as a good while others consider it as a bad. It has already been shown in the literature that it is not possible to escape from the

Gibbard-Satterthwaite impossibility in a domain where the set of admissible preferences of each agent includes all single-peaked and all single-dipped preferences, so the domain needs to be restricted further. We analyze a new domain in which the social planner has information about the kind of preference of each agent (single-peaked or single-dipped) but is uncertain as to where each agent's peak or dip is located and how each agent ranks the rest of the alternatives. This domain fits well with situations in which even though the social planner knows a location for each agent, that location may not necessarily coincide with her peak or dip. Consider for instance that the facility to be built is a nursery. Parents with children may consider this facility desirable, but it might be undesirable for others without children or for those who prefer to live in a quiet neighborhood. The location of the house of each agent may be known, but taking into account the high number of hours people spend at work, parents may prefer to have a nursery close to their workplace rather than to their house. Note that such situations cannot be accommodated in the domain of Alcalde-Unzu and Vorsatz (2018), where the peak or the dip of the preference of each agent corresponds to the unique location of the agent, which is known by the social planner. In contrast, to allow each agent total flexibility as to the location of her peak or dip, the social planner needs to have full information as to the type of preference of each agent, while in the domain of Alcalde-Unzu and Vorsatz (2018) that information is private. We characterize all strategy-proof rules on this new domain and show that they are all also group strategy-proof. Finally, we restrict the family characterized by imposing Pareto efficiency and show that this property implies a strong restriction on the range of strategy-proof rules.

The family of strategy-proof rules characterized here shows some similarities with and differences from the family of strategy-proof rules characterized in Alcalde-Unzu and Vorsatz (2018). All strategy-proof rules on the domain of Alcalde-Unzu and Vorsatz (2018) also follow a two-step procedure. In their domain, the location of the peak or dip of each agent is known, so the first step of their rules asks which agents have single-peaked preferences. As a result of the first step, both the type of preference of each agent and the location of the peaks and dips are known. In the domain analyzed here, the type of preference of each agent is public information and in the first step we ask agents with single-peaked preferences about their peaks. As a result, the type of preference of each agent and the location of all peaks are known. Note that even though the social planner in our domain has less information after the first step, at most two alternatives are preselected in both settings. If two alternatives are preselected, the second step of Alcalde-Unzu and Vorsatz (2018) asks all agents their ordinal preference between the two alternatives to make the final decision. Unlike them, we only need to ask agents with single-dipped preferences about their dips to choose the final location. Thus, less information is required for the strategy-proof rules on our domain to be implemented than the strategy-proof rules on the domain of Alcalde-Unzu and Vorsatz (2018): we only need to know the location of the peaks and dips, while they need to know the type of preferences of each agent and the ordinal preference between any pair of preselected alternatives that may appear as the outcome of the first step. Moreover, we provide a closed-form characterization of the strategy-proof rules on our domain while Alcalde-Unzu and Vorsatz (2018) do not. We therefore consider that our family of rules is easier to implement in real-life situations than the family of strategy-proof rules on that other domain.

Considerations for further research may include constraining the family characterized by imposing other axioms such as anonymity or extending the model by allowing for indifferences. Note that it is not possible to apply the classical definition of anonymity because the set of admissible preferences differs from one agent to another. However, we can define a new property, "type-anonymity", which allows for permutations only between agents with the same set of admissible preferences (*i.e.*, an agent with single-peaked preferences can only permute with another agent with single-peaked preferences and an agent with single-dipped preferences can only permute with another agent with single-dipped preferences). Moreover, note that we assume throughout the chapter that preferences are linear orders. Allowing for indifferences would necessitate extending the results to the domains of single-plateau (Berga, 1998; Moulin, 1984) and single-basined (Bossert and Peters, 2014) preferences.

Appendix

Proof of Proposition 3.3.1

We first need to introduce some notation and definitions. For any $x \in X$ and any $R_i \in \mathcal{R}_i$, we define the lower contour set of R_i at x as $L(R_i, x) = \{y \in X : x\, R_i\, y\}$ and the strict lower contour set of R_i at x as $\bar{L}(R_i, x) = \{y \in X : x\, P_i\, y\}$. We also define, for each $y, z \in X$ and each profile $R \in \mathcal{R}$, the set $S(R; y, z) = \{i \in N : y\, P_i\, z\}$. A binary relation $\succsim_{(R;y,z)}$ on $S(R; y, z)$ is defined in the following way: for each $i, j \in S(R; y, z)$, $i \succsim_{(R;y,z)} j$ if $L(R_i, z) \subset \bar{L}(R_j, y)$. We can define the strict and indifference binary relations associated to $\succsim_{(R;y,z)}$. Formally, for each $i, j \in S(R; y, z)$, we say that $i \sim_{(R;y,z)} j$ if $[L(R_i, z) \subset \bar{L}(R_j, y)$ and $L(R_j, z) \subset \bar{L}(R_i, y)]$, and we say that $i \succ_{(R;y,z)} j$ if $[L(R_i, z) \subset \bar{L}(R_j, y)$ and $L(R_j, z) \not\subset \bar{L}(R_i, y)]$. Finally, we say that a profile R' is a strict monotonic transformation of R at an alternative $z \in X$ whenever for each $i \in N$ such that $R'_i \neq R_i$ and each $x \in X \setminus \{z\}$ such that $z\, R_i\, x$, then $z\, P'_i\, x$.

After introducing all these concepts, we are now ready to prove the proposition. The proof is based on the following lemma.

Lemma 3.6.1. *For each $R \in \mathcal{R}$ and each $y, z \in X$, there exists a strict monotonic transformation of R at z, $R' \in \mathcal{R}$ (possibly $R' = R$), such that $y \, P'_i \, z$ for each $i \in S(R; y, z)$ and $\succsim_{(R';y,z)}$ is complete and acyclic.*

Proof: Consider any profile $R \in \mathcal{R}$ and any pair of alternatives $y, z \in X$. Without loss of generality, assume that $y < z$.

If $D = \emptyset$ (respectively, $A = \emptyset$), then $\mathcal{R}$ is the domain in which all agents have single-peaked (respectively, single-dipped) preferences. Consider $R' = R$. Since, by Barberà et al. (2010), $\succsim_{(R;y,z)}$ is complete and acyclic, then $\succsim_{(R';y,z)}$ is complete and acyclic.

Suppose from now on that $A \neq \emptyset \neq D$ and consider $R' = (R'_A, R_D) \in \mathcal{R}$ such that, for each $i \in A$, $p(R'_i) = y$ and $z \, P'_i \, w$ for each $w < y$. By construction, $S(R; y, z) \subseteq S(R'; y, z)$, and then, $y \, P'_i \, z$ for each $i \in S(R; y, z)$. Moreover, it is easy to see that for each $x \in X \setminus \{z\}$ such that $z \, R_i \, x$, it is satisfied that $z \, P'_i \, x$. Therefore, R' is a strict monotonic transformation of R at z such that $y \, P'_i \, z$ for each $i \in S(R; y, z)$.

We prove now that $\succsim_{(R';y,z)}$ is complete and acyclic. To show that $\succsim_{(R';y,z)}$ is complete we divide the proof in three parts.

(i) *We show that $i \sim_{(R';y,z)} j$ for each $i, j \in A \cap S(R'; y, z)$.*

Since $p(R'_k) = y$ for each $k \in \{i, j\}$, we have that $\bar{L}(R'_k, y) = X \setminus \{y\}$. Moreover, given that, for each $k \in \{i, j\}$, $k \in S(R'; y, z)$, then $y \notin L(R'_k, z)$. Therefore, $L(R'_i, z) \subset \bar{L}(R'_j, y)$ and $L(R'_j, z) \subset \bar{L}(R'_i, y)$, and, thus, $i \sim_{(R';y,z)} j$.

(ii) *We show that $i \sim_{(R';y,z)} j$ for each $i, j \in D \cap S(R'; y, z)$.*

Observe that, for each $k \in \{i, j\}$, since $k \in D \cap S(R'; y, z)$, then $d(R'_k) > y$. We define first $L(R'_k, z)$ and $\bar{L}(R'_k, y)$ for each $k \in \{i, j\}$ depending on the location of $d(R'_k)$:

(ii.1) Suppose that $y < d(R'_k) \leq z$. Then, $L(R'_k, z) = [v, z] \cap X$ with $y < v \leq z$. Similarly, we have that $[\bar{L}(R'_k, y) = (y, w] \cap X$ with $w > z$ if $w < +\infty]$ or $[\bar{L}(R'_k, y) = (y, w) \cap X$ if $w = +\infty]$.

(ii.2) Suppose that $d(R'_k) > z$. Then, $[L(R'_k, z) = [z, v] \cap X$ with $v > z$ if $v < +\infty]$ or $[L(R'_k, z) = [z, v) \cap X$ if $v = +\infty]$. Similarly, we have that $[\bar{L}(R'_k, y) = (y, w] \cap X$ with $w > v$ if $w < +\infty]$ or $[\bar{L}(R'_k, y) = (y, w) \cap X$ if $w = +\infty]$.

It can be checked that for any $i, j \in D \cap S(R'; y, z)$, $L(R'_i, z) \subset \bar{L}(R'_j, y)$ and $L(R'_j, z) \subset \bar{L}(R'_i, y)$ by $(ii.1)$ and $(ii.2)$. Therefore, $i \sim_{(R';y,z)} j$.

(iii) *We show that $j \succsim_{(R';y,z)} i$ for each $i \in A \cap S(R'; y, z)$ and each $j \in D \cap S(R'; y, z)$.*

Since $p(R'_i) = y$, $\bar{L}(R'_i, y) = X \setminus \{y\}$. Given that $j \in S(R'; y, z)$, then $y \notin L(R'_j, z)$. Therefore, $L(R'_j, z) \subset \bar{L}(R'_i, y)$ and we conclude that $j \succsim_{(R';y,z)} i$.

Hence, the binary relation $\succsim_{(R';y,z)}$ is complete.

Finally, we show that $\succsim_{(R';y,z)}$ is acyclic. Since, by (i), we have that for any two agents in $A \cap S(R;y,z)$, they are indifferent by $\succsim_{(R;y,z)}$, then no cycle can be formed between agents of $A \cap S(R;y,z)$. A similar argument can be applied to the agents of $D \cap S(R;y,z)$ by (ii). In (iii), we have shown that any agent of $D \cap S(R';y,z)$ is ranked by $\succsim_{(R';y,z)}$ above any agent of $A \cap S(R';y,z)$. Therefore, no cycle can be formed between agents of $A \cap S(R';y,z)$ and $D \cap S(R';y,z)$. Thus, $\succsim_{(R';y,z)}$ is acyclic and the proof is finished. □

The implication of the lemma is exactly one of the conditions that Barberà et al. (2010) provide to guarantee the equivalence between SP and GSP: indirect sequential inclusion. Then, Proposition 3.3.1 is proved.

Proof of Proposition 3.3.2

Let $R, R' \in \mathcal{R}$ be such that, for each $i \in N$ and each $x, y \in r_f$, $x\, P_i\, y \Leftrightarrow x\, P'_i\, y$. Suppose by contradiction that $f(R) \neq f(R')$. Starting at R, construct a sequence of profiles in which the preferences of all agents $i \in N$ are changed one-by-one from R_i to R'_i such that the sequence ends at R'. Since $f(R) \neq f(R')$, the outcome of the function must change along this sequence. Let $S \subset N$ be the set of agents that have changed preferences in the sequence the last time the rule selects $f(R)$. That is, $f(R'_S, R_{-S}) = f(R)$. Let $i \in N \setminus S$ be the next agent changing preferences in the sequence. Then, by construction, $f(R'_{S\cup\{i\}}, R_{-(S\cup\{i\})}) \neq f(R'_S, R_{-S})$. If $f(R'_{S\cup\{i\}}, R_{-(S\cup\{i\})})\, P_i\, f(R'_S, R_{-S})$, then agent i manipulates f at (R'_S, R_{-S}) via R'_i. If, however, $f(R'_S, R_{-S})\, P_i\, f(R'_{S\cup\{i\}}, R_{-(S\cup\{i\})})$, we have that $f(R'_S, R_{-S})\, P'_i\, f(R'_{S\cup\{i\}}, R_{-(S\cup\{i\})})$ (because these two alternatives belong to r_f) and agent i manipulates f at $(R'_{S\cup\{i\}}, R_{-(S\cup\{i\})})$ via R_i.

A remark

Proposition 3.3.2 implies that the outcome of f only depends on the preferences over the alternatives in the range of f. Therefore, from now on, it would be sufficient to consider only the preferences over the alternatives of r_f. However, some proofs refer to the preferences over X. Then, sometimes we will make still use of the notation $p(R_i)$ and $d(R_i)$ instead of $p_f(R_i)$ and $d_f(R_i)$, respectively.

For the remaining proofs, we denote the vector of peaks and dips of the agents of S over the entire set of alternatives X at profile R by $o(R_S)$. We define, for each $R \in \mathcal{R}$, $\omega^*(o(R)) = \{x \in r_f : \exists R' \in \mathcal{R}$ such that $o(R') = o(R)$ and $f(R') = x\}$. That is, $\omega * (o(R))$ gives the set of alternatives in the range of f that appear as the outcome of f for the profiles with vector of peaks and dips equal to $o(R)$. Observe that, by definition, $\omega^*(o(R)) \supseteq \omega(p(R))$ for all $R \in \mathcal{R}$.

Proof of Proposition 3.3.3

Consider any $R \in \mathcal{R}$. The proof is divided in six steps.

Step 1: We prove that $|\omega^*(o(R))| \leq 2$.

Alcalde-Unzu and Vorsatz (2018) analyzes the SP rules in a model in which the location of each agent is public information and it is assumed that the peak or dip of her preference is in her location, but the social planner does not know if each agent has single-peaked or single-dipped preferences. Proposition 1 of that paper analyzes the subdomain in which all agents have declared their type of preference (single-peaked or single-dipped) and, then, the social planner already knows the type of preference of each agent and the location of all peaks and dips. That result shows that, after knowing this information, the range is reduced to at most two alternatives. In our model, the social planner already knows the type of preference (single-peaked or single-dipped) of each agent. Observe that, then, the subdomain that arises after knowing the information about the location of the peaks and dips (*i.e.*, $o(R)$) in our model is exactly the same as the one analyzed in Proposition 1 of Alcalde-Unzu and Vorsatz (2018). Then, we can apply that result and we obtain that $|\omega^*(o(R))| \leq 2$. From now on, we denote, for each $R \in \mathcal{R}$, $\min \omega(p_f(R))$ by $\underline{\omega}(p_f(R))$ and $\max \omega(p_f(R))$ by $\overline{\omega}(p_f(R))$.

Step 2: We prove that if $|\omega^*(o(R))| = 2$*, then* $\omega^*(o(R'_i, R_{-i})) = \omega^*(o(R))$ *for each* $i \in D$ *and each* $R'_i \in \mathcal{R}_i$.

We start introducing some notation. We denote, for any $R \in \mathcal{R}$, $N(R) \equiv \{i \in N : o(R_i) \in (\underline{\omega}^*(o(R)), \overline{\omega}^*(o(R)))\}$. The following lemma shows that, fixing a profile of peaks and dips $o(R)$, only the preferences of the agents in $N(R)$ can affect the outcome of f.

Lemma 3.6.2. *Let* f *be SP. For each* $R, R' \in \mathcal{R}$ *such that* $o(R) = o(R')$*, if* $R_{N(R)} = R'_{N(R)}$*, then* $f(R) = f(R')$.

Proof: Suppose by contradiction that there are two profiles $R, R' \in \mathcal{R}$ such that $o(R) = o(R')$ and $R_{N(R)} = R'_{N(R)}$, but $f(R) \neq f(R')$. Assume without loss of generality that $f(R) < f(R')$. Observe that, since $o(R) = o(R')$, $\omega^*(o(R)) = \omega^*(o(R'))$. Given that $f(R) \neq f(R')$, we have that $|\omega^*(o(R))| \neq 1$. Therefore, by Step 1, $|\omega^*(o(R))| = 2$. Then, $f(R) = \underline{\omega}^*(o(R))$ and $f(R') = \overline{\omega}^*(o(R))$. Starting at R, construct the sequence of profiles in which the preferences of all agents $i \in N$ are changed one-by-one from R_i to R'_i so that the sequence ends at R'. Observe that in all profiles of the sequence, the vector of peaks and dips of the agents is the same. Therefore, for each profile of the sequence, the outcome of f is either $\underline{\omega}^*(o(R)))$ or $\overline{\omega}^*(o(R))$. Since $f(R) \neq f(R')$, the outcome must have changed along the sequence. So, let $S \subset N$ be the set of agents that have changed preferences in the sequence the last time the rule selects

$f(R)$, and let $i \in N$ be the next agent changing preferences in the sequence. Then, $f(R) = f(R'_S, R_{-S}) = \underline{\omega}^*(o(R)) \neq \overline{\omega}^*(o(R)) = f(R'_{S\cup\{i\}}, R_{-(S\cup\{i\})}) = f(R')$. Since $R_{N(R)} = R'_{N(R)}$, we have that $i \notin N(R)$. If [$i \in A$ and $p(R_i) \leq \underline{\omega}^*(o(R))$] or [$i \in D$ and $d(R_i) \geq \overline{\omega}^*(o(R))$], then $\underline{\omega}^*(o(R))\, P'_i\, \overline{\omega}^*(o(R))$ and, therefore, agent i manipulates f at $(R'_{S\cup\{i\}}, R_{-(S\cup\{i\})})$ via R_i. Otherwise, if [$i \in A$ and $p(R_i) \geq \overline{\omega}^*(o(R))$] or [$i \in D$ and $d(R_i) \leq \underline{\omega}^*(o(R))$], then $\overline{\omega}^*(o(R))\, P_i\, \underline{\omega}^*(o(R))$ and, therefore, agent i manipulates f at (R'_S, R_{-S}) via R'_i. □

Lemma 3.6.2 implies that if the outcome of ω^* in some profile contains two alternatives, then at least one agent should have her peak or dip in that profile strictly between the two preselected alternatives. Therefore, we deduce that there is at least one alternative of X between the two alternatives of ω^* in that profile. This fact is summed up in the following corollary:

Corollary 3.6.3. *Let f be SP. For each $R \in \mathcal{R}$ such that $|\omega^*(o(R))| = 2$, then $(\underline{\omega}^*(o(R)), \overline{\omega}^*(o(R))) \cap X \neq \emptyset$.*

We now introduce another lemma that shows some restrictions on $\omega^*(o(R'_i, R_{-i}))$ in relation with $\omega^*(o(R))$ for each $R \in \mathcal{R}$, each $i \in D$, and each $R'_i \in \mathcal{R}_i$. We will use in the lemma and in the proof of Proposition 3.3.3 the following notation: $\underline{\omega}^*(o(R))$ and $\overline{\omega}^*(o(R))$ will be denoted by l and r, respectively. Similarly, $\underline{\omega}^*(o(R'_i, R_{-i}))$ and $\overline{\omega}^*(o(R'_i, R_{-i}))$ will be denoted by l' and r', respectively.

Lemma 3.6.4. *Let f be SP and consider any $R \in \mathcal{R}$, $i \in D$ and $R'_i \in \mathcal{R}_i$.*

(i) If $d(R_i) \leq l$, then $l' \in [d(R_i), l]$ and $r' \leq r$.

(ii) If $d(R_i) \geq r$, then $r' \in [r, d(R_i)]$ and $l' \geq l$.

(iii) If $i \in N(R)$, then $\{l', r'\} \subseteq [l, r]$.

Proof: We start proving part (i). Assume that $d(R_i) \leq l$. Suppose first by contradiction that $r' > r$. Consider a preference profile $R'' \in \mathcal{R}$ such that $o(R'') = o(R'_i, R_{-i})$ and $f(R'') = r'$.[7] Since $o(R_i, R''_{-i}) = o(R)$, $f(R_i, R''_{-i}) \in \{l, r\}$ and agent i manipulates f at this profile via R''_i to obtain r'. Then, $r' \leq r$. Suppose now by contradiction that $l' \notin [d(R_i), l]$. If $l' > l$, consider $\hat{R} \in \mathcal{R}$ such that $o(\hat{R}) = o(R)$ and $f(\hat{R}) = l$. Observe that $o(R'_i, \hat{R}_{-i}) = o(R'_i, R_{-i})$ and, therefore, $f(R'_i, \hat{R}_{-i}) \in \{l', r'\}$. Agent i manipulates f at $\hat{R}$ via R'_i. Finally, if $l' < d(R_i)$, consider $\tilde{R}' \in \mathcal{R}$ such that $o(\tilde{R}') = o(R'_i, R_{-i})$ and $f(\tilde{R}') = l'$. Consider also a preference $\tilde{R}_i \in \mathcal{R}_i$ with $d(\tilde{R}_i) = d(R_i)$ and $l'\, \tilde{P}_i\, r$. Observe that $o(\tilde{R}_i, \tilde{R}'_{-i}) = o(R)$ and, therefore, $f(\tilde{R}_i, \tilde{R}'_{-i}) \in \{l, r\}$. Therefore, agent i manipulates f at this profile via $\tilde{R}'_i$ to obtain l'.

[7] It can be checked that all preference profiles and preference rankings introduced in the proofs exist. We omit these parts of the proofs, but they can be provided upon request.

The proof of part (ii) is similar and thus omitted. We prove now part (iii). Assume that $i \in N(R)$ and suppose by contradiction that $\{l', r'\} \not\subseteq [l, r]$. Assume without loss of generality that $l' \notin [l, r]$. Consider $\bar{R}' \in \mathcal{R}$ such that $o(\bar{R}') = o(R'_i, R_{-i})$ and $f(\bar{R}') = l'$. Consider also $\bar{R}_i \in \mathcal{R}_i$ such that $d(\bar{R}_i) = d(R_i)$ and $l' \, \bar{P}_i \, w$ for each $w \in \{l, r\}$. Since $o(\bar{R}_i, \bar{R}'_{-i}) = o(R)$, we have that $f(\bar{R}_i, \bar{R}'_{-i}) \in \{l, r\}$ and agent i manipulates f at this profile via $\bar{R}'_i$ to obtain l'. □

We are now ready to prove Step 2. Consider any $R \in \mathcal{R}$ such that $|\omega^*(o(R))| = 2$. We show that for each $i \in D$ and each $R'_i \in \mathcal{R}_i$, $\{l', r'\} = \{l, r\}$. We divide the proof into five cases.

<u>Case 1</u>: [$d(R_i) \leq l$ and $d(R'_i) \leq l$] or [$d(R_i) \geq r$ and $d(R'_i) \geq r$].

We only prove the case in which $d(R_i) \leq l$ and $d(R'_i) \leq l$ because the other is similar and thus omitted. Assume without loss of generality that $d(R'_i) < d(R_i)$. Given that $d(R_i) \leq l$, we can deduce by part (i) of Lemma 3.6.4 that $l' \in [d(R_i), l]$ and $r' \leq r$. Since $d(R'_i) < d(R_i)$ and we have already deduced that $d(R_i) \leq l'$, we have that $d(R'_i) < l'$. Then, we can apply part (i) of Lemma 3.6.4 (with (R'_i, R_{-i}) and R_i playing the roles of R and R'_i, respectively) to obtain that $l \in [d(R'_i), l']$ and $r \leq r'$. Therefore, $l = l'$ and $r = r'$.

<u>Case 2</u>: $d(R_i) \in (l, r)$ and $d(R'_i) \in (l, r)$.

Given that $i \in N(R)$, we can deduce by part (iii) of Lemma 3.6.4 that $\{l', r'\} \subseteq [l, r]$. If $d(R'_i) \leq l'$, then by part (i) of Lemma 3.6.4 (with (R'_i, R_{-i}) and R_i playing the roles of R and R'_i, respectively) we deduce that $l \in [d(R'_i), l']$. Since $d(R'_i) \in (l, r)$, a contradiction is reached. If $d(R'_i) \geq r'$, we obtain a similar contradiction applying part (ii) of Lemma 3.6.4. Finally, if $i \in N(R')$, we can apply part (iii) of Lemma 3.6.4 (with (R'_i, R_{-i}) and R_i playing the roles of R and R'_i, respectively) to obtain that $\{l, r\} \subseteq [l', r']$. Therefore, $l = l'$ and $r = r'$.

<u>Case 3</u>: $d(R_i) \notin (l, r)$ and $d(R'_i) \in (l, r)$.

Assume without loss of generality that $d(R_i) \leq l$. By part (i) of Lemma 3.6.4 to obtain that $l' \in [d(R_i), l]$ and $r' \leq r$. If $r' \leq d(R'_i)$, we have by part (ii) of Lemma 3.6.4 (with (R'_i, R_{-i}) and R_i playing the roles of R and R'_i, respectively) that $r \in [r', d(R'_i)]$, but this contradicts that $d(R'_i) \in (l, r)$. Then, we have that $r' > d(R'_i)$ and, therefore, $i \in N(R')$. We can now apply part (iii) of Lemma 3.6.4 (with (R'_i, R_{-i}) and R_i playing the roles of R and R'_i, respectively) to obtain that $\{l, r\} \subseteq [l', r']$. Then, we have that $l' = l$ and $r' = r$.

<u>Case 4</u>: $d(R_i) \in (l, r)$ and $d(R'_i) \notin (l, r)$.

Assume without loss of generality that $d(R'_i) \leq l$. Since $i \in N(R)$, we can apply part (iii) of Lemma 3.6.4 to obtain that $\{l', r'\} \subseteq [l, r]$. Given that $d(R'_i) \leq l$ and we have already deduced that $l \leq l'$, then $d(R'_i) \leq l'$. Then, by part (i) of Lemma

3.6.4 (with (R'_i, R_{-i}) and R_i playing the roles of R and R'_i, respectively) we obtain that $r \leq r'$ and, thus, $r' = r$. If $l' \in [l, d(R_i))$, then we have the situation of Case 3 (with (R'_i, R_{-i}) and R_i playing the roles of R and R'_i, respectively) and, thus, $l' = l$ and $r' = r$. Otherwise, if $l' \in [d(R_i), r]$, then we have the situation of Case 1 (with (R'_i, R_{-i}) and R_i playing the roles of R and R'_i, respectively) and, thus, $l' = l$ and $r' = r$.

Case 5: [$d(R_i) \leq l$ and $d(R'_i) \geq r$] or [$d(R_i) \geq r$ and $d(R'_i) \leq l$].

We only prove the case in which $d(R_i) \leq l$ and $d(R'_i) \geq r$ because the other is similar and thus omitted. Since $|\omega^*(o(R))| = 2$, we have that, by Corollary 3.6.3, $(l, r) \cap X \neq \emptyset$. Consider then $\bar{R}_i \in \mathcal{R}_i$ such that $d(\bar{R}_i) \in (l, r)$. Then, by Case 3 (with $\bar{R}_i$ playing the role of R'_i) we obtain that $\omega^*(o(\bar{R}_i, R_{-i})) = \{l, r\}$. Then, applying Case 4 (with $(\bar{R}_i, R_{-i})$ playing the role of R), we obtain that $l' = l$ and $r' = r$.

Step 3: We prove that if $|\omega^*(o(R))| = 1$, then $|\omega^*(o(R'_i, R_{-i}))| = 1$.

Suppose by contradiction that $|\omega^*(o(R))| = 1$ and $|\omega^*(o(R'_i, R_{-i}))| \neq 1$. Then, by Step 1, $|\omega^*(o(R'_i, R_{-i}))| = 2$. Therefore, by Step 2 (with (R'_i, R_{-i}) and R_i playing the roles of R and R'_i, respectively) we deduce that $\omega^*(o(R)) = \omega^*(o(R'_i, R_{-i}))$, which contradicts $|\omega^*(o(R))| = 1$.

Step 4: We prove that if $|\omega^*(o(R))| = 1$, then there is $x \in r_f \setminus \{\omega^*(o(R))\}$ such that $\omega^*(o(R'_S, R_{-S})) = \omega^*(o(R))$ or $\omega^*(o(R'_S, R_{-S})) = x$ for each $S \subseteq D$ and each $R'_S \in \mathcal{R}^S$.

Consider any $S \subseteq D$ and any $R'_S \in \mathcal{R}^S$. Starting at R, construct the sequence of profiles in which the preferences of all agents $i \in S$ are changed one-by-one from R_i to R'_i so that the sequence ends at (R'_S, R_{-S}). Then, since $|\omega^*(o(R))| = 1$, we can apply successive times Step 3 to conclude that the outcome of ω^* in all profiles of the sequence is a singleton. Therefore, $|\omega^*(o(R'_S, R_{-S}))| = 1$.

Consider now the mapping $h_f : \mathcal{R}^D \to r_f$ such that for each $R_D \in \mathcal{R}^D$, $h_f(R_D) = \omega^*(o(R_D, R_A)) = f(R_D, R_A)$. Observe that h_f is well-defined because $|\omega^*(o(R_D, R_A))| = 1$ for each $R_D \in \mathcal{R}^D$ by the previous paragraph. Given that f is SP, we have that h_f is SP too. Note that the domain of h_f is the set of all profiles of single-dipped preferences. Then, we can apply the result in Barberà et al. (2012) to obtain that the range of h_f cannot contain more than two alternatives. Since $\omega^*(o(R))$ is already in the range of h_f, we have that there exists $x \in r_f \setminus \omega^*(o(R))$ such that for each $R_D \in \mathcal{R}^D$, $h_f(R_D) = \omega^*(o(R))$ or $h_f(R_D) = x$. Then, we deduce that there exists $x \in r_f \setminus \omega^*(o(R))$ such that $\omega^*(o(R'_S, R_{-S})) = \omega^*(o(R))$ or $\omega^*(o(R'_S, R_{-S})) = x$ for each $S \subseteq D$ and each $R'_S \in \mathcal{R}^S$.

Step 5: We prove that $|\omega(p(R))| \leq 2$.

We have, by Step 1, that $|\omega^*(o(R))| \leq 2$. Suppose first that $|\omega^*(o(R))| = 2$. For this case, we define $\omega(p(R)) = \omega^*(o(R))$. To prove that $|\omega(p(R))| \leq 2$, we need to prove that $\omega^*(o(R)) = \omega^*(o(R'))$ for each $R' \in \mathcal{R}$ such that $p(R') = p(R)$. Consider then any profile $R' \in \mathcal{R}$ such that $p(R') = p(R)$. Starting at with R, construct the sequence of profiles in which the preferences of the agents $i \in D$ are changed one-by-one from R_i to R'_i so that the sequence ends at (R'_D, R_A). Then, by successive applications of Step 2, we obtain that the outcome of ω^* in all profiles of the sequence is equal to $\omega^*(o(R))$. Therefore, $\omega^*(o(R'_D, R_A)) = \omega^*(o(R))$. Since $p(R') = p(R)$, we have that $o(R') = o(R'_D, R_A)$ and, therefore, $\omega^*(o(R')) = \omega^*(o(R'_D, R_A))$. Thus, $\omega^*(o(R')) = \omega^*(o(R))$.

Suppose now that $|\omega^*(o(R))| = 1$. To prove that $|\omega(p(R))| \leq 2$, we need to prove that there exists $x \in r_f \setminus \omega^*(o(R))$ such that $[\omega^*(o(R')) = \omega^*(o(R))$ or $\omega^*(o(R')) = x]$ for each $R' \in \mathcal{R}$ such that $p(R') = p(R)$. Consider then any profile $R' \in \mathcal{R}$ such that $p(R') = p(R)$. Starting at R, construct the sequence of profiles in which the preferences of the agents $i \in D$ are changed one-by-one from R_i to R'_i so that the sequence ends at (R'_D, R_A). Then, by successive applications of Step 4, we obtain that there exists $x \in r_f \setminus \omega^*(o(R))$ such that the outcome of ω^* in all profiles of the sequence is equal to $\omega^*(o(R))$ or x. Therefore, $\omega^*(o(R'_D, R_A)) = \omega^*(o(R))$ or $\omega^*(o(R'_D, R_A)) = x$. Since $p(R') = p(R)$, we have that $o(R') = o(R'_D, R_A)$ and, therefore, $\omega^*(o(R')) = \omega^*(o(R'_D, R_A))$. Thus, $\omega^*(o(R')) = \omega^*(o(R))$ or $\omega^*(o(R')) = x$. If $\omega^*(o(R')) = x$ for some $R' \in \mathcal{R}$ with $p(R') = p(R)$, then we define $\omega(p(R)) = \{\omega^*(o(R)), x\}$. Otherwise, we define $\omega(p(R)) = \omega^*(o(R))$.

Step 6: We prove that $|\omega(p_f(R))| \leq 2$.

We have, by Step 5, that $|\omega(p(R))| \leq 2$. Then, applying Proposition 2 we obtain that $|\omega(p_f(R))| \leq 2$.

Proof of Proposition 3.3.4

Let f be SP. We define a function ω on r_f^A such that for each $(\alpha_1, \ldots, \alpha_a) \in r_f^A$, $\omega(\alpha_1, \ldots, \alpha_a) = \{x \in r_f : \exists R \in \mathcal{R} \text{ such that } p(R) = (\alpha_1, \ldots, \alpha_a) \text{ and } f(R) = x\}$. By Proposition 3.3.3, we have that for each $(\alpha_1, \ldots, \alpha_a) \in r_f^A$, $\omega(\alpha_1, \ldots, \alpha_a) \in r_f^2$. We now define for each $\{x, y\} \in r_f^2$, a binary decision function $g_{\{x,y\}} : \mathcal{R} \to \{l, r\}$ such that for each $R \in \mathcal{R}$,

$$g_{\{x,y\}}(R) = l \text{ if } \omega(p_f(R)) \neq \{x, y\},$$

$$g_{\{x,y\}}(R) = l \text{ if } \omega(p_f(R)) = \{x, y\} \text{ and } x = y, \text{ and}$$

$$g_{\{x,y\}}(R) = \begin{cases} l & \text{if } \omega(p_f(R)) = \{x, y\},\ x \neq y, \text{ and } f(R) = \underline{\omega}(p_f(R)) \\ r & \text{if } \omega(p_f(R)) = \{x, y\},\ x \neq y, \text{ and } f(R) = \overline{\omega}(p_f(R)). \end{cases}$$

Then, these structures of functions ω and $\{g_{\{x,y\}}\}_{\{x,y\}\in r_f^2}$ allow us to define rule f as follows:

For each $R \in \mathcal{R}$,

$$f(R) = \begin{cases} \underline{\omega}(p_f(R)) & \text{if } g_{\omega(p_f(R))}(R) = l, \\ \overline{\omega}(p_f(R)) & \text{if } g_{\omega(p_f(R))}(R) = r. \end{cases}$$

Necessary results for the remaining proofs

Proposition 3.3.3 allows us to deduce that any SP rule can be decomposed in two steps. In the first step, each agent of A declares her peak and, depending on them, at most two alternatives are preselected. If one alternative is preselected, this is the final outcome of the rule. Otherwise, in the second step, the rule selects between the two preselected alternatives one of them as the final winner. This fact can be summarized in the following proposition.

Proposition 3.6.5. *If f is SP, there is a function $\omega : r_f^A \to r_f^2$ and a set of functions $\{g_{\vec{p}_f} : \mathcal{R} \to \{l,r\}\}_{\vec{p}_f\in r_f^A}$ such that for each $R \in \mathcal{R}$,*

$$f(R) = \begin{cases} \underline{\omega}(p_f(R)) & \text{if } g_{p_f(R)}(R) = l \\ \overline{\omega}(p_f(R)) & \text{if } g_{p_f(R)}(R) = r. \end{cases}$$

Proof: Let f be SP. We define a function ω on r_f^A such that for each $(\alpha_1,\ldots,\alpha_a) \in r_f^A$, $\omega(\alpha_1,\ldots,\alpha_a) = \{x \in r_f : \exists R \in \mathcal{R} \text{ such that } p(R) = (\alpha_1,\ldots,\alpha_a) \text{ and } f(R) = x\}$. By Proposition 3.3.3, we have that for each $(\alpha_1,\ldots,\alpha_a) \in r_f^A$, $\omega(\alpha_1,\ldots,\alpha_a) \in r_f^2$. We now define for each $(\alpha_1,\ldots,\alpha_a) \in r_f^A$, a binary decision function $g_{(\alpha_1,\ldots,\alpha_a)} : \mathcal{R} \to \{l,r\}$ such that for each $R \in \mathcal{R}$,

$$g_{(\alpha_1,\ldots,\alpha_a)}(R) = l \text{ if } p_f(R) \neq (\alpha_1,\ldots,\alpha_a),$$

$$g_{(\alpha_1,\ldots,\alpha_a)}(R) = l \text{ if } p_f(R) = (\alpha_1,\ldots,\alpha_a) \text{ and } \underline{\omega}(p_f(R)) = \overline{\omega}(p_f(R)) \text{ and,}$$

$$g_{(\alpha_1,\ldots,\alpha_a)}(R) = \begin{cases} l & \text{if } p_f(R) = (\alpha_1,\ldots,\alpha_a),\ \underline{\omega}(p_f(R)) \neq \overline{\omega}(p_f(R)), \text{ and } f(R) = \underline{\omega}(p_f(R)) \\ r & \text{if } p_f(R) = (\alpha_1,\ldots,\alpha_a),\ \underline{\omega}(p_f(R)) \neq \overline{\omega}(p_f(R)), \text{ and } f(R) = \overline{\omega}(p_f(R)). \end{cases}$$

Then, these structures of functions ω and $\{g_{(\alpha_1,\ldots,\alpha_a)}\}_{(\alpha_1,\ldots,\alpha_a)\in r_f^A}$ allow us to define rule f as follows:

For each $R \in \mathcal{R}$,

$$f(R) = \begin{cases} \underline{\omega}(p_f(R)) & \text{if } g_{p_f(R)}(R) = l \\ \overline{\omega}(p_f(R)) & \text{if } g_{p_f(R)}(R) = r. \end{cases}$$

□

Proposition 3.6.5 explains the structure of any SP rule in a slight different way than Proposition 3.3.4 in the main text. While in the main text a function g is defined for each $(x, y) \in r_f^2$, here we define a function g for each possible vector of peaks $\vec{p}_f \in r_f^A$. This new definition is less simple but more helpful for the proof of Proposition 3.4.1. The idea is that each SP rule f depends on a set of functions. The first of them, ω, determines the set of alternatives that are preselected when the agents of A have declared their peaks. That is, $\omega(p_f(R))$ gives the alternatives that can be selected by f when the vector of peaks is equal to $p_f(R)$. We know, by Proposition 3.3.3, that this set of preselected alternatives includes at most two alternatives. To choose the final winner, a binary decision function $g_{p_f(R)} : \mathcal{R} \to \{l, r\}$ is applied. If the outcome of $g_{p_f(R)}$ at R is l, then $\underline{\omega}(p_f(R))$ is chosen by the rule and, otherwise, $\overline{\omega}(p_f(R))$ is selected.

To prove Propositions 3.4.1 and 3.4.2, we need to introduce a lemma that imposes some restrictions on the functions $g_{\vec{p}_f}$. We first define, for each $R \in \mathcal{R}$, $L(R) = \{i \in (N(R) \cap A) \cup D : \underline{\omega}(p_f(R)) \, P_i \, \overline{\omega}(p_f(R))\}$. The lemma will show that any binary decision function, associated with a vector of peaks $\vec{p}_f \in r_f^A$, $g_{\vec{p}_f}$, can be defined by specifying a set of coalitions $W(g_{\vec{p}_f}) \subseteq 2^N$, called left-decisive sets, such that $g_{\vec{p}_f}$ chooses l in a profile $R \in \mathcal{R}$ with $p_f(R) = \vec{p}_f$ if $L(R)$ belongs to $W(g_{\vec{p}_f})$, and r otherwise. We introduce a formal definition of these binary decision functions.

Definition 3.6.6. *Given $\vec{p}_f \in r_f^A$, the binary decision function $g_{\vec{p}_f}$ is called a voting by collections of left-decisive sets if there is a set of coalitions $W(g_{\vec{p}_f}) \subseteq 2^N$ such that for each $R \in \mathcal{R}$ with $p_f(R) = \vec{p}_f$,*

$$g_{\vec{p}_f}(R) = \begin{cases} l & \textit{if } L(R) \in W(g_{\vec{p}_f}) \\ r & \textit{otherwise,} \end{cases}$$

and the following conditions are satisfied:

- $W(g_{p_f(R)}) \subseteq 2^{(N(R) \cap A) \cup D}$.
- *If $B \in W(g_{p_f(R)})$ and $B \subset C \subseteq [(N(R) \cap A) \cup D]$, then $C \in W(g_{p_f(R)})$.*
- *If $|\omega(\vec{p}_f)| = 2$, then $\emptyset \notin W(g_{\vec{p}_f}) \neq \emptyset$.*

Observe that Definition 3.6.6 imposes some conditions on the left-decisive sets. The first condition requires that the left-decisive sets of a binary decision function $g_{p_f(R)}$ have to be subsets of $(N(R) \cap A) \cup D$. This condition implies that the decision between $\underline{\omega}(p_f(R))$ and $\overline{\omega}(p_f(R))$ should depend only on the opinion of the agents with single-dipped preferences and those agents with single-peaked preferences whose peaks at R are located between the two preselected alternatives. Observe that Lemma 3.6.2 already established that, after knowing the location of the peaks and dips of all agents, the final decision does not depend on those agents with peak or dip outside the interval between the preselected alternatives. Given that we only know the location of all peaks, then, according to Lemma 3.6.2, we also need to know the location of all dips to choose the final alternative. The second condition, a monotonicity property, says that all supersets of a left-decisive set are also left-decisive sets. Finally, the non-emptiness condition guarantees that both l and r appears as the outcome of $g_{\vec{p}_f}$ at some profiles.

Lemma 3.6.7. *If f is SP, there is a function $\omega : r_f^A \to r_f^2$ and a set of voting by collections of left-decisive sets $\{g_{\vec{p}_f} : \mathcal{R} \to \{l, r\}\}_{\vec{p}_f \in r_f^A}$ such that for each $R \in \mathcal{R}$,*

$$f(R) = \begin{cases} \underline{\omega}(p_f(R)) & \text{if } g_{p_f(R)}(R) = l \\ \overline{\omega}(p_f(R)) & \text{if } g_{p_f(R)}(R) = r. \end{cases}$$

Proof: By Proposition 3.6.5, it only remains to be shown that $g_{\vec{p}_f}$ can be defined as a voting by collections of left-decisive sets. This is equivalent to show that there is a set of coalitions $W(g_{\vec{p}_f}) \subseteq 2^N$ that satisfies the conditions in Definition 3.6.6. Given $\vec{p}_f$, we define, for each $g_{\vec{p}_f}$, a set $W(g_{\vec{p}_f}) \subseteq 2^N$ in the following way: $B \in W(g_{\vec{p}_f})$ if there is $R \in \mathcal{R}$ such that $p_f(R) = \vec{p}_f$, $L(R) = B$, and $g_{\vec{p}_f}(R) = l$. Observe that, by definition, $W(g_{p_f(R)}) \subseteq 2^{(N(R) \cap A) \cup D}$ for each $R \in \mathcal{R}$.

Step 1: We show that if $B \in W(g_{\vec{p}_f})$ for some $\vec{p}_f \in r_f^A$, then for each $R' \in \mathcal{R}$ such that $p_f(R') = \vec{p}_f$ and $L(R') = B$, we can assume that $g_{\vec{p}_f}(R') = l$.

Consider any $\vec{p}_f \in r_f^A$ and any $B \in W(g_{\vec{p}_f})$. Then, there is $R \in \mathcal{R}$ such that $p_f(R) = \vec{p}_f$, $L(R) = B$, and $g_{\vec{p}_f}(R) = l$.

If $|\omega(p_f(R))| = 1$, then consider any $R' \in \mathcal{R}$ such that $p_f(R') = \vec{p}_f$ and $L(R') = B$. Since in this case the outcome of f does not depend on the outcome of $g_{\vec{p}_f}$, we can assume that $g_{\vec{p}_f}(R') = l$. Then, assume from now on that $|\omega(p_f(R))| = 2$. Suppose by contradiction that there exists $\bar{R} \in \mathcal{R}$ such that $p_f(\bar{R}) = \vec{p}_f$, $L(\bar{R}) = B$, but $g_{\vec{p}_f}(\bar{R}) = r$. Starting at R, construct the sequence of profiles in which the preferences of all agents $i \in [(N(R) \cap A) \cup D]$ are changed one-by-one from R_i to $\bar{R}_i$ so that the sequence ends at $(\bar{R}_{(N(R) \cap A) \cup D}, R_{-((N(R) \cap A) \cup D)})$. Since $g_{\vec{p}_f}(R) \neq g_{\vec{p}_f}(\bar{R})$, the outcome must have changed along the sequence. So, let $S \subset [(N(R) \cap A) \cup D]$ be the set of agents that have changed preferences in the sequence the last time $g_{\vec{p}_f}$ selects l, and let $i \in [(N(R) \cap A) \cup D]$ be the next agent changing preferences in the sequence. Then, $g_{\vec{p}_f}(R) = g_{\vec{p}_f}(\bar{R}_S, R_{-S}) = l \neq r = g_{\vec{p}_f}(\bar{R}_{S\cup\{i\}}, R_{-(S\cup\{i\})}) = g_{\vec{p}_f}(\bar{R}_{(N(R)\cap A)\cup D}, R_{-((N(R)\cap A)\cup D)})$. Therefore, $f(R) = f(\bar{R}_S, R_{-S}) = \underline{\omega}(p_f(R)) \neq \overline{\omega}(p_f(R)) = f(\bar{R}_{S\cup\{i\}}, R_{-(S\cup\{i\})}) = f(\bar{R}_{(N(R)\cap A)\cup D}, R_{-((N(R)\cap A)\cup D)})$. If $i \in B$, $\underline{\omega}(p_f(R))\, \bar{P}_i\, \overline{\omega}(p_f(R))$ and agent i manipulates f at $(\bar{R}_{S\cup\{i\}}, R_{-(S\cup\{i\})})$ via R_i. Otherwise, if $i \notin B$, $\overline{\omega}(p_f(R))\, P_i\, \underline{\omega}(p_f(R))$ and agent i manipulates f at $(\bar{R}_S, R_{-S})$ via $\bar{R}_i$.

Step 2: We show that if $B \in W(g_{p_f(R)})$ and $B \subset C \subseteq [(N(R) \cap A) \cup D]$ for some $R \in \mathcal{R}$, then we can assume that $C \in W(g_{p_f(R)})$.

Consider any $R \in \mathcal{R}$, any $B \in W(g_{p_f(R)})$, and any $C \subseteq [(N(R) \cap A) \cup D]$ such that $B \subset C$.

If $|\omega(p_f(R))| = 1$, then we have that in all profiles $\bar{R} \in \mathcal{R}$ such that $p_f(\bar{R}) = p_f(R)$, the outcome of f does not depend on the outcome of $g_{p_f(R)}$ and, thus, we can assume that $g_{p_f(R)}(\bar{R}) = l$ for each $\bar{R} \in \mathcal{R}$ such that $p_f(\bar{R}) = p_f(R)$. Then, there is $\bar{R}' \in \mathcal{R}$ such that $p_f(\bar{R}') = p_f(R)$, $L(\bar{R}') = C$ and $g_{p_f(R)}(\bar{R}') = l$. By definition of $W(g_{p_f(R)})$, we obtain that $C \in W(g_{p_f(R)})$. Assume from now on that $|\omega(p_f(R))| = 2$ and suppose by contradiction that $C \notin W(g_{p_f(R)})$. Consider $R' \in \mathcal{R}$ such that $p_f(R') = p_f(R)$ and $L(R') = B$. Note that since $B \in W(g_{p_f(R)}) = W(g_{p_f(R')})$, then, by Step 1, $g_{p_f(R)}(R') = l$ and, therefore, $f(R') = \underline{\omega}(p_f(R))$. Consider now $R''_{C\setminus B} \in \mathcal{R}^{C\setminus B}$ such that $p_f(R''_{C\setminus B}) = p_f(R_{C\setminus B})$ and $\underline{\omega}(p_f(R))\, P''_j\, \overline{\omega}(p_f(R))$ for each $j \in C \setminus B$. Observe that $p_f(R''_{C\setminus B}, R'_{-(C\setminus B)}) = p_f(R)$ and, then $\omega(p_f((R''_{C\setminus B}, R'_{-(C\setminus B)}) = \omega(p_f(R))$. Since $L(R''_{C\setminus B}, R'_{-(C\setminus B)}) = C \notin W(g_{p_f(R)}) = W(g_{p_f(R''_{C\setminus B}, R'_{-(C\setminus B)})})$, we have that $g_{p_f(R)}(R''_{C\setminus B}, R'_{-(C\setminus B)}) = r$. Therefore, $f(R''_{C\setminus B}, R'_{-(C\setminus B)}) = \overline{\omega}(p_f(R))$. However, the agent set $C \setminus B$ manipulates f at this profile via $R'_{C\setminus B}$ to obtain $\underline{\omega}(p_f(R))$ and f is not GSP. By Proposition 3.3.1, f is not SP and this is a contradiction.

Step 3: We show that if $|\omega(\vec{p}_f)| = 2$ for some $\vec{p}_f \in r_f^A$, then $\emptyset \notin W(g_{\vec{p}_f}) \neq \emptyset$.

Consider any $\vec{p}_f \in r_f^A$ such that $|\omega(\vec{p}_f)| = 2$. We only prove that $W(g_{\vec{p}_f}) \neq \emptyset$ because the other part is similar and thus omitted. To do it, we are going to show that, given $R \in \mathcal{R}$ such that $p_f(R) = \vec{p}_f$, and $L(R) = [(N(R) \cap A) \cup D]$, then $g_{\vec{p}_f}(R) = l$ and, therefore, $[(N(R) \cap A) \cup D] \in W(g_{\vec{p}_f})$. Suppose otherwise that $g_{\vec{p}_f}(R) = r$ and, therefore, $f(R) = \overline{\omega}(p_f(R))$. Since $\underline{\omega}(p_f(R)) \in \omega(p_f(R))$, there is $R' \in \mathcal{R}$ such that $p_f(R') = \vec{p}_f$ and $f(R') = \underline{\omega}(p_f(R))$. Then, starting at R, consider the sequence of profiles in which the preferences of all agents $i \in N$ are changed one-by-one from R_i to R'_i so that the sequence ends at R'. Observe that all profiles of the sequence have the same vector of peaks than R and, therefore, f chooses in all of them $\underline{\omega}(p_f(R))$ or $\overline{\omega}(p_f(R))$. Since $f(R) \neq f(R')$, the outcome must have changed along the sequence. So, let $S \subset N$ be the set of agents that have changed preferences in the sequence the last time the rule selects $f(R)$, and let $i \in N$ be the next agent changing preferences in the sequence. Then, $f(R) = f(R'_S, R_{-S}) = \overline{\omega}(p_f(R)) \neq \underline{\omega}(p_f(R)) = f(R'_{S\cup\{i\}}, R_{-(S\cup\{i\})}) = f(R')$. If $i \in L(R)$, then $\underline{\omega}(p_f(R))\, P_i\, \overline{\omega}(p_f(R))$ and agent i manipulates f at (R'_S, R_{-S}) via R'_i. Otherwise, if $i \notin L(R)$, then $i \in A \setminus (N(R) \cap A)$. If $p(R_i) \leq \underline{\omega}(p_f(R))$, then agent i manipulates f at (R'_S, R_{-S}) via R'_i. If $p(R_i) \geq \overline{\omega}(p_f(R))$, then $p(R'_i) \geq \overline{\omega}(p_f(R))$ and agent i manipulates f at $(R'_{S\cup\{i\}}, R_{-(S\cup\{i\})})$ via R_i. □

Proof of Proposition 3.4.1

Consider any $R \in \mathcal{R}$ such that $|\omega(p_f(R))| = 2$. To prove the proposition, we need a set of lemmas. For all of them, consider any $i \in A$ and any $R'_i \in \mathcal{R}_i$. We denote $\underline{\omega}(p_f(R))$ and $\overline{\omega}(p_f(R))$ by l and r, respectively. Similarly, we denote $\underline{\omega}(p_f(R'_i, R_{-i}))$ and $\overline{\omega}(p_f(R'_i, R_{-i}))$ by l' and r', respectively. Observe that, since $|\omega(p_f(R))| = 2$, $l \neq r$. The objective of this set of lemmas is to show, in Lemma 3.6.11, that if $p_f(R'_i) \in (l, r)$, then $\{l', r'\} = \{l, r\}$.

Lemma 3.6.8. *Let f be SP. If $p_f(R_i) \in (l, r)$, then $\{l', r'\} \cap (l, r) = \emptyset$.*

Proof: Suppose by contradiction that $i \in N(R)$ but $l' \in (l, r)$ (if $r' \in (l, r)$, the proof is similar and thus omitted). Consider $\bar{R}' \in \mathcal{R}$ such that $p_f(\bar{R}') = p_f(R'_i, R_{-i})$ and $f(\bar{R}') = l'$. Consider now $\bar{R}_i \in \mathcal{R}_i$ such that $p_f(\bar{R}_i) = p_f(R_i)$ and $l'\, \bar{P}_i\, v$ for each $v \in \{l, r\}$. Note that $p_f(\bar{R}_i, \bar{R}'_{-i}) = p_f(R)$ and, then, $\omega(p_f(\bar{R}_i, \bar{R}'_{-i})) = \{l, r\}$. Therefore, $f(\bar{R}_i, \bar{R}'_{-i}) \in \{l, r\}$ and agent i manipulates f at this profile via $\bar{R}'_i$ to obtain l'. □

Lemma 3.6.9. *Let f be SP. If $[p_f(R_i) \leq l$, $p_f(R'_i) \in (l, r)$ and $r' = r]$ or $[p_f(R_i) \geq r$, $p_f(R'_i) \in (l, r)$ and $l' = l]$, then $\{l', r'\} = \{l, r\}$.*

Proof: We only show the case in which $p_f(R_i) \leq l$, $p_f(R'_i) \in (l, r)$ and $r' = r$ because the other is similar and thus omitted. Suppose by contradiction that $l' \neq l$.

Step 1: We show that $l' \in (l, p_f(R'_i)]$.

Suppose by contradiction that $l' \notin (l, p_f(R'_i)]$. If $l' < l$, we can apply Lemma 3.6.8 (with R'_i playing the role of R_i and vice versa) to obtain that $l \notin (l', r')$, which is a contradiction. If $l' > p_f(R'_i)$, consider $\bar{R} \in \mathcal{R}$ such that $p_f(\bar{R}) = p_f(R)$ and $f(\bar{R}) = l$. Consider also $\bar{R}'_i \in \mathcal{R}_i$ such that $p_f(\bar{R}'_i) = p_f(R'_i)$ and $l\, \bar{P}'_i\, l'$. Observe that $p_f(\bar{R}'_i, \bar{R}_{-i}) = p_f(R'_i, R_{-i})$ and then, $f(\bar{R}'_i, \bar{R}_{-i}) \in \{l', r'\}$. Therefore, agent i manipulates f at this profile via $\bar{R}_i$ to obtain l.

Step 2: We show that for each $C \in W(g_{p_f(R'_i, R_{-i})})$, $C \cap D \in W(g_{p_f(R)})$.

Suppose by contradiction that there is some $C \in W(g_{p_f(R'_i, R_{-i})})$, but $C \cap D \notin W(g_{p_f(R)})$. Consider a profile $\hat{R} \in \mathcal{R}$ such that:

(*i*) $p_f(\hat{R}) = p_f(R'_i, R_{-i})$,

(ii) for all $j \in [(N(\hat{R}) \cap A) \cup D]$, $l' \, \hat{P}_j \, r \Leftrightarrow j \in C$, and

(iii) for all $k \in [N(R) \cap A] \cup [D \setminus (C \cap D)]$, $r \, \hat{P}_k \, l$.

It can be checked that $L(\hat{R}) = C$. Observe that for each $j \in D$ such that $l' \, \hat{P}_j \, r$, we also have that $l \, \hat{P}_j \, r$. Then, we can deduce that $L(R_i, \hat{R}_{-i}) = C \cap D$. Observe that $p_f(R_i, \hat{R}_{-i}) = p_f(R)$ and then, $\omega(p_f(R_i, \hat{R}_{-i})) = \{l, r\}$. Since $C \in W(g_{p_f(R'_i, R_{-i})}) = W(g_{p_f(\hat{R})})$ and $C \cap D \notin W(g_{p_f(R)}) = W(g_{p_f(R_i, \hat{R}_{-i})})$, we have that $f(\hat{R}) = l'$ and $f(R_i, \hat{R}_{-i}) = r$. Then, since $p_f(R_i) < l' < r$, we have that $l' \, P_i \, r$ and agent i manipulates f at $(R_i, \hat{R}_{-i})$ via $\hat{R}_i$.

Step 3: We show that there exists $B \in W(g_{p_f(R'_i, R_{-i})})$ such that $B \cap D = \emptyset$.

We know, by Step 1, that $l' \in (l, p_f(R_i)]$. The proof is divided into two cases depending on l'.

- Suppose that $l' = p_f(R'_i)$. We know, by Lemma 3.6.7 (exactly by the third point in Definition 3.6.6), that $W(g_{p_f(R)}) \neq \emptyset$. Consider then a coalition $C \in W(g_{p_f(R)})$. Observe that $[C \cap N(R'_i, R_{-i}) \cap A] \cap D = \emptyset$. If we prove that $[C \cap N(R'_i, R_{-i}) \cap A] \in W(g_{p_f(R'_i, R_{-i})})$, then Step 3 will be proved by setting $B = [C \cap N(R'_i, R_{-i}) \cap A]$. Suppose by contradiction that $[C \cap N(R'_i, R_{-i}) \cap A] \notin W(g_{p_f(R'_i, R_{-i})})$. Consider now a profile $\tilde{R} \in \mathcal{R}$ such that

  (i) $p_f(\tilde{R}) = p_f(R)$,

  (ii) for all $j \in (N(\tilde{R}) \cap A)$, $l \, \tilde{P}_j \, r \Leftrightarrow j \in C$,

  (iii) for all $k \in D$, $d(\tilde{R}_k) = l'$ and $l \, \tilde{P}_k \, r \Leftrightarrow k \in C$, and

  (iv) for all $m \in [N(R'_i, \tilde{R}_{-i}) \cap (A \setminus (C \cap A))]$, $r \, \tilde{P}_m \, l'$.

  Consider also $\tilde{R}'_i \in \mathcal{R}_i$ such that $p_f(\tilde{R}'_i) = p_f(R'_i)$ and $l \, \tilde{P}'_i \, r'$. Observe that $p_f(\tilde{R}'_i, \tilde{R}_{-i}) = p_f(R'_i, R_{-i})$ and, then $\omega(p_f(\tilde{R}'_i, \tilde{R}_{-i})) = \{l', r'\}$. It can be checked that $L(\tilde{R}) = C$. Observe that for each $j \in N(R'_i, R_{-i}) \cap A$ such that $l \, \tilde{P}_j \, r$, we also have that $l' \, \tilde{P}_j \, r'$. Then, we can deduce that $L(\tilde{R}'_i, \tilde{R}_{-i}) = C \cap N(R'_i, R_{-i}) \cap A$. Since $C \in W(g_{p_f(R)}) = W(g_{p_f(\tilde{R})})$, but $C \cap N(R'_i, R_{-i}) \cap A \notin W(g_{p_f(R'_i, R_{-i})}) = W(g_{p_f(\tilde{R}'_i, \tilde{R}_{-i})})$, we have that $f(\tilde{R}) = l$ and $f(\tilde{R}'_i, \tilde{R}_{-i}) = r'$. Therefore, agent i manipulates f at $(\tilde{R}'_i, \tilde{R}_{-i})$ via $\tilde{R}_i$.

- Suppose that $l' \in (l, p_f(R'_i))$. Given that $\{i\} \cap D = \emptyset$, if we show that $\{i\} \in W(g_{p_f(R'_i, R_{-i})})$, then Step 3 will be proved by setting $B = \{i\}$. Suppose then by contradiction that $\{i\} \notin W(g_{p_f(R'_i, R_{-i})})$. Consider a profile $R'' \in \mathcal{R}$ such that

  (i) $p_f(R'') = p_f(R'_i, R_{-i})$,

(ii) $l' \, P''_i \, l \, P''_i \, r$,

(iii) for all $j \in D$, $l \, P''_j \, r \, P''_j \, l'$,

(iv) for all $k \in [(N(R'_i, R_{-i}) \cap A) \setminus \{i\}]$, $r \, P''_k \, l'$, and

(v) for all $m \in [(N(R) \setminus N(R'_i, R_{-i})) \cap A]$, $l \, P''_m \, r$.

Observe that $p_f(R_i, R''_{-i}) = p_f(R)$ and then, $\omega(p_f(R_i, R''_{-i})) = \{l, r\}$. It can be checked that $L(R'') = \{i\}$ and $L(R_i, R''_{-i}) = [(N(R) \setminus N(R'_i, R_{-i})) \cap A] \cup D$. Since $\{i\} \notin W(g_{p_f(R'_i, R_{-i})}) = W(g_{p_f(R'')})$, we have that $f(R'') = r'$. We know, by Lemma 3.6.7 (exactly by the second and third points in Definition 3.6.6), that $[(N(R'_i, R_{-i}) \cap A) \cup D] \in W(g_{p_f(R'')}) = W(g_{p_f(R'_i, R_{-i})})$. By Step 2, $[(N(R'_i, R_{-i}) \cap A) \cup D] \cap D = D \in W(g_{p_f(R)}) = W(g_{p_f(R_i, R''_{-i})})$. Given that $D \subseteq [(N(R) \setminus N(R'_i, R_{-i})) \cap A] \cup D$, we have, by Lemma 3.6.7 (exactly by the second point in Definition 3.6.6), that $[(N(R) \setminus N(R'_i, R_{-i})) \cap A] \cup D \in W(g_{p_f(R_i, R''_{-i})})$. Then, $f(R_i, R''_{-i}) = l$. Therefore, agent i manipulates f at R'' via R_i.

Step 4: We find a contradiction.

By Step 3, there exists $B \in W(g_{p_f(R'_i, R_{-i})})$ such that $B \cap D = \emptyset$. Then, by Step 2 (with $B \cap D$ playing the role of C), we have that $(B \cap D) \cap D \in W(g_{p_f(R)})$. Therefore, $\emptyset \in W(g_{p_f(R)})$. Since $|\omega(p_f(R))| = 2$, this contradicts Lemma 3.6.7 (exactly the third point in Definition 3.6.6). □

Lemma 3.6.10. *Let f be SP.*

(i) *If $p_f(R_i) \leq l$, then $l' \geq l$ and $r' \geq r$.*

(ii) *If $p_f(R_i) \geq r$, then $l' \leq l$ and $r' \leq r$.*

Proof: We only prove part (i) because the other is similar and thus omitted. Suppose then that $p_f(R_i) \leq l$. We divide the proof into two steps.

Step 1: We show that if $l \leq l'$, then $r' \geq r$.

Suppose by contradiction that $p_f(R_i) \leq l \leq l'$ and $r' < r$. Consider $\bar{R} \in \mathcal{R}$ such that $p_f(\bar{R}) = p_f(R)$ and $f(\bar{R}) = r$. Observe that $p_f(R'_i, \bar{R}_{-i}) = p_f(R'_i, R_{-i})$ and, therefore, $f(R'_i, \bar{R}_{-i}) \in \{l', r'\}$. Since $p_f(\bar{R}_i) \leq l' \leq r' < r$, $w \, \bar{P}_i \, r$ for all $w \in \{l', r'\}$, and, therefore, agent i manipulates f at $\bar{R}$ via R'_i.

Step 2: We show that $l \leq l'$.

Suppose by contradiction that $l' < l$. First, if $p_f(R_i) < l$, then consider $\bar{R}' \in \mathcal{R}$ such that $p_f(\bar{R}') = p_f(R'_i, R_{-i})$ and $f(\bar{R}') = l'$. Consider also $\bar{R}_i \in \mathcal{R}_i$ such that $p_f(\bar{R}_i) = p_f(R_i)$ and $l' \, \bar{P}_i \, l$. Observe that $p_f(\bar{R}_i, \bar{R}'_{-i}) = p_f(R)$ and, then, $\omega(p_f(\bar{R}_i, \bar{R}'_{-i})) = \{l, r\}$. Therefore, $f(\bar{R}_i, \bar{R}'_{-i}) \in \{l, r\}$ and agent i manipulates f at this profile via $\bar{R}'_i$ to obtain l'. Then, assume from now on that $p_f(R_i) = l$.

If $r' < r$, then consider $\hat{R} \in \mathcal{R}$ such that

(i) $p_f(\hat{R}) = p_f(R)$,

(ii) $l' \, \hat{P}_i \, r$,

(iii) For all $j \in [(N(R) \cap A) \cup D]$, $r \, \hat{P}_j \, l$.

It can be checked that $L(\hat{R}) = \emptyset$ and, then, by Lemma 3.6.7 (exactly by the third point in Definition 3.6.6), $f(\hat{R}) = r$. Observe that $p_f(R'_i, \hat{R}_{-i}) = p_f(R'_i, R_{-i})$ and then, $\omega(p_f(R'_i, \hat{R}_{-i})) = \{l', r'\}$. Therefore, $f(R'_i, \hat{R}_{-i}) \in \{l', r'\}$. Since $w \, \hat{P}_i \, r$ for all $w \in \{l', r'\}$, agent i manipulates f at $\hat{R}$ via R'_i. Then, we have deduced that $r' \geq r$.

Observe that, if $p_f(R'_i) \in (l', r')$, we can deduce by Lemma 3.6.8 (with R'_i playing the role of R_i and vice versa) that $\{l, r\} \cap (l', r') = \emptyset$. However, this contradicts that $l \in (l', r')$. Then, $p_f(R'_i) \notin (l', r')$.

If $p_f(R'_i) \geq r'$, then consider $\hat{R}' \in \mathcal{R}$ such that $p_f(\hat{R}') = p_f(R'_i, R_{-i})$ and $f(\hat{R}') = l'$. Observe that $p_f(R_i, \hat{R}'_{-i}) = p_f(R)$ and then, $\omega(p_f(R_i, \hat{R}'_{-i})) = \{l, r\}$. Therefore, $f(R_i, \hat{R}'_{-i}) \in \{l, r\}$. Since $v \, \hat{P}'_i \, l'$ for each $v \in \{l, r\}$, agent i manipulates f at $\hat{R}'$ via R_i.

Finally, if $p_f(R'_i) \leq l'$, we have that $p_f(R'_i) \leq l' < l$. Then, by Step 1 (with R'_i playing the role of R_i and vice versa), we have that $r \geq r'$. Thus, $r' = r$. Observe that we have $p_f(R'_i) \leq l'$, $p_f(R_i) \in (l', r')$ and $r = r'$. Then, applying Lemma 3.6.9 (with R'_i playing the role of R_i and vice versa), we have that $\{l, r\} = \{l', r'\}$, which contradicts that $l' < l$. □

Lemma 3.6.11. *Let f be SP. If $p_f(R'_i) \in (l, r)$, then $\{l', r'\} = \{l, r\}$.*

Proof: First, if $r' \leq l$, we can apply Lemma 3.6.10 (exactly part (ii) with R'_i playing the role of R_i and vice versa) to obtain that $l \leq l'$ and $r \leq r'$. However, this contradicts that $r' < r$. If $l' \geq r$, a similar contradiction is reached applying part (i) of Lemma 3.6.10. Then, we can assume from now on that $l' < r$ and $r' > l$.

If $i \in N(R)$, by Lemma 3.6.8, we have that $\{l', r'\} \cap (l, r) = \emptyset$ and, thus, $l' \leq l$ and $r' \geq r$. Observe then that $i \in N(R'_i, R_{-i})$ and we can apply Lemma 3.6.8 (with R'_i playing the role of R_i and vice versa) to obtain $\{l, r\} \cap (l', r') = \emptyset$. Therefore, $\{l', r'\} = \{l, r\}$.

If $i \notin N(R)$, we can assume without loss of generality that $p_f(R_i) \leq l$. By Lemma 3.6.10 (exactly part (i)), $l' \geq l$ and $r' \geq r$. Therefore, $l' \in [l, r)$ and $r' \geq r$. We distinguish two cases. If $l' \geq p_f(R'_i)$, we can apply Lemma 3.6.10 (exactly part (i) with R'_i playing the role of R_i and vice versa) to obtain that $l \geq l'$ and $r \geq r'$. However, this contradicts that $l < p_f(R'_i) \leq l'$. Then, we can assume from now on that $l' \in [l, p_f(R'_i))$. Observe then that $i \in N(R'_i, R_{-i})$. By Lemma 3.6.8 (with R'_i playing the role of R_i and vice versa), $\{l, r\} \cap (l', r') = \emptyset$. Since we already know that $r' \geq r$, we conclude that $r' = r$. Observe that $p_f(R_i) \leq l$, $p_f(R'_i) \in (l, r)$ and $r' = r$. Then, applying Lemma 3.6.9, we obtain that $\{l', r'\} = \{l, r\}$. □

We are now ready to prove Proposition 4. Remember that $|\omega(p_f(R))| = 2$. Suppose by contradiction that there exists $x \in r_f \cap (l, r)$. Then, there exists a profile $\bar{R} \in \mathcal{R}$ such that $x \in \omega(p_f(\bar{R}))$. By Proposition 3.3.3, $|\omega(p_f(\bar{R}))| \leq 2$. Denote $\omega(p_f(\bar{R})) = \{x, y\}$, with $x \leq y$ without loss of generality.

Consider a subprofile $\hat{R}_A \in \mathcal{R}^A$ such that $p_f(\hat{R}_i) = x$ for all $i \in A$. Starting at R, construct the sequence of profiles in which the preferences of all agents $i \in A$ are changed one-by-one from R_i to $\hat{R}_i$ so that the sequence ends at $(\hat{R}_A, R_D)$. By successive applications of Lemma 3.6.11 we obtain that $\omega(p_f(\hat{R}_A, R_D)) = \{l, r\}$. Consider now a profile $\tilde{R} \in \mathcal{R}$ such that $p_f(\tilde{R}) = p_f(\bar{R})$ and $f(\tilde{R}) = x$. Since $p_f(\hat{R}_A, \tilde{R}_D) = p_f(\hat{R}_A, R_D)$, we have that $\omega(p_f(\hat{R}_A, \tilde{R}_D)) = \{l, r\}$ and, then $f(\hat{R}_A, \tilde{R}_D) \in \{l, r\}$. Therefore, the agent set A manipulates f at this profile via $\tilde{R}_A$ to obtain x. Then f is not GSP and, by Proposition 3.3.1, f is not SP, which is a contradiction.

Proof of Proposition 3.4.2

Consider any $x \in r_f \setminus \{\min r_f, \max r_f\}$ and suppose by contradiction that $\omega(p_f(R)) \neq x$ for each $R \in \mathcal{R}$. Since $x \in r_f$, there exists $R' \in \mathcal{R}$ such that $\omega(p_f(R')) = \{x, y\}$ for some $y \in r_f \setminus \{x\}$. Assume w.l.o.g. that $x < y \leq \max r_f$, that is, $\omega(p_f(R')) = (x, y)$. By Proposition 3.4.1, $(x, y) \cap r_f = \emptyset$. Since $x > \min r_f$ and r_f is countable (because X is countable), there is $z \in r_f$ such that $\min r_f \leq z < x$ and $(z, x) \cap r_f = \emptyset$. Then, there exists $\bar{R} \in \mathcal{R}$ such that $z \in \omega(p_f(\bar{R}))$.

Consider also $\tilde{R} \in \mathcal{R}$ such that

(i) for all $i \in A$, $p_f(\tilde{R}_i) = x$ and $w \, \tilde{P}_i \, y$ for each $w \leq x$, and

(ii) for all $j \in D$, $d_f(\tilde{R}_j) = y$.

We first show that $x \in \omega(p_f(\tilde{R}))$. Suppose by contradiction that $x \notin \omega(p_f(\tilde{R}))$ and, therefore, $f(\tilde{R}) \neq x$. Consider profile $(R'_A, \tilde{R}_D) \in \mathcal{R}$. Since $p_f(R'_A, \tilde{R}_D) = p_f(R')$, we have that $\omega(p_f(R'_A, \tilde{R}_D)) = (x, y)$. Given that, by Proposition 3.4.1, $N(R'_A, \tilde{R}_D) \cap A = \emptyset$, then, by construction, $L(R'_A, \tilde{R}_D) = D$ and, by Lemma 3.6.7 (exactly by a combination of the second and third points in Definition 3.6.6), we have that $f(R'_A, \tilde{R}_D) = x$. Then, the agent set A manipulates f at $\tilde{R}$ via R'_A to obtain x and f is not GSP. By Proposition 3.3.1, f is not SP, which is a contradiction. Then, we have deduced that $x \in \omega(p_f(\tilde{R}))$.

Since $\omega(p_f(\tilde{R})) \neq x$ by assumption, we have, by Proposition 3.4.1, that $\omega(p_f(\tilde{R})) \in \{(z, x), (x, y)\}$. If $\omega(p_f(\tilde{R})) = (z, x)$ and, given that, by Proposition 3.4.1, $N(\tilde{R}) \cap A = \emptyset$, then, by construction, we have that $L(\tilde{R}) = D$. Therefore, by Lemma 3.6.7 (exactly by a combination of the second and third points in Definition 3.6.6), that $f(\tilde{R}) = z$. Consider now profile $(R'_A, \tilde{R}_D) \in \mathcal{R}$. Since $p_f(R'_A, \tilde{R}_D) = p_f(R')$, we have that $\omega(p_f(R'_A, \tilde{R}_D)) = (x, y)$. By Proposition 3.4.1, $N(R'_A, \tilde{R}_D) \cap A = \emptyset$ and, then, by construction, we have that $L(R'_A, \tilde{R}_D) = D$. Therefore, by Lemma 3.6.7 (exactly by a combination of the second and third points in Definition 3.6.6), we have that $f(R'_A, \tilde{R}_D) = x$. Since $x\,\tilde{P}_i\,z$ for each $i \in A$, then the agent set A manipulates f at $\tilde{R}$ via R'_A to obtain x and f is not GSP. By Proposition 3.3.1, f is not SP, which is a contradiction.

Finally, suppose that $\omega(p(\tilde{R})) = (x, y)$. Consider a profile $\tilde{R}' \in \mathcal{R}$ such that

(i) $p_f(\tilde{R}') = p_f(\tilde{R})$, and

(ii) for all $j \in D$, $d_f(\tilde{R}'_j) = x$.

Since, by Proposition 3.4.1, $N(\tilde{R}') \cap A = \emptyset$, we have that, by construction, $L(\tilde{R}') = \emptyset$. Therefore, by Lemma 3.6.7 (exactly by the third point in Definition 3.6.6), $f(\tilde{R}') = y$. Consider now $(\bar{R}_A, \tilde{R}'_D) \in \mathcal{R}$. Since $p_f(\bar{R}_A, \tilde{R}'_D) = p_f(\bar{R})$, then $\omega(p_f(\bar{R}_A, \tilde{R}'_D)) = \omega(p_f(\bar{R}))$. Given that $z \in \omega(p_f(\bar{R}))$, we have that $\omega(p_f(\bar{R}_A, \tilde{R}'_D)) \in \{z, (u, z), (z, x)\}$ with $u < z$. If $\omega(p_f(\bar{R}_A, \tilde{R}'_D)) = z$, we have that $|\omega(p_f(\bar{R}_A, \tilde{R}'_D)) = z$. If $\omega(p_f(\bar{R})) = (u, z)$ with $u < z$, observe that, by construction, $L(\bar{R}_A, \tilde{R}'_D) = D$ and, then, by Lemma 3.6.7 (exactly the third point in Definition 3.6.6), $f(\bar{R}_A, \tilde{R}'_D) = u$. Finally, if $\omega(p_f(\bar{R}_A, \tilde{R}'_D)) = (z, x)$ and, since, by Proposition 3.4.1, $N(\bar{R}_A, \tilde{R}'_D) \cap A = \emptyset$, then, by construction, $L(\bar{R}_A, \tilde{R}'_D) = D$. Therefore, by Lemma 3.6.7 (exactly by a combination of the second and third points in Definition 3.6.6), we have that $f(\bar{R}_A, \tilde{R}'_D) = z$. Since for each $i \in A$, $w\,\tilde{P}'_i\,y$ for each $w \leq x$, the agent set A manipulates f at $\tilde{R}'$ via $\bar{R}_A$ to obtain u or z and f is not GSP. By Proposition 3.3.1, f is not SP, which is a contradiction.

Proof of Proposition 3.4.5

Given a SP rule f, we include $\min r_f$ in T_f if for all $R \in \mathcal{R}$ such that $p_f(R_i) = \min r_f$ for all $i \in A$, $f(R) = \min r_f$. Similarly, we include $\max r_f$ in T_f if for all $R \in \mathcal{R}$ such that $p_f(R_i) = \max r_f$ for all $i \in A$, $f(R) = \max r_f$. Once we have T_f defined, we define a correspondence $\mathcal{L} : T_f \to 2^A$ such that for each $\alpha \in T_f$, $C \in \mathcal{L}(\alpha)$ if [$C \in \mathcal{L}(\beta)$ for some $\beta <^* \alpha$] or [there is $R \in \mathcal{R}$ such that $\omega(p_f(R)) = \alpha$ and $(p_f(R_i) \leq^* \alpha \Leftrightarrow i \in C)$].

We now prove a lemma.

Lemma 3.6.12. *Let f be SP and consider any $\alpha \in T_f$ and any $C \in \mathcal{L}(\alpha) \setminus \mathcal{L}(\beta)$ for each $\beta <^* \alpha$. Then:*

(i) *for each $R' \in \mathcal{R}$ such that [if $i \in C$, then $p_f(R'_i) \leq^* \alpha$], we have that $\omega(p_f(R')) \leq^* \alpha$.*

(ii) *for each $R' \in \mathcal{R}$ such that [if $p_f(R'_i) \leq^* \alpha$, then $i \in C$], we have that $\omega(p_f(R')) \geq^* \alpha$*

Proof: We only prove (i) because (ii) is similar and thus omitted. Let $\alpha \in T_f$ and $C \in \mathcal{L}(\alpha)\setminus\mathcal{L}(\beta)$ for each $\beta <^* \alpha$. Then, there is $R \in \mathcal{R}$ such that $[p_f(R_i) \leq^* \alpha \Leftrightarrow i \in C]$ and $\omega(p_f(R)) = \alpha$. Consider $R' \in \mathcal{R}$ such that [if $i \in C$, then $p_f(R'_i) \leq^* \alpha$] and suppose by contradiction that $\omega(p_f(R')) >^* \alpha$. Let $C' \equiv \{i \in A : p_f(R'_i) \leq^* \alpha\}$. Note that $C \subseteq C'$. Consider now $\bar{R}, \bar{R}', \hat{R}' \in \mathcal{R}$ such that

(i) $p_f(\bar{R}) = p_f(R)$ and $p_f(\bar{R}') = p_f(\hat{R}') = p_f(R')$,

(ii) $f(\bar{R}) = \underline{\omega}(p_f(R))$ and $f(\hat{R}') = \overline{\omega}(p_f(R'))$,

(iii) for each $j \in A$ and each $v, w \in r_f$ with $v \leq^* \alpha <^* w$, $[v\, \bar{P}_j\, w \Leftrightarrow j \in C]$, and

(iv) for each $j \in A$ and each $v, w \in r_f$ with $v \leq^* \alpha <^* w$, $[v\, \bar{P}'_j\, w \Leftrightarrow v\, \hat{P}'_j\, w \Leftrightarrow j \in C']$.

We first show that $f(\bar{R}'_A, \bar{R}_D) \leq^* \underline{\omega}(p_f(R))$. Suppose by contradiction that $f(\bar{R}'_A, \bar{R}_D) >^* \underline{\omega}(p_f(R))$. Starting at $\bar{R}$, construct a sequence of profiles in which the preferences of all agents $i \in A$ are changed one-by-one from $\bar{R}_i$ to $\bar{R}'_i$ such that the sequence ends at $(\bar{R}'_A, \bar{R}_D)$. Since $f(\bar{R}) = \underline{\omega}(p_f(R)) <^* f(\bar{R}'_A, \bar{R}_D)$, the outcome of the function must change from $\underline{\omega}(p_f(R))$ or an alternative to the left of $\underline{\omega}(p_f(R))$ to an alternative to the right of $\underline{\omega}(p_f(R))$ along this sequence. Let $S \subset A$ be the set of agents that have changed preferences in the sequence the last time f selects an alternative to the left or equal to $\underline{\omega}(p_f(R))$. That is, $f(\bar{R}'_S, \bar{R}_{-S}) \leq^* \underline{\omega}(p_f(R))$. Let $i \in A \setminus S$ be the next agent changing preferences in the sequence. Then, by construction, $f(\bar{R}'_{S\cup\{i\}}, \bar{R}_{-(S\cup\{i\})}) >^* \underline{\omega}(p_f(R))$. If $i \in C'$, then $f(\bar{R}'_S, \bar{R}_{-S})\, \bar{P}'_i\, f(\bar{R}'_{S\cup\{i\}}, \bar{R}_{-(S\cup\{i\})})$ given that, by construction, $[v\, \bar{P}'_i\, w \Leftrightarrow i \in C']$ for each $v, w \in r_f$ with $v \leq^* \alpha <^* w$. Therefore, agent i manipulates f at $(\bar{R}'_{S\cup\{i\}}, \bar{R}_{-(S\cup\{i\})})$ via $\bar{R}_i$. Otherwise, if $i \in A \setminus C'$, then $f(\bar{R}'_{S\cup\{i\}}, \bar{R}_{-(S\cup\{i\})})\, \bar{P}_i\, f(\bar{R}'_S, \bar{R}_{-S})$ given that $[v\, \bar{P}_i\, w \Leftrightarrow i \in C]$ for each $v, w \in r_f$ with $v \leq^* \alpha <^* w$ and $C \subseteq C'$. Therefore, agent i manipulates f at $(\bar{R}'_S, \bar{R}_{-S})$ via $\bar{R}'_i$. Hence, $f(\bar{R}'_A, \bar{R}_D) \leq^* \underline{\omega}(p_f(R))$.

Since $\omega(p_f(R')) >^* \alpha$ and $p_f(\bar{R}'_A, \bar{R}_D) = p_f(R')$, then we have $\omega(p_f(\bar{R}'_A, \bar{R}_D)) >^* \alpha$. If $[|\omega(p_f(R))| = 2]$ or $[|\omega(p_f(R))| = 1$ and $\omega(p_f(R)) \notin \omega(p_f(\bar{R}'_A, \bar{R}_D))]$, then $f(\bar{R}'_A, \bar{R}_D) >^* \underline{\omega}(p_f(R))$ and this is a contradiction. Suppose then from now on that $|\omega(p_f(R))| = 1$ and $\omega(p_f(R)) \in \omega(p_f(\bar{R}'_A, \bar{R}_D))$. Then we have that $\omega(p_f(R)) = \alpha$ and $\omega(p_f(\bar{R}'_A, \bar{R}_D)) = (\alpha, \gamma)$, with $\gamma > \alpha$. Since $f(\hat{R}') = \overline{\omega}(p_f(R'))$ and $p_f(\hat{R}') = p_f(R') = p_f(\bar{R}'_A, \bar{R}_D)$, then $f(\hat{R}') = \gamma$.

We now show that $f(\bar{R}_A, \hat{R}'_D) \geq^* \gamma$. Suppose by contradiction that $f(\bar{R}_A, \hat{R}'_D) <^* \gamma$. Starting at $\hat{R}'$, construct a sequence of profiles in which the preferences of all agents $i \in A$ are changed one-by-one from $\hat{R}'_i$ to $\bar{R}_i$ such that the sequence ends at $(\bar{R}_A, \hat{R}'_D)$. Since $f(\hat{R}') = \gamma >^* f(\bar{R}_A, \hat{R}'_D)$, the outcome of the function must change from γ or an alternative to the right of γ to an alternative to the left of γ

along this sequence. Let $S \subset A$ be the set of agents that have changed preferences in the sequence the last time f selects an alternative to the right or equal to γ. That is, $f(\bar{R}_S, \hat{R}'_{-S}) \geq^* \gamma$. Let $i \in A \setminus S$ be the next agent changing preferences in the sequence. Then, by construction, $f(\bar{R}_{S\cup\{i\}}, \hat{R}'_{-(S\cup\{i\})}) <^* \gamma$. If $i \in C'$, then $f(\bar{R}_{S\cup\{i\}}, \hat{R}'_{-(S\cup\{i\})})\, \hat{P}'_i\, f(\bar{R}_S, \hat{R}'_{-S})$ given that, by construction, $[v\, \hat{P}'_i\, w \Leftrightarrow i \in C']$ for each $v, w \in r_f$ with $v \leq^* \alpha <^* w$. Therefore, agent i manipulates f at $(\bar{R}_S, \hat{R}'_{-S})$ via $\bar{R}_i$. Otherwise, if $i \in A \setminus C'$, then $f(\bar{R}_S, \hat{R}'_{-S})\, \bar{P}_i\, f(\bar{R}_{S\cup\{i\}}, \hat{R}'_{-(S\cup\{i\})})$ given that $[v\, \bar{P}_i\, w \Leftrightarrow i \in C]$ for each $v, w \in r_f$ with $v \leq^* \alpha <^* w$ and $C \subseteq C'$. Therefore, agent i manipulates f at $(\bar{R}_{S\cup\{i\}}, \hat{R}'_{-(S\cup\{i\})})$ via $\hat{R}'_i$. Hence, $f(\bar{R}_A, \hat{R}'_D) \geq^* \gamma$ and, therefore $f(\bar{R}_A, \hat{R}'_D) >^* \alpha$.

Finally, since $\omega(p_f(R)) = \alpha$ and $p_f(\bar{R}_A, \hat{R}'_D) = p_f(R)$, we have that $f(\bar{R}_A, \hat{R}'_D) = \alpha$, which is a contradiction. □

We now show that Definition 3.4.4 is satisfied.

$\Leftarrow$) Observe that the union of conditions (i) and (ii) of Lemma 3.6.12 implies that, for each $R \in \mathcal{R}$ and each $\alpha \in T_f$, if $\{i \in A : p_f(R_i) \leq^* \alpha\} \in \mathcal{L}(\alpha) \setminus \mathcal{L}(\beta)$ for all $\beta <^* \alpha$, then $\omega(p_f(R)) = \alpha$.

$\Rightarrow$) Consider now any $\alpha \in T_f$, any $\vec{p}_f \in r_f^A$ such that $\omega(\vec{p}_f) = \alpha$ and any $R \in \mathcal{R}$ with $p_f(R) = \vec{p}_f$. We have to show that $\{i \in A : p_f(R_i) \leq^* \alpha\} \in \mathcal{L}(\alpha)$ and $\{i \in A : p_f(R_i) \leq^* \beta\} \notin \mathcal{L}(\beta)$ for all $\beta <^* \alpha$. The facts that $\omega(\vec{p}_f) = \alpha$ and $p_f(R) = \vec{p}_f$ imply that $\omega(p_f(R)) = \alpha$. Then, by definition of $\mathcal{L}$, we have that $\{i \in A : p_f(R_i) \leq^* \alpha\} \in \mathcal{L}(\alpha)$. Suppose now by contradiction that $\{i \in A : p_f(R_i) \leq^* \beta\} \in \mathcal{L}(\beta)$ for some $\beta <^* \alpha$. Then, $\omega(p_f(R)) \leq^* \beta$ and this is a contradiction.

Finally, we show that $\mathcal{L}$ satisfies the conditions in Definition 3.4.3 and then, it is a left coalition system.

Observe first that condition (i) of Lemma 3.6.12 implies that if $C \in \mathcal{L}(\alpha)$ for some $\alpha \in T_f$, then $C' \in \mathcal{L}(\alpha)$ for each $C' \supseteq C$. Then, condition (i) in Definition 3.4.3 is satisfied. Moreover, by definition of $\mathcal{L}$, if $C \in \mathcal{L}(\alpha)$ for some $\alpha \in T_f$, then $C \in \mathcal{L}(\beta)$ for all $\beta >^* \alpha$. Therefore, condition (ii) in Definition 3.4.3 is also satisfied.

Suppose now that r_f has a maximum and $\max r_f \notin T_f$ and we show that $\emptyset \in \mathcal{L}(\max T_f) \setminus \mathcal{L}(\alpha)$ for each $\alpha \in T_f \setminus \{\max T_f\}$. We first show that $\emptyset \in \mathcal{L}(\alpha)$ for some $\alpha \in T_f$. Suppose by contradiction that $\emptyset \notin \mathcal{L}(\alpha)$ for any $\alpha \in T_f$. Consider $R \in \mathcal{R}$ such that $p_f(R_i) = \max r_f$ for each $i \in A$. Then, $\{i \in A : p_f(R_i) \leq^* \alpha\} = \emptyset$ for each $\alpha \in T_f$. Since $\emptyset \notin \mathcal{L}(\alpha)$ for any $\alpha \in T_f$, we have that $\omega(p_f(R)) \notin T_f$, which is a contradiction. Hence, $\emptyset \in \mathcal{L}(\alpha)$ for some $\alpha \in T_f$. Second, we show that $\emptyset \notin \mathcal{L}(\alpha)$ for each $\alpha \in T_f \setminus \{\max T_f\}$. Suppose by contradiction that $\emptyset \in \mathcal{L}(\alpha)$ for some $\alpha \in T_f \setminus \{\max T_f\}$. Then, by condition ($i$) in Definition 3.4.3, we have that $\mathcal{L}(\alpha) = 2^A$. Then, $\omega(p_f(R)) \leq^* \alpha$ for each $R \in \mathcal{R}$. Since $\max r_f \notin T_f$, we have

$\max T_f = (y, \max r_f)$ with $y < \max r_f$. Given that $\alpha \in T_f \setminus \{\max T_f\}$, we have that $\alpha \leq^* y$. Therefore, $\omega(p_f(R)) \leq^* y$ for each $R \in \mathcal{R}$, which contradicts that $\max r_f$ belongs to r_f. Hence, condition (iii) in Definition 3.4.3 is also satisfied.

If, in contrast, r_f does not have a maximum, then we show that $\emptyset \notin \mathcal{L}(\alpha)$ for each $\alpha \in T_f$. Suppose by contradiction that $\emptyset \in \mathcal{L}(\alpha)$ for some $\alpha \in T_f$. Then, by condition (i) in Definition 3.4.3, we have that $\mathcal{L}(\alpha) = 2^A$. Then, we have that $\omega(p_f(R)) \leq^* \alpha$ for each $R \in \mathcal{R}$. Note that, since we also know that X is countable, we can deduce that T_f has a maximum. Therefore, r_f also has a maximum, which is a contradiction. Then, condition (iv) in Definition 3.4.3 is also satisfied.

Proof of Proposition 3.4.7

Given a generalized median voter function ω on a set T_f, with $[V_f \cup (r_f \setminus \{\min r_f, \max r_f\})] \subseteq T_f \subseteq V_f \cup r_f$, we have to prove that $g_{(x,y)}$ is a voting by collections of left-decisive sets for each $(x, y) \in V_f$. This is equivalent to show that for each $(x, y) \in V_f$, there is a minimal set of coalitions $W(g_{(x,y)}) \in 2^N$ that satisfies the conditions in Definition 3.4.6. We define first a set $W^*(g_{(x,y)}) \subseteq 2^N$ in the following way: $C \in W^*(g_{(x,y)})$ if there is $R \in \mathcal{R}$ such that $\omega(p_f(R)) = (x, y)$, $L_{(x,y)}(R) = C$ and $g_{(x,y)}(R) = l$ (and thus $f(R) = x$). Observe that, by definition, for each $C \in W^*(g_{(x,y)})$, $C \cap A \in \mathcal{L}(x, y) \setminus \mathcal{L}(x)$. Now, we define $W(g_{(x,y)})$ as the set of the minimal coalitions of $W^*(g_{(x,y)})$. Then, we have that $W(g_{(x,y)})$ is a minimal set of coalitions and that for each $C \in W(g_{(x,y)})$, $C \cap A \in \mathcal{L}(x, y) \setminus \mathcal{L}(x)$.

Step 1: We show that if $C \in W(g_{(x,y)})$, then for each $R' \in \mathcal{R}$ such that $\omega(p_f(R')) = (x, y)$ and $C \subseteq L_{(x,y)}(R')$, we have that $g_{(x,y)}(R') = l$.

Suppose by contradiction that $C \in W(g_{(x,y)})$ but there is $\bar{R} \in \mathcal{R}$ such that $\omega(p_f(\bar{R})) = (x, y)$, $C \subseteq L_{(x,y)}(\bar{R})$ and $g_{(x,y)}(\bar{R}) = r$ (and thus $f(\bar{R}) = y$). Since $C \in W(g_{(x,y)})$, there is $R \in \mathcal{R}$ such that $\omega(p_f(R)) = (x, y)$, $L_{(x,y)}(R) = C$ and $g_{(x,y)}(R) = l$ (and thus $f(R) = x$). Suppose first that $L_{(x,y)}(\bar{R}) = C$ and consider $(\bar{R}_C, R_{-C}) \in \mathcal{R}$. Note that $\{i \in A : p_f((\bar{R}_C, R_{-C})_i) \leq^* (x, y)\} = \{i \in A : p_f(R_i) \leq^* (x, y)\} \in \mathcal{L}(x, y) \setminus \mathcal{L}(x)$. Then, we have by Proposition 3.4.5 that $\omega(p_f(\bar{R}_C, R_{-C})) = (x, y)$. If $g_{(x,y)}(\bar{R}_C, R_{-C}) = l$, then $f(\bar{R}_C, R_{-C}) = x$ and the agent set $N \setminus C$ manipulates f at this profile via $\bar{R}_{-C}$. Otherwise, if $g_{(x,y)}(\bar{R}_C, R_{-C}) = r$, then $f(\bar{R}_C, R_{-C}) = y$ and the agent set C manipulates f at this profile via R_C. In both cases, f is not GSP and, by Proposition 3.3.1, f is not SP, which is a contradiction. Hence, observe that we have deduced until now that for each $R' \in \mathcal{R}$ such that $\omega(p_f(R')) = (x, y)$ and $L_{(x,y)}(R') = C$, we have that $g_{(x,y)}(R') = l$ (and thus $f(R') = x$).

Suppose now that $L_{(x,y)}(\bar{R}) \neq C$, that is, $C \subset L_{(x,y)}(\bar{R})$. Denote $B = L_{(x,y)}(\bar{R})$ and observe that $C \subset B$. Consider $(\bar{R}_C, R_{-C}) \in \mathcal{R}$. Note that $\{i \in A : p_f((\bar{R}_C, R_{-C})_i) \leq^*$

$(x,y)\} = \{i \in A : p_f(R_i) \leq^* (x,y)\} \in \mathcal{L}(x,y) \setminus \mathcal{L}(x)$. Then, we have by Proposition 3.4.5 that $\omega(p_f(\bar{R}_C, R_{-C})) = (x,y)$. Given that $L_{(x,y)}(\bar{R}_C, R_{-C}) = C$, we have, by the previous paragraph, that $g_{(x,y)}(\bar{R}_C, R_{-C}) = l$ and thus, $f(\bar{R}_C, R_{-C}) = x$. Consider now $(\bar{R}_B, R_{-B}) \in \mathcal{R}$. Note that $\{i \in A : p_f((\bar{R}_B, R_{-B})_i) \leq^* (x,y)\} = \{i \in A : p_f(\bar{R}_i) \leq^* (x,y)\} \in \mathcal{L}(x,y) \setminus \mathcal{L}(x)$. Then, we have by Proposition 3.4.5 that $\omega(p_f(\bar{R}_B, R_{-B})) = (x,y)$. If $g_{(x,y)}(\bar{R}_B, R_{-B}) = l$, then $f(\bar{R}_B, R_{-B}) = x$ and the agent set $N \setminus B$ manipulates f at this profile via $\bar{R}_{N\setminus B}$. Otherwise, if $g_{(x,y)}(\bar{R}_B, R_{-B}) = r$, then $f(\bar{R}_B, R_{-B}) = y$ and the agent set $B \setminus C$ manipulates f at this profile via $R_{B\setminus C}$. In both cases, f is not GSP and, by Proposition 3.3.1, f is not SP, which is a contradiction.

Step 2: We show that for each $C \in W(g_{(x,y)})$, $C \cap D \neq \emptyset$.

Suppose by contradiction that there is $C \in W(g_{(x,y)})$ such that $C \cap D = \emptyset$. Then, by definition, $C \cap A = C \in \mathcal{L}(x,y) \setminus \mathcal{L}(x)$. Consider any $\vec{p}_f \in r_f^A$ such that for each $i \in A$, $(\vec{p}_f)_i \leq^* (x,y) \Leftrightarrow i \in C$. Since $C \in \mathcal{L}(x,y) \setminus \mathcal{L}(x)$, there is $R \in \mathcal{R}$ such that $p_f(R) = \vec{p}_f$ and $f(R) = y$. However, observe that $C \subseteq L_{(x,y)}(R)$ and then, by Step 2, we have that $g_{(x,y)}(R) = l$ and thus, $f(R) = x$, which is a contradiction.

Step 3: We show that for each minimal coalition B of $\mathcal{L}(x,y) \setminus \mathcal{L}(x)$, there is $C \in W(g_{(x,y)})$ such that $C \cap A = B$.

Suppose by contradiction that for some minimal coalition B of $\mathcal{L}(x,y) \setminus \mathcal{L}(x)$, there is no $C \in W(g_{(x,y)})$ such that $C \cap A = B$. Consider any $\vec{p}_f \in r_f^A$ such that for each $i \in A$, $(\vec{p}_f)_i \leq^* (x,y) \Leftrightarrow i \in B$. Since $B \in \mathcal{L}(x,y) \setminus \mathcal{L}(x)$, there is $R \in \mathcal{R}$ such that $p_f(R) = \vec{p}_f$ and $f(R) = x$. Observe that $L_{(x,y)}(R) \cap A = B$. Since for each $C \in W(g_{(x,y)})$, $C \cap A \neq B$, we have that there is no $C \in W(g_{(x,y)})$ such that $C \subseteq L_{(x,y)}(R)$. Therefore, $g_{(x,y)}(R) = r$ and thus, $f(R) = y$, which is a contradiction.

Proof of Theorem 3.4.8

First, note that the equivalence between (i) and (ii) is provided in Proposition 3.3.1. Then, it only remains to be shown the equivalence between (i) and (iii).

If (i) is satisfied, then the structure of the rule f is as described in (iii) given the results of Sections 3.3 and 3.4. To prove that (iii) implies (i), consider any rule f such that it can be decomposed as described in (iii) with a generalized median voter function ω on a set T_f, with $[V_f \cup (r_f \setminus \{\min r_f, \max r_f\})] \subseteq T_f \subseteq V_f \cup r_f$, and a set of voting by collections of left-decisive sets $\{g_{(x,y)} : \mathcal{R} \to \{l,r\}\}_{(x,y)\in V_f}$. Suppose by contradiction that there is a profile $R \in \mathcal{R}$ and an agent $i \in N$ with the alternative preference $R'_i \in \mathcal{R}_i$ such that $f(R'_i, R_{-i}) \, P_i \, f(R)$. We assume without loss of generality that $f(R) < f(R'_i, R_{-i})$.

Suppose first that $\omega(p_f(R)) = \omega(p_f(R'_i, R_{-i}))$. If $|\omega(p_f(R))| = 1$, then $f(R) = f(R'_i, R_{-i})$, which contradicts $f(R'_i, R_{-i}) \, P_i \, f(R)$. Otherwise, if $|\omega(p_f(R))| = 2$, then

$f(R) = \underline{\omega}(p_f(R))$ and $f(R'_i, R_{-i}) = \overline{\omega}(p_f(R))$. Given that $f(R) = \underline{\omega}(p_f(R))$, we know by definition that $g_{\omega(p_f(R))}(R) = l$ and therefore, $C \subseteq L_{\omega(p_f(R))}(R)$ for some $C \in W(g_{\omega(p_f(R))})$.[8] Since $f(R'_i, R_{-i})\, P_i\, f(R)$, $i \notin L_{\omega(p_f(R))}(R)$. Observe then that $L_{\omega(p_f(R))}(R) \subseteq L_{\omega(p_f(R))}(R'_i, R_{-i})$ and thus, $C \subset L_{\omega(p_f(R))}(R'_i, R_{-i})$. Therefore, $g_{\omega(p_f(R))}$ $(R'_i, R_{-i}) = l$ and consequently, $f(R'_i, R_{-i}) = \underline{\omega}(p_f(R))$, which is a contradiction.

Suppose now that $\omega(p_f(R)) \neq \omega(p_f(R'_i, R_{-i}))$. Then, $i \in A$ and given that $f(R) < f(R'_i, R_{-i})$, we have that $\omega(p_f(R)) <^* \omega(p_f(R'_i, R_{-i}))$. Since $f(R'_i, R_{-i})\, P_i\, f(R)$, we have that if $|\omega(p_f(R))| = 1$, then $p_f(R_i) > \omega(p_f(R))$ and if $|\omega(p_f(R))| = 2$, then $p_f(R_i) \geq \overline{\omega}(p_f(R))$. Thus, in any case $p_f(R_i) >^* \omega(p_f(R))$ and then, $i \notin \{j \in A : p_f(R_j) \leq^* \omega(p_f(R))\}$. Observe then that $\{j \in A : p_f(R_j) \leq^* \omega(p_f(R))\} \subseteq \{j \in A : p_f((R'_i, R_{-i})_j) \leq^* \omega(p_f((R'_i, R_{-i}))\}$. Therefore, by Proposition 3.4.5, $\omega(p_f(R'_i, R_{-i})) \leq^* \omega(p_f(R))$, which is a contradiction.

Proof of Proposition 3.5.1

We first prove (i). Take any SP rule f such that $r_f = X$. Suppose by contradiction that f is not PE. Then, there is $x \in X$ and $R \in \mathcal{R}$ such that $x\, P_i\, f(R)$ for each $i \in N$. Since $r_f = X$, $x \in r_f$ and then there is a profile $R' \in \mathcal{R}$ such that $f(R') = x$. Therefore, the agent set N manipulates f at R via R' and f is not GSP. By Proposition 3.3.1, f is not SP, which is a contradiction.

Second, we show (ii). Take any SP rule f such that $r_f \notin \{X, \{\min X, \max X\}\}$. Therefore, $X \backslash r_f \neq \emptyset$. Suppose by contradiction that f is PE. First, we prove the result when $\{\min X, \max X\} \not\subset r_f$. Suppose without loss of generality that $\min X \notin r_f$. Consider $R \in \mathcal{R}$ such that for each $i \in A$, $p(R_i) = \min X$ and for each $i \in D$, $d(R_i) = \max X$. Observe that, by construction, $\min X\, P_i\, x$ for each $x \in r_f$ and each $i \in N$. Then, $\min X$ Pareto dominates $f(R)$ because $f(R) \in r_f$. Now, we prove the case in which $\{\min X, \max X\} \subset r_f$. Let $x \in X \setminus r_f$ and let $y \in r_f \setminus \{\min r_f, \max r_f\}$ be the closest alternative to x in $r_f \setminus \{\min r_f, \max r_f\}$. Observe that x and y always exist because $r_f \notin \{X, \{\min X, \max X\}\}$. Consider a profile $R' \in \mathcal{R}$ such that for each $i \in A$, $[p(R'_i) = x$ and $p_f(R'_i) = y]$, and for each $i \in D$, $d_f(R'_i) = y$. Then, $\{j \in A : p_f(R'_j) \leq^* y\} = A$ and for each $\alpha <^* y$, $\{j \in A : p_f(R'_j) \leq^* \alpha\} = \emptyset$. Therefore, by Theorem 3.4.8, $\omega(p_f(R')) = y$, and $f(R') = y$. However, $x\, P'_i\, y$ for each $i \in N$ and a contradiction is reached.

Finally, we prove (iii). Consider any SP rule f such that $r_f = \{\min X, \max X\} \neq X$. We divide the proof into three steps:

Step 1: We show that if $A = \emptyset$, then f is PE.

[8] With a slight abuse of notation, we write for each $R \in \mathcal{R}$, $L_{\omega(p_f(R))}(R)$ to refer to $L_{(\underline{\omega}(p_f(R)), \overline{\omega}(p_f(R)))}(R)$.

Since $A = \emptyset$, $N = D \neq \emptyset$. Observe then that for any agent, either $\min X$ or $\max X$ is the most preferred alternative of X. Consider first any profile $R \in \mathcal{R}$ such that all agents have the same most preferred alternative of X. Assume, without loss of generality, that this most preferred alternative is $\min X$. Then, $L_{(\min X, \max X)}(R) = N$ and, by Theorem 3.4.8, $g_{(\min X, \max X)}(R) = l$ and $f(R) = \min X$. Consider now any profile $R' \in \mathcal{R}$ such that not all agents have the same most preferred alternative. Then, there are some agents, say $S \subset D$, whose most preferred alternative is $\min X$, while for the remaining agents, $N \setminus S$, $\max X$ is the most preferred alternative. Since $r_f = \{\min X, \max X\}$, we have $f(R') \in \{\min X, \max X\}$. If, on the one hand, $f(R') = \min X$, then $f(R')\, P'_i\, x$ for each $i \in S$ and each $x \in X \setminus \{\min X\}$. If, on the other hand, $f(R') = \max X$, then $f(R')\, P'_i\, x$ for each $i \in N \setminus S$ and each $x \in X \setminus \{\max X\}$. Hence, f is PE.

Step 2: We show that if $A \neq \emptyset$ and $T_f = \{(\min X, \max X)\}$, then f is PE.

Since we know, by Theorem 3.4.8 (exactly by point (i) in Definition 3.4.6), that for each $C \in W(g_{(\min X, \max X)})$, $C \cap D \neq \emptyset$, we have that $D \neq \emptyset$. Additionally, since r_f has a maximum at $\max X$ and $\max X \notin T_f$, we know, by Theorem 3.4.8 (exactly by point (iii) in Definition 3.4.3) that $\emptyset \in \mathcal{L}(\min X, \max X)$. Then, $\emptyset$ is a minimal coalition of $\mathcal{L}(\min X, \max X)$ and by Theorem 3.4.8 (exactly by point (ii) in Definition 3.4.6), there is $C \in W(g_{(\min X, \max X)})$ such that $C \subseteq D$. Observe that for any agent of D, either $\min X$ or $\max X$ is the most preferred alternative of X. Observe also that since $T_f = \{(\min X, \max X)\}$, then for each $R \in \mathcal{R}$, $\omega(p_f(R)) = (\min X, \max X)$. Consider first any profile $R' \in \mathcal{R}$ such that $C \subseteq L_{(\min X, \max X)}(R')$. Then, by Theorem 3.4.8, $g_{(\min X, \max X)}(R') = l$ and $f(R') = \min X$. Since $C \subseteq L_{(\min X, \max X)}(R')$ and $C \subseteq D$, we have that $\min X\, P'_i\, x$ for each $i \in C$ and each $x \in X \setminus \{\min X\}$. Consider now any profile $R'' \in \mathcal{R}$ such that $C \nsubseteq L_{(\min X, \max X)}(R'')$. Then, by Theorem 3.4.8, $g_{(\min X, \max X)}(R'') = r$ and $f(R'') = \max X$. Since $C \nsubseteq L_{(\min X, \max X)}(R'')$ and $C \subseteq D$, we have that $\max X\, P''_i\, x$ for some $i \in C$ and each $x \in X \setminus \{\max X\}$. Hence, f is PE.

Step 3: We show that if $A \neq \emptyset$ and $T_f \neq \{(\min X, \max X)\}$, then f is not PE.

Since $r_f = \{\min X, \max X\}$ and $T_f \neq \{(\min X, \max X)\}$, we have that $\min X \in T_f$ and/or $\max X \in T_f$. Suppose without loss of generality that $\min X \in T_f$. Let $x \in X \setminus r_f$ and consider a profile $R \in \mathcal{R}$ such that for each $i \in A$, $[p(R_i) = x$ and $p_f(R_i) = \min X]$, and for each $i \in D$, $d_f(R_i) = \min X$. Since $\min X \in T_f$ and $\{j \in A : p_f(R_j) \leq^* \min X\} = A$ then, by Theorem 3.4.8, we have $\omega(p_f(R)) = \min X$ and $f(R) = \min X$. However, $x\, P_i \min X$ for each $i \in N$ and thus, f is not PE.

Bibliography

Achuthankutty, G. and Roy, S. (2018), 'Dictatorship on top-circular domains', *Theory and Decision* **85**(3-4), 479–493.

Alcalde-Unzu, J. and Vorsatz, M. (2018), 'Strategy-proof location of public facilities', *Games and Economic Behavior* **112**, 21–48.

Barberà, S. (2011), Strategyproof social choice, *in* 'Handbook of social choice and welfare', Vol. 2, Elsevier, pp. 731–831.

Barberà, S., Berga, D. and Moreno, B. (2010), 'Individual versus group strategy-proofness: When do they coincide?', *Journal of Economic Theory* **145**(5), 1648–1674.

Barberà, S., Berga, D. and Moreno, B. (2012), 'Domains, ranges and strategy-proofness: the case of single-dipped preferences', *Social Choice and Welfare* **39**(2-3), 335–352.

Barberà, S. and Jackson, M. (1994), 'A characterization of strategy-proof social choice functions for economies with pure public goods', *Social Choice and Welfare* **11**(3), 241–252.

Berga, D. (1998), 'Strategy-proofness and single-plateaued preferences', *Mathematical Social Sciences* **35**(2), 105–120.

Berga, D. and Serizawa, S. (2000), 'Maximal domain for strategy-proof rules with one public good', *Journal of Economic Theory* **90**(1), 39–61.

Black, D. (1948*a*), 'The decisions of a committee using a special majority', *Econometrica: Journal of the Econometric Society* pp. 245–261.

Black, D. (1948*b*), 'On the rationale of group decision-making', *Journal of political economy* **56**(1), 23–34.

Bossert, W. and Peters, H. (2014), 'Single-basined choice', *Journal of Mathematical Economics* **52**, 162–168.

Feigenbaum, I. and Sethuraman, J. (2015), Strategyproof mechanisms for one-dimensional hybrid and obnoxious facility location models, *in* 'Workshops at the twenty-ninth AAAI conference on artificial intelligence'.

Gibbard, A. et al. (1973), 'Manipulation of voting schemes: a general result', *Econometrica* **41**(4), 587–601.

Manjunath, V. (2014), 'Efficient and strategy-proof social choice when preferences are single-dipped', *International Journal of Game Theory* **43**(3), 579–597.

Moulin, H. (1980), 'On strategy-proofness and single peakedness', *Public Choice* **35**(4), 437–455.

Moulin, H. (1984), 'Generalized condorcet-winners for single peaked and single-plateau preferences', *Social Choice and Welfare* **1**(2), 127–147.

Satterthwaite, M. A. (1975), 'Strategy-proofness and arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions', *Journal of economic theory* **10**(2), 187–217.

Thomson, W. (2008), Where should your daughter go to college? an axiomatic analysis, Technical report, Tech. rep., Mimeo University of Rochester.

Conclusions and further research

This thesis focuses on analyzing some aspects of two fields of economics, Cooperative Game Theory and Social Choice Theory, where agents' preferences or opinions play a crucial role in the final outcome. Regarding Cooperative Game Theory, we analyze how agents' preferences over the coalitions in which they may take part lead to stable partitions (Chapter 1). We also study the specific structure of those stable partitions (Chapter 2). Regarding Social Choice Theory, we analyze a location problem in which agents' preferences over alternatives give rise to a final collective decision that cannot be manipulated by any agent (Chapter 3). Our main findings are summed up below, and some questions and extensions, which have arisen in the course of our work, are pointed out.

Chapter 1, written in collaboration with Elena Iñarra of the University of the Basque Country, introduces a generalized claims problem to deal with coalition formation in a bankruptcy situation, bringing together two branches of the literature that have been analyzed separately until now: Claims problems and coalition formation problems. In this new setting, we analyze the core stability of the coalition formation problems that emerge from bankruptcy situations. The main result, Theorem 2, characterizes all rules that guarantee the non-emptiness of the core. The properties required for such a characterization are continuity, resource monotonicity, and consistency. We show, for instance, that the Random Arrival rule (Shapley, 1953), which fails to satisfy consistency, may induce coalition formation problems with an empty core. By contrast, we analyze a well-known class of rules, parametric rules (Young, 1987), which satisfy the required properties so that the existence of stable partitions is guaranteed. Chapter 1 also contributes to the literature of coalition formation problems by introducing a new class of games, called "regular coalition formation problems", which guarantee the existence of stability. These problems satisfy the properties of weak pairwise alignment and lack of rings. This new class includes the coalition formation problems that satisfy the common ranking property (Farrell and Scotchmer, 1988) and is contained in the class of stable coalition formation problems that satisfy the top coalition property (Banerjee et al., 2001).

Several questions emerge from this new model. An initial extension in the short run is the application of other concepts of stability in addition to core stability. For there to be core stability there must be no coalition such that all agents strictly prefer that coalition to the one in which they are taking part in the current partition. However, we could require there to be no coalition such that only one agent strictly prefers that coalition to her coalition in the current partition while the others are not worse off. The strong core notion Roth and Postlewaite (1977) can then be applied. Further, if the idea is to focus on the analysis of partitions in which no agent can benefit from moving from her current coalition to another existing coalition, Nash stability Bogomolnaia et al. (2002) should be applied. Bogomolnaia et al. (2002) also define other stability concepts related to individual deviations such as individual stability and contractual individual stability. Recently, Karakaya (2011) has proposed a new stability notion based on the so-called "free exit-free entry membership rights", referring to it as strong Nash stability.

Looking again at the input of a generalized claims problem (a vector of claims and a set of endowments), two natural extensions emerge. With respect to claims, observe that we have assumed that they are the same across coalitions, which reflects a kind of objectivity. However, claims may be subjective and may depend, for instance, on the identities of the members of the coalitions. The introduction of subjective claims may make the analysis cumbersome and it is thus left for future research. Endowments might depend on how the remaining agents are organized, which may lead to externalities across coalitions and may call for the analysis of a game in partition function.

Pycia (2012) proves that the Kalai-Smorodinsky bargaining solution Kalai and Smorodinsky (1975) does not always induce stability. Similarly, we prove in Chapter 1 that the random arrival rule does not always induce coalition formation problems with stable partitions. Therefore, the study of the domains of coalition formation problems in which these and other well-known rules induce stability is another possibly extension that merits further analysis.

Another issue that arises naturally from the results provided in Chapter 1 is the analysis of the structure of the stable partitions. This question is addressed in Chapter 2, written in collaboration with Bettina Klaus of the University of Lausanne. Once the non-emptiness of the core is guaranteed, the study of how agents sort themselves into coalitions to form stable partitions becomes a relevant issue. Note that payoffs may differ considerably depending on the rule used to divide the endowments, which induces different agents' preferences so different stable partitions may emerge under different rules. We analyze a particular generalized claims problem where the endowment of each group is a fixed proportion of the sum of its members' claims and singleton coalitions receive zero endowment. Proportional cuts are frequently applied in many real-life situations, so they seem to be a very natural constraint to consider. We show that when any continuous, strict resource monotonic, and consistent rule is applied, each stable partition contains at most one singleton and for any coalition with a size larger than two, each agent receives a proportional payoff. For the weak notion of resource monotonicity, we do not characterize all stable partitions but we still guarantee the existence of a pairwise stable partition with at most one singleton coalition if the set of agents is odd. We also provide two algorithms for constructing pairwise stable partitions for the CEA and CEL rules, respectively. For CEA, a pairwise stable partition is obtained by assortatively pairing off either the two highest claim agents (assortative coalition) or the highest and the lowest claim agent (extremal coalition). For CEL, an assortative stable partition is obtained by sequentially pairing off the two lowest claim agents.

One possible extension of Chapter 2 is briefly explained in its last section, which outlines how the model can be generalized to more real-life situations. Recall the example of the call for funding research teams in which the government has a budget to invest in projects. Those projects can be carried out by research groups which must be formed by researchers with the aim of obtaining a joint profit to divide among them. When the budget cannot be directly assigned to agents, a two-step procedure is needed. The process can thus be sketched as follows: First, the budget is divided among the groups and, second, the amount for each group is divided among its members. In this case, the formation of groups will depend on both the rule that divides the budget among the different groups and the rule that is used to distribute the endowment of each group among its members. This two-step model would enable the stable partitions which emerge from different combinations of rules to be analyzed.

Finally, Chapter 3, written in collaboration with Jorge Alcalde-Unzu of the Universidad Pública de Navarra and Marc Vorsatz of the Universidad Nacional de Educación a Distancia (UNED), analyzes the problem of locating a public facility taking into account agents' preferences over the possible locations. In particular, we consider the location of a facility that is considered a good by part of society and a bad by the rest. Even though this context has been already analyzed, we propose a new domain in which the kind of preferences of each agent (single-peaked or single-dipped preferences) is known by the social planner but there is no public information about the location of the peak or dip and the rest of the preference of each agent. This model allows each agent to have her peak or dip at any point such as her house, her workplace or her children's school. In this setting, we look for social choice rules that induce agents to reveal their preferences truthfully. The main result characterizes all strategy-proof rules on this domain and shows that they are all also group strategy-proof. We also analyze which of these rules are Pareto efficient.

A primary extension of the results of this chapter is the study of which strategy-proof rules satisfy the axiom of anonymity. It must be taken into account that it is not possible to apply the classical definition of anonymity because the set of admissible preferences differs from one agent to another. We therefore opt to the property of type-anonymity, which allows for permutations only between those agents with the same set of admissible preferences (*i.e.*, an agent with single-peaked preferences can only swap with another agent with single-peaked preferences, and an agent with single-dipped preferences can only swap with another agent with single-dipped preferences). This extension is already in progress.

Another straightforward question concerns indifferences. The model in Chapter 3 only considers preferences that are linear orders. If indifferences are allowed for, then the domains need to be extended to single-plateau Berga (1998); Moulin (1984) and single-basined Bossert and Peters (2014) preferences, respectively.

A natural question that comes to mind is how the results can be extended for the case of k units of the same facility. This extension needs to take into account several technical issues that complicate it. A particular agent could, for instance, have single-peaked preferences over the location of each unit of the facility, but there are many possible ways of extending those preferences to vectors of locations of the k units of the facility (see, for instance, Heo (2013) and Lahiri and Pramanik (2019) for the location of two public goods and several public bads, respectively, on an interval).

One last possible extension for the results of Chapter 3 could be the analysis of the case in which the possible locations are described by a k-dimensional vector (see Barberà et al. (1993) for a similar analysis of the case of only single-peaked preferences).

Bibliography

Banerjee, S., Konishi, H. and Sönmez, T. (2001), 'Core in a simple coalition formation game', *Social Choice and Welfare* **18**(1), 135–153.

Barberà, S., Gul, F. and Stacchetti, E. (1993), 'Generalized median voter schemes and committees', *Journal of Economic Theory* **61**(2), 262–289.

Berga, D. (1998), 'Strategy-proofness and single-plateaued preferences', *Mathematical Social Sciences* **35**(2), 105–120.

Bogomolnaia, A., Jackson, M. O. et al. (2002), 'The stability of hedonic coalition structures', *Games and Economic Behavior* **38**(2), 201–230.

Bossert, W. and Peters, H. (2014), 'Single-basined choice', *Journal of Mathematical Economics* **52**, 162–168.

Farrell, J. and Scotchmer, S. (1988), 'Partnerships', *The Quarterly Journal of Economics* **103**(2), 279–297.

Heo, E. J. (2013), 'Strategy-proof rules for two public goods: double median rules', *Social Choice and Welfare* **41**(4), 895–922.

Kalai, E. and Smorodinsky, M. (1975), 'Other solutions to nash's bargaining problem', *Econometrica: Journal of the Econometric Society* pp. 513–518.

Karakaya, M. (2011), 'Hedonic coalition formation games: A new stability notion', *Mathematical Social Sciences* **61**(3), 157–165.

Lahiri, A. and Pramanik, A. (2019), 'On strategy-proof social choice between two alternatives', *Social Choice and Welfare* pp. 1–27.

Moulin, H. (1984), 'Generalized condorcet-winners for single peaked and single-plateau preferences', *Social Choice and Welfare* **1**(2), 127–147.

Pycia, M. (2012), 'Stability and preference alignment in matching and coalition formation', *Econometrica* **80**(1), 323–362.

Roth, A. E. and Postlewaite, A. (1977), 'Weak versus strong domination in a market with indivisible goods', *Journal of Mathematical Economics* **4**(2), 131–137.

Shapley, L. S. (1953), 'A value for n-person games', *Contributions to the Theory of Games* **2**(28), 307–317.

Young, H. P. (1987), 'On dividing an amount according to individual claims or liabilities', *Mathematics of Operations Research* **12**(3), 398–414.