# AN UNCERTAINTY PRINCIPLE FOR SOLUTIONS OF THE SCHRÖDINGER EQUATION ON $H$-TYPE GROUPS

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**Abstract**

In this paper we consider uncertainty principles for solutions of certain PDEs on $H$-type groups. We first prove that, on $H$-type groups, the heat kernel is an average of gaussians in the central variable so that it does not satisfy a certain reformulation of Hardy's uncertainty principle.

We then prove the analogue of Hardy's Uncertainty Principle for solutions of the Schrödinger equation with potential on $H$-type groups. This extends the free case considered by Ben Saïd, Dogga and Thangavelu [BTD] and by Ludwig and Müller [LM].

*Keywords and phrases*: Uncertainty Principle; $H$-type group; Schrödinger equation; heat kernel.

## 1. Introduction

The aim of this paper is to prove a version of Hardy's Uncertainty Principles (UP) on $H$-type groups. Let us recall that Hardy's uncertainty principle states that a function $f$ and its Fourier transform

$$\hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(x)\, \mathrm{d}x, \xi \in \mathbb{R}^d,$$

cannot both have fast decay simultaneously:

THEOREM 1.1 (Hardy [Ha]). *Let $f$ be a function such that $f(x) = O(e^{-|x|^2/\beta^2})$ and $\hat{f}(\xi) = O(e^{-4|\xi|^2/\alpha^2})$. If $1/\alpha\beta > 1/4$ then $f \equiv 0$ while if $1/\alpha\beta = 1/4$, $f$ is a constant multiple of $e^{-|x|^2/\beta^2}$.*

The case $1/\alpha\beta > 1/4$ is sometimes refered to as the super-critical case and $1/\alpha\beta = 1/4$ as the critical one. The gaussian function that provides the critical decay is ubiquitous in harmonic analysis and one may then ask why it also appears here. One may for instance argue that the gaussian is the heat kernel on $\mathbb{R}^d$ and it is indeed possible to reformulate Hardy's Theorem in terms of solutions of the free heat equation :

*Let $u$ be a solution of $\partial_t u = \Delta u$. If $u(x,0)$ and $e^{|x|^2/\delta^2}u(x,1)$ are in $L^2(\mathbb{R}^d)$ for some $\delta \leq 2$, then $f \equiv 0$.*

*On the other hand, if $u(x,0)$ is a finite measure and $e^{|x|^2/4}u(x,1)$ is in $L^\infty(\mathbb{R}^d)$ then $u(x,t) = cp_t$ where $h_t$ is the heat kernel on $\mathbb{R}^d$ and $c$ is a constant.*

Recall that $h_t(x) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$. Let us reproduce the argument, apparently due to E. Zuazua, that can be found in the introduction of [EKPV1] (see also [Ja]):

Consider $u_0 = u(x,0)$ and $f = e^{\Delta}u_0 = u(x,1)$. Then $e^{|x|^2/\delta^2}f \in L^2(\mathbb{R}^d)$ and $e^{4|x|^2/2^2}\hat{f} = \hat{u}_0 \in L^2(\mathbb{R}^d)$ so that, from Cowling and Price's $L^2$ extension of Hardy's Theorem, $f = 0$. Backward uniqueness of the heat evolution then implies that $u_0 = 0$. On the other hand, when $\delta = 2$ and $u_0$ is a bounded measure, then $e^{|x|^2}\hat{f} \in L^\infty(\mathbb{R}^d)$. Thus, if we also assume that $e^{|x|^2/\delta^2}f \in L^\infty(\mathbb{R}^d)$, then Hardy's Theorem implies that $f = ch_1$.

Thus on $\mathbb{R}^d$, the heat kernel has a critical decay rate at two different times and super-critical decay rate implies that the solution vanishes. The question then arizes whether the heat kernel has some optimal decay rate on non-abelian Lie groups *i.e.* is also an optimizer of an uncertainty principle in that setting.

There is indeed a vast literature concerning extensions Hardy's uncertainty principle that would characterize the heat kernel in other setting. Among the references related to this paper, let us mention the extensive work by Baklouti, Kaniuth, Thangavelu and coauthors, [BK,BT,KK,PT,SST,Th1,Th2], where extensions to various Lie groups have been proven (see also the survey [FS] and the book [Th3]). Let us summarize the kind of results contained in those papers. On most Lie groups, rather optimal decay rates are known for the heat kernel. One then reformulates Hardy's theorem in a way that makes sense in the Lie group considered. It then turns out that the super-critical case of Hardy's theorem is valid in many settings *i.e.* only 0 has faster decay than the heat-kernel. However, so far, no version of Hardy's Theorem on Lie groups allows to characterize the heat kernel in the critical case. On the other hand, there is also no version that shows that the heat kernel is *not* characterized as having critical decay.

In this paper, we will focus on a family of nilpotent Lie groups known as $H$-type groups and that includes the Heisenberg group. Those groups can be seen as $\mathbb{R}^n \times \mathbb{R}^m$ with a non-euclidean structure and the center is $\mathbb{R}^m$ ($m = 1$ for the Heisenberg group). We will show that the heat kernel on $H$-type groups does not satisfy a certain reformulation of the uncertainty principle in terms of the (Euclidean) Fourier transform on the center.

In order to state this reformulation, let us first recall that a function $f \in L^2(\mathbb{R}^m)$ is positive definite if its Fourier transform is positive. We can then define the cone

$$\mathcal{C} = \{f \in L^2(\mathbb{R}^d) \, : f \geq 0, \; \hat{f} \geq 0\}$$

of positive, positive definite functions. This is a convex cone. An extremal of $\mathcal{C}$ is any $f \in \mathcal{C}$, $f \neq 0$ such that, if $f = f_1 + f_2$ with $f_1, f_2 \in \mathcal{C}$, then there exists $\lambda \in [0,1]$ such that $f_1 = \lambda f$ and $f_2 = (1-\lambda)f$. The cone $\mathcal{C}$ and its extremals have been studied in [JMR] and a full characterization of its extremals is still open. However, the following is a reformulation of Hardy's Uncertainty principle in the critical case:

*For every $t>0$, the heat kernel on $\mathbb{R}^m$, $h_t$ is an extremal of $\mathcal{C}$.*

We will show that this is not the case on $H$-type groups. More precisely, as said above, we identify an $H$-type group $G$ with $\mathbb{R}^n\times\mathbb{R}^m$ where $\mathbb{R}^m$ is the center. Then the heat kernel $G$ is a function of 2 variables, $p_t(x,z)$ where $z$ is the central variable. Of course, $p_t$ is positive. It turns out that $z\to p_t(x,z)$ is also positive definite so that $p_t(x,\cdot)\in\mathcal{C}$. However, we will show that, when $x$ is fixed, this function is an average of Gaussians $p_t(x,z)=\int_0^{+\infty}e^{-t|z|^2}\,\mathrm{d}\mu_x(t)$ and that the measure $\mu_x$ has a support that is not a single point. As a consequence, $p_t(x,\cdot)$ is not an extremal, for any $x$. We consider this as an indication that the heat kernel might not be characterized via an uncertainty principle, due to its lack of concentration in the central variable.

Another possible direction is to restate Hardy's Uncertainty Principle in terms of solutions of the free Schrödinger equation. This direction has first been investigated by Chanillo [Ch] who noticed that Hardy's UP can be reformulated as follows:

*Let $u$ be a solution of $i\partial_t u+\Delta u=0$. If $u(x,0)=O(e^{-|x|^2/\beta^2})$, $u(x,1)=O(e^{-|x|^2/\alpha^2})$ and $\alpha\beta<4$, then $u\equiv 0$. Also, if $\alpha\beta=4$, $u$ has as initial datum a constant multiple of $e^{-(1/\beta^2+i/4)|x|^2}$.*

Chanillo then skillfully transfers this result to solutions of the Schrödinger equation on complex solvable Lie groups (up to the end point $\alpha\beta=4$). This strategy of proof has also been adopted by Ben Saïd, Dogga and Thangavelu in [BTD] to extend Chanillo's version of Hardy's UP to $H$-type groups and, independently and almost simultaneously, by Ludwig, Müller [LM] for general step 2 nilpotent Lie groups. Our aim here is to further extend the result in [BTD] to cover solutions of linear Schrödinger equations of the form

$$i\partial_t u(g,t)+\mathcal{L}_G u(g,t)+V(g,t)u(g,t)=0,\ \ (g,t)\in G\times[0,T] \tag{1.1}$$

where $G$ is an $H$-type group and with some restrictions on the potential $V$.

In the Euclidean case, this extension was given by Escauriaza, Kenig, Ponce and Vega [EKPV1, EKPV2] where the authors developed a machinery based on real variable calculus in order to prove Hardy's Uncertainty Principle, which typically is proved via complex analysis. This method has then been adapted to different settings, including to the Magnetic Schrödinger equation [BFGRV, CF1, CF2]. By using a Radon transform we have been able to reduce the setting of of the Schrödinger equation on $H$-type groups to the one in [CF2]. Doing so our main result concerning solutions of Schrödinger equations is then the following:

Theorem 1.2. *Let $G$ be an $H$-type group. Let $T>0$ and $s,\sigma>0$. Let $V$ be a bounded real-valued potential that is independent on time and on the central variable. Let $u\in C^1([0,T],H^1(G))$ be a solution of (1.1). Assume that there is a $C>0$ such that*

$$|u(x,z,0)|\le Cp_s(x,z),$$
$$|u(x,z,T)|\le Cp_\sigma(x,z).$$

*If $s\sigma<T^2$, then $u\equiv 0$.*

The actual result is more general as it allows some complex and time-dependent potentials, provided they go sufficiently fast to 0 at infinity. For the precise statement that requires the introduction of several notations, see Theorem 3.1. In the case $V = 0$ we recover the result in [BTD].

The remaining of the paper is organized as follows: In Section 2 we introduce $H$-type groups as well as the different concepts we need to explain in order to state and prove our results. This section concludes with the proof that the heat kernel is not an extremal positive positive definite function in the central variable. In Section 3 we turn to the study of the linear Schrödinger equation using real variable methods. By using different reductions first to the Heisenberg group and then to the Euclidean setting, we give sufficient decrease conditions on a solution to vanish identically. Then, we prove our main result in this section by relating these decrease conditions to the decrease of the heat kernel.

## 2. $H$-type groups

In this section, we gather the necessary information we will need on $H$-type groups. $H$-type groups were first introduced in [Ka]. [BLU, Chapter 18] contains an extended development of their fundamental properties; some of them being further extended in [El]. We follow closely the presentation of this last paper here, and refer the reader to [BLU, El] for further details. Note that, for briefness of presentation, we present as definition some results that actually need proofs.

For elements of $\mathbb{R}^k$ we denote by $|\cdot|$ the Euclidean norm and by $\langle\cdot,\cdot\rangle$ the Euclidean scalar product. We write $\mathbb{N} = \{0, 1, 2, \ldots\}$ for the non-negative integers. For $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ we use the classical multi-index notations $|\alpha| = \alpha_1 + \cdots + \alpha_k$.

If $f, g$ are two functions $X \to \mathbb{R}$, we write $f(x) \lesssim g(x)$ —resp. $f(x) \simeq g(x)$— to mean there exist finite positive constants $C_1$, $C_2$ such that $f(x) \le C_2 g(x)$ —resp. $C_1 g(x) \le f(x) \le C_2 g(x)$— for all $x \in X$.

### 2.1. Generalities

Definition 2.1. Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra with center $\mathfrak{z} \neq 0$. We say $\mathfrak{g}$ is of *$H$-type* (or Heisenberg type) if $\mathfrak{g}$ is equipped with an inner product $\langle\cdot,\cdot\rangle$ such that:

1. $[\mathfrak{z}^\perp, \mathfrak{z}^\perp] = 0$ and;
2. for each $z \in \mathfrak{z}$, define $J_z : \mathfrak{z}^\perp \to \mathfrak{z}^\perp$ by
$$\langle J_z x, y\rangle = \langle z, [x, y]\rangle$$
whenever $x, y \in \mathfrak{z}^\perp$. Then $J_z$ is orthogonal whenever $\langle z, z\rangle = 1$.

An *$H$-type group* is a connected, simply connected Lie group whose Lie algebra is of $H$-type.

It can be shown that an $H$-type group is a stratified 2-step nilpotent Lie group. In particular, we have an orthogonal decomposition $\mathfrak{g} = \mathfrak{z}^\perp \oplus \mathfrak{z}$ such that $[\mathfrak{z}^\perp, \mathfrak{z}^\perp] = \mathfrak{z}$. Moreover,

$\mathfrak{z}^\perp$ has even dimension. We will denote by $2n = \dim \mathfrak{z}^\perp$ and $m = \mathfrak{z}$ and we identify $\mathfrak{z}^\perp \simeq \mathbb{R}^{2n}$ and $\mathfrak{z} = \mathbb{R}^m$. It turns out that $m$ and $n$ can not be arbitrary. Indeed, writing $2n = a2^{4p+q}$ where $a$ is odd and $0 \le q \le 3$, then $\mathbb{R}^{2n} \times \mathbb{R}^m$ can be endowed with a Lie algebra structure of $H$-type as above if and only if $m < \rho(2n) := 8p + 2^q$.

As $G$ is step 2 nilpotent, $G$ can be identified with $\mathfrak{g} = \mathbb{R}^{2n+m}$ as set. An element $g \in G$ is thus of the form $g = (x, z)$ with $x \in \mathbb{R}^{2n}$, $z \in \mathbb{R}^m$. According to the Baker–Campbell–Hausdorff formula, the group operation is then given by

$$(x, z) \cdot (y, z') = (x + y, z + z' + \frac{1}{2}[x, y]).$$

The identity of $G$ is $(0, 0)$, and the inverse operation is given by $(x, z)^{-1} = (-x, -z)$. The maps $\{J_z : z \in \mathbb{R}^m\}$ are identified with $2n\times 2n$ skew-symmetric matrices which are orthogonal when $|z| = 1$. In particular, $|J_z x| = |x|\,|z|$.

Next, for $a \in \mathbb{R}\setminus\{0\}$, we define the dilations $\varphi_a(x, z) = (ax, a^2z)$. Note that this is both a group and a Lie algebra automorphism. We also denote by $\varphi_a$ the action of $\varphi_a$ on functions: for a function $f$ on $\mathbb{R}^{2n+m}$ we denote $\varphi_a \cdot f(x, z) = f(ax, a^2z)$.

We let $\{e_1, \dots, e_{2n}\}$ denote the standard basis for $\mathbb{R}^{2n}$, and $\{u_1, \dots, u_m\}$ denote the standard basis for $\mathbb{R}^m$.

The Haar measure on $G$ is simply the Lebesgue measure on $\mathbb{R}^{2n+m}$ and convolution of two functions is given by

$$f * g(x, z) = \int_G f(\omega) g\big(\omega^{-1}(x, z)\big)\,\mathrm{d}\omega = \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^m} f(u, v) g(x - u, z - v + \frac{1}{2}[x, u])\,\mathrm{d}u\,\mathrm{d}v.$$

We can now identify $\mathfrak{g}$ with the set of left-invariant vector fields on $G$, where $X_i(0) = \dfrac{\partial}{\partial x_i}$ and $Z_i(0) = \dfrac{\partial}{\partial z_i}$; then span$\{X_1, ..., X_{2n}\} = \mathfrak{z}^\perp$, span$\{Z_1, ..., Z_m\} = \mathfrak{z}$. A computation shows that

$$X_i = \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{j=1}^m \langle J_{u_j} x, e_i\rangle \frac{\partial}{\partial z_j} \quad \text{and} \quad Z_j = \frac{\partial}{\partial z_j}.$$

The elementary Lie brackets are given by

$$[X_i, X_j] = \sum_{k=1}^m \langle J_{u_k} e_i, e_j\rangle Z_k$$

and all other elementary brackets are 0.

Note that the $X_i$'s are homogeneous of degree 1 and the $Z_j$ are homogeneous of degree 2 with respect to $\varphi_a$: $X_i(\varphi_a \cdot f) = a\varphi_a \cdot (X_i f)$, $Z_j(\varphi_a \cdot f) = a^2\varphi_a \cdot (Z_j f)$. We will use the following notation: if $\alpha = (\alpha_1, \dots, \alpha_{2n}, \alpha_{2n+1}, \dots, \alpha_{2n+m}) \in \mathbb{N}^{2n+m}$ then $\mathcal{X}^\alpha = X_1^{\alpha_1} \cdots X_{2n}^{\alpha_{2n}} Z_1^{\alpha_{2n+1}} \cdots Z_m^{\alpha_{2n+m}}$ and $w(\alpha) = \alpha_1 + \cdots + \alpha_{2n} + 2\alpha_{2n+1} + \cdots + 2\alpha_{2n+m}$.

Definition 2.2. The *sublaplacian* $\mathcal{L}$ for $G$ is the operator given by:

$$\mathcal{L} = \sum_{j=1}^{2n} X_j^2. \tag{2.2}$$

**2.2. Carnot-Carathéodory distance** Recall (see, e.g., [VSCC]) that the Carnot-Carathéodory distance to the origin associated to the sum of squares operator $\mathcal{L}$ is defined by

$$d(x,z) := \inf_{\gamma} |\gamma|$$

where the infimum is taken over all absolutely continuous curves $\gamma : [0,1] \to G$ which are horizontal and connect 0 with $(x,z)$ *i.e.* $\gamma(0) = 0$, $\gamma(1) = (x,z)$ and which are horizontal, that is

$$\gamma'(t) = \sum_{j=1}^{2n} a_j(t) X_j\big(\gamma(t)\big)$$

for a.e. $t \in [0,1]$. Here, $|\gamma|$ denotes the length of $\gamma$ given by

$$|\gamma| = \int_0^1 \left(\sum_{j=1}^{2n} |a_j(t)|^2\right)^{\frac{1}{2}} \mathrm{d}t$$

We then define the *Carnot-Carathéodory distance* $d$ via the formula $d(g,h) = d(g^{-1}h)$ where we use the same letter $d$ to designate the distance and the distance to 0. Note that, by definition, $d$ is left-invariant, *i.e.* $d(g,h) = d(kg,kh)$ for every $g,k,h \in G$

The Carnot-Carathéodory distance $d$ can be computed explicitly on $G$ as follows: Define the function $\nu : [0,\pi) \to \mathbb{R}$ by $\nu(0) = 0$ and, for $\theta \in (0,\pi)$,

$$\nu(\theta) = -\frac{\mathrm{d}}{\mathrm{d}\theta}[\theta \cot\theta] = \frac{\theta - \sin\theta\cos\theta}{\sin^2\theta} = \frac{2\theta - \sin 2\theta}{1 - \cos 2\theta}.$$

Then [El, Theorem 3.5]

$$d(x,z) = \begin{cases} |x|\frac{\theta}{\sin\theta} & \text{if } x \neq 0, z \neq 0 \\ |x| & \text{if } z = 0 \\ \sqrt{4\pi|z|} & \text{if } x = 0 \end{cases}$$

where $\theta$ is the unique solution in $[0,\pi)$ of the equation $\nu(\theta) = \dfrac{4|z|}{|x|^2}$.

Note that $d\big(\varphi_a(x,z)\big) = ad(x,z)$ from which it is not difficult to show that $d(x,z) \simeq |x| + |z|^{1/2}$. Equivalently, $d(x,z)^2 \simeq |x|^2 + |z|$. One can get a more precise result:

Lemma 2.3. *For every $(x,z) \in G$,*

$$\frac{\pi}{4}(|x|^2 + 4|z|) \leq d(x,z)^2 \leq \pi(|x|^2 + 4|z|). \tag{2.3}$$

*Moreover, the constants in this inequality are optimal.*

*Further, for every $\varepsilon \in (0,1)$, there exists a constant $C_\varepsilon$ such that, for every $(x,z) \in G$,*

$$(1-\varepsilon)|x|^2 + \pi\varepsilon|z| \leq d(x,z) \leq (1+\varepsilon)|x|^2 + \frac{20\pi}{\varepsilon}|z|. \tag{2.4}$$

The first part of this lemma *i.e.* (2.3), is well known [VSCC]. We here take the opportunity to find the best constants. The estimate (2.4) can be found in [LM] in a less precise form.

*Proof.* The inequalities are obvious when $x=0$ or $z=0$ so we assume that $x,z\neq 0$.

Let $F:[0,\pi]\to\mathbb{R}$ be defined by $F(0)=1$ and, for $\theta\in(0,\pi]$,

$$F(\theta)=\frac{\theta^2}{\sin^2\theta+\theta-\sin\theta\cos\theta}=\frac{\theta^2}{\theta+\sin\theta(\sin\theta-\cos\theta)}.$$

Note that, if $\theta$ is the unique solution in $[0,\pi)$ of $\nu(\theta)=\dfrac{4|z|}{|x|^2}$, then

$$F(\theta):=\frac{(\theta/\sin\theta)^2}{1+\nu(\theta)}=\frac{d(x,z)^2}{|x|^2+4|z|}.$$

It is therefore enough to check that $\pi/4=F(\pi/4)\leq F(\theta)\leq F(\pi)=\pi$.

But

$$\begin{aligned}F'(\theta)=&\frac{2\theta}{\theta+\sin\theta(\sin\theta-\cos\theta)}\\&-\frac{\theta^2}{\big(\theta+\sin\theta(\sin\theta-\cos\theta)\big)^2}\big(1+\cos\theta(\sin\theta-\cos\theta)+\sin\theta(\sin\theta+\cos\theta)\big)\\=&\frac{2\theta}{(\theta+\sin\theta(\sin\theta-\cos\theta))^2}\times\\&\times\big[2\big(\theta+\sin\theta(\sin\theta-\cos\theta)\big)-\theta\big(1+\cos\theta(\sin\theta-\cos\theta)+\sin\theta(\sin\theta+\cos\theta)\big)\big]\end{aligned}$$

which has same sign as

$$\psi(\theta):=\theta\big(1-\sin\theta(\sin\theta+\cos\theta)\big)+(2\sin\theta-\theta\cos\theta)(\sin\theta-\cos\theta).$$

Notice that $\psi(\pi/4)=0$ and that

$$\begin{aligned}\psi'(\theta)&=1+\theta+(\cos\theta+\theta\sin\theta)(\sin\theta-\cos\theta)+(\sin\theta-\theta\cos\theta)(\sin\theta+\cos\theta)\\&=\theta(1-\cos 2\theta-\sin 2\theta)+1-\cos 2\theta+\sin 2\theta.\end{aligned}$$

Now $\psi'(\pi/4)=0$ and

$$\psi''(\theta)=1+\cos 2\theta+\sin 2\theta+2\theta(\sin 2\theta-\cos 2\theta)=1+\cos(2\theta)(1-2\theta)+\sin 2\theta(1+2\theta).$$

From this, it is obvious that $\psi''\geq 2-\pi/2$ on $[0,\pi/4]$, thus $\psi'\leq 0$ on $[0,\pi/4]$ therefore $\psi$ thus $F$ decreases on $[0,\pi/4]$. In particular, $\pi/4=F(\pi/4)\leq F(\theta)\leq F(0)=1$ for $\theta\in[0,\pi/4]$.

For $\theta\in[\pi/4,3\pi/4]$ one easily checks that $1-\cos 2\theta-\sin 2\theta\geq 0$ and $1-\cos 2\theta+\sin 2\theta\geq 0$ thus $\psi'\geq 0$, thus $\psi$ increases and is therefore non-negative. This in turn means that $F'$ is non negative, thus $\pi/4=F(\pi/4)\leq F(\theta)\leq F(3\pi/4)$ for $\theta\in[\pi/4,3\pi/4]$.

Finally, on $[3\pi/4,\pi]$, $1+\cos(2\theta)+\sin 2\theta\leq 1$ while $\cos(2\theta)-\sin(2\theta)\leq -1$ thus $\psi''(\theta)\leq 2(1-\theta)\leq 0$. Therefore $\psi'(\theta)\geq\psi'(\pi)=0$ and $\psi$ is increasing, $\psi(\theta)\geq\psi(3\pi/4)=3\pi/4+2>0$.

It follows that $F$ is increasing on $[3\pi/4, \pi]$ thus $F(\pi/4) \leq F(3\pi/4) \leq F(\theta) \leq F(\pi) = \pi$ for $\theta \in [3\pi/4, \pi]$.

The proof of (2.4) then follows the steps of [LM]: on one hand,

$$d(x,z)^2 \geq \max\big(d(x,0)^2, d(0,z)^2\big) \geq \max(|x|^2, \pi|z|) \geq (1-\varepsilon)|x|^2 + \pi\varepsilon|z|.$$

On the other hand

$$d(x,z)^2 \leq \big(d(x,0) + d(0,z)\big)^2 \leq (|x| + \sqrt{4\pi|z|})^2 \leq (1+\varepsilon)|x|^2 + \frac{20\pi}{\varepsilon}|z|.$$

as claimed. □

### 2.3. Heat kernel

Definition 2.4. The heat kernel $p_t$ for $G$ is the unique fundamental solution to the corresponding heat equation $\left(\mathcal{L} - \dfrac{\partial}{\partial t}\right) u = 0$ that is, $p_t = e^{t\mathcal{L}}\delta_0$, where $\delta_0$ is the Dirac delta distribution supported at 0.

In particular, if $\left(\mathcal{L} - \dfrac{\partial}{\partial t}\right) u = 0$ and $u(x,z,0) = u_0(x,z) \in L^2(\mathbb{R}^{2n+m})$ then $u(x,z,t) = u_0 * p_t(x,z)$.

Our next step is to record an explicit formula for $p_t(x,z)$. Various derivations of this formula appear in the literature. For general step 2 nilpotent groups, [Ga] derived such a formula probabilistically from a formula in [Le] regarding the Lévy area process. Another common approach, worked out in [DP], involves expressing $p_t$ as the Fourier transform of the Mehler kernel. Taylor [Ta] has a similar computation. Other approaches have involved complex Hamiltonian mechanics [BGG], magnetic field heat kernels [Kl], and approximation of Brownian motion by random walks [Hu]. Of particular interest to us is the approach by Randall [Ra] who obtains the formula for $H$-type groups as the Radon transform of the heat kernel for the Heisenberg group. In our notation, we find that

$$p_t(x,z) = t^{-n-m} \int_{\mathbb{R}^m} e^{-\frac{\pi}{2}|\lambda|\coth(2\pi|\lambda|)|t^{-1/2}x|^2} \left(\frac{|\lambda|/2\pi}{2\sinh(2\pi|\lambda|)}\right)^n e^{2i\pi\langle\lambda, t^{-1}z\rangle}\,\mathrm{d}\lambda \tag{2.5}$$

Also this is not obvious from the above formula, $p_t$ is non-negative and $p_t * p_{t'} = p_{t+t'}$. Note that $p_t(x,z) = t^{-m-n} p_1(t^{-1/2}x, t^{-1}z)$ *i.e.* $p_t = t^{-m-n}\varphi_{t^{-1/2}} p_1$.

Using Harnack inequalities one can show that

$$p_t(x,z) \lesssim C(\varepsilon) t^{-n-m} e^{-\frac{1}{(4+\varepsilon)t}d(x,z)^2}. \tag{2.6}$$

A similar statement actually holds for general nilpotent Lie groups. More precise estimates have been obtained in [DP,El] but the one above is sufficient for our needs. Further, Randall [Ra] has shown that $p_t$ admits an analytic extension with respect to the time parameter $t$ to the half-complex plane $\mathrm{Re}(t) > 0$. The estimate (2.6) can be extended to complex time $t$:

Lemma 2.5 (Estimate of the heat kernel). *For every $\varepsilon \in (0,1)$, there exists $C = C(\varepsilon)$ such that, for every $t > 0$*

$$p_t(x,z) \le \frac{C}{t^{n+m}} \exp\left(-\frac{1}{4t}\big((1-\varepsilon)|x|^2 + \pi\varepsilon|z|\big)\right).$$

*Moreover, if $t \in \mathbb{C}$ with $\mathrm{Re}(t) > 0$,*

$$p_t(x,z) \le \frac{C}{\mathrm{Re}(t)^{n+m}} \exp\left(-\,\mathrm{Re}\,\frac{1}{4t}\big((1-\varepsilon)|x|^2 + \pi\varepsilon|z|\big)\right).$$

*Proof.* For $t$ real, this formula follows from incorporating (2.4) into (2.6) and is well known up to the numerical constants (see [VSCC, p 50]).

For complex $t$, we use the fact that $p_t$ admits an analytic extension and the result follows by an application of Phragmén-Lindelöf as in Theorem 3.4.8 of [Da]. □

We are now in position to prove our first result:

Theorem 2.6. *For every $x \in \mathbb{R}^n$, there exists a non-negative finite measure $\nu_x$ on $[0,+\infty)$ such that*

*i.* $\displaystyle\int_0^{+\infty} \frac{1}{\tau}\, d\nu_x(\tau) < +\infty;$

*ii. for every $a > 0$, $\nu_x([0,a]) > 0$ and $\nu_x([a,+\infty]) > 0$;*

*iii. for every $z \in \mathbb{R}^m$,*

$$p_1(x,z) = \int_0^{+\infty} \frac{1}{\tau} e^{-\pi|z|^2/\tau^2}\, \mathrm{d}\nu_x(\tau).$$

*In particular, for $x$ fixed, $z \to p_1(x,z) \in \mathcal{C}$ but is not an extremal of $\mathcal{C}$.*

*Proof.* Let us write

$$P_{n,m}(x,z) = \int_{\mathbb{R}^m} e^{-\frac{\pi}{2}|\lambda|\coth(2\pi|\lambda|)|x|^2} \left(\frac{|\lambda|/2\pi}{2\sinh(2\pi|\lambda|)}\right)^n e^{2i\pi\langle\lambda,z\rangle}\, \mathrm{d}\lambda.$$

In particular, $P_{n,1}$ is the heat kernel of the Heisenberg group, as shown by Hulanicki [Hu]. Randall [Ra] showed that

$$P_{n,m}(x,z) = c_{n,m} \int_{\mathbb{S}^{m-1}} P_{n,1}(\langle z,\theta\rangle)\, \mathrm{d}\sigma(\theta) \tag{2.7}$$

where $\sigma$ is the normalized Lebesgue measure on the unit sphere $\mathbb{S}^{m-1}$ of $\mathbb{R}^m$ and $c_{n,m}$ is a normalization constant. Further, when $m < \rho(2n)$, Randall has shown that $P_{n,m}$ is the heat kernel on $\mathbb{R}^{2n} \times \mathbb{R}^m$ with the $H$-type group structure mentioned above.

One consequence of (2.7) is that, for every $m$, $P_{n,m}$ is a non-negative function. On the other hand, let us denote by $\varphi_x$ the function defined on $\mathbb{R}$ by

$$\varphi_x(u) = e^{-\frac{\pi}{2}u\coth(2\pi u)|x|^2} \left(\frac{u/2\pi}{2\sinh(2\pi u)}\right)^n.$$

Then $P_{n,m}$ is the Fourier transform of the radial extension to $\mathbb{R}^m$ of $\varphi_x$, *i.e.* $P_{n,m}(x,z) = \mathcal{F}[\varphi_x(|\lambda|)](z)$ where $\mathcal{F}$ is the usual Fourier transform on $\mathbb{R}^m$. It then follows from a celebrated theorem of Schoenberg [Sc] (see also [SvP] for a more modern proof and further references), that $\varphi_x$ is an average of Gaussians with respect to some finite non-negative measure $\nu_x$

$$\varphi_x(u) = \int_0^{+\infty} e^{-\pi\tau^2u^2}\,\mathrm{d}\nu_x(\tau).$$

One easily checks that $\int_0^\infty \varphi_x(u)\,\mathrm{d}u$ is finite, thus Fubini's theorem yelds

$$\int_0^\infty \varphi_x(u)\,\mathrm{d}u = \int_0^{+\infty}\int_0^\infty e^{-\pi\tau^2u^2}\,\mathrm{d}u\,\mathrm{d}\nu_x(\tau) = \int_0^{+\infty}\frac{1}{\tau}\,\mathrm{d}\nu_x(\tau) < +\infty.$$

In particular, $\nu_x$ has essentially no mass at 0.

Next, it follows from Fubini's theorem again that

$$p_1(x,z) = \int_{\mathbb{R}^m}\int_0^{+\infty} e^{-\pi\tau^2|\lambda|^2}\,\mathrm{d}\nu_x(\tau)e^{2i\pi\langle\lambda,z\rangle}\,\mathrm{d}\lambda = \int_0^{+\infty}\frac{1}{\tau}e^{-\pi|z|^2/\tau^2}\,\mathrm{d}\nu_x(\tau)$$

as claimed.

If there were an $a_x$ such that $\nu_x([0,a_x]) = 0$ then we would have $\varphi_x \lesssim e^{-\pi a_x^2|u|^2}$ which is obviously not the case. On the other hand, if the support of $\nu_x$ where included in $[0,a_x]$, then $\varphi_x$ would extend into an entire function of order 2, $|\varphi_x(u+iv)| \lesssim e^{\pi a_x^2 v^2}$. But this is again not the case since

$$|\varphi_x(iu)| = \left(\frac{|u|/2\pi}{|\sin 2\pi u|}\right)^n \to +\infty$$

when $u \to k/2$, $k \in \mathbb{Z}\setminus\{0\}$.

For $x$ fixed, we can now write $p_1(x,z) = g(z) + h(z)$ with

$$g(z) = \int_0^{+\infty}\frac{1}{\tau}e^{-\pi|z|^2/\tau^2}\mathbf{1}_{[0,1)}(\tau)\,\mathrm{d}\nu_x(\tau) \quad\text{and}\quad h(z) = \int_0^{+\infty}\frac{1}{\tau}e^{-\pi|z|^2/\tau^2}\mathbf{1}_{[1,+\infty)}\,\mathrm{d}\nu_x(\tau).$$

As $\nu_x([0,1]), \nu_x([1,+\infty)) > 0$, $g,h \neq 0$. Further, using the same reasoning as above, one can show that $g$ has faster decay than $p_1(x,\cdot)$ and that $h$ extends holomorphically to $\mathbb{C}$ while $p_1(x,\cdot)$ does not. Therefore $g,h$ are not multiples of $p_1(x,\cdot)$. □

Remark 2.7. A similar result holds for $p_t$ thanks to the scaling property

$$p_t(x,z) = t^{-m-n}p_1(t^{-1/2}x, t^{-1}z).$$

It would be nice if the measure $\nu_x$ could be determined explicitly. Our attempts to do so have not succeeded.

## 3. The Schrödinger equation in $H$-type groups: A "real" approach

In this section we deal with a solution of the Schrödinger equation for the sub-Laplacian of an $H$-type group $G$ (isomorphic to $\mathbb{R}^{2n} \times \mathbb{R}^m$),

$$i\partial_t u(g,t) + \mathcal{L}_G u(g,t) + V(g,t)u(g,t) = 0, \quad (g,t) \in G \times [0,T], \tag{3.8}$$

where $g = (x,z),\ x \in \mathbb{R}^{2n},\ z \in \mathbb{R}^m$. We assume that the potential satisfies the following hypothesis.

HYPOTHESIS 1. The potential $V$ is independent of the central variable and can be written in the form $V(x,z,t) = V_1(x) + V_2(x,t)$ where,

- $V_1$ is a real-valued bounded potential;
- for some $a, b, T > 0$,

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^{2n}} |e^{\frac{T^2|x|^2}{(at+b(T-t))^2}} V_2(x,t)| < +\infty. \tag{3.9}$$

Note that, if $V$ satisfies Hypothesis 1 for some $a, b$ then it also satisfies the hypothesis for $a' \geq a$ and $b' \geq b$. Also (3.9) implies that

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^{2n}} |\mathrm{Im}\, V_2(x,t)| < +\infty.$$

THEOREM 3.1. *Let $a, b, T > 0$ and let $V$ be a potential satisfying Hypothesis 1 with parameters $a, b, T$. Let $u \in C^1([0,T], H^1(G))$ be a solution of (3.8). Assume that there are $c, C > 0$ and a function $g \in L^1(\mathbb{R}^m)$ such that*

$$|u(x,z,0)| \leq Ce^{-c|z|-|x|^2/b^2}, \tag{3.10}$$

$$|u(x,z,T)| \leq g(z)e^{-|x|^2/a^2} \tag{3.11}$$

*If $ab < 4T$, then $u \equiv 0$.*

REMARK 3.2. The condition $s\sigma < 4T$ is essentially optimal in the sense that the result does not hold when $s\sigma > 4T$. Indeed, take $V = 0$, $\varepsilon, T > 0$ and $u(x,z,0) = p_{\varepsilon T}(x,z)$ so that $u(x,z,T) = p_{(\varepsilon+i)T}(x,z)$. It follows from Lemma 2.5 that

$$|u(x,z,0)| \leq C_\varepsilon e^{-\frac{\pi}{4T}|z|} e^{-|x|^2 \frac{(1-\varepsilon)}{4\varepsilon T}}$$

while

$$|u(x,z,T)| \leq C_{\varepsilon,T} e^{-\frac{\pi\varepsilon^2}{4(1+\varepsilon^2)T}|z|} e^{-|x|^2 \frac{(1-\varepsilon)\varepsilon}{4(1+\varepsilon^2)T}}$$

Thus (3.10) is satisfied with $b^2 = \frac{4\varepsilon T}{(1-\varepsilon)}$ while (3.10) is satisfied with $a^2 = \frac{4(1+\varepsilon^2)T}{(1-\varepsilon)\varepsilon}$. It follows that

$$a^2b^2 = 16T^2 \frac{1+\varepsilon^2}{(1-\varepsilon)^2}$$

and any number $> 16T^2$ can be written in this form.

Note that if $|u(x,z,0)| \leq Ce^{-d(x,z)^2/\beta^2}$ and $|u(x,z,T)| \leq Ce^{-d(x,z)^2/\alpha^2}$ then, with (2.4), we get that (3.10)-(3.11) hold with $a = (1+\varepsilon)\alpha$, $b = (1+\varepsilon)\beta$ and $\varepsilon > 0$ arbitrarily small. It follows that, if $\alpha\beta < 4T$ then $u \equiv 0$. The above example again shows that this is false if $\alpha\beta > 4T$.

*Proof.* The proof of this theorem is then done in several steps.

**Step 1.** *Reduction to $T = 1$.*

Note that if $u$ is a solution of (3.8) on $G \times [0,T]$ and $U(x,z,t) = u(\sqrt{T}x, Tz, Tt)$ then $U$ is a solution of

$$i\partial_t U = \mathcal{L}_G U + V_T U \tag{3.12}$$

in $G \times [0,1]$ where $V_T(x,z,t) = TV(\sqrt{T}x, Tz, Tt)$. Note that $V_T$ satisfies Hypothesis 1 with parameters $\tilde{a} = a/T, \tilde{b} = b/T, 1$. Moreover, (3.10) implies that

$$|U(x,z,0)| \leq Ce^{-cT|z| - T|x|^2/b^2} = e^{-cT|z| - |x|^2/\tilde{b}^2}$$

while (3.11) implies that $|U(x,z,1)| \leq g(Tz)e^{-|x|^2/\tilde{a}^2}$. Therefore, it is enough to prove Theorem (3.1) for $T = 1$.

**Step 2.** *Reduction to the Heisenberg group.*

This reduction has been pointed out in [BTD, Ra] and is done via a partial Radon transform:

Proposition 3.3. *Let $u$ and $V$ be as in Theorem 3.1. Let $\eta \in \mathbb{S}^{m-1}$ be a unit vector in $\mathbb{R}^m$ and $\eta^\perp = \{\xi \in \mathbb{R}^m : \langle \xi, \eta \rangle = 0\}$. Define,*

$$U_\eta(x,s,t) = \int_{\eta^\perp} u(x, s\eta + \tilde{z}, t)\,\mathrm{d}\tilde{z}, \quad x \in \mathbb{R}^{2n},\ s \in \mathbb{R},\ t \in [0,1].$$

*Then $U_\eta$ is a solution of the following Schrödinger equation*

$$i\partial_t U_\eta + \mathcal{L}_{\mathbb{H}^n} U_\eta + V(x,t)U_\eta = 0, \quad (x,s,t) \in \mathbb{H}^n \times [0,1], \tag{3.13}$$

*where $\mathcal{L}_{\mathbb{H}^n}$ stands for the sub-Laplacian on the Heisenberg group $\mathbb{H}^n$. Moreover, there exists $C_1$ and $c_1$ positive constants depending only on $C$ and $c$ such that, for $(x,s) \in \mathbb{H}^n$*

$$|U_\eta(x,s,0)| \leq C_1 e^{-c_1|s| - |x|^2/b^2},$$
$$|U_\eta(x,s,1)| \leq \tilde{g}(s)e^{-|x|^2/a^2}$$

*where $\tilde{g}(s) := \displaystyle\int_{\eta^\perp} g(s\eta + \tilde{z})\,\mathrm{d}\tilde{z}$ is the Radon transform of $g$.*

Note that $\tilde{g}(s)$ is well defined for almost every $s, \eta$ and that $\tilde{g} \in L^1(\mathbb{R})$ for almost every $\eta$.

*Proof.* The fact that $U_\eta$ satisfies (3.13) is established in [BTD, Ra]. It remains to establish the Gaussian bounds on $U$. Since $U_\eta(x,s,t) = \int_{\eta^\perp} u(x, s\eta + \tilde{z}, t)\,\mathrm{d}\tilde{z}$, we have

$$|U_\eta(x,s,0)| \leq \int_{\eta^\perp} |u(x, s\eta+\tilde{z}, 0)|\,\mathrm{d}\tilde{z} \leq Ce^{-|x|^2/b^2} \int_{\eta^\perp} e^{-c|s\eta+\tilde{z}|}\,\mathrm{d}\tilde{z}.$$

Taking into account that $\tilde{z} \in \eta^\perp$, and that $\eta$ is a unit vector, we get

$$|s\eta + \tilde{z}|^2 = s^2 + |z|^2 \Rightarrow |s\eta+\tilde{z}| \geq \frac{|s| + |\tilde{z}|}{\sqrt{2}}.$$

Using this estimate leads to the desired decay for $U_\eta(x,s,0)$. Moreover,

$$|U_\eta(x,s,1)| \leq \int_{\eta^\perp} |u(x, s\eta+\tilde{z}, 1)|\,\mathrm{d}\tilde{z} \leq e^{-|x|^2/a^2} \int_{\eta^\perp} g(s\eta+\tilde{z})\,\mathrm{d}\tilde{z}$$

as claimed. □

**Step 3.** *Reduction to the magnetic Laplacian on $\mathbb{R}^n$.*

We now show that Schrödinger equations on the Heisenberg group can be seen as magnetic Schrödinger equations.

Proposition 3.4. *Let $u$ and $V$ be as in Theorem 3.1. Let $U_\eta$ be defined in Proposition 3.3.*

*For $\xi \in \mathbb{R}$, consider the partial Fourier transform of $U$ in the central variable:*

$$f_\xi(x,t) = \int_{\mathbb{R}} U_\eta(x,z,t)e^{i\xi z}\,\mathrm{d}z.$$

*Then $f_\xi$ is a solution of the following magnetic Schrödinger equation*

$$i\partial_t f_\xi + \Delta_C f_\xi + V f_\xi = 0,$$

*where $\Delta_C = (\nabla - iC_\xi)^2$ and $C_\xi = \frac{M_\xi x}{2}$ with $M_\xi = \xi \begin{pmatrix} 0 & -I_{n\times n} \\ I_{n\times n} & 0 \end{pmatrix}$.*

*Moreover $f_\xi(x,0)$ admits a holomorphic extension to $f_{\xi+i\eta}(x,0)$ in $\{|\eta| < c\}$. Further,*

$$\begin{array}{rcl} |f_\xi(x,0)| & \leq & Ce^{-|x|^2/b^2}, \\ |f_\xi(x,1)| & \leq & \|\tilde{g}\|_{L^1} e^{-|x|^2/a^2}. \end{array} \tag{3.14}$$

*Proof.* We recall that

$$\mathcal{L} = \sum_{j=1}^{2n} X_j^2 = \Delta_{\mathbb{R}^{2n}} + \frac{1}{4}|x|^2\frac{\partial^2}{\partial z^2} + \sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial x_{n+j}} - x_{n+j}\frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial z}.$$

Then,

$$
\begin{aligned}
-i\partial_t f_\xi &= -\int_{\mathbb{R}} i\partial_t U e^{i\xi z}\,\mathrm{d}z \\
&= \int_{\mathbb{R}} \Delta_{\mathbb{R}^{2n}} U e^{i\xi z}\,\mathrm{d}z + \frac{1}{4}|x|^2 \int_{\mathbb{R}} \partial_{zz} U e^{i\xi z}\,\mathrm{d}z + \sum_{j=1}^{n} \int_{\mathbb{R}} \left(x_j \partial_{n+j}\partial_z U - x_{n+j}\partial_j\partial_z U\right) e^{i\xi z}\,\mathrm{d}z \\
&\quad + V(x,t) \int_{\mathbb{R}} U e^{i\xi z}\,\mathrm{d}z \\
&= \Delta_{\mathbb{R}^{2n}} f_\xi - \frac{\xi^2}{4}|x|^2 f_\xi - i\xi \sum_{j=1}^{n} \left(x_j \partial_{n+j} f_\xi - x_{n+j}\partial_j f_\xi\right) + V f_\xi = \Delta_C f_\xi + V f_\xi
\end{aligned}
$$

as claimed.

The last part of the proposition is immediate using the definition of $f_\xi$ as a partial Fourier transform in the central variable. □

**Step 4.** *Conclusion*

Using the previous step, $f_\xi$ is a solution of

$$i\partial_t f_\xi + \Delta_C f_\xi + V f_\xi = 0,$$

that satisfies the Gaussian estimates (3.14). Further, since $ab < 4$, there exists $\delta > 0$ such that $ab < 4\frac{\sin\delta}{\delta}$. Now, using Theorem 1.11 in [CF2] (in the case $A \equiv 0$, which implies that the result is valid in any even dimension) combined with the previous results that give information about the evolution and decay of $f_\xi$, we conclude that, for $0 < \xi < \delta$, $t \in [0,1]$, $f_\xi(x,t) = 0$. In particular, for $t = 0$ we get that $f_\xi(x,0) = 0$ for $0 < \xi < \delta$. But, from Proposition 3.4, we know that $f_\xi(x,0)$ admits an holomorphic extension to a strip $\{\xi + i\zeta : |\zeta| < c\}$. It follows that $f_\xi(x,0) = 0$ on the whole real line ($\zeta = 0$).

As $\xi$ is arbitrary and $f_\xi(x,0)$ is the partial Fourier transform of $U_\eta(x,s,0)$, we conclude that $U_\eta(x,s,0) = 0$.

Finally, $U_\eta$ is the Radon transform of $u$, we conclude that $u = 0$, which is the desired result. □

As an immediate corollary, we can now obtain the following result, which is stated in a slightly more restrictive form as Theorem 1.2 in the introduction:

Corollary 3.5. *Let $T > 0$ and $s, \sigma > 0$. Let $V$ be a potential satisfying Hypothesis 1 with parameters $2\sqrt{\sigma}, 2\sqrt{s}, T$. Let $u \in C^1([0,T], H^1(G))$ be a solution of (3.8). Assume that there is a $C > 0$ such that*

$$
\begin{aligned}
|u(x,z,0)| &\le C p_s(x,z). \\
|u(x,z,T)| &\le C p_\sigma(x,z).
\end{aligned}
$$

*If $s\sigma < T^2$, then $u \equiv 0$.*

Remark 3.6. The case $V = 0$ of this theorem has been obtained by Ben Saïd, Thangavelu, Dogga [BTD] for $H$-type groups and by Ludwig and Müller [LM] for general step 2 nilpotent Lie groups.

*Proof of Corollary 3.5.* From Lemma 2.5 we get

$$|u(x,z,0)| \leq C_\varepsilon \exp\left(-\frac{1}{4s}\big((1-\varepsilon)|x|^2 + \pi\varepsilon|z|\big)\right)$$
$$|u(x,z,T)| \leq C_\varepsilon \exp\left(-\frac{1}{4\sigma}\big((1-\varepsilon)|x|^2 + \pi\varepsilon|z|\big)\right).$$

Let $a_\varepsilon = 2\sqrt{\frac{\sigma}{1-\varepsilon}}$ and $b_\varepsilon = 2\sqrt{\frac{s}{1-\varepsilon}}$. As $s\sigma < T^2$, for $\varepsilon$ small enough, $a_\varepsilon b_\varepsilon = \frac{4}{1-\varepsilon}\sqrt{s\sigma} < 4T$.

Further $V$ satisfies Hypothesis 1 with parameters $a_\varepsilon, b_\varepsilon, T$. Applying Theorem 3.1 we thus get $u = 0$. □


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## References


[BK] A. Baklouti & E. Kaniuth, 'On Hardy's uncertainty principle for connected nilpotent Lie groups', *Math. Z.* **259** (2008), 233–247.

[BT] A. Baklouti & S. Thangavelu, 'Variants of Miyachi's theorem for nilpotent Lie groups', *J. Aust. Math. Soc.* **88** (2010), 1–17.

[BFGRV] J. A. Barceló, L. Fanelli, S. Gutiérrez, A. Ruiz & M.C. Vilela, 'Hardy uncertainty principle and unique continuation properties of covariant Schrödinger flows', *J. Funct. Anal.* **264** no. 10 (2013), 2386–2425.

[BGG] R. Beals, B. Gaveau & P. C. Greiner, 'Hamilton-Jacobi theory and the heat kernel on Heisenberg groups', *J. Math. Pures Appl. (9)* **79** (2000), 633–689.

[BLU] A. Bonfiglioli, E. Lanconelli & F. Uguzzoni, *Stratified Lie Groups and Potential Theory for Their Sub-Laplacians* Springer Monographs in Mathematics (Springer, Berlin, 2007).

[BTD] S. Ben Saïd, S. Thangavelu & V. N. Dogga, 'Uniqueness of solutions to Schrödinger equatinos on $H$-type groups', *J. Aust. Math. Soc. (3)* **95** (2013), 297–314.

[CF1] B. Cassano & L. Fanelli, 'Sharp Hardy uncertainty principle and Gaussian profiles of covariant Schrödinger evolutions', *Trans. Amer. Math. Soc.* **367** (2015), 2213–2233.

[CF2] B. Cassano & L. Fanelli, 'Gaussian decay of harmonic oscillators and related models', *J. Math. Anal. Appl.* **456** (2017), 214–228.

[Ch] S. Chanillo, 'Uniqueness of solutions to Schrödinger equations on complex semi-simple Lie groups,' *Proc. Indian Acad. Sci. Math. Sci.* **117** (2007), 325–331.

[CP] M. G. Cowling & J. F. Price, 'Generalizations of Heisenberg's inequality', in: *Harmonic Analysis* (eds. G. Mauceri, F. Ricci and G. Weiss), Lecture Notes in Math., 992 (Springer, Berlin, 1983), 443–449.

[Da] E. B. Davies, *Heat kernel and spectral theory.* (Cambridge University Press, Cambridge, 1989).

[DP] E. B. Davies & M. M. H. Pang, 'Sharp heat kernel bounds for some Laplace operators', *Quart. J. Math. Oxford Ser. (2)* **40** (1989), 281–290.

[El] N. Eldredge, 'Precise estimates for the subelliptic heat kernel on $H$-type groups', *J. Math. Pures Appl.* **92** (2009), 52–85.

[EKPV1] L. Escauriaza, C. E. Kenig, G. Ponce & L. Vega, Hardy's uncertainty principle, convexity and Schrödinger evolutions', *J. Eur. Math. Soc.* **10** (2008), 883–907.

[EKPV2] L. Escauriaza, C. E. Kenig, G. Ponce & L. Vega, 'The sharp Hardy uncertainty principle for Schrödinger evolutions', *Duke. Math. J.* **155** (2010), 163–187.

[FS] G. B. Folland & A. Sitaram, 'The uncertainty principle: a mathematical survey', *J. Fourier Anal. Appl.* **3** (1997), 207–238.

[Ga] B. Gaveau, 'Principe de moindre action, propagation de la chaleur et estimées sous-elliptiques sur certains groupes nilpotents', *Acta Math.* **139** (1977), 95–153.

[Ha] G. H. Hardy, 'A theorem concerning Fourier transforms', *Journal London Math. Soc.* **8**, (1933) 227–231.

[Hu] A. Hulanicki, 'The distribution of energy in the Brownian motion in the Gaussian field and analytic-hypoellipticity of certain subelliptic operators on the Heisenberg group', *Studia Math.* **56** (1976), 165–173.

[Ja] Ph. Jaming, 'Uncertainty principles for orthonormal bases', *Séminaire d'équations aux dérivées partielles, 2006, Palayseau, France.* expose XV, 21 fevrier 2006.

[JMR] Ph. Jaming, M. Matolcsi & S. Révész, 'On the extremal rays of the cone of positive, positive definite functions', *J. Fourier Anal. Appl.* **15** (2009), 561–582.

[KK] E. Kaniuth & A. Kumar, 'Hardy's theorem for simply connected nilpotent Lie groups', *Math. Proc. Cambridge Philos. Soc.* **131** (2001), 487–494.

[Ka] A. Kaplan, 'Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms', *Trans. Amer. Math. Soc.* **258** (1980), 147–153.

[Kl] A. Klingler, 'New derivation of the Heisenberg kernel', *Comm. Partial Differential Equations* **22** (1997), 2051–2060.

[Le] P. Lévy, 'Wiener's random function, and other Laplacian random functions', in: *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950* (University of California Press, Berkeley and Los Angeles, 1951) 171–187.

[LM] J. Ludwig & D. Müller, 'Uniqueness of solutions to Schrödinger equations on 2-step nilpotent Lie groups', *Proc. Amer. Math. Soc.* **142** (2014), 2101–2118.

[PT] S. Parui & S. Thangavelu, 'On theorems of Beurling and Hardy for certain step two nilpotent groups', *Integral Transforms Spec. Funct.* **20** (2009), 127–145.

[Ra] J. Randall, 'The heat kernel for generalized Heisenberg groups', *J. Geom. Anal.* **6** (1996), 287–316.

[Sc] I. J. Schoenberg, 'Metric spaces and completely monotone functions', *Ann. of Math.* **39** (1938), 811–841.

[SST] A. Sitaram, M. Sundari & S. Thangavelu, 'Uncertainty principles on certain Lie groups', *Proc. Indian Acad. Sci. Math. Sci.* **105** (1995), 135–151.

[SvP] A .G. M. Steerneman & F. van Perlo-ten Kleij, 'Spherical distributions: Schoenberg (1938) revisited', *Expo. Math.* **23** (2005) 281–287.

[Ta] T. Taylor, 'A parametrix for step-two hypoelliptic diffusion equations', *Trans. Amer. Math. Soc.* **296** (1986), 191–215.

[Th1] S. Thangavelu, 'An analogue of Hardy's theorem for the Heisenberg group.' *Colloq. Math.* **87** (2001), 137–145.

[Th2] S. Thangavelu, 'Revisiting Hardy's theorem for the Heisenberg group.' *Math. Z.* **242** (2002), 761–779.

[Th3] S. Thangavelu, *An introduction to the uncertainty principle. Hardy's theorem on Lie groups. With a foreword by Gerald B. Folland* Progress in Mathematics, 217 (Birkhauser Boston, Inc., Boston, MA, 2004).

[VSCC] N. Th. Varopoulos, L. Saloff-Coste & T. Coulhon, *Analysis and geometry on groups* Cambridge Tracts in Mathematics, 100, (Cambridge University Press, Cambridge, 1992).

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