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Coupled Optimal Results with an Application to Nonlinear Integral Equations

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Abstract: In the present work, we consider the best proximal problem related to a coupled mapping, which we define using control functions and weak inequalities. As a consequence, we obtain some results on coupled fixed points. Our results generalize some recent results in the literature. Also, as an application of the results obtained, we present the solution to a system of a coupled Fredholm nonlinear integral equation. Our work is supported by several illustrations.

Keywords: partially ordered set; control function; best proximity point; coupled best proximity point; integral equation

MSC: 47H10; 54H10; 54H25; 41A50; 46TXX

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1. Introduction

The contraction mapping principle is one of the pioneering ideas of mathematics associated with physical as well as mathematical endeavors. It was first investigated by S. Banach [1] and shows us the root of the fixed point discussions in much of the existing literature, such as [2,3].

We have used weak contraction to prove our results. The idea of weak contraction in Hilbert spaces given by Alber et al. [4] and extended by Rhoades [3]. In this connection one can see the work mentioned in [5]. Later on, Berinde [6] introduced weak contraction in metric spaces also known by 'almost contraction'. Weak contractions were investigated and generalized in metric spaces and in ordered metric spaces by various researchers (see [7–16] and references cited therein).

It is possible to find a point where we can find an approximation of the fixed point equation $d(\varpi, \mathfrak{T}\varpi) = 0$ and how? The answer to this question is affirmative and the research can be observed in Eldred et al. [17] and Kirk et al. [18]. In short, the methodology to obtain such result adopts non-self mapping in between two non intersecting sets, which has a distance mentioned as $d(\mathscr{A}, \mathscr{B})$ where $\mathscr{A}, \mathscr{B}$ are two sets such that $\mathscr{A} \cap \mathscr{B} = \emptyset$.

Our point of discussion deals with a problem of optimization which is at par to the approximate solution of a fixed point equation $d(\varpi, \mathfrak{T}\varpi) = 0$. The problem is of global minima which has nothing to do with the establishment of such theory of best approximation theorem while we are inclined to investigate best proximity theorems. Some of the works deal with best approximation issues can be mentioned through [19–21]. The result is as follows:

Theorem 1 ([19]). *Let $\mathscr{A}$ be a non-empty compact convex subset of a normed linear space $\mathscr{X}$ and $\mathfrak{T} : \mathscr{A} \to \mathscr{X}$ be a continuous function. Then there exists $\varpi \in \mathscr{A}$ such that*

$$\| \varpi - \mathfrak{T}\varpi \| = d(\mathfrak{T}\varpi, \mathscr{A}) = \inf\{\| \mathfrak{T}\varpi - \varkappa \| : \varkappa \in \mathscr{A}\}.$$

The point ϖ does not ensure the extremum of $\| \varpi - \mathfrak{T}\varpi \|$.

The results discussed in the paper are associated with the equation

$$d(\varpi,\, \mathfrak{T}\varpi) = d(\mathscr{A},\, \mathscr{B}),$$

where the required identification of $\mathscr{A}$, $\mathscr{B}$, $d(\mathscr{A},\, \mathscr{B})$ has been done already. The minima are realized through a mapping $\mathfrak{T} : \mathscr{A} \to \mathscr{B}$. It is better to mention that a fixed point of the mapping $\mathfrak{T}$ can be there with the condition $\mathscr{A} \cap \mathscr{B} \neq \emptyset$.

The idea of contraction using coupling of mappings first seen in Bhaskar et al. [22] though first realized in Guo et al. [23]. Couple best proximity results are also discussed in some of the work of [24–26]. V. Sankar Raj [27] obtained an interesting result on best proximity for weakly contractive non-self mappings. Many discussions related with the existence of fixed point through the consideration of order relation with the underneath metric and of best approximation are investigated in [2,20,27–38]. Contraction mapping procedures have been also continuously employing in differential equations and integral equations as cornerstone instruments to prove the existence of related solutions (see [39–41]). A large number of initial and boundary value problems can be converted to nonlinear integral equations (both Fredholm and its special case-Volterra nonlinear equations). Sidorov et al. [42] constructed the solution of nonlinear Volterra operator-integral equations in the sense of Kantorovich.

In this paper, we investigate the coupled proximity point in ordered metric spaces associated with a weak inequality. Inspired by the work of Luong and Thuan [43], in Section 2, we discuss some of the prerequisites for the mathematical approach towards our results. In Section 3, two propositions and two theorems are the points of discussion in which the blending of partial order and weak inequalities can be found. As a consequence of Section 3, we obtain some coupled fixed point results in Section 4. As an application of the results obtained, we investigate the existence of solution to Fredholm nonlinear integral equation in Section 5. In the last section, we provide a suitable illustration which satisfies the coupled best proximity point result.

2. Preliminaries

Some fundamental discussions to reach our main results are as follows:

Let (Ω, ϱ) be a partially ordered metric space (POMS), where $\Omega = (\mathscr{X},\, \preceq)$, $\mathscr{X}$ is a non-empty set endowed with a partial order $\preceq$ and ϱ is a metric induced on $\mathscr{X}$.

Unless otherwise specified, it is assumed throughout this article that $\mathscr{A}$ and $\mathscr{B}$ are two non-empty subsets of the metric space.

$$\begin{aligned}
\varrho(\mathscr{A},\, \mathscr{B}) &= \inf\{\varrho(\varpi,\, \vartheta) : \varpi \in \mathscr{A} \text{ and } \vartheta \in \mathscr{B}\},\\
\mathscr{A}_0 &= \{\varpi \in \mathscr{A} : \varrho(\varpi,\, \vartheta) = \varrho(\mathscr{A},\, \mathscr{B}) \text{ for some } \vartheta \in \mathscr{B}\},\\
\mathscr{B}_0 &= \{\vartheta \in \mathscr{B} : \varrho(\varpi,\, \vartheta) = \varrho(\mathscr{A},\, \mathscr{B}) \text{ for some } \varpi \in \mathscr{A}\}.
\end{aligned}$$

It is to be noted that, for every $\varpi \in \mathscr{A}$, there exists $\vartheta \in \mathscr{B}_0$ such that $\varrho(\varpi,\, \vartheta) = \varrho(\mathscr{A},\, \mathscr{B})$ and conversely, for every $\vartheta \in B_0$ there exists $\varpi \in A_0$ such that $\varrho(\varpi,\, \vartheta) = \varrho(\mathscr{A},\, \mathscr{B})$.

In the following we give some notation and notions:

- Best Proximity Point: *BPP*
- Coupled Best Proximity Point: *CBPP*
- Coupled fixed Point: *CFP*
- Proximally generalized coupled weal contraction: *PGCWC*

Definition 1 ([27]). *Let $\mathscr{A}$ and $\mathscr{B}$ be two non-empty subsets of a metric space $(\mathscr{X}, \varrho)$ with $\mathscr{A}_0 \neq \emptyset$. Then the pair $(\mathscr{A}, \mathscr{B})$ is said to have the P-property if, for any $\varpi_1, \varpi_2 \in \mathscr{A}_0$ and $\vartheta_1, \vartheta_2 \in \mathscr{B}_0$,*

$$\left.\begin{array}{rcl} \varrho(\varpi_1, \vartheta_1) & = & \varrho(\mathscr{A}, \mathscr{B}) \\ \varrho(\varpi_2, \vartheta_2) & = & \varrho(\mathscr{A}, \mathscr{B}) \end{array}\right\} \Rightarrow \varrho(\varpi_1, \varpi_2) = \varrho(\vartheta_1, \vartheta_2).$$

In [28], Abkar and Gabeleh show that every non-empty, bounded, closed and convex pair of subsets of a uniformly convex Banach space has the P-property. Some non-trivial examples of a non-empty pair of subsets that satisfies the P-property are given in [28].

Definition 2. *A mapping $\mathfrak{T} : \mathscr{A} \to \mathscr{A}$ is said to be increasing if for all $\varpi_1, \varpi_2 \in \mathscr{A}$,*

$$\varpi_1 \preceq \varpi_2 \implies \mathfrak{T}\varpi_1 \preceq \mathfrak{T}\varpi_2.$$

Definition 3 ([31]). *A mapping $\mathfrak{T} \colon \mathscr{A} \longrightarrow \mathscr{B}$ is said to be proximally increasing if for all $u_1, u_2, \varpi_1, \varpi_2 \in A$,*

$$\left.\begin{array}{l} \varpi_1 \preceq \varpi_2 \\ \varrho(u_1, \mathfrak{T}\varpi_1) = \varrho(\mathscr{A}, \mathscr{B}), \\ \varrho(u_2, \mathfrak{T}\varpi_2) = \varrho(\mathscr{A}, \mathscr{B}) \end{array}\right\} \Rightarrow u_1 \preceq u_2.$$

One can see that, for a self-mapping, the notion of proximally increasing reduces to that of increasing mapping.

Definition 4. *A mapping $\mathfrak{T} \colon \mathscr{A} \longrightarrow \mathscr{B}$ is said to be proximally increasing on $\mathscr{A}_0$ if for all $u_1, u_2, \varpi_1, \varpi_2 \in \mathscr{A}_0$,*

$$\left.\begin{array}{l} \varpi_1 \preceq \varpi_2 \\ \varrho(u_1, \mathfrak{T}\varpi_1) = \varrho(\mathscr{A}, \mathscr{B}), \\ \varrho(u_2, \mathfrak{T}\varpi_2) = \varrho(\mathscr{A}, \mathscr{B}) \end{array}\right\} \Rightarrow u_1 \preceq u_2.$$

Definition 5. *An element $\varpi^* \in \mathscr{A}$ is said to be BPP of the mapping $\mathfrak{T} \colon \mathscr{A} \longrightarrow \mathscr{B}$ if $\varrho(\varpi^*, \mathfrak{T}\varpi^*) = \varrho(\mathscr{A}, \mathscr{B})$.*

Definition 6 ([22]). *A mapping $\mathfrak{T} \colon \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$ is said to have the mixed monotone property if $\mathfrak{T}$ is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument; that is, if*

$$\varpi_1, \varpi_2 \in \mathscr{A}, \ \varpi_1 \preceq \varpi_2 \implies \mathfrak{T}(\varpi_1, \vartheta) \preceq \mathfrak{T}(\varpi_2, \vartheta), \quad \text{for all } \vartheta \in \mathscr{A};$$

and

$$\vartheta_1, \vartheta_2 \in A, \ \vartheta_1 \preceq \vartheta_2 \implies \mathfrak{T}(\varpi, \vartheta_1) \succeq \mathfrak{T}(\varpi, \vartheta_2), \quad \text{for all } \varpi \in \mathscr{A}.$$

Definition 7 ([25]). *A mapping $\mathfrak{T} \colon \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{B}$ is said to have proximal mixed monotone property if $\mathfrak{T}(\varpi, \vartheta)$ is proximally non-decreasing in ϖ and is proximally non-increasing in ϑ; that is, for all $\varpi, \vartheta \in \mathscr{A}$*

$$\left.\begin{array}{l} \varpi_1 \preceq \varpi_2 \\ \varrho(u_1, \mathfrak{T}(\varpi_1, \vartheta)) = \varrho(\mathscr{A}, \mathscr{B}), \\ \varrho(u_2, \mathfrak{T}(\varpi_2, \vartheta)) = \varrho(\mathscr{A}, \mathscr{B}) \end{array}\right\} \Rightarrow u_1 \preceq u_2$$

and

$$\left.\begin{array}{l} \vartheta_1 \preceq \vartheta_2 \\ \varrho(v_1, \mathfrak{T}(\varpi, \vartheta_1)) = \varrho(\mathscr{A}, \mathscr{B}), \\ \varrho(v_2, \mathfrak{T}(\varpi, \vartheta_2)) = \varrho(\mathscr{A}, \mathscr{B}) \end{array}\right\} \Rightarrow v_2 \preceq v_1,$$

where $\varpi_1,\ \varpi_2,\ \vartheta_1,\ \vartheta_2,\ u_1,\ u_2,\ v_1,\ v_2 \in \mathscr{A}$.

One can see that, if $\mathscr{A} = \mathscr{B}$ in the above definition, the notion of the proximal mixed monotone property reduces to that of the mixed monotone property.

Definition 8. *A mapping* $\mathfrak{T}\colon \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{B}$ *is said to have proximal mixed monotone property on* $\mathscr{A}_0 \times \mathscr{A}_0$ *if for all* $\varpi,\ \vartheta \in \mathscr{A}_0$

$$\left.\begin{array}{l}\varpi_1 \preceq \varpi_2\\ \varrho(u_1,\ \mathfrak{T}(\varpi_1,\ \vartheta)) = \varrho(\mathscr{A},\ \mathscr{B}),\\ \varrho(u_2,\ \mathfrak{T}(\varpi_2,\ \vartheta)) = \varrho(\mathscr{A},\ \mathscr{B})\end{array}\right\} \Rightarrow u_1 \preceq u_2$$

and

$$\left.\begin{array}{l}\vartheta_1 \preceq \vartheta_2\\ \varrho(v_1,\ \mathfrak{T}(\varpi,\ \vartheta_1)) = \varrho(\mathscr{A},\ \mathscr{B}),\\ \varrho(v_2,\ \mathfrak{T}(\varpi,\ \vartheta_2)) = \varrho(\mathscr{A},\ \mathscr{B})\end{array}\right\} \Rightarrow v_2 \preceq v_1,$$

where $\varpi_1,\ \varpi_2,\ \vartheta_1,\ \vartheta_2,\ u_1,\ u_2,\ v_1,\ v_2 \in \mathscr{A}_0$.

Definition 9 ([26])**.** *An element* $(\varpi^*,\ \vartheta^*) \in \mathscr{A} \times \mathscr{A}$, *is called a CBPP of the mapping* $\mathfrak{T}\colon \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{B}$ *if* $\varrho(\varpi^*,\ \mathfrak{T}(\varpi^*,\ \vartheta^*)) = \varrho(\mathscr{A},\ \mathscr{B})$ *and* $\varrho(\vartheta^*,\ \mathfrak{T}(\vartheta^*,\ \varpi^*) = \varrho(\mathscr{A},\ \mathscr{B})$.

The following results of [25] are required in the sequel.

Lemma 1 ([25])**.** *Let* (Ω, ϱ) *be a POMS and* $\mathscr{A}, \mathscr{B}$ *be non-empty subsets of* $\mathscr{X}$. *Assume* $\mathscr{A}_0 \neq \phi$. *A mapping* $\mathfrak{T} : \mathscr{A} \times \mathscr{A} \to \mathscr{B}$ *has the proximal mixed monotone property with* $\mathfrak{T}(\mathscr{A}_0 \times \mathscr{A}_0) \subseteq \mathscr{B}_0$ *such that*

$$\left.\begin{array}{l}\varpi_0 \preceq \varpi_1 \text{ and } \vartheta_0 \succeq \vartheta_1\\ \varrho(\varpi_1,\ \mathfrak{T}(\varpi_0,\ \vartheta_0)) = \varrho(\mathscr{A},\ \mathscr{B}),\\ \varrho(\varpi_2,\ \mathfrak{T}(\varpi_1,\ \vartheta_1)) = \varrho(\mathscr{A},\ \mathscr{B})\end{array}\right\} \Rightarrow \varpi_1 \preceq \varpi_2,$$

where $\varpi_0,\ \varpi_1,\ \varpi_2,\ \vartheta_0,\ \vartheta_1 \in \mathscr{A}_0$.

Lemma 2 ([25])**.** *Let* (Ω, ϱ) *be a POMS and* $\mathscr{A}, \mathscr{B}$ *be non-empty subsets of* $\mathscr{X}$. *Assume* $\mathscr{A}_0 \neq \phi$. *A mapping* $\mathfrak{T} : \mathscr{A} \times \mathscr{A} \to \mathscr{B}$ *has the proximal mixed monotone property with* $\mathfrak{T}(\mathscr{A}_0 \times \mathscr{A}_0) \subseteq \mathscr{B}_0$ *such that*

$$\left.\begin{array}{l}\varpi_0 \preceq \varpi_1 \text{ and } \vartheta_0 \succeq \vartheta_1\\ \varrho(\vartheta_1,\ \mathfrak{T}(\vartheta_0,\ \varpi_0)) = \varrho(\mathscr{A},\ \mathscr{B}),\\ \varrho(\vartheta_2,\ \mathfrak{T}(\vartheta_1,\ \varpi_1)) = \varrho(\mathscr{A},\ \mathscr{B})\end{array}\right\} \Rightarrow \vartheta_1 \succeq \vartheta_2,$$

where $\varpi_0,\ \varpi_1,\ \vartheta_0,\ \vartheta_1,\ \vartheta_2 \in A_0$.

3. Main Results

In our results, we use the following class of functions.

Our assumption is that the set of all functions $\phi : [0,\infty) \to [0,\infty)$ denoted by Υ, which satisfy

(i_ϕ) ϕ is assumed to be continuous and $\phi(t) = 0$ iff $t = 0$
(ii_ϕ) ϕ satisfied subadditivity property for all $t, s \in [0,\infty)$.

The set of all functions $\psi\colon [0,\ \infty) \longrightarrow [0,\ \infty)$ denoted by Ξ satisfies the following property

(i_ψ) ψ holds continuity and $\psi(t) = 0$ iff $t = 0$.

Γ denotes the set of all functions $\beta\colon [0,\ \infty) \longrightarrow [0,\ \infty)$ such that
(ii_β) β is bounded on any bounded interval in $[0,\ \infty)$,
(iii_β) β is continuous at 0 and $\beta(0) = 0$.

To prove our main result, we introduce the proximally generalized coupled weak contraction mapping as follows:

Definition 10. *Let (Ω, ϱ) be a POMS and $\mathscr{A}, \mathscr{B}$ be non-empty subsets of $\mathscr{X}$. A mapping $\mathfrak{T} : \mathscr{A} \times \mathscr{A} \to \mathscr{B}$ is said to be proximally generalized coupled weak contraction on $\mathscr{A}$, satisfying $\varpi_1 \preceq \varpi_2$ and $\vartheta_1 \succeq \vartheta_2$*

$$\zeta(\varrho(\mathfrak{T}(\varpi_1,\ \vartheta_1),\ \mathfrak{T}(\varpi_2,\ \vartheta_2)) \leq \frac{1}{2}\chi(\varrho(\varpi_1, \varpi_2) + \varrho(\vartheta_1,\ \vartheta_2)) - \xi(\frac{d(\varpi_1,\ \varpi_2) + \varrho(\vartheta_1, \vartheta_2)}{2}), \tag{1}$$

where $\varpi_1, \varpi_2, \vartheta_1, \vartheta_2, u, v \in \mathscr{A}$ and $\zeta \in \Upsilon, \chi \in \Xi$ and $\xi \in \Gamma$.

Example 1. *Suppose that $\mathscr{X} = \{(0,1), (0,-1), (1,0), (-1,0)\}$ and $\varrho = \sqrt{(\varpi_2 - \varpi_1)^2 + (\vartheta_2 - \vartheta_1)^2}$ with usual order.*

Take $\mathscr{A} = \{(0,1), (1,0)\}$ and $\mathscr{B} = \{(0,-1), (-1,0)\}$. Define $\mathfrak{T} : \mathscr{A} \times \mathscr{A} \to \mathscr{B}$ as $\mathfrak{T}(\varpi_1,\ \varpi_i) = (0,\ -1)$ and $\mathfrak{T}(\varpi_2,\ \varpi_i) = (-1,\ 0)$ for $i = 1,2$ and $\varpi_1 = (0,\ 1), \varpi_2 = (1,0)$.

Take $\zeta(t) = t$, $\chi(t) = t^2$ and $\xi(t) = t$. Here it is not difficult to see that $\mathfrak{T}$ is PGCWC on $\mathscr{A}$, satisfying $\varpi_1 \preceq \varpi_2$ and $\vartheta_1 \succeq \vartheta_2$.

Example 2. *Suppose that $\mathscr{X} = \mathbb{R}$ and $\varrho = |\varpi - \vartheta|$ with usual order.*

Take $\mathscr{A} = \{-1,1\}$ and $\mathscr{B} = \{-2,2\}$. Define $\mathfrak{T} : \mathscr{A} \times \mathscr{A} \to \mathscr{B}$ as $\mathfrak{T}(\varpi_1,\ \varpi_i) = 2$ and $\mathfrak{T}(\varpi_2,\ \varpi_i) = -2$ for $i = 1,2$ and $\varpi_1 = -1, \varpi_2 = 1$.

Take $\zeta(t) = t$, $\chi(t) = t^4$ and $\xi(t) = t$. Here it is not difficult to see that $\mathfrak{T}$ is PGCWC on $\mathscr{A}$, satisfying $\varpi_1 \preceq \varpi_2$ and $\vartheta_1 \succeq \vartheta_2$.

Firstly, we are presenting two propositions which will help us to prove our theorems.

Proposition 1. *Let (Ω, ϱ) be a POMS and $\mathscr{A}$, $\mathscr{B}$ be non-empty closed subsets of $\mathscr{X}$ induced by metric ϱ such that $\mathscr{A}_0 \neq \emptyset$ closed and $(\mathscr{A}, \mathscr{B})$ satisfies P-property. Suppose that $\mathfrak{T} : \mathscr{A} \times \mathscr{A} \to \mathscr{B}$ such that $\mathfrak{T}(\mathscr{A}_0 \times \mathscr{A}_0) \subseteq \mathscr{B}_0$ and $\mathfrak{T}$ is satisfying proximally mixed monotone property and $\mathfrak{T}$ is PGCWC on $\mathscr{A}$. Suppose that*

$$\zeta(\varpi) \leq \chi(\vartheta) \implies \varpi \leq \vartheta, \tag{2}$$

for any sequence $\{\varpi_n\}$ in $[0,\ \infty)$ with $\varpi_n \longrightarrow t > 0$,

$$\zeta(t) - \overline{\lim}\ \chi(\varpi_n) + 2\ \underline{\lim}\ \xi(\varpi_n) > 0, \tag{3}$$

where $\zeta \in \Upsilon, \chi \in \Xi$ and $\xi \in \Gamma$.

Further, suppose that there exist sequences $\{\varpi_n\}$ and $\{\vartheta_n\}$ in $\mathscr{A}_0$ defined as $\varpi_{n+1} = \mathfrak{T}(\varpi_n,\ \vartheta_n), \vartheta_{n+1} = \mathfrak{T}(\vartheta_n,\ \varpi_n)$ such that

$$\varrho(\varpi_{n+1}, \mathfrak{T}(\varpi_n,\ \vartheta_n)) = \varrho(\mathscr{A},\ \mathscr{B}) \text{ with } \varpi_n \preceq \varpi_{n+1}$$

and

$$\varrho(\vartheta_{n+1}, \mathfrak{T}(\vartheta_n,\ \varpi_n)) = \varrho(\mathscr{A},\ \mathscr{B}) \text{ with } \vartheta_n \succeq \vartheta_{n+1}$$

for all $n \geq 0$. Then

$$L_n = \delta_n = \varrho(\varpi_n,\ \varpi_{n+1}) + \varrho(\vartheta_n,\ \vartheta_{n+1}) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{4}$$

Proof. By our assumption in the proposition, there exist sequences $\{\varpi_n\}$ and $\{\vartheta_n\}$ in $\mathscr{A}_0$ such that

$$\varrho(\varpi_{n+1}, \mathfrak{T}(\varpi_n,\ \vartheta_n)) = \varrho(\mathscr{A},\ \mathscr{B}) \text{ with } \varpi_n \preceq \varpi_{n+1}$$

and

$$\varrho(\vartheta_{n+1}, \mathfrak{T}(\vartheta_n,\ \varpi_n)) = \varrho(\mathscr{A},\ \mathscr{B}) \text{ with } \vartheta_n \succeq \vartheta_{n+1}$$

for all $n \geq 0$.

As, $(\mathscr{A},\ \mathscr{B})$ satisfies P-property, we have

$$\varrho(\varpi_{n+1},\ \varpi_n) = \varrho(\mathfrak{T}(\varpi_{n-1}, \vartheta_{n-1}),\ \mathfrak{T}(\varpi_n, \vartheta_n)).$$

Now, $\mathfrak{T}$ is *PGCWC* on $\mathscr{A}$, we have

$$\zeta(\varrho(\varpi_{n+1},\varpi_n)) = \zeta(\varrho(\mathfrak{T}(\varpi_n,\ \vartheta_n),\ \mathfrak{T}(\varpi_{n-1},\vartheta_{n-1}))) \le \frac{1}{2}\chi(\varrho((\varpi_n,\varpi_{n-1}) + (\vartheta_n,\vartheta_{n-1}))) - \xi(\frac{\varrho(\varpi_n,\varpi_{n-1}) + \varrho(\vartheta_n,\vartheta_{n-1})}{2}), \text{ for all } n \in \mathbb{N} \tag{5}$$

and

$$\zeta(\varrho(\vartheta_{n+1},\vartheta_n)) = \zeta(\varrho(\mathfrak{T}(\vartheta_n,\varpi_n),\ \mathfrak{T}(\vartheta_{n-1},\ \varpi_{n-1}))) \le \frac{1}{2}\chi(\varrho(\vartheta_{n-1},\vartheta_n) + \varrho(\varpi_{n-1},\varpi_n))) - \xi(\frac{\varrho(\vartheta_{n-1},\vartheta_n) + \varrho(\varpi_{n-1},\varpi_n)}{2}), \text{ for all } n \in \mathbb{N}. \tag{6}$$

Adding (5) and (6), we have

$$\zeta(\varrho(\varpi_{n+1},\ \varpi_n)) + \zeta(\varrho(\vartheta_{n+1},\ \vartheta_n)) \le \chi(\varrho(\varpi_n,\varpi_{n-1}) + \varrho(\vartheta_n,\vartheta_{n-1})) - 2\xi(\frac{\varrho(\varpi_n,\varpi_{n-1}) + \varrho(\vartheta_n,\vartheta_{n-1})}{2}), \text{ for all } n \in \mathbb{N}. \tag{7}$$

By the 2nd property of the set of functions denoted by Y, we have

$$\zeta(\varrho(\varpi_n,\ \varpi_{n+1}) + \varrho(\vartheta_n,\ \vartheta_{n+1})) \le \zeta(\varrho(\varpi_n,\ \varpi_{n+1})) + \zeta(\varrho(\vartheta_n,\ \vartheta_{n+1})). \tag{8}$$

From (7) and (8), we have

$$\zeta(\varrho(\varpi_n,\ \varpi_{n+1}) + \varrho(\vartheta_n,\ \vartheta_{n+1})) \le \chi(\varrho(\varpi_n,\varpi_{n-1}) + \varrho(\vartheta_n,\vartheta_{n-1})) - 2\xi(\frac{\varrho(\varpi_n,\varpi_{n-1}) + \varrho(\vartheta_n,\vartheta_{n-1})}{2}), \text{ for all } n \in \mathbb{N}. \tag{9}$$

Take $L_n = \varrho(\varpi_n,\ \varpi_{n+1}) + \varrho(\vartheta_n,\ \vartheta_{n+1})$ for all $n \ge 0$. Using (9), we have

$$\zeta(L_n) \le \chi(L_{n-1}) - 2\,\xi(\frac{L_{n-1}}{2}). \tag{10}$$

Since $\xi(t) \ge 0$, we have $\zeta(L_n) \le \chi(L_{n-1})$. By (2), we get $L_n \le L_{n-1}$, that is, $\{L_n\}$ is a monotone decreasing sequence for all positive integer n. Hence there exists an $r \ge 0$ such that

$$L_n = \varrho(\varpi_n,\ \varpi_{n+1}) + \varrho(\vartheta_n,\ \vartheta_{n+1}) \longrightarrow r \text{ as } n \longrightarrow \infty. \tag{11}$$

Taking limit supremum in both sides of (10), using (11), the properties of χ and ξ, and the continuity of ζ, we obtain

$$\zeta(r) \le \overline{\lim}\,\chi(L_{n-1}) + 2\,\overline{\lim}\,(-\,\xi(\frac{L_{n-1}}{2})).$$

Since

$$2\,\overline{\lim}\,(-\,\xi(L_{n-1})) = -2\ \underline{\lim}\,\xi(\frac{L_{n-1}}{2}),$$

it follows that

$$\zeta(r) \le \overline{\lim}\,\chi(L_{n-1}) - 2\,\underline{\lim}\,\xi(\frac{L_{n-1}}{2}),$$

that is,

$$\zeta(r) - \overline{\lim}\chi(L_{n-1}) + 2\,\underline{\lim}\xi(\frac{L_{n-1}}{2}) \le 0,$$

which by (3), is a contradiction unless $r = 0$. Therefore,

$$L_n = \delta_n = \varrho(\varpi_n, \varpi_{n+1}) + \varrho(\vartheta_n, \vartheta_{n+1}) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

Hence the result. □

Proposition 2. *In addition to the hypotheses of Proposition 1 assume that $\mathscr{X}$ is complete. Then the sequences $\{\varpi_n\}$ and $\{\vartheta_n\}$ defined in Proposition 1 are Cauchy sequences in $\mathscr{A}_0$.*

Proof. Using Proposition 1, we have that $\{\delta_n\}$ is a monotone decreasing sequence and $\delta_n \to 0$.

Now, to prove $\{\varpi_n\}$ and $\{\vartheta_n\}$ are Cauchy sequences in $\mathscr{A}_0$.

Suppose that one of the sequences $\{\varpi_n\}$ or $\{\vartheta_n\}$ is not a Cauchy sequence. So that there exists $\epsilon > 0$ for which we can find subsequences $\{\varpi_{n(k)}\}, \{\varpi_{m(k)}\}$ of $\{\varpi_n\}$ and $\{\vartheta_{n(k)}\}, \{\vartheta_{m(k)}\}$ of $\{\vartheta_n\}$ respectively can be found considering $n(k)$ the smallest integer for which $n(k) > m(k) \geq k$, such that

$$\varrho(\varpi_{n(k)}, \varpi_{m(k)}) + \varrho(\vartheta_{n(k)}, \vartheta_{m(k)}) \geq \epsilon.$$

which means that,

$$\varrho(\varpi_{n(k)-1}, \varpi_{m(k)}) + \varrho(\vartheta_{n(k)-1}, \vartheta_{m(k)}) < \epsilon.$$

$$\begin{aligned}
\epsilon \leq & r_k = \varrho(\varpi_{n(k)}, \varpi_{m(k)}) + \varrho(\vartheta_{n(k)}, \vartheta_{m(k)}) \\
&\leq \varrho(\varpi_{n(k)}, \varpi_{n(k)-1}) + \varrho(\varpi_{n(k)-1}, \varpi_{m(k)}) + \varrho(\vartheta_{n(k)}, \vartheta_{n(k)-1}) + \varrho(\vartheta_{n(k)-1}, \vartheta_{m(k)}) \\
&\leq \varrho(\varpi_{n(k)}, \varpi_{n(k)-1}) + (\vartheta_{n(k)}, \vartheta_{n(k)-1}) + \epsilon.
\end{aligned}$$

Putting $k \longrightarrow \infty$ in the above inequality and applying (4), we have

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} [\varrho(\varpi_{n(k)}, \varpi_{m(k)}) + \varrho(\vartheta_{n(k)}, \vartheta_{m(k)})] = \epsilon. \tag{12}$$

Now,

$$\begin{aligned}
\epsilon \leq & r_k = \%(\pounds_{n(k)}, \pounds_{m(k)}) + (\#_{n(k)}, \#_{m(k)}) \\
&\leq \varrho(\varpi_{n(k)}, \varpi_{n(k)+1}) + \varrho(\varpi_{n(k)+1}, \varpi_{m(k)+1}) + \varrho(\varpi_{m(k)+1}, \varpi_{m(k)}) \\
&\quad + \varrho(\vartheta_{n(k)}, \vartheta_{n(k)+1}) + \varrho(\vartheta_{n(k)+1}, \vartheta_{m(k)+1}) + \varrho(\vartheta_{m(k)+1}, \vartheta_{m(k)}) \\
&\leq \varrho(\varpi_{n(k)+1}, \varpi_{m(k)+1}) + (\vartheta_{n(k)+1}, \vartheta_{m(k)+1}) + \delta_{n(k)} + \delta_{m(k)}.
\end{aligned}$$

where $\delta_{n(k)} = \varrho(\varpi_{n(k)}, \varpi_{n(k)+1}) + \varrho(\vartheta_{n(k)}, \vartheta_{n(k)+1})$ and $\delta_{m(k)} = \varrho(\varpi_{m(k)+1}, \varpi_{m(k)}) + \varrho(\vartheta_{m(k)+1}, \vartheta_{m(k)})$. Using 2nd property of the set of functions denoted by Υ, we get,

$$\begin{aligned}
\zeta(r_k) =& \zeta(\varrho(\varpi_{n(k)+1}, \varpi_{m(k)+1}) + \varrho(\vartheta_{n(k)+1}, \vartheta_{m(k)+1}) + \delta_{n(k)} + \delta_{m(k)}) \\
&\leq \zeta(\varrho(\varpi_{n(k)+1}, \varpi_{m(k)+1})) + \zeta(\varrho(\vartheta_{n(k)+1}, \vartheta_{m(k)+1})) + \zeta(\delta_{n(k)}) + \zeta(\delta_{m(k)}).
\end{aligned} \tag{13}$$

As $\varpi_{n(k)} \preceq \varpi_{m(k)}$ and $\vartheta_{n(k)} \succeq \vartheta_{m(k)}$ and $\mathfrak{T}$ is *PGCWC* on $\mathscr{A}$, we get

$$\zeta(\varrho(\varpi_{n(k)+1}, \varpi_{m(k)+1})) \leq \frac{1}{2}\chi(\varrho(\varpi_{n(k)}, \varpi_{m(k)}) + \varrho(\vartheta_{n(k)}, \vartheta_{m(k)})) - \xi(\frac{\varrho(\varpi_{n(k)}, \varpi_{m(k)}) + \varrho(\vartheta_{n(k)}, \vartheta_{m(k)})}{2}). \tag{14}$$

and

$$\zeta(\varrho(\vartheta_{n(k)+1}, \vartheta_{m(k)+1})) \leq \frac{1}{2}\chi(\varrho(\vartheta_{n(k)}, \vartheta_{m(k)}) + \varrho(\varpi_{n(k)}, \varpi_{m(k)})) - \xi(\frac{\varrho(\vartheta_{n(k)}, \vartheta_{m(k)}) + \varrho(\varpi_{n(k)}, \varpi_{m(k)})}{2}). \tag{15}$$

Using the 2nd property of the set of all functions denoted by Υ, (13), (14) and (15), we have

$$\begin{aligned}\zeta(\varrho(\varpi_{n(k)},\,\varpi_{m(k)})+\varrho(\vartheta_{n(k)},\,\vartheta_{m(k)})) &\leq \chi(\delta_{n(k)}+\delta_{m(k)})+\chi(\varrho(\varpi_{n(k)},\,\varpi_{m(k)})+\varrho(\vartheta_{n(k)},\,\vartheta_{m(k)}))\\ &\quad -2\,\xi(\frac{\varrho(\varpi_{n(k)},\,\varpi_{m(k)})+\varrho(\vartheta_{n(k)},\,\vartheta_{m(k)})}{2})+\zeta(\delta_{n(k)})+\zeta(\delta_{n(k)})\\ &\leq \chi(\delta_{n(k)})+\chi(\delta_{m(k)})+\chi(\varrho(\varpi_{n(k)},\,\varpi_{m(k)})+\varrho(\vartheta_{n(k)},\,\vartheta_{m(k)}))\\ &\quad -2\,\xi(\frac{\varrho(\varpi_{n(k)},\,\varpi_{m(k)})+\varrho(\vartheta_{n(k)},\,\vartheta_{m(k)})}{2})-\xi(\frac{\delta_{n(k)}}{2})-\xi(\frac{\delta_{m(k)}}{2}).\end{aligned}$$

Taking limit supremum in both sides of the above inequality, using (12) and (13), the properties of χ and ξ, contiunuity of ζ, we have

$$\begin{aligned}\zeta(\epsilon) \leq&\overline{\lim}\,\chi(0)+\overline{\lim}\,\chi(\varrho(\varpi_{n(k)},\,\varpi_{m(k)})\\ &+\varrho(\vartheta_{n(k)},\,\vartheta_{m(k)}))+2\,\overline{\lim}\,(-\,\xi(\frac{\varrho(\varpi_{n(k)},\,\varpi_{m(k)})+\varrho(\vartheta_{n(k)},\,\vartheta_{m(k)})}{2})\\ \leq&\overline{\lim}\,\chi(\varrho(\varpi_{n(k)},\,\varpi_{m(k)})+\varrho(\vartheta_{n(k)},\,\vartheta_{m(k)}))+2\,\overline{\lim}\,(-\,\xi(\frac{\varrho(\varpi_{n(k)},\,\varpi_{m(k)})+\varrho(\vartheta_{n(k)},\,\vartheta_{m(k)})}{2})\\ &+\overline{\lim}\,(-\,\xi(0))+\overline{\lim}\,(-\,\xi(0)).\end{aligned}$$

Since

$$2\,\overline{\lim}\,(-\,\xi(\varrho(\varpi_{n(k)},\,\varpi_{m(k)})+\varrho(\vartheta_{n(k)},\,\vartheta_{m(k)})))=-2\,\underline{\lim}\,\xi(\frac{\varrho(\varpi_{n(k)},\,\varpi_{m(k)})+\varrho(\vartheta_{n(k)},\,\vartheta_{m(k)})}{2}),$$

it follows that,

$$\zeta(\epsilon)\leq\overline{\lim}\,\chi(\varrho(\varpi_{n(k)},\,\varpi_{m(k)})+\varrho(\vartheta_{n(k)},\,\vartheta_{m(k)}))-2\,\underline{\lim}\,\xi(\frac{\varrho(\varpi_{n(k)},\,\varpi_{m(k)})+\varrho(\vartheta_{n(k)},\,\vartheta_{m(k)})}{2}),$$

that is,

$$\zeta(\epsilon)-\overline{\lim}\,\chi(\varrho(\varpi_{n(k)},\,\varpi_{m(k)})+\varrho(\vartheta_{n(k)},\,\vartheta_{m(k)}))+2\,\underline{\lim}\,\xi(\frac{\varrho(\varpi_{n(k)},\,\varpi_{m(k)})+\varrho(\vartheta_{n(k)},\,\vartheta_{m(k)})}{2})\leq 0,$$

which is a contradiction due to (3). Therefore, $\{\varpi_n\}$ and $\{\vartheta_n\}$ are Cauchy sequences in $\mathscr{A}_0$. □

Theorem 2. *Let (Ω,ϱ) be a POMS and $\mathscr{A}$, $\mathscr{B}$ be non-empty closed subsets of complete set $\mathscr{X}$ induced with metric ϱ such that $\mathscr{A}_0\neq\emptyset$ closed and $(\mathscr{A},\,\mathscr{B})$ satisfies P-property. Suppose that $\mathfrak{T}:\mathscr{A}\times\mathscr{A}\to\mathscr{B}$ such that $\mathfrak{T}(\mathscr{A}_0\times\mathscr{A}_0)\subseteq\mathscr{B}_0$ and $\mathfrak{T}$ satisfies the proximal mixed monotone property and $\mathfrak{T}$ is PGCWC on $\mathscr{A}$. Suppose that*

$$\zeta(\varpi)\leq\psi(\vartheta)\implies\varpi\leq\vartheta, \tag{16}$$

for any sequence $\{\varpi_n\}$ in $[0,\,\infty)$ with $\varpi_n\longrightarrow t>0$,

$$\zeta(t)-\overline{\lim}\,\chi(\varpi_n)+2\,\underline{\lim}\,\xi(\varpi_n)>0, \tag{17}$$

where $\zeta\in\Upsilon,\chi\in\Xi$ and $\xi\in\Gamma$.

Assume that there exist (ϖ_0,ϑ_0) and (ϖ_1,ϑ_1)in $\mathscr{A}\times\mathscr{A}$ such that $\varpi_1=\mathfrak{T}(\varpi_0,\,\vartheta_0)$ with $\varpi_0\preceq\varpi_1$ and $\vartheta_1=\mathfrak{T}(\vartheta_0,\,\varpi_0)$ with $\vartheta_0\succeq\vartheta_1$.

Further, suppose that either

(a) *$\mathfrak{T}$ is continuous or*

(*b*) *if* $\{\varpi_n\}$, $\{\vartheta_n\}$ *are non-decreasing sequences in* $\mathscr{X}$ *such that* $\varpi_n \longrightarrow \varpi$ *and* $\vartheta_n \longrightarrow \vartheta$ *then* $\varpi_n \preceq \varpi$, $\vartheta \preceq \vartheta_n$ *for all* $n \geq 0$.
Then, $\mathfrak{T}$ *has a CBPP, that is, there exists* $(\varpi, \vartheta) \in \mathscr{A}_0 \times \mathscr{A}_0$ *such that*

$$\varrho(\varpi,\ \mathfrak{T}(\varpi,\ \vartheta)) = \varrho(\mathscr{A},\ \mathscr{B}) \text{ and } \varrho(\vartheta,\ \mathfrak{T}(\vartheta,\ \varpi)) = (\mathscr{A},\ \mathscr{B}).$$

Proof. By the conditions of the Theorem 2, there exist elements $(\varpi_0,\ \vartheta_0), (\varpi_1,\ \vartheta_1) \in \mathscr{A}_0 \times \mathscr{A}_0$ such that

$$\varrho(\varpi_1,\ \mathfrak{T}(\varpi_0,\ \vartheta_0)) = \varrho(\mathscr{A},\ \mathscr{B}) \quad \text{with} \quad \varpi_0 \preceq \varpi_1, \text{ and}$$

$$\varrho(\vartheta_1,\ \mathfrak{T}(\vartheta_0,\ \varpi_0)) = \varrho(\mathscr{A},\ \mathscr{B}) \quad \text{with} \quad \vartheta_1 \preceq \vartheta_0.$$

As $\mathfrak{T}(\mathscr{A}_0 \times \mathscr{A}_0) \subseteq \mathscr{B}_0$, there exists an element $(\varpi_2, \vartheta_2) \in \mathscr{A}_0 \times \mathscr{A}_0$ such that

$$\varrho(\varpi_2,\ \mathfrak{T}(\varpi_1, \vartheta_1)) = \varrho(\mathscr{A},\ \mathscr{B}), \text{ and}$$
$$\varrho(\vartheta_2,\ \mathfrak{T}(\vartheta_1, \varpi_1)) = \varrho(\mathscr{A},\ \mathscr{B}).$$

By the use of Lemmas 1 and 2, we obtain $\varpi_1 \preceq \varpi_2$ and $\vartheta_2 \succeq \vartheta_1$. Iterating in the same way, we can construct the sequences $\{\varpi_n\}$ and $\{\vartheta_n\}$ in $\mathscr{A}_0$ such that

$$\varrho(\varpi_{n+1},\ \mathfrak{T}(\varpi_n,\ \vartheta_n)) = \varrho(\mathscr{A},\ \mathscr{B}) \text{ for all } n \geq 0 \text{ with } \varpi_0 \preceq \varpi_1 \preceq \varpi_2 \preceq \varpi_3 \preceq \ldots \preceq \varpi_n. \tag{18}$$

$$\varrho(\vartheta_{n+1},\ \mathfrak{T}(\vartheta_n,\ \varpi_n)) = \varrho(\mathscr{A},\ \mathscr{B}) \text{ for all } n \geq 0 \text{ with } \vartheta_0 \succeq \vartheta_1 \succeq \vartheta_2 \succeq \vartheta_3 \succeq \ldots \succeq \vartheta_n. \tag{19}$$

Then

$$\varrho(\varpi_n, \mathfrak{T}(\varpi_{n-1}, \vartheta_{n-1})) = \varrho(\mathscr{A},\ \mathscr{B}), \text{ with } \varpi_{n-1} \preceq \varpi_n$$

and

$$\varrho(\varpi_{n+1}, \mathfrak{T}(\varpi_n, \vartheta_n)) = \varrho(\mathscr{A},\ \mathscr{B}) \text{ with } \vartheta_{n-1} \succeq \vartheta_n,$$

for all $n \in \mathbb{N}$.

Using Propositions 1 and 2, we have that $\{L_n\}$ is a monotone decreasing sequence, $L_n \to 0$ and $\{\varpi_n\}$ and $\{\vartheta_n\}$ are Cauchy sequences in $\mathscr{A}_0$.

As $\mathscr{X}$ is complete, $\mathscr{A}_0 \subset \mathscr{X}$ and $\mathscr{A}_0$ is closed, hence $\mathscr{A}_0$ is also complete. So, by the completeness of $\mathscr{A}$, there are elements ϖ^*, ϑ^* such that $\varpi_n \to \varpi^*$ and $\vartheta_n \to \vartheta^*$ as $n \to \infty$.

Therefore,

$$\lim_{n\to\infty} \varrho((\varpi_n,\ \vartheta_n),\ (\varpi^*,\ \vartheta^*)) = 0, \tag{20}$$

and

$$\lim_{n\to\infty} \varrho((\vartheta_n,\ \varpi_n),\ (\vartheta^*,\ \varpi^*)) = 0. \tag{21}$$

Let the condition (a) hold.
So, by the continuity of $\mathfrak{T}$,

$$\mathfrak{T}(\varpi_n,\ \vartheta_n) \to \mathfrak{T}(\varpi^*,\ \vartheta^*), \qquad \mathfrak{T}(\vartheta_n,\ \varpi_n) \to \mathfrak{T}(\vartheta^*,\ \varpi^*).$$

Now, from (3), (18) and the continuity of the metric ϱ, we get

$$\varrho(\varpi^*,\ \mathfrak{T}(\varpi,\ \vartheta)) = \varrho(\mathscr{A},\ \mathscr{B}) \text{ and } \varrho(\vartheta^*,\ \mathfrak{T}(\vartheta,\ \varpi)) = \varrho(\mathscr{A},\ \mathscr{B}).$$

Let the condition (b) hold.

Now,

$$\varrho(\varpi_{n+1}, \mathfrak{T}(\varpi_n, \vartheta_n)) = \varrho(\mathscr{A}, \mathscr{B}) \text{ with } \varpi_n \preceq \varpi_{n+1} \tag{22}$$

and

$$\varrho(\vartheta_{n+1}, \mathfrak{T}(\vartheta_n, \varpi_n)) = \varrho(\mathscr{A}, \mathscr{B}) \text{ with } \vartheta_n \succeq \vartheta_{n+1}. \tag{23}$$

Also, $\varpi_n \to \varpi$ and $\vartheta_n \to \vartheta$ with $\varpi_n \preceq \varpi$ and $\vartheta_n \succeq \vartheta$ and $\mathscr{A}_0$ is closed. Therefore, $(\varpi, \vartheta) \in \mathscr{A}_0 \times \mathscr{A}_0$. Since $\mathfrak{T}(\mathscr{A}_0 \times \mathscr{A}_0) \subseteq \mathscr{B}_0$, there exist elements $\mathfrak{T}(\varpi, \vartheta), \mathfrak{T}(\vartheta, \varpi) \in \mathscr{B}_0$. So, there is $(\varpi^*, \vartheta^*) \in \mathscr{A}_0 \times \mathscr{A}_0$, such that

$$\varrho(\varpi^*, \mathfrak{T}(\varpi, \vartheta)) = \varrho(\mathscr{A}, \mathscr{B}) \tag{24}$$

and

$$\varrho(\vartheta^*, \mathfrak{T}(\vartheta, \varpi)) = \varrho(\mathscr{A}, \mathscr{B}). \tag{25}$$

By P-property of $(\mathscr{A}, \mathscr{B})$, (22), (23), (24) and (25) respectively, we have

$$\varrho(\varpi_{n+1}, \varpi^*) = \varrho(\mathfrak{T}(\varpi_n, \vartheta_n), \mathfrak{T}(\varpi, \vartheta)) \tag{26}$$

and

$$\varrho(\vartheta^*, \vartheta_{n+1}) = \varrho(\mathfrak{T}(\vartheta, \varpi), \mathfrak{T}(\vartheta_n, \varpi_n)). \tag{27}$$

Since $\varpi_n \to \varpi$ and $\vartheta_n \to \vartheta$, using $PGCWC$ property of $\mathfrak{T}$, we have

$$\begin{aligned} \zeta(\varrho(\varpi_{n+1}, \varpi^*)) &= \zeta(\varrho(\mathfrak{T}(\varpi_n, \vartheta_n), \mathfrak{T}(\varpi, \vartheta))) \\ &\leq \frac{1}{2}\chi(\varrho(\varpi_n, \varpi) + \varrho(\vartheta_n, \vartheta)) - \xi(\frac{\varrho(\varpi_n, \varpi) + \varrho(\vartheta_n, \vartheta)}{2}) \text{ for all } n, \end{aligned} \tag{28}$$

and

$$\begin{aligned} \zeta(\varrho(\vartheta^*, \vartheta_{n+1})) &= \zeta(\mathfrak{T}(\vartheta, \varpi), \varrho(\mathfrak{T}(\vartheta_n, \varpi_n)) \\ &\leq \frac{1}{2}\chi(\varrho(\vartheta, \vartheta_n) + \varrho(\varpi, \varpi_n)) - \xi(\frac{\varrho(\vartheta, \vartheta_n) + \varrho(\varpi, \varpi_n)}{2}) \text{ for all } n. \end{aligned} \tag{29}$$

Again, using the 2nd property of the set of all functions denoted by Υ, we get

$$\begin{aligned} \zeta(\varrho(\varpi_{n+1}, \varpi^*) + \varrho(\vartheta^*, \vartheta_{n+1})) &\leq \zeta(\varrho(\varpi_{n+1}, \varpi^*)) + \zeta(\varrho(\vartheta^*, \vartheta_{n+1})) \\ &\leq \chi(\varrho(\varpi_n, \varpi) + \varrho(\vartheta_n, \vartheta)) - 2\,\xi(\frac{\varrho(\varpi_n, \varpi) + \varrho(\vartheta, \vartheta_n)}{2}). \end{aligned}$$

Taking $n \to \infty$ in the above inequality, we have

$$\begin{aligned} \lim_{n\to\infty}\zeta(\varrho(\varpi_{n+1}, \varpi^*) + \varrho(\vartheta^*, \vartheta_{n+1})) &\leq \lim_{n\to\infty}\zeta(\varrho(\varpi_{n+1}, \varpi^*)) + \lim_{n\to\infty}\zeta(\varrho(\vartheta^*, \vartheta_{n+1})) \\ &\leq \lim_{n\to\infty}\chi(\varrho(\varpi_n, \varpi) + \varrho(\vartheta_n, \vartheta)) - 2\lim_{n\to\infty}\xi(\frac{\varrho(\varpi_n, \varpi) + \varrho(\vartheta, \vartheta_n)}{2}), \end{aligned}$$

that is,

$$\zeta(\varrho(\varpi_{n+1}, \varpi)) + \zeta(\varrho(\vartheta, \vartheta_{n+1})) \leq 0.$$

It implies $\zeta(\varrho(\varpi_{n+1}, \varpi)) \leq 0$ and $\zeta(\varrho(\vartheta, \vartheta_{n+1})) \leq 0$. Therefore, $\varpi = \varpi^*$ and $\vartheta = \vartheta^*$ Now, using (24) and (25), we have

$$\varrho(\vartheta^*, \mathfrak{T}(\varpi, \vartheta)) = \varrho(\mathscr{A}, \mathscr{B}) \text{ and } \varrho(\vartheta^*, \mathfrak{T}(\vartheta, \varpi)) = \varrho(\mathscr{A}, \mathscr{B}).$$

Hence the result. □

Theorem 3. *In addition to the hypotheses of Theorem 2, assume that for any two elements* (ϖ, ϑ) *and* (ϖ^*, ϑ^*) *in* $\mathscr{A}_0 \times \mathscr{A}_0$*, there exists* $(u_1, v_1) \in \mathscr{A}_0 \times \mathscr{A}_0$ *such that* (u_1, v_1) *is comparable to* (ϖ, ϑ) *and* (ϖ^*, ϑ^*)*, then* $\mathfrak{T}$ *has a unique CBPP.*

Proof. From Theorem 2, the set of coupled best proximity points of $\mathfrak{T} \neq \emptyset$. Assume that there exist (ϖ, ϑ) and (ϖ^*, ϑ^*) in $\mathscr{A} \times \mathscr{A}$ which are coupled best proximity points.

So,

$$\varrho(\varpi, \mathfrak{T}(\varpi, \vartheta)) = \varrho(\mathscr{A}, \mathscr{B}), \ \varrho(\vartheta, \mathfrak{T}(\vartheta, \varpi)) = \varrho(\mathscr{A}, \mathscr{B}) \text{ and}$$

$$\varrho(\varpi^*, \mathfrak{T}(\varpi^*, \vartheta^*)) = \varrho(\mathscr{A}, \mathscr{B}), \ d(\vartheta^*, \mathfrak{T}(\vartheta^*, \varpi^*)) = \varrho(\mathscr{A}, \mathscr{B}).$$

The following two cases arise:

Case I:

With the assumption of comparability of (ϖ, ϑ), say (ϖ, ϑ) is comparable to (ϖ^*, ϑ^*) where the ordering prevails in $\mathscr{A} \times \mathscr{A}$. As $\mathfrak{T}$ is *PGCWC* on $\mathscr{A}$ to $\varrho(\varpi, \mathfrak{T}(\varpi, \vartheta)) = \varrho(\mathscr{A}, \mathscr{B})$ and $\varrho(\varpi^*, \mathfrak{T}(\varpi^*, \vartheta^*)) = \varrho(\mathscr{A}, \mathscr{B})$, we have

$$\zeta(\varrho(\varpi, \varpi^*)) \leq \frac{1}{2}\chi(\varrho(\varpi, \varpi^*) + \varrho(\vartheta, \vartheta^*)) - \xi(\frac{\varrho(\varpi, \varpi^*) + \varrho(\vartheta, \vartheta^*)}{2}). \tag{30}$$

Similarly, it can be proved that

$$\zeta(\varrho(\vartheta, \vartheta^*)) \leq \frac{1}{2}\chi(\varrho(\vartheta, \vartheta^*) + \varrho(\varpi, \varpi^*)) - \xi(\frac{\varrho(\vartheta, \vartheta^*) + \varrho(\varpi, \varpi^*)}{2}). \tag{31}$$

Adding (30) and (31), we get

$$\zeta(\varrho(\varpi, \varpi^*)) + \zeta(\varrho(\vartheta, \vartheta^*)) \leq \chi(\varrho(\varpi, \varpi^*) + \varrho(\vartheta, \vartheta^*)) - 2\xi(\frac{\varrho(\varpi, \varpi^*) + \varrho(\vartheta, \vartheta^*)}{2}). \tag{32}$$

Applying the 2nd property of the set of all functions denoted by Υ, we have

$$\zeta(\varrho(\varpi, \varpi^*) + \varrho(\vartheta, \vartheta^*)) \leq \zeta(\varrho(\varpi, \varpi^*)) + \zeta(\varrho(\vartheta, \vartheta^*)). \tag{33}$$

Using (32) and (33), we have

$$\zeta(\varrho(\varpi, \varpi^*) + \varrho(\vartheta, \vartheta^*)) \leq \chi(\varrho(\varpi, \varpi^*) + \varrho(\vartheta, \vartheta^*)) - 2\xi(\frac{\varrho(\varpi, \varpi^*) + \varrho(\vartheta, \vartheta^*)}{2}). \tag{34}$$

Imposing limit supremum in both sides of the above inequality, the properties of χ and ξ, contiunuity of ζ, we have

$$\zeta(\varrho(\varpi, \varpi^*) + \varrho(\vartheta, \vartheta^*)) \leq \overline{\lim}\, \chi(\varrho(\varpi, \varpi^*) + \varrho(\vartheta, \vartheta^*)) + \overline{\lim}\, 2\xi(-(\frac{\varrho(\varpi, \varpi^*) + \varrho(\vartheta, \vartheta^*)}{2})). \tag{35}$$

Since,

$$\overline{\lim}\, 2\xi(-(\frac{\varrho(\varpi, \varpi^*) + \varrho(\vartheta, \vartheta^*)}{2})) = -\underline{\lim}\, 2\xi(\frac{\varrho(\varpi, \varpi^*) + \varrho(\vartheta, \vartheta^*)}{2}),$$

From (35), we have

$$\zeta(\varrho(\varpi, \varpi^*) + \varrho(\vartheta, \vartheta^*)) - \overline{\lim}\, \chi(\varrho(\varpi, \varpi^*) + \varrho(\vartheta, \vartheta^*)) + \underline{\lim}\, 2\xi(-(\frac{\varrho(\varpi, \varpi^*) + \varrho(\vartheta, \vartheta^*)}{2}) \leq 0,$$

which lead us to a contradiction and consequently, $\varrho(\varpi, \varpi^*) + \varrho(\vartheta, \vartheta^*) = 0$, that is, $\varrho(\varpi, \varpi^*) = 0$ and $\varrho(\vartheta, \vartheta^*) = 0$. Hence $\varpi = \varpi^*$ and $\vartheta = \vartheta^*$.

Case II:

This case arises when $(\varpi,\ \vartheta)$ is not comparable to $(\varpi^*,\ \vartheta^*)$. So, on the assumption of existence of an element $(u_1,\ v_1) \in \mathscr{A}_0 \times \mathscr{A}_0$ which is comparable to $(\varpi,\ \vartheta)$ and $\mathfrak{T}(\mathscr{A}_0 \times \mathscr{A}_0) \subseteq \mathscr{B}_0$, there is $(u_2,\ v_2) \in \mathscr{A}_0 \times \mathscr{A}_0$ such that

$$\varrho(u_2,\ \mathfrak{T}(u_1,\ v_1)) = \varrho(\mathscr{A},\ \mathscr{B}) \text{ and } \varrho(v_2,\ \mathfrak{T}(v_1,\ u_1)) = \varrho(\mathscr{A},\ \mathscr{B}).$$

From Lemmas 1 and 2, we have

$$\left.\begin{array}{l} u_1 \preceq \varpi \text{ and } v_1 \succeq \vartheta \\ \varrho(u_2,\ \mathfrak{T}(u_1,\ v_1) = \varrho(\mathscr{A},\ \mathscr{B}), \\ \varrho(\varpi,\ \mathfrak{T}(\varpi,\ \vartheta)) = \varrho(\mathscr{A},\ \mathscr{B}) \end{array}\right\} \Rightarrow u_2 \preceq \varpi$$

and

$$\left.\begin{array}{l} u_1 \preceq \varpi \text{ and } v_1 \succeq \vartheta \\ \varrho(v_2,\ \mathfrak{T}(v_1,\ u_1) = \varrho(\mathscr{A},\ \mathscr{B}), \\ \varrho(\vartheta,\ \mathfrak{T}(\vartheta,\ \varpi)) = \varrho(\mathscr{A},\ \mathscr{B}). \end{array}\right\} \Rightarrow v_2 \succeq \vartheta.$$

From the above inequalities, we have $u_2 \preceq \varpi$ and $v_2 \succeq \vartheta$. Iterating in the same way, we get sequences $\{u_n\}, \{v_n\}$ such that

$$\begin{array}{l} \varrho(u_{n+1},\ \mathfrak{T}(u_n,\ v_n)) = \varrho(\mathscr{A},\ \mathscr{B}) \\ \text{and} \\ \varrho(v_{n+1},\ \mathfrak{T}(v_n,\ u_n)) = \varrho(\mathscr{A},\ \mathscr{B}), \end{array}$$

with $u_n \preceq \varpi,\ v_n \succeq \vartheta$ for all $n \in \mathbb{N}$. Now,

$$\varrho(u_{n+1},\ \mathfrak{T}(u_n,\ v_n)) = \varrho(\mathscr{A},\ \mathscr{B})$$

and

$$\varrho(\varpi,\ \mathfrak{T}(\varpi,\ \vartheta)) = \varrho(\mathscr{A},\ \mathscr{B}).$$

So, applying P-property, we have

$$\zeta(\varrho(u_{n+1},\ \varpi)) = \zeta(\mathfrak{T}(u_n,\ v_n),\ \mathfrak{T}(\varpi,\ \vartheta)).$$

Now, using the fact that $\mathfrak{T}$ is $PGCWC$ on $\mathscr{A}$, we have

$$\zeta(\varrho(u_{n+1},\ \varpi)) \leq \frac{1}{2}\chi(\varrho(u_n,\ \varpi) + \varrho(v_n,\ \vartheta)) - \xi(\frac{\varrho(u_n,\ \varpi) + \varrho(v_n,\ \vartheta)}{2}). \quad (36)$$

Similarly, we have

$$\zeta(\varrho(\vartheta,\ v_{n+1})) \leq \frac{1}{2}\chi(\varrho(\vartheta,\ v_n) + \varrho(\varpi,\ u_n)) - \xi(\frac{\varrho(\vartheta,\ v_n) + \varrho(\varpi,\ u_n)}{2}). \quad (37)$$

Adding (36) and (37), we have

$$\zeta(\varrho(u_{n+1},\ \varpi) + \varrho(\vartheta,\ v_{n+1})) \leq \chi(\varrho(\varpi,\ u_n) + \varrho(\vartheta,\ v_n)) - 2\xi(\frac{\varrho(\vartheta,\ v_n) + \varrho(\varpi,\ u_n)}{2}). \quad (38)$$

It implies

$$\zeta(\varrho(u_{n+1},\ \varpi) + \varrho(\vartheta,\ v_{n+1})) \leq \chi(\varrho(\varpi,\ u_n) + \varrho(\vartheta,\ v_n)).$$

Using (2) in the above inequality, we have

$$\varrho(u_{n+1},\ \varpi) + \varrho(\vartheta,\ v_{n+1}) \leq \varrho(\varpi,\ u_n) + \varrho(\vartheta,\ v_n).$$

This shows that the sequence $\{\varrho(\varpi,\ u_n) + \varrho(\vartheta,\ v_n)\}$ is a decreasing sequence. Therefore, there exists $l \geq 0$ such that

$$\lim_{n\to\infty} \varrho(\varpi,\ u_n) + \varrho(\vartheta,\ v_n) = l.$$

Now, to prove $l = 0$. On the contrary, assume that $l > 0$. Imposing limit supremum in both sides of (38), the properties of χ and ξ, contiunuity of ζ, we have

$$\zeta(l) \leq \overline{\lim}\ \chi(\varrho(\vartheta,\ v_n) + \varrho(\varpi,\ u_n)) + \overline{\lim}\ 2\xi(-(\frac{\varrho(\varpi,\ u_n) + \varrho(\vartheta,\ v_n)}{2})).$$

But $\overline{\lim}\ 2\xi(-(\frac{\varrho(\varpi,\ u_n) + \varrho(\vartheta,\ v_n)}{2})) = -\underline{\lim}\ 2\xi(\frac{\varrho(\varpi,\ u_n) + \varrho(\vartheta,\ v_n)}{2})$ and as a consequence,

$$\zeta(l) - \overline{\lim}\ \chi(\varrho(\vartheta,\ v_n) + \varrho(\varpi,\ u_n)) + \underline{\lim}\ 2\xi(\frac{\varrho(\varpi,\ u_n) + \varrho(\vartheta,\ v_n)}{2}) \leq 0,$$

which is a contradiction. Therefore, $l = 0$, that is,

$$\lim_{n\to\infty} \varrho(\varpi,\ u_n) + \varrho(\vartheta,\ v_n) = 0.$$

It implies $u_n \to \varpi$ and $v_n \to \vartheta$. In a similar way, we can prove that $u_n \to \varpi^*$ and $v_n \to \vartheta^*$. Consequently, $\varpi = \varpi^*$ and $\vartheta = \vartheta^*$. Hence the theorem. □

4. Consequences Related to Fixed Point Results

The results discussed in the previous section have the following consequences in the fixed point category.

If we assume $\varrho(\mathscr{A},\ \mathscr{B}) = 0$, that is, $\mathscr{A} = \mathscr{B} = \mathscr{X}$, we have the following theorem.

Theorem 4. *Let (Ω, ϱ) be a POMS and $\mathfrak{T} : \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ be a mapping having the mixed monotone property on $\mathscr{X}$ such that there exist two elements $\varpi_0, \vartheta_0 \in \mathscr{X}$ with*

$$\varpi_0 \preceq \mathfrak{T}(\varpi_0,\ \vartheta_0),\ \vartheta_0 \succeq \mathfrak{T}(\vartheta_0,\ \varpi_0).$$

Suppose that

$$\zeta(\varpi) \leq \chi(\vartheta) \implies \varpi \leq \vartheta, \tag{39}$$

for any sequence $\{\varpi_n\}$ in $[0,\ \infty)$ with $\varpi_n \longrightarrow t > 0$,

$$\zeta(t) - \overline{\lim}\ \chi(\varpi_n) + 2\ \underline{\lim}\ \xi(\varpi_n) > 0, \tag{40}$$

and

$$\zeta(\varrho(\mathfrak{T}(\varpi,\vartheta),\mathfrak{T}(u,v)) \leq \frac{1}{2}\chi(\varrho(\varpi,u) + \varrho(\vartheta,v)) - \xi(\frac{\varrho(\varpi,u) + \varrho(\vartheta,v)}{2}), \tag{41}$$

where $\zeta \in \Upsilon, \chi \in \Xi$ and $\xi \in \Gamma, \varpi, \vartheta, u, v \in \mathscr{X}$.

Further suppose that $\mathscr{X}$ is complete and any of the following conditions holds:

(a) $\mathfrak{T}$ is continuous or

(b) If $\{\varpi_n\}$, $\{\vartheta_n\}$ are non-decreasing sequences in X such that $\varpi_n \longrightarrow \varpi$ and $\vartheta_n \longrightarrow \vartheta$ then $\varpi_n \preceq \varpi$, $\vartheta \preceq \vartheta_n$ for all $n \geq 0$.

Then $\mathfrak{T}$ has a CFP in X, if there exist $\varpi, \vartheta \in \mathscr{X}$, that is, $\varpi = \mathfrak{T}(\varpi,\vartheta)$ and $\vartheta = \mathfrak{T}(\vartheta,\varpi)$.

Proof. By the statement of the theorem, $\varpi_0, \vartheta_0 \in \mathscr{X}$ such that

$$\varpi_0 \preceq \mathfrak{T}(\varpi_0,\ \vartheta_0),\ \vartheta_0 \succeq \mathfrak{T}(\vartheta_0,\ \varpi_0).$$

Construct two sequences, $\{\varpi_n\}$, $\{\vartheta_n\}$ in $\mathscr{X}$ defined as follows

$$\varpi_{n+1} = \mathfrak{T}(\varpi_n,\ \vartheta_n) \text{ and } \vartheta_{n+1} = \mathfrak{T}(\vartheta_n,\ \varpi_n) \text{ for all } n \geq 0. \tag{42}$$

We have to show that

$$\varpi_n \preceq \varpi_{n+1} \tag{43}$$

and

$$\vartheta_n \succeq \vartheta_{n+1}. \tag{44}$$

To prove this, we use mathematical induction. Let $n = 0$. As $\varpi_0 \preceq \mathfrak{T}(\varpi_0,\ \vartheta_0)$, $\vartheta_0 \succeq (\vartheta_0,\ \varpi_0)$ and $\varpi_1 = \mathfrak{T}(\varpi_0,\ \vartheta_0)$, $\vartheta_1 = \mathfrak{T}(\vartheta_0,\ \varpi_0)$, we have

$$\varpi_0 \preceq \varpi_1 \text{ and } \vartheta_0 \succeq \vartheta_1.$$

So from (43) and (44), we can say that mathematical induction holds for $n = 0$.
Now, assume that (43) and (44) hold for for some fixed $n \geq 0$.
By mixed monotone property of $\mathfrak{T}$ and $\varpi_n \preceq \varpi_{n+1}$ and $\vartheta_n \succeq \vartheta_{n+1}$, we get

$$\varpi_{n+2} = \mathfrak{T}(\varpi_{n+1},\ \vartheta_{n+1}) \succeq \mathfrak{T}(\varpi_n,\ \vartheta_{n+1}) \succeq \mathfrak{T}(\varpi_n,\ \vartheta_n) = \varpi_{n+1} \tag{45}$$

and

$$\vartheta_{n+2} = \mathfrak{T}(\vartheta_{n+1},\ \varpi_{n+1}) \preceq \mathfrak{T}(\vartheta_n,\ \varpi_{n+1}) \preceq \mathfrak{T}(\vartheta_n,\ \varpi_n) = \vartheta_{n+1}. \tag{46}$$

So, by (45) and (46), we get

$$\varpi_{n+1} \preceq \varpi_{n+2} \text{ and } \vartheta_{n+1} \succeq \vartheta_{n+2}.$$

So, by mathematical induction we can conclude that (43) and (44) hold for all $n \geq 0$.
Therefore,

$$\varpi_0 \preceq \varpi_1 \preceq \varpi_2 \ldots \varpi_n \preceq \varpi_{n+1} \ldots$$

and

$$\vartheta_0 \succeq \vartheta_1 \succeq \vartheta_2 \ldots \succeq \vartheta_n \succeq \vartheta_{n+1} \ldots.$$

Since $\varpi_{n+1} \preceq \varpi_n$ and $\varpi_{n+1} \succeq \varpi_n$, from (39), we get

$$\begin{aligned}\zeta(\varrho(\varpi_{n+1},\varpi_n)) &= \zeta(\varrho(\mathfrak{T}(\varpi_n,\ \vartheta_n),\ \mathfrak{T}(\varpi_{n-1},\vartheta_{n-1}))) \leq \frac{1}{2}\chi(\varrho((\varpi_n,\varpi_{n-1}) + (\vartheta_n,\vartheta_{n-1}))) \\ &\quad -\xi(\frac{\varrho(\varpi_n,\varpi_{n-1}) + \varrho(\vartheta_n,\vartheta_{n-1})}{2}),\ \text{ for all } n \in \mathbb{N}\end{aligned} \tag{47}$$

and

$$\begin{aligned}\zeta(\varrho(\vartheta_{n+1},\vartheta_n)) &= \zeta(\varrho(\mathfrak{T}(\vartheta_n,\varpi_n),\ \mathfrak{T}(\vartheta_{n-1},\ \varpi_{n-1}))) \\ &\leq \frac{1}{2}\chi(\varrho(\vartheta_{n-1},\vartheta_n) + \varrho(\varpi_{n-1},\varpi_n))) - \xi(\frac{\varrho(\vartheta_{n-1},\vartheta_n) + \varrho(\varpi_{n-1},\varpi_n)}{2}),\ \text{ for all } n \in \mathbb{N}.\end{aligned} \tag{48}$$

Adding (47) and (48), we have

$$\zeta(\varrho(\varpi_{n+1}, \varpi_n)) + \zeta(\varrho(\vartheta_{n+1}, \vartheta_n)) \leq \chi(\varrho(\varpi_n, \varpi_{n-1}) + \varrho(\vartheta_n, \vartheta_{n-1})) - 2\xi(\frac{\varrho(\varpi_n, \varpi_{n-1}) + \varrho(\vartheta_n, \vartheta_{n-1})}{2}), \text{ for all } n \in \mathbb{N}. \quad (49)$$

By the 2nd property of the set of functions denoted by Υ, we have

$$\zeta(\varrho(\varpi_n, \varpi_{n+1}) + \varrho(\vartheta_n, \vartheta_{n+1})) \leq \zeta(\varrho(\varpi_n, \varpi_{n+1})) + \zeta(\varrho(\vartheta_n, \vartheta_{n+1})). \quad (50)$$

From (49) and (50), we have

$$\zeta(\varrho(\varpi_n, \varpi_{n+1}) + \varrho(\vartheta_n, \vartheta_{n+1})) \leq \chi(\varrho(\varpi_n, \varpi_{n-1}) + \varrho(\vartheta_n, \vartheta_{n-1})) - 2\xi(\frac{\varrho(\varpi_n, \varpi_{n-1}) + \varrho(\vartheta_n, \vartheta_{n-1})}{2}), \text{ for all } n \in \mathbb{N}. \quad (51)$$

Take $L_n = \varrho(\varpi_n, \varpi_{n+1}) + \varrho(\vartheta_n, \vartheta_{n+1})$ for all $n \geq 0$. Using (51), we have

$$\zeta(L_n) \leq \chi(L_{n-1}) - 2\,\xi(\frac{L_{n-1}}{2}). \quad (52)$$

Since $\xi(t) \geq 0$, we have $\zeta(L_n) \leq \chi(L_{n-1})$. By (2), we get $L_n \leq L_{n-1}$, that is, $\{L_n\}$ is a monotone decreasing sequence for all positive integer n. Hence there exists an $r \geq 0$ such that

$$L_n = \varrho(\varpi_n, \varpi_{n+1}) + \varrho(\vartheta_n, \vartheta_{n+1}) \longrightarrow r \text{ as } n \longrightarrow \infty. \quad (53)$$

Taking limit supremum in both sides of (52), using (53), the properties of χ and ξ, and the continuity of ζ, we obtain

$$\zeta(r) \leq \overline{\lim}\,\chi(L_{n-1}) + 2\,\overline{\lim}\,(-\,\xi(\frac{L_{n-1}}{2})).$$

Since

$$2\,\overline{\lim}\,(-\,\xi(L_{n-1})) = -2\,\underline{\lim}\,\xi(\frac{L_{n-1}}{2}),$$

it follows that

$$\zeta(r) \leq \overline{\lim}\,\chi(L_{n-1}) - 2\,\underline{\lim}\,\xi(\frac{L_{n-1}}{2}),$$

that is,

$$\zeta(r) - \overline{\lim}\,\chi(L_{n-1}) + 2\,\underline{\lim}\,\xi(\frac{L_{n-1}}{2}) \leq 0,$$

which by (3), is a contradiction unless $r = 0$. Therefore,

$$L_n = \delta_n = \varrho(\varpi_n, \varpi_{n+1}) + \varrho(\vartheta_n, \vartheta_{n+1}) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

Now, we have to prove that the sequences $\{\varpi_n\}$ and $\{\vartheta_n\}$ are Cauchy which is directly following from the proof of the Proposition 2 of the Section 3. Next we prove the existence of the couple fixed point.

Since $\mathscr{X}$ is complete, there exist $\varpi, \vartheta \in \mathscr{X}$ such that

$$\lim_{n\to\infty} \varpi_n = \varpi \text{ and } \lim_{n\to\infty} \vartheta_n = \vartheta. \quad (54)$$

Now, assuming condition (a) and taking $n \to \infty$ in (42) and by (54), we have

$$\varpi = \lim_{n\to\infty} \varpi_n = \mathfrak{T}(\varpi_{n-1}, \vartheta_{n-1}) = \mathfrak{T}(\lim_{n\to\infty} \varpi_{n-1}, \lim_{n\to\infty} \vartheta_{n-1}) = \mathfrak{T}(\varpi, \vartheta),$$

$$\vartheta = \lim_{n\to\infty} \vartheta_n = \mathfrak{T}(\vartheta_{n-1},\, \varpi_{n-1}) = \mathfrak{T}(\lim_{n\to\infty} \vartheta_{n-1},\, \lim_{n\to\infty} \varpi_{n-1}) = \mathfrak{T}(\vartheta,\, \varpi).$$

Therefore, $\varpi = \mathfrak{T}(\varpi,\, \vartheta)$ and $\vartheta = \mathfrak{T}(\vartheta,\, \varpi)$.

Finally, suppose that condition (b) holds.

As $\{\varpi_n\}$ is non-decreasing, $\varpi_n \to \varpi$ and as $\{\vartheta_n\}$ is non-increasing, $\vartheta_n \to \vartheta$, by our assumption, we have

$$\varpi_n \preceq \varpi \text{ and } \vartheta_n \succeq \vartheta.$$

Since

$$\varrho(\varpi,\, \mathfrak{T}(\varpi,\, \vartheta)) \le \varrho(\varpi,\, \varpi_{n+1}) + \varrho(\varpi_{n+1},\, \mathfrak{T}(\varpi,\, \vartheta)) = \varrho(\varpi,\, \varpi_{n+1}) + \varrho(\mathfrak{T}(\varpi_n,\, \vartheta_n),\, \mathfrak{T}(\varpi,\, \vartheta)).$$

So

$$\begin{aligned} \zeta(\varrho(\varpi,\, \mathfrak{T}(\varpi,\, \vartheta))) &\le \zeta(\varrho(\varpi,\, \varpi_{n+1}) + \varrho(\mathfrak{T}(\varpi_n,\, \vartheta_n),\, \mathfrak{T}(\varpi,\, \vartheta))) \\ &\le \zeta(\varrho(\varpi,\, \varpi_{n+1})) + \zeta(\varrho(\mathfrak{T}(\varpi_n,\, \vartheta_n),\, \mathfrak{T}(\varpi,\, \vartheta)) \\ &\le \zeta(\varrho(\varpi,\, \varpi_{n+1})) + \chi(\varrho(\varpi_n,\, \varpi) + \varrho(\vartheta_n,\, \vartheta)) - \xi(\frac{\varrho(\varpi_n,\, \varpi) + \varrho(\vartheta_n,\, \vartheta)}{2}). \end{aligned} \tag{55}$$

Taking $n \to \infty$ in (55) and using (54) and the properties of χ, ξ, we have

$$\zeta(\varrho(\varpi,\, \mathfrak{T}(\varpi,\, \vartheta))) = 0.$$

So, $\varrho(\varpi,\, \mathfrak{T}(\varpi,\, \vartheta)) = 0$.

Consequently,

$$\varpi = \mathfrak{T}(\varpi,\, \vartheta)).$$

Similarly, we can establish that

$$\vartheta = \mathfrak{T}(\vartheta,\, \varpi).$$

□

If we assume $\zeta = \chi$ in Theorem 4, we have the following result of Luong et al. [43].

Corollary 1. *Let (Ω, ϱ) be a POMS and $\mathfrak{T} : \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ be a mapping having the mixed monotone property on $\mathscr{X}$ such that there exist two elements $\varpi_0, \vartheta_0 \in \mathscr{X}$ with*

$$\varpi_0 \preceq \mathfrak{T}(\varpi_0,\, \vartheta_0),\; \vartheta_0 \succeq \mathfrak{T}(\vartheta_0,\, \varpi_0).$$

Suppose that

$$\zeta(\varrho(\mathfrak{T}(\varpi,\vartheta),\mathfrak{T}(u,v)) \le \frac{1}{2}\zeta(\varrho(\varpi,u) + \varrho(\vartheta,v)) - \xi(\frac{\varrho(\varpi,u) + \varrho(\vartheta,v)}{2}),$$

where $\zeta \in \Upsilon$ and $\xi \in \Gamma$, $\varpi, \vartheta, u, v \in \mathscr{X}$.

Further suppose that $\mathscr{X}$ is complete and any of the following conditions holds:

(a) *$\mathfrak{T}$ is continuous or*

(b) *If $\{\varpi_n\}$, $\{\vartheta_n\}$ are non-decreasing sequences in $\mathscr{X}$ such that $\varpi_n \longrightarrow \varpi$ and $\vartheta_n \longrightarrow \vartheta$ then $\varpi_n \preceq \varpi$, $\vartheta \preceq \vartheta_n$ for all $n \ge 0$.*

Then $\mathfrak{T}$ has a coupled fixed point in $\mathscr{X}$, if there exist $\varpi, \vartheta \in \mathscr{X}$, that is, $\varpi = \mathfrak{T}(\varpi, \vartheta)$ and $\vartheta = \mathfrak{T}(\vartheta, \varpi)$.

If we consider $\zeta \in \Upsilon$ as an identity mapping, the following corollary occurs.

Corollary 2. *Let (Ω, ϱ) be a POMS and $\mathfrak{T} : \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ be a mapping having the mixed monotone property on $\mathscr{X}$ such that there exist two elements $\varpi_0, \vartheta_0 \in X$ with*

$$\varpi_0 \preceq \mathfrak{T}(\varpi_0,\ \vartheta_0),\ \vartheta_0 \succeq \mathfrak{T}(\vartheta_0,\ \varpi_0).$$

Suppose there exists $\xi \in \Gamma$ such that

$$\varrho(\mathfrak{T}(\varpi,\ \vartheta), \mathfrak{T}(u,\ v) \leq \frac{1}{2}(\varrho(\varpi, u) + \varrho(\vartheta, v)) - \theta(\frac{\varrho(\varpi, u) + \varrho(\vartheta, v)}{2}),$$

where $\varpi, \vartheta, u, v \in \mathscr{X}$. Further suppose that $\mathscr{X}$ is complete, and that any of the following conditions holds:

(a) *$\mathfrak{T}$ is continuous or*
(b) *If $\{\varpi_n\},\ \{\vartheta_n\}$ are non-decreasing sequences in X such that $\varpi_n \longrightarrow \varpi$ and $\vartheta_n \longrightarrow \vartheta$ then $\varpi_n \preceq \varpi,\ \vartheta \preceq \vartheta_n$ for all $n \geq 0$.*

Then $\mathfrak{T}$ has a coupled fixed point in $\mathscr{X}$, if there exist $\varpi, \vartheta \in \mathscr{X}$, that is, $\varpi = \mathfrak{T}(\varpi, \vartheta)$ and $\vartheta = \mathfrak{T}(\vartheta, \varpi)$.

If we take $\xi(t) = \dfrac{(1-k)}{2})t$ in the Corollary 1, we have the following result.

Corollary 3. *Let (Ω, ϱ) be a POMS and $\mathfrak{T} : \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ be a mapping having the mixed monotone property on $\mathscr{X}$ such that there exist two elements $\varpi_0, \vartheta_0 \in X$ with*

$$\varpi_0 \preceq \mathfrak{T}(\varpi_0,\ \vartheta_0),\ \vartheta_0 \succeq \mathfrak{T}(\vartheta_0,\ \varpi_0),$$

such that

$$\varrho(\mathfrak{T}(\varpi, \vartheta),\ \mathfrak{T}(u, v) \leq \frac{k}{2}(\varrho(\varpi, u) + \varrho(\vartheta, v)),$$

where $\varpi, \vartheta, u, v \in \mathscr{X}$. Further suppose that $\mathscr{X}$ is complete and any of the following conditions holds:

(a) *$\mathfrak{T}$ is continuous or*
(b) *If $\{\varpi_n\},\ \{\vartheta_n\}$ are non-decreasing sequences in $\mathscr{X}$ such that $\varpi_n \longrightarrow \varpi$ and $\vartheta_n \longrightarrow \vartheta$ then $\varpi_n \preceq \varpi,\ \vartheta \preceq \vartheta_n$ for all $n \geq 0$.*

Then $\mathfrak{T}$ has a CFP in $\mathscr{X}$, if there exist $\varpi, \vartheta \in \mathscr{X}$, that is, $\varpi = \mathfrak{T}(\varpi,\ \vartheta)$ and $\vartheta = \mathfrak{T}(\vartheta,\ \varpi)$.

Corollary 4. *In addition to hypotheses of Corollary 1, assume that for every $(\varpi,\ \vartheta), (m,\ n) \in \mathscr{X} \times \mathscr{X}$, there exists a $(u,\ v)$ in $\mathscr{X} \times \mathscr{X}$ that is, comparable to $(\varpi,\ \vartheta)$ and $(m,\ n)$, then $\mathfrak{T}$ has a unique CFP.*

Corollary 5. *In addition to hypotheses of Theorem 4, if ϖ_0, ϑ_0 are comparable then $\mathfrak{T}$ has a unique CFP.*

Corollary 6. *In addition to hypotheses of Corollary 2, if ϖ_0, ϑ_0 are comparable then $\mathfrak{T}$ has a unique CFP.*

5. Application

The contextual discussion on the results lead us to following integral application.
Now, we study the solution of following Fredholm nonlinear integral equation:

$$\begin{aligned} \varpi(p) &= \int_b^a \mathscr{K}_1(p,\ q)[f(q, \varpi(q)) + g(q,\ \vartheta(q))]dq + \int_b^a \mathscr{K}_2(p,\ q)[f(q, \vartheta(q)) + g(q,\ \varpi(q))]dq + h(p) \\ \vartheta(p) &= \int_b^a \mathscr{K}_1(p,\ q)[f(q, \vartheta(q)) + g(q,\ \varpi(q))]dq + \int_b^a \mathscr{K}_2(p,\ q)[f(q, \varpi(q)) + g(q,\ \vartheta(q))]dq + h(p) \end{aligned} \tag{56}$$

for all $p, q \in [a,\ b]$.

We assume that $\mathscr{K}_1, \mathscr{K}_2, f, g$ satisfy the following conditions

Assumption 1.
- $\mathscr{K}_1(p,\ q) \geq 0$ *and* $\mathscr{K}_2(p,\ q) \leq 0$ *for all* $p, q \in [a, b]$.
- *There exist* $\lambda, \mu > 0$ *such that for all* $\varpi, \vartheta \in \mathbb{R}, \varpi \preceq \vartheta$
$$0 \leq f(p,\ \varpi) - f(p,\ \vartheta) \leq \lambda(\varpi - \vartheta),$$
and
$$-\mu(\varpi - \vartheta) \leq g(p,\ \varpi) - g(q,\ \vartheta) \leq 0.$$
- $\max\{\lambda,\ \mu\} \sup_{p\in[a,\ b]} \{\int_b^a \mathscr{K}_1(p,\ q) - \int_b^a \mathscr{K}_2(p,\ q)\} \leq \frac{1}{2}.$

Definition 11. *An element* $(\varsigma,\ \tau) \in C(I,\ R) \times C(I,\ \mathbb{R})$ *is called a coupled lower and upper solution of the integral equation (56) if* $\varsigma(q) \preceq \tau(q)$ *and*

$$\varsigma(p) \leq \int_b^a \mathscr{K}_1(p,\ q)[f(q,\ \varsigma(q)) + g(q,\ \tau(q))]dq + \int_b^a \mathscr{K}_2(p,\ q)[f(q,\ \tau(q)) + g(q,\ \varsigma(q))]dq + h(p)$$

and

$$\tau(p) \geq \int_b^a \mathscr{K}_1(p,\ q)[f(q,\ \tau(q)) + g(q,\ \varsigma(q))]dq + \int_b^a \mathscr{K}_2(p,\ q)[f(q,\ \varsigma(q)) + g(q,\ \tau(q))]dq + h(p)$$

for all $p \in [a, b]$.

Theorem 5. *Consider the integral Equation (56) where* $\mathscr{K}_1, \mathscr{K}_2 \in C(I \times I,\ \mathbb{R}), f, g \in C(I \times I,\ \mathbb{R})$ *and* $h \in C(I,\ \mathbb{R})$. *Suppose that Assumption 1 is satisfied. Then the existence of a coupled lower and upper solution for (56) provides the unique solution of (56) in* $C(I,\ \mathbb{R})$.

Proof. Let $\mathscr{X} = C(I,\ \mathbb{R})$. $\mathscr{X}$ is a partially ordered set if we define the following order relation in $\mathscr{X}$: $\varpi, \vartheta \in C(I,\ \mathbb{R}), \varpi \preceq \vartheta \Leftrightarrow \varpi(p) \leq \vartheta(p)$, for all $p \in [a,\ b]$. Assume that $(\mathscr{X},\ \varrho)$ is a complete metric space with metric

$$\varrho(\varpi,\ \vartheta) = \sup_{p\in[a,\ b]} |\varpi(p) - \vartheta(p)|, \varpi, \vartheta \in C(I,\ \mathbb{R}).$$

Suppose $\{u_n\}$ is a monotone non-decreasing in $\mathscr{X}$ that converges to $u \in \mathscr{X}$. Then, for every $p \in [a,\ b]$, the sequence of real numbers

$$u_1(p) \preceq u_2(p) \preceq \ldots \preceq u_n(p) \preceq \ldots$$

converges to $u(p)$. Therefore, for all $p \in [a,\ b], n \in \mathbb{N}, u_n(p) \preceq u(p)$. Hence $u_n \preceq u$ for all n. Similarly, we can verify that limit $v(p)$ of a monotone non-increasing sequence $v_n(p) \in X$ is a lower bound for all the elements in the sequence. That is, $v \preceq v_n$ for all n. Therefore, condition (b) of Theorem 4 holds.

Now, $\mathscr{X} \times \mathscr{X} = C(I,\ \mathbb{R}) \times C(I,\ \mathbb{R})$ is a partially ordered set if we define the following order relation in $\mathscr{X} \times \mathscr{X}$ by $(\varpi,\ \vartheta), (u,\ v) \in \mathscr{X} \times \mathscr{X}, (\varpi,\ \vartheta) \preceq (u,\ v) \Leftrightarrow \varpi(p) \leq u(p)$ and $\vartheta(p) \geq v(p)$, for all $p \in [a,\ b]$. For any $\varpi, \vartheta \in \mathscr{X}$, $\max\{\varpi(p), \vartheta(p)\}$ and $\min\{\varpi(p), \vartheta(p)\}$, for each $p \in [a,\ b]$, are in X and are the upper and lower bounds of ϖ, ϑ, respectively. Therefore, for every $(\varpi,\ \vartheta), (u,\ v) \in \mathscr{X} \times \mathscr{X}$, there exists a $(\max\{\varpi,\ u\}, \min\{\vartheta,\ v\}) \in \mathscr{X} \times \mathscr{X}$ that is, comparable to (ϖ, ϑ) and $(u,\ v)$.

Define $\mathfrak{T} : \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ by

$$\mathfrak{T}(\varpi,\ \vartheta)(p) = \int_b^a \mathscr{K}_1(p,\ q)[f(q, \varpi(q)) + g(q,\ \vartheta(q))]dq + \int_b^a \mathscr{K}_2(p,\ q)[f(q, \vartheta(q)) + g(q,\ \varpi(q))]dq + h(p)$$

for all $p \in [a,\ b]$. Now we shall show that $\mathfrak{T}$ has the mixed monotone property. Now, for $\varpi_1 \preceq \varpi_2$, that is, $\varpi_1(p) \leq \varpi_2(p)$, for all $p \in [a, b]$, we have

$$\begin{aligned}
& \mathfrak{T}(\varpi_1,\ \vartheta)(p) - \mathfrak{T}(\varpi_2,\ \vartheta)(p) \\
= & \int_b^a \mathscr{K}_1(p,\ q)[f(q,\ \varpi_1(q)) + g(q,\ \vartheta(q))]dq + \int_b^a \mathscr{K}_2(p,\ q)[f(q,\ \vartheta(q)) + g(q,\ \varpi_1(q))]dq + h(p) \\
- & \int_b^a \mathscr{K}_1(p,\ q)[f(q, \varpi_2(q)) + g(q,\ \vartheta(q))]dq - \int_b^a \mathscr{K}_2(p,\ q)[f(q,\ \vartheta(q)) + g(q,\ \varpi_2(q))]dq - h(p) \\
= & \int_b^a \mathscr{K}_1(p,\ q)[f(q, \varpi_1(q)) - f(q,\ \varpi_2(q))]dq + \int_b^a \mathscr{K}_2(p,\ q)[g(q,\ \varpi_1(q)) - g(q,\ \varpi_2(q))]dq \\
\leq & \ 0, \text{ by Assumption 1.}
\end{aligned}$$

Therefore, $\mathfrak{T}(\varpi_1,\ \vartheta) \leq \mathfrak{T}(\varpi_2,\ \vartheta)$, for all $p \in [a, b]$, that is, $\mathfrak{T}(\varpi_1,\ \vartheta) \preceq \mathfrak{T}(\varpi_2,\ \vartheta)$. Similar cases can be proved when $\vartheta_1 \succeq \vartheta_2$, that is, $\vartheta_1 \geq \vartheta_2$, for all $p \in [a, b]$, we have

$$\begin{aligned}
& \mathfrak{T}(\varpi,\ \vartheta_1)(p) - \mathfrak{T}(\varpi,\ \vartheta_2)(p) \\
= & \int_b^a \mathscr{K}_1(p,\ q)[f(q,\ \varpi(q)) + g(q,\ \vartheta_1(q))]dq + \int_b^a \mathscr{K}_2(p,\ q)[f(q,\ \vartheta_1(q)) + g(q,\ \varpi(q))]dq + h(p) \\
- & \int_b^a \mathscr{K}_1(p,\ q)[f(q, \varpi(q)) + g(q,\ \vartheta_2(q))]dq - \int_b^a \mathscr{K}_2(p,\ q)[f(q,\ \vartheta_2(q)) + g(q,\ \varpi(q))]dq - h(p) \\
= & \int_b^a \mathscr{K}_1(p,\ q)[g(q, \vartheta_1(q)) - g(q,\ \vartheta_2(q))]dq + \int_b^a \mathscr{K}_2(p,\ q)[f(q,\ \vartheta_1(q)) - f(q,\ \vartheta_2(q))]dq \\
\leq & \ 0, \text{ by Assumption 1.}
\end{aligned}$$

Therefore, $\mathfrak{T}(\varpi,\ \vartheta_1) \leq \mathfrak{T}(\varpi,\ \vartheta_2)$, for all $p \in [a, b]$, that is, $\mathfrak{T}(\varpi,\ \vartheta_1) \preceq \mathfrak{T}(\varpi,\ \vartheta_2)$. Thus, $\mathfrak{T}(\varpi,\ \vartheta)$ is monotone non-decreasing in ϖ and monotone non-increasing in ϑ. Now, for $\varpi \succeq u$ and $\vartheta \preceq v$, that is, $\varpi(p) \geq u(p)$, $\vartheta(p) \leq v(p)$ for all $p \in [a,\ b]$, we have

$$\begin{aligned}
\varrho(\mathfrak{T}(\varpi,\ \vartheta),\ \mathfrak{T}(u,\ v)) &= \sup_{p\in[a,\ b]} |\mathfrak{T}(\varpi, \vartheta)(p) - \mathfrak{T}(u, v)(p)| \\
&= \sup_{p\in[a,\ b]} | \int_b^a \mathscr{K}_1(p,\ q)[f(q,\ \varpi(q)) + g(q,\ \vartheta(q))]dq \\
&+ \int_b^a \mathscr{K}_2(p,\ q)[f(q,\ \vartheta(q)) + g(q,\ \varpi(q))]dq + h(p) \\
&- (\int_b^a \mathscr{K}_1(p,\ q)[f(q, u(q)) + g(q,\ v(q))]dq \\
&+ \int_b^a \mathscr{K}_2(p,\ q)[f(q,\ v(q)) + g(q,\ u(q))]dq + h(p))| \\
&= \sup_{p\in[a,\ b]} | \int_b^a \mathscr{K}_1(p,\ q)[f(q,\ \varpi(q)) - f(q,\ u(q)) + g(q,\ \vartheta(q)) - g(q,\ v(q))]dq \\
&+ \int_b^a \mathscr{K}_2(p,\ q)[(f(q,\ \vartheta(q)) - f(q,\ v(q))) + (g(q,\ \varpi(q)) - g(q,\ u(q)))]dq| \\
&= \sup_{p\in[a,\ b]} | \int_b^a \mathscr{K}_1(p,\ q)[f(q,\ \varpi(q)) - f(q,\ u(q)) - (g(q,\ v(q)) - g(q,\ \vartheta(q))]dq \\
&- \int_b^a \mathscr{K}_2(p,\ q)[(f(q,\ v(q)) - f(q,\ \vartheta(q))) - (g(q,\ \varpi(q)) - g(q,\ u(q)))]dq| \\
&\leq \sup_{p\in[a,\ b]} | \int_b^a \mathscr{K}_1(p,\ q)[\lambda(\varpi(q) - u(q)) + \mu(v(q) - \vartheta(q))]dq \\
&- \int_b^a \mathscr{K}_2(p,\ q)[\lambda(v(q) - \vartheta(q)) + \mu(\varpi(q) - u(q))]dq| \\
&\leq \max\{\lambda,\ \mu\} \sup_{p\in[a,\ b]} \int_b^a (\mathscr{K}_1(p,\ q) - \mathscr{K}_2(p,\ q))[(\varpi(q) - u(q)) + (v(q) - \vartheta(q))]dq.
\end{aligned}$$

Therefore, $\varpi(q) - u(q) \leq \varrho(\varpi, u), v(q) - \vartheta(q) \leq \varrho(v, \vartheta)$, for all $q \in [a, b]$, we obtain,

$$\begin{aligned}
\varrho(\mathfrak{T}(\varpi, \vartheta), (u, v)) &= \max\{\lambda, \mu\} \sup_{p\in[a, b]} \int_b^a (\mathscr{K}_1(p, q) - \mathscr{K}_2(p, q))[\varpi(q) - u(q) + v(q) - \vartheta(q)]dq \\
&\leq \max\{\lambda, \mu\}[\varrho(\varpi, u) + \varrho(\vartheta, v)] \sup_{p\in[a, b]} \int_b^a (\mathscr{K}_1(p, q) - \mathscr{K}_2(p, q))dq \\
&\leq \frac{\varrho(\varpi, u) + \varrho(\vartheta, v)}{2}.
\end{aligned}$$

Let $\zeta, \chi, \xi : [0, \infty) \to [0, \infty)$ be defined by

$$\begin{aligned}
\zeta(p) &= \frac{p}{4} \text{ for all } p \in [0, \infty), \\
\chi(p) &= \frac{3}{2}p \text{ for all } p \in [0, \infty), \\
\xi(p) &= \frac{5}{8}p \text{ for all } p \in [0, \infty).
\end{aligned}$$

Therefore, for all $\varpi \succeq u, \vartheta \preceq v$, we have

$$\begin{aligned}
\zeta(\mathfrak{T}(\varpi, \vartheta), \mathfrak{T}(u, v)) &= \frac{\varrho(\mathfrak{T}(\varpi, \vartheta) + \mathfrak{T}(u, v))}{4} \\
&\leq \frac{\varrho(\varpi, u) + (\vartheta, v)}{8} \\
&= \frac{3}{4}\varrho(\varpi, u) + (\vartheta, v) - \frac{5}{8}(\varrho(\varpi, u) + (\vartheta, v)) \\
&= \frac{1}{2}\chi(\varrho(\varpi, u) + (\vartheta, v)) - \frac{5}{4} \times \frac{1}{2}(\varrho(\varpi, u) + (\vartheta, v)) \\
&= \frac{1}{2}\chi(\varrho(\varpi, u) + (\vartheta, v)) - \xi(\frac{\varrho(\varpi, u) + (\vartheta, v)}{2}).
\end{aligned}$$

So,

$$\begin{aligned}
\varpi(p) = \quad & \mathfrak{T}(\varpi, \vartheta)(p) = \int_b^a \mathscr{K}_1(p, q)[f(q,\varpi(q)) + g(q, \vartheta(q))]dq + \\
& \int_b^a \mathscr{K}_2(p, q)[f(q,\vartheta(q)) + g(q, \varpi(q))]dq + h(p)
\end{aligned}$$

and

$$\begin{aligned}
\vartheta(p) = \quad & \mathfrak{T}(\vartheta, \varpi)(p) = \int_b^a \mathscr{K}_1(p, q)[f(q,\vartheta(q)) + g(q, \varpi(q))]dq + \\
& \int_b^a \mathscr{K}_2(p, q)[f(q,\varpi(q)) + g(q, \vartheta(q))]dq + h(p).
\end{aligned}$$

Hence, the system of coupled integral equation (56) possesses a unique solution. □

6. Illustration

Example 3. *Assume that* $(X = \mathbb{R}^2, \varrho)$ *is a complete metric space, where the metric* ϱ *is defined as* $\varrho(\varpi, \vartheta) =| \varpi_1 - \varpi_2 | + | \vartheta_1 - \vartheta_2 |$, *for* $\varpi = (\varpi_1, \vartheta_1)$, $\vartheta = (\varpi_2, \vartheta_2) \in \mathscr{X}$. *We define a partial order* $\preceq$ *on* $\mathscr{X}$ *such that* $(\varpi_1, \vartheta_1) \preceq (u_1, v_1)$ *and* $(\varpi_2, \vartheta_2) \succeq (u_2, v_2)$ *if and only if* $\varpi_1 \leq u_1$ *and* $\varpi_2 \leq u_2$, *for all* $(\varpi_1, \varpi_2), (u_1, v_1), (\vartheta_1, \vartheta_2), (u_2, v_2) \in \mathscr{X}$. *Let* $\mathscr{A} = \{(\varpi, 1) : 0 \leq \varpi \leq 1\} \cup \{0, \varpi) : 1 \leq \varpi < 2\}$, $\mathscr{B} = \{(\varpi, -1) : 0 \leq \varpi \leq 1\} \cup \{0, \varpi) : -2 < \varpi \leq -1\}$, $\mathscr{A}_0 = \{(\varpi, 1) : 0 \leq \varpi \leq 1\}$ *and* $\mathscr{B}_0 = \mathscr{B} = \{(\varpi, -1) : 0 \leq \varpi \leq 1\}$.

Let $\mathfrak{T} : \mathscr{A} \times \mathscr{A} \to \mathscr{B}$ *be defined as*

$$\mathfrak{T}((\varpi_1, 1), (\varpi_2, 1)) = (\frac{\varpi_1 + \varpi_2}{2}, -1) \text{ if } t = ((\varpi_1, 1), (\varpi_2, 1)) \in \mathscr{A}_0 \times \mathscr{A}_0.$$

Therefore, it is clear that $\varrho(\mathscr{A}, \mathscr{B}) = 2$, $\mathscr{A}_0 \subseteq \mathscr{A}$ *and* $\mathfrak{T}(\mathscr{A}_0 \times \mathscr{A}_0) \subseteq \mathscr{B}_0$.
Now, we show that $\mathfrak{T}$ *is satisfying the proximal mixed monotone property:*
Take $(\varpi_0, 1) \preceq (\varpi_1, 1)$ *and* $(\vartheta_0, 1) \succeq (\vartheta_1, 1)$ *in* $\mathscr{A}_0 \times \mathscr{A}_0$ *with*

$$\varrho((\varpi_1, 1), \mathfrak{T}((\varpi_0, 1), (\vartheta_0, 1))) = \varrho(\mathscr{A}, \mathscr{B})$$

and

$$\varrho((\varpi_2, 1), \mathfrak{T}((\varpi_1, 1), (\vartheta_1, 1))) = \varrho(\mathscr{A}, \mathscr{B}),$$

which implies,

$$\varpi_1 = \frac{\varpi_0 + \vartheta_0}{2} \text{ and } \varpi_2 = \frac{\varpi_1 + \vartheta_1}{2}.$$

Now, we get from the order relation,

$$\frac{\varpi_0 + \vartheta_0}{2} \leq \frac{\varpi_1 + \vartheta_1}{2}.$$

Therefore,

$$\varpi_1 \leq \varpi_2, \text{ that is, } (\varpi_1, 1) \preceq (\varpi_2, 1).$$

Again, take $(\vartheta_0, 1) \succeq (\vartheta_1, 1)$ *and* $(\varpi_0, 1) \preceq (\varpi_1, 1)$ *in* $\mathscr{A}_0 \times \mathscr{A}_0$ *with*

$$\varrho((\vartheta_1, 1), \mathfrak{T}((\vartheta_0, 1), (\varpi_0, 1))) = \varrho(\mathscr{A}, \mathscr{B})$$

and

$$\varrho((\vartheta_2, 1), \mathfrak{T}((\vartheta_1, 1), (\varpi_1, 1))) = \varrho(\mathscr{A}, \mathscr{B}),$$

which also implies,

$$\vartheta_1 = \frac{\vartheta_0 + \varpi_0}{2} \text{ and } \vartheta_2 = \frac{\vartheta_1 + \varpi_1}{2}.$$

Again, we get from the order relation,

$$\frac{\vartheta_0 + \varpi_0}{2} \leq \frac{\vartheta_1 + \varpi_1}{2}.$$

Therefore,

$$\vartheta_1 \leq \vartheta_2, \text{ that is, } (\vartheta_1, 1) \succeq (\vartheta_2, 1).$$

So, $\mathfrak{T}$ *satisfies the proximal mixed monotone property.*
Define $\zeta \in \Upsilon, \chi \in \Xi, \xi \in \Gamma$ *as*

$$\zeta(t) = t^2, \qquad \chi(t) = \begin{cases} \frac{t^2}{2}, t \in [0,1] \\ \frac{t^3}{2}, otherwise \end{cases} \qquad \text{and } \xi(t) = \begin{cases} 0, t \in [0,1] \\ \frac{t^3}{2}, otherwise. \end{cases}$$

Here, it is not difficult to see that $\zeta(t) - \overline{\lim}\, \chi(\varpi_n) + 2\, \underline{\lim}\, \xi(\varpi_n) > 0$.

So, all the postulates of Theorems 2 and 3 are satisfied and we can draw a conclusion that $((0, 1), (0, 1)) \in \mathscr{A} \times \mathscr{A}$ *is the unique coupled best proximity point of* $\mathfrak{T}$.

Note 1. *As the sets* $\mathscr{A}$ *and* $\mathscr{B}$ *are not closed in the illustration, we may relax the closure property of the sets* $\mathscr{A}$ *and* $\mathscr{B}$ *in our theorems.*

Remark 1. *The control functions, we have used in our results show the more general form of the theorems mentioned in Luong and Thuan [43].*

Only the fixed point results are extracted here to represent the application for the existence of solution of an integral equation. Some best proximity point results related to earlier publications in the literature may also be obtained through our results.

Remark 2. *The results related to fixed point proved here, are not using P-property as the property is not needed to proved fixed point results. The space considered in our example in Section 5, is also not satisfying P-property.*

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