Asymptotic structure of space-time and gravitational radiation in the presence of a non-negative cosmological constant

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Resumen

(In Spanish)

Esta memoria recoge los resultados del proyecto de investigación de tesis doctoral que lleva por título “Asymptotic structure of space-time and gravitational radiation with a non-negative cosmological constant” , supervisada y tutelada por el prof. José María Martín Senovilla y llevada a cabo en el departamento de Física Teórica e Historia de la Ciencia, dentro del grupo de Gravitación y Cosmología, con el apoyo económico de las ayudas n. FIS2017-85076-P (MINECO/AEI/FEDER, EU) y n. IT956-16 (Gobierno Vasco).

La investigación se centra en el estudio de la radiación gravitatoria desde un punto de vista formal, así como de la estructura de las regiones asintóticas del espacio-tiempo en presencia de una constante cosmólogica no negativa. A continuación, se presenta un resumen que incluye antecedentes, estado actual de las líneas de investigación y una breve compilación de los resultados obtenidos durante el periodo de desarrollo de la tesis doctoral.

Antecedentes

La tesis se enmarca dentro de la teoría de la Relatividad General fundada por Einstein [1]. A día de hoy, esta es la mejor teoría que tenemos para describir la gravedad a bajas energías (no se espera que sea así a altas energías, veáse por ejemplo la introducción de [2]). El estudio llevado a cabo se centra en la estructura asintótica del espacio-tiempo y en la radiación gravitatoria (véase [3] para una revisión histórica sobre este tema).

La constante cosmológica

En 1917 Einstein [4] introdujo la constante cosmológica Λ en sus ecuaciones de campo. Su motivación original era la de conseguir una solución homogénea y estática, para lo cual era necesario ajustar de manera precisa el valor de la constante cosmológica. Sin embargo, esta idea resulta poco acertada desde el punto de vista físico, mientras que, desde el punto de vista matemático, el término de constante cosmológica en las ecuaciones las hace más generales. Ese mismo año, de Sitter [5] encontró una solución a las ecuaciones de campo con constante cosmológica positiva que, de hecho, incluía un régimen dinámico. El descubrimiento observacional de Hubble [6] mostraba un universo en expansión, lo que estaba en sintonía con los modelos de Fridman [7] y Lemaître [8]. Setenta años después, mediciones de Λ usando supernovas [9, 10] evidenciaron que habitamos un universo con constante cosmológica positiva, con un valor cercano a cero. El efecto de una Λ positiva (sin importar cómo de grande es su valor) en nuestro universo es que este se encuentre en una fase de expansión acelerada.

El pequeño valor de la constante cosmológica aún carece de una explicación ultrerior. Ya en 1987, Weinberg usó un argumento antrópico basado en la formación de sistemas gravitatorios ligados para predecir una cota superior en el valor de Λ [11]. Sin embargo, los argumentos antrópicos aportan pistas para la búsqueda de una explicación subyacente pero no constituyen por sí mismos una teoría. Desde el punto de vista de física de partículas, se puede interpretar Λ como efectiva –en oposición a pura– identificándola con una densidad de energía de vacío. No obstante, el valor de Λ calculado en teoría de partículas difiere en muchos órdenes de magnitud del valor observado [12]. Desde cierto punto de vista, este es el llamado *problema de la constante cosmológica*. Todo ello motiva el estudio de las consecuencias físicas que tiene una constante cosmológica en las ecuaciones de campo de Einstein.

Para revisiones en esta materia véase [13, 14].

Estructura asintótica

Nociones físicas de importancia, como la energía del campo gravitatorio, se entienden mejor si se consideran sistemas gravitatorios aislados [15–19]. Intuitivamente, tal sistema puede representar una estrella después de haber «vaciado» el universo de todo lo demás. Una idea complementaria es que hay que alejarse muy lejos para poder considerar un sistema como un todo. En esta descripción pictórica, «muy lejos» significa *en el infinito*, que matemáticamente está bien defino. Es allí donde ciertas cantidades físicas, e incluso nociones como la de onda gravitatoria, cobran más sentido [20, 21]. Una descripción formal del infinito fue presentada por Penrose en 1963 [22]. Uno de los aspectos de su idea es traer el infinito a una distancia coordenada finita o, en otras palabras, hacer del infinito una región local. Técnicamente, se le adhiere una frontera al espacio-tiempo representando el infinito, tal que los campos físicos puedan ser evaluados allí [23]. Es por ello que la frontera codifica la física del las regiones asintóticas del espacio-tiempo. De manera notable, la naturaleza de dicha frontera depende de si la constante cosmológica es cero, positiva o negativa (pasando su carácter causal de ser luminoso a espacial o temporal). Por la misma razón, la física del infinito se ve afectada por Λ. Es por ello que el pequeño valor de Λ es suficiente para cambiar el escenario asintótico de manera abrupta con respecto a $\Lambda = 0$ [24].

Radiación gravitatoria

La naturaleza dinámica de la geometría espacio-temporal permite que se produzcan cambios en el campo gravitatorio que afectan de manera causal distintos puntos del espacio-tiempo. Este fenómeno de propagación se denomina comúnmente ondas gravitatorias (o más genéricamente radiación gravitatoria) y es un efecto completamente no lineal. En las primeras etapas de la teoría de la Relatividad General, el concepto de onda no estaba bien definido, ni si quiera en 1918, cuando Einstein publicó su famosa fórmula del cuadrupolo [25] en el límite de campo débil, estableciendo así el primer paso hacia una teoría de radiación gravitatoria. Su construcción estaba guiada por las soluciones de tiempo retardado de la electrodinámica clásica. El hecho de que su fórmula solo fuera válida en el régimen lineal de la teoría sembraba dudas sobre la veracidad de las ondas gravitatorias en la teoría completa. De hecho, Einstein mismo y Rosen llegaron a la conclusión de que las ondas no existían, después de encontrar una solución de onda plana que contenía, según ellos, singularidades (estas no eran verdaderas singularidades, sino problemas en la elección de coordenadas) [26, 27].

En los años 50 y 60 llegaron nuevos avances. Un papel fundamental lo desempeñó Pirani, quien buscó una caracterización algebraica de la radiación gravitatoria habiéndose inspirado en el libro de electrodinámica de Synge [28]. Su caracterización se basó en un problema de valores propios del tensor de curvatura [29, 30], adecuándose de forma natural a la clasificación previa de Petrov [31]. Obsérvese que, a falta de un tensor de energía momento para el campo gravitatorio, Pirani hizo uso de la curvatura. Sin embargo, quien llevó a su extremo este paralelismo fueron Bel [32, 33] y Robinson. Dieron la definición de un tensor, cuadrático en la curvatura, que compartía una cantidad asombrosa de propiedades con el tensor energía-momento del campo electromagnético, a excepción de las unidades físicas. Paralelamente, Trautman trató la cuestión de la radiación estudiando condiciones de frontera en el infinito [34, 35], una manera de abordar el problema reminiscente de la condición de radiación de Sommerfeld (véase [36]). La colaboración entre Trautman, Robinson [37, 38], Pirani y Bondi [39] culminó con una serie de artículos [21, 40, 41] (véase también [42]) en los que los distintos enfoques convergían. Ejemplos de ello son el postulado de una condición de radiación en el infinito en los trabajos de Bondi y su equipo o la caracterización algebraica del tensor de Weyl en el denominado *comportamiento de pelado*. Además, el tratamiento geométrico del infinito de Penrose llevó a resultados similares [15, 43, 44], y en la década siguiente todo fue condensado de manera geométrica y robusta por Geroch [17]. Algunos de estos hitos incluyen el llamado tensor de Bondi y la energía-momento de Bondi-Trautman, así como el descubrimiento del grupo de simetrías asintótico BMS (Bondi, Metzner y Sachs). Así fue como se demostró que la radiación gravitatoria *es un elemento intrínseco de la teoría completa* y se despejaron las dudas sobre su existencia desde el punto de vista teórico (véase [45] para una revisión histórica).

Más progresos llegaron en la década de 1970 y 1980. Ha de mencionarse la introducción de métodos simplécticos [46, 47] y la identificación y caracterización de los grados de libertad radiativos del campo gravitatorio en términos de clases de equivalencia de conexiones en el infinito [48]. También hubo avances hacia la definición de momento angular [49–51], y un estudio más profundo de la frontera asintótica y del grupo de simetría BMS [52–55]. Todo esto acompañado del descumbrimiento del pulsar binario PSR B1913+16 [56] cuyo análisis brindó la primera prueba observacional (indirecta) de la existencia de ondas gravitatorias (véase, por ejemplo, [57]).

Salvo la geometrización conforme de Penrose, los avances teóricos fundamentales arriba citados *solo son válidos cuando la constante cosmológica es cero.*

Otros avances recientes

Actualmente, tenemos certeza de que hay objetos astrofísicos que pierden energía por emisión de ondas gravitatorias. Con el anuncio de la primera detección directa en 2016 [58], estas ondas se convirtieron en un hecho observacional robusto. Además, la constante cosmológica (pura o efectiva) está determinada con precisión y es positiva [59]. Por tanto, el escenario es un tanto paradójico: hay grandes logros teóricos en el campo de la radiación gravitatoria en el infinito que no son directamente aplicables al universo que habitamos.

En [60] se puso de nuevo la atención en esta cuestión y más tarde se expuso la problemática de manera detallada en [61] desde el punto de vista de restricciones geométricas y topológicas de la frontera conforme. Durante los últimos años se ha alcanzado cierta comprensión del problema por distintas vías que incluyen una fórmula de cuadrupolo en el régimen lineal [62, 63], una definición espinorial de masa [64], caracterizaciones basadas en coordenadas o coeficientes de spin [65, 66], métodos Hamiltonianos [67] y otros enfoques [68–73] (véase [74] como revisión de algunos de estos trabajos). En [75] se propuso una condición de radiación covariante en el infinito en presencia de una constante cosmológica positiva; la idea se puso a prueba en el caso asintóticamente plano en [76]. Aún así, la ausencia de una estructura universal, un grupo asintótico de simetrías y el problema de la dependencia direccional cuando uno se aproxima al infinito [77] muestra la incompletitud existente en la caracterización de la radiación gravitatoria en el infinito con constante cosmológica positiva.

Resultados

Esta tesis trata de resolver algunos de estos problemas abiertos. Se estudia tanto el caso con constante cosmológica positiva como el caso asintóticamente plano. En este último, se consigue un refinamiento de propiedades ya conocidas (tales como la energía-momento del campo gravitatorio y el comportamiento de pelado) y también resultados nuevos (una condición de radiación geométrica o el comportamiento de pelado para el denominado tensor de Bel-Robinson). De gran relevancia es el análisis de la estructura asintótica con constante cosmológica positiva, que se estudia desde su base, revisitando conceptos básicos y dando las expresiones relevantes de los campos físicos en la frontera conforme. Se emplean objetos definidos con el tensor de Bel-Robinson (lo que comúnmente se conoce bajo el nombre de cantidades de 'superenergía'), ya que se adaptan de forma natural a las propiedades de marea del campo gravitatorio. La condición de radiación encontrada es válida en los dos escenarios (con constante cosmológica positiva y sin ella), se construye en el infinito y también tiene en cuenta las energías de marea. Además, se demuestra una relación geométrica entre las cantidades a nivel superenergético (o de marea) y otras cantidades a nivel energético, siendo las primeras 'fuente' de las segundas. Esta relación sirve para formular una clase de tensores de tipo 'news', usando un resultado geométrico hallado que generaliza un teorema de Geroch [17]. Además, la condición de radiación y los tensores de tipo news y la estructura direccional del campo gravitatorio se ponen en conexión con la formulación de un criterio de 'no radiación entrante', equipando la frontera conforme con una congruencia de curvas. Esto lleva a la definición de un grupo asintótico de simetrías que preserva las nuevas estructuras, y al estudio de cantidades conservadas. Además, la correspondencia entre el caso con constante cosmológica positiva y el caso cero es nítida, pudiéndose tomar el límite de la primera a la segunda situación. Finalmente, se aplican los resultados a varios ejemplos de soluciones exactas a las ecuaciones de campo de Einstein, obteniéndose la respuesta esperada y demostrando así la validez de de los resultados.

Más allá de lo especificado arriba, tres de las principales ideas que se destacan en la tesis son las siguientes:

1. Las cantidades de superenergía están naturalmente adaptadas al problema de la caracterización de la radación gravitatoria.
2. En el caso con constante cosmológica positiva, cualquier dinámica del campo gravitatorio en la frontera conforme tiene que estar codificada en la tríada $(\mathscr{J}, h_{ab}, D_{ab})$, donde $\mathscr{J}$ es una variedad Riemanniana en tres dimensiones (la frontera conforme), h_{ab} es la métrica sobre dicha variedad y D_{ab} es un tensor simétrico y sin traza.
3. Los tensores de tipo news están asociados, al menos parcialmente, a variedades de dimensión 2 en el infinito.

Mientras que futuras líneas de investigación incluyen:

- Definición de una energía-momento en el caso con constante cosmológica positiva asociado a la estructura introducida en el infinito.

- Empleo de métodos simplécticos.
- Generalización de la construcción del teorema de pelado a curvas generales.
- Aplicación de los métodos de marea propuestos al caso con constante cosmológica negativa.

List of publications and contributions to conferences

Publications

- F. Fernández-Álvarez and J. M. M. Senovilla. "Novel characterization of gravitational radiation in asymptotically flat spacetimes," Phys. Rev. D 101, 024060 (2020). DOI: 10.1103/PhysRevD.101.024060.
- F. Fernández-Álvarez and J. M. M. Senovilla. "Gravitational radiation condition at infinity with a positive cosmological constant," Phys. Rev. D 102, 101502 (2020). DOI: 10.1103/PhysRevD.102.101502.
- F. Fernández-Álvarez and J. M. M. Senovilla. *Asymptotic structure with vanishing cosmological constant*. 2021. arXiv: 2105.09166 [gr-qc].
- F. Fernández-Álvarez and J. M. M. Senovilla. *Asymptotic structure with a positive cosmological constant*. 2021. arXiv: 2105.09167 [gr-qc].

Conferences

- Oral communication EREP2018 (2018): "Gravitational radiation at null infinity: The Bondi News criterion from a superenergy point of view".
- Oral communication GR22 (2019): "A novel characterisation of gravitational radiation in asymptotically flat space-times".
- Oral communication CERS 10 (2020): "A novel characterisation of gravitational radiation in asymptotically flat space-times".

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Lastly, Lucía. There are hidden universes in our letters, paths to the past and to the future. Places to go, to stay and to come back. They have been always there. Thank you for being home to me.

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A la memoria de los profesores

Armando García-Mendoza
Miguel Ángel Ramos Osorio

Preface

Lee, mira y manipula este libro interactivo que te irá desvelando las maravillas del universo. Coloca las pegatinas en el lugar apropiado para transformar las ilustraciones. Gira el disco para conocer la posición de las constelaciones en cada época del año y aprender a identificarlas. Usa el astrolabio, como los navegantes del pasado, para saber la hora por la posición de las estrellas. Despliega las páginas para admirar el Sol y sus planetas. Ponte las gafas para ver el firmamento en relieve. Descubre las dimensiones del espacio leyendo el librito desplegable. Averigua cómo son las galaxias, qué es una enana blanca, un agujero negro, un púlsar y otros muchos datos interesantes acerca de nuestro mundo maravilloso.[*]

[*] *Viaja por el universo*. SM saber. Biblioteca interactiva. SM, 1993. isbn: 8434841088.

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Theorems and propositions

1 | Introduction

Gravity affects everything and everything is a source of gravity. This would be a very condensed way[1] to convey one of the main ideas behind General Relativity, a theory that was born with Einstein field equations [1]. The understanding of gravity as geometry of the space-time is the best description of the gravitational interaction we have nowadays –at least at the low-energy regime, since quantum effects are expected to appear at the Planck scale, see e. g. the introduction of [2]. In a general situation, the gravitational field is dynamical, and this is the same phenomenon by which the curvature of the space-time can be altered. This thesis is a study of two particular features of the theory that were formally immature or conceptually misleading until the renaissance[2] –to put it in the words of C. M. Will– of General Relativity: infinity and gravitational radiation. Both objects of study are closely related, as an adequate treatment of infinity allows to consider isolated systems whose emission of gravitational waves can be characterised precisely in the asymptotic regions of the space-time. Basic physical concepts for the gravitational field, as energy-momentum, are defined at infinity when the cosmological constant Λ vanishes. In contrast, the presence a positive Λ alters the situation drastically.

The cosmological constant: observational fact and conundrum

In 1917, the cosmological constant was introduced by Einstein [4] who pursued a homogeneous static universe that could fit Mach's philosophy. Although his original motivation, which required the value of Λ to be fine-tuned, was not fortunate from the physical point of view, the presence of Λ in the field equations makes them more general, and hence one *has to consider it*. That very same year, de Sitter [5] found a solution to the field equations for a positive cosmological constant –which indeed included dynamical features.

[1]This introduction aims at simplicity and non-technical descriptions; more detailed preambles are found at the beginning of each chapter.

[2]Where to put the historic limits of this bright epoch of developments and achievements can be ambiguous, although fundamental progress was made in the 1950s and 1960s. For a historical review focusing on gravitational radiation see [3].

The observational discovery of Hubble [6] pictures an expanding universe, in agreement with the previous models of Fridman [7] and Lemaître [8]. In spite of this, the actual value of the cosmological constant (if present) could not be inferred at the time. It was not after 70 years later that measurements of Λ using supernovae [9, 10] evinced that we inhabit a universe with a positive cosmological constant. Its value, although only slightly above zero, served to establish that the universe undergoes an accelerated expansion –with a striking impact on the physics at infinity.

The tiny value of the cosmological constant has no ulterior explanation yet. As soon as in 1987, Weinberg used an anthropic argument based on the formation of gravitational bound systems in order to predict an upper limit for the value of the cosmological constant [11]. However, anthropic arguments drop hints for the search of an explanation but do not provide us with a fundamental description or theory by themselves. From the point of view of particle physics, one can think of the cosmological constant as an *effective* –in contrast to *bare*– term by identifying it with a *vacuum energy*. The particle theory calculations give an outcome that differs from the observed value of Λ by several orders of magnitude though –see [12]. This is, in a way, the cosmological constant problem.

Hence, even though we know that Λ has to enter Einstein field equations, its role in cosmological dynamics and its value in our universe, truth is that the underlying nature of this constant is still a conundrum. For reviews on the matter, see e.g. [13, 14].

Infinity is reachable

Important physical notions, such a gravitational mass, are better understood if one considers isolated (gravitational) systems [15–19]. Intuitively, such a system can represent a star of the universe after removing everything else. A complementary idea is the requirement of going far away from a system to see it as a 'whole'. In this pictorical description, 'far away' means *at infinity*. There, physical quantities, and even notions such as gravitational waves, become clearer [20, 21]. Hence, a formal description of infinity is needed and was put forward in 1963 by Penrose [22]. One aspect of his idea is to bring infinity to a finite coordinate distance or, in other words, to make infinity a local place. Penrose's elegant treatment of infinity allows to attach a boundary to the space-time, so that physical fields are actually evaluated there [23]. The boundary encodes the physics of the asymptotic regions of the space-time. Remarkably, it is affected by the sign of the cosmological constant in a profound way, and thus the same can be said of the asymptotic physics. From the geometric point of view, the direct consequence of a negative, positive or vanishing Λ makes infinity a timelike, spacelike or lightlike hypersurface, independently of how big (small) Λ is. Hence, the tiny value of the cosmological constant in our universe is enough to change the asymptotic arena abruptly from that with $\Lambda = 0$ [24].

More than ripples of space-time

The dynamical nature of the space-time geometry allows for changes of the gravitational field that affect causally different points of the space-time. This propagation phenomena, commonly denominated gravitational waves –or more generically gravitational radiation–, is a fully non-linear effect. The concept itself was fuzzy in the early stages of the theory. In 1918, Einstein published his quadrupole formula [25] in the weak-field limit establishing the first step towards a theory of gravitational radiation. His approach was guided by the way one constructs retarded-time solutions in classical electrodynamics. Nevertheless, the fact that his formula applies only to the linearised theory cast doubts upon the feature of gravitational radiation in the non-linear regime. Indeed, Einstein himself and Rosen came to the conclusion that gravitational waves did not exist, after finding a gravitational plane wave solution [26, 27] which contained 'singularities' –these were not true singularities, but a coordinate illness of the solution. Hence, the concept of gravitational wave was not clear at all at the time.

Advances came in the 1950-60's. A key role was played by Pirani, who searched for an algebraic characterisation of gravitational radiation, inspired by Synge's book [28] for the case of the electromagnetic interaction. He based his characterisation in a problem of eigenvalues of the curvature tensor [29, 30] in harmony with previous Petrov's classification [31]. Observe that, in lack of an 'energy momentum-tensor' for the gravitational field, Pirani made use of the curvature tensor. However, the parallelism with electromagnetism of the algebraic method ran deeper with the work of Bel [32, 33] and Robinson. They gave the definition of a tensor field, cuadratic in the curvature, and sharing an astonishing number of properties with the energy-momentum tensor of the electromagnetism –except the physical units. Parallelly, Trautman treated the problem of gravitational radiation by studying *boundary conditions* at infinity [34, 35], an approach that is reminiscent of Sommeferld's radiation condition –see [36]–. The common collaboration of Trautman , Robinson [37, 38], Pirani and Bondi [39] ended up with a series of papers [21, 40, 41] (see also [42]) in which the different approaches converged – e. g., the postulation of a radiation condition at infinity in Bondi's metric-based approach or the algebraic characterisation of the curvature tensor and its interpretation in the so called *peeling behaviour*. Moreover, Penrose's geometrical treatment of infinity led to similar outcomes [15, 43, 44], and in the next decade all the results were condensed in a solid geometrical description by Geroch [17]. Some of those landmarks include the so called Bondi news tensor and Bondi-Trautman momentum and the discovery of the asymptotic group of symmetries BMS (named after Bondi, Metzner and Sachs). Gravitational radiation was shown to be an *intrinsic feature of the full theory* and cleared up foregoing uncertainties on the theoretical side –for a brief historical review see [45].

More progress was made in the late 1970s and 1980s. One has to mention the introduction of simplectic methods [46, 47] and the identification and characterisation of the radiative degrees of freedom of the gravitational field in terms of classes of equivalence of connections at infinity [48]. Advances towards the definition of angular momentum [49–51], and further study of the asymptotic boundary symmetries and momentum [52–55] were made. All this was accompanied with the discovery of the binary pulsar PSR B1913+16 [56] whose analysis provided the first observational evidence of the existence of such waves –see e.g. [57].

Among these theoretical successful developments, the fundamental ones but the conformal geometrisation of Penrose *are valid only when the cosmological constant vanishes*.

The scenario

At the present time, we know certainly that astrophysical objects can loose energy by the emission of gravitational waves. With the announce of the first direct detection in 2016 [58], these waves became a robust observational fact. Also, the cosmological constant (bare or effective) is accurately determined as positive [59]. Thus, the scenario is a bit of paradoxical: we have a lot of great theoretical achievements in the understanding of gravitational radiation at infinity which cannot be applied to the universe we inhabit.

Attention to the problem of asymptotic characterisation of gravitational radiation was revived in [60] and later the problematic was exposed accurately in [61] from the point of view of restricting the topology and geometry of the conformal boundary: if one does not restrict the asymptotic structure, it is not possible to identify an asymptotic symmetry group but if the constraints are too strong one may loose too much information –asking for conformal flatness, for instance, removes half the components of the gravitational field–. During the last years some understanding has been achieved and different approaches explored, which include a quadrupole formula and the study of the linear regime [62, 63], a spinorial definition of mass [64], spin-coefficient/coordinate-based approaches [65, 66], Hamiltonian methods [67] and others [68–73] – see [74] for a review of some of the works. A geometric and covariant radiation condition at infinity in the presence of a positive cosmological constant was proposed [75] and tested in the asymptotically flat case [76]. Still, the absence of a universal structure together with a general group of asymptotic symmetries and the problem of the directional-dependence as one approaches infinity [77] makes the whole picture incomplete.

This thesis aims at solving some of the open problems in the characterisation of radiation at infinity with a positive cosmological constant. The task involves the study of the vanishing-Λ case; refinement of old features –such as energy-momentum of the gravitational field and the peeling behaviour– and new results –a geometric radiation condition or the peeling behaviour of the Bel-Robinson tensor– are given. Of greatest relevance is the analysis of the asymptotic structure with a positive cosmological constant, which is studied from its ground, reviewing some basic ideas and giving the expressions of the relevant fields on the conformal boundary. Objects defined upon the Bel-Robinson tensor are used (commonly called 'superenergy' quantities), which suit the tidal nature of the gravitational field. The radiation condition –valid in both scenarios– at infinity is of a tidal nature too and ruled by the asymptotic geometry itself, and a detailed analysis of it is carried out. Moreover, a geometric relation between quantities at the superenergy level ($MT^{-2}L^{-3}$) and others at the energy level is established, being the latter 'the source' of the former. This relation serves to formulate a news-like class of tensor fields using a general geometric result –which generalises a theorem by Geroch. In addition to that, the radiation condition and the news-like tensors and the directional structure of the gravitational field at infinity are put in connection, formulating a criterion of no-incoming radiation at infinity and equipping the conformal boundary with a selected congruence of curves. This led to the definition of a group of asymptotic symmetries preserving the new structures and to the study of conserved charges. Also, the limit of the cosmological constant to zero shows that the main ideas in both scenarios exhibit a clear correspondence. Finally, application of the results to exact solutions of Einstein Field Equations are given to illustrate their validity.

Conventions and notation

Throughout the memoir, 4 space-time dimensions are considered and quantities in physical space-time $\hat{M}$ are distinguished from those in conformal space-time M by using hats. Frequently used abbreviations include: KVF (Killing vector field), CKVF (conformal Killing vector field), EFE (Einstein field equations) and PND (principal null direction). Part of the notation is summarised in table 1.1.

The following conventions are used:

- Space-time metric signature: $(-,+,+,+)$.
- Space-time indices: α,β,γ, etc; three dimensional space-like hypersurfaces indices: a,b,c, etc; surfaces indices: A,B,C, etc.
- Riemann tensor, Ricci tensor and scalar curvature: $R_{\alpha\beta\gamma}{}^{\delta}v_{\delta} := \left(\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha}\right)v_{\gamma}$, $R_{\alpha\beta} := R_{\alpha\mu\beta}{}^{\mu}$, $R := R_{\mu\nu}g^{\mu\nu}$.
- Choice of orientation: $\eta_{0123} = 1$, $\epsilon_{123} = 1$, $\mathring{\epsilon}_{23} = 1$.
- Symmetrisation and antisymmetrisation: $2T_{[\alpha\beta]} := \left(T_{\alpha\beta} - T_{\beta\alpha}\right)$, $2T_{(\alpha\beta)} := \left(T_{\alpha\beta} + T_{\beta\alpha}\right)$.
- Commutator of two vector fields: $[v,w]^{\alpha} := v^{\mu}\nabla_{\mu}w^{\alpha} - w^{\mu}\nabla_{\mu}v^{\alpha}$.

- Commutator of two endomorphisms or two $(1,1)$-tensors: $[A,B]_\alpha{}^\beta := \left(A_\alpha{}^\mu B_\mu{}^\beta - B_\alpha{}^\mu A_\mu{}^\beta\right)$.
- Hodge dual operation on space-time two-forms: $2\overset{*}{\omega}_{\alpha\beta} := \eta^{\mu\nu}{}_{\alpha\beta}\omega_{\mu\nu}$.
- D'Alambert operator: $\Box := g^{\mu\nu}\nabla_\mu\nabla_\nu$.

	Physical space-time $\hat{M}$	Conformal space-time M	Spacelike hypersurfaces $\mathcal{I}$	Surfaces $\mathcal{S}$
Metric	$\hat{g}_{\alpha\beta}$	$g_{\alpha\beta}$	h_{ab}	q_{AB}
Volume form	$\hat{\eta}_{\alpha\beta\gamma\delta}$	$\eta_{\alpha\beta\gamma\delta}$	ϵ_{abc}	$\mathring{\epsilon}_{AB}$
Covariant derivative	$\hat{\nabla}_\alpha$	∇_α	$\overline{\nabla}_a$	$\mathcal{D}_A$
Curvature tensor	$\hat{R}_{\alpha\beta\gamma}{}^\delta$	$R_{\alpha\beta\gamma}{}^\delta$	$\overline{R}_{abc}{}^d$	$\mathring{R}_{ABC}{}^D$
Projector	–	–	$P^\alpha{}_\beta$	$\mathring{P}^a{}_b$
Bases	–	–	$\{e^\alpha{}_a\}$ $\{\omega_\alpha{}^a\}$	$\left\{E^a{}_A\right\}$ $\left\{W_a{}^A\right\}$

Table 1.1: General notation. The first two columns corresponds to the physical and conformal space-time $\hat{M}$ and M, the third one to any 3-dimensional hypersurface (in particular this notation is used for $\mathscr{J}$) and the last one to any 2-dimensional surface $\mathcal{S}$ (like codimension-1 manifolds on $\mathscr{J}$).

2 | Superenergy

The impossibility of defining a local energy-momentum tensor for the gravitational field does not rule out the possibility of describing the field's strength. Apart from the asymptotic definitions of energy mentioned in chapter 1, a local notion of *energy density divided by area* exists. This particular weighting of the energy is naturally associated to tidal forces and is described by a rank-four tensor, quadratic in the Weyl curvature, called the Bel-Robinson tensor [32]. Importantly, this object is defined locally and shows lots of analogies with the energy-momentum tensor of the electromagnetic field [78]. As a matter of fact, it can not have physical units of energy density and this led to assign the name *supernergy quantities* to objects defined with this tensor. Since its formulation, the superenergy turned out to be a useful tool in a variety of studies of gravity, such as the causal propagation of gravity [79], the algebraic characterisation of the Weyl tensor [80], the formation of black holes [81] or the global non-linear stability of Minkowski spacetime [82]. In addition, it emerges in the quasilocal formulations of energy [83] and can be formulated for fields other than gravity [84] exhibiting interchanges of superenergy quantities and conserved mixed currents of different fields [85–87].

Superenergy is an indispensable tool in this thesis. Particularly inspiring for us is Bel's definition of 'intrinsic radiative states'[33] –in agreement with Pirani's ideas [30]; also there is a more recent definition of intrinsic radiative states and an analogue to the Poynting theorem in electromagnetism [88]. The tidal nature of actual gravitational-wave detections makes the Bel-Robinson an appealing candidate for grounding the study of gravitational radiation –for other recent studies in which the tidal nature of gravitational waves plays an important role see e.g. [89, 90]. As it becomes manifest in chapters 4, 5 and 8, one advantage of our superenergy-based methods is their easy and straightforward application.

There exists a general definition of superenergy tensor [85] and, within that broader class, the Bel-Robinson tensor is the *basic superenergy tensor* constructed with the Weyl tensor. Next subsections apply to any basic superenergy tensor built with a Weyl-tensor candidate, i. e., a traceless tensor $W_{\alpha\beta\gamma}{}^{\delta}$ sharing all the algebraic symmetries of the Weyl tensor.

2.1 Basic superenergy tensors

The basic superenergy tensor $\mathcal{T}_{\alpha\beta\gamma\delta}\,\{W\}$ constructed with a Weyl-tensor candidate $W_{\alpha\beta\gamma\delta}$ is defined in 4 dimensions as

$$\mathcal{T}_{\alpha\beta\gamma\delta} := W_{\alpha\mu\gamma}{}^{\nu}W_{\delta\nu\beta}{}^{\mu}+\overset{*}{W}_{\alpha\mu\gamma}{}^{\nu}\overset{*}{W}_{\delta\nu\beta}{}^{\mu} = W_{\alpha\mu\gamma\nu}W_{\beta}{}^{\mu}{}_{\delta}{}^{\nu}+W_{\alpha\mu\delta\nu}W_{\beta}{}^{\mu}{}_{\gamma}{}^{\nu}-\frac{1}{8}g_{\alpha\beta}g_{\gamma\delta}W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}. \tag{2.1}$$

Observe that, being quadratic in the Weyl-tensor candidate, its geometrised dimensions are L^{-4}. However, the physical dimensions are energy-density per area, $ML^{-3}T^{-2}$ [83–85, 91, 92], and yet another simple proof was presented in [76] —see section 7.5. The properties of this tensor include:

i) it is completely symmetric $\mathcal{T}_{(\alpha\beta\gamma\delta)} = \mathcal{T}_{\alpha\beta\gamma\delta}$,

ii) it is traceless $\mathcal{T}^{\mu}_{\ \beta\gamma\mu} = 0$,

iii) obeys a dominant superenergy property, $\mathcal{T}_{\mu\nu\rho\sigma}\, v^{\mu}w^{\nu}u^{\rho}q^{\sigma} \geq 0$, where the four vectors are causal and future oriented. In particular,

iv) $\mathcal{T}^{\alpha}_{\ \nu\rho\sigma}\, k^{\nu}\ell^{\rho}z^{\sigma}$ is future pointing and lightlike, if k^{ν}, ℓ^{ρ}, z^{σ} are lightlike and future oriented. [93, 94].

In addition to these algebraic properties, let us include a differential one,

$$\nabla_{\mu}\mathcal{T}_{\alpha\beta\gamma}{}^{\mu} = 2W_{\mu\gamma\nu\alpha}Y_{\beta}{}^{\nu\mu} + 2W_{\mu\gamma\nu\beta}Y_{\alpha}{}^{\nu\mu} + g_{\alpha\beta}W^{\mu\nu\rho}{}_{\gamma}Y_{\mu\nu\rho} \ , \tag{2.2}$$

where

$$Y_{\alpha\beta\gamma} := -\nabla_{\mu}W_{\alpha\beta\gamma}{}^{\mu}, \tag{2.3}$$

and point out the following case:

$$Y_{\alpha\beta\gamma} = 0 \implies \nabla_{\mu}\mathcal{T}_{\alpha\beta\gamma}{}^{\mu} = 0 \ . \tag{2.4}$$

At this point, it is worth mentioning that for the particular case with $\mathcal{T}_{\alpha\beta\gamma\delta}$ constructed with the Weyl tensor, $Y_{\alpha\beta\gamma}$ is the Cotton tensor and, if Einstein's field equations hold, the absence of matter fields implies $Y_{\alpha\beta\gamma} = 0$ and $\mathcal{T}_{\alpha\beta\gamma\delta}$ is divergence-free.

Later, we will be interested in relating the algebraic classification of $W_{\alpha\beta\gamma\delta}$ with $\mathcal{T}_{\alpha\beta\gamma\delta}$ in a precise way. A characterisation of the Petrov type of $W_{\alpha\beta\gamma}{}^{\delta}$ and its repeated (or degenerated) principal null directions (PND) is [80, 95]

- $\mathcal{T}_{\alpha\beta\gamma\mu}\, k^\mu = 0$ and $\mathcal{T}_{\alpha\beta\gamma\delta}\, \neq 0 \Longleftrightarrow$ Petrov type N and k^α is a quadruple PND.
- $\mathcal{T}_{\alpha\beta\nu\mu}\, k^\nu k^\mu = 0$ and $\mathcal{T}_{\alpha\beta\gamma\mu}\, k^\mu \neq 0 \Longleftrightarrow$ Petrov type III and k^α is a triple PND.
- $\mathcal{T}_{\alpha\rho\nu\mu}\, k^\rho k^\nu k^\mu = 0$ and $\mathcal{T}_{\alpha\beta\nu\mu}\, k^\nu k^\mu \neq 0 \Longleftrightarrow$ Petrov type II or D and k^α is a double PND.
- $\mathcal{T}_{\sigma\rho\nu\mu}\, k^\sigma k^\rho k^\nu k^\mu = 0$ and $\mathcal{T}_{\alpha\rho\nu\mu}\, \ell^\rho \ell^\nu \ell^\mu \neq 0 \quad \forall$ lightlike $\ell^\alpha \Longleftrightarrow$ Petrov type I and k^α is a non-degenerate PND.
- $\mathcal{T}_{\alpha\rho\nu\mu}\, k^\rho k^\nu k^\mu = 0$, $\mathcal{T}_{\alpha\rho\nu\mu}\, \ell^\rho \ell^\nu \ell^\mu = 0$ and $\mathcal{T}_{\alpha\beta\gamma\delta}\, \neq 0 \Longleftrightarrow$ Petrov type D and k^α, ℓ^α are double PND.

For a detailed description of these and more general properties, see [80, 85] and references therein.

2.1.1 Orthogonal decomposition

Choose a unit timelike future-pointing vector field u^α. At each point, introduce a basis $\{e^\alpha{}_a\}$ spanning the vector space whose elements are all the vectors orthogonal to u^α at that point. Also, define a basis $\{\omega_\alpha{}^a\}$ for the dual space of one-forms such that they are orthogonal to u^α. We call these objects orthogonal to u^α 'spatial with respect to u^α' or, for simplicity, spatial —Latin indices denote spatial components and run from 1 to 3. Let $q^\alpha = e^\alpha{}_a q^a$ be any spatial vector, $q^\alpha u_\alpha = 0$. A projector[1] can be defined,

$$P^\alpha{}_\beta = e^\alpha{}_i \omega_\beta{}^i = \delta^\alpha{}_\beta + u^\alpha u_\beta, \quad P^\alpha{}_\mu u^\mu = 0, \quad P^\alpha{}_\mu q^\mu = q^\alpha \; . \tag{2.5}$$

In this way, any space-time vector w^α can be decomposed as a spatial part with respect to u^α together with another one tangent to u^α,

$$w^\alpha = -u^\alpha u_\mu w^\mu + \overline{w}^\alpha, \text{ with } \overline{w}^\alpha = \overline{w}^p e^\alpha{}_p \; , \tag{2.6}$$

such that

$$P^\alpha{}_\mu w^\mu = \overline{w}^\alpha \; . \tag{2.7}$$

This is generalised to higher-rank tensors in an obvious way. Latin indices are raised and lowered with

$$h_{ab} := e^\alpha{}_a e^\beta{}_b g_{\alpha\beta}, \quad h^{ab} := \omega_\alpha{}^a \omega_\beta{}^b g^{\alpha\beta} \; . \tag{2.8}$$

With these tools, it is possible to decompose the Weyl-tensor candidate and $\mathcal{T}^\alpha{}_{\beta\gamma\delta}$ into tangent and spatial parts with respect to u^α. The former is fully determined by its *electric*

[1]Throughout the text, projectors are denoted by a (decorated or not) $P^\alpha{}_\beta$ and defined according to the context.

and *magnetic* parts with respect to u^α

$$D_{\alpha\beta} := u^\mu u^\nu W_{\mu\alpha\nu\beta} \ , \tag{2.9}$$
$$C_{\alpha\beta} := u^\mu u^\nu \overset{*}{W}_{\mu\alpha\nu\beta} \ . \tag{2.10}$$

which are symmetric, traceless, spatial fields,

$$D_{\alpha\beta} = D_{ab}{\omega_\alpha}^a{\omega_\beta}^b \ , \tag{2.11}$$
$$C_{\alpha\beta} = C_{ab}{\omega_\alpha}^a{\omega_\beta}^b \ . \tag{2.12}$$

The full splitting of $\mathcal{T}_{\alpha\beta\gamma\delta}$ is

$$\mathcal{T}_{\alpha\beta\gamma\delta} = \mathcal{W}u_\alpha u_\beta u_\gamma u_\delta + 4\overline{\mathcal{P}}_{(\alpha}u_\beta u_\gamma u_{\delta)} + 4u_{(\alpha}Q_{\beta\gamma\delta)} + 6t_{(\alpha\beta}u_\gamma u_{\delta)} + t_{\alpha\beta\gamma\delta} \ . \tag{2.13}$$

We are interested in the first three terms on the right-hand side. They have obvious definitions in terms of the projector and u^α. In particular, $\overline{\mathcal{P}}^\alpha$ and $Q_{\alpha\beta\gamma}$ are spatial, i. e., $\overline{\mathcal{P}}^\alpha = {e^\alpha}_a\overline{\mathcal{P}}^a$, $Q_{\alpha\beta\gamma} = {\omega_\alpha}^a{\omega_\beta}^b{\omega_\gamma}^c Q_{abc}$. The first two are called the *superenergy density* and *super-Poynting* vector field, and have the following expressions in terms of the electric and magnetic parts of the Weyl-tensor candidate ([33, 78]):

$$\mathcal{W} = D_{ab}D^{ab} + C_{ab}C^{ab} \ , \tag{2.14}$$
$$\overline{\mathcal{P}}^a = [C, D]_{rs}\,\epsilon^{rsa} = 2{C_r}^t D_{ts}\epsilon^{rsa} \ , \tag{2.15}$$

where we have defined an alternating 3-dimensional tensor.

$$-u_\alpha\epsilon_{abc} := \eta_{\alpha\beta\gamma\delta}{e^\alpha}_a{e^\alpha}_b{e^\alpha}_c \tag{2.16}$$

Also, the third one can be expressed as [88]

$$Q_{\alpha\beta\gamma} = P_{\alpha\beta}\,\overline{\mathcal{P}}_\gamma - 2\left(D_{\alpha\mu}C_{\beta\nu} + D_{\beta\mu}C_{\alpha\nu}\right)u^\rho{\eta_{\rho\gamma}}^{\nu\mu} \ , \quad {Q^\mu}_{\mu\gamma} = \overline{\mathcal{P}}_\alpha \ . \tag{2.17}$$

The superenergy density and the super-Poynting vector field inherit their name from the analogy with the electromagnetism when ${W_{\alpha\beta\gamma}}^\delta$ is the Weyl tensor: the former represents energy density per unit area and the latter, the spatial direction of propagation of supernergy with respect to u^α. Following this analogy, one can also define a supermomentum

$$\mathcal{P}^\alpha := -u^\mu u^\nu u^\rho \mathcal{T}^\alpha_{\mu\nu\rho} = \mathcal{W}u^\alpha + \overline{\mathcal{P}}^\alpha \ , \tag{2.18}$$

which is non-spacelike and future pointing. One important feature is that

$${W_{\alpha\beta\gamma}}^\delta = 0 \Longleftrightarrow \mathcal{T}_{\alpha\beta\gamma\delta} = 0 \Longleftrightarrow \mathcal{W} = 0 \ . \tag{2.19}$$

Particularly inspiring for us is the following definition by Bel [33]:

Definition 2.1.1. There is a state of intrinsic gravitational radiation at a point p when $\overline{\mathcal{P}}^{\alpha}|_p \neq 0$ for all unit timelike u^α.

This classical definition agrees with the discussion by Pirani [30] and is based on the analogy with null electromagnetic fields which are precisely the fields with a non-zero Poynting vector for all possible observers. More recently, another similar characterization was put forward in [88] in the following terms

Definition 2.1.2. There is a superenergy state of intrinsic gravitational radiation at a point p when $Q_{\alpha\beta\gamma}|_p \neq 0$ for all unit timelike u^α.

Using (2.17) it is easy to check that every state of intrinsic gravitational radiation is also a superenergy such state, but there are more cases of the latter in general.

2.2 Lightlike projections

Let u^α be the unit timelike vector field, as defined in previous subsection, and r^α a unit, vector (field) –non necessarily defined everywhere– spatial with respect to u^α, i. e., $r_\alpha u^\alpha = 0$, $r^\alpha = r^a e^\alpha{}_a$. There are two (up to a boost) independent lightlike directions coplanar with u^α and r^α:

$$\overset{+}{k}{}^\alpha := \frac{1}{\sqrt{2}}\left(u^\alpha + r^\alpha\right) \ , \tag{2.20}$$

$$\overset{-}{k}{}^\alpha := \frac{1}{\sqrt{2}}\left(u^\alpha - r^\alpha\right) \tag{2.21}$$

such that $\overset{+}{k}{}^\alpha \overset{-}{k}{}_\alpha = -1$. At each point, introduce a basis for the vector space constituted by all vectors orthogonal to $\overset{+}{k}{}_\alpha$, $\{\overset{+}{e}{}^\alpha{}_{\hat{a}}\} = \{\overset{+}{k}{}^\alpha, E^\alpha{}_A\}$, and do the same with respect to $\overset{-}{k}{}_\alpha$, $\{\overset{-}{e}{}^\alpha{}_{\tilde{a}}\} = \{\overset{-}{k}{}^\alpha, E^\alpha{}_A\}$. Notice that, as these vector fields are lightlike, $\overset{-}{k}{}^\alpha = \overset{-}{k}{}^{\tilde{1}}\,\overset{-}{e}{}^\alpha{}_{\tilde{1}}$ and $\overset{+}{k}{}^\alpha = \overset{+}{k}{}^{\hat{1}}\,\overset{+}{e}{}^\alpha{}_{\hat{1}}$. One can introduce dual bases $\{\overset{+}{\omega}{}_\alpha{}^{\hat{a}}\} = \{-\overset{-}{k}{}_\alpha, W_\alpha{}^A\}$, $\{\overset{-}{\omega}{}_\alpha{}^{\tilde{a}}\} = \{-\overset{+}{k}{}_\alpha, W_\alpha{}^A\}$, such that $\overset{+}{k}{}^\alpha \overset{-}{\omega}{}_\alpha{}^{\tilde{a}} = 0$, $\overset{-}{k}{}^\alpha \overset{+}{\omega}{}_\alpha{}^{\hat{a}} = 0$, $\overset{-}{k}{}^\alpha \overset{-}{\omega}{}_\alpha{}^{\tilde{a}} = \overset{-}{k}{}^{\tilde{a}}$, $\overset{+}{k}{}^\alpha \overset{+}{\omega}{}_\alpha{}^{\hat{a}} = \overset{+}{k}{}^{\hat{a}}$. Here, $\{E^\alpha{}_A\}$, $\{W_\alpha{}^A\}$ are bases spanning the two-dimensional vector space orthogonal to r^α and u^α –equivalently, orthogonal to $\overset{\pm}{k}{}^\alpha$. Any space-time vector w^α decomposes into a part tangent to u^α and a spatial part, $\overline{w}^\alpha$, which splits into a component tangent to r^α and another one that is orthogonal to both vector fields u^α and r^α, $\mathring{w}^\alpha = \mathring{w}^A E^\alpha{}_A$,

$$w^\alpha = -w_\mu u^\mu u^\alpha + \overline{w}^\alpha = -w_\mu u^\mu u^\alpha + w_\mu r^\mu r^\alpha + \mathring{w}^\alpha \ . \tag{2.22}$$

The object

$$\mathring{P}^\alpha{}_\beta = \delta^\alpha{}_\beta + u^\alpha u_\beta - r^\alpha r_\beta = \delta^\alpha{}_\beta + \overset{-}{k}{}^\alpha \overset{+}{k}{}_\beta + \overset{+}{k}{}^\alpha \overset{-}{k}{}_\beta \tag{2.23}$$

is the projector orthogonal to n^α and r^α

$$\mathring{P}^{\alpha}{}_{\beta}\, w^{\beta} = \mathring{w}^{\beta}, \quad \mathring{P}^{\alpha}{}_{\beta}\, r_{\alpha} = \mathring{P}^{\alpha}{}_{\beta}\, u_{\alpha} = 0 \ . \tag{2.24}$$

Again, all this is generalised to higher-rank tensors in a natural way.

There are some useful lightlike projections of the Weyl-tensor candidate we are interested in. Some of them are expressed in terms of $\{\overset{+}{e}{}^{\alpha}{}_{\hat{a}}\}$ and $\{\overset{+}{\omega}{}_{\alpha}{}^{\hat{a}}\}$ only, and are orthogonal to $\overset{+}{k}_{\alpha}$ in their contravariant indices and to $\overset{-}{k}{}^{\alpha}$ in the covariant ones:

$$\overset{+}{D}{}^{\alpha\beta} := \overset{+}{k}{}^{\mu}\overset{+}{k}{}^{\nu} W_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta} = \overset{+}{D}{}^{\hat{a}\hat{b}}\,\overset{+}{e}{}^{\alpha}{}_{\hat{a}}\,\overset{+}{e}{}^{\beta}{}_{\hat{b}} \ , \tag{2.25}$$

$$\overset{+}{C}{}^{\alpha\beta} := \overset{+}{k}{}^{\mu}\overset{+}{k}{}^{\nu}\, {}^{*}W_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta} = \overset{+}{C}{}^{\hat{a}\hat{b}}\,\overset{+}{e}{}^{\alpha}{}_{\hat{a}}\,\overset{+}{e}{}^{\beta}{}_{\hat{b}} \ , \tag{2.26}$$

$$\overset{-+}{D}{}_{\alpha}{}^{\beta} := \overset{-}{k}{}^{\mu}\overset{+}{k}{}^{\nu} W_{\mu\alpha\nu}{}^{\beta} = \overset{-+}{D}{}_{\hat{a}}{}^{\hat{b}}\,\overset{+}{\omega}{}_{\alpha}{}^{\hat{a}}\,\overset{+}{e}{}^{\beta}{}_{\hat{b}} \ , \tag{2.27}$$

$$\overset{-+}{C}{}_{\alpha}{}^{\beta} := \overset{-}{k}{}^{\mu}\overset{+}{k}{}^{\nu}\, {}^{*}W_{\mu\alpha\nu}{}^{\beta} = \overset{-+}{C}{}_{\hat{a}}{}^{\hat{b}}\,\overset{+}{\omega}{}_{\alpha}{}^{\hat{a}}\,\overset{+}{e}{}^{\beta}{}_{\hat{b}} \ , \tag{2.28}$$

whereas others are written in terms of $\{\overset{-}{e}{}^{\alpha}{}_{\tilde{a}}\}$ and $\{\overset{-}{\omega}{}_{\alpha}{}^{\tilde{a}}\}$, and are orthogonal to $\overset{-}{k}_{\alpha}$ in their contravariant indices and to $\overset{+}{k}{}^{\alpha}$ in the convariant ones:

$$\overset{-}{D}{}^{\alpha\beta} := \overset{-}{k}{}^{\mu}\overset{-}{k}{}^{\nu} W_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta} = \overset{-}{D}{}^{\tilde{k}\tilde{l}}\,\overset{-}{e}{}^{\alpha}{}_{\tilde{k}}\,\overset{-}{e}{}^{\beta}{}_{\tilde{l}} \ , \tag{2.29}$$

$$\overset{-}{C}{}^{\alpha\beta} := \overset{-}{k}{}^{\mu}\overset{-}{k}{}^{\nu}\, {}^{*}W_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta} = \overset{-}{C}{}^{\tilde{k}\tilde{l}}\,\overset{-}{e}{}^{\alpha}{}_{\tilde{k}}\,\overset{-}{e}{}^{\beta}{}_{\tilde{l}} \ , \tag{2.30}$$

$$\overset{+-}{D}{}_{\alpha}{}^{\beta} := \overset{+}{k}{}^{\mu}\overset{-}{k}{}^{\nu} W_{\mu\alpha\nu}{}^{\beta} = \overset{+-}{D}{}_{\tilde{k}}{}^{\tilde{l}}\,\overset{-}{\omega}{}_{\alpha}{}^{\tilde{k}}\,\overset{-}{e}{}^{\beta}{}_{\tilde{l}} \ , \tag{2.31}$$

$$\overset{+-}{C}{}_{\alpha}{}^{\beta} := \overset{+}{k}{}^{\mu}\overset{-}{k}{}^{\nu}\, {}^{*}W_{\mu\alpha\nu}{}^{\beta} = \overset{+-}{C}{}_{\tilde{k}}{}^{\tilde{l}}\,\overset{-}{\omega}{}_{\alpha}{}^{\tilde{k}}\,\overset{-}{e}{}^{\beta}{}_{\tilde{l}} \ . \tag{2.32}$$

Notice that these quantities are not completely independent from each other but all the information of $W_{\alpha\beta\gamma}{}^{\delta}$ is contained in the first pair of the upper set plus the first pair of the lower one. In terms of the Weyl scalars, the first set of equations contains $\phi_{2,3,4}$; the second, $\phi_{2,0,1}$ — see appendix D.2.

It will turn out to be very practical to introduce the following notation for any symmetric tensor $B_{\mu\nu}$:

$$B_{\alpha\beta} = u^{\mu}u^{\nu}B_{\mu\nu}u_{\alpha}u_{\beta} + u^{\mu}\mathring{P}^{\nu}{}_{(\alpha}u_{\beta)}B_{\mu\nu} + 2B_{\mu}u^{\mu}r_{(\alpha}u_{\beta)} + r_{\alpha}r_{\beta}B + 2\mathring{B}_{(\alpha}r_{\beta)} + \mathring{B}_{\alpha\beta} \ , \tag{2.33}$$

and

$$\grave{B}_{\alpha\beta} := \mathring{B}_{\alpha\beta} - \frac{1}{2}\mathring{P}_{\alpha\beta}\,\mathring{P}^{\mu\nu}\mathring{B}_{\mu\nu} \ , \quad \grave{B}_{\mu}{}^{\mu} = 0 \ , \tag{2.34}$$

where

$$\mathring{B}_{\alpha\beta} := \mathring{P}^{\mu}{}_{\alpha}\,\mathring{P}^{\nu}{}_{\beta}\,B_{\mu\nu}, \quad B_{\alpha} := r^{\mu}B_{\mu\alpha}, \quad \mathring{B}_{\alpha} := \mathring{P}^{\nu}{}_{\alpha}\,r^{\mu}B_{\mu\nu}, \quad B := r^{\mu}r^{\nu}B_{\mu\nu}. \tag{2.35}$$

Obviously, $\mathring{B}_{\alpha\beta}$ and $\mathring{B}_{\alpha}$ are orthogonal to r^{α} and u^{α}. The same notation will be used with uppercase indices. Define

$$q_{AB} := E^{\alpha}{}_{A}E^{\beta}{}_{B}\mathring{P}_{\alpha\beta} = E^{\alpha}{}_{A}E^{\beta}{}_{B}\mathring{g}_{\alpha\beta} \ , \tag{2.36}$$

$$q^{AB} := W_{\alpha}{}^{A}W_{\beta}{}^{B}\mathring{P}^{\alpha\beta} = W_{\alpha}{}^{A}W_{\beta}{}^{B}\mathring{P}^{\alpha\beta} \tag{2.37}$$

to lower and raise capital Latin indices. Then,

$$\mathring{B}_{AB} := E^{\alpha}{}_{A}E^{\beta}{}_{B}\mathring{B}_{\alpha\beta} \ , \quad \grave{B}_{AB} := \mathring{B}_{AB} - \frac{1}{2}q_{AB}\mathring{B}^{C}{}_{C} \ , \quad \mathring{B}_{A} := E^{A}{}_{\alpha}r^{\mu}B_{\mu\alpha} \ , \quad \grave{B}_{A}{}^{A} = 0 \tag{2.38}$$

An alternating two-dimensional tensor can be defined by

$$r_{m}\mathring{\epsilon}_{AB} = \epsilon_{mab}E^{a}{}_{A}E^{b}{}_{B} \ . \tag{2.39}$$

A list of properties of these quantities has been placed in appendix D.

The electric and magnetic parts of the Weyl-tensor candidate read:

$$D_{ab} = D\left(r_{a}r_{b} - \frac{1}{2}\mathring{P}_{ab}\right) + 2r_{(a}W_{b)}{}^{B}\mathring{D}_{B} + W_{a}{}^{A}W_{b}{}^{B}\grave{D}_{AB} \quad , \tag{2.40}$$

$$C_{ab} = C\left(r_{a}r_{b} - \frac{1}{2}\mathring{P}_{ab}\right) + 2r_{(a}W_{b)}{}^{B}\mathring{C}_{B} + W_{a}{}^{A}W_{b}{}^{B}\grave{C}_{AB} \tag{2.41}$$

and equivalently, in terms of the lightlike components that we have just presented,

$$\begin{aligned} D_{ab} = & \overset{\mp}{k}{}^{\mu}\overset{\mp}{k}{}^{\nu}\overset{\pm}{D}_{\mu\nu}r_{a}r_{b} + 2r_{(a}W_{b)}{}^{B}\left(\overset{+}{\mathring{D}}_{B} + \overset{-}{\mathring{D}}_{B}\right) + \\ & \frac{1}{2}W_{a}{}^{A}W_{b}{}^{B}\left(\overset{+}{\mathring{D}}_{AB} + \overset{-}{\mathring{D}}_{AB}\right) - \frac{1}{2}\overset{\mp}{k}{}^{\mu}\overset{\mp}{k}{}^{\nu}\overset{\pm}{D}_{\mu\nu}\mathring{P}_{ab} \ , \end{aligned} \tag{2.42}$$

$$\begin{aligned} C_{ab} = & \overset{\mp}{k}{}^{\mu}\overset{\mp}{k}{}^{\nu}\overset{\pm}{C}_{\mu\nu}r_{a}r_{b} + 2r_{(a}W_{b)}{}^{B}\left(\overset{+}{\mathring{C}}_{B} + \overset{-}{\mathring{C}}_{B}\right) + \\ & \frac{1}{2}W_{a}{}^{A}W_{b}{}^{B}\left(\overset{+}{\mathring{C}}_{AB} + \overset{-}{\mathring{C}}_{AB}\right) - \frac{1}{2}\overset{\mp}{k}{}^{\mu}\overset{\mp}{k}{}^{\nu}\overset{\pm}{C}_{\mu\nu}\mathring{P}_{ab} \ . \end{aligned} \tag{2.43}$$

Another quantity that will appear later on is:

$$\overset{\pm}{\mathring{t}}_{ABC} := E^{\alpha}{}_{A}E^{\beta}{}_{B}E^{\gamma}{}_{C}\overset{\pm}{k}{}^{\mu}W_{\alpha\beta\gamma\mu} = \pm 2\sqrt{2}q_{C[A}\overset{\pm}{\mathring{D}}_{B]} \ , \tag{2.44}$$

where

$$q_{AB} := E^{\alpha}{}_{A}E^{\beta}{}_{B}g_{\alpha\beta} \ . \tag{2.45}$$

2.3 Radiant superenergy

Now, we introduce a new kind of superenergy quantities associated to lightlike directions. The use of these objects helps in identifying the radiative sectors of the superenergy tensor. For such reason, we refer to them as *radiant superenergy* quantities. Given a future-oriented lightlike vector field ℓ^α we define its associated *radiant supermomentum*,

$$ {}^{\ell}\mathcal{Q}^{\alpha} := -\ell^{\mu}\ell^{\nu}\ell^{\sigma}\mathcal{T}^{\alpha}{}_{\mu\nu\sigma} \ . \tag{2.46} $$

Given any lightlike vector field k^α such that $k_\alpha \ell^\alpha = -1$, $\mathcal{Q}^\alpha$ decomposes as[2]

$$ {}^{\ell}\mathcal{Q}^{\alpha} = {}^{\ell}\mathcal{W} k^{\alpha} + {}^{\ell}\underline{\mathcal{Q}}^{\alpha} = {}^{\ell}\mathcal{W} k^{\alpha} + {}^{\ell}\mathcal{Z}\ell^{\alpha} + \left(\delta^{\alpha}_{\mu} + \ell^{\alpha}k_{\mu} + k^{\alpha}\ell_{\mu}\right) {}^{\ell}\underline{\mathcal{Q}}^{\mu} \ , \tag{2.47} $$

where ${}^{\ell}\underline{\mathcal{Q}}^{\alpha}$ is the *radiant super-Poynting vector* and ${}^{\ell}\mathcal{W}$ and ${}^{\ell}\mathcal{Z}$ the corresponding transverse and longitudinal[3] *radiant superenergy densities*. Note that ${}^{\ell}\mathcal{Z}$ and $(\delta^{\alpha}_{\mu} + \ell^{\alpha}k_{\mu} + k^{\alpha}\ell_{\mu}) {}^{\ell}\underline{\mathcal{Q}}^{\mu}$ depend on the choice of k^α. In particular, for the previously defined ${}^{\pm}k^{\alpha}$, the supermomenta read

$$ {}^{+}\mathcal{Q}^{\alpha} := -{}^{+}k^{\mu}\,{}^{+}k^{\nu}\,{}^{+}k^{\rho}\mathcal{T}^{\alpha}{}_{\mu\nu\rho} = {}^{+}\mathcal{W}\,{}^{-}k^{\alpha} + {}^{+}\underline{\mathcal{Q}}^{\alpha} = {}^{+}\mathcal{W}\,{}^{-}k^{\alpha} + {}^{+}\underline{\mathcal{Q}}^{a}\,{}^{+}e^{\alpha}{}_{a} \ , \tag{2.48} $$

$$ {}^{-}\mathcal{Q}^{\alpha} := -{}^{-}k^{\mu}\,{}^{-}k^{\nu}\,{}^{-}k^{\rho}\mathcal{T}^{\alpha}{}_{\mu\nu\rho} = {}^{-}\mathcal{W}\,{}^{+}k^{\alpha} + {}^{-}\underline{\mathcal{Q}}^{\alpha} = {}^{-}\mathcal{W}\,{}^{+}k^{\alpha} + {}^{-}\underline{\mathcal{Q}}^{k}\,{}^{-}e^{\alpha}{}_{k} \ . \tag{2.49} $$

where

$$ {}^{+}\underline{\mathcal{Q}}^{a} = {}^{+}\mathcal{Z}\,{}^{+}k^{a} + {}^{+}\underline{\mathcal{Q}}^{A}E^{a}{}_{A} \ , \tag{2.50} $$

$$ {}^{-}\underline{\mathcal{Q}}^{k} = {}^{-}\mathcal{Z}\,{}^{-}k^{k} + {}^{-}\underline{\mathcal{Q}}^{A}E^{k}{}_{A} \ , \tag{2.51} $$

Also, the following formulae hold[4]

$$ {}^{+}\mathcal{W} = -{}^{+}k_{\mu}\,{}^{+}\mathcal{Q}^{\mu} = 2\,{}^{+}C_{\mu\nu}\,{}^{+}C^{\mu\nu} = 2\,{}^{+}D_{\mu\nu}\,{}^{+}D^{\mu\nu} = 2\,{}^{+}\mathring{C}_{AB}\,{}^{+}\mathring{C}^{AB} = 2\,{}^{+}\mathring{D}_{AB}\,{}^{+}\mathring{D}^{AB} \geq 0, \tag{2.52} $$

$$ {}^{-}\mathcal{W} = -{}^{-}k_{\mu}\,{}^{-}\mathcal{Q}^{\mu} = 2\,{}^{-}C_{\mu\nu}\,{}^{-}C^{\mu\nu} = 2\,{}^{-}D_{\mu\nu}\,{}^{-}D^{\mu\nu} = 2\,{}^{-}\mathring{C}_{AB}\,{}^{-}\mathring{C}^{AB} = 2\,{}^{-}\mathring{D}_{AB}\,{}^{-}\mathring{D}^{AB} \geq 0, \tag{2.53} $$

$$ {}^{+}\mathcal{Z} = -{}^{-}k_{\mu}\,{}^{+}\mathcal{Q}^{\mu} = 2\,{}^{-+}C_{\mu\nu}\,{}^{+}C^{\mu\nu} = 2\,{}^{-+}D_{\mu\nu}\,{}^{+}D^{\mu\nu} = 4\,{}^{+}\mathring{C}_{A}\,{}^{+}\mathring{C}^{A} \geq 0, \tag{2.54} $$

$$ {}^{-}\mathcal{Z} = -{}^{+}k_{\mu}\,{}^{-}\mathcal{Q}^{\mu} = 2\,{}^{+-}C_{\mu\nu}\,{}^{-}C^{\mu\nu} = 2\,{}^{+-}D_{\mu\nu}\,{}^{-}D^{\mu\nu} = 4\,{}^{-}\mathring{C}_{A}\,{}^{-}\mathring{C}^{A} \geq 0, \tag{2.55} $$

$$ {}^{+}\mathcal{Q}^{A} = 4\sqrt{2}\,{}^{+}\mathring{C}_{P}\,{}^{+}\mathring{C}^{AP}, \tag{2.56} $$

$$ {}^{-}\mathcal{Q}^{A} = -4\sqrt{2}\,{}^{-}\mathring{C}_{P}\,{}^{-}\mathring{C}^{AP} \ . \tag{2.57} $$

The expressions on the right-hand side can be derived by direct computation, using properties i), viii), ix), xv), xxxi) and xxxii) on page 211 . In addition, we define the

[2]The underlining used here should not be confused with the short bar placed under quantities associated to a congruence (compare to the notation of appendix A.3).

[3]For the Bel-Robinson tensor, the 'transverse' and 'longitudinal' modes of the gravitational radiation determine completely ${}^{\ell}\mathcal{W}$ and ${}^{\ell}\mathcal{Z}$, respectively. This can be easily inferred from the expressions of these quantities in terms of the Weyl-tensor candidate scalars in appendix D.2.

[4]This can be done for any lightlike vector fields ℓ^α and k^α as defined above. Just for convenience, we present them for ${}^{\pm}k^{\alpha}$.

Coulomb superenergy density as[5]:

$$\mathcal{V} := {}^{+}k^{\mu}\,{}^{-}k^{\nu}\,{}^{+}k^{\rho}\,{}^{-}k^{\sigma}\mathcal{D}_{\mu\nu\rho\sigma} = {}^{+-}C^{AB}\,{}^{+-}C_{AB} + {}^{+-}D^{AB}\,{}^{+-}D_{AB} = C^2 + D^2 \geq 0. \tag{2.58}$$

Obser fove the non-negativity of eqs. (2.52) to (2.55) and (2.58). Contracting eq. (2.13) four times with u^α in the form

$$u^\alpha = \frac{1}{\sqrt{2}}\left({}^{+}k^\alpha + {}^{-}k^\alpha\right), \tag{2.59}$$

one gets the relation

$$\mathcal{W} = \frac{1}{4}\left[{}^{+}\mathcal{W} + 4\,{}^{+}\mathcal{Z} + 6\mathcal{V} + 4\,{}^{-}\mathcal{Z} + {}^{-}\mathcal{W}\right] . \tag{2.60}$$

Indeed, it is easy to generalise this formula for any kind of coplanarity and to obtain the following lemma

Lemma 2.3.1. Let $\mathcal{W}$ be the superenergy density associated to a unit timelike vector field u^α, and ${}^{\pm}\mathcal{W}$, ${}^{\pm}\mathcal{Z}$, $\mathcal{V}$, the superenergy densities associated to a couple of lightlike vector fields ${}^{\pm}k^\alpha$ such that

$$u^\alpha = \left(a\,{}^{-}k^\alpha + b\,{}^{+}k^\alpha\right), \quad \text{with } ab = \frac{1}{2} . \tag{2.61}$$

Then,

$$\mathcal{W} = \left[b^4\,{}^{+}\mathcal{W} + 4b^3a\,{}^{+}\mathcal{Z} + 6a^2b^2\mathcal{V} + 4ba^3\,{}^{-}\mathcal{Z} + a^4\,{}^{-}\mathcal{W}\right] , \tag{2.62}$$

$$\mathcal{W} = 0 \Longleftrightarrow \left\{\mathcal{V} = 0, \quad {}^{\pm}\mathcal{W} = 0, \quad {}^{\pm}\mathcal{Z} = 0\right\} . \tag{2.63}$$

Any radiant supermomentum ${}^{\ell}\mathcal{Q}^\alpha$ constructed with a future-pointing lightlike vector field as in eq. (2.46) has some basic properties,

i) ${}^{\ell}\mathcal{Q}^\alpha$ is lightlike, ${}^{\ell}\mathcal{Q}^\mu\,{}^{\ell}\mathcal{Q}_\mu = 0$, and future pointing. This follows by the dominant superenergy condition in the version of property iv) on page 8.

ii) $\left(\delta^\nu_\mu + \ell^\nu k_\mu + k^\nu \ell_\mu\right){}^{\ell}\underline{\mathcal{Q}}^\mu\,{}^{\ell}\underline{\mathcal{Q}}_\nu = 2\,{}^{\ell}\mathcal{Z}\,{}^{\ell}\mathcal{W}$, which can be shown applying property i) to ${}^{\ell}\mathcal{Q}^\alpha$ and using eq. (2.47). From these same equations, it follows that

iii) ${}^{\ell}\underline{\mathcal{Q}}^\alpha = 0 \Longleftrightarrow {}^{\ell}\mathcal{Z} = 0$. And, also, ${}^{\ell}\mathcal{Q}^\alpha = 0 \Longleftrightarrow {}^{\ell}\mathcal{W} = 0 = {}^{\ell}\mathcal{Z}$.

If we contract the radiant supermomenta with $P^\alpha{}_\beta$, we obtain their parts orthogonal to

[5]For the Bel-Robinson tensor, this is completely determined by the Coulomb part of the gravitational field –see the expressions in terms of the Weyl scalars in appendix D.2.

u^α,

$$P^\alpha{}_\mu\, {}^{+}\!\mathcal{Q}^\mu = \frac{1}{\sqrt{2}}\left({}^{+}\!\mathcal{Z} - {}^{+}\!\mathcal{W}\right) r^\alpha + {}^{+}\!\underline{\mathcal{Q}}^A E^\alpha{}_A = {}^{+}\!\underline{\mathcal{Q}}^\alpha \ , \tag{2.64}$$

$$P^\alpha{}_\mu\, {}^{-}\!\mathcal{Q}^\mu = -\frac{1}{\sqrt{2}}\left({}^{-}\!\mathcal{Z} - {}^{-}\!\mathcal{W}\right) r^\alpha + {}^{-}\!\underline{\mathcal{Q}}^A E^\alpha{}_A = {}^{-}\!\underline{\mathcal{Q}}^\alpha \ . \tag{2.65}$$

Using properties xv), xvii), xxi) and xxii) on page 212, we can write the radiant decomposition in terms of the electric and magnetic parts of the Weyl-tensor candidate

$$ {}^{+}\!\mathcal{Z} = \left(\mathring{D}_A + \mathring{\epsilon}_A{}^E \mathring{C}_E\right)\left(\mathring{D}^A + \mathring{\epsilon}^{AD}\mathring{C}_D\right) \quad , \tag{2.66}$$

$$ {}^{-}\!\mathcal{Z} = \left(\mathring{D}_A - \mathring{\epsilon}_A{}^E \mathring{C}_E\right)\left(\mathring{D}^A - \mathring{\epsilon}^{AD}\mathring{C}_D\right) \quad , \tag{2.67}$$

$$ {}^{+}\!\mathcal{W} = 2\left(\grave{C}_{AB} - \grave{D}^T{}_{(B}\mathring{\epsilon}_{A)T}\right)\left(\grave{C}^{AB} - \grave{D}_M{}^{(B}\mathring{\epsilon}^{A)M}\right) \quad , \tag{2.68}$$

$$ {}^{-}\!\mathcal{W} = 2\left(\grave{C}_{AB} + \grave{D}^T{}_{(B}\mathring{\epsilon}_{A)T}\right)\left(\grave{C}^{AB} + \grave{D}_M{}^{(B}\mathring{\epsilon}^{A)M}\right) \quad , \tag{2.69}$$

$$ {}^{+}\!\mathcal{Q}^A = 2\sqrt{2}\left(\grave{C}^{AB} - \grave{D}_M{}^{(B}\mathring{\epsilon}^{A)M}\right)\left(\mathring{C}_B - \mathring{\epsilon}_B{}^E\mathring{D}_E\right) \quad , \tag{2.70}$$

$$ {}^{-}\!\mathcal{Q}^A = -2\sqrt{2}\left(\grave{C}^{AB} + \grave{D}_M{}^{(B}\mathring{\epsilon}^{A)M}\right)\left(\mathring{C}_B + \mathring{\epsilon}_B{}^E\mathring{D}_E\right) \quad . \tag{2.71}$$

And with these relations it is straightforward to compute

$$ {}^{+}\!\mathcal{Z} - {}^{-}\!\mathcal{Z} = 4\mathring{D}^A\mathring{\epsilon}_{AB}\mathring{C}^B \quad , \tag{2.72}$$

$$ {}^{+}\!\mathcal{Z} + {}^{-}\!\mathcal{Z} = 2\left(\mathring{C}_A\mathring{C}^A + \mathring{D}_A\mathring{D}^A\right) \quad , \tag{2.73}$$

$$ {}^{+}\!\mathcal{W} - {}^{-}\!\mathcal{W} = 8\grave{D}^{TB}\mathring{\epsilon}_{TA}\grave{C}^A{}_B \ , \tag{2.74}$$

$$ {}^{+}\!\mathcal{W} + {}^{-}\!\mathcal{W} = 4\left(\grave{D}_{AB}\grave{D}^{AB} + \grave{C}_{AB}\grave{C}^{AB}\right) \ , \tag{2.75}$$

$$ \sqrt{2}\left({}^{+}\!\mathcal{Q}^A - {}^{-}\!\mathcal{Q}^A\right) = 8\left(\mathring{C}_P\grave{C}^{PA} + \mathring{D}_E\grave{D}^{EA}\right) \ , \tag{2.76}$$

$$ \sqrt{2}\left({}^{+}\!\mathcal{Q}^A + {}^{-}\!\mathcal{Q}^A\right) = -8\mathring{\epsilon}_T{}^A\left(\mathring{D}_P\grave{C}^{PT} - \mathring{C}_E\grave{D}^{ET}\right) \ . \tag{2.77}$$

We would like to have a complete relation between the radiant and the standard super-momentum, and we already have the relation between superenergy densitites eq. (2.60). Thus, it only remains to find an expression for the super-Poynting vector in terms of the new quantities. For that purpose, substitute $\overset{\pm}{k}{}^\alpha$ in terms of u^α and r^α in eqs. (2.48) and (2.49),

$$ {}^{+}\!\mathcal{Q}^\alpha = \frac{1}{2\sqrt{2}}\left(\mathcal{Q}^\alpha - 3\mathcal{D}^\alpha{}_{\mu\nu\rho}u^\mu r^\nu r^\rho - 3\mathcal{D}^\alpha{}_{\mu\nu\rho}r^\mu u^\nu u^\rho - \mathcal{D}^\alpha{}_{\mu\nu\rho}r^\mu r^\nu r^\rho\right) \ , \tag{2.78}$$

$$ {}^{-}\!\mathcal{Q}^\alpha = \frac{1}{2\sqrt{2}}\left(\mathcal{Q}^\alpha - 3\mathcal{D}^\alpha{}_{\mu\nu\rho}u^\mu r^\nu r^\rho + 3\mathcal{D}^\alpha{}_{\mu\nu\rho}r^\mu u^\nu u^\rho + \mathcal{D}^\alpha{}_{\mu\nu\rho}r^\mu r^\nu r^\rho\right) \ . \tag{2.79}$$

Thus,

$$ {}^{+}\!\mathcal{Q}^\alpha + {}^{-}\!\mathcal{Q}^\alpha = \frac{1}{\sqrt{2}}\left(\mathcal{Q}^\alpha - 3\mathcal{D}^\alpha{}_{\mu\nu\rho}u^\mu r^\nu r^\rho\right) \ . \tag{2.80}$$

If we contract this equation with $P^{\alpha}{}_{\beta}$, the right-hand side is determined by eq. (2.17),

$$P^{\alpha}{}_{\beta}\mathcal{D}_{\alpha\mu\nu\rho}u^{\mu}r^{\nu}r^{\rho} = -Q_{\beta\nu\rho}r^{\nu}r^{\rho} = -\overline{\mathcal{P}}_{\beta} + 4r_{\beta}\mathring{\epsilon}_{AB}C^{A}D^{B} + 4W_{\beta}{}^{E}\mathring{\epsilon}_{EA}\left(\mathring{C}^{A}D - \mathring{D}^{A}C\right) , \tag{2.81}$$

which, after inserting eq. (2.72), becomes

$$P^{\alpha}{}_{\beta}\mathcal{D}_{\alpha\mu\nu\rho}u^{\mu}r^{\nu}r^{\rho} = -\overline{\mathcal{P}}_{\beta} - \left({}^{+}\!\mathcal{Z} - {}^{-}\!\mathcal{Z}\right)r_{\beta} + 4W_{\beta}{}^{E}\mathring{\epsilon}_{EA}\left(\mathring{C}^{A}D - \mathring{D}^{A}C\right) . \tag{2.82}$$

The left-hand side is known by eqs. (2.64) and (2.65), therefore

$$\begin{aligned} &\frac{1}{\sqrt{2}}\left({}^{+}\!\mathcal{Z} - {}^{-}\!\mathcal{Z} - {}^{+}\!\mathcal{W} + {}^{-}\!\mathcal{W}\right)r^{\alpha} + \left({}^{+}\!\underline{\mathcal{Q}}^{A} + {}^{-}\!\underline{\mathcal{Q}}^{A}\right)E^{\alpha}{}_{A} \\ = &\frac{1}{\sqrt{2}}\left[4\overline{\mathcal{P}}^{\alpha} + 3\left({}^{+}\!\mathcal{Z} - {}^{-}\!\mathcal{Z}\right)r^{\alpha} - 12E^{\alpha}{}_{E}\mathring{\epsilon}^{E}{}_{A}\left(\mathring{C}^{A}D - \mathring{D}^{A}C\right)\right] . \end{aligned} \tag{2.83}$$

After recombining the terms, one finds the following relation:

$$4\overline{\mathcal{P}}^{a} = \left(2\,{}^{-}\!\mathcal{Z} - 2\,{}^{+}\!\mathcal{Z} - {}^{+}\!\mathcal{W} + {}^{-}\!\mathcal{W}\right)r^{a} + \left[\sqrt{2}\left({}^{+}\!\underline{\mathcal{Q}}^{A} + {}^{-}\!\underline{\mathcal{Q}}^{A}\right) + 12\mathring{\epsilon}^{A}{}_{E}\left(\mathring{C}^{E}D - \mathring{D}^{E}C\right)\right]E^{a}{}_{A} . \tag{2.84}$$

The whole supermomentum is determined by eqs. (2.60) and (2.84),

$$4\mathcal{Q}^{\alpha} = \left[{}^{+}\!\mathcal{W} + 4\,{}^{+}\!\mathcal{Z} + 6\mathcal{V} + 4\,{}^{-}\!\mathcal{Z} + {}^{-}\!\mathcal{W}\right]u^{\alpha} + \left(2\,{}^{-}\!\mathcal{Z} - 2\,{}^{+}\!\mathcal{Z} - {}^{+}\!\mathcal{W} + {}^{-}\!\mathcal{W}\right)r^{\alpha} + \tag{2.85}$$

$$+ \left[\sqrt{2}\left({}^{+}\!\underline{\mathcal{Q}}^{A} + {}^{-}\!\underline{\mathcal{Q}}^{A}\right) + 12\mathring{\epsilon}^{A}{}_{E}\left(\mathring{C}^{E}D - \mathring{D}^{E}C\right)\right]E^{\alpha}{}_{A} . \tag{2.86}$$

The radiant components contain no information about the traces $D^{A}{}_{A}$ and $C^{A}{}_{A}$, whose squares determine the Coulomb superenergy density $\mathcal{V}$. Besides, the longitudinal radiant superenergy densities ${}^{\pm}\!\mathcal{Z}$ are controlled by $\mathring{C}_{A}$, $\mathring{D}_{A}$. Note that Equations (2.84) and (2.85) contain a 'mixed' term

$$d^{A} := \mathring{\epsilon}^{A}{}_{E}\left(\mathring{C}^{E}D - \mathring{D}^{E}C\right) . \tag{2.87}$$

Let us finish this section with some results. The first one follows from the Petrov classification on page 8

Lemma 2.3.2. A radiant supermomentum, ${}^{\ell}\!\mathcal{Q}^{\alpha}$, constructed with a lightlike vector ℓ^{α} vanishes if and only if ℓ^{α} is a repeated PND of the corresponding Weyl-tensor candidate.

Lemma 2.3.3. Consider the lightlike projections of the Weyl-tensor candidate tensor for a couple of lighlike vectors ${}^{\pm}\!k^{\alpha}$ as in eq. (2.59). Let one of the associated radiant supermomenta vanish, ${}^{\pm}\!\mathcal{Q}^{\alpha} = 0$. Then,

i) ${}^{\pm}\!\mathring{D}_{A} = 0 = {}^{\pm}\!\mathring{C}_{A}$ $\quad(\Longleftrightarrow \mathring{D}_{A}\mathring{\epsilon}^{AB} = \mp\mathring{C}^{B})$ and then $\mathring{C}_{A} = {}^{\mp}\!\mathring{C}_{A}$, $\mathring{D}_{A} = {}^{\mp}\!\mathring{D}_{A}$,

ii) $\grave{D}_{T(B}\mathring{\epsilon}_{A)}{}^{T} = \pm\grave{C}_{AB}$, iii)

$$ {}^{\mp}\mathcal{Z} = 4\mathring{C}_A\mathring{C}^A = 4\mathring{D}_A\mathring{D}^A \ , \tag{2.88}$$
$$ {}^{\mp}\mathcal{W} = \mp 16\grave{D}^{TB}\mathring{\epsilon}_{TA}\grave{C}^A{}_B = 16\grave{C}_{AB}\grave{C}^{AB} = 16\grave{D}_{AB}\grave{D}^{AB} \ , \tag{2.89}$$
$$ {}^{\mp}\mathcal{Q}^A = 8\sqrt{2}\grave{C}^{AB}C_B = \pm 8\sqrt{2}\grave{D}_T{}^{(B}\mathring{\epsilon}^{A)T}C_B \ , \tag{2.90}$$
$$ d_A = \mathring{\epsilon}_A{}^E\left(\delta_E{}^B D \pm C\mathring{\epsilon}_E{}^B\right)\mathring{C}_B \ . \tag{2.91}$$

Proof. All the points above follow directly from eqs. (2.66) to (2.77), using property iii) on page 15. □

The next result follows by inspection of eq. (2.84),

Lemma 2.3.4. Consider the super-Poynting $\overline{\mathcal{P}}^a$ associated to a timelike, unit vector u^α and a couple of independent lightlike vectors as in eq. (2.59). Let ${}^{\pm}\mathcal{Q}^\alpha$ be their associated radiant supermomenta. Then, the necessary and sufficient conditions on the radiant quantities that make the super-Poynting vanish are

$$\left.\begin{array}{rcl} 2\left({}^{-}\mathcal{Z} - {}^{+}\mathcal{Z}\right) - {}^{+}\mathcal{W} + {}^{-}\mathcal{W} & = & 0 \\ \sqrt{2}\left({}^{+}\underline{\mathcal{Q}}^A + {}^{-}\underline{\mathcal{Q}}^A\right) + 12d^A & = & 0 \end{array}\right\} \Longleftrightarrow \overline{\mathcal{P}}^a = 0 \ . \tag{2.92}$$

Corollary 2.3.1. If one of the radiant supermomenta considered in lemma 2.3.4 vanishes, say ${}^{-}\mathcal{Q}^\alpha = 0$, then

$$ {}^{+}\mathcal{Q}^\alpha = 0 \Longleftrightarrow \overline{\mathcal{P}}^a = 0 \ . \tag{2.93}$$

The same holds true by interchanging the $+$ with the $-$ sign.

Proof. By property iii) on page 15, ${}^{-}\mathcal{Q}^\alpha = 0 \Longleftrightarrow {}^{-}\mathcal{Z} = {}^{-}\mathcal{W} = 0$. Now, by that same property and lemma 2.3.3, ${}^{+}\mathcal{Q}^\alpha = 0 \Longleftrightarrow {}^{+}\mathcal{Z} = {}^{+}\mathcal{W} = 0 \implies d_A = 0$. Then, eq. (2.92) is trivially satisfied and $\overline{\mathcal{P}}^a = 0$. For the converse, if $\overline{\mathcal{P}}^a = 0$, by eq. (2.92) we get ${}^{+}\mathcal{Z} = -{}^{+}\mathcal{W}$, but the only possibility is ${}^{+}\mathcal{Z} = {}^{+}\mathcal{W} = 0$ because both quantities are non negative (eqs. (2.52) and (2.54)). By property iii) on page 15, ${}^{+}\mathcal{Q}^\alpha = 0$. □

Proposition 2.3.1. Consider the super-Poynting $\overline{\mathcal{P}}^a$ associated to a timelike, unit vector u^α and a couple of independent lightlike vectors as in eqs. (2.20) and (2.21). Let ${}^{\pm}\mathcal{Q}^\alpha$ be the two associated radiant supermomenta. The following conditions are all equivalent:

1. ${}^{-}\mathcal{Q}^\alpha = {}^{+}\mathcal{Q}^\alpha = 0$.

2. ${}^{-}\mathcal{Q}^\alpha = 0$ and $\overline{\mathcal{P}}^a = 0$.

3. ${}^{-}\mathcal{Q}^\alpha = 0$ and $\overline{\mathcal{P}}^\alpha r_\alpha = 0$.

4. $\grave{D}_{AB} = \grave{C}_{AB} = 0$ and $\mathring{D}_A = \mathring{C}_A = 0$.

5. In the basis $\{r^a, E^a{}_A\}$,

$$(D_{ab}) = D \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}, \quad (C_{ab}) = C \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}. \tag{2.94}$$

Remark 2.3.1. This case corresponds to the situation where the Weyl-tensor candidate tensor has Petrov type D at those points where the above conditions hold and $\overset{\pm}{k}{}^{\alpha}$ are the two double principal null directions.

Proof.

- 1 $\Longleftrightarrow$ 2: it follows from corollary 2.3.1.
- 2 $\Longleftrightarrow$ 3: point 2 implies 3 trivially; by eq. (2.84), point 3 implies that the first line of eq. (2.92) in lemma 2.3.4 holds and, noting property iii) on page 15, $\overset{+}{\mathcal{W}} = 0 = \overset{+}{\mathcal{Z}}$. But, then, lemma 2.3.3 tells us that $d_A = 0 = \overset{+}{\mathcal{Q}}_A$. Altogether, by lemma 2.3.4, we have $\overline{\mathcal{P}}^a = 0$.
- 1 $\Longleftrightarrow$ 4: point 1 implies 4 by property iii) on page 15 and lemma 2.3.3. The converse is shown noting that 4 implies $\overset{\pm}{\mathcal{W}} = 0 = \overset{\pm}{\mathcal{Z}}$ by eqs. (2.72), (2.73) and (2.75) which, by property iii) on page 15, implies 1.
- 4 $\Longleftrightarrow$ 5: point 4 is saying explicitly that in the basis of point 5, the tensors C_{ab}, D_{ab} have precisely the form displayed in eq. (2.94).

□

Desde el final.
Al borde
de mí mismo.
Tan lejos.
En donde las ventanas
encendidas, son sólo otra pieza de la noche.

Benjamín Prado, *Límite*. *Todos nosotros*, 1998.

3 | Conformal geometry and infinity

Studying isolated systems, among other aspects of gravity, requires investigating the asymptotic properties of the space-time. The first strong theoretical evidences for the existence of gravitational waves in full General Relativity with *vanishing cosmological constant* were grounded on metric-based methods [21, 34, 40] or on the –now usually referred to as– Newman-Penrose (NP) formalism [96] –for reviews see [97–99]. Typically, one defines a suitable radial coordinate –either an 'areal coordinate' or an affine parameter along outgoing null geodesics– and then make expansions of the metric or curvature coefficients towards infinity (as described by asymptotic values of such radial coordinate). Some of the results achieved on those works include: formulations of energy-momentum of the gravitational field at null infinity in full General Relativity, an energy-loss formula of a system emitting gravitational radiation or the discovery of an asymptotic group of symmetries and the peeling behaviour of the Weyl tensor. Little time after, Penrose had the innovative idea of describing infinity as a hypersurface avoiding the use of limits and facilitating the employment of covariant methods [15, 22]. Schematically, one associates to a given space-time an unphysical –or conformal– space-time with a boundary $\mathscr{J}$ –the precise idea is presented in the upcoming section. Indeed, this boundary 'attached' to the space-time is the suitable arena for describing the gravitational radiation escaping from –or entering into– the space-time. For a vanishing cosmological constant, Geroch [17] studied the geometry of the conformal boundary setting the bases for the covariant asymptotic characterisation of gravitational radiation and Ashtekar used it as the 'kinematical arena' of the radiative degrees of freedom of the gravitational field [48]. In view of the mathematical elegance and rich physics that arise at the conformal boundary, our approach to the problem of the characterisation of gravitational radiation in chapters 4 to 7 is based on fields on $\mathscr{J}$.

Most of the ideas introduced in this chapter are well known and can be found in the literature –see e.g. [100, 101]. Nevertheless, we derive them from scratch with two purposes:

1. Set the basic material of conformal structure in our conventions and present all the

needed formulae in one place,

2. Derive the relations between physical geometric quantities and matter fields at infinity with our assumptions on the energy-momentum tensor.

3.1 Conformal completion

The physical space-times $\left(\hat{M}, \hat{g}_{\alpha\beta}\right)$ that we consider admit a conformal completion (unphysical space-time) $\left(M, g_{\alpha\beta}\right)$ *à la* Penrose with boundary $\mathscr{J}$ [100, 101]:

i) There exists an embedding $\phi : \hat{M} \to M$ such that $\phi(\hat{M}) = M \setminus \mathscr{J}$, and the physical metric is related to the conformal one as

$$g_{\alpha\beta} = \Omega^2 \hat{g}_{\alpha\beta}, \tag{3.1}$$

where, abusing notation, we refer to the pullback of the conformal metric to the physical space-time, $(\phi^* g)_{\alpha\beta}$, by $g_{\alpha\beta}$.

ii) $\Omega > 0$ in $M \setminus \mathscr{J}$, $\Omega = 0$ on $\mathscr{J}$ and $N_\alpha := \nabla_\alpha \Omega$ (the normal to $\mathscr{J}$) is non-vanishing there.

iii) $\hat{g}_{\alpha\beta}$ is a solution to EFE (3.12) with $\Lambda \geq 0$.

iv) The energy-momentum tensor, $\hat{T}_{\alpha\beta}$, vanishes at $\mathscr{J}$ and $T_{\alpha\beta} := \Omega^{-1}\hat{T}_{\alpha\beta}$ is smooth there.

Depending on the specific matter content, property iv) can be strengthen. Indeed for many relevant matter fields, $\Omega^{-2}\hat{T}_{\alpha\beta}$ is smooth at $\mathscr{J}$. Notice that $T := T^\mu{}_\mu = \Omega^{-3}\hat{T}^\mu{}_\mu := \Omega^{-3}\hat{T}$. Also, $\mathscr{J}$ is not connected, in general; it is divided into 'future' and 'past' components, denoted by $\mathscr{J}^\pm$ respectively. In this section, we use $\mathscr{J}$ generically to refer to any of them. In some of the subsequent sections we will work with $\mathscr{J}^+$, though. The connection of the unphysical space-time can be written as

$$\Gamma^\alpha{}_{\beta\gamma} = \hat{\Gamma}^\alpha{}_{\beta\gamma} + \gamma^\alpha{}_{\beta\gamma} \ , \tag{3.2}$$

where $\hat{\Gamma}^\alpha{}_{\beta\gamma}$ is the connection of the physical space-time and

$$\gamma^\alpha{}_{\beta\gamma} = \Omega^{-1}\left(2\delta^\alpha{}_{(\beta}\nabla_{\gamma)}\Omega - g_{\gamma\beta}\nabla^\alpha\Omega\right) \ . \tag{3.3}$$

Accordingly, the Ricci tensor and the scalar curvature are given by

$$R_{\alpha\beta} = \hat{R}_{\alpha\beta} - 2\Omega^{-1}\nabla_\alpha N_\beta + 3\Omega^{-2} g_{\alpha\beta} N_\mu N^\mu - \Omega^{-1} g_{\alpha\beta}\nabla_\mu N^\mu \ , \tag{3.4}$$

$$R = \Omega^{-2}\hat{R} - 6\Omega^{-1}\nabla_\mu N^\mu + 12\Omega^{-2} N_\mu N^\mu \ . \tag{3.5}$$

The conformal completion is not unique: given $\left(\hat{M}, \hat{g}_{\alpha\beta}\right)$ there is a conformal class of completions related by

$$\Omega \to \tilde{\Omega} = \omega\Omega \quad \text{with } \omega > 0 \text{ on } M \ . \tag{3.6}$$

This rescaling of the conformal factor is a gauge freedom and, as such, can be used to simplify matters. In the upcoming subsection we will fix it partially. A gauge transformation changes the unphysical space-time geometry as

$$\tilde{g}_{\alpha\beta} = \omega^2 g_{\alpha\beta} \ , \tag{3.7}$$

$$\tilde{\Gamma}^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} + C^{\alpha}{}_{\beta\gamma} \ , \quad C^{\alpha}{}_{\beta\gamma} = \omega^{-1} g^{\alpha\tau}\left(2g_{\tau(\beta}\omega_{\gamma)} - g_{\gamma\beta}\omega_{\tau}\right) \ , \tag{3.8}$$

$$\tilde{R}_{\alpha\beta} = R_{\alpha\beta} - 2\omega^{-1}\nabla_{\alpha}\omega_{\beta} - \omega^{-2}g_{\alpha\beta}\omega_{\mu}\omega^{\mu} - \omega^{-1}g_{\alpha\beta}\nabla_{\mu}\omega^{\mu} + 4\omega^{-2}\omega_{\alpha}\omega_{\beta} \ , \tag{3.9}$$

$$\tilde{R} = \omega^{-2}R - 6\omega^{-3}\nabla_{\mu}\omega^{\mu} \ , \tag{3.10}$$

$$\tilde{N}_{\alpha} = \omega N_{\alpha} + \Omega\omega_{\alpha} \tag{3.11}$$

where $\omega_{\alpha} := \nabla_{\alpha}\omega$. Here we have written the equations in terms of the original connection and metric. Further gauge changes can be found in appendix C.

3.2 Fields at infinity

Einstein field equations in the presence of a cosmological constant (here assumed to be non-negative) read

$$\hat{R}_{\alpha\beta} - \frac{1}{2}\hat{R}\hat{g}_{\alpha\beta} + \Lambda\hat{g}_{\alpha\beta} = \varkappa\hat{T}_{\alpha\beta} \ , \tag{3.12}$$

with $\varkappa := 8\pi G c^{-4}$, where G is the gravitational constant and c the speed of light. Thus, the left-hand side extends smoothly to infinity. Using eqs. (3.4) and (3.5) we get

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + 3\Omega^{-2}g_{\alpha\beta}N_{\mu}N^{\mu} + 2\Omega^{-1}\left(\nabla_{\alpha}N_{\beta} - g_{\alpha\beta}\nabla_{\mu}N^{\mu}\right) + \Omega^{-2}\Lambda g_{\alpha\beta} = \varkappa\hat{T}_{\alpha\beta} \ . \tag{3.13}$$

If we multiply by Ω^2 and evaluate at $\mathscr{J}$ –i.e. set $\Omega = 0$–, we obtain

$$3g_{\alpha\beta}N_{\mu}N^{\mu} + \Lambda g_{\alpha\beta} \stackrel{\mathscr{J}}{=} 0 \ , \tag{3.14}$$

from where

$$N_{\mu}N^{\mu} \stackrel{\mathscr{J}}{=} -\frac{\Lambda}{3} \ . \tag{3.15}$$

This formula indicates the causal character of the conformal boundary; in this thesis, either spacelike ($\Lambda > 0$) or lightlike ($\Lambda = 0$). Now, multiply eq. (3.13) by Ω to get

$$\Omega R_{\alpha\beta} - \frac{1}{2}\Omega R g_{\alpha\beta} + 2\left(\nabla_\alpha N_\beta - g_{\alpha\beta}\nabla_\mu N^\mu\right) = \Omega\varkappa\hat{T}_{\alpha\beta} - \Omega^{-1}\left(3N_\mu N^\mu + \Lambda\right)g_{\alpha\beta} \tag{3.16}$$

and observe that this equation is regular at $\mathscr{J}$, see eq. (3.14). Take its trace and evaluate at $\Omega = 0$:

$$6\nabla_\mu N^\mu \overset{\mathscr{J}}{=} \Omega^{-1}4\left(3N_\mu N^\mu + \Lambda\right) . \tag{3.17}$$

Then, insert this back into the previous equation. After evaluation at $\mathscr{J}$, we derive

$$\nabla_\alpha N_\beta \overset{\mathscr{J}}{=} \frac{1}{4}g_{\alpha\beta}\nabla_\mu N^\mu . \tag{3.18}$$

Sometimes eq. (3.18) is referred to as 'asymptotic Einstein condition' [102]. It is well known [17, 102–104] that the gauge can be chosen such that

$$\nabla_\mu N^\mu \overset{\mathscr{J}}{=} 0. \tag{3.19}$$

To show this, compute the gauge change using eqs. (3.8) and (3.11)

$$\tilde{\nabla}_\mu\tilde{N}^\mu = 2\omega^{-2}N^\mu\nabla_\mu\omega + \omega^{-1}\nabla_\mu N^\mu - \omega^{-1}g^{\alpha\beta}C^\mu{}_{\alpha\beta}\left(N_\mu + \omega^{-1}\Omega\nabla_\mu\omega\right) + \Omega\omega^{-2}\Box\omega , \tag{3.20}$$

evaluate at $\mathscr{J}$ and multiply by ω^2

$$\begin{aligned}\omega^2\tilde{\nabla}_\mu\tilde{N}^\mu &\overset{\mathscr{J}}{=} 2N^\mu\nabla_\mu\omega + \omega\nabla_\mu N^\mu - \omega g^{\alpha\beta}N_\mu C^\mu{}_{\alpha\beta}\\ &\overset{\mathscr{J}}{=} \omega\nabla_\mu N^\mu + 4N^\mu\nabla_\mu\omega .\end{aligned} \tag{3.21}$$

Equating $\tilde{\nabla}_\mu\tilde{N}^\mu$ to zero gives a differential equation for the possible gauge factors

$$4N^\mu\partial_\mu\omega + \omega\Box\Omega \overset{\mathscr{J}}{=} 0 \tag{3.22}$$

which always has non-trivial solutions. From now on, we *adopt this gauge fixing* that we call *divergence-free* gauge. Nevertheless, note that the freedom is still large and one can change from one conformal gauge to another by keeping eq. (3.19) with the additional restriction (given a solution $\grave{\omega}$ of eq. (3.22), $\tilde{\omega} := \omega\grave{\omega}$ is a new solution)

$$\pounds_{\vec{N}}\omega \overset{\mathscr{J}}{=} 0 . \tag{3.23}$$

From eq. (3.18), this gauge implies

$$\nabla_\alpha N_\beta \overset{\mathscr{J}}{=} 0 . \tag{3.24}$$

Consider the combination[1]

$$S_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{6} R g_{\alpha\beta} \ , \tag{3.25}$$

and eq. (3.13) as an equation for $\nabla_\alpha N_\beta$. Substitute $\kappa \hat{T} = -\hat{R} + 4\Lambda$ together with $\hat{T} = \Omega^{-3} T$ and eq. (3.5),

$$\nabla_\alpha N_\beta = -\frac{1}{2}\Omega R_{\alpha\beta} + \frac{1}{8}\Omega R g_{\alpha\beta} + \frac{1}{4} g_{\alpha\beta} \nabla_\mu N^\mu + \frac{1}{2}\varkappa \Omega^2 \left(T_{\alpha\beta} - \frac{1}{4} T g_{\alpha\beta} \right) \ , \tag{3.26}$$

and replace $R_{\alpha\beta}$ with $S_{\alpha\beta}$,

$$\nabla_\alpha N_\beta = -\frac{1}{2}\Omega S_{\alpha\beta} + \frac{1}{24}\Omega R g_{\alpha\beta} + \frac{1}{4} g_{\alpha\beta} \nabla_\mu N^\mu + \frac{1}{2}\varkappa \Omega^2 \left(T_{\alpha\beta} - \frac{1}{4} T g_{\alpha\beta} \right) \ . \tag{3.27}$$

It is convenient to define

$$\underline{T}_{\alpha\beta} := T_{\alpha\beta} - \frac{1}{4} T g_{\alpha\beta} \tag{3.28}$$

and introduce the scalar [105]

$$f := \frac{1}{4} \nabla_\mu N^\mu + \frac{\Omega}{24} R \ , \tag{3.29}$$

which in our gauge vanishes at $\mathscr{J}$

$$f \overset{\mathscr{J}}{=} 0 \ . \tag{3.30}$$

In terms of f, eq. (3.27) becomes

$$\nabla_\alpha N_\beta = -\frac{1}{2}\Omega S_{\alpha\beta} + f g_{\alpha\beta} + \frac{1}{2}\Omega^2 \varkappa \underline{T}_{\alpha\beta} \ . \tag{3.31}$$

Or course, from this equation one deduces directly eq. (3.18). Once again, consider eq. (3.13), this time as an equation for $\nabla_\mu N^\mu$. Take its trace and multiply by Ω:

$$N_\mu N^\mu = \frac{\Omega^3}{12}\varkappa T - \frac{\Lambda}{3} + \frac{\Omega^2}{12} R + \frac{\Omega}{2} \nabla_\mu N^\mu \ . \tag{3.32}$$

As a check, one recovers eq. (3.15) after evaluating at $\mathscr{J}$. Introducing f in eq. (3.32), we get

$$N_\mu N^\mu = \frac{\Omega^3}{12}\varkappa T - \frac{\Lambda}{3} + 2\Omega f \ . \tag{3.33}$$

If we contract eq. (3.31) with N^β,

$$N^\mu \nabla_\alpha N_\mu = \frac{1}{2}\nabla_\alpha \left(N_\mu N^\mu \right) = -\frac{1}{2}\Omega S_{\alpha\mu} N^\mu + f N_\alpha + \frac{1}{2}\varkappa \Omega^2 N^\mu \underline{T}_{\alpha\mu} \ , \tag{3.34}$$

[1] Note that this is twice the Schouten tensor, whose standard definition in 4 dimensions is $\frac{1}{2}\left(R_{\alpha\beta} - \frac{1}{6} R g_{\alpha\beta} \right)$.

and take the covariant derivative of eq. (3.33),

$$\frac{1}{2}\nabla_\alpha \left(N_\mu N^\mu\right) = \frac{1}{24}\Omega^3 \varkappa \nabla_\alpha T + \frac{1}{8}\Omega^2 \varkappa N_\alpha T + \Omega \nabla_\alpha f + f N_\alpha \tag{3.35}$$

we arrive, equating both expressions, at

$$\nabla_\alpha f = -\frac{1}{2} S_{\alpha\mu} N^\mu + \frac{1}{2}\Omega \varkappa N^\mu \underline{T}_{\alpha\mu} - \frac{1}{24}\Omega^2 \varkappa \nabla_\alpha T - \frac{1}{8}\Omega \varkappa N_\alpha T \ . \tag{3.36}$$

If we want to extract information about the complete orthogonal component of $S_{\alpha\beta}$ at $\mathscr{J}$, we have to contract eq. (3.31) with N^α and substitute $S_{\alpha\mu}N^\mu$ in terms of eq. (3.36)

$$\begin{aligned} N^\mu \nabla_\mu N_\alpha &= -\frac{1}{2}\Omega N^\mu S_{\mu\alpha} + f N_\alpha + \frac{1}{2}\Omega^2 \varkappa N^\mu \underline{T}_{\mu\alpha} \\ &= \Omega \nabla_\alpha f - \frac{1}{2}\Omega^2 \varkappa N^\mu \underline{T}_{\alpha\mu} + \frac{1}{24}\Omega^3 \varkappa \nabla_\alpha T + \frac{1}{8}\Omega^2 \varkappa T N_\alpha \\ &+ f N_\alpha + \frac{1}{2}\Omega^2 \varkappa N^\mu \underline{T}_{\alpha\mu} \\ &= \Omega \nabla_\alpha f + f N_\alpha + \frac{1}{8}\Omega^2 \varkappa T N_\alpha + \frac{1}{24}\Omega^3 \varkappa \nabla_\alpha T \ . \end{aligned} \tag{3.37}$$

After this, contract eq. (3.36) with N^μ to find

$$N^\mu \nabla_\mu f = N^\mu N^\rho \left(-\frac{1}{2} S_{\mu\rho} + \frac{1}{2}\Omega \varkappa \underline{T}_{\mu\rho}\right) - N^\mu \left(\frac{1}{24}\varkappa \Omega^2 \nabla_\mu T + \frac{1}{8}\Omega \varkappa T N_\mu\right) \tag{3.38}$$

then take the covariant derivative of eq. (3.31) along N^ρ taking into account the last two equations

$$N^\rho \nabla_\rho \left(\nabla_\alpha N_\beta\right) = -\frac{1}{2}\Omega N^\rho \nabla_\rho S_{\alpha\beta} + \frac{1}{2} N^2 S_{\alpha\beta} + g_{\alpha\beta} N^\rho \nabla_\rho f + \frac{1}{2}\varkappa \Omega^2 N^\rho \nabla_\rho \underline{T}_{\alpha\beta} + \Omega \varkappa N^2 \underline{T}_{\alpha\beta} \ , \tag{3.39}$$

$$\begin{aligned} &= -\frac{1}{2}\Omega N^\rho \nabla_\rho S_{\alpha\beta} + \frac{1}{2} N^2 S_{\alpha\beta} - \frac{1}{2} S_{\mu\nu} N^\mu N^\nu g_{\alpha\beta} + \frac{1}{2}\Omega \varkappa N^\mu N^\nu \underline{T}_{\mu\nu} g_{\alpha\beta} \\ &- \frac{1}{24}\Omega^2 \varkappa N^\mu \nabla_\mu T g_{\alpha\beta} + \frac{1}{8}\Omega \varkappa N^2 \varkappa T g_{\alpha\beta} + \frac{1}{2}\Omega^2 \varkappa N^\rho \nabla_\rho \underline{T}_{\alpha\beta} + \Omega \varkappa N^2 \underline{T}_{\alpha\beta} \ . \end{aligned} \tag{3.40}$$

Therefore, at $\mathscr{J}$

$$N^\rho \nabla_\rho \left(\nabla_\alpha N_\beta\right) \overset{\mathscr{J}}{=} \frac{\Lambda}{6} S_{\alpha\beta} - \frac{1}{2} N^\mu N^\nu S_{\mu\nu} g_{\alpha\beta} \ . \tag{3.41}$$

According to eqs. (3.4), (3.5) and (3.25),

$$\hat{S}_{\alpha\beta} = S_{\alpha\beta} + 2\Omega^{-1} \nabla_\alpha N_\beta - \Omega^{-2} g_{\alpha\beta} N_\mu N^\mu \ . \tag{3.42}$$

Take the covariant derivative and antisymmetrise the first pair of indices,

$$\nabla_{[\alpha}\hat{S}_{\beta]\gamma} = \nabla_{[\alpha}S_{\beta]\gamma} - 2\Omega^{-2}N_{[\alpha}\nabla_{\beta]}N_{\gamma} + 2\Omega^{-1}\nabla_{[\alpha}\nabla_{\beta]}N_{\gamma} + 2\Omega^{-3}N_{[\alpha}g_{\beta]\gamma}N^{\mu}N_{\mu} - 2\Omega^{-2}N^{\mu}g_{\gamma[\beta}\nabla_{\alpha]}N_{\mu} \ , \tag{3.43}$$

multiply by Ω,

$$\nabla_{[\alpha}\left(\Omega\hat{S}_{\beta]\gamma}\right) - N_{[\alpha}\hat{S}_{\beta]\gamma} = \Omega\nabla_{[\alpha}S_{\beta]\gamma} - 2\Omega^{-1}N_{[\alpha}\nabla_{\beta]}N_{\gamma} + 2\nabla_{[\alpha}\nabla_{\beta]}N_{\gamma} + 2\Omega^{-2}N_{[\alpha}g_{\beta]\gamma}N^{\mu}N_{\mu} - 2\Omega^{-1}N^{\mu}g_{\gamma[\beta}\nabla_{\alpha]}N_{\mu} \ , \tag{3.44}$$

and replace the third term on the right-hand side using the Ricci identity and the Riemann tensor decomposition,

$$R_{\alpha\beta\gamma\mu} = C_{\alpha\beta\gamma\mu} + g_{\alpha[\gamma}S_{\mu]\beta} - g_{\beta[\gamma}S_{\mu]\alpha} \ , \tag{3.45}$$

to obtain

$$\nabla_{[\alpha}\left(\Omega\hat{S}_{\beta]\gamma}\right) - N_{[\alpha}\hat{S}_{\beta]\gamma} = \Omega\nabla_{[\alpha}S_{\beta]\gamma} - 2\Omega^{-1}N_{[\alpha}\nabla_{\beta]}N_{\gamma} + C_{\alpha\beta\gamma}{}^{\mu}N_{\mu} + g_{\alpha[\gamma}S_{\mu]\beta}N^{\mu} - g_{\beta[\gamma}S_{\mu]\alpha}N^{\mu} + 2\Omega^{-2}N_{[\alpha}g_{\beta]\gamma}N^{\mu}N_{\mu} - 2\Omega^{-1}N^{\mu}g_{\gamma[\beta}\nabla_{\alpha]}N_{\mu} \ . \tag{3.46}$$

Notice that

$$\begin{aligned} g_{\alpha[\gamma}S_{\mu]\beta}N^{\mu} - g_{\beta[\gamma}S_{\mu]\alpha}N^{\mu} &= g_{\gamma[\alpha}S_{\beta]\mu}N^{\mu} - N_{[\alpha}S_{\beta]\gamma} \\ &= g_{\gamma[\alpha}\hat{S}_{\beta]\mu}N^{\mu} - 2\Omega^{-1}N^{\mu}g_{\gamma[\alpha}\nabla_{\beta]}N_{\mu} + \Omega^{-2}g_{\gamma[\alpha}N_{\beta]}N_{\mu}N^{\mu} \\ &+ 2\Omega^{-1}N_{[\alpha}\nabla_{\beta]}N_{\gamma} - \Omega^{-2}N_{[\alpha}g_{\beta]\gamma}N_{\mu}N^{\mu} \ , \end{aligned} \tag{3.47}$$

which substituted in eq. (3.46) produces

$$\nabla_{[\alpha}\left(\Omega\hat{S}_{\beta]\gamma}\right) = \Omega\nabla_{[\alpha}S_{\beta]\gamma} + C_{\alpha\beta\gamma}{}^{\mu}N_{\mu} + g_{\gamma[\alpha}\hat{S}_{\beta]\mu}N^{\mu}. \tag{3.48}$$

To see how the energy-momentum tensor enters into this equation, plug

$$\hat{S}_{\alpha\beta} = \hat{R}_{\alpha\beta} - \frac{1}{6}\hat{R}\hat{g}_{\alpha\beta} = \varkappa\hat{T}_{\alpha\beta} + \frac{1}{3}\left(\Lambda - \varkappa\hat{T}\right)\hat{g}_{\alpha\beta} = \varkappa\Omega T_{\alpha\beta} + \frac{1}{3}\left(\Omega^{-2}\Lambda - \Omega\varkappa\hat{T}\right)g_{\alpha\beta} \tag{3.49}$$

into eq. (3.48),

$$\frac{1}{3}\Omega^{-2}\Lambda N_{[\beta}g_{\alpha]\gamma} + \varkappa\nabla_{[\alpha}\left(\Omega^{2}T_{\beta]\gamma}\right) - \frac{1}{3}\varkappa\nabla_{[\alpha}\left(\Omega^{2}T\right)g_{\beta]\gamma} = \Omega\nabla_{[\alpha}S_{\beta]\gamma} + C_{\alpha\beta\gamma}{}^{\mu}N_{\mu} + \Omega\varkappa g_{\gamma[\alpha}T_{\beta]\mu}N^{\mu} - \frac{1}{3}\varkappa N^{\mu}g_{\gamma[\alpha}g_{\beta]\mu}\Omega T + \frac{1}{3}\Lambda\Omega^{-2}g_{\gamma[\alpha}N_{\beta]} \ , \tag{3.50}$$

and rearrange the terms to get

$$2\Omega\varkappa N_{[\alpha}\left(T_{\beta]\gamma}\right)+\Omega^2\varkappa\nabla_{[\alpha}T_{\beta]\gamma}-\Omega\varkappa N_{[\alpha}g_{\beta]\gamma}T-\frac{1}{3}\Omega^2\varkappa\nabla_{[\alpha}\left(T\right)g_{\beta]\gamma}$$
$$=\Omega\nabla_{[\alpha}S_{\beta]\gamma}+C_{\alpha\beta\gamma}{}^{\mu}N_{\mu}+\Omega\varkappa g_{\gamma[\alpha}T_{\beta]\mu}N^{\mu}\ , \tag{3.51}$$

where we have taken into account property iv) on page 22[2].

The Cotton tensor, both of the physical and conformal space-time, is defined[3] as

$$\hat{Y}_{\alpha\beta\gamma}:=\hat{\nabla}_{[\alpha}\hat{S}_{\beta]\gamma}\ , \tag{3.52}$$
$$Y_{\alpha\beta\gamma}:=\nabla_{[\alpha}S_{\beta]\gamma}\ . \tag{3.53}$$

At the end of this chapter, we are going to show that both the physical Cotton tensor $\hat{Y}_{\alpha\beta\gamma}$ and the Weyl tensor $C_{\alpha\beta\gamma}{}^{\delta}$ vanish at $\mathscr{J}$. Thus, it is natural to introduce a couple of tensor fields regular at $\mathscr{J}$; the *rescaled Cotton tensor*

$$y_{\alpha\beta\gamma}:=\Omega^{-1}\hat{Y}_{\alpha\beta\gamma}\ , \tag{3.54}$$

–notice that it is defined in terms of the physical Cotton tensor, which is not a conformal-invariant object– and the *rescaled Weyl tensor*

$$d_{\alpha\beta\gamma}{}^{\delta}:=\Omega^{-1}C_{\alpha\beta\gamma}{}^{\delta}\ . \tag{3.55}$$

This last tensor field features the algebraic symmetries of the Weyl tensor and plays a major role in the asymptotic study of the gravitational field and the properties of space-times from the point of view of their conformal extensions — [17, 48, 106–109] are just a few examples. The rescaled Weyl tensor (3.55) at $\mathscr{J}$ is completely determined by its electric and magnetic parts –see section 2.1.1 and note that the notation used there for a Weyl-tensor candidate is used now for the rescaled Weyl tensor–,

$$D_{\alpha\beta}\overset{\mathscr{J}}{:=}n^{\mu}n^{\nu}d_{\mu\alpha\nu\beta}\ , \tag{3.56}$$
$$C_{\alpha\beta}\overset{\mathscr{J}}{:=}n^{\mu}n^{\nu}\,{}^{*}d_{\mu\alpha\nu\beta}\ . \tag{3.57}$$

[2]The vanishing of $C_{\alpha\beta\gamma}{}^{\delta}$ at $\mathscr{J}$ can be derived from this equation, depending on the matter content. At the end of the chapter, we will give the proof under certain assumptions on the physical Cotton-York tensor at $\mathscr{J}$, instead.

[3]Keep in mind the extra factor 2 with respect to the standard definition due to our definition of $S_{\alpha\beta}$.

From the contracted Bianchi identity one can show that

$$\nabla_\mu \left(C_{\alpha\beta\gamma}{}^{\mu}\right) + Y_{\alpha\beta\gamma} = 0 \ , \tag{3.58}$$
$$\hat{\nabla}_\mu \left(\hat{C}_{\alpha\beta\gamma}{}^{\mu}\right) + \hat{Y}_{\alpha\beta\gamma} = 0 \ . \tag{3.59}$$

The first of these two equations can be rewritten in terms of the rescaled Weyl tensor,

$$\Omega\nabla_\mu d_{\alpha\beta\gamma}{}^{\mu} + d_{\alpha\beta\gamma}{}^{\mu} N_\mu + Y_{\alpha\beta\gamma} = 0, \tag{3.60}$$

and evaluated at $\mathscr{J}$,

$$d_{\alpha\beta\gamma}{}^{\mu} N_\mu + Y_{\alpha\beta\gamma} \stackrel{\mathscr{J}}{=} 0 \ . \tag{3.61}$$

We can multiply now eq. (3.51) by Ω^{-1},

$$\begin{aligned} 2\varkappa N_{[\alpha} T_{\beta]\gamma} - \varkappa N_{[\alpha} g_{\beta]\gamma} T + \Omega\varkappa\nabla_{[\alpha} T_{\beta]\gamma} - \frac{1}{3}\Omega\varkappa g_{\gamma[\beta}\nabla_{\alpha]} T \\ = \nabla_{[\alpha} S_{\beta]\gamma} + d_{\alpha\beta\gamma}{}^{\mu} N_\mu + \varkappa g_{\gamma[\alpha} T_{\beta]\mu} N^\mu \end{aligned} \tag{3.62}$$

and evaluate it at $\mathscr{J}$ using eq. (3.61),

$$2\varkappa N_{[\alpha} T_{\beta]\gamma} - \varkappa N_{[\alpha} g_{\beta]\gamma} T - \varkappa g_{\gamma[\alpha} T_{\beta]\mu} N^\mu \stackrel{\mathscr{J}}{=} 0 \ . \tag{3.63}$$

The energy-momentum tensor determines, through the field equations, the Cotton tensor. This appears explicitly from definition eq. (3.52) and eq. (3.49),

$$\hat{Y}_{\alpha\beta\gamma} = \kappa\hat{\nabla}_{[\alpha}\hat{T}_{\beta]\gamma} - \frac{1}{3}\varkappa\hat{g}_{\gamma[\beta}\nabla_{\alpha]}\hat{T} \ . \tag{3.64}$$

In order to write this formula in terms of quantities in M, note that

$$\hat{\nabla}_{[\alpha}\hat{T}_{\beta]\gamma} = \Omega\nabla_{[\alpha} T_{\beta]\gamma} - N^\lambda T_{\lambda[\beta} g_{\alpha]\gamma} + 2N_{[\alpha} T_{\beta]\gamma} \ , \tag{3.65}$$

$$\hat{g}_{\gamma[\beta}\nabla_{\alpha]} T = 3N_{[\alpha} g_{\beta]\gamma} T + \Omega g_{\gamma[\beta}\nabla_{\alpha]} T \ . \tag{3.66}$$

Then,

$$\frac{1}{\varkappa}\hat{Y}_{\alpha\beta\gamma} = \Omega\nabla_{[\alpha} T_{\beta]\gamma} - N^\lambda T_{\lambda[\beta} g_{\alpha]\gamma} + 2N_{[\alpha} T_{\beta]\gamma} - N_{[\alpha} g_{\beta]\gamma} T - \frac{1}{3}\Omega g_{\gamma[\beta}\nabla_{\alpha]} T \tag{3.67}$$

The relation between the conformal and physical connections gives

$$\nabla_{[\mu} C_{\alpha\beta]\gamma}{}^{\delta} = \hat{\nabla}_{[\mu} C_{\alpha\beta]\gamma}{}^{\delta} + \Omega^{-1}\left(g_{\gamma[\mu} C_{\alpha\beta]\rho}{}^{\delta} N^\rho - \delta^\delta_{[\mu} C_{\alpha\beta]\rho\gamma} N^\rho\right) \tag{3.68}$$

which, after taking the trace, yields the relation

$$\nabla_\mu \left(\Omega^{-1} C_{\alpha\beta\gamma}{}^{\mu}\right) = \Omega^{-1} \hat{\nabla}_\mu C_{\alpha\beta\gamma}{}^{\mu} \ . \tag{3.69}$$

which allows us to write equation eq. (3.59) as

$$y_{\alpha\beta\gamma} + \nabla_\mu d_{\alpha\beta\gamma}{}^{\mu} = 0 \ . \tag{3.70}$$

To see which conditions on the matter content make the rescaled Weyl tensor divergence-free at $\mathscr{I}$, multiply eq. (3.67) by Ω^{-1}

$$\frac{1}{\varkappa} y_{\alpha\beta\gamma} = \nabla_{[\alpha} T_{\beta]\gamma} - \Omega^{-1} N^{\lambda} T_{\lambda[\beta}\, g_{\alpha]\gamma} + 2\Omega^{-1} N_{[\alpha} T_{\beta]\gamma} - \Omega^{-1} N_{[\alpha}\, g_{\beta]\gamma} T - \frac{1}{3} g_{\gamma[\beta} \nabla_{\alpha]} T \tag{3.71}$$

so that the following implication holds:

$$\hat{T}_{\alpha\beta}\,|_{\mathscr{I}} \sim \mathcal{O}\left(\Omega^{p}\right) \ \text{with } p > 2 \implies y_{\alpha\beta\gamma} \overset{\mathscr{I}}{=} 0 \overset{\mathscr{I}}{=} \nabla_\mu d_{\alpha\beta\gamma}{}^{\mu} = 0 \ . \tag{3.72}$$

Substitution of eq. (3.70) into eq. (3.60) produces

$$d_{\alpha\beta\gamma}{}^{\mu} N_\mu + Y_{\alpha\beta\gamma} - \Omega y_{\alpha\beta\gamma} = 0 \ . \tag{3.73}$$

To end this section, we summarise the relevant equations which are eqs. (3.31), (3.33), (3.36), (3.70) and (3.73),

$$\nabla_\alpha N_\beta = -\frac{1}{2}\Omega S_{\alpha\beta} + f g_{\alpha\beta} + \frac{1}{2}\Omega^2 \varkappa \underline{T}_{\alpha\beta} \ , \tag{3.74}$$

$$N_\mu N^\mu = \frac{\Omega^3}{12}\varkappa T - \frac{\Lambda}{3} + 2\Omega f \ , \tag{3.75}$$

$$\nabla_\alpha f = -\frac{1}{2} S_{\alpha\mu} N^\mu + \frac{1}{2}\Omega \varkappa N^\mu \underline{T}_{\alpha\mu} - \frac{1}{24}\Omega^2 \varkappa \nabla_\alpha T - \frac{1}{8}\Omega \varkappa N_\alpha T \ , \tag{3.76}$$

$$d_{\alpha\beta\gamma}{}^{\mu} N_\mu + \nabla_{[\alpha}\left(S_{\beta]\gamma}\right) - \Omega y_{\alpha\beta\gamma} = 0 \ , \tag{3.77}$$

$$y_{\alpha\beta\gamma} + \nabla_\mu d_{\alpha\beta\gamma}{}^{\mu} = 0 \ , \tag{3.78}$$

$$R_{\alpha\beta\gamma\delta} = \Omega d_{\alpha\beta\gamma\delta} + g_{\alpha[\gamma} S_{\delta]\beta} - g_{\beta[\gamma} S_{\delta]\alpha} \ . \tag{3.79}$$

whose evaluation at $\mathscr{J}$ is

$$\nabla_\alpha N_\beta \stackrel{\mathscr{J}}{=} 0 \ , \tag{3.80}$$

$$N_\mu N^\mu \stackrel{\mathscr{J}}{=} -\frac{\Lambda}{3} \ , \tag{3.81}$$

$$\nabla_\alpha f \stackrel{\mathscr{J}}{=} -\frac{1}{2} S_{\alpha\mu} N^\mu \ , \tag{3.82}$$

$$d_{\alpha\beta\gamma}{}^{\mu} N_\mu + \nabla_{[\alpha} \left(S_{\beta]\gamma} \right) \stackrel{\mathscr{J}}{=} 0 \ , \tag{3.83}$$

$$y_{\alpha\beta\gamma} + \nabla_\mu d_{\alpha\beta\gamma}{}^{\mu} \stackrel{\mathscr{J}}{=} 0 \ , \tag{3.84}$$

$$R_{\alpha\beta\gamma\delta} \stackrel{\mathscr{J}}{=} g_{\alpha[\gamma} S_{\delta]\beta} - g_{\beta[\gamma} S_{\delta]\alpha} \ . \tag{3.85}$$

Equations (3.74) to (3.79) [110–112] in vacuum – $T_{\alpha\beta} = 0 = y_{\alpha\beta\gamma}$ – constitute the so called Metric Conformal Field Equations (MCFE) [107] which, when considered as a system of differential equations for the variables $\left(\Omega, d_{\alpha\beta\gamma}{}^{\mu}, f, g_{\alpha\beta}, S_{\alpha\beta}\right)$, are equivalent to the physical vacuum EFE —the Riemann components $R_{\alpha\beta\gamma\delta}$ are considered as functions of the metric components $g_{\alpha\beta}$.

Observe that the PND of $d_{\alpha\beta\gamma}{}^{\delta}$ and $C_{\alpha\beta\gamma}{}^{\delta}$ coincide on a neighbourhood U outside $\mathscr{J}$ and thus, the number and multiplicity of PND of $d_{\alpha\beta\gamma}{}^{\delta}$ at the boundary of U, given by $\mathscr{J}$, will be equal or greater than that number for $C_{\alpha\beta\gamma}{}^{\delta}$ on U. For instance, the Weyl tensor may be algebraically general in a neighbourhood of $\mathscr{J}$ and the rescaled Weyl tensor, algebraically special at $\mathscr{J}$. This follows by a simple argument: in an expansion of $C_{\alpha\beta\gamma}{}^{\delta}$ around $\Omega = 0$ the first-order term is given by $d_{\alpha\beta\gamma}{}^{\delta}$ at $\Omega = 0$. Since k^α is a PND of $C_{\alpha\beta\gamma}{}^{\delta}$ around $\Omega = 0$ if and only if it is a PND at every order in the expansion, it has to be a PND of $d_{\alpha\beta\gamma}{}^{\delta}$ at $\Omega = 0$, and with (at least) the same multiplicity.

Finally, let us introduce *the rescaled Bel-Robinson tensor*:

$$\mathcal{D}_{\alpha\beta\gamma\delta} := d_{\alpha\mu\gamma}{}^{\nu} d_{\delta\nu\beta}{}^{\mu} + \overset{*}{d}_{\alpha\mu\gamma}{}^{\nu} \overset{*}{d}_{\delta\nu\beta}{}^{\mu} \ , \tag{3.86}$$

which is the basic superenergy tensor of the rescaled Weyl tensor. It is regular and, in general, non-vanishing at $\mathscr{J}$, and plays a central role in our study of the asymptotic structure of space-time. Its divergence is easily computed using eqs. (2.2) and (3.78) and reads

$$\nabla_\mu \mathcal{D}_{\alpha\beta\gamma}{}^{\mu} = 2 d_{\mu\gamma\nu\alpha} y_\beta{}^{\nu\mu} + 2 d_{\mu\gamma\nu\beta} y_\alpha{}^{\nu\mu} + g_{\alpha\beta} d^{\mu\nu\rho}{}_{\gamma} y_{\mu\nu\rho} \ , \tag{3.87}$$

3.2.1 Matter content and the vanishing of the Weyl tensor at infinity

The components of the energy-momentum tensor of the matter fields at infinity and the proof of the vanishing of the Weyl tensor at $\mathscr{J}$ are presented more clearly if the $\Lambda > 0$

and $\Lambda = 0$ cases are treated separately.

Positive Λ

Let us introduced the normalised version of N_α

$$n_\alpha := \frac{1}{N} N_\alpha \, , \tag{3.88}$$

with $N := \sqrt{-N^\mu N_\mu}$. In general, this definition is valid only on a neighbourhood of $\mathscr{I}$, where N_α is timelike. In that neighbourhood we can introduce the projector to $\mathscr{I}$ (see appendix A):

$$P^\alpha{}_\beta := \delta^\alpha{}_\beta - n^\alpha n_\beta \; . \tag{3.89}$$

Note that the explicit form of n^α reads

$$n_\alpha = \frac{N_\alpha}{\sqrt{\frac{\Lambda}{3} - \frac{\Omega^3}{12}\varkappa T - 2\Omega f}} \stackrel{\mathscr{I}}{=} \sqrt{\frac{3}{\Lambda}} N_\alpha \tag{3.90}$$

and contracting with N^α eq. (3.80) one gets

$$\nabla_\alpha N \stackrel{\mathscr{I}}{=} 0 \; , \tag{3.91}$$

which implies that the normalised n_α is covariantly constant at $\mathscr{I}$ as well

$$\nabla_\alpha n_\beta \stackrel{\mathscr{I}}{=} 0 \; . \tag{3.92}$$

Before showing the vanishing of the Weyl tensor, we give the components of the tensor $S_{\alpha\beta}$ at $\mathscr{I}$. Using eq. (3.30) and comparing with eq. (3.76)

$$\nabla_\alpha f \stackrel{\mathscr{I}}{=} -N_\alpha N^\rho \nabla_\rho f \stackrel{\mathscr{I}}{=} \frac{3}{2\Lambda} N_\alpha N^\rho N^\mu S_{\mu\rho} \; , \tag{3.93}$$

from where we also deduce that

$$P^\alpha{}_\beta N^\mu S_{\alpha\mu} \stackrel{\mathscr{I}}{=} 0 \; . \tag{3.94}$$

The contraction of this equation with $N^\alpha N^\beta$ shows that

$$n^\mu n^\nu S_{\mu\nu} \stackrel{\mathscr{I}}{=} N^\mu N^\nu N^\rho \nabla_\rho \left(\nabla_\mu N_\nu \right) \; . \tag{3.95}$$

Observe that contracting eq. (3.63) with $N^\alpha N^\gamma P^\beta{}_\delta$ gives

$$n^\mu P^\rho{}_\alpha T_{\mu\rho} \overset{\mathscr{J}}{=} 0\ . \tag{3.96}$$

Furthermore, if one contracts with $N^\alpha P^\gamma{}_\chi P^\beta{}_\delta$,

$$N^\mu N_\mu P^\gamma{}_\chi P^\beta{}_\delta T_{\beta\gamma} + \frac{1}{2} P_{\chi\delta} N^\mu N^\rho T_{\mu\rho} + \frac{1}{2} N^\mu N_\mu T h_{\chi\delta} \overset{\mathscr{J}}{=} 0\ , \tag{3.97}$$

and uses here eq. (3.89),

$$N^\mu N_\mu P^\gamma{}_\chi P^\beta{}_\delta T_{\beta\gamma} + \frac{1}{2} N^\tau N_\tau P_{\chi\delta} P^{\rho\mu} T_{\mu\rho} \overset{\mathscr{J}}{=} 0\ . \tag{3.98}$$

Then, contract one more time with $P^{\chi\delta}$ to get

$$P^{\mu\rho} T_{\mu\rho} \overset{\mathscr{J}}{=} 0\ . \tag{3.99}$$

Inserting this last line into eq. (3.98) the result is

$$P^\mu{}_\alpha P^\nu{}_\beta T_{\mu\nu} \overset{\mathscr{J}}{=} 0\ . \tag{3.100}$$

Finally, apply $N^\alpha P^{\beta\gamma}$ to eq. (3.63) to derive

$$\frac{3}{2} N^\alpha N^\mu T_{\alpha\mu} \overset{\mathscr{J}}{=} \frac{3}{2} N_\nu N^\nu T - N_\rho N^\rho P^{\beta\gamma} T_{\beta\gamma}\,, \tag{3.101}$$

or, equivalently,

$$n^\rho n^\mu T_{\mu\rho} \overset{\mathscr{J}}{=} -T\ . \tag{3.102}$$

We have just shown that, at $\mathscr{J}$, the tensor $T_{\alpha\beta}$ has only one non-vanishing component in general:

$$T_{\alpha\beta} \overset{\mathscr{J}}{=} -T n_\alpha n_\beta\ . \tag{3.103}$$

Equation (3.67), evaluated on $\mathscr{J}$ and using eq. (3.103), reduces to

$$\hat{Y}_{\alpha\beta\gamma} \overset{\mathscr{J}}{=} 0\ . \tag{3.104}$$

Recall that $\mathscr{J}$ is spacelike, which implies that the Weyl tensor is completely determined by the electric and magnetic parts in the standard 3+1 decomposition —as generally described for a Weyl-tensor candidate tensor in section 2.1.1,

$$E_{\alpha\beta} \overset{\mathscr{J}}{:=} n^{\mu}n^{\nu}C_{\mu\alpha\nu\beta}, \tag{3.105}$$

$$B_{\alpha\beta} \overset{\mathscr{J}}{:=} n^{\mu}n^{\nu}{}^{*}C_{\mu\alpha\nu\beta} \ . \tag{3.106}$$

Lemma 3.2.1 (Vanishing of the Weyl tensor at $\mathscr{J}$ with a positive cosmological constant)**.** Assume that

1. $\Omega\hat{Y}_{\alpha\beta\gamma} \overset{\mathscr{J}}{=} 0$,

2. $C_{\alpha\beta\gamma}{}^{\delta}$ is regular at $\mathscr{J}$ and $\Omega\nabla_{\mu}C_{\alpha\beta\gamma}{}^{\delta} \overset{\mathscr{J}}{=} 0$.

Then,

$$C_{\alpha\beta\gamma}{}^{\delta} \overset{\mathscr{J}}{=} 0 \ . \tag{3.107}$$

Proof. We will use the lightlike decomposition of a Weyl-tensor candidate presented in section 2.2. First, from eqs. (3.59) and (3.69) one has

$$\Omega\nabla_{\mu}C_{\alpha\beta\gamma}{}^{\mu} - N_{\mu}C_{\alpha\beta\gamma}{}^{\mu} + \Omega\hat{Y}_{\alpha\beta\gamma} = 0. \tag{3.108}$$

This equation evaluated at $\mathscr{J}$ gives

$$N_{\mu}C_{\alpha\beta\gamma}{}^{\mu} = 0. \tag{3.109}$$

But eq. (3.109) clearly implies $E_{ab} \overset{\mathscr{J}}{=} 0 \overset{\mathscr{J}}{=} B_{ab}$, i. e.,

$$C_{\alpha\beta\gamma}{}^{\delta} \overset{\mathscr{J}}{=} 0 \ . \tag{3.110}$$

$\square$

Remark 3.2.1. Since all the terms in eq. (3.51) are regular at $\mathscr{J}$, including $C_{\alpha\beta\gamma}{}^{\delta}$ which by construction is regular there,

$$C_{\alpha\beta\gamma}{}^{\mu}N_{\mu} \overset{\mathscr{J}}{=} 0 \ , \tag{3.111}$$

from where eq. (3.110) is derived. Equation (3.51) depends on property iv) of page 22, which by means of EFEs implies the vanishing of the Cotton-York tensor $\hat{Y}_{\alpha\beta\gamma}$, as it has been shown previously.

Continuing the analysis, take equation eq. (3.62), apply $\Omega^{-1}P^{\beta}{}_{\delta}N^{\alpha}N^{\gamma}$ and evaluate at $\mathscr{J}$ using eqs. (3.71) and (3.73):

$$\begin{aligned}&\frac{1}{6}\varkappa P^{\beta}{}_{\delta}N^{\alpha}N^{\gamma}\nabla_{\beta}Tg_{\alpha\gamma}+\varkappa P^{\beta}{}_{\delta}N^{\alpha}N^{\gamma}\nabla_{[\alpha}T_{\beta]\gamma}\\ &\overset{\mathscr{J}}{=}-P^{\beta}{}_{\delta}N^{\alpha}N^{\gamma}\nabla_{\mu}d_{\alpha\beta\gamma}{}^{\mu}-\frac{1}{2}\varkappa N_{\mu}N^{\mu}P^{\beta}{}_{\delta}N^{\rho}\Omega^{-1}T_{\beta\rho}\ .\end{aligned}\tag{3.112}$$

(We know, by eq. (3.96), that $P^{\beta}{}_{\delta}N^{\rho}\Omega^{-1}T_{\beta\rho}$ is regular at $\mathscr{J}$). Dividing by N^2,

$$-\frac{1}{6}\varkappa P^{\beta}{}_{\delta}\nabla_{\beta}T+\varkappa P^{\beta}{}_{\delta}n^{\alpha}n^{\gamma}\nabla_{[\alpha}T_{\beta]\gamma}\overset{\mathscr{J}}{=}-P^{\beta}{}_{\delta}n^{\alpha}n^{\gamma}\nabla_{\mu}d_{\alpha\beta\gamma}{}^{\mu}+\frac{1}{2}\varkappa P^{\beta}{}_{\delta}N^{\rho}\Omega^{-1}T_{\beta\rho}.\tag{3.113}$$

Using our choice of gauge, the second term on the left-hand side can be rewritten as

$$\begin{aligned}2P^{\beta}{}_{\delta}n^{\alpha}n^{\gamma}\nabla_{[\alpha}T_{\beta]\gamma}&\overset{\mathscr{J}}{=}P^{\beta}{}_{\delta}n^{\alpha}n^{\gamma}\nabla_{\alpha}T_{\beta\gamma}-P^{\beta}{}_{\delta}n^{\alpha}n^{\gamma}\nabla_{\beta}T_{\alpha\gamma}\\ &\overset{\mathscr{J}}{=}n^{\alpha}\nabla_{\alpha}\left(P^{\beta}{}_{\delta}n^{\gamma}T_{\beta\gamma}\right)-n^{\alpha}\underbrace{\nabla_{\alpha}\left(P^{\beta}{}_{\delta}n^{\gamma}\right)}_{\overset{\mathscr{J}}{=}0}+P^{\beta}{}_{\delta}\nabla_{\beta}T\\ &\overset{\mathscr{J}}{=}n^{\alpha}\nabla_{\alpha}\left(\Omega\underbrace{\Omega^{-1}P^{\beta}{}_{\delta}n^{\gamma}T_{\beta\gamma}}_{\text{Regular at }\mathscr{J}}\right)+P^{\beta}{}_{\delta}\nabla_{\beta}T\\ &\overset{\mathscr{J}}{=}\Omega^{-1}P^{\beta}{}_{\delta}n^{\gamma}T_{\beta\gamma}n^{\alpha}N_{\alpha}+P^{\beta}{}_{\delta}\nabla_{\beta}T.\end{aligned}\tag{3.114}$$

Then, eq. (3.113) reads

$$\frac{1}{3}\varkappa P^{\beta}{}_{\delta}\nabla_{\beta}T-\varkappa\Omega^{-1}P^{\beta}{}_{\delta}N^{\gamma}T_{\beta\gamma}\overset{\mathscr{J}}{=}-P^{\beta}{}_{\delta}n^{\gamma}n^{\alpha}\nabla_{\mu}d_{\alpha\beta\gamma}{}^{\mu}\ .\tag{3.115}$$

Now, contract eq. (3.62) with $\Omega^{-1}\eta^{\alpha\beta}{}_{\rho\sigma}N^{\gamma}N^{\rho}$ and evaluate at $\mathscr{J}$

$$\varkappa\eta^{\alpha\beta}{}_{\rho\sigma}N^{\rho}N^{\gamma}\nabla_{[\alpha}T_{\beta]\gamma}\overset{\mathscr{J}}{=}-\eta^{\alpha\beta}{}_{\rho\sigma}N^{\rho}N^{\gamma}\nabla_{\mu}d_{\alpha\beta\gamma}{}^{\mu}\ ,\tag{3.116}$$

where the rest of the terms vanish because $\eta^{\alpha\beta}{}_{\rho\sigma}N^{\rho}N^{\sigma}=0$. Notice, also, that

$$\begin{aligned}&\eta^{\alpha\beta}{}_{\rho\sigma}N^{\rho}N^{\gamma}\nabla_{[\alpha}T_{\beta]\gamma}\overset{\mathscr{J}}{=}\eta_{\nu\mu\rho\sigma}N^{\rho}N^{\gamma}P^{\mu\beta}P^{\nu\alpha}\nabla_{[\alpha}T_{\beta]\gamma}\\ &\overset{\mathscr{J}}{=}\frac{1}{2}\eta_{\nu\mu\rho\sigma}\left[N^{\rho}P^{\nu\alpha}\underbrace{\nabla_{\alpha}\left(N^{\gamma}P^{\mu\beta}T_{\beta\gamma}\right)}_{\propto N_{\alpha}}-T_{\beta\gamma}N^{\rho}P^{\nu\alpha}\underbrace{\nabla_{\alpha}\left(N^{\gamma}P^{\mu\beta}\right)}_{\overset{\mathscr{J}}{=}0}-(\beta\leftrightarrow\alpha)\right]\overset{\mathscr{J}}{=}0.\end{aligned}\tag{3.117}$$

Then,

$$n^{\alpha}n^{\gamma}\nabla_{\mu}{}^{*}d_{\alpha\beta\gamma}{}^{\mu}\overset{\mathscr{J}}{=}0\ .\tag{3.118}$$

Equations (3.115) and (3.118) give us information about the divergence of D_{ab} and C_{ab} at $\mathscr{J}$. From the first it is easy to see that[4]

$$\omega_{\delta}{}^{d}\overline{\nabla}_m D_d{}^m \stackrel{\mathscr{J}}{=} \varkappa\Omega^{-1}P^{\beta}{}_{\delta}N^{\gamma}T_{\beta\gamma} - \frac{1}{3}\varkappa P^{\beta}{}_{\delta}\nabla_{\beta}T \tag{3.119}$$

and from the latter,

$$\overline{\nabla}_m C_a{}^m \stackrel{\mathscr{J}}{=} 0\ . \tag{3.120}$$

In particular, $D_d{}^m$ is divergence free too in vacuum.

Vanishing Λ

Similarly to the $\Lambda > 0$ case, from eq. (3.62) one can deduce that at $\mathscr{J}$

$$T_{\alpha\beta} \stackrel{\mathscr{J}}{=} \mu N_{\alpha}N_{\beta}, \tag{3.121}$$

for some function μ. Using eq. (3.121) into eq. (3.67), the physical Cotton tensor vanishes at $\mathscr{J}$

$$\hat{Y}_{\alpha\beta\gamma} \stackrel{\mathscr{J}}{=} 0\ . \tag{3.122}$$

In fact, eq. (3.122) is one of the conditions involved in the vanishing of the Weyl tensor at $\mathscr{J}$ (see [100])

Lemma 3.2.2 (Vanishing of the Weyl tensor at $\mathscr{J}$ with vanishing cosmological constant)**.** Assume that $\mathscr{J}$ has $\mathbb{R}\times\mathbb{S}^2$ topology and that

1. $\hat{Y}_{\alpha\beta\gamma} \stackrel{\mathscr{J}}{=} 0$ and $\Omega\nabla_{\sigma}\hat{Y}_{\alpha\beta\gamma} \stackrel{\mathscr{J}}{=} 0$,
2. $C_{\alpha\beta\gamma}{}^{\delta}$ and $\nabla_{\mu}C_{\alpha\beta\gamma}{}^{\delta}$ are regular at $\mathscr{J}$.

Then,

$$C_{\alpha\beta\gamma}{}^{\delta} \stackrel{\mathscr{J}}{=} 0\ . \tag{3.123}$$

Proof. We will use the lightlike decomposition of a Weyl-tensor candidate presented in section 2.2. First, from eqs. (3.59) and (3.69) one has

$$\Omega\nabla_{\mu}C_{\alpha\beta\gamma}{}^{\mu} - N_{\mu}C_{\alpha\beta\gamma}{}^{\mu} + \Omega\hat{Y}_{\alpha\beta\gamma} = 0. \tag{3.124}$$

This equation evaluated at $\mathscr{J}$ gives

$$N_{\mu}C_{\alpha\beta\gamma}{}^{\mu} \stackrel{\mathscr{J}}{=} 0, \tag{3.125}$$

[4]The intrinsic covariant derivative on $\mathscr{J}$ is denoted by $\overline{\nabla}_a$, while $(\omega_{\alpha}{}^{a})$ is a basis of linearly independent one-forms on $\mathscr{J}$ orthogonal to n^{α}. For further details, see appendix A.1, where we introduce this notation for a general 3-dimensional hypersurface.

immediately implying that $C_{\alpha\beta\gamma}{}^{\mu}$ has Petrov type N at $\mathscr{J}$ with N^{α} the repeated principal null direction. But this condition shows that the only components that survive at $\mathscr{J}$ are those of the symmetric traceless rank-2 tensor field

$$\overset{\ell}{\underline{E}}_{\alpha\beta} := \overline{\ell}^{\mu}\overline{\ell}^{\nu}\underline{P}^{\rho}{}_{\alpha}\underline{P}^{\sigma}{}_{\beta}C_{\rho\mu\sigma\nu}\ , \tag{3.126}$$

where $\overline{\ell}^{\alpha}$ is any lightlike vector field on $\mathscr{J}$, *which we choose to be orthogonal to cuts*, such that $\overline{\ell}^{\mu}N_{\mu} \overset{\mathscr{J}}{=} -1$ and

$$\underline{P}^{\alpha}{}_{\beta} := \delta^{\alpha}_{\beta} + N^{\alpha}\overline{\ell}_{\beta} + \overline{\ell}^{\alpha}N_{\beta} \tag{3.127}$$

is the projector to the two dimensional space orthogonal to N^{α} and $\overline{\ell}^{\beta}$. Now, take the derivative of eq. (3.124) and evaluate it at $\mathscr{J}$

$$N_{\sigma}\nabla_{\mu}C_{\alpha\beta\gamma}{}^{\mu} - N_{\mu}\nabla_{\sigma}C_{\alpha\beta\gamma}{}^{\mu} \overset{\mathscr{J}}{=} 0. \tag{3.128}$$

Contract this equation with $\overline{\ell}^{\sigma}\overline{\ell}^{\alpha}\overline{\ell}^{\gamma}$ to obtain

$$\underline{P}^{\mu\rho}\nabla_{\mu}\left(\overline{\ell}^{\sigma}\overline{\ell}^{\nu}C_{\sigma\beta\nu\rho}\right) - C_{\alpha\beta\gamma\rho}\underline{P}^{\rho\mu}\left(\underline{P}^{\alpha}{}_{\tau}\overline{\ell}^{\gamma}\nabla_{\mu}\overline{\ell}^{\tau} + \underline{P}^{\gamma}{}_{\tau}\overline{\ell}^{\alpha}\nabla_{\mu}\overline{\ell}^{\tau}\right) \overset{\mathscr{J}}{=} 0, \tag{3.129}$$

where we have used eqs. (3.125) and (3.127). If we contract now with $\underline{P}^{\beta}{}_{\delta}$, we find

$$\underline{P}^{\mu\rho}\nabla_{\mu}\left(\overset{\ell}{\underline{E}}_{\beta\rho}\right) + \left(\overset{\ell}{\underline{s}}_{\gamma\delta}{}^{\mu} + \overset{\ell}{\underline{s}}_{\gamma}{}^{\mu}{}_{\delta}\right)\nabla_{\mu}\overline{\ell}^{\alpha} \overset{\mathscr{J}}{=} 0, \tag{3.130}$$

where $\overset{\ell}{\underline{s}}_{\alpha\beta\gamma} := \overline{\ell}^{\mu}\underline{P}^{\rho}{}_{\alpha}\underline{P}^{\sigma}{}_{\beta}\underline{P}^{\nu}{}_{\gamma}C_{\rho\sigma\nu\mu}$. But by the properties in appendix D and eq. (3.125) this tensor field vanishes at $\mathscr{J}$, and therefore

$$\underline{P}^{\mu\rho}\nabla_{\mu}\left(\overset{\ell}{\underline{E}}_{\beta\rho}\right) \overset{\mathscr{J}}{=} 0\ . \tag{3.131}$$

This equation is equivalently written by means of the intrinsic connection on each cut $\mathcal{S}$ defined by $\overline{\ell}^{\alpha}$ as

$$\mathcal{D}_{M}\left(\overset{\ell}{\underline{E}}_{A}{}^{M}\right) \overset{\mathcal{S}}{=} 0\ . \tag{3.132}$$

By assumption, the topology of the cuts is $\mathbb{S}^2$, and then $\overset{\ell}{\underline{E}}_{AB}$ is a traceless divergence-free symmetric tensor on $\mathbb{S}^2$ and must vanish [113],

$$\overset{\ell}{\underline{E}}_{AB} \overset{\mathcal{S}}{=} 0\ . \tag{3.133}$$

Since this happens on any $\mathcal{S} \subset \mathscr{J}$ for all cuts transversal to N^{α} –and $\mathscr{J}$ can be foliated by these cuts, it holds everywhere on $\mathscr{J}$, implying

$$C_{\alpha\beta\gamma}{}^{\delta} \overset{\mathscr{J}}{=} 0\ . \tag{3.134}$$

□

Remark 3.2.2. Instead of assuming $\hat{Y}_{\alpha\beta\gamma} \overset{\mathscr{I}}{=} 0$ in the proof, we could have started from eq. (3.51), which makes use of property iv) on page 22. Both paths are equivalent since, as have been shown above, the assumption iv) on the energy-momentum tensor implies the vanishing of the physical Cotton-York tensor.

Detrás.
Abajo.
Al límite.
En el sitio en que todo se reúne en nosotros
igual que dentro
de un sólo hombre suena
el bosque entero.

Benjamín Prado, *Límite. Todos nosotros*, 1998.

4 | Asymptotic structure with vanishing cosmological constant

The geometry and physics present at infinity with a vanishing cosmological constant differ notably from the ones with a positive Λ. Those differences are discussed thoroughly in chapters 5 to 7. Let us just mention here that the fundamental distinctions emerge as a consequence of the change in the causal character of $\mathscr{J}$, no matter *how tiny* the cosmological constant is [24]. This chapter is devoted to the $\Lambda = 0$ scenario, in which the unphysical space-time has a lightlike conformal boundary according to eq. (3.75). In this context, $\mathscr{J}$ is endowed with a conformal class of degenerate metrics and null generators which constitute a *universal structure*. This structure underlies many of the favourable features in the asymptotically flat situation.

Asymptotics with $\Lambda = 0$ can be tackled in the old metric-based approach [21, 40, 41] –see also [114]–, in the NP formalism [15, 96] or by employing covariant methods and studying the intrinsic structure of $\mathscr{J}$ [17, 48] in a gauge and coordinate-independent way. For instance, one can derive the asymptotic symmetry group by first writing in coordinate form the degenerate metric on $\mathscr{J}$ and fixing the conformal gauge (3.6) such that the degenerate metric is that of a round two-sphere –resulting in 'Bondi gauge or system'[115][1]–; then, restricting the allowed coordinate transformations to those preserving the form of the round metric. Alternatively, one can define it by determining those transformations which leave invariant the universal sctructure of $\mathscr{J}$. As another example, the classical criterion that determines the presence of gravitational radiation arriving at $\mathscr{J}$ is based on the so called *news tensor*, which is a rank-2 symmetric traceless tensor field on $\mathscr{J}$ orthogonal to the generators; this tensor field can be treated as what it is, or instead consider a complex function –i.e., the news function– which after gauge fixing is determined by the shear of a conveniently selected lighlike vector field on $\mathscr{J}$. However that is not the most general picture. In contrast, the covariant approach does not fix the conformal gauge, neither it needs of the introduction of coordinates. We incorporate this philosophy to our

[1]For the study of asymptotics in alternative coordinate systems, see [116].

new method for characterising gravitational radiation. They are the kind of techniques that we find more appropriate; partly because they are geometrically meaningful and also because they can be compared more easily to the $\Lambda > 0$ scenario to show why *one can not simply adapt the known* $\Lambda = 0$ *results as shortcuts to the new scenario.* All these ideas are made explicit in the course of the next sections and in chapter 5. For a review of previously known results see e.g. [117, 118].

Although the main aim in this chapter is the characterisation of gravitational radiation, there are other results that are worth remarking. After giving the grounds and deriving the basic intrinsic geometry of $\mathscr{J}$ based on Geroch's ideas [17], an endomorphism at the tangent space of any point in $\mathscr{J}$ is found which provides the asymptotic behaviour of physical fields approaching $\mathscr{J}$ along null geodesics. Its application to the physical Weyl tensor provides the so called peeling behaviour [40, 96], which is presented in form of a theorem –see theorem 1. Not only that but we also use it to obtain the peeling behaviour of the physical Bel-Robinson tensor –theorem 4– and, as a consequence of this, an alignment of physical supermomenta towards infinity occurs. The second part of the chapter is devoted to the characterisation of gravitational radiation at infinity, putting forward the new –superenergy-based– criterion for determining the presence of gravitational radiation at infinity and comparing it with the classical condition. Beautifully, the criterion is in correspondence with the asymptotic alignment of supermomenta and the superenergy at infinity can be understood as sourcing the so called news tensor field. All these features provide a test of our approach towards the characterisation of gravitational radiation by means of the rescaled Bel-Robinson tensor (3.86). The core of these ideas is applied to the $\Lambda > 0$ scenario in chapters 5 and 6.

4.1 Asymptotic geometry and fields

Let us begin by studying the intrinsic geometry of $\mathscr{J}$ and its relation to the physical fields at infinity.

4.1.1 Some basic geometry of $\mathscr{J}$

Let $\{e^{\alpha}{}_{a}\}$ be a basis of the set of vector fields tangent to $\mathscr{J}$, i.e., orthogonal to N_{α}, with $a = 1, 2, 3$, and let $\{\omega_{\alpha}{}^{a}\}$ be a dual basis. In particular, $N^{\alpha} = N^{a} e^{\alpha}{}_{a}$ is collinear with the generators of $\mathscr{J}$ and N^{a} is the degeneration vector field of the induced first fundamental form

$$\overline{g}_{ab} := e^{\alpha}{}_{a} e^{\beta}{}_{b} g_{\alpha\beta} \ , \quad N^{a}\overline{g}_{ab} = 0 \ . \tag{4.1}$$

Equation (3.80) implies that the second fundamental form of $\mathscr{J}$ vanishes

$$K_{ab} := e^{\mu}{}_{a} e^{\nu}{}_{b} \nabla_{\mu} N_{\nu} = 0 \ , \tag{4.2}$$

hence the intrinsic Lie derivative on $\mathscr{J}$ of $\overline{g}_{ab}$ along N^a is zero

$$\pounds_{\vec{N}}\overline{g}_{ab} = 0 \tag{4.3}$$

and the *induced connection*

$$\forall\ X^\alpha = X^a e^\alpha{}_a\ , \quad \forall\ Y^\alpha = Y^a e^\alpha{}_a\ , \quad X^a\overline{\nabla}_a Y^b := \omega_\beta{}^b X^\alpha \nabla_\alpha Y^\beta \tag{4.4}$$

is torsion-free and 'metric',

$$\overline{\nabla}_a \overline{g}_{bc} = 0. \tag{4.5}$$

The induced connection coefficients $\overline{\Gamma}^c{}_{ab}$ are thus given by

$$e^\mu{}_a \nabla_\mu e^\gamma{}_b = \overline{\Gamma}^c{}_{ab} e^\gamma{}_c\ . \tag{4.6}$$

One can introduce a volume three-form ϵ_{abc} on $\mathscr{J}$ by means of the space-time volume four-form $\eta_{\alpha\beta\gamma\delta}$,

$$-N_\alpha\, \epsilon_{abc} \overset{\mathscr{J}}{=} \eta_{\alpha\mu\nu\sigma} e^\mu{}_a e^\nu{}_b e^\sigma{}_c\ , \tag{4.7}$$

and a contravariant version determined by $\epsilon^{abc}\epsilon_{abc} = 6$. We fix the corresponding orientations to $\epsilon_{123} = \eta_{0123} = 1$. The choice of gauge also implies that the induced connection is volume preserving,

$$\overline{\nabla}_a \epsilon_{bcd} = 0\ . \tag{4.8}$$

Although the metric is degenerate, one can define a contravariant object that 'raises indices' by

$$\overline{g}_{ea}\overline{g}^{ed}\overline{g}_{db} := \overline{g}_{ab}. \tag{4.9}$$

There is a freedom in adding to $\overline{g}^{ab}$ any term of the form $N^a v^b + N^b v^a$. One can make a choice, however, by picking out a dual basis $\{\omega_\alpha{}^a\}$ and instead defining $\overline{g}^{ab}$ as

$$\overline{g}^{ab} := \omega_\alpha{}^a \omega_\beta{}^b g^{\alpha\beta}\ , \quad \overline{g}^{ef}\overline{g}_{ef} = 2\ , \tag{4.10}$$

from where eq. (4.9) follows. Due to its topology, $\mathscr{J}$ admits a natural definition of *cuts* $\mathcal{S}$, i. e., any closed spacelike surface transversal to the generators everywhere. Every $\mathcal{S}$ is a topological two-sphere $\mathbb{S}^2$ with a positive-definite metric inherited from –and which essentially is– $\overline{g}_{ab}$,

$$q_{AB} \overset{\mathcal{S}}{:=} E^a{}_A E^b{}_B \overline{g}_{ab}\ , \tag{4.11}$$

where $\{E^a{}_A\}$ is a basis of the set $\mathfrak{X}_\mathcal{S}$ of tangent vector fields on $\mathcal{S}$, with $A = 1, 2$. A basis of tangent vector fields to $\mathcal{S}$ considered within M is $\{E^\alpha{}_A\}$ where $E^\alpha{}_A = e^\alpha{}_a E^a{}_A$. Similarly, we introduce bases $\left\{W_a{}^A\right\}$ and $\left\{W_\alpha{}^A\right\}$ of the dual space $\Lambda_\mathcal{S}$. At each cut $\mathcal{S}$, there is a unique lightlike vector field ℓ^α other than N^α such that

$$\ell^\mu \ell_\mu \overset{\mathcal{S}}{=} 0 \ , \quad \ell_\alpha \overset{\mathcal{S}}{=} \ell_a \omega_\alpha{}^a \ , \quad \ell^\alpha \omega_\alpha{}^a \overset{\mathcal{S}}{=} 0 \ , \quad \ell_a E^a{}_A \overset{\mathcal{S}}{=} 0, \quad \ell^\alpha N_\alpha \overset{\mathcal{S}}{=} -1 \ . \tag{4.12}$$

This set of vector fields can be used to complete the bases on $\mathscr{J}$ *at* $\mathcal{S}$, $\left\{-\ell_a, W_a{}^A\right\}$, $\left\{N^a, E^a{}_A\right\}$, and to write the projector to $\mathcal{S}$:

$$\mathring{P}^\alpha{}_\beta = E^\alpha{}_M W_\beta{}^M \overset{\mathcal{S}}{=} \delta^\alpha_\beta + \ell_\beta N^\alpha + N_\beta \ell^\alpha \ . \tag{4.13}$$

Also, one has

$$\overline{g}^{mb} e^\mu{}_m g_{\mu\alpha} \overset{\mathcal{S}}{=} \omega_\alpha{}^b + \ell_\alpha N^b, \quad \ell_m \overline{g}^{am} \overset{\mathcal{S}}{=} 0 \tag{4.14}$$

and the projector to a given cut within $\mathscr{J}$ takes the form

$$\mathring{P}^a{}_b = \delta^a_b + N^a \ell_b = \overline{g}^{ac}\overline{g}_{bc} \ . \tag{4.15}$$

The intrinsic volume two-form of $(\mathcal{S}, q_{AB})$ reads

$$-\ell_a \mathring{\epsilon}_{AB} \overset{\mathcal{S}}{=} \epsilon_{amn} E^m{}_A E^n{}_B \ , \tag{4.16}$$

$$N^a \mathring{\epsilon}^{AB} \overset{\mathcal{S}}{=} \epsilon^{amn} W_m{}^A W_n{}^B \ , \tag{4.17}$$

where the orientation is chosen such that $\mathring{\epsilon}_{23} = \overline{\epsilon}_{123} = 1$, and the inherited connection

$$\forall \ U^a \overset{\mathcal{S}}{=} E^a{}_A U^A \ , \quad \forall \ V^a \overset{\mathcal{S}}{=} E^a{}_A V^A \ , \quad V^M \mathcal{D}_M U^A \overset{\mathcal{S}}{:=} W_m{}^A V^n \overline{\nabla}_n U^m \tag{4.18}$$

is metric and volume preserving –ergo this is the intrinsic Levi-Civita connection on $(\mathcal{S}, q_{AB})$–

$$\mathcal{D}_A q_{AB} = 0 \ , \quad \mathcal{D}_A \mathring{\epsilon}_{BC} = 0 \ . \tag{4.19}$$

Equation (3.80) implies that

$$\overline{\nabla}_a N^b = 0 \ . \tag{4.20}$$

The relation between the space-time covariant derivative and the induced derivative on $\mathscr{J}$ for any tensor field $T^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_q}$ defined *at least* on $\mathscr{J}$ is

$$\begin{aligned} &\omega_{\mu_1}{}^{a_1} \dots \omega_{\mu_r}{}^{a_r} e^{\nu_1}{}_{b_q} \dots e^{\nu_q}{}_{b_q} e^\rho{}_c \nabla_\rho T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_q} \overset{\mathscr{J}}{=} \overline{\nabla}_c \overline{T}^{a_1 \dots a_r}{}_{b_1 \dots b_q} \\ &- \sum_{i=1}^{r} T^{a_1 \dots a_{i-1} \sigma a_{i+1} \dots a_r}{}_{b_1 \dots b_q} N_\sigma \Psi^{a_i}{}_c \end{aligned} \tag{4.21}$$

where we have defined

$$\overline{T}^{a_1 \dots a_r}{}_{b_1 \dots b_q} \overset{\mathscr{J}}{:=} \omega_{\mu_1}{}^{a_1} \dots \omega_{\mu_r}{}^{a_r} e^{\nu_1}{}_{b_q} \dots e^{\nu_q}{}_{b_q} T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_q} \ , \quad \Psi^a{}_c := \omega^a{}_\mu e^\nu{}_c \nabla_\nu \overline{\ell}^\mu \ , \tag{4.22}$$

with $\overline{\ell}^{\alpha}$ any vector field on $\mathscr{J}$ satisfying $\overline{\ell}^{\alpha}N_{\alpha} = -1$ and $\overline{\ell}^{\alpha}\omega_{\alpha}{}^{a} = 0$. We also have used that $K_{ab} = 0$ –for general formulae see [119]–, whereas the relation between the induced covariant derivative on $\mathscr{J}$ and the intrinsic covariant derivative on $\mathcal{S}$ for a tensor field $T^{a_1...a_r}{}_{b_1...b_q}$ defined *at least* on $\mathcal{S}$ reads

$$
\begin{aligned}
&W_{m_1}{}^{A_1}...W_{m_r}{}^{A_r}E^{n_1}{}_{B_q}...E^{n_q}{}_{B_q}E^{r}{}_{C}\overline{\nabla}_r T^{m_1...m_r}{}_{n_1...n_q} \stackrel{\mathcal{S}}{=} \mathcal{D}_C \mathring{T}^{A_1...A_r}{}_{B_1...B_q}\\
&-\sum_{i=1}^{q} T^{A_1...A_r}{}_{B_1...B_{i-1}sB_{i+1}...B_q}N^{s}H_{CB_i}
\end{aligned}
\tag{4.23}
$$

with

$$
\mathring{T}^{A_1...A_r}{}_{B_1...B_q} \stackrel{\mathcal{S}}{:=} W_{m_1}{}^{A_1}...W_{m_r}{}^{A_r}E^{n_1}{}_{B_q}...E^{n_q}{}_{B_q}T^{m_1...m_r}{}_{n_1...n_q}\ ,\quad H_{AB} \stackrel{\mathcal{S}}{:=} E^{a}{}_{A}E^{b}{}_{B}\overline{\nabla}_a \ell_b\ . \tag{4.24}
$$

Observe that under conformal gauge transformations, the following changes apply

$$
\tilde{\overline{g}}_{ab} = \omega^2 \overline{g}_{ab}\ , \tag{4.25}
$$

$$
\tilde{q}_{AB} \stackrel{\mathcal{S}}{=} \omega^2 q_{AB}\ . \tag{4.26}
$$

The curvature tensor associated to the induced connection satisfies

$$
\left(\overline{\nabla}_a\overline{\nabla}_b - \overline{\nabla}_b\overline{\nabla}_a\right)v^d = -\overline{R}_{abc}{}^{d}v^c \quad \forall v^c \in T_{\mathscr{J}}\ , \tag{4.27}
$$

it is related to the space-time curvature through the 'Gauss equation'

$$
e^{\alpha}{}_{a}e^{\beta}{}_{b}e^{\gamma}{}_{c}\omega_{\delta}{}^{d}R_{\alpha\beta\gamma}{}^{\delta} \stackrel{\mathscr{J}}{=} \overline{R}_{abc}{}^{d} \tag{4.28}
$$

and has the properties

$$
\overline{R}_{abc}{}^{d} = -\overline{R}_{bac}{}^{d}\ ,\quad \overline{R}_{[abc]}{}^{d} = 0\ ,\quad \overline{R}_{abc}{}^{c} = 0\ ,\quad \overline{\nabla}_{[e}\overline{R}_{ab]c}{}^{d} = 0\ . \tag{4.29}
$$

Its non-vanishing trace constitutes a symmetric tensor field

$$
\overline{R}_{ab} := \overline{R}_{adb}{}^{d} = \overline{R}_{ba}\ . \tag{4.30}
$$

The curvature tensor can be expressed as

$$
\overline{R}_{abc}{}^{d} = 2\overline{g}_{c[a}\overline{S}_{b]}{}^{d} - 2\delta^{d}_{[a}\overline{S}_{b]c}\ , \tag{4.31}
$$

where the tensor fields $\overline{S}_{a}{}^{b}$ and $\overline{S}_{ab} := \overline{g}_{am}\overline{S}_{b}{}^{m} = \overline{S}_{ba}$ will be shown to coincide with pullbacks of the space-time Schouten tensor to $\mathscr{J}$ –see section 4.1.2. Of course, one can lower the contravariant index of the curvature tensor with the degenerate metric $\overline{g}_{ab}$,

$$\overline{R}_{abcd} := \overline{g}_{ed}\overline{R}_{abc}{}^{e} \ , \tag{4.32}$$

however, information is lost in this process and one has to treat the fully covariant version as a different tensor. Using the 'metricity' of the induced connection, it follows that $\overline{R}_{abcd}$ has all the symmetries of a Riemann tensor, including

$$\overline{R}_{abcd} = \overline{R}_{cdab} = -\overline{R}_{abdc} \ . \tag{4.33}$$

In considering the action of $\overline{R}_{abc}{}^{d}$ on N^a, via eq. (4.27) and using eq. (4.20), one finds

$$\overline{R}_{abc}{}^{d}N^c = 0 \ . \tag{4.34}$$

This implies that

$$N^a\overline{R}_{abcd} = 0 \ , \tag{4.35}$$

hence the lower-index version of the curvature tensor is orthogonal to N^a in all its indices. This property makes it effectively a two-dimensional tensor field with the symmetries of a Riemann tensor, thus we can write it as

$$\overline{R}_{abcd} = \frac{1}{2}\overline{R}\left(\overline{g}_{ac}\overline{g}_{db} - \overline{g}_{bc}\overline{g}_{da}\right) \tag{4.36}$$

for some scalar field $\overline{R}$. Using the properties presented so far, it follows that

$$\pounds_{\vec{N}}\overline{R}_{abcd} = N^e\overline{\nabla}_e\overline{R}_{abcd} = 0 \ , \quad N^e\overline{\nabla}_e\overline{R} = 0 \ . \tag{4.37}$$

Using eq. (4.31), $\overline{R}_{ab}$ can be expressed as

$$\overline{R}_{ab} = \overline{S}_{ab} + \overline{g}_{ab}\overline{S}_{m}{}^{m} \ , \tag{4.38}$$

and $\overline{R}_{abcd}$ as

$$\overline{R}_{abcd} = 2\overline{g}_{c[a}\overline{S}_{b]d} - 2\overline{g}_{d[a}\overline{S}_{b]c} \ . \tag{4.39}$$

Because of eq. (4.35), one can take the traces of this tensor field with $\overline{g}^{ab}$. In doing so, if one compares eqs. (4.36), (4.38) and (4.39), it follows that

$$\overline{S}_{mn}\overline{g}^{mn} = \frac{\overline{R}}{2} \ , \tag{4.40}$$

$$2\overline{S}_{m}{}^{m} + \frac{1}{2}\overline{R} = \overline{g}^{rs}\overline{R}_{rs} \ . \tag{4.41}$$

Hence, the following expression holds

$$\overline{S}_{ab} = \overline{R}_{ab} - \frac{1}{2}\overline{g}_{ab}\left(\overline{g}^{mn}\overline{R}_{mn} - \frac{1}{2}\overline{R}\right) \ . \tag{4.42}$$

From eq. (4.23) it is easily deduced the 'Gauss relation' between the intrinsic curvature $\mathring{R}_{ABC}{}^{D}$ of any cut $\mathcal{S}$ and the curvature of $\mathscr{J}$,

$$E^{a}{}_{A}E^{b}{}_{B}E^{c}{}_{C}\overline{R}_{abc}{}^{d}W_{d}{}^{D}v_{D} \overset{\mathcal{S}}{=} \left(\mathcal{D}_{A}\mathcal{D}_{B} - \mathcal{D}_{B}\mathcal{D}_{A}\right)v_{C} \overset{\mathcal{S}}{:=} \mathring{R}_{ABC}{}^{D}v_{D} \ , \quad \forall v_{A} \in \Lambda_{\mathcal{S}} \ . \tag{4.43}$$

One can readily show that

$$\mathring{R}_{AB} := \mathring{R}_{AMB}{}^{M} \overset{\mathcal{S}}{=} q_{AB}\overline{S}_{mn}\overline{g}^{mn} \overset{\mathcal{S}}{=} \frac{1}{2}\overline{R}q_{AB} \ , \tag{4.44}$$

$$\mathring{R} := \mathring{R}_{M}{}^{M} \overset{\mathcal{S}}{=} \overline{R} \overset{\mathcal{S}}{=} 2K \ , \tag{4.45}$$

where K is the Gaussian curvature of $(\mathcal{S}, q_{AB})$. Instead of single cuts $\mathcal{S}$, one can consider a generic *foliation* where each leaf $\mathcal{S}_{C}$ is defined by a different constant value C of a function F such that

$$\dot{F} := N^{m}\overline{\nabla}_{m}F \neq 0 \ . \tag{4.46}$$

Each leaf is a cut, by definition transversal to N^{a}. Then, associated to a given foliation there is a one-form

$$\overline{\ell}_{a} := -\frac{1}{\dot{F}}\overline{\nabla}_{a}F, \quad N^{m}\overline{\ell}_{m} = -1 \ . \tag{4.47}$$

We set univocally $\overline{\ell}_{\alpha} := \omega_{\alpha}{}^{a}\overline{\ell}_{a}$ and require $\overline{\ell}_{\mu}\overline{\ell}^{\mu} = 0$, which implies that $\overline{\ell}_{a} \overset{\mathcal{S}}{=} \ell_{a}$ on each cut $\mathcal{S}$. The restriction of $\overline{\ell}_{a}$ to each cut $\mathcal{S}_{C}$ of the foliation defines a ℓ_{a} there, as the field uniquely defined by eq. (4.12). One can introduce couples of vector fields $\left\{\underline{E}^{a}{}_{A}\right\}$ and $\left\{\underline{W}_{a}{}^{A}\right\}$ –with $A = 2, 3$– serving as bases for the set of vector fields and forms on $\mathscr{J}$ orthogonal to $\overline{\ell}_{a}$ and N^{a}. Also, on each leaf $\mathcal{S}_{C}$ they constitute bases for the vector fields and forms that are orthogonal to N^{a} and ℓ_{a} there. Let us introduce the projector

$$\underline{P}^{a}{}_{b} := \delta^{a}_{b} + N^{a}\overline{\ell}_{b} \ , \quad \underline{P}^{m}{}_{b}\overline{\ell}_{m} = 0 = \underline{P}^{a}{}_{m}N^{m} \ , \quad \underline{P}^{a}{}_{b} \overset{\mathcal{S}_{C}}{=} \mathring{P}^{a}{}_{b} \ . \tag{4.48}$$

We will distinguish quantities projected to a single cut $\mathcal{S}_{C}$ from those projected with $\underline{P}^{a}{}_{b}$ by using the following notation

$$\underline{v}_{b} := \underline{P}^{m}{}_{b}v_{m} \ , \tag{4.49}$$

$$\mathring{v}_{b} \overset{\mathcal{S}_{C}}{:=} \mathring{P}^{m}{}_{b}v_{m} \ , \tag{4.50}$$

and similarly

$$\underline{v}_B := \underline{E}^m{}_B v_m \ , \tag{4.51}$$

$$\mathring{v}_B \overset{\mathcal{S}_C}{:=} E^m{}_B v_m \ . \tag{4.52}$$

Of course, given any one-form field v_a on $\mathscr{J}$,

$$\underline{v}_a \overset{\mathcal{S}_C}{=} \mathring{v}_a \ . \tag{4.53}$$

A simple calculation leads to

$$\pounds_{\vec{N}} \bar{\ell}_a = N^e \overline{\nabla}_e \bar{\ell}_a = -\underline{P}^m{}_a \overline{\nabla}_m \ln \dot{F} \tag{4.54}$$

and the next relations hold

$$\underline{P}^m{}_a \overline{\nabla}_m F = 0 \quad \Big(\Longleftrightarrow \underline{E}^m{}_A \bar{\ell}_m = 0 \Big) \ , \tag{4.55}$$

$$\pounds_{\vec{N}} \underline{E}^a{}_A = -N^a \underline{E}^m{}_A \overline{\nabla}_m \ln \dot{F} \ , \quad \pounds_{\vec{N}} \underline{W}_a{}^A = 0 \ . \tag{4.56}$$

In addition, one can define

$$\underline{q}_{AB} := \underline{E}^a{}_A \underline{E}^b{}_B \bar{g}_{ab} \ , \quad \underline{q}^{AB} := \underline{W}_a{}^A \underline{W}_b{}^B \bar{g}^{ab} \ , \tag{4.57}$$

where $\underline{q}_{AB}$ is such that it coincides with the metric q_{AB} of each leaf $\mathcal{S}_C$. All cuts are isometric, though, as a quick calculation taking into account the above relations and eq. (4.3) yields

$$\pounds_N \underline{q}_{AB} = 0 \ , \tag{4.58}$$

hence $\underline{q}_{AB}$ and q_{AB} are essentially the same object. Hence, the curvature (4.44) is basically the same for every cut of the foliation, in agreement with eq. (4.37) and, indeed, all cuts are isometric, even if they do not belong to the same foliation.

There is a special sort of foliations that we call *adapted* to N^a. These are defined by functions F fulfilling

$$\pounds_{\vec{N}} \dot{F} = 0 \ . \tag{4.59}$$

Given any adapted foliation, an appropriate gauge fixing ($\omega = \dot{F}$) allows, via the transformations of appendix C, to set

$$\bar{\ell}_a = -\overline{\nabla}_a F \ , \quad \overline{\nabla}_{[a} \bar{\ell}_{b]} = 0 \ , \quad \pounds_{\vec{N}} \bar{\ell}_a = 0 \ , \quad \pounds_{\vec{N}} \underline{E}^a{}_A = 0 \ . \tag{4.60}$$

We refer to this kind of foliations as *canonically adapted to* N^a.

Finally, let us introduce the kinematical quantities

$$\underline{\Theta}_{ab} := \underline{P}^r{}_{(a}\underline{P}^s{}_{b)}\overline{\nabla}_r\overline{\ell}_s \ , \tag{4.61}$$

$$\underline{\sigma}_{ab} := \underline{\Theta}_{ab} - \frac{1}{2}\overline{g}_{ab}\overline{g}^{rs}\underline{\Theta}_{rs} \ . \tag{4.62}$$

Observe that on every cut $\mathcal{S}_C$ of the foliation –see definition (4.24)–

$$\underline{E}^a{}_A\underline{E}^b{}_B\underline{\Theta}_{ab} \overset{\mathcal{S}_C}{=} H_{AB} \ . \tag{4.63}$$

Also, definition (4.62) is nothing but the shear of the one-form $\overline{\ell}_a$ orthogonal to each cut $\mathcal{S}_C$ of the foliation.

A deeper characterisation of the curvature and the interplay between the induced connection, the choice of foliation and the space-time fields is given in section 4.1.2.

4.1.2 Curvature on $\mathscr{J}$ and its relation to space-time fields

If one considers the Gauss relation (4.28) and uses eq. (3.85), eq. (4.31) is obtained with

$$\overline{S}_{ab} \overset{\mathscr{J}}{:=} \frac{1}{2}e^\alpha{}_a e^\beta{}_b S_{\alpha\beta} \ , \tag{4.64}$$

$$\overline{S}_a{}^b \overset{\mathscr{J}}{:=} \frac{1}{2}e^\alpha{}_a\omega_\beta{}^b S_\alpha{}^\beta. \tag{4.65}$$

For simplicity, consider a foliation given by F with $\overline{\ell}_a$ as in eq. (4.47) and $\overline{\ell}_\alpha := \omega_\alpha{}^a\overline{\ell}_a$ determined by $\overline{\ell}_\mu\overline{\ell}^\mu = 0$ –see section 4.1.1–. Now, since $f \overset{\mathscr{J}}{=} 0$, eq. (3.82) gives

$$N^r\overline{S}_r{}^a = N^a \pounds_{\vec{\ell}} f \ , \tag{4.66}$$

$$N^r\overline{S}_{ra} = 0 \ . \tag{4.67}$$

Provided eq. (3.80), a general formula [120] gives

$$\pounds_{\vec{N}}\overline{\Gamma}^a{}_{bc} = \overline{R}_{cdb}{}^a N^d \tag{4.68}$$

which in conjunction with eqs. (4.31), (4.40) and (4.41) and eqs. (4.66) and (4.67) provides us with

$$\pounds_{\vec{N}}\overline{\Gamma}^a{}_{bc} = N^a\overline{S}_{bc} + \overline{g}_{bc}N^m\overline{S}_m{}^a = N^a\left(\overline{S}_{bc} + \overline{g}_{bc}\pounds_{\vec{\ell}} f\right) \ , \tag{4.69}$$

$$\pounds_{\vec{\ell}} f = \overline{S}_m{}^m - \overline{S}_{mn}\overline{g}^{mn} = \overline{S}_m{}^m - \frac{\overline{R}}{2} \ . \tag{4.70}$$

In a middle step in deriving the second formula, we have contracted the second and fourth indices of eq. (4.31) with $\delta^a_b = -N^a\overline{\ell}_b + \underline{P}^a{}_b$. Also, a direct calculation using eqs. (4.39) and (4.66) yields

$$N^c\overline{R}_{cab}{}^d\overline{\ell}_d = \overline{g}_{ab}\pounds_{\vec{\ell}}f + \overline{S}_{ab}\ , \tag{4.71}$$

whereas application of the 'Ricci identity' leads to

$$N^c\overline{\nabla}_c\underline{\Theta}_{ab} - \pounds_{\vec{N}}\overline{\ell}_a\pounds_{\vec{N}}\overline{\ell}_b - \underline{P}^m{}_{(a}\underline{P}^n{}_{b)}\overline{\nabla}_m\pounds_{\vec{N}}\overline{\ell}_n = \overline{g}_{ab}\pounds_{\vec{\ell}}f + \overline{S}_{ab}\ , \tag{4.72}$$

where we have symmetrised the free indices, introduced (4.61) and taken into account that

$$N^d\overline{\nabla}_d\underline{P}^a{}_b = N^aN^d\overline{\nabla}_d\overline{\ell}_b\ . \tag{4.73}$$

One can take the trace-free part of eq. (4.72),

$$\begin{aligned}N^c\overline{\nabla}_c\underline{\sigma}_{ab} = \overline{S}_{ab} - \frac{1}{2}\overline{g}_{ab}\overline{g}^{mn}\overline{S}_{mn} + \pounds_{\vec{N}}\overline{\ell}_a\pounds_{\vec{N}}\overline{\ell}_b + \underline{P}^m{}_{(a}\underline{P}^n{}_{b)}\overline{\nabla}_m\pounds_{\vec{N}}\overline{\ell}_n\\ - \frac{1}{2}\overline{g}_{ab}\overline{g}^{mn}\left(\pounds_{\vec{N}}\overline{\ell}_m\pounds_{\vec{N}}\overline{\ell}_n + \underline{P}^e{}_n\underline{P}^f{}_m\overline{\nabla}_f\pounds_{\vec{N}}\overline{\ell}_e\right)\ ,\end{aligned} \tag{4.74}$$

which in terms of the function F giving the foliation –see eqs. (4.47) and (4.54)– reads

$$\begin{aligned}N^c\overline{\nabla}_c\underline{\sigma}_{ab} = \overline{S}_{ab} - \frac{1}{2}\overline{g}_{ab}\overline{g}^{mn}\overline{S}_{mn} + \underline{P}^m{}_a\overline{\nabla}_m\left(\ln\dot{F}\right)\underline{P}^n{}_b\overline{\nabla}_n\left(\ln\dot{F}\right) - \underline{P}^m{}_{(a}\underline{P}^n{}_{b)}\overline{\nabla}_n\left(\underline{P}^r{}_m\overline{\nabla}_r\ln\dot{F}\right)\\ - \frac{1}{2}\overline{g}_{ab}\overline{g}^{ef}\left[\underline{P}^m{}_e\overline{\nabla}_m\left(\ln\dot{F}\right)\underline{P}^n{}_f\overline{\nabla}_n\left(\ln\dot{F}\right) - \underline{P}^m{}_e\underline{P}^n{}_f\overline{\nabla}_n\left(\underline{P}^r{}_m\overline{\nabla}_r\ln\dot{F}\right)\right]\ .\end{aligned} \tag{4.75}$$

On each cut, one can take the pullback with $\{E^a{}_A\}$ to find

$$\begin{aligned}N^c\overline{\nabla}_c\underline{\sigma}_{AB} \overset{\mathcal{S}}{=} \mathring{S}_{AB} - \frac{1}{2}q_{AB}\mathring{S}_M{}^M + \mathcal{D}_A\left(\ln\dot{F}\right)\mathcal{D}_B\left(\ln\dot{F}\right) + \mathcal{D}_A\mathcal{D}_B\ln\dot{F}\\ - \frac{1}{2}q_{AB}\left[\mathcal{D}_M\mathcal{D}^M\ln\dot{F} + \mathcal{D}_M\left(\ln\dot{F}\right)\mathcal{D}^M\left(\ln\dot{F}\right)\right]\ ,\end{aligned} \tag{4.76}$$

where eq. (4.56) has been used and we have introduced

$$\mathring{S}_{AB} \overset{\mathcal{S}}{:=} E^a{}_AE^b{}_B\overline{S}_{ab},\ \ \mathring{S}_M{}^M \overset{\mathcal{S}}{=} \overline{S}_{mn}\overline{g}^{mn} \overset{\mathcal{S}}{=} \frac{\overline{R}}{2} \overset{\mathcal{S}}{=} \frac{\mathring{R}}{2}\ . \tag{4.77}$$

Now, let us consider the following lightlike projections –see section 2.2– of the rescaled

Weyl tensor[2],

$$ {}^{N}\!D^{\alpha\beta} \overset{\mathscr{J}}{:=} N^{\mu}N^{\nu}d_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta} = {}^{N}\!D^{ab}e^{\alpha}{}_{a}e^{\beta}{}_{b} \ , \tag{4.78} $$

$$ {}^{N}\!C^{\alpha\beta} \overset{\mathscr{J}}{:=} N^{\mu}N^{\nu}\,{}^{*}\!d_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta} \overset{\mathscr{J}}{=} {}^{N}\!C^{ab}e^{\alpha}{}_{a}e^{\beta}{}_{b} \ , \tag{4.79} $$

where

$$ {}^{*}\!d_{\alpha\beta\gamma}{}^{\delta} := \frac{1}{2}\eta_{\alpha\beta}{}^{\mu\nu}d_{\mu\nu\gamma}{}^{\delta}. \tag{4.80} $$

see section 2.2 for this kind of decomposition in general; some of the properties listed therein will be used too. Contract eq. (3.83) with N^{β}, raise the index γ and contract with $e^{\alpha}{}_{a}\omega_{\gamma}{}^{b}$ to get

$$ {}^{N}\!D_{a}{}^{b} := \overline{g}_{ma}\,{}^{N}\!D^{mb} = N^{m}\overline{\nabla}_{m}\overline{S}_{a}{}^{b} - N^{b}\overline{\nabla}_{a}\left(\pounds_{\vec{N}}f\right) \ . \tag{4.81} $$

One may lower the contravariant index with $\overline{g}_{ab}$ so that

$$ {}^{N}\!D_{ab} := \overline{g}_{mb}\,{}^{N}\!D_{a}{}^{m} = N^{m}\overline{\nabla}_{m}\overline{S}_{ab} \ . \tag{4.82} $$

Notice that ${}^{N}\!D_{ab}$ is symmetric and effectively two-dimensional $N^{m}\,{}^{N}\!D_{am} = 0$. In addition, if firstly one takes the Hodge dual of eq. (3.83) with $\eta^{\alpha\beta\gamma\delta}$ and contract once with N^{α} and the remaining two free indices with $\omega_{\alpha}{}^{a}$, then

$$ {}^{N}\!C^{ab} = \epsilon^{rsa}\overline{\nabla}_{r}\overline{S}_{s}{}^{b} \tag{4.83} $$

follows. Also, lowering an index,

$$ {}^{N}\!C^{a}{}_{b} := \overline{g}_{mb}\,{}^{N}\!C^{am} = \epsilon^{rsa}\overline{\nabla}_{r}\overline{S}_{sb} \ . \tag{4.84} $$

It will become useful to consider the component

$$ -\sqrt{2}\,{}^{N}\!\underline{C}_{a} \overset{\mathscr{J}}{:=} \overline{\ell}_{r}\,{}^{N}\!C^{r}{}_{a} = \overline{\ell}_{r}\epsilon^{mpr}\overline{\nabla}_{m}\overline{S}_{pa} \ . \tag{4.85} $$

On each cut, projecting with $E^{a}{}_{A}$, one has

$$ -\sqrt{2}\,{}^{N}\!\underline{C}_{A} \overset{\mathcal{S}}{=} \mathring{\epsilon}^{MP}\mathcal{D}_{M}\mathring{S}_{PA} \ . \tag{4.86} $$

By general properties presented in appendix D, one has ${}^{N}\!\underline{D}_{A} = \mathring{\epsilon}_{AB}\,{}^{N}\!\underline{C}^{B}$, hence

$$ -\sqrt{2}\,{}^{N}\!\underline{D}_{A} \overset{\mathcal{S}}{:=} E^{a}{}_{A}\overline{\ell}_{m}\,{}^{N}\!D^{m}{}_{a} \overset{\mathcal{S}}{=} 2\mathcal{D}_{[M}\mathring{S}_{A]}{}^{M}. \tag{4.87} $$

[2]Notation used in section 2.2 for lightlike projections associated to ${}^{+}\!k^{\alpha}$ applies here to N^{α} simply by changing a '+' by an 'N' in the upper-right indices .

From eq. (3.84), contracting twice with N^{α}, one arrives at

$$\overline{\nabla}_m {}^{N}\underline{D}^{bm} = -y_m{}^b{}_f N^m N^f, \tag{4.88}$$

which by means of the rescaled energy momentum tensor $T_{\alpha\beta}$ reads –see eq. (3.67)–

$$\overline{\nabla}_m {}^{N}\underline{D}^{bm} \overset{\mathscr{J}}{=} \varkappa N^{\mu} N^{\nu} {}_{0}T_{\mu\nu} N^b , \tag{4.89}$$

where

$$N^{\mu} N^{\nu} {}_{0}T_{\mu\nu} \overset{\mathscr{J}}{:=} \Omega^{-1} \varkappa N^{\mu} N^{\nu} T_{\mu\nu} \tag{4.90}$$

is regular at $\mathscr{J}$ because $T_{\mu\nu} N^{\mu} N^{\nu} \overset{\mathscr{J}}{=} 0$ due to eq. (3.121). Equation (4.88) may be expanded and contracted with $\overline{\ell}_b$ to get on any cut

$$\overline{\ell}_m \overline{\ell}_n N^p \overline{\nabla}_p {}^{N}\underline{D}^{nm} \overset{\mathcal{S}}{=} -\sqrt{2}\mathcal{D}_M {}^{N}\underline{D}^M - \underline{\sigma}_{AB} N^m \overline{\nabla}_m \mathring{S}^{AB} + y_m{}^b{}_f N^m \overline{\ell}_b N^f . \tag{4.91}$$

4.2 News, BMS and asymptotic energy-momentum

This section is devoted to the study of the asymptotic group of symmetries at $\mathscr{J}$, the isolation of the radiative degrees of freedom of the gravitational field and the definition of an asymptotic energy-momentum, which are closely related tasks.

4.2.1 Geroch's tensor rho and news tensor

A result by Geroch [17] gives the existence and uniqueness of a symmetric tensor field ρ_{ab} on $\mathscr{J}$ whose gauge behaviour and differential properties play a fundamental role in finding the so called 'news' tensor, N_{ab} – in the classical characterisation, the tensor field which determines the presence of outgoing gravitation radiation at $\mathscr{J}$. In section 6.2, related general results for two dimensional Riemannian manifolds are proven –see corollaries 6.2.2 and 6.2.3. Those results can be particularised for the present case, leading to Geroch's tensor. However, we take a different approach here due to the particular structure of the three-dimensional manifold $\mathscr{J}$.

Lemma 4.2.1. Let t_{ab} be any symmetric tensor field on $\mathscr{J}$, orthogonal to N^a, whose behaviour under conformal rescalings (4.25) is

$$\tilde{t}_{ab} = t_{ab} - a\frac{1}{\omega}\overline{\nabla}_a \overline{\omega}_b + \frac{2a}{\omega^2}\overline{\omega}_a\overline{\omega}_b - \frac{a}{2\omega^2}\overline{\omega}_c\overline{\omega}^c \overline{g}_{ab} \tag{4.92}$$

for some fixed constant $a \in \mathbb{R}$, where $\overline{\omega}_a \overset{\mathcal{S}}{:=} \overline{\nabla}_a \omega$. Then,

$$\tilde{\overline{\nabla}}_{[c}\tilde{t}_{a]b} = \overline{\nabla}_{[c}t_{a]b} + \frac{1}{\omega}\left(a\frac{\overline{R}}{2} - \overline{g}^{ed}t_{ed}\right)\overline{\omega}_{[c}\overline{g}_{a]b} \quad , \tag{4.93}$$

In particular, for any symmetric gauge-invariant tensor field B_{ab} on $\mathscr{I}$ orthogonal to N^a,

$$\tilde{\overline{\nabla}}_{[c}\tilde{B}_{a]b} = \overline{\nabla}_{[c}B_{a]b} - \frac{1}{\omega}B_{ed}\overline{g}^{ed}\overline{\omega}_{[c}\overline{g}_{a]b} \quad . \tag{4.94}$$

Proof. A direct calculations yields

$$\tilde{\overline{\nabla}}_{[c}\tilde{t}_{a]b} = \overline{\nabla}_{[c}t_{a]b} + \frac{1}{\omega}t_{b[c}\overline{\omega}_{a]} + \frac{1}{\omega}\overline{g}_{b[c}t_{a]e}\overline{g}^{de}\overline{\omega}_d + \frac{1}{\omega}\frac{a\overline{R}}{2}\overline{\omega}_{[c}\overline{g}_{a]b} \, . \tag{4.95}$$

Observe that the term

$$\frac{1}{\omega}t_{b[c}\overline{\omega}_{a]} + \frac{1}{\omega}\overline{g}_{b[c}t_{a]e}\overline{g}^{de}\overline{\omega}_d \tag{4.96}$$

is effectively two-dimensional (it is orthogonal to N^a). Hence, one can use the two-dimensional identity [121]

$$A_{CAE} = 2q_{E[A}A_{C]DM}q^{DM}, \text{ for any tensor such that } A_{CAE} = -A_{ACE} \tag{4.97}$$

in order to write

$$\frac{1}{\omega}t_{b[c}\overline{\omega}_{a]} + \frac{1}{\omega}\overline{g}_{b[c}t_{a]e}\overline{g}^{de}\overline{\omega}_d = -\frac{1}{\omega}t_{ed}\overline{g}^{ed}\overline{\omega}_{[c}q_{a]b} \quad , \tag{4.98}$$

arriving at the final result. For a gauge invariant tensor $a = 0$ in eq. (4.92), therefore one only has to set this value in eq. (4.93) to obtain eq. (4.94). □

Corollary 4.2.1. A symmetric gauge-invariant tensor field m_{ab} on $\mathscr{I}$, orthogonal to N^a, satisfies

$$\overline{\nabla}_{[c}\tilde{m}_{b]a} = \overline{\nabla}_{[c}m_{b]a} \tag{4.99}$$

if and only if $m_{ed}\overline{g}^{ed} = 0$.

Corollary 4.2.2 (The tensor ρ)**.** There is a unique symmetric tensor field ρ_{ab} on $\mathscr{I}$ orthogonal to N^a whose behaviour under conformal rescalings (4.25) is as in (4.92) and satisfies the equation

$$\overline{\nabla}_{[c}\rho_{a]b} = 0 \tag{4.100}$$

in any conformal frame. This tensor field must have a trace $\rho_{ed}\overline{g}^{ed} = a\overline{R}/2$ and is given in the gauge where the cuts of $\mathscr{I}$ are endowed with the round metric by $\rho_{ab} = \overline{g}_{ab}a\overline{R}/4$.

Proof. Existence is proved by noticing that $\rho_{ab} = \overline{g}_{ab}a\overline{R}/4$ fulfils $\overline{\nabla}_a\rho_{bc} = 0$ in the round metric sphere. Concerning uniqueness, notice that lemma 4.2.1 fixes the trace of ρ_{ab} to $\rho_{ed}\overline{g}^{ed} = a\overline{R}/2$, and recall the assumption that eq. (4.100) holds in any gauge. Then, if two different solutions ${}_1\rho_{ab}$ and ${}_2\rho_{ab}$ exist, $\overline{\nabla}_{[c}\left({}_1\rho_{a]b} - {}_2\rho_{a]b}\right) = 0$. However, in that case and since $\rho_{ad}N^d = 0$, the difference ${}_1\rho_{ab} - {}_2\rho_{ab}$ is traceless, Codazzi tensor on $\mathbb{S}^2$ and, as a consequence of the uniqueness of this kind of tensors [113], ${}_1\rho_{ab} - {}_2\rho_{ab} = 0$. □

Remark 4.2.1. Geroch's original tensor corresponds to $a = 1$.

Remark 4.2.2. Since these results are in essence two-dimensional, one could have taken a different path for the proof. Namely, use the general results for tensors ρ on two-dimensional Riemannian manifolds presented in [122] that will be studied in section 6.2.

Remark 4.2.3. Applying the results of [122], one also finds that the Lie derivative on a cut of the projection to that cut of ρ_{ab} along any conformal Killing vector field (CKVF) χ^A is proportional to $\mathcal{D}_A\mathcal{D}_B\mathcal{D}_C\chi^C$, and in particular vanishes for Killing vector fields (KVF).

Remark 4.2.4. Contraction of eq. (4.100) with N^a gives

$$\pounds_{\vec{N}}\rho_{ab} = N^e\overline{\nabla}_e\rho_{ab} = 0 \ . \tag{4.101}$$

Therefore, $\overline{\rho}_{ab}$ is 'constant' along the generators of $\mathscr{J}$. In particular, this feature makes ρ_{ab} invariant under the so called supertranslations –see section 4.2.

A direct calculation for determining the gauge behaviour of $\overline{S}_{ab}$ shows that

$$\tilde{\overline{S}}_{ab} = \overline{S}_{ab} - \frac{1}{\omega}\overline{\nabla}_a\overline{\omega}_b + \frac{2}{\omega^2}\overline{\omega}_a\overline{\omega}_b - \frac{1}{2\omega^2}\overline{\omega}_c\overline{\omega}^c\overline{g}_{ab} \ , \tag{4.102}$$

with $\overline{\omega}_a \overset{\mathscr{J}}{:=} \overline{\nabla}_a\omega$. Also, this can be projected to any cut $\mathcal{S}$

$$\tilde{\mathring{S}}_{AB} \overset{\mathcal{S}}{=} \mathring{S}_{AB} - \frac{1}{\omega}\mathcal{D}_A\mathring{\omega}_B + \frac{2}{\omega^2}\mathring{\omega}_A\mathring{\omega}_B - \frac{1}{2\omega^2}\mathring{\omega}_C\mathring{\omega}^C q_{AB} \ , \tag{4.103}$$

where we have used that in our gauge $N^e\overline{\omega}_e = 0$. That is, $\overline{S}_{ab}$ has the adequate gauge behaviour and trace (4.77) that imply by lemma 4.2.1

$$\overline{\nabla}_{[c}\tilde{\mathring{S}}_{a]b} = \overline{\nabla}_{[c}\mathring{S}_{a]b} \tag{4.104}$$

and allows to write the following result

Proposition 4.2.1 (News tensor)**.** The tensor field on $\mathscr{J}$

$$N_{ab} := \overline{S}_{ab} - \rho_{ab} \ , \tag{4.105}$$

is symmetric, traceless, gauge invariant, orthogonal to N^a and satisfies the gauge-invariant equation

$$\overline{\nabla}_{[a}\overline{S}_{b]c} = \overline{\nabla}_{[a}N_{b]c} \ , \tag{4.106}$$

where ρ_{ab} is the tensor field of corollary 4.2.2 (for $a = 1$). Besides, N_{ab} is unique with these properties.

Proof. The tensor field N_{ab} is symmetric, traceless, gauge invariant and orthogonal to N^a as a consequence of eqs. (4.67) and (4.102) and corollary 4.2.2. That eq. (4.106) is gauge invariant follows from corollary 4.2.1. The uniqueness of N_{ab} is a consequence of corollary 4.2.2 too and eq. (4.106). □

The tensor field N_{ab} is the *news tensor*, and can be projected on any cut $\mathcal{S}$

$$N_{AB} := \underline{E}^a{}_A \underline{E}^b{}_B N_{ab} \ , \quad q^{MN} N_{MN} = 0 \ , \tag{4.107}$$

and the same can be done with ρ_{ab}

$$\rho_{AB} := \underline{E}^a{}_A \underline{E}^b{}_B \rho_{ab} \tag{4.108}$$

Note that contraction of eq. (4.106) with N^c and $E^c{}_C E^a{}_A E^b{}_B$, respectively, yields

$$\pounds_{\vec{N}} \overline{S}_{ab} = N^c \overline{\nabla}_c \overline{S}_{ab} = N^c \overline{\nabla}_c N_{ab} = \pounds_{\vec{N}} N_{ab} \ , \quad \mathcal{D}_{[C} \mathring{S}_{A]B} \overset{\mathcal{S}}{=} \mathcal{D}_{[C} N_{A]B} \tag{4.109}$$

and also observe that in general, one has

$$\pounds_{\vec{N}} N_{ab} \neq 0 \tag{4.110}$$

and the notation

$$\dot{N}_{AB} := \underline{E}^a{}_A \underline{E}^b{}_B \pounds_{\vec{N}} N_{ab} \tag{4.111}$$

will be used. From eqs. (4.82) and (4.83) and eq. (4.109), one gets

$$\overset{N}{C}{}^a{}_b = \epsilon^{rsa} \overline{\nabla}_r N_{sb} \ , \tag{4.112}$$

$$\overset{N}{D}_{ab} = N^m \overline{\nabla}_m N_{ab} \tag{4.113}$$

and from eq. (4.85)

$$-\sqrt{2} \overset{N}{\underline{C}}_a = \overline{\ell}_r \epsilon^{mpr} \overline{\nabla}_m N_{pa} \ . \tag{4.114}$$

On each cut

$$\overset{N}{\underline{C}}{}^A{}_B :\overset{\mathcal{S}}{=} W_a{}^A E^b{}_B \overset{N}{C}{}^a{}_b \overset{\mathcal{S}}{=} \dot{N}_{BM} \mathring{\epsilon}^{AM} \ , \tag{4.115}$$

$$\overset{N}{\underline{C}}_A :\overset{\mathcal{S}}{=} E^a{}_A \overset{N}{\underline{C}}_a = -\frac{1}{\sqrt{2}} \mathring{\epsilon}^{RP} \mathcal{D}_R N_{PA} \ , \tag{4.116}$$

$$\overset{N}{\underline{D}}_{AB} :\overset{\mathcal{S}}{=} E^a{}_A E^b{}_B \overset{N}{D}_{ab} \overset{\mathcal{S}}{=} \dot{N}_{AB}, \tag{4.117}$$

$$\overset{N}{\underline{D}}_A \overset{\mathcal{S}}{=} -\frac{1}{\sqrt{2}} \mathcal{D}_M \mathring{N}_A{}^M \ , \tag{4.118}$$

where we have used eq. (4.17), eq. (4.56). Some of these formulae will be used in section 4.4.

Looking back to eq. (4.75), inserting decomposition (4.105), it follows that

$$\dot{\sigma}_{ab} := N^c\overline{\nabla}_c\underline{\sigma}_{ab} = N_{ab} + \rho_{ab} - \frac{\overline{R}}{4}\overline{g}_{ab} + \underline{P}^m{}_a\overline{\nabla}_m\left(\ln\dot{F}\right)\underline{P}^n{}_b\overline{\nabla}_n\left(\ln\dot{F}\right) - \underline{P}^m{}_{(a}\underline{P}^n{}_{b)}\overline{\nabla}_n\left(\underline{P}^r{}_m\overline{\nabla}_r\ln\dot{F}\right)$$
$$- \frac{1}{2}\overline{g}_{ab}\overline{g}^{ef}\left[\underline{P}^m{}_e\overline{\nabla}_m\left(\ln\dot{F}\right)\underline{P}^n{}_f\overline{\nabla}_n\left(\ln\dot{F}\right) - \underline{P}^m{}_e\underline{P}^n{}_f\overline{\nabla}_n\left(\underline{P}^r{}_m\overline{\nabla}_r\ln\dot{F}\right)\right] \ , \tag{4.119}$$

or on each cut

$$\dot{\sigma}_{AB} := N^c\overline{\nabla}_c\underline{\sigma}_{AB} \overset{\mathcal{S}}{=} N_{AB} + \rho_{AB} - \frac{\overline{R}}{4}q_{AB} + \mathcal{D}_A\left(\ln\dot{F}\right)\mathcal{D}_B\left(\ln\dot{F}\right) - \mathcal{D}_A\left(\mathcal{D}_B\ln\dot{F}\right)$$
$$- \frac{1}{2}q_{AB}q^{EF}\left[\mathcal{D}_E\left(\ln\dot{F}\right)\mathcal{D}_F\left(\ln\dot{F}\right) - \mathcal{D}_E\left(\mathcal{D}_F\ln\dot{F}\right)\right] \ . \tag{4.120}$$

This equation relates the news tensor to the 'time' derivative of the shear tensor $\underline{\sigma}_{ab}$. Observe that in general *only for canonically adapted foliations –eq. (4.60)– in which the gauge fixing*[3] *gives the round metric on the cuts* one obtains

$$\dot{\sigma}_{ab} = N_{ab} \ . \tag{4.121}$$

4.2.2 Symmetries and universal structure

The conformal boundary for vanishing cosmological constant presents a universal structure [17] which gives rise to an asymptotic symmetry group know as the BMS group –named after Bondi, Metzner and Sachs [21, 40, 43]–, which has been widely studied [16, 17, 48, 123] –see also [103, 115]– and recently has attracted renewed attention with proposals of generalisations and extensions [124–128]. The BMS group admits different characterisations, from coordinate-based methods, to covariant ones. We focus on the latter and, particularly, in Geroch's approach –see also [2] for more details. The asymptotic infinitesimal symmetries are those vector fields preserving the *universal structure*

Definition 4.2.1 (Universal structure)**.** Let N^a be the tangent vector field to the generators of $\mathscr{J}$ and $\overline{g}_{ab}$ its degenerate metric. Then, the universal structure of $\mathscr{J}$ consists of the conformal family of pairs

$$(\overline{g}_{ab}, N^a) \ .$$

Two pairs belong to the same conformal family if and only if $\left(\overline{g'}_{ab}, N'^a\right) = (\Psi^2\overline{g}_{ab}, \Psi^{-1}N^a)$, where Ψ is a positive function on $\mathscr{J}$.

Remark 4.2.5. An equivalent formulation is to consider the gauge-invariant object [17]

$$\overline{g}_{ab}N^cN^d \ . \tag{4.122}$$

[3] The so called 'Bondi gauge' usually refers to this kind of gauge-fixing.

and the infinitesimal symmetries are those that leave it invariant, that is

$$\pounds_{\vec{\xi}}\left(\overline{g}_{ab}N^{c}N^{d}\right)=0\ . \tag{4.123}$$

The alegbra $\mathfrak{bms}$ is characterised by the infinitesimal symmetries ξ^a defined by

$$\pounds_{\vec{\xi}}N^{a}=-\psi N^{a}\ , \tag{4.124}$$

$$\pounds_{\vec{\xi}}\overline{g}_{ab}=2\psi\overline{g}_{ab}. \tag{4.125}$$

The infinitesimal symmetries τ^a proportional to N^a,

$$\tau^{a}=\alpha\xi^{a} \tag{4.126}$$

are called *supertranslations*. They form an infinite-dimensional subalgebra $\mathfrak{t}$ of $\mathfrak{bms}$ and the group of supertranslations $\mathcal{T}$ is a Lie ideal of BMS. One has

$$\pounds_{\vec{\tau}}N^{a}=0\ , \tag{4.127}$$

$$\pounds_{\vec{\tau}}\overline{g}_{ab}=0 \tag{4.128}$$

and

$$\pounds_{\vec{N}}\alpha=0\ . \tag{4.129}$$

The resulting symmetry group BMS consists of the semidirect product [129] of the Lorentz group $\mathsf{SO}(1,3)$ with the normal subgroup of supertranslations T,

$$\mathsf{BMS}=\mathsf{T}\ltimes\mathsf{SO}(1,3)\ . \tag{4.130}$$

Geroch identified a 4-dimensional subspace of infinitesimal translations, given by those elements of $\mathfrak{t}$ satisfying

$$\overline{\nabla}_{a}\overline{\nabla}_{b}\alpha+\alpha\rho_{ab}=\frac{1}{2}\left(\overline{g}^{mn}\overline{\nabla}_{m}\overline{\nabla}_{n}\alpha+\alpha\frac{\overline{R}}{4}\right)\overline{g}_{ab}\ . \tag{4.131}$$

An interpretation of this equation is given in section 6.2. These infinitesimal symmetries enter into the definition of the Bondi-Trautman energy-momentum.

4.2.3 Asymptotic energy-momentum of the gravitational field

Any weakly asymptotically simple $\Lambda=0$ space-time features the existence of a total energy-momentum at $\mathscr{J}$ –the so called Bondi-Trautman momentum [21, 34, 115]. This four-vector field includes the energy of the gravitational field and yields the notion of energy-loss due to the presence of gravitational waves. Geroch [17] presented it as a particular case of a generalised momentum built upon a vector field associated to any supertranslation $\tau^a = \alpha N^a$ and given by –see [51] for the spin-coefficient version–

$$\underset{\tau}{\mathcal{M}}^a := \alpha\, {}^N\!D^{am}\overline{\ell}_m - 2\left(\alpha\overline{\nabla}_m\overline{\ell}_n + \overline{\ell}_m\overline{\nabla}_n\alpha\right)\overline{g}^{np}N_{pq}\overline{g}^{q[m}N^{a]}\ , \tag{4.132}$$

where $\overline{\ell}_a$ is any one-form satisfying $\overline{\ell}_m N^m = -1$. It is possible to include Geroch's approach into our formalism of foliations –section 4.1.1. Let $\overline{\ell'}_a$ be any field associated to a foliation, as in eq. (4.47), with the defining function F giving the cuts $\mathcal{S}_C$ at constant values $F = C =$ constant and lightlike extension $\overline{\ell'}_\alpha$. The two-form $\underset{\tau}{\mathbf{M}} := \underset{\tau}{\mathcal{M}}^a\epsilon_{abc}\mathrm{d}x^b\wedge\mathrm{d}x^c$ integrated over any cut $\mathcal{S}_C$ gives a charge associated to that cut and the supertranslation τ^a

$$\underset{\tau}{\mathcal{E}}\left[\mathcal{S}_C\right] := -\frac{1}{8\pi}\int_{\mathcal{S}_C}\underset{\tau}{\mathbf{M}} = -\frac{1}{8\pi}\int_{\mathcal{S}_C}\underset{\tau}{\mathcal{M}}^a\overline{\ell'}_a\mathring{\epsilon}' \tag{4.133}$$

This formula can be shown to be independent of the choice of $\overline{\ell}^a$ in eq. (4.132), thus without loss of generality, let us write $\overline{\ell'}_a = \overline{\ell}_a$. Using eq. (4.132) and introducing eq. (4.62), the charge can be rewritten as

$$\underset{\tau}{\mathcal{E}}\left[\mathcal{S}_C\right] = -\frac{1}{8\pi}\int_{\mathcal{S}_C}\left(\alpha\, {}^N\!D^{rs}\overline{\ell}_r\overline{\ell}_s + \alpha\underline{\sigma}_{rs}N^{rs}\right)\mathring{\epsilon}\ . \tag{4.134}$$

The first term contains essentially a Coulomb contribution from the gravitational field – see eq. (D.10) where ${}^N\!D^{rs}\overline{\ell}_r\overline{\ell}_s$ corresponds to D. The difference of the quantity (4.134) for any two cuts $\mathcal{S}_2$ and $\mathcal{S}_1$, with the former to the future of the latter, is derived by computing the divergence of eq. (4.132) and integrating over the three-dimensional portion $\Delta\subset\mathscr{J}$ bounded by the two cuts:

$$\underset{\tau}{\mathcal{E}}\left[\mathcal{S}_2\right] - \underset{\tau}{\mathcal{E}}\left[\mathcal{S}_1\right] = -\frac{1}{8\pi}\int_\Delta\left(\alpha y_m{}^f{}_p N^m\overline{\ell}_f N^p + \alpha\overline{S}_{rs}N^{rs} + N^{rs}\overline{\nabla}_r\overline{\nabla}_s\alpha\right)\epsilon\ . \tag{4.135}$$

Indeed, it is possible to differentiate along the foliation to obtain the infinitesimal change in the charge (4.134) on a cut $\mathcal{S}_C$,

$$\begin{aligned}\frac{\mathrm{d}\underset{\tau}{\mathcal{E}}\left[\mathcal{S}_C\right]}{\mathrm{d}C} = &-\frac{1}{8\pi}\int_{\mathcal{S}_C}\frac{\alpha}{\dot{F}}\Big[N^{PQ}\mathring{S}_{PQ} + N^{PQ}\mathcal{D}_P\mathcal{D}_Q\ln\dot{F} - 2\mathcal{D}_P\left(N^{QP}\right)\mathcal{D}_Q\ln\dot{F}\\ &+\mathcal{D}_Q\mathcal{D}_P N^{QP} + y_m{}^f{}_p N^m\overline{\ell}_f N^p + N^{AB}\mathcal{D}_A\left(\ln\dot{F}\right)\mathcal{D}_B\left(\ln\dot{F}\right)\Big]\mathring{\epsilon}\end{aligned} \tag{4.136}$$

where eqs. (4.91), (4.117), (4.118) and (4.120) have been used. Since the cuts are topological spheres, total divergences integrate out and one can simplify the expression above to reach the nice formula

$$\frac{\mathrm{d}\underset{\tau}{\mathcal{E}}\left[\mathcal{S}_C\right]}{\mathrm{d}C} = -\frac{1}{8\pi}\int_{\mathcal{S}_C}\frac{1}{\dot{F}}\left[\alpha N^{AB}\mathring{S}_{AB} + N^{AB}\mathcal{D}_A\mathcal{D}_B\alpha + \alpha\varkappa N^\mu N^\nu\, {}_0T_{\mu\nu}\right]\mathring{\epsilon}\ , \tag{4.137}$$

where the matter fields enter the integral through (see eqs. (3.67) and (4.90))

$$y_m{}^f{}_p N^m \overline{\ell}_f N^p \overset{\mathscr{J}}{=} \varkappa N^\mu N^\nu \; {}_0T_{\mu\nu} \; . \tag{4.138}$$

If one chooses τ^a as an infinitesimal translation –i.e., α satisfying eq. (4.131)–, (4.134) gives the *Bondi-Trautman energy-momentum*. In that case, eq. (4.135) and eq. (4.137) yield

$$\underset{\tau}{\mathcal{E}}\left[\mathcal{S}_2\right] - \underset{\tau}{\mathcal{E}}\left[\mathcal{S}_1\right] = -\frac{1}{8\pi}\int_\Delta \alpha\left(N^{qp}N_{qp} + \varkappa N^\mu N^\nu \; {}_0T_{\mu\nu}\right)\epsilon \; . \tag{4.139}$$

$$\frac{\mathrm{d}\,\underset{\tau}{\mathcal{E}}\left[\mathcal{S}_C\right]}{\mathrm{d}C} = -\frac{1}{8\pi}\int_{\mathcal{S}_C} \frac{\alpha}{\dot{F}}\left(N^{PQ}N_{PQ} + \varkappa N^\mu N^\nu \; {}_0T_{\mu\nu}\right)\mathring{\epsilon} \; . \tag{4.140}$$

The second one is the *Bondi-Trautman energy-momentum-loss formula.* Both eq. (4.135) and eq. (4.140) feature the same property: in the absence of the matter-field contribution $N^\mu N^\nu \; {}_0T_{\mu\nu}$, a non-vanishing news tensor diminishes the total energy-momentum. To our knowledge, it is the first time eq. (4.140) is presented including the matter term and the factors associated to the choice of foliation $\dot{F}$ and translation α; in the literature, either just eq. (4.139) is given [17], even without the matter contribution (see [130] for a recent derivation), or just eq. (4.140) is considered, typically without the matter term and the factor $\dot{F}$ corresponding to the choice of foliation [16, 22, 44, 100]. Moreover, when α is set to a constant –equivalently, a 'time' translation is selected– to get the total energy-loss, the dimensional analysis of eq. (4.140) becomes obscure –see section 4.4.2 for the discussion of the units including α and $\dot{F}$. For later convenience, let us define the energy-momentum loss associated to gravitational waves only (i.e., excluding the matter term)

$$\frac{\mathrm{d}\,\overset{G}{\underset{\tau}{\mathcal{E}}}\left[\mathcal{S}_C\right]}{\mathrm{d}C} = -\frac{1}{8\pi}\int_{\mathcal{S}_C} \frac{\alpha}{\dot{F}} N^{PQ}N_{PQ}\,\mathring{\epsilon} \; . \tag{4.141}$$

Observe that eq. (4.140) is the general energy-momentum loss, whereas the commonly presented energy-loss formulae involving the square of the Lie derivative along N^a (4.120) of the shear tensor (4.62) arise by making all or some of the following elections: $\alpha = 1$, a canonical foliation (4.60), a round-metric gauge and the absence of the asymptotic matter term (4.90).

4.3 Asymptotic propagation of physical fields and the peeling property revisited

We deal now with the behaviour of physical fields when they are parallelly transported along null geodesics. The outcome of this process when applied to the physical Weyl tensor, typically receives the name of *peeling* property or behaviour [17, 40, 100, 103, 115]. We adopt Geroch's approach and refine it to define an endomorphism at the tangent space of every point of $\mathscr{I}^+$, represented by the endpoint in $\left(M, g_{\alpha\beta}\right)$ of the chosen future-pointing null geodesic.

Let $\gamma(\lambda)$ be a curve parametrised by $\lambda \in [-1, 0]$, with one endpoint at $p_0 \in \mathscr{I}^+$ (corresponding to $\lambda_0 := \lambda\Big|_{p_0} = 0$) and the past endpoint at $p_1 \in \hat{M}$ (with $\lambda_1 := \lambda\Big|_{p_1} = -1$). Points belonging to γ corresponding to fixed values $\lambda = \lambda_i$ will be labelled by p_i,

$$\begin{aligned} \gamma(\lambda) : \ [-1, 0] &\longrightarrow M \\ \lambda &\longrightarrow p \ . \end{aligned}$$

Denote the tangent vector field to the curve by ℓ^α and choose the parametrisation such that

$$\ell^\mu N_\mu = \frac{\mathrm{d}\Omega}{\mathrm{d}\lambda} \overset{\mathscr{I}^+}{=} -1 \ . \tag{4.142}$$

At first order around $\lambda_0 = 0$, $\Omega \approx -\lambda$. Observe that we do not require at this stage ℓ^α to be lightlike, though we have chosen it to be future-pointing. Next, denote by

$$\begin{aligned} &{t_\alpha}^\beta(\lambda_i, \lambda_j) \ , \text{ the parallel propagator w.r.t. } {\Gamma^a}_{bc} \ , \\ &{\hat{t}_\alpha}{}^\beta(\lambda_i, \lambda_j) \ , \text{ the parallel propagator w.r.t. } {\hat{\Gamma}^a}{}_{bc} \ , \end{aligned}$$

such that given any one-form v_α defined at p_j, the result of parallel-transporting it along ℓ^α from p_j to p_i results on the new one-form $\underset{*}{v}_\alpha(\lambda_i)$ at p_i given by

$$\underset{*}{v}_\alpha(\lambda_i) = {t_\alpha}^\mu(\lambda_i, \lambda_j)\, v_\mu(\lambda_j) \ , \tag{4.143}$$

and introduce the notation

$$\underset{*}{v}^\alpha(\lambda_i) = g^{\alpha\mu}(\lambda_i)\, \underset{*}{v}_\mu(\lambda_i) \ . \tag{4.144}$$

Observe that indices α and μ in this relation belong to different tangent spaces. The propagator ${t_\alpha}^\beta$ is a 'bi-tensor' [131] which is defined by the differential equation

$$\frac{\mathrm{d}{t_\alpha}^\beta(\lambda, \lambda_j)}{\mathrm{d}\lambda} = \ell^\rho {\Gamma^\mu}_{\alpha\rho}(\lambda)\, {t_\mu}^\beta(\lambda, \lambda_j) \ , \tag{4.145}$$

$$\frac{\mathrm{d}{\hat{t}_\alpha}{}^\beta(\lambda, \lambda_j)}{\mathrm{d}\lambda} = \ell^\rho {\hat{\Gamma}^\mu}{}_{\alpha\rho}(\lambda)\, {\hat{t}_\mu}{}^\beta(\lambda, \lambda_j) \ , \tag{4.146}$$

with 'initial' condition

$${t_\alpha}^\beta(\lambda_j, \lambda_j) = \delta^\alpha_\beta \ , \quad {\hat{t}_\alpha}{}^\beta(\lambda_j, \lambda_j) = \delta^\alpha_\beta \ , \tag{4.147}$$

and satisfies

$${t_\alpha}^\mu(\lambda, \lambda_i)\, {t_\mu}^\beta(\lambda_i, \lambda) = \delta^\beta_\alpha \ , \quad {t_\alpha}^\mu(\lambda_j, \lambda_i) = {t^\mu}_\alpha(\lambda_i, \lambda_j) \tag{4.148}$$

where

$$t^{\mu}{}_{\alpha}\left(\lambda_i,\lambda_j\right) := g^{\mu\sigma}\left(\lambda_i,\lambda_i\right) g_{\nu\alpha}\left(\lambda_j,\lambda_j\right) t_{\sigma}{}^{\nu}\left(\lambda_i,\lambda_j\right) \ . \tag{4.149}$$

The main idea [17] is to perform 3 different parallel transports of any covariant tensor field:

1. From p_0 to $p\left(\lambda\right)$ with $t_{\alpha}{}^{\beta}\left(\lambda,\lambda_0\right)$,

2. From $p(\lambda)$ to p_1 with $\hat{t}_{\alpha}{}^{\beta}\left(\lambda_1,\lambda\right)$,

3. From p_1 to p_0 with $t_{\alpha}{}^{\beta}\left(\lambda_0,\lambda_1\right)$.

That is, a transport along γ back and forth, departing from $\mathscr{J}^+$ and interchanging the conformal connection by the physical one for a stretch of γ. If one chains one operation after another, the result is an endomorphism on the co-tangent space at p_0:

$$\overset{\gamma}{\underset{\lambda}{}}L_{\alpha}{}^{\beta} := t_{\alpha}{}^{\mu}\left(\lambda_0,\lambda_1\right) \hat{t}_{\mu}{}^{\rho}\left(\lambda_1,\lambda\right) t_{\rho}{}^{\beta}\left(\lambda,\lambda_0\right). \tag{4.150}$$

The upper and lower indices on the left-hand side of $\overset{\gamma}{\underset{\lambda}{}}L_{\beta}{}^{\alpha}$ indicate its dependence on the curve γ and the point $p(\lambda)$. Since the notation may become cumbersome, we drop this two labels in most of the calculations and recover them only when doing so happens to be convenient. Since $L_{\beta}{}^{\alpha}$ is a tensor at p_0, acting on covariant objects, we introduce the notation

$$\overleftarrow{T}_{\alpha_1\ldots\alpha_r} := L_{\alpha_1}{}^{\mu_1}\ldots L_{\alpha_r}{}^{\mu_r} T_{\mu_1\ldots\mu_r}\,. \tag{4.151}$$

The action on the metric at p_0 gives

$$\overleftarrow{g}_{\alpha\beta} = \underset{\lambda}{\Xi}^2 g_{\alpha\beta}, \tag{4.152}$$

where one uses that the connections are metric-compatible with respect to $g_{\alpha\beta}$ and $\hat{g}_{\alpha\beta}$, respectively, and introduces the definition

$$\underset{\lambda}{\Xi} := \frac{\Omega(\lambda)}{\Omega(\lambda_1)} \tag{4.153}$$

–we will drop the label on the left-hand side. Equation (4.152) implies that the endomorphism $L_{\alpha}{}^{\beta}$ preserves the null cone (and obviously also the future orientation) and therefore it is proportional to a Lorentz transformation at p_0. Recalling the first of eq. (4.148), it is easy to verify that

$$\overset{-1}{L}_{\beta}{}^{\alpha} := t_{\beta}{}^{\mu}\left(\lambda_0,\lambda\right) \hat{t}_{\mu}{}^{\rho}\left(\lambda,\lambda_1\right) t_{\rho}{}^{\alpha}\left(\lambda_1,\lambda_0\right) \tag{4.154}$$

is the inverse operator, that is,

$$L_{\alpha}{}^{\rho}\,\overset{-1}{L}_{\rho}{}^{\beta} = \delta_{\alpha}^{\beta}\ . \tag{4.155}$$

The version of $L_\beta{}^\alpha$ that acts on contravariant fields is defined as

$$ {}^{\gamma}_{\lambda}\tilde{L}^{\beta}{}_{\alpha} := t_\alpha{}^\mu\left(\lambda_0,\lambda\right)\hat{t}_\mu{}^\rho\left(\lambda,\lambda_1\right)t_\rho{}^\beta\left(\lambda_1,\lambda_0\right), \tag{4.156} $$

and a simple calculation using the second of eq. (4.148) shows that

$$ {}^{-1}L_\alpha{}^\beta = \tilde{L}^\beta{}_\alpha = \frac{1}{\Xi^2}L^\beta{}_\alpha \ , \tag{4.157} $$

where $L^\beta{}_\alpha = g^{\beta\mu}L_\mu{}^\nu g_{\nu\alpha}$. Therefore, taking into account eq. (4.157) one can work only with $L_\beta{}^\alpha$. Some useful relations are

$$ \overleftarrow{\eta}_{\alpha\beta\gamma\delta} = \Xi^4\eta_{\alpha\beta\gamma\delta} \ , \tag{4.158} $$

$$ {}^{-1}L_\rho{}^{\alpha}\,{}^{-1}L_\sigma{}^\beta g^{\rho\sigma} = \frac{1}{\Xi^2}g^{\alpha\beta} \ , \tag{4.159} $$

$$ \overleftarrow{v}_\mu\overleftarrow{w}^\mu = \Xi^2 v_\mu w^\mu \quad \forall\ v^\alpha,\ w^\alpha \ , \tag{4.160} $$

where

$$ \overleftarrow{w}^\alpha := g^{\alpha\mu}\overleftarrow{w}_\mu \ . \tag{4.161} $$

The next task to be addressed is to find the explicit form of the operator $L_\alpha{}^\beta$. We believe that this could be done for arbitrary curves, however the most relevant case – and easiest to deal with– is when ℓ^α is geodesic and lightlike with λ an affine parameter,

$$ \ell_\mu\ell^\mu = 0 \ , \quad \ell^\rho\nabla_\rho\ell_\alpha = 0. \tag{4.162} $$

We assume this restriction from now on. Observe that for null geodesics (e.g. [103]) one can always write

$$ \hat{\ell}_\alpha = \ell_\alpha \tag{4.163} $$

where $\hat{\ell}_\alpha$ is lightlike and geodesic with respect to the physical metric. This fact allows to deduce the action of $L_\alpha{}^\beta$ on ℓ_α at p_0,

$$ \overleftarrow{\ell}_\alpha = \ell_\alpha \ . \tag{4.164} $$

Observe that $L_\alpha{}^\beta$ has at most 16 independent components. It can be expressed in the bases $\{-N_\alpha, -\ell_\alpha, q_\alpha, r_\alpha\}$ and $\{\ell^\alpha, N^\alpha, q^\alpha, r^\alpha\}$, with q_α and r_α arbitrary unit one-forms orthogonal to N^α and ℓ^α at p_0, as

$$ \begin{aligned} {}^{\gamma}_{\lambda}L_\beta{}^\alpha = {}& {}^{\gamma}_{\lambda}AN^\alpha N_\beta + {}^{\gamma}_{\lambda}BN^\alpha\ell_\beta + {}^{\gamma}_{\lambda}C\ell^\alpha\ell_\beta + {}^{\gamma}_{\lambda}D\ell^\alpha N_\beta + {}^{\gamma}_{\lambda}Fq^\alpha r_\beta + {}^{\gamma}_{\lambda}Gr^\alpha q_\beta + {}^{\gamma}_{\lambda}Hq^\alpha q_\beta + {}^{\gamma}_{\lambda}Ir^\alpha r_\beta \\ & + N^\alpha\,{}^{\gamma}_{\lambda}\mathring{v}_\beta + \ell^\alpha\,{}^{\gamma}_{\lambda}\mathring{w}_\beta + {}^{\gamma}_{\lambda}\mathring{x}^\alpha N_\beta + {}^{\gamma}_{\lambda}\mathring{y}^\alpha\ell_\beta \ . \end{aligned} \tag{4.165} $$

The dependence on the curve γ and the point λ is contained in the 8 scalars and 4 2-dimensional vector fields correspondingly labelled in the formula above. By eqs. (4.157) and (4.163) direct simplifications take place:

$$A = 0\ , \quad B = -1\ , \quad D = -\Xi^2\ , \quad \mathring{v}_\beta = 0\ , \quad \mathring{x}^\alpha = 0\ . \tag{4.166}$$

Projecting eq. (4.152) with the elements of the bases, one arrives at the expression

$$L_\beta{}^\alpha = -N^\alpha \ell_\beta + C\ell^\alpha \ell_\beta - \Xi^2 \ell^\alpha N_\beta + F\left(q^\alpha r_\beta - r^\alpha q_\beta\right) + H\mathring{P}^\alpha{}_\beta + \ell^\alpha \mathring{w}_\beta + \mathring{y}^\alpha \mathring{\ell}_\beta, \tag{4.167}$$

with

$$2\Xi^2 C = -\Xi^2 \mathring{y}_\mu \mathring{y}^\mu = -\mathring{w}_\mu \mathring{w}^\mu\ , \quad F^2 + H^2 = \Xi^2 \tag{4.168}$$

and

$$0 = -\mathring{w}_r - F\mathring{y}_q - H\mathring{y}_r\ , \tag{4.169}$$
$$0 = -\mathring{w}_q + F\mathring{y}_r - H\mathring{y}_q\ , \tag{4.170}$$
$$0 = -\Xi^2\mathring{y}_q - F\mathring{w}_r - H\mathring{w}_q\ , \tag{4.171}$$
$$0 = -\Xi^2\mathring{y}_r + F\mathring{w}_q - H\mathring{w}_r\ . \tag{4.172}$$

By construction, one has

$${}_{\lambda_1}L_\beta{}^\alpha = \delta^\beta_\alpha, \tag{4.173}$$

which implies

$${}_{\lambda_1}C = {}_{\lambda_1}F = 0\ , \quad {}_{\lambda_1}H = 1\ , \quad {}_{\lambda_1}\mathring{w}_\beta = 0\ , \quad {}_{\lambda_1}\mathring{y}_\beta = 0\ . \tag{4.174}$$

Since $L_\alpha{}^\beta$ depends on λ, it makes sense to search for a differential equation for it. To that purpose, notice that another version of eq. (4.145) can be written for $t_\alpha{}^\beta(\lambda, \lambda_j)$ by using eq. (4.148),

$$\frac{\mathrm{d}t_\alpha{}^\beta(\lambda_j, \lambda)}{\mathrm{d}\lambda} = -k^\rho \Gamma^\beta{}_{\rho\mu} t_\alpha{}^\mu(\lambda_j, \lambda)\ . \tag{4.175}$$

The final differential formula for $L_\alpha{}^\beta$ reads

$$\frac{\mathrm{d}L_\beta{}^\alpha}{\mathrm{d}\lambda} = \frac{1}{\Omega(\lambda)} L_\beta{}^\mu \Lambda_\mu{}^\alpha \tag{4.176}$$

where

$$\Lambda_\beta{}^\alpha = {}_\lambda\Lambda_\beta{}^\alpha := \Omega(\lambda)\,\ell^\sigma(\lambda)\,\gamma^\nu{}_{\mu\sigma}(\lambda)\,t_\nu{}^\alpha(\lambda, \lambda_0)\,t_\beta{}^\mu(\lambda_0, \lambda) \tag{4.177}$$
$$= \frac{\mathrm{d}\Omega}{\mathrm{d}\lambda}\delta^\alpha_\beta + \ell^\alpha \underset{*}{N}_\beta - \ell_\beta \underset{*}{N}^\alpha\ , \tag{4.178}$$

with

$$\underset{*}{N}_\beta = {}_\lambda\underset{*}{N}_\beta := t_\beta{}^\mu(\lambda_0, \lambda)\,N_\mu(\lambda)\ , \quad {}_{\lambda_0}\underset{*}{N}_\alpha = N_\alpha\ , \tag{4.179}$$

and $\gamma^{\alpha}{}_{\beta\gamma}$ giving the difference between the unphysical and physical connection –see the formulae of conformal transformations in appendix C. Observe that $\Lambda_{\beta}{}^{\alpha}$ is a tensor at p_0 that depends on λ. Equation (4.176) is a Fuchsian system with a regular singular point at $\lambda = \lambda_0$ –recall that $\Omega(\lambda_0) = 0$. In components, one has the following non-trivial equations

$$\frac{\mathrm{d}F}{\mathrm{d}\lambda} = \frac{F}{\Omega(\lambda)}\frac{\mathrm{d}\Omega}{\mathrm{d}\lambda} , \tag{4.180}$$

$$\frac{\mathrm{d}H}{\mathrm{d}\lambda} = \frac{H}{\Omega(\lambda)}\frac{\mathrm{d}\Omega}{\mathrm{d}\lambda} , \tag{4.181}$$

$$\frac{\mathrm{d}C}{\mathrm{d}\lambda} = \frac{1}{\Omega(\lambda)}\left[2C\frac{\mathrm{d}\Omega}{\mathrm{d}\lambda} + \underset{*}{N}_{\rho}\mathring{y}^{\rho}\right] , \tag{4.182}$$

$$\frac{\mathrm{d}\mathring{y}_r}{\mathrm{d}\lambda} = \frac{1}{\Omega(\lambda)}\left[\mathring{y}_r\frac{\mathrm{d}\Omega}{\mathrm{d}\lambda} - \mathring{r}_{\rho}\underset{*}{N}^{\rho}\right] , \tag{4.183}$$

$$\frac{\mathrm{d}\mathring{y}_q}{\mathrm{d}\lambda} = \frac{1}{\Omega(\lambda)}\left[\mathring{y}_q\frac{\mathrm{d}\Omega}{\mathrm{d}\lambda} - \mathring{q}_{\rho}\underset{*}{N}^{\rho}\right] , \tag{4.184}$$

$$\frac{\mathrm{d}\mathring{w}_r}{\mathrm{d}\lambda} = \frac{1}{\Omega(\lambda)}\left[F\underset{*}{N}_{\mu}q^{\mu} + H\underset{*}{N}_{\mu}r^{\mu} + 2\mathring{w}_r\frac{\mathrm{d}\Omega}{\mathrm{d}\lambda}\right] , \tag{4.185}$$

$$\frac{\mathrm{d}\mathring{w}_q}{\mathrm{d}\lambda} = \frac{1}{\Omega(\lambda)}\left[H\underset{*}{N}_{\mu}q^{\mu} - F\underset{*}{N}_{\mu}r^{\mu} + 2\mathring{w}_q\frac{\mathrm{d}\Omega}{\mathrm{d}\lambda}\right] . \tag{4.186}$$

Using eq. (4.173) as initial condition, eqs. (4.180) and (4.181) yield

$$\overset{\gamma}{\underset{\lambda}{}}F = 0 , \quad \overset{\gamma}{\underset{\lambda}{}}H = \underset{\lambda}{\Xi} , \tag{4.187}$$

and then, from eqs. (4.169) and (4.170),

$$\mathring{w}_{\alpha} = -\Xi\mathring{y}_{\alpha} . \tag{4.188}$$

Since C is determined by $\mathring{y}_{\alpha}$ through eq. (4.168), it only remains to solve for $\mathring{y}_{\alpha}$. Equations (4.183) and (4.184) are two uncoupled linear ODEs, whose solution with the initial condition (4.174) reads

$$\mathring{y}^{\alpha} = -\Omega(\lambda)\int_{\lambda_1}^{\lambda}\frac{1}{\Omega^2(\lambda')}\,\underset{\lambda'}{}\underset{*}{N}^{\alpha}\mathrm{d}\lambda' . \tag{4.189}$$

This solution is smooth in the limit $\lambda = \lambda_0$. Taking this into account, if one multiplies eq. (4.182) by Ω and evaluates at λ_0, it follows that

$$-2\underset{\lambda_0}{}C = \underset{\lambda_0}{}\mathring{y}_{\mu}\,\underset{\lambda_0}{}\mathring{y}^{\mu} = 0 . \tag{4.190}$$

All in all, the final expression of $L_{\beta}{}^{\alpha}$ is

$$L_{\beta}{}^{\alpha} = -N^{\alpha}\ell_{\beta} - \frac{1}{2}\mathring{y}^2\ell^{\alpha}\ell_{\beta} - \Xi^2\ell^{\alpha}N_{\beta} + \Xi\mathring{P}^{\alpha}{}_{\beta} - \Xi\ell^{\alpha}\mathring{y}_{\beta} + \mathring{y}^{\alpha}\mathring{\ell}_{\beta}, \tag{4.191}$$

with $\mathring{y}^2 = \mathring{y}_\mu \mathring{y}^\mu$ and $\mathring{y}_\alpha$ determined by eq. (4.189), depending on the choice of curve through $\underset{*}{N}^\alpha$ –given in eq. (4.179)– and on λ. Consider the decomposition

$$\underset{\lambda}{\overset{\gamma}{L}}{}_\beta{}^\alpha = \underset{\lambda}{\overset{\gamma}{p}}{}_\beta{}^\mu \underset{\lambda}{\overset{\gamma}{K}}{}_\mu{}^\alpha \, , \tag{4.192}$$

$$\underset{\lambda}{\overset{\gamma}{p}}{}_\beta{}^\mu := -N^\mu \ell_\beta - \underset{\lambda}{\overset{\gamma}{\Xi}}{}^2 \ell^\mu N_\beta + \underset{\lambda}{\overset{\gamma}{\Xi}} \mathring{P}^\mu{}_\beta \, , \tag{4.193}$$

$$\underset{\lambda}{\overset{\gamma}{K}}{}_\mu{}^\alpha := \delta^\alpha_\mu - \frac{1}{2}\mathring{y}^2 \ell^\alpha \ell_\mu - \ell^\alpha \mathring{y}_\mu + \mathring{y}^\alpha \ell_\mu \, . \tag{4.194}$$

The interest of this decomposition is that $\underset{\lambda}{\overset{\gamma}{K}}{}_\mu{}^\alpha$ carries mostly details of the curve γ, whereas $\underset{\lambda}{\overset{\gamma}{p}}{}_\beta{}^\mu$ contains essentially powers of Ω and no information about the curve γ: just the value of Ω at the chosen point p_1 –see (4.153).

We are mainly interested in the *asymptotic behaviour* of $L_\beta{}^\alpha$, i.e. when $\lambda \to \lambda_0$. It is

Weyl-tensor candidate	Non-vanishing ${}^{(a)}\Psi_i$	${}^{(a)}\psi_i$ when $\lambda = \lambda_0$	PND
${}^{(4)}C_{\alpha\beta\gamma\delta}$	${}^{(4)}\psi_4$	$\Omega(\lambda_1)^{-2}\phi_4$	$(\ell^\alpha, \ell^\alpha, \ell^\alpha, \ell^\alpha)$
${}^{(3)}C_{\alpha\beta\gamma\delta}$	${}^{(3)}\psi_3$	$\Omega(\lambda_1)^{-3}\phi_3$	$(\ell^\alpha, \ell^\alpha, \ell^\alpha, N^\alpha)$
${}^{(2)}C_{\alpha\beta\gamma\delta}$	${}^{(2)}\psi_2$	$\Omega(\lambda_1)^{-4}\phi_2$	$(\ell^\alpha, \ell^\alpha, N^\alpha, N^\alpha)$
${}^{(1)}C_{\alpha\beta\gamma\delta}$	${}^{(1)}\psi_1$	$\Omega(\lambda_1)^{-5}\phi_1$	$(\ell^\alpha, N^\alpha, N^\alpha, N^\alpha)$
${}^{(0)}C_{\alpha\beta\gamma\delta}$	${}^{(0)}\psi_0$	$\Omega(\lambda_1)^{-6}\phi_0$	$(N^\alpha, N^\alpha, N^\alpha, N^\alpha)$

Table 4.1: The asymptotic propagation of the physical Weyl tensor (4.198) is composed by the five terms listed above. Each one has the symmetries of a Weyl tensor and one non-vanishing Weyl scalar which in the limit $\lambda \to \lambda_0 = 0$ coincides up to a multiplicative constant with one of the scalars of the rescaled Weyl tensor $d_{\alpha\beta\gamma}{}^\delta$. The repeated principal null directions are listed in the last column.

very interesting the fact that details on the choice of γ become irrelevant at zeroth order in this regime because

$$\underset{\lambda_0}{\overset{\gamma}{K}}{}_\beta{}^\alpha = \delta^\alpha_\beta \, . \tag{4.195}$$

In other words, *the asymptotic behaviour is ruled by* $\underset{\lambda}{\overset{\gamma}{p}}{}_\beta{}^\mu$ which we come to call *the asymptotic propagator*. In order to derive this behaviour for *any* physical field, one has to follow the next steps:

1) Propagate the physical field from $p(\lambda)$ to p_0 using $t_\beta{}^\alpha(\lambda_0, \lambda)$ –hence defining a new tensor of the same type at p_0 on $\mathscr{J}^+$.

2) Apply to the covariant version of the new tensor at p_0 as many copies of $L_\beta{}^\alpha$ as free indices has the field.

3) Expand the expression obtained previously in terms of λ near $\lambda_0 = 0$.

Note that this program 'compares' the parallel propagation of the physical tensor field in $\hat{M}$ from the point p_1 to $p(\lambda)$, with the propagation in M between the two points. Expanding λ around the limit value $\lambda_0 = 0$, one takes this comparison towards infinity of $\hat{M}$.

The canonical example is the application to the physical Weyl tensor. Consider $\hat{C}_{\alpha\beta\gamma\delta}(\lambda)$, i.e. the physical Weyl tensor at $p(\lambda)$. Now, take step 1) to define a tensor at p_0

$$\underset{*}{\hat{C}}_{\alpha\beta\gamma\delta} = \frac{1}{\Omega^2(\lambda)} \underset{*}{C}_{\alpha\beta\gamma\delta} = \frac{1}{\Omega(\lambda)} \underset{*}{d}_{\alpha\beta\gamma\delta} \,. \tag{4.196}$$

Notice that

$$\underset{*}{d}_{\alpha\beta\gamma\delta}\Big|_{\lambda=\lambda_0=0} = d_{\alpha\beta\gamma\delta}\Big|_{p_0} \,, \tag{4.197}$$

thus $\underset{*}{\hat{C}}_{\alpha\beta\gamma\delta}$ contains a pole of order 1 in the limit $\lambda \to \lambda_0 = 0$. Nevertheless, this divergence is overcome in step 2),

$$\underset{*}{\overleftarrow{\hat{C}}}_{\alpha\beta\gamma\delta} = \Omega\,{}^{(4)}C_{\alpha\beta\gamma\delta} + \Omega^2\,{}^{(3)}C_{\alpha\beta\gamma\delta} + \Omega^3\,{}^{(2)}C_{\alpha\beta\gamma\delta} + \Omega^4\,{}^{(1)}C_{\alpha\beta\gamma\delta} + \Omega^5\,{}^{(0)}C_{\alpha\beta\gamma\delta} \,, \tag{4.198}$$

where ${}^{(a)}C_{\alpha\beta\gamma\delta}$ with $a = 0, 1, 2, 3, 4$ are Weyl-tensor candidates, regular in the limit to $\lambda_0 = 0$ and with algebraic properties listed in table 4.1. They depend on λ and, assuming that Ω admits a Taylor expansion around $\lambda_0 = 0$, we write them near $\lambda_0 = 0$ as

$${}^{(a)}C_{\alpha\beta\gamma\delta} = {}^{(a,0)}C_{\alpha\beta\gamma\delta} + \sum_{i=1}^{\infty} {}^{(a,i)}C_{\alpha\beta\gamma\delta}\lambda^i \,. \tag{4.199}$$

Their explicit expressions are expressed as

$$ {}^{(4)}C_{\alpha\beta\gamma\delta} = \frac{4}{\Omega^2(\lambda_1)} \underset{*}{d}_{\tau\omega\chi\eta} K_\mu{}^\tau K_\nu{}^\omega K_\rho{}^\chi K_\sigma{}^\eta N^\nu N^\sigma \mathring{P}_{[\alpha}{}^\mu \ell_{\beta]} \mathring{P}_{[\gamma}{}^\rho \ell_{\delta]} \ , \tag{4.200} $$

$$ \begin{aligned} {}^{(3)}C_{\alpha\beta\gamma\delta} &= \frac{1}{\Omega^3(\lambda_1)} \underset{*}{d}_{\tau\omega\chi\eta} K_\mu{}^\tau K_\nu{}^\omega K_\rho{}^\chi K_\sigma{}^\eta \Big[-4\mathring{P}^\mu{}_{[\alpha} N^\nu \ell_{\beta]} N^\rho \ell_{[\gamma} N_{\delta]} \ell^\sigma - 4 N^\mu \ell^\nu \ell_{[\alpha} N_{\beta]} \mathring{P}^\rho{}_{[\gamma} \ell_{\rho]} N^\sigma \\ &\quad - 4\mathring{P}^\mu{}_{[\alpha} \mathring{P}^\nu{}_{\beta]} N^\sigma \ell_{[\delta} \mathring{P}^\rho{}_{\gamma]} - 4 N^\mu \ell_{[\alpha} \mathring{P}^\nu{}_{\beta]} \mathring{P}^\rho{}_{[\gamma} \mathring{P}^\sigma{}_{\delta]} \Big] \ , \end{aligned} \tag{4.201} $$

$$ \begin{aligned} {}^{(2)}C_{\alpha\beta\gamma\delta} &= \frac{1}{\Omega^4(\lambda_1)} \underset{*}{d}_{\tau\omega\chi\eta} K_\mu{}^\tau K_\nu{}^\omega K_\rho{}^\chi K_\sigma{}^\eta \Big[\mathring{P}^\mu{}_\alpha \mathring{P}^\nu{}_\beta \mathring{P}^\rho{}_\gamma \mathring{P}^\sigma{}_\delta - \mathring{P}^\mu{}_\alpha \mathring{P}^\nu{}_\beta \left(\mathring{P}^\rho{}_\gamma N_\delta + \mathring{P}^\sigma{}_\delta N_\gamma \right) - \\ &\quad - \mathring{P}^\rho{}_\gamma \mathring{P}^\sigma{}_\delta \left(\mathring{P}^\mu{}_\alpha \ell^\nu \mathring{N}_\beta + \mathring{P}^\nu{}_\beta N^\mu \ell_\alpha \right) + 4\ell^\mu N^\nu \ell^\rho N^\sigma N_{[\alpha} \ell_{\beta]} N_{[\gamma} \ell_{\delta]} + 4\mathring{P}^\mu{}_{[\alpha} N_{\beta]} \mathring{P}^\rho{}_{[\gamma} \ell_{\delta]} N^\sigma \ell^\nu \\ &\quad + 4\mathring{P}^\mu{}_{[\alpha} \ell_{\beta]} \mathring{P}^\rho{}_{[\gamma} N_{\rho]} \ell^\sigma N^\nu + 4\mathring{P}^\mu{}_{[\alpha} \mathring{P}^\nu{}_{\beta]} N_{[\gamma} \ell_{\delta]} \ell^\rho N^\sigma + 4\mathring{P}^\rho{}_{[\gamma} \mathring{P}^\sigma{}_{\delta]} N_{[\alpha} \ell_{\beta]} \ell^\mu N^\nu \Big] \ , \end{aligned} \tag{4.202} $$

$$ \begin{aligned} {}^{(1)}C_{\alpha\beta\gamma\delta} &= \frac{1}{\Omega^5(\lambda_1)} \underset{*}{d}_{\tau\omega\chi\eta} K_\mu{}^\tau K_\nu{}^\omega K_\rho{}^\chi K_\sigma{}^\eta \Big[-4\mathring{P}^\mu{}_{[\alpha} \ell^\nu N_{\beta]} \ell^\rho N_{[\gamma} \ell_{\delta]} N^\sigma - 4\ell^\mu N^\nu N_{[\alpha} \ell_{\beta]} \mathring{P}^\rho{}_{[\gamma} N_{\rho]} \ell^\sigma \\ &\quad - 4\mathring{P}^\mu{}_{[\alpha} \mathring{P}^\nu{}_{\beta]} \ell^\sigma N_{[\delta} \mathring{P}^\rho{}_{\gamma]} - 4\ell^\nu N_{[\beta} \mathring{P}^\mu{}_{\alpha]} \mathring{P}^\rho{}_{[\gamma} \mathring{P}^\sigma{}_{\delta]} \Big] \ , \end{aligned} \tag{4.203} $$

$$ {}^{(0)}C_{\alpha\beta\gamma\delta} = \frac{4}{\Omega^6(\lambda_1)} \underset{*}{d}_{\tau\omega\chi\eta} K_\mu{}^\tau K_\nu{}^\omega K_\rho{}^\chi K_\sigma{}^\eta \ell^\nu \ell^\sigma \mathring{P}_{[\alpha}{}^\mu N_{\beta]} \mathring{P}_{[\gamma}{}^\rho N_{\delta]} \ . \tag{4.204} $$

Observe that the leading-order term of eq. (4.200) reads

$$ {}^{(4,0)}C_{\alpha\beta\gamma\delta} = \frac{4}{\Omega^2(\lambda_1)} d_{\mu\nu\rho\sigma} N^\nu N^\sigma \mathring{P}_{[\alpha}{}^\mu \ell_{\beta]} \mathring{P}_{[\gamma}{}^\rho \ell_{\delta]} \tag{4.205} $$

and is determined by the rescaled Weyl tensor $d_{\alpha\beta\gamma}{}^\delta$ projected to a Petrov-type N Weyl-candidate tensor. Now one can perform step 3), finally arriving at the next result:

Theorem 1 (Peeling of the Weyl tensor)**.** *Let $\left(M, g_{\alpha\beta}\right)$ be a conformal completion of a physical space-time with $\Lambda = 0$ as presented on page 22 and let γ be a lightlike geodesic with affine parameter λ and tangent vector field ℓ^α as in eq. (4.142). Also, let one end point p_0 ($\lambda = \lambda_0 = 0$) of γ be at $\mathscr{J}^+$ and the other one, p_1 ($\lambda = \lambda_1 = -1$), in $\hat{M}$. Then, the asymptotic behaviour of the physical Weyl tensor $\hat{C}_{\alpha\beta\gamma\delta}$ along γ follows by application of steps 1) to 3) on page 64 and reads*

$$ \underset{*}{\overleftarrow{\hat{C}}}_{\alpha\beta\gamma\delta} = \lambda\, {}^{(N)}d_{\alpha\beta\gamma\delta} + \lambda^2\, {}^{(III)}e_{\alpha\beta\gamma\delta} + \lambda^3\, {}^{(II/D)}f_{\alpha\beta\gamma\delta} + \lambda^4\, {}^{(I)}g_{\alpha\beta\gamma\delta} + \lambda^5\, {}^{(I)}h_{\alpha\beta\gamma\delta} + \mathcal{O}\left(\lambda^6\right) \ , \tag{4.206} $$

near $\lambda = \lambda_0 = 0$, where the tensors

$$ {}^{(N)}d_{\alpha\beta\gamma\delta} := -\, {}^{(4,0)}C_{\alpha\beta\gamma\delta} \ , \tag{4.207} $$

$$
{}^{(III)}e_{\alpha\beta\gamma\delta} := {}^{(3,0)}C_{\alpha\beta\gamma\delta} + \frac{\Omega_2}{2}\,{}^{(4,0)}C_{\alpha\beta\gamma\delta} - {}^{(4,1)}C_{\alpha\beta\gamma\delta} \ , \tag{4.208}
$$

$$
\begin{aligned}
{}^{(II/D)}f_{\alpha\beta\gamma\delta} :=& - {}^{(2,0)}C_{\alpha\beta\gamma\delta} - \Omega_2\,{}^{(3,0)}C_{\alpha\beta\gamma\delta} + \frac{\Omega_3}{6}\,{}^{(4,0)}C_{\alpha\beta\gamma\delta} + {}^{(3,1)}C_{\alpha\beta\gamma\delta} - {}^{(4,2)}C_{\alpha\beta\gamma\delta} \\
&+ \frac{\Omega_2}{2}\,{}^{(4,1)}C_{\alpha\beta\gamma\delta} \ ,
\end{aligned} \tag{4.209}
$$

$$
\begin{aligned}
{}^{(I)}g_{\alpha\beta\gamma\delta} :=& {}^{(1,0)}C_{\alpha\beta\gamma\delta} + \frac{3\Omega_2}{2}\,{}^{(2,0)}C_{\alpha\beta\gamma\delta} + \left(\frac{\Omega_2^2}{4} - \frac{\Omega_3}{3}\right){}^{(3,0)}C_{\alpha\beta\gamma\delta} + \frac{\Omega_4}{4!}\,{}^{(4,0)}C_{\alpha\beta\gamma\delta} \\
&- {}^{(2,1)}C_{\alpha\beta\gamma\delta} + {}^{(3,2)}C_{\alpha\beta\gamma\delta} - {}^{(4,3)}C_{\alpha\beta\gamma\delta} + \frac{\Omega_2}{2}\,{}^{(4,2)}C_{\alpha\beta\gamma\delta} - \Omega_2\,{}^{(3,1)}C_{\alpha\beta\gamma\delta} \\
&+ \frac{\Omega_3}{6}\,{}^{(4,1)}C_{\alpha\beta\gamma\delta} \ ,
\end{aligned} \tag{4.210}
$$

$$
\begin{aligned}
{}^{(I)}h_{\alpha\beta\gamma\delta} :=& - {}^{(0,0)}C_{\alpha\beta\gamma\delta} - 2\Omega_2\,{}^{(1,0)}C_{\alpha\beta\gamma\delta} + \left(\frac{\Omega_3}{2} - \frac{3}{4}\Omega_2^2\right){}^{(2,0)}C_{\alpha\beta\gamma\delta} + \left(\frac{1}{6}\Omega_2\Omega_3 - \frac{\Omega_4}{12}\right){}^{(3,0)}C_{\alpha\beta\gamma\delta} \\
&+ \frac{\Omega_5}{5!}\,{}^{(4,0)}C_{\alpha\beta\gamma\delta} + {}^{(1,1)}C_{\alpha\beta\gamma\delta} - {}^{(2,2)}C_{\alpha\beta\gamma\delta} + {}^{(3,3)}C_{\alpha\beta\gamma\delta} - {}^{(4,4)}C_{\alpha\beta\gamma\delta} \\
&+ \frac{3}{2}\Omega_2\,{}^{(2,1)}C_{\alpha\beta\gamma\delta} - \Omega_2\,{}^{(3,2)}C_{\alpha\beta\gamma\delta} + \frac{\Omega_2}{2}\,{}^{(4,3)}C_{\alpha\beta\gamma\delta} + \frac{\Omega_2^2}{4}\,{}^{(3,1)}C_{\alpha\beta\gamma\delta} \\
&- \frac{\Omega_3}{3}\,{}^{(3,1)}C_{\alpha\beta\gamma\delta} + \frac{\Omega_3}{6}\,{}^{(4,2)}C_{\alpha\beta\gamma\delta} + \frac{\Omega_4}{4!}\,{}^{(4,1)}C_{\alpha\beta\gamma\delta} \ .
\end{aligned} \tag{4.211}
$$

are Weyl-tensor candidates labelled with their Petrov type, respectively; Ω_i*, with* $i = 1, 2, 3, 4, 5$*, is the* i*-th derivative of* Ω *w.r.t.* λ *evaluated at* $\lambda = \lambda_0 = 0$*, and* ${}^{(a)}C_{\alpha\beta\gamma\delta}$*, with* $a = 0, 1, 2, 3, 4$*, are the Weyl-tensor candidates of table 4.1 each one having one non-vanishing Weyl scalar* ${}^{(a)}\Psi_a$ *in the tetrad containing* ℓ^α *and* N^α*.*

Proof. The asymptotic propagation along γ of the physical Weyl tensor is given in eq. (4.198). Then, one expands around $\lambda_0 = 0$ and rearranges the terms by powers of λ. The algebraic structure of the first 5 terms of eqs. (4.207) to (4.211) follows from the properties listed in table 4.1. □

Remark 4.3.1. The Weyl-tensor candidates of eqs. (4.207) to (4.211) have the algebraic structure specified in table 4.2. Notice that though this constitutes the so called *peeling* property, the present derivation is purely geometric, showing neatly that we derive the behaviour of the physical field (the Weyl tensor in this case) as it approaches $\mathscr{J}$ along null geodesics, thereby providing a solid foundation for the so-called peeling behaviour. Notice, further, that once this construction has been performed, it can be applied to any physical field whatsoever by just following the steps 1) to 3) on page 64 and using the explicit form of $L_\beta{}^\alpha$ eq. (4.191).

Remark 4.3.2. The Weyl scalars of the first three elements in eq. (4.206) have the following expressions:

$$\eta_4 = -\frac{1}{\Omega^2(\lambda_1)}\phi_4 \ , \qquad \tau_4 = -\frac{\Omega_3}{6\Omega^2(\lambda_1)}\phi_4 - {}^{(4,2)}\psi_4 + \frac{\Omega_2}{2}{}^{(4,1)}\psi_4 \ , \tag{4.212}$$

$$\chi_4 = \frac{\Omega_2}{2\Omega^2(\lambda_1)}\phi_4 - {}^{(4,1)}\psi_4 \ , \qquad \tau_3 = -\frac{\Omega_2}{\Omega^3(\lambda_1)}\phi_3 + {}^{(3,1)}\psi_3 \ , \tag{4.213}$$

$$\chi_3 = \frac{1}{\Omega^3(\lambda_1)}\phi_3 \ , \qquad \tau_2 = -\frac{1}{\Omega^4(\lambda_1)}\phi_2 \ , \tag{4.214}$$

where ϕ_i with $i = 2, 3, 4$ are the scalars of the rescaled Weyl tensor $d_{\alpha\beta\gamma}{}^{\delta}$ and ${}^{(a,i)}\psi_i$ are the scalars corresponding to the tensors ${}^{(a,i)}C_{\alpha\beta\gamma\delta}$ of eq. (4.199).

Weyl-tensor candidate	Non-vanishing scalars	degeneracy of ℓ^α as PND
${}^{(N)}d_{\alpha\beta\gamma\delta}$	η_4	4
${}^{(III)}e_{\alpha\beta\gamma\delta}$	$\chi_3\ \chi_4$	3
${}^{(II/D)}f_{\alpha\beta\gamma\delta}$	$\tau_4\ \tau_3\ \tau_2$	2
${}^{(I)}g_{\alpha\beta\gamma\delta}$	$\nu_4\ \nu_3\ \nu_2\ \nu_1$	1
${}^{(I)}h_{\alpha\beta\gamma\delta}$	$\mu_4\ \mu_3\ \mu_2\ \mu_1\ \mu_0$	0

Table 4.2: The vector ℓ^α, tangent to γ, is a principal null direction of the first four terms in the asymptotic propagation of the physical Weyl tensor. The degree of degeneracy decreases towards higher order terms; this effect is commonly referred to as the *peeling* property of the Weyl tensor.

4.4 Asymptotic radiant supermomentum

As it has been exposed in the introduction, we give a characterisation of the gravitational radiation grounded on the rescaled version (3.86) of the Bel-Robinson tensor. One constructs the *asymptotic radiant supermomentum* as

$$\mathcal{Q}^\alpha := -N^\mu N^\nu N^\rho \mathcal{D}^\alpha{}_{\mu\nu\rho} \ . \tag{4.215}$$

The definition and description of general *radiant supermomenta* are studied in section 2.3; the following fundamental properties were presented in [76]

i) $\mathcal{Q}^\mu$ is lightlike $\mathcal{Q}^\mu \mathcal{Q}_\mu \overset{\mathscr{J}}{=} 0$ and future pointing at $\mathscr{J}$, which follows from the causal character of N^α and known properties of superenergy tensors [85, 93].

ii) Under gauge transformations it changes as

$$\mathcal{Q}^\alpha \to \omega^{-7}\left(\mathcal{Q}^\alpha - 3\frac{\Omega}{\omega}\mathcal{D}^\alpha{}_{\beta\rho\tau}N^\beta N^\rho \nabla^\tau \omega\right) + \mathcal{O}(\Omega^2). \tag{4.216}$$

iii) It is divergence-free at $\mathscr{J}$, independently of the matter content,

$$\nabla_\mu \mathcal{Q}^\mu \overset{\mathscr{J}}{=} 0 \ . \tag{4.217}$$

The last property is easily verified by noting (3.87), so that, recalling eq. (3.80), one can write

$$\nabla_\mu \mathcal{Q}^\mu \overset{\mathscr{J}}{=} 4N^\alpha\, {}^N\!D^{\beta\gamma} y_{\alpha\beta\gamma} \overset{\mathscr{J}}{=} {}^N\!\underline{D}^{BC}\underline{E}^\beta{}_B \underline{E}^\gamma{}_C N^\alpha y_{\alpha\beta\gamma} + \sqrt{2}\,{}^N\!\underline{D}^B \underline{E}^\beta{}_B N^\alpha N^\gamma y_{\alpha\beta\gamma} \ , \tag{4.218}$$

where in the last equality we have exploited the fact that ${}^N\!D^{\alpha\beta} = {}^N\!D^{ab} e^\alpha{}_a e^\beta{}_b$ and expanded in the bases $\{N^a, \underline{E}^\alpha{}_A\}$, $\left\{-\bar{\ell}_\alpha, \underline{W}_\alpha{}^A\right\}$. From eq. (3.67), taking into account eq. (3.121), it follows that

$$\underline{E}^\beta{}_B \underline{E}^\gamma{}_C N^\alpha y_{\alpha\beta\gamma} \overset{\mathscr{J}}{=} \frac{1}{2}\varkappa \Omega^{-1} \underline{q}_{BC} N^\mu N^\nu T_{\mu\nu} \ , \tag{4.219}$$

$$\underline{E}^\beta{}_B N^\alpha N^\gamma y_{\alpha\beta\gamma} \overset{\mathscr{J}}{=} 0 \ . \tag{4.220}$$

By properties listed in appendix D , ${}^N\!\underline{D}^{BC}\underline{q}_{BC} = 0$ and then property iii) follows[4].

The asymptotic radiant supermomentum $\mathcal{Q}^\alpha$ is geometrically well defined, as it is built only with the generators of $\mathscr{J}$ and the rescaled Weyl tensor $d_{\alpha\beta\gamma}{}^\delta$. Moreover, it has a good gauge behaviour at $\mathscr{J}$, $\tilde{\mathcal{Q}}^\alpha = \omega^{-7}\mathcal{Q}^\alpha$. These facts, together with the close relation with the intrinsic fields on $\mathscr{J}$ exhibited by the rescaled Weyl tensor –see section 4.1.2–, suggests a link between $\mathcal{Q}^\alpha$ and the news tensor N_{ab} of eq. (4.107). To show that this is the case, first decompose the asymptotic radiant supermomentum as

$$\mathcal{Q}^\alpha \overset{\mathscr{J}}{=} \mathcal{W}\bar{\ell}^\alpha + \overline{\mathcal{Q}}^\alpha = \mathcal{W}\bar{\ell}^\alpha + \overline{\mathcal{Q}}^a e^\alpha{}_a \quad , \tag{4.221}$$

where $\bar{\ell}_\alpha = \omega_\alpha{}^a \bar{\ell}_a$ is a lightlike field at $\mathscr{J}$ associated to a foliation as in eq. (4.47) whose

[4] In [76], property iii) was presented in a less general situation. As we have shown, $\mathcal{Q}^\alpha$ is divergence-free at $\mathscr{J}$ independently of the matter content.

restriction on each cut gives the ℓ_α of eq. (4.12). The quantity

$$\mathcal{W} := -N_\mu \mathcal{Q}^\mu \geq 0 \tag{4.222}$$

is the *asymptotic radiant superenergy* and the vector field

$$\overline{\mathcal{Q}}^a := \mathcal{Z} N^a + \underline{\mathcal{Q}}^A \underline{E}^a{}_A \quad \text{with} \quad \mathcal{Z} := -\overline{\ell}_\mu \mathcal{Q}^\mu \geq 0 \tag{4.223}$$

is the *asymptotic radiant super-Poynting* [76]. Observe that $\mathcal{W}$ is invariant under the choice of $\overline{\ell}_\alpha$, whereas $\mathcal{Z}$ and $\underline{\mathcal{Q}}^A$ depend on the choice of foliation –one can consider this decomposition on a single cut $\mathcal{S}$ only, and then these quantities depend on the choice of that cut. From the general formulae of section 2.3, the relation between these quantities and the lightlike projections of $d_{\alpha\beta\gamma}{}^\delta$ is

$$\mathcal{W} = 2\,{}^N\!\underline{C}_{AB}\,{}^N\!\underline{C}^{AB} = 2\,{}^N\!\underline{D}_{AB}\,{}^N\!\underline{D}^{AB} \geq 0, \tag{4.224}$$

$$\mathcal{Z} = 4\,{}^N\!\underline{C}_A\,{}^N\!\underline{C}^A = 4\,{}^N\!\underline{D}_A\,{}^N\!\underline{D}^A \geq 0, \tag{4.225}$$

$$\underline{\mathcal{Q}}^A = 4\sqrt{2}\,{}^N\!\underline{C}_P\,{}^N\!\underline{C}^{AP}\;. \tag{4.226}$$

Then, eqs. (4.115) to (4.118) bring forth the connection between the asymptotic radiant supermomentum and the news tensor

$$\mathcal{W} = 2\dot{N}^{RT}\dot{N}_{RT} \geq 0 \quad , \tag{4.227}$$

$$\mathcal{Z} \overset{\mathcal{S}}{=} 2\mathcal{D}_R N^R{}_T \mathcal{D}_M N^{MT} \geq 0 \quad , \tag{4.228}$$

$$\underline{\mathcal{Q}}^A \overset{\mathcal{S}}{=} -4\dot{N}^{MA}\mathcal{D}_E N^E{}_M \quad . \tag{4.229}$$

4.4.1 Radiation condition

In [76] a new criterion to determine the presence of radiation at $\mathscr{J}$ escaping from the space-time was presented. The criterion holds in the $\Lambda > 0$ case [75] too and is analysed in chapter 5. It translates into the following results

Theorem 2 (Radiation condition on a cut)**.** *There is no gravitational radiation on a given cut $\mathcal{S} \subset \mathscr{J}$ if and only if the radiant super-Poynting $\overline{\mathcal{Q}}^a$ vanishes on that cut:*

$$N_{AB} = 0 \quad \Longleftrightarrow \quad \overline{\mathcal{Q}}^a \overset{\mathcal{S}}{=} 0 \quad (\Longleftrightarrow \quad \mathcal{Z} = 0).$$

Proof. Consider equation (4.228). Since the right-hand side is a square, it follows that $\mathcal{Z} = 0 \Longleftrightarrow \mathcal{D}_{[A}N_{B]C} = 0$. Using now property iii) on page 15 this happens if and only if ${}^N\!\mathcal{Q}^a = 0$. But $\mathcal{D}_{[A}N_{B]C} = 0$ —which is equivalent to $D_A N^A{}_B = 0$— states that N_{AB} is a symmetric and traceless Codazzi tensor on the compact 2-dimensional $\mathcal{S}$, and then it necessarily vanishes (e.g. [113] and references therein). Equivalently, N_{AB} is a traceless symmetric divergence-free tensor on the closed $\mathcal{S}$, which implies that $N_{AB} = 0$. Hence $N_{AB} = 0 \Longleftrightarrow {}^{N}\!\mathcal{Q}^{a} = 0$ on $\mathcal{S}$. $\square$

Remark 4.4.1. Equivalently: there is no gravitational radiation on a given cut $\mathcal{S} \subset \mathscr{J}$ if and only if the radiant supermomentum is orthogonal to $\mathcal{S}$ everywhere and not collinear with N^{α}, i. e.,

$$\mathcal{Q}^{\alpha} \overset{\mathcal{S}}{=} \mathcal{W}\bar{\ell}^{\alpha} \Longleftrightarrow N_{AB} \overset{\mathcal{S}}{=} 0 \tag{4.230}$$

Remark 4.4.2. The topology of $\mathscr{J}$ plays a key role in the proof. If the cuts do not have $\mathbb{S}^2$-topology,

$$\mathcal{D}_M N_B{}^M \overset{\mathcal{S}}{=} 0 \implies \overline{\mathcal{Q}}^{a} \overset{\mathcal{S}}{=} 0 \ ,$$

even if $N_{AB} \neq 0$. In any case, this does not pose a problem when considering portions of $\mathscr{J}$, instead of single cuts –see remark 4.4.4.

Theorem 3 (No radiation on Δ)**.** *There is no gravitational radiation on the open portion $\Delta \subset \mathscr{J}$ with the same topology of $\mathscr{J}$ if and only if the radiant supermomentum $\mathcal{Q}^{\alpha}$ vanishes on Δ:*

$$N_{ab} \overset{\Delta}{=} 0 \quad \Longleftrightarrow \quad \mathcal{Q}^{\alpha} \overset{\Delta}{=} 0.$$

Proof. According to remark 4.4.1 of theorem 2, absence of radiation on Δ requires that $\mathcal{Q}^{\alpha} \overset{\mathcal{S}}{=} \mathcal{W}\ell^{\alpha}$ on *every possible* $\mathcal{S}$ included in Δ. But this is only possible if $\mathcal{Q}^{\alpha} \overset{\Delta}{=} 0$. Another route to derive this result is to note that $N_{AB} = 0$ on every cut within Δ, and thus $N_{ab} \overset{\Delta}{=} 0$. In particular $\pounds_{\vec{n}} N_{ab} \overset{\Delta}{=} 0$ so that $\dot{N}_{AB}$ vanishes too at any cut within Δ. $\square$

Remark 4.4.3. The two following re-statements are equivalent to that of theorem 3:

- No gravitational radiation on $\Delta \subset \mathscr{J} \Longleftrightarrow \mathcal{Q}^{\alpha}$ is orthogonal to all surfaces within Δ.
- No gravitational radiation on $\Delta \subset \mathscr{J} \Longleftrightarrow N^{\alpha}\Big|_{\Delta}$ is a principal null vector of $d_{\alpha\beta\gamma}{}^{\delta}\Big|_{\Delta}$.

The first point follows by remark 4.4.1, particularising to any possible cut within Δ – hence, implying that $\mathcal{Q}^{\alpha} \overset{\Delta}{=} 0$. The second statement follows by lemma 2.3.2.

Remark 4.4.4. Regarding remark 4.4.2, it may be the case that even if one foliates Δ by topological non-spheres, a different choice of foliation gives topological-$\mathbb{S}^2$ cuts. Hence, theorem 2 applies to those new cuts within Δ. It may be also the case that a foliation by topological-$\mathbb{S}^2$ cuts of a given Δ is not possible –as it happens in the C-metric [106]–, hence the situation described in remark 4.4.2 has to be considered. However, if that is the case, theorem 3 requires the whole supermomentum $\mathcal{Q}^{\alpha}$, and not just $\overline{\mathcal{Q}}^{a}$, to vanish, which involves $\dot{N}_{AB}$ as well –see eq. (4.227). Therefore, even if $\mathcal{D}_C N_A{}^C \overset{\mathcal{S}}{=} 0$ on every cut $\mathcal{S} \subset \Delta$ of a given foliation, the criterion still detects gravitational radiation whenever $\dot{N}_{AB} \overset{\Delta}{\neq} 0$. In principle, one could also choose a different interval Δ' that can be foliated by topological spheres and such that $\Delta' \cap \Delta \neq \emptyset$, and then apply theorem 2 and theorem 3 to the region $\Delta' \cap \Delta \subset \mathscr{J}$ to determine the presence of radiation there.

4.4.2 Balance law

It is possible to write a balance law describing the outgoing superenergy flux and the news tensor. Begin by considering a connected portion $\Delta \subset \mathscr{J}$, with $\mathbb{R} \times \mathbb{S}^2$ topology. Let it be bounded by two (non-intersecting) cuts, $\mathcal{S}_1$ and $\mathcal{S}_2$, the latter to the future of the former, and orthogonal lightlike vector fields (other than N^α) $\underset{1}{\ell}^\alpha$ and $\underset{2}{\ell}^\alpha$ as in eq. (4.12), respectively. Consider any lightlike field $\overline{\ell}_\alpha = \omega_\alpha{}^a \overline{\ell}_a$ in Δ with the properties of eq. (4.47), such that

$$\overline{\ell}_\alpha \overset{\mathcal{S}_1}{=} \underset{1}{\ell}_\alpha \; , \quad \overline{\ell}_\alpha \overset{\mathcal{S}_2}{=} \underset{2}{\ell}_\alpha \; . \tag{4.231}$$

Equation (4.217) decomposes as

$$\overline{\ell}^\mu \nabla_\mu \mathcal{W} + \mathcal{W} \Psi^m{}_m \overset{\Delta}{=} -\overline{\nabla}_m \overline{\mathcal{Q}}^m \; , \tag{4.232}$$

where eqs. (4.21) and (4.22) were used. Using the quantities and notation introduced in section 4.1.1, integration of eq. (4.232) leads to a Gauss-law formula

$$\int_\Delta \left(\overline{\ell}^\mu \nabla_\mu \mathcal{W} + \mathcal{W} \psi^m{}_m \right) \epsilon = \Phi\left[\mathcal{S}_2\right] - \Phi\left[\mathcal{S}_1\right] \; , \tag{4.233}$$

where $\Phi\left[\mathcal{S}\right]$ is the *radiant superenergy density flux*, defined as

$$\Phi\left[\mathcal{S}\right] := \int_{\mathcal{S}} \mathcal{Z} \mathring{\epsilon} \geq 0 \; , \quad \Phi\left[\mathcal{S}\right] = 0 \Longleftrightarrow N_{AB} \overset{\mathcal{S}}{=} 0 \; . \tag{4.234}$$

Equation (4.233) shows that *the change of the asymptotic radiant superenergy density $\mathcal{W}$ along any outgoing lightlike direction $\overline{\ell}^\alpha$ in a volume Δ is balanced by the flux of radiant superenergy density on the boundary of* Δ –constituted by the two cuts $\mathcal{S}_{1,2}$. Let us remark that this formula is valid in the presence of arbitrary matter fields –with the general assumption iv) on page 22. In other words, eq. (4.233) contains purely geometric terms. The choice of $\overline{\ell}^\alpha$ does not change eq. (4.233), as the difference between one choice and another can be checked to be a total divergence that integrates out [76]. Moreover, eq. (4.232) is gauge invariant. After some manipulation of the integrand and using eq. (4.228), the radiant superenergy density flux reads

$$\Phi\left[\mathcal{S}\right] = \int_{\mathcal{S}} N_{RS} \left(2K N^{RS} - \mathcal{D}_M \mathcal{D}^M N^{RS} \right) \mathring{\epsilon} \; , \tag{4.235}$$

where K denotes the Gaussian curvature of the cuts. Observe that, although it does not manifest itself explicitly so, the integral on the right-hand side of eq. (4.235) is positive. It shows that the flux of radiant superenergy is indeed associated to the presence of gravitational waves and sourced, ultimately, by the news tensor –as one could already expect from eq. (4.227). The first term on the right-hand side of eq. (4.235) reminds us of the energy-momentum loss due to gravitational waves of eq. (4.141). Without loss of generality , one can consider a foliation containing $\mathcal{S}$ and select the function F that appears in eq. (4.141) such that it fulfills eq. (4.59) at least on $\mathcal{S}$. If one does so, it is always possible to choose the conformal gauge in order to set

$$K\frac{\dot{F}}{\alpha} = \text{constant} \ , \quad \text{at } \mathcal{S}_{1,2} \ . \tag{4.236}$$

Under such gauge choice, eq. (4.235) reads

$$\Phi\left[\mathcal{S}\right] = -\left[16\pi K\frac{\dot{F}}{\alpha}\frac{\mathrm{d}\overset{G}{\underset{\tau}{\mathcal{E}}}\left[\mathcal{S}_C\right]}{\mathrm{d}C} + \int_{\mathcal{S}} N_{RS}\mathcal{D}_M\mathcal{D}^M N^{RS}\mathring{\epsilon}\right] \ , \tag{4.237}$$

and eq. (4.233) can be rewritten as

$$\int_\Delta \left(\bar{\ell}^\mu \nabla_\mu \mathcal{W} + \mathcal{W}\psi^m{}_m\right)\epsilon = -\left[16\pi K\frac{\dot{F}}{\alpha}\frac{\mathrm{d}\overset{G}{\underset{\tau}{\mathcal{E}}}\left[\mathcal{S}_C\right]}{\mathrm{d}C} + \int_{\mathcal{S}} N_{RS}\mathcal{D}_M\mathcal{D}^M N^{RS}\mathring{\epsilon}\right]\Bigg|_{\mathcal{S}_1}^{\mathcal{S}_2} \ . \tag{4.238}$$

The interpretation of this formulae is essentially the same as eq. (4.233). Even so, let us point out that for fixed $\mathcal{S}_{1,2}$, the change in the radiant superenergy density in the volume Δ depends only on the initial and final evaluation of the news tensor N_{ab}, i.e., on $N_{ab}\Big|_{\mathcal{S}_{1,2}}$. In a way, the integral on the left-hand side of eqs. (4.233) and (4.238) measures the failure of the system to recover its initial state. From another point of view, consider a gravitational system that is initially in equilibrium in the sense of having

$$N_{AB}\Big|_{\mathcal{S}_1} = 0 \ . \tag{4.239}$$

Then, the rate of change in the Bondi-Trautman energy at a later retarded time, i.e., on $\mathcal{S}_2$, can be expressed as the change of $\mathcal{W}$ in the volume Δ plus an additional term whose interpretation is not clear to us and that vanishes if and only if[5] so does N_{AB}

$$\frac{\mathrm{d}\overset{G}{\underset{\tau}{\mathcal{E}}}\left[\mathcal{S}_2\right]}{\mathrm{d}C} = -\frac{\alpha}{K\dot{F}16\pi}\left[\int_\Delta \left(\bar{\ell}^\mu \nabla_\mu \mathcal{W} + \mathcal{W}\psi^m{}_m\right)\epsilon + \int_{\mathcal{S}_2} N_{RS}\mathcal{D}_M\mathcal{D}^M N^{RS}\mathring{\epsilon}\right] \ . \tag{4.240}$$

As a final remark, notice that when eq. (4.239) holds, eq. (4.234) implies

$$\int_\Delta \left(\bar{\ell}^\mu \nabla_\mu \mathcal{W} + \mathcal{W}\psi^m{}_m\right)\epsilon = 0 \iff N_{AB}\Big|_{\mathcal{S}_2} = 0 \ , \tag{4.241}$$

and that $N_{AB}\Big|_{\mathcal{S}_{1,2}} = 0$ is a reasonable initial and final condition for any physical system that at first is in equilibrium, then undergoes a change that takes it out of equilibrium and finally settles down.

[5] Double implication holds true whenever $\mathcal{S}$ has $\mathbb{S}^2$-topology.

As a last remark, observe that eq. (4.238) allows us to perform a quick check of the physical units[6]: $[\underset{\tau}{\overset{G}{\mathcal{E}}}\left[\mathcal{S}_C\right] = ML^2T^{-2}$ so that $[\mathrm{d}\,\underset{\tau}{\overset{G}{\mathcal{E}}}\left[\mathcal{S}_C\right]/\mathrm{d}C] = ML^2T^{-2}[C]^{-1}$. As $[\varphi] = [C]L^{-2}$ and taking into account $[\alpha] = L$ the right-hand side of (4.238) has dimensions of $[K\varphi/\alpha]ML^2T^{-2}[C]^{-1} = ML^{-3}T^{-2}$. Concerning the left-hand side, using that $[L^\mu] = L$ and that $[\mathcal{W}] = \left[\mathcal{T}_{\alpha\beta\gamma\delta}\right]L^{-4}$, we need to know the units of the volume integral on $\mathscr{J}$ but, according to (4.7), these are $[\boldsymbol{\epsilon}] = L^4$. Hence, $\left[\mathcal{D}_{\alpha\beta\gamma\delta}\right] = MT^{-2}L^{-3}$ and the physical units of the Bel-Robinson tensor are

$$\left[\mathcal{T}_{\alpha\beta\gamma\delta}\right] = MT^{-2}L^{-3} \quad . \tag{4.242}$$

4.4.3 Alignment of supermomenta and the peeling property of the BR-tensor

The tools presented in section 4.3, namely the asymptotic propagation of fields along null geodesics, can be applied to the physical Bel-Robinson tensor

$$\hat{\mathcal{T}}_{\alpha\beta\gamma\delta} := \hat{C}_{\alpha\mu\gamma}{}^{\nu}\hat{C}_{\delta\nu\beta}{}^{\mu} + {}^{*}\hat{C}_{\alpha\mu\gamma}{}^{\nu}\,{}^{*}\hat{C}_{\delta\nu\beta}{}^{\mu} \,. \tag{4.243}$$

Consider this tensor field at point $p(\lambda)$ and parallel propagate it along the curve γ defined as in section 4.3 to p_0. This process defines a new tensor at p_0 which we denote by $\underset{*}{\hat{\mathcal{T}}}_{\alpha\beta\gamma\delta}$. Application of $L_{\beta}{}^{\alpha}$ gives

$$\underset{*}{\overset{\leftarrow}{\hat{\mathcal{T}}}}_{\alpha\beta\gamma\delta} := \Omega^4\left(\lambda_1\right)\left(\underset{*}{\overset{\leftarrow}{\hat{C}}}_{\alpha\mu\gamma}{}^{\nu}\underset{*}{\overset{\leftarrow}{\hat{C}}}_{\delta\nu\beta}{}^{\mu} + {}^{*}\underset{*}{\overset{\leftarrow}{\hat{C}}}_{\alpha\mu\gamma}{}^{\nu}\,{}^{*}\underset{*}{\overset{\leftarrow}{\hat{C}}}_{\delta\nu\beta}{}^{\mu}\right) \,, \tag{4.244}$$

where $\underset{*}{\overset{\leftarrow}{\hat{C}}}_{\alpha\mu\gamma}{}^{\nu}$ and ${}^{*}\underset{*}{\overset{\leftarrow}{\hat{C}}}_{\delta\nu\beta}{}^{\mu}$ are the asymptotic propagated physical Weyl tensor (4.198) and its Hodge dual, respectively. In order to arrive at eq. (4.244) one has to use eqs. (4.148), (4.158) and (4.160). Then, it is possible to derive the *peeling* property of the Bel-Robinson tensor,

Theorem 4 (Peeling of the Bel-Robinson tensor)**.** *Let $\left(M, g_{\alpha\beta}\right)$ be a conformal completion of a physical space-time with $\Lambda = 0$ as presented on page 22 and let γ be a lightlike geodesic with affine parameter λ and tangent vector field ℓ^α as in eq. (4.142). Also, let one end point p_0 ($\lambda = \lambda_0 = 0$) of γ be at $\mathscr{J}$ and the other one, p_1 ($\lambda = \lambda_1 = -1$), in $\hat{M}$. Then, the asymptotic behaviour of the physical Bel-Robinson tensor $\hat{\mathcal{T}}_{\alpha\beta\gamma\delta}$ along γ follows by*

[6]We use the notation $[P]$ to denote the *physical* units of any object P; our choice is that the conformal factors Ω and ω are dimensionless.

application of steps 1) to 3) on page 64 and reads

$$\begin{aligned}\frac{1}{\Omega^4(\lambda_1)}\overset{\leftarrow}{\hat{\mathcal{T}}}_{*\alpha\beta\gamma\delta} &= \lambda^{2\,(N)}\mathcal{D}_{\alpha\beta\gamma\delta} + \lambda^{3\,(1)}\mathcal{X}_{\alpha\beta\gamma\delta} + \lambda^{4\,(2)}\mathcal{X}_{\alpha\beta\gamma\delta} + \lambda^{4\,(III)}\mathcal{E}_{\alpha\beta\gamma\delta} + \lambda^{5\,(3)}\mathcal{X}_{\alpha\beta\gamma\delta} \\ &+ \lambda^{6\,(4)}\mathcal{X}_{\alpha\beta\gamma\delta} + \lambda^{6\,(II/D)}\mathcal{F}_{\alpha\beta\gamma\delta} + \lambda^{7\,(5)}\mathcal{X}_{\alpha\beta\gamma\delta} + \lambda^{8\,(6)}\mathcal{X}_{\alpha\beta\gamma\delta} + \lambda^{8\,(I)}\mathcal{G}_{\alpha\beta\gamma\delta} \\ &+ \lambda^{9\,(7)}\mathcal{X}_{\alpha\beta\gamma\delta} + \lambda^{10\,(8)}\mathcal{X}_{\alpha\beta\gamma\delta} + \lambda^{10\,(I)}\mathcal{H}_{\alpha\beta\gamma\delta} + \mathcal{O}\left(\lambda^{11}\right)\ , \end{aligned} \tag{4.245}$$

near $\lambda = \lambda_0$, where

$$^{(N)}\mathcal{D}_{\alpha\beta\gamma\delta} := {}^{(N)}d_{\alpha\mu\gamma}{}^{\nu\,(N)}d_{\delta\nu\beta}{}^{\mu} + {}^{(N)*}d_{\alpha\mu\gamma}{}^{\nu\,(N)*}d_{\delta\nu\beta}{}^{\mu}\ , \tag{4.246}$$

$$^{(III)}\mathcal{E}_{\alpha\beta\gamma\delta} := {}^{(III)}e_{\alpha\mu\gamma}{}^{\nu\,(III)}e_{\delta\nu\beta}{}^{\mu} + {}^{(III)*}e_{\alpha\mu\gamma}{}^{\nu\,(III)*}e_{\delta\nu\beta}{}^{\mu}\ , \tag{4.247}$$

$$^{(II/D)}\mathcal{F}_{\alpha\beta\gamma\delta} := {}^{(II/D)}f_{\alpha\mu\gamma}{}^{\nu\,(II/D)}f_{\delta\nu\beta}{}^{\mu} + {}^{(II/D)*}f_{\alpha\mu\gamma}{}^{\nu\,(II/D)*}f_{\delta\nu\beta}{}^{\mu}\ , \tag{4.248}$$

$$^{(I)}\mathcal{G}_{\alpha\beta\gamma\delta} := {}^{(I)}g_{\alpha\mu\gamma}{}^{\nu\,(I)}g_{\delta\nu\beta}{}^{\mu} + {}^{(I)*}g_{\alpha\mu\gamma}{}^{\nu\,(I)*}g_{\delta\nu\beta}{}^{\mu}\ , \tag{4.249}$$

are basic superenergy tensors labelled with the Petrov type of the Weyl-tensor candidate they are built with, respectively; the Weyl-tensor candidates are the ones of theorem 1 described in table 4.2. The tensor fields $^{(a)}\mathcal{X}_{\alpha\beta\gamma\delta}$ with $a = 1, 2, 3, 4, 5, 6$ are symmetric and traceless, and contain cross terms:

$$^{(1)}\mathcal{X}_{\alpha\beta\gamma\delta} := {}^{(N)}d_{\alpha\mu\gamma}{}^{\nu\,(III)}e_{\delta\nu\beta}{}^{\mu} + {}^{(N)*}d_{\alpha\mu\gamma}{}^{\nu\,(N)*}e_{\delta\nu\beta}{}^{\mu} + {}^{(III)}e_{\alpha\mu\gamma}{}^{\nu\,(N)}d_{\delta\nu\beta}{}^{\mu} + {}^{(III)*}e_{\alpha\mu\gamma}{}^{\nu\,(N)*}d_{\delta\nu\beta}{}^{\mu}\ , \tag{4.250}$$

$$^{(2)}\mathcal{X}_{\alpha\beta\gamma\delta} := {}^{(N)}d_{\alpha\mu\gamma}{}^{\nu\,(II/D)}f_{\delta\nu\beta}{}^{\mu} + {}^{(N)*}d_{\alpha\mu\gamma}{}^{\nu\,(II/D)*}f_{\delta\nu\beta}{}^{\mu} + {}^{(II/D)}f_{\alpha\mu\gamma}{}^{\nu\,(N)}d_{\delta\nu\beta}{}^{\mu} + {}^{(II/D)*}f_{\alpha\mu\gamma}{}^{\nu\,(N)*}d_{\delta\nu\beta}{}^{\mu}\ , \tag{4.251}$$

$$\begin{aligned}^{(3)}\mathcal{X}_{\alpha\beta\gamma\delta} :=\ & {}^{(N)}d_{\alpha\mu\gamma}{}^{\nu\,(I)}g_{\delta\nu\beta}{}^{\mu} + {}^{(N)*}d_{\alpha\mu\gamma}{}^{\nu\,(I)*}g_{\delta\nu\beta}{}^{\mu} + {}^{(I)}g_{\alpha\mu\gamma}{}^{\nu\,(N)}d_{\delta\nu\beta}{}^{\mu} + {}^{(I)*}g_{\alpha\mu\gamma}{}^{\nu\,(N)*}d_{\delta\nu\beta}{}^{\mu} \\ &+ {}^{(II/D)}f_{\alpha\mu\gamma}{}^{\nu\,(III)}e_{\delta\nu\beta}{}^{\mu} + {}^{(II/D)*}f_{\alpha\mu\gamma}{}^{\nu\,(N)*}e_{\delta\nu\beta}{}^{\mu} + {}^{(III)}e_{\alpha\mu\gamma}{}^{\nu\,(II/D)}f_{\delta\nu\beta}{}^{\mu} + {}^{(III)*}e_{\alpha\mu\gamma}{}^{\nu\,(II/D)*}f_{\delta\nu\beta}{}^{\mu}\ ,\end{aligned} \tag{4.252}$$

$$\begin{aligned}^{(4)}\mathcal{X}_{\alpha\beta\gamma\delta} :=\ & {}^{(N)}d_{\alpha\mu\gamma}{}^{\nu\,(I)}h_{\delta\nu\beta}{}^{\mu} + {}^{(N)*}d_{\alpha\mu\gamma}{}^{\nu\,(I)*}h_{\delta\nu\beta}{}^{\mu} + {}^{(I)}h_{\alpha\mu\gamma}{}^{\nu\,(N)}d_{\delta\nu\beta}{}^{\mu} + {}^{(I)*}h_{\alpha\mu\gamma}{}^{\nu\,(N)*}d_{\delta\nu\beta}{}^{\mu} \\ &+ {}^{(I)}g_{\alpha\mu\gamma}{}^{\nu\,(III)}e_{\delta\nu\beta}{}^{\mu} + {}^{(I)*}g_{\alpha\mu\gamma}{}^{\nu\,(N)*}e_{\delta\nu\beta}{}^{\mu} + {}^{(III)}e_{\alpha\mu\gamma}{}^{\nu\,(I)}g_{\delta\nu\beta}{}^{\mu} + {}^{(III)*}e_{\alpha\mu\gamma}{}^{\nu\,(I)*}g_{\delta\nu\beta}{}^{\mu}\ .\end{aligned} \tag{4.253}$$

Proof. Application of steps 1) and 2) leads to eq. (4.244). Then, a direct calculation of the Taylor series around λ_0 yields eq. (4.245). □

The interest of the above result lies in the following remarkable property of supermomenta:

Corollary 4.4.1. Let conditions of theorem 4 hold and $\hat{\mathcal{P}}_{\alpha}$ be the supermomentum associated with a causal vector field $\hat{u}^{\alpha}$, constructed with the physical Bel-Robinson tensor

$\hat{\mathcal{T}}_{\alpha\beta\gamma\delta}$. Then, the asymptotic behaviour of the supermomentum along γ follows by application of steps 1) to 3) on page 64 and reads

$$\underset{*}{\overleftarrow{\hat{\mathcal{P}}}}_{\alpha} = \Omega^{6}\left(\lambda_{1}\right)\left(\ell_{\mu}\underset{*}{\overleftarrow{\hat{u}}}^{\mu}\right)^{3}\mathcal{W}\lambda^{2}\ell_{\alpha} + \mathcal{O}\left(\lambda^{3}\right) \ , \tag{4.254}$$

where $\mathcal{W}$ is the asymptotic radiant superenergy (4.222) and $\underset{*}{\overleftarrow{\hat{u}}}^{\mu} := g^{\nu\mu}L_{\nu}{}^{\rho}\underset{*}{\hat{u}}_{\rho}\left(\lambda_{0}\right)$.

Proof. Step 1) together with eq. (4.148) provides us with

$$\underset{*}{\hat{\mathcal{P}}}_{\alpha} = \Omega^{6}\left(\lambda\right)\underset{*}{\hat{u}}^{\beta}\underset{*}{\hat{u}}^{\gamma}\underset{*}{\hat{u}}^{\delta}\underset{*}{\hat{\mathcal{T}}}_{\alpha\beta\gamma\delta} \ . \tag{4.255}$$

Next, one applies step 2) and uses eq. (4.160),

$$\underset{*}{\overleftarrow{\hat{\mathcal{P}}}}_{\alpha} = \frac{\Omega^{6}\left(\lambda\right)}{\Xi^{6}\left(\lambda\right)}\underset{*}{\overleftarrow{\hat{u}}}^{\beta}\underset{*}{\overleftarrow{\hat{u}}}^{\gamma}\underset{*}{\overleftarrow{\hat{u}}}^{\delta}\underset{*}{\overleftarrow{\hat{\mathcal{T}}}}_{\alpha\beta\gamma\delta} \ . \tag{4.256}$$

This last expression has a Taylor expansion around λ_0 that reads

$$\underset{*}{\overleftarrow{\hat{\mathcal{P}}}}_{\alpha} = \Omega^{10}\left(\lambda_{1}\right)\underset{*}{\overleftarrow{\hat{u}}}^{\beta}\underset{*}{\overleftarrow{\hat{u}}}^{\gamma}\underset{*}{\overleftarrow{\hat{u}}}^{\delta}\,{}^{(N)}\mathcal{D}_{\alpha\beta\gamma\delta}\lambda^{2} + \mathcal{O}\left(\lambda^{3}\right) \ . \tag{4.257}$$

where eq. (4.245) was used. Finally, using eqs. (4.205) and (4.207) together with eq. (4.222), the result follows. □

Remark 4.4.5. Observe that eq. (4.254) is well behaved at $\lambda = \lambda_0$ if and only if $\ell_{\mu}\underset{*}{\overleftarrow{\hat{u}}}^{\mu}$ does not diverge there. The equation, when regular, shows that at leading order only the N^α component of $\hat{u}^\alpha$ contributes to the physical supermomentum transported along a null geodesic reaching $\mathscr{J}$. This is in natural agreement with (3), which bases the determination of outgoing gravitational radiation precisely on the asymptotic radiant supermomentum eq. (4.215), i.e., a radiant supermomentum for the 'observer' N^α.

Remark 4.4.6. Notice that $\hat{u}^\alpha$ has to be causal, and in particular can be lightlike. Hence the result applies to physical *radiant* supermomenta too.

La tarde forma pájaros sobre las azoteas.
Del color rojo sale una manzana.
En el perro que ladra
se van acumulando los tablones.
Salta un delfín
y es, durante un segundo,
parte del cielo.

Benjamín Prado, *Límite*. *Todos nosotros*, 1998.

5 | Asymptotic structure with a positive cosmological constant

Observational data [9, 10] reveal that we inhabit an accelerated-expanding universe. This empirical fact evince the presence of a positive (bare or effective) cosmological constant. This scenario differs drastically from the asymptotically flat case, for $\mathscr{J}$ is a spatial hypersurface –see eq. (3.81)– and its topology is not determined by universal constraints [61]. Not only that, but an intrinsic notion of evolution is lacking, as the natural geometric observer n^{α} (3.88) which is timelike at infinity is also normal to $\mathscr{J}$. Hence, there is no notion of a privileged congruence of curves –as in the case of the lightlike generators for $\Lambda = 0$. This last feature is studied in section 5.4 and chapter 7 too.

As a consequence, while the conformal completion –see section 3.1– can be built for any value of the cosmological constant Λ, its relationship with the news tensor and Bondi-Trautman energy-momentum has only been established in the asymptotically flat case with $\Lambda = 0$ –see chapter 4. Thus, a rigorous theoretical description of radiation escaping to infinity in the presence of a positive Λ, *no matter how tiny* Λ *may be*, is necessary. Signs of attention to this situation date back to [60], and were amplified in [61] where the predicament was clearly presented. Some advances have been made [62, 64–66, 68–71] (see [24, 74] for reviews), usually trying to adapt techniques from the $\Lambda = 0$ case to the new scenario. One of the challenging difficulties is to understand and describe unambiguously the directional dependence that emerges when one approaches infinity in different lightlike directions [77]. Not to mention the absence of an asymptotic universal structure of infinity. In summary, until recently [122] the next question had remained open: *How to tell when a space-time with positive cosmological constant contains gravitational radiation arriving at infinity?* This fundamental question underlies any other hypothetical deeper characterisation, such as a formula for the energy carried away by the waves from an isolated source or the definition of a mass-energy. We answered the question taking a fully new perspective of the problem [75], different from the methods used previously in the literature. As it has been emphasised in previous chapters, our investigation is grounded in studying tidal effects, motivated by the nature of the gravitational field and of actual gravitational-wave measurements. Our approach is already supported by its successful application to the well-established asymptotically flat case [76] –see chapter 4.

Summarising, this chapter contains the study of the intrinsic asymptotic structure and its relation to the space-time fields. The approach followed is to treat $\mathscr{J}$ as a hypersurface and apply the formulae of appendix A –also, the notation for 3-dimensional hypersurfaces introduced there is used for $\mathscr{J}$ here–. The intrinsic curvature is connected to the kinematics of the congruence of timelike curves tangent to the vector field n_α. Afterwards, a new satisfactory *radiation condition at infinity* in the presence of a positive cosmological constant is presented and compared with the $\Lambda = 0$-limiting case. To our knowledge, it is the first such criterion.

5.1 Infinity and its intrinsic geometry

In the present scenario, $\mathscr{J}$ is a space-like three-dimensional hypersurface –see fig. 5.1. Its topology is not fixed in general and typical cases include $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{R}^3$ –for some examples see [61, 109]. Hence, one can always think of $\mathscr{J}$ as $\mathbb{S}^3$ or $\mathbb{S}^3$ after removing a set of points. Also, an important element in chapters 6 and 7 is the introduction of cuts; a cut $(\mathcal{S}, q_{AB})$ on $\mathscr{J}$ is a two-dimensional Riemannian manifold $\mathcal{S} \subset \mathscr{J}$ equipped with a metric q_{AB}.

Begin by noting that, in view of eq. (3.92), the second fundamental form of $(\mathscr{J}, h_{ab})$ –eq. (A.3)– vanishes,

$$\kappa_{ab} = 0 \ . \tag{5.1}$$

This can be used to simplify the Gauss equation relating the space-time Riemann tensor $R_{\alpha\beta\gamma}{}^{\delta}$ and the intrinsic curvature $\overline{R}_{abc}{}^{d}$ – eqs. (A.13) to (A.15)–, yielding

$$\overline{R}_{abc}{}^{d} \overset{\mathscr{J}}{=} e^{\alpha}{}_{a} e^{\beta}{}_{b} e^{\gamma}{}_{c} R_{\alpha\beta\gamma}{}^{\delta} \omega_{\delta}{}^{d} \ , \tag{5.2}$$

$$\overline{R}_{ac} \overset{\mathscr{J}}{=} e^{\alpha}{}_{a} e^{\gamma}{}_{c} R_{\alpha\gamma} + n^{\beta} n_{\delta} e^{\alpha}{}_{a} e^{\gamma}{}_{c} R_{\alpha\beta\gamma}{}^{\delta} \ , \tag{5.3}$$

$$\overline{R} \overset{\mathscr{J}}{=} R + 2 n^{\alpha} n^{\gamma} R_{\alpha\gamma} \ , \tag{5.4}$$

The intrinsic Schouten tensor in three dimensions is defined as

$$\overline{S}_{ab} := \overline{R}_{ab} - \frac{1}{4} \overline{R} h_{ab} \ , \tag{5.5}$$

and we can use the equations above in order to write it in terms of the space-time curvature

$$\overline{S}_{ab} \overset{\mathscr{J}}{=} e^{\mu}{}_{a} e^{\nu}{}_{b} S_{\mu\nu} + n^{\rho} n^{\sigma} e^{\mu}{}_{a} e^{\nu}{}_{b} R_{\rho\mu\sigma\nu} - \frac{1}{12} R h_{ab} - \frac{1}{2} n^{\mu} n^{\nu} R_{\mu\nu} h_{ab} \ . \tag{5.6}$$

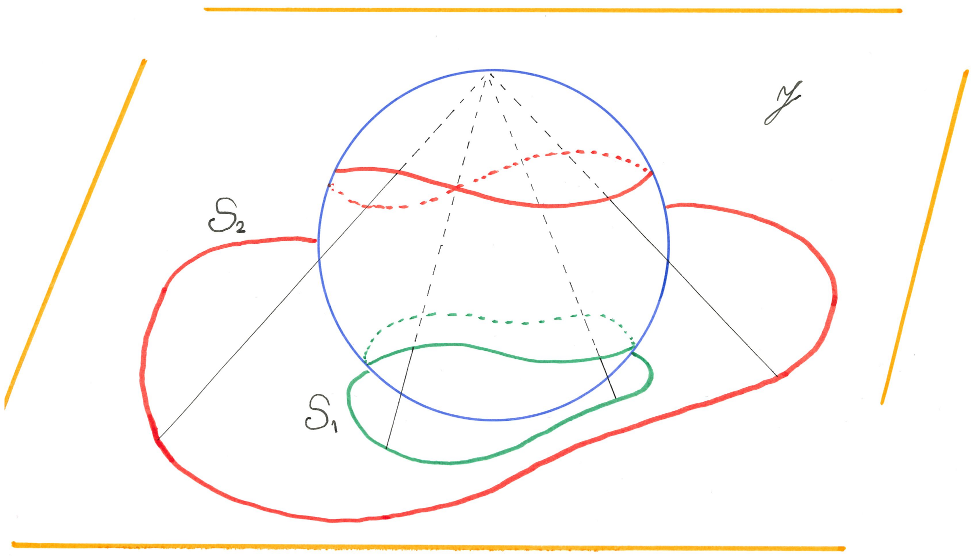

Figure 5.1: In the presence of a positive cosmological constant, $\mathscr{I}$ usually has $\mathbb{S}^3$-topology or $\mathbb{S}^3$ without a set of points. Also, one can consider Riemannian surfaces, or cuts, denoted by $\mathcal{S}$. The figure shows –with one dimension suppressed– the stereographic projection of $\mathscr{I}$ to the plane, including a couple of cuts labelled by $\mathcal{S}_1$ and $\mathcal{S}_2$. Thus, one can picture $\mathscr{I}$ as $\mathbb{R}^3$, which is how it is represented in the rest of the figures.

On $\mathscr{I}$ the space-time curvature is determined by $S_{\alpha\beta}$ completely, see eq. (3.85), and it is possible to write

$$n^\rho n^\sigma e^\mu{}_a e^\nu{}_b R_{\rho\mu\sigma\nu} \overset{\mathscr{I}}{=} -\frac{1}{2} e^\mu{}_a e^\nu{}_b S_{\mu\nu} + \frac{1}{2} h_{ab} n^\rho n^\sigma R_{\rho\sigma} + \frac{1}{12} R h_{ab}, \tag{5.7}$$

and use this to arrive at

$$\overline{S}_{ab} \overset{\mathscr{I}}{=} \frac{1}{2} e^\mu{}_a e^\nu{}_b S_{\mu\nu} \ . \tag{5.8}$$

Indeed, by eqs. (3.85), (5.2) and (5.8) one can write

$$\overline{R}_{abcd} = 2h_{a[c}\overline{S}_{d]b} - 2h_{b[c}\overline{S}_{d]a} \ , \tag{5.9}$$

which is valid in general for dimension 3. Note that on a neighbourhood of $\mathscr{I}$ where n_α is well defined, since $P^\alpha{}_\beta$ is defined there too, we can consider

$$\overline{S}_{\alpha\beta} = \frac{1}{2} P^\mu{}_\alpha P^\mu{}_\beta S_{\alpha\beta} \ . \tag{5.10}$$

Also, we introduce the intrinsic Cotton tensor:

$$\overline{Y}_{abc} := 2\overline{\nabla}_{[a}\overline{S}_{b]c} \ , \tag{5.11}$$

together with the Cotton-York tensor,

$$\overline{Y}_{ab} := -\frac{1}{2}\epsilon_a{}^{pq}\overline{Y}_{pqb} \ . \tag{5.12}$$

The electric and magnetic parts of the rescaled Weyl tensor can be written explicitly in terms of $\overline{S}_{\alpha\beta}$,

$$\begin{aligned} C_{ab} &\stackrel{\mathscr{J}}{=} \frac{1}{2}e^\alpha{}_a e^\beta{}_b \eta^{\rho\sigma}{}_{\alpha\nu} n^\lambda n^\nu d_{\rho\sigma\beta\lambda} \stackrel{\mathscr{J}}{=} -\frac{1}{2}e^\alpha{}_a e^\beta{}_b \epsilon^{\rho\sigma}{}_\alpha n^\lambda d_{\rho\sigma\beta\lambda} \\ &\stackrel{\mathscr{J}}{=} -\frac{1}{2N}e^\alpha{}_a e^\beta{}_b e^\rho{}_p e^\sigma{}_q \epsilon^{pq}{}_\alpha N^\lambda d_{\rho\sigma\beta\lambda} \stackrel{\mathscr{J}}{=} \frac{1}{2N}e^\alpha{}_a e^\beta{}_b e^\rho{}_p e^\sigma{}_q \epsilon^{pq}{}_\alpha \nabla_{[\rho}S_{\sigma]\beta} \ . \end{aligned} \tag{5.13}$$

In the second line we have used eq. (3.73). A similar computation can be performed to write an equation for D_{ab}, and we end up with two important formulae:

$$C_{ab} = \sqrt{\frac{3}{\Lambda}}\epsilon^{pq}{}_a \overline{\nabla}_{[p}\overline{S}_{q]b} \ , \tag{5.14}$$

$$D_{ab} \stackrel{\mathscr{J}}{=} -\sqrt{\frac{3}{\Lambda}}e^\alpha{}_a e^\beta{}_b n^\mu \nabla_{[\alpha}S_{\mu]\beta} \stackrel{\mathscr{J}}{=} \sqrt{\frac{3}{\Lambda}}e^\alpha{}_a e^\beta{}_b n^\mu \nabla_\mu \overline{S}_{\alpha\beta} \ . \tag{5.15}$$

where in the last line we have used eq. (3.94). Remarkably, eq. (5.14) tell us that *the magnetic part of the rescaled Weyl tensor is completely determined by the geometry of* $\mathscr{J}$. In contrast, eq. (5.15) shows that *the electric part is unknown from the intrinsic point of view*[1]. These two conclusions have direct implications in the search of the asymptotic radiative degrees of freedom with a positive cosmological constant and must be taken fully into account.

To see what implication a vanishing C_{ab} would have on the geometry of $\mathscr{J}$, use eq. (5.11) to write it as

$$C_{ab} \stackrel{\mathscr{J}}{=} \frac{1}{2}\sqrt{\frac{3}{\Lambda}}\epsilon^{pq}{}_a \overline{Y}_{pqb} \stackrel{\mathscr{J}}{=} -\sqrt{\frac{3}{\Lambda}}\overline{Y}_{ab} \ . \tag{5.16}$$

It is well known (see [100], for instance) that the Cotton-York tensor of the metric of a three dimensional manifold vanishes if and only if the metric is locally conformally flat. Thus, the vanishing of the magnetic part of the rescaled Weyl tensor strongly constraints the intrinsic geometry and the would-be degrees of freedom of the gravitational field —for a discussion on this matter, see [62].

[1]Note that we are able to write D_{ab} in terms of $\overline{S}_{\alpha\beta}$ because it is defined on a neighbourhood of $\mathscr{J}$ and we can compute its derivative along n^α.

5.2 Kinematics of the normal to $\mathscr{J}$

It can be elucidatory to connect the kinematic quantities –shear, acceleration and expansion[2]– of n_α to the Schouten tensor and with the electric and magnetic parts of the rescaled Weyl tensor through eqs. (5.14) and (5.15). The relationship gives us an intuitive idea of the impact that $C_{\alpha\beta}$ and $D_{\alpha\beta}$ have on the congruence of curves that an asymptotic observer would follow.

To start with, we have to compute the covariant derivative of n_α using eqs. (3.31), (3.35) and (3.36),

$$\begin{aligned}\nabla_\alpha n_\beta &= \frac{1}{N}\nabla_\alpha N_\beta - \frac{1}{N^2}N_\beta\nabla_\alpha N = -\frac{\Omega}{2N}S_{\alpha\beta} + \frac{f}{N}g_{\alpha\beta} + \frac{1}{2}\Omega^2\frac{\varkappa}{N}\underline{T}_{\alpha\beta}\\ &\quad - \frac{1}{2N^3}N_\beta\left(-2N_\alpha f - 2\Omega\nabla_\alpha f - \frac{1}{4}\Omega^2\varkappa T N_\alpha - \frac{1}{12}\Omega^3\varkappa\nabla_\alpha T\right)\\ &= -\frac{1}{2N}\Omega S_{\alpha\mu}P^\mu{}_\beta + \frac{1}{N}P_{\alpha\beta}f + \frac{1}{2N}\Omega^2\varkappa P^\mu{}_\beta\underline{T}_{\alpha\mu}\ . \end{aligned} \tag{5.17}$$

It is easy to see that this vanishes at $\mathscr{J}$, as it must, given our choice of gauge. In other words, the kinematic quantities vanish at $\mathscr{J}$. Nevertheless, their 'time derivatives' –along n^α– may be non-vanishing at $\mathscr{J}$. To begin with, consider the acceleration,

$$a_\alpha = n^\mu\nabla_\mu n_\alpha = -\frac{1}{2N}\Omega P^\nu{}_\alpha n^\mu S_{\mu\nu} + \frac{1}{2N}\Omega^2\varkappa P^\nu{}_\alpha n^\mu\underline{T}_{\nu\mu} \overset{\mathscr{J}}{=} 0\ , \tag{5.18}$$

$$\begin{aligned}\dot{a}_\alpha := n^\mu\nabla_\mu a_\alpha &= \frac{1}{2N}P^\nu{}_\alpha n^\mu S_{\nu\mu} - \Omega n^\rho\nabla_\rho\left(\frac{1}{2N}P^\nu{}_\alpha n^\mu S_{\nu\mu}\right)\\ &\quad - \Omega\varkappa P^\nu{}_\alpha n^\mu\underline{T}_{\nu\mu} + \frac{1}{2}\Omega^2\varkappa n^\rho\nabla_\rho\left(\frac{1}{N}P^\nu{}_\alpha n^\mu\underline{T}_{\nu\mu}\right)\ ,\end{aligned} \tag{5.19}$$

and from eq. (3.94) we deduce that $\dot{a}_\alpha \overset{\mathscr{J}}{=} 0$. Next, consider the expansion

$$\theta := \nabla_\rho n^\rho = -\frac{1}{2N}\Omega h^{\mu\rho}S_{\mu\rho} + \frac{3}{N}f + \frac{1}{2N}\Omega^2\varkappa h^{\mu\rho}\underline{T}_{\mu\rho} \overset{\mathscr{J}}{=} 0\ . \tag{5.20}$$

[2]By definition, n_α is proportional to an exact differential and, therefore, has vanishing rotation

$$\dot{\theta} := n^\rho \nabla_\rho \theta = \frac{1}{2} h^{\mu\rho} S_{\mu\rho} - \Omega n^\rho \nabla_\rho \left(\frac{1}{2N} h^{\mu\rho} S_{\mu\rho} \right) + \frac{3}{N} n^\rho \nabla_\rho f - \frac{3}{N^2} f n^\rho \nabla_\rho N$$
$$- \Omega \varkappa h^{\mu\rho} \underline{T}_{\mu\rho} + \frac{1}{2} \varkappa \Omega^2 n^\rho \nabla_\rho \left(\frac{1}{N} h^{\mu\rho} \underline{T}_{\mu\rho} \right) = \frac{1}{2} P^{\mu\nu} S_{\mu\rho} + \tag{5.21}$$
$$+\Omega \left[-n^\rho \nabla_\rho \left(\frac{1}{2N} P^{\mu\nu} S_{\mu\nu} \right) + \varkappa P^{\mu\nu} \underline{T}_{\mu\nu} + \frac{3}{2} n^\mu n^\nu \underline{T}_{\mu\nu} + \frac{3}{8} \varkappa T \right]$$
$$+ 3 f n^\rho \nabla_\rho \left(N^{-1} \right) - \frac{3}{2} S_{\mu\nu} n^\mu n^\nu + \Omega^2 \left[\frac{1}{2} \varkappa n^\rho \nabla_\rho \left(\frac{1}{N} P^{\mu\nu} \underline{T}_{\mu\nu} \right) - \frac{1}{8N} \varkappa n^\mu \nabla_\mu T \right]$$
$$\overset{\mathscr{J}}{=} \frac{1}{2} S^\mu{}_\mu - S_{\rho\mu} n^\rho n^\mu \ . \tag{5.22}$$

Finally, the shear,

$$\sigma_{\alpha\beta} := \left(P^\mu{}_\alpha P^\nu{}_\beta - \frac{1}{3} P_{\alpha\beta} h^{\nu\mu} \right) \nabla_\mu n_\nu = -\frac{1}{2N} \Omega P^\mu{}_\alpha P^\nu{}_\beta S_{\mu\nu} + \frac{1}{N} P_{\alpha\beta} f$$
$$+ \frac{1}{2N} \Omega^2 P^\mu{}_\alpha P^\nu{}_\beta \underline{T}_{\mu\nu} + \left(\frac{1}{6N} \Omega P^{\mu\nu} S_{\mu\nu} - \frac{1}{N} f - \frac{1}{6N} \Omega^2 \varkappa P^{\mu\nu} \underline{T}_{\mu\nu} \right) P_{\alpha\beta} =$$
$$= -\frac{1}{2N} \Omega P^\mu{}_\alpha P^\nu{}_\beta S_{\mu\nu} + \frac{1}{2N} \Omega^2 P^\mu{}_\alpha P^\nu{}_\beta \underline{T}_{\mu\nu} + \left(\frac{1}{6N} \Omega P^{\mu\nu} S_{\mu\nu} - \frac{1}{6N} \Omega^2 \varkappa P^{\mu\nu} \underline{T}_{\mu\nu} \right) P_{\alpha\beta}$$
$$\overset{\mathscr{J}}{=} 0 \ . \tag{5.23}$$

$$\dot{\sigma}_{\alpha\beta} := n^\rho \nabla_\rho \sigma_{\alpha\beta} = \frac{1}{2} P^\mu{}_\alpha P^\nu{}_\beta S_{\mu\nu} - \frac{1}{6} P_{\alpha\beta} P^{\mu\nu} S_{\mu\nu}$$
$$+ \Omega \left[-n^\rho \nabla_\rho \left(\frac{1}{2N} P^\mu{}_\alpha P^\nu{}_\beta S_{\mu\nu} \right) + \frac{1}{6} n^\rho \nabla_\rho \left(\frac{1}{N} P_{\alpha\beta} P^{\mu\nu} S_{\mu\nu} \right) - \varkappa P^\mu{}_\beta P^\nu{}_\alpha \underline{T}_{\mu\nu} + \frac{1}{3} \varkappa P^{\mu\nu} \underline{T}_{\mu\nu} \right]$$
$$+ \Omega^2 n^\rho \nabla_\rho \left(\frac{1}{2N} \varkappa P^\mu{}_\beta P^\nu{}_\alpha \underline{T}_{\mu\nu} - \frac{1}{6N} \varkappa P^{\mu\nu} \underline{T}_{\mu\nu} P_{\alpha\beta} \right)$$
$$\overset{\mathscr{J}}{=} \frac{1}{2} P^\mu{}_\alpha P^\nu{}_\beta S_{\mu\nu} - \frac{1}{6} P_{\alpha\beta} P^{\mu\nu} S_{\mu\nu} \ . \tag{5.24}$$

Note that this quantity is different from zero (in general), completely tangent to $\mathscr{J}$ and coincides with the traceless part of the intrinsic Schouten tensor

$$\dot{\sigma}_{ab} \overset{\mathscr{J}}{:=} \overline{S}_{ab} - \frac{1}{3} h_{ab} \overline{S}^c{}_c \ . \tag{5.25}$$

It will be necessary, as we will see shortly, to have the second derivative too,

$$\ddot{\sigma}_{\alpha\beta} := n^\rho \nabla_\rho \dot{\sigma}_{\alpha\beta} = \frac{1}{2} n^\rho \nabla_\rho s_{\alpha\beta} - \frac{1}{6} P_{\alpha\beta} n^\mu \nabla_\mu s^\nu{}_\nu$$
$$- N \Big(- \frac{1}{2N} n^\rho \nabla_\rho s_{\alpha\beta} + \frac{1}{6N} P_{\alpha\beta} n^\rho \nabla_\rho s^\mu{}_\mu + \frac{1}{2N^2} n^\rho \nabla_\rho N s_{\alpha\beta} - \frac{1}{6N^2} n^\rho \nabla_\rho \left(N \right) s^\alpha{}_\beta P_{\alpha\beta}$$
$$- \varkappa P^\mu{}_\beta P^\nu{}_\alpha \underline{T}_{\mu\nu} + \frac{1}{3} \varkappa n^\mu n^\nu \underline{T}_{\mu\nu} P_{\alpha\beta} \Big) + \Omega A_{\alpha\beta} + \Omega^2 B_{\alpha\beta} \ , \tag{5.26}$$

where A and B are regular (non-vanishing in general) symmetric tensors. Notice that contracting with $n^{\alpha}P^{\beta\gamma}$ eq. (3.61) and using eq. (3.94)

$$n^{\mu}\nabla_{\mu}\overline{S}^{\nu}{}_{\nu} \overset{\mathscr{J}}{=} 0. \tag{5.27}$$

Observe, also, that by eqs. (3.100) and (3.102) we have

$$P^{\mu}{}_{\beta}P^{\nu}{}_{\alpha}\underline{T}_{\mu\nu} \overset{\mathscr{J}}{=} -\frac{1}{4}TP_{\alpha\beta}\ , \tag{5.28}$$

$$n^{\mu}n^{\nu}\underline{T}_{\mu\nu} \overset{\mathscr{J}}{=} -\frac{3}{4}T\ . \tag{5.29}$$

Taking into account these last equations we arrive at

$$\ddot{\sigma}_{ab} \overset{\mathscr{J}}{=} e^{\alpha}{}_{a}e^{\beta}{}_{b}\ddot{\sigma}_{\alpha\beta} \overset{\mathscr{J}}{=} 2e^{\alpha}{}_{a}e^{\beta}{}_{b}n^{\rho}\nabla_{\rho}\overline{S}_{\alpha\beta}\ . \tag{5.30}$$

From eqs. (5.14), (5.15), (5.25) and (5.30), we get the desired relations:

$$C_{ab} = \sqrt{\frac{3}{\Lambda}}\left[\epsilon^{pq}{}_{a}\overline{\nabla}_{[p}\dot{\sigma}_{q]b} + \frac{1}{2}\epsilon^{p}{}_{ba}\overline{\nabla}_{c}\dot{\sigma}^{c}{}_{p}\right]\ , \tag{5.31}$$

$$D_{ab} = \frac{1}{2}\sqrt{\frac{3}{\Lambda}}\ddot{\sigma}_{ab}\ . \tag{5.32}$$

For the first equation, we have used another interesting relation that can be obtained if one considers eq. (5.25) and takes the trace in eq. (3.61),

$$\overline{\nabla}_{c}\overline{S}^{c}{}_{a} \overset{\mathscr{J}}{=} \overline{\nabla}_{a}\overline{S}^{c}{}_{c} \overset{\mathscr{J}}{=} \frac{3}{2}\overline{\nabla}_{c}\dot{\sigma}_{a}{}^{c}\ . \tag{5.33}$$

5.3 Characterisation of gravitational radiation at $\mathscr{J}$

At this stage, we have presented the basic asymptotic structure with a positive cosmological constant (section 5.1) and the superenergy formalism (chapter 2). Thus, we are ready to tackle the problem of gravitational radiation at infinity. In this section we formulate a radiation condition, and expand the contents originally presented in [75]. Let us remark that to our knowledge, it is the first covariant, gauge-invariant criterion formulated in the presence of a positive cosmological constant.

The obvious choice of superenergy tensor at infinity is *the rescaled Bel-Robinson tensor* (3.86) which is regular and, in general, non-vanishing at $\mathscr{J}$. In order to define a supermomentum, one needs to select an observer. Since we aim at an observer-independent characterisation of radiation, the optimal way would be to have a natural privileged 'asymptotic observer'. But this is indeed given by the asymptotic geometry itself: the normal $N_{\alpha}|_{\mathscr{J}}$ is the suitable vector field. Hence, a natural definition of *asymptotic supermomentum* is

–see eq. (2.18) for the general definition–

$$p^\alpha := -N^\mu N^\nu N^\rho \mathcal{D}^\alpha{}_{\mu\nu\rho} \ , \tag{5.34}$$

or its *canonical* version

$$\mathcal{P}^\alpha := -n^\mu n^\nu n^\rho \mathcal{D}^\alpha{}_{\mu\nu\rho} \ . \tag{5.35}$$

In a neighbourhood of $\mathscr{J}$, where n^α is well defined, these two vector fields are collinear, $\mathcal{P}^\alpha = N^{-3} p^\alpha$, and have the same causal orientation. The reason why we introduce them both is that eq. (5.34) has a good behaviour in the limit $\Lambda \to 0$ – if the limit exists– in contrast to eq. (5.35). This issue will be analysed in section 5.5. Apart from this, the properties that will be listed next apply to both versions of the asymptotic supermomentum, unless explicitly said otherwise.

The orthogonal splitting of $\mathcal{P}^\alpha$ at $\mathscr{J}$ is given by

$$\mathcal{P}^\alpha = -\mathcal{W} n^\alpha + e^\alpha{}_a \overline{\mathcal{P}}^a \ . \tag{5.36}$$

which defines

- the *asymptotic canonical superenergy density*, $\mathcal{W} := -n_\mu \mathcal{P}^\mu \geq 0$,
- and the *asymptotic canonical super-Poynting vector*, $\overline{\mathcal{P}}^\alpha := P^\alpha{}_\mu \mathcal{P}^\mu = e^\alpha{}_a \overline{\mathcal{P}}^a$ —see eq. (2.18)—, which is a vector field tangent to $\mathscr{J}$.

From the general properties presented in section 2.1, it follows that

i) $\mathcal{P}^\alpha$ is causal and future pointing at and around $\mathscr{J}$, –see property iii) on page 8.

ii) Using eqs. (3.71), (3.87) and (3.103), the divergence of $\mathcal{P}^\alpha$ at $\mathscr{J}$ reads

$$\nabla_\mu \mathcal{P}^\mu \overset{\mathscr{J}}{=} N \varkappa \, {}_1T_{ab} D^{ab} \ , \tag{5.37}$$

where ${}_1T_{ab} \overset{\mathscr{J}}{:=} \Omega^{-1} T_{\mu\nu} \bar{e}^\mu{}_a \bar{e}^\nu{}_b$. In particular, if the energy-momentum tensor of the physical space-time $(\hat{M}, \hat{g}_{\mu\nu})$ behaves near $\mathscr{J}$ as $\hat{T}_{\alpha\beta}|_{\mathscr{J}} \sim \mathcal{O}(\Omega^3)$ (which includes the vacuum case $\hat{T}_{\alpha\beta} = 0$), then

$$\nabla_\mu \mathcal{P}^\mu \overset{\mathscr{J}}{=} 0. \tag{5.38}$$

This follows from eq. (3.84), recalling eq. (3.92) and eq. (3.72).

iii) Under gauge transformations, they change as

$$\begin{aligned}\mathcal{P}^{\alpha} \rightarrow & \frac{\omega^{-7}}{\left(1-2\Omega N^{-2}N^{\tau}\omega_{\tau}-\Omega^{2}\omega^{-2}N^{-2}\omega_{\tau}\omega^{\tau}\right)^{3/2}}\Big[\mathcal{P}^{\alpha} \\ & -\left(3\omega^{-1}\Omega N^{-1}n^{\rho}n^{\nu}\omega^{\mu}+3\omega^{-2}\Omega^{2}N^{-2}n^{\rho}\omega^{\nu}\omega^{\mu}+\omega^{-3}N^{-3}\Omega^{3}\omega^{\rho}\omega^{\mu}\omega^{\nu}\right)\mathcal{D}^{\alpha}{}_{\mu\nu\rho}\Big].\end{aligned} \tag{5.39}$$

$$p^{\alpha} \rightarrow \omega^{-7}\Big[p^{\alpha}-\left(3\omega^{-1}\Omega N^{\rho}N^{\nu}\omega^{\mu}+3\omega^{-2}\Omega^{2}N^{\rho}\omega^{\nu}\omega^{\mu}+\omega^{-3}\Omega^{3}\omega^{\rho}\omega^{\mu}\omega^{\nu}\right)\mathcal{D}^{\alpha}{}_{\mu\nu\rho}\Big]. \tag{5.40}$$

This behaviour is deduced using eqs. (3.11), (3.55) and (C.8), and the fact that the Weyl tensor is conformally invariant. At $\mathscr{J}$, the asymptotic supermomentum has good gauge-behaviour

$$\mathcal{P}^{\alpha} \overset{\mathscr{J}}{\rightarrow} \omega^{-7}\mathcal{P}^{\alpha}\ , \tag{5.41}$$

$$p^{\alpha} \overset{\mathscr{J}}{\rightarrow} \omega^{-7}p^{\alpha}\ . \tag{5.42}$$

The divergence property of the canonical supermomentum can be expressed as

$$\overline{\nabla}_{e}\overline{\mathcal{P}}^{e}+n^{\mu}\nabla_{\mu}\left(\mathcal{W}\right) \overset{\mathscr{J}}{=} N\varkappa\,{}_{1}T_{ab}D^{ab}\ . \tag{5.43}$$

Under appropriate conditions, this expression leads to an integral balance-law —see section 7.5. Typically, kinematic terms associated to n^{α} enter this kind of equation [88], however, due to our partial gauge-fixing they vanish at $\mathscr{J}$. Nevertheless, it is possible to write $\overline{\mathcal{P}}^{\alpha}$ in terms of the derivatives of the shear by using eqs. (5.31) and (5.32),

$$\overline{\mathcal{P}}^{a} \overset{\mathscr{J}}{=} -\frac{6}{\Lambda}\overline{\nabla}^{[a}\left(\dot{\sigma}^{s]t}\right)\ddot{\sigma}_{ts}+\frac{3}{2\Lambda}\ddot{\sigma}^{a}{}_{s}\overline{\nabla}_{c}\left(\dot{\sigma}^{cs}\right)\ . \tag{5.44}$$

Or, using eqs. (5.14) and (5.15), in terms of the Schouten tensor,

$$\overline{\mathcal{P}}^{a} \overset{\mathscr{J}}{=} \frac{12}{\Lambda}e^{\alpha}{}_{t}e^{\beta}{}_{s}\overline{\nabla}^{[s}\left(\overline{S}^{a]t}\right)n^{\mu}\nabla_{\mu}\overline{S}_{\alpha\beta}\ . \tag{5.45}$$

Our asymptotic gravitational-radiation condition is built upon this object. In order to characterise the presence of gravitational radiation at infinity, we aim at a criterion with the following features:

i) Gauge-invariant, as any physical statement should not depend on the choice of the representative within the conformal class of metrics.

ii) Observer-independent.

iii) *Strictly* asymptotic, i. e., defined at $\mathscr{J}$.

iv) With the necessary and sufficient information encoded in $(\mathscr{J}, h_{ab}, D_{ab})$. This is justified from the point of view of a fundamental result by Friedrich [112, 132] which states that a solution of the Λ-vacuum Einstein field equations is fully determined by initial/final data consisting of the conformal class of a 3-dimensional Riemmanian manifold plus a traceless and divergence-free tensor D_{ab}.

According to the justification of point iv), *one cannot aspire to describe gravitational radiation at $\mathscr{J}$ without taking D_{ab} into account.*

Our proposal, presented in [75], reads

Criterion 1 (Asymptotic gravitational-radiation condition with $\Lambda > 0$)**.** Consider a 3-dimensional open connected subset $\Delta \subset \mathscr{J}$. There is no radiation on Δ if and only if the asymptotic super-Poynting vanishes there

$$\overline{\mathcal{P}}^{\alpha} \overset{\Delta}{=} 0 \Longleftrightarrow \text{No gravitational radiation on } \Delta.$$

Remark 5.3.1. An equivalent statement is that in absence of gravitational radiation, and only in that case, the supermomentum[3] points along the normal N^{α} at $\mathscr{J}$, or:

- No gravitational radiation on $\Delta \subset \mathscr{J} \Longleftrightarrow p^{\alpha}$ is orthogonal to all surfaces within Δ .
- No gravitational radiation on $\Delta \subset \mathscr{J} \Longleftrightarrow N^{\alpha}|_{\Delta}$ is a principal vector (in the sense of Pirani, i. e., those lying in the intersection of two principal planes, see [30, 33, 133]) of $d_{\alpha\beta\gamma}{}^{\delta}|_{\Delta}$.

Remark 5.3.2. The criterion fulfils property i) as follows from eq. (5.41); property ii), according to the discussion on the geometric nature of N^{α} at the beginning of this section; property iii), by definition; property iv), since by eq. (2.15) the presence of radiation is completely given by the interplay of D_{ab} and C_{ab}, the latter being fully determined by the intrinsic geometry – see eq. (5.14).

Remark 5.3.3. According to the previous remark, the presence/absence of radiation cannot be determined by the intrinsic geometry of $\mathscr{J}$ exclusively in general —with the exception of the trivial cases of a conformally flat metric h_{ab} or a vanishing D_{ab}.

Remark 5.3.4. From eq. (2.15), the radiation condition is equivalent to the vanishing of the commutator of D_{ab} and C_{ab}, and this is only possible if $d^{\alpha}{}_{\beta\gamma\delta}|_{\mathscr{J}}$ has Petrov-type I or D [33, 88]. In accordance with remark 5.3.1, the Petrov type-D situation arises when $n^{\alpha}|_{\mathscr{J}}$ is coplanar with the two multiple PND.

[3] The same applies to the canonical supermomentum. However, the characterisation in terms of p^{α} can be compared with the $\Lambda = 0$ case, as we will see.

Remark 5.3.5. Our criterion 1 is different, but gained some influence, from definition 2.1.1. Had we chosen to inspire our criterion on definition 2.1.2, we would have had to use $Q_{abc}|_{\mathscr{J}}$ constructed with $n^{\mu}|_{\mathscr{J}}$ instead of the asymptotic super-Poynting. The vanishing of $Q_{abc}|_{\mathscr{J}}$ is equivalent to the electric and magnetic parts being proportional [88], that is

$$AC_{ab} + BD_{ab} \stackrel{\mathscr{J}}{=} 0 \tag{5.46}$$

for some A and B. This is always the case for Petrov type D. Thus, the small difference between both possibilities is that using $Q_{abc}|_{\mathscr{J}}$ there will be more radiative situations: those with the electric and magnetic parts commuting but not proportional to each other.

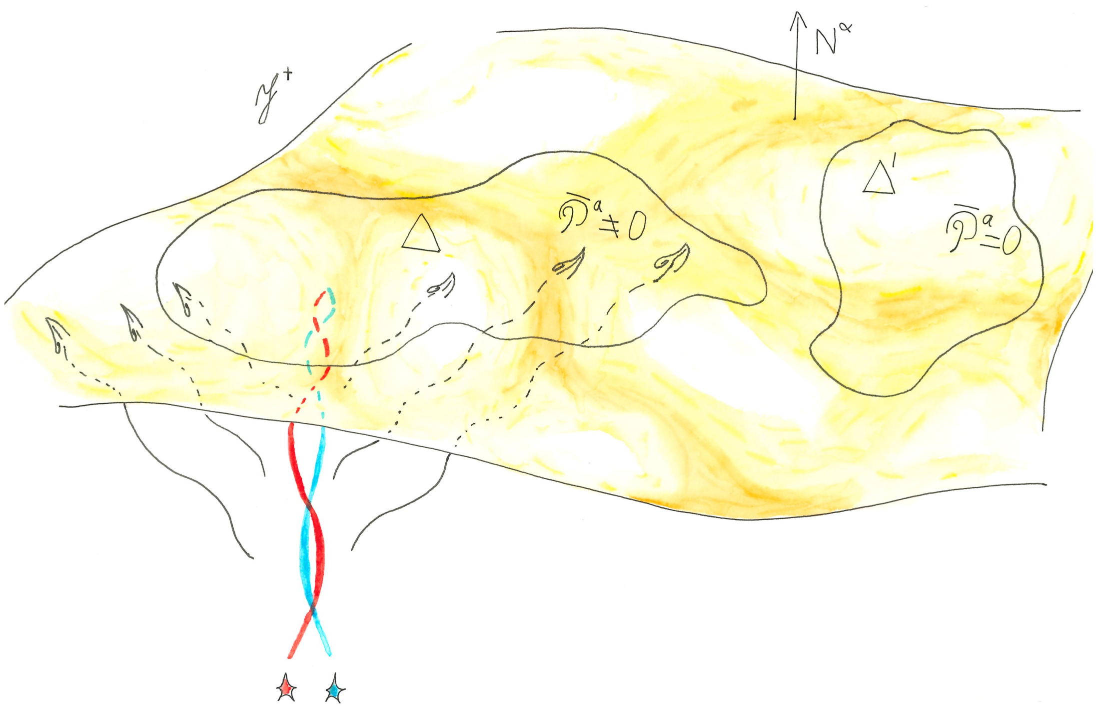

Figure 5.2: Gravitational radiation arrives at an open region Δ on $\mathscr{J}^+$ but does not at the open region Δ'. Our criterion states that the asymptotic super-Poynting is different from zero on Δ and vanishes on Δ'.

Examples illustrating the soundness of this criterion were presented in [75] and some of them will be expanded in chapter 8, as well as new ones presented. Furthermore, the criterion has an equivalent formulation in the asymptotically flat scenario, see chapter 4. In that case, it has been proved to be successful and equivalent to the traditional one in terms of the so called news tensor. More details on the limit to $\Lambda = 0$ will be given in section 5.5 but, before that, we investigate the relation between the radiation condition and the radiant quantities.

5.4 Lightlike approach and the directional-dependence problem

We have presented a reliable condition that tells if gravitational waves arrive at infinity or not. Not only it is of special relevance by itself, but it constitutes a crucial first step towards a deeper characterisation of gravitational radiation at $\mathscr{J}$. One of the biggest challenges is the directional dependence that emerges when one approaches infinity in different lightlike directions [77]. Our criterion 1 already bypasses this difficulty. Even more, it states that the presence of radiation cannot be determined by the rescaled Weyl scalar ϕ_4 only, as it is sometimes assumed in the literature –we are going to show this presently. A better understanding of this directional dependence in the presence/absence of radiation is needed. In our formalism, the logical way to proceed is to understand the role of the lightlike projections of the rescaled Bel-Robinson tensor, by defining –see eqs. (2.20) and (2.21)–

$$\overset{+}{k}{}^{\alpha} := \frac{1}{\sqrt{2}}\left(n^{\alpha} + m^{\alpha}\right) \tag{5.47}$$
$$\overset{-}{k}{}^{\alpha} := \frac{1}{\sqrt{2}}\left(n^{\alpha} - m^{\alpha}\right) \quad ,$$

for some unit spacelike vector field tangent to $\mathscr{J}$, $m^{\alpha} \overset{\mathscr{J}}{=} e^{\alpha}{}_{a} m^{a}$. In these definitions

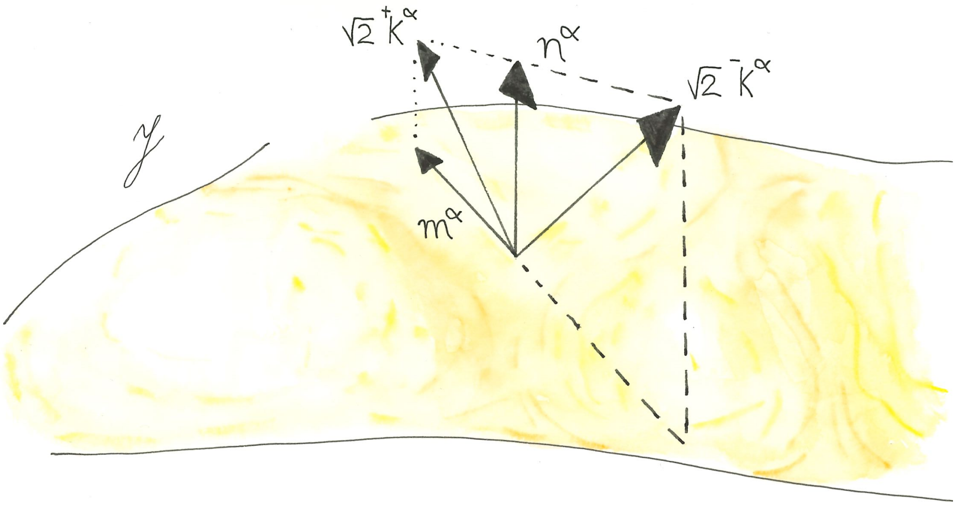

Figure 5.3: The lightlike decomposition (5.47) on $\mathscr{J}$. Given a unit spacelike vector field m^{α} tangent to $\mathscr{J}$ and the unit normal n^{α}, two coplanar lightlike directions are determined.

n_{α} plays the role of u_{α} in section 2.2 and we denote by $\{\underline{E}^{\alpha}{}_{A}\}$ the basis spanning the two dimensional space of vectors orthogonal to m^{α} and n^{α} –see appendix A.3 for more details on that. The algebraic, lightlike decomposition of section 2.2 applies the same now, though we substitute the over ring by an underbar in quantities projected with $\underline{E}^{\alpha}{}_{A}$

in order to distinguish them from objects projected with $E^{\alpha}{}_{A}$[4]–e.g., given any one-form v_α on $\mathscr{J}$, $\underline{v}_A := \underline{E}^{\alpha}{}_{A} v_\alpha$ whereas $\mathring{v}_A := E^{\alpha}{}_{A} v_\alpha$. Also, we define the radiant supermomenta –see eq. (2.46)– associated to the vector fields of eq. (5.47) using the rescaled Bel-Robinson tensor:

$$\overset{+}{\mathcal{Q}}{}^{\alpha} := -\overset{+}{k}{}^{\mu}\overset{+}{k}{}^{\nu}\overset{+}{k}{}^{\rho}\mathcal{D}^{\alpha}{}_{\mu\nu\rho} = \overset{+}{\mathcal{W}}\,\overset{-}{k}{}^{\alpha} + \overset{+}{\underline{\mathcal{Q}}}{}^{\alpha} = \overset{+}{\mathcal{W}}\,\overset{-}{k}{}^{\alpha} + \overset{+}{\underline{\mathcal{Q}}}{}^{a}\,\overset{+}{e}{}^{\alpha}{}_{a} \,, \tag{5.48}$$

$$\overset{-}{\mathcal{Q}}{}^{\alpha} := -\overset{-}{k}{}^{\mu}\overset{-}{k}{}^{\nu}\overset{-}{k}{}^{\rho}\mathcal{D}^{\alpha}{}_{\mu\nu\rho} = \overset{-}{\mathcal{W}}\,\overset{+}{k}{}^{\alpha} + \overset{-}{\underline{\mathcal{Q}}}{}^{\alpha} = \overset{-}{\mathcal{W}}\,\overset{+}{k}{}^{\alpha} + \overset{-}{\underline{\mathcal{Q}}}{}^{k}\,\overset{-}{e}{}^{\alpha}{}_{k} \,. \tag{5.49}$$

Thus, the first step is to write criterion 1 in terms of the radiant quantities.

Lemma 5.4.1 (Radiant formulation of the asymptotic gravitational-radiation condition)**.** Consider a three-dimensional open connected subset $\Delta \subset \mathscr{J}$, then

$$\left.\begin{aligned} 2\left(\overset{-}{\mathcal{Z}} - \overset{+}{\mathcal{Z}}\right) - \overset{+}{\mathcal{W}} + \overset{-}{\mathcal{W}} &\triangleq 0 \\ \sqrt{2}\left(\overset{+}{\underline{\mathcal{Q}}}{}^{A} + \overset{-}{\underline{\mathcal{Q}}}{}^{A}\right) + 12 d^{A} &\triangleq 0 \end{aligned}\right\} \Longleftrightarrow \text{No gravitational radiation on } \Delta \,. \tag{5.50}$$

Proof. It follows directly by application of lemma 2.3.4. □

Remark 5.4.1. In terms of Weyl scalars, the no-radiation condition in eq. (5.50) reads:

$$8\phi_1\bar{\phi}_1 - 8\phi_3\bar{\phi}_3 - 4\phi_4\bar{\phi}_4 + 4\phi_0\bar{\phi}_0 = 0 \,, \tag{5.51}$$

$$\phi_3\bar{\phi}_4 + \phi_0\bar{\phi}_1 - 3\phi_1\bar{\phi}_2 - 3\phi_2\bar{\phi}_3 = 0 \,. \tag{5.52}$$

This is easily deduced using the formulae of appendix D.2.

The directional freedom translates into the choice of m^a, which then automatically gives $\overset{\pm}{k}{}^{\alpha}$ by eq. (5.47). Indeed, this vector field may serve to define an intrinsic 'evolution' direction on $\mathscr{J}$, if selected properly. Thus, one needs some physical criteria underlying one choice or another. We propose two choices of increasing specialisation that we call *orientations*,

Definition 5.4.1 (Weak orientation)**.** We say that m^a defines a weak orientation when $\overset{-}{k}{}^{\alpha}$ is aligned with a PND of the rescaled Weyl tensor.

Remark 5.4.2. For Petrov-type I $d_{\alpha\beta\gamma}{}^{\delta}$ there are four possible, non-equivalent, weak orientations; one for each PND. For type II, there are 3; for type D and III, 2; for type N, just 1.

[4]The point of making this change of notation is to distinguish the quantities associated to r^a of section 2.2, which is not in general a field on $\mathscr{J}$, from those associated with m^α, which is a vector field on $\mathscr{J}$.

Remark 5.4.3. The vector m^a defines a weak orientation if and only if $^{-}\mathcal{W} = 0$. See eq. (2.53) and the Petrov characterisation on page 8.

Definition 5.4.2 (Strong orientation)**.** We say that m^a defines a strong orientation when $^{-}k^{\alpha}$ is aligned with a PND of highest multiplicity of the rescaled Weyl tensor $\mathscr{J}$.

Remark 5.4.4. The strong orientation is a particular case of weak orientation. If $d_{\alpha\beta\gamma}{}^{\delta}$ has Petrov type I, any strong orientation is a weak orientation too, hence there are four non-equivalent possibilities; for type II, III and N, there is one single strong orientation; for type D, there are two.

Remark 5.4.5. The vector m^a defines a strong orientation if and only if $^{-}\mathcal{W} = 0 = {}^{-}\mathcal{Z}$. This follows by lemma 2.3.2 recalling property iii) on page 15.

An immediate result that follows by applying these definitions is the characterisation of the Petrov type of $d^{\alpha}{}_{\beta\gamma\delta}$ in the absence of radiation at infinity by means of the radiant superenergy quantities:

Lemma 5.4.2 (Radiation condition and Petrov types)**.** Consider a three-dimensional open connected subset $\Delta \subset \mathscr{J}$. Choose m^a defining a weak orientation according to definition 5.4.1 and define $^{\pm}k^{\alpha}$ as in eq. (5.47). Let $\overline{\mathcal{P}}^{a}$ and $^{\pm}\mathcal{Q}^{\alpha}$ be the canonical asymptotic super-Poynting vector and the radiant supermomenta associated to $^{\pm}k^{\alpha}$, respectively. Then,

$$\begin{Bmatrix} 2\left({}^{-}\mathcal{Z} - {}^{+}\mathcal{Z}\right) - {}^{+}\mathcal{W} \overset{\Delta}{=} 0 \\ \sqrt{2}\,{}^{+}\underline{\mathcal{Q}}^{A} + 12 d^{A} \overset{\Delta}{=} 0 \\ {}^{+}\mathcal{Q}^{\alpha} \overset{\Delta}{\neq} 0 \overset{\Delta}{\neq} {}^{-}\mathcal{Q}^{\alpha} \end{Bmatrix} \iff \left\{\overline{\mathcal{P}}^{a} \overset{\Delta}{=} 0 \quad \text{and } d^{\alpha}{}_{\beta\gamma\delta} \text{ Petrov type I on } \Delta\right\}, \tag{5.53}$$

$$\left\{{}^{+}\mathcal{Q}^{\alpha} \overset{\Delta}{=} 0 \overset{\Delta}{=} {}^{-}\mathcal{Q}^{\alpha}\right\} \iff \left\{\overline{\mathcal{P}}^{a} \overset{\Delta}{=} 0 \quad \text{and } d^{\alpha}{}_{\beta\gamma\delta} \text{ Petrov type D on } \Delta\right\}. \tag{5.54}$$

Proof. For $d_{\alpha\beta\gamma}{}^{\delta}$ of Petrov type I, set $^{-}\mathcal{W} = 0$ in eq. (5.50) which, by property ii) on page 15, gives the first two lines in eq. (5.53). If $^{-}Z = 0$, then $^{-}k^{\alpha}$ is a repeated principal null direction of $d_{\alpha\beta\gamma}{}^{\delta}$, which is incompatible with Petrov-type I. The same occurs if $^{+}\mathcal{Q}^{\alpha} = 0$. Thus, the third line in eq. (5.53) follows. The case of Petrov type-D $d_{\alpha\beta\gamma}{}^{\delta}$ is a consequence of weak orientation, together with what it is said at the end of remark 5.3.4. □

More can be said on the direction of propagation of the superenergy, in this case applying strong orientation,

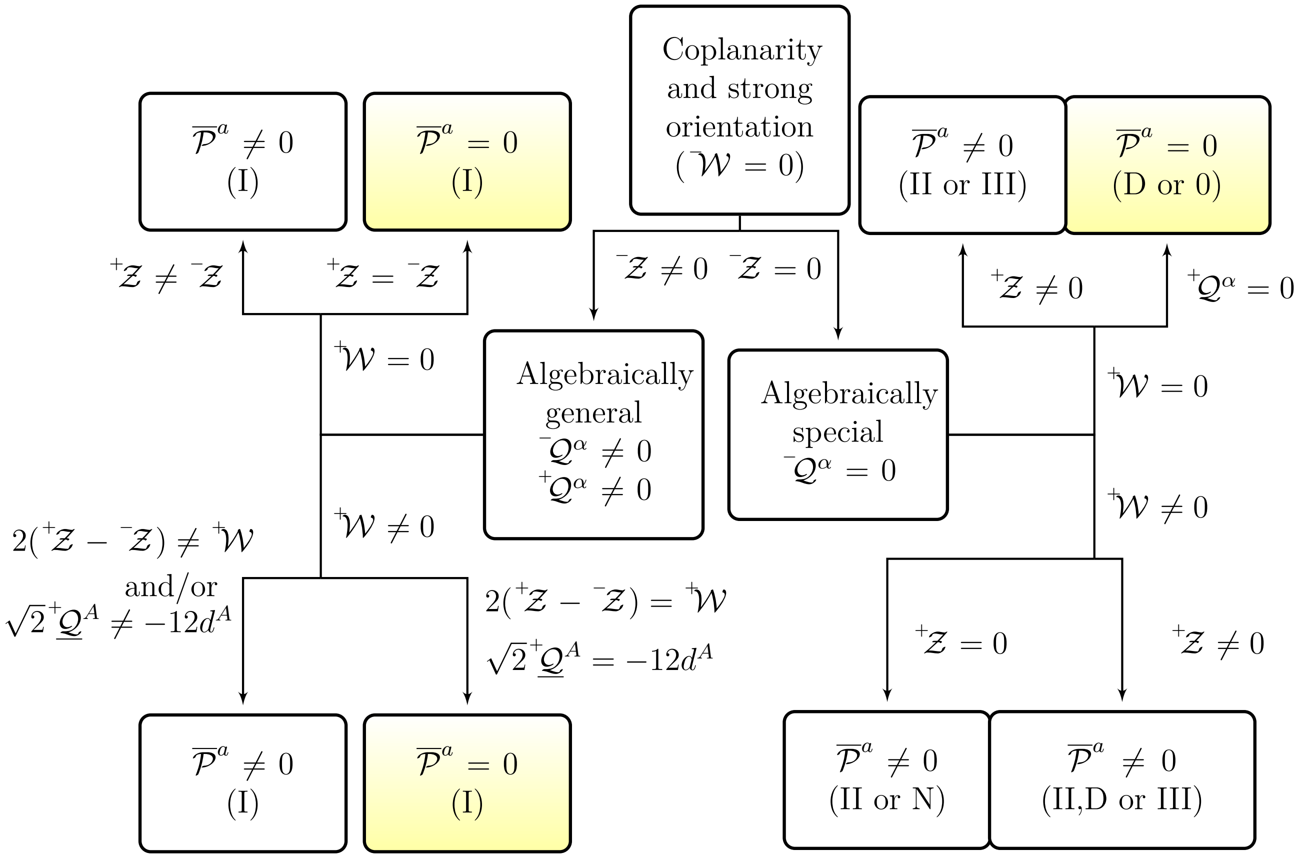

Figure 5.4: **Flow of the asymptotic superenergy quantities**. One starts from the above middle node: strong orientation is chosen ($-m^a$ points along the spatial projection to $\mathscr{J}$ of a PND of the rescaled Weyl tensor with highest multiplicity). Then, either the rescaled Weyl tensor is algebraically general (left-hand side of the diagram) or it is special (right-hand side of the diagram). Moving to the left, either the radiant superenergy $\overset{+}{\mathcal{W}}$ vanishes (above left-hand side) or not (below left-hand side). Thus, for an algebraically general rescaled Weyl tensor on $\mathscr{J}$, there are four configurations of asymptotic radiant superenergy: in two of them, there is gravitational radiation (one with $\overset{+}{\mathcal{W}} \neq 0$, the other one with $\overset{+}{\mathcal{W}} = 0$); in the other two there is no gravitational radiation (the shaded nodes). Moving to the right, one finds the algebraically special cases. There are four possibilities, from which just one corresponds to no radiation (the shaded node, for Petrov type D or 0, the only case in which both radiant supermomenta vanish).

Lemma 5.4.3. Choose m^a defining a strong orientation according to definition 5.4.2, and define $\overset{\pm}{k}^\alpha$ as in eq. (5.47). Let $\overline{\mathcal{P}}^a$ and $\overset{\pm}{\mathcal{Q}}^\alpha$ be the canonical asymptotic super-Poynting vector and the radiant supermomenta associated to $\overset{\pm}{k}^\alpha$, respectively. Then, the canonical asymptotic super-Poynting vector takes the form

$$\overline{\mathcal{P}}^a \stackrel{\mathscr{J}}{=} -\left(\frac{1}{2}\,{}^{+}\!\mathcal{Z} + \frac{1}{4}\,{}^{+}\!\mathcal{W}\right) m^a + \left(\frac{1}{2\sqrt{2}}\,{}^{+}\!\underline{\mathcal{Q}}^A + 3d^A\right) \underline{E}^a{}_A \ , \tag{5.55}$$

hence, the superenergy flux cannot propagate in directions orthogonal to m^a on $\mathscr{J}$. Furthermore,

$$m_s \overline{\mathcal{P}}^s \leq 0 \ , \tag{5.56}$$

equality holding if and only if $\overline{P}^a = 0$.

Proof. For the first part, one only has to plug ${}^{-}\!\mathcal{W} = 0 = {}^{-}\!\mathcal{Z}$ in eq. (2.84). For the second part, on the one hand, if $m_s \overline{\mathcal{P}}^s = 0$, then ${}^{+}\!\mathcal{W} = 0 = {}^{+}\!\mathcal{Z}$ and, by property ii) on page 15 and eq. (2.48), ${}^{+}\!\mathcal{Q}^\alpha = 0$. But then, since strong orientation requires ${}^{-}\!\mathcal{Q}^\alpha = 0$ (remark 5.4.5) using lemma 2.3.4 $\overline{\mathcal{P}}^a = 0$ follows. In that case there is no radiation according to criterion 1. On the other hand, if $m_s \overline{\mathcal{P}}^s \neq 0$, by the positivity of ${}^{+}\!\mathcal{Z}$ and ${}^{+}\!\mathcal{W}$, $m_s \overline{\mathcal{P}}^s < 0$ necessarily. □

Remark 5.4.6. Equation (5.56) supports the idea of considering any m^a defining a strong orientation as a good candidate for intrinsic 'evolution' direction. The reason is that directions orthogonal to m^a are *transversal* to the flux of superenergy, which can be thought to be associated with 'changes in the gravitational system'.

Remark 5.4.7. For $d_{\alpha\beta\gamma}{}^{\delta}$ of Petrov type I, the sign of $m_s \overline{\mathcal{P}}^s$ is not defined because this case is not strongly orientable and, by positivity of the radiant superenergy quantities, it is not determined in the general expression of eq. (2.84).

The Petrov characterisation of the rescaled Weyl tensor at $\mathscr{J}$ in terms of the asymptotic superenergy quantities is summarised in fig. 5.4.

The idea of having a preferred, intrinsic, 'evolution' direction, m^a, at $\mathscr{J}$ is conceptually important. Indeed, the existence of a congruence of curves intrinsic to $\mathscr{J}$ will serve to define further structure related to *absence of incoming radiation* and to the novel definition of symmetries in chapter 7. Write the decomposition of the covariant derivative of this vector field as –see appendix A.3 for details–

$$\overline{\nabla}_a m_b = m_a \underline{a}_b + \underline{\kappa}_{ab} + \underline{\omega}_{ab} \ , \tag{5.57}$$

where the shear of m_a is defined as the traceless part of $\underline{\kappa}_{ab}$,

$$\underline{\Sigma}_{ab} := \underline{\kappa}_{ab} - \frac{1}{2} \underline{P}_{ab} \underline{P}^{cd} \underline{\kappa}_{cd} \ . \tag{5.58}$$

Also, define the symmetric, traceless part of the symmetrised, projected derivative of $\bar{k}_{\alpha}$,

$$\bar{\underline{\sigma}}_{\alpha\beta} \overset{\mathscr{J}}{:=} \underline{P}^{\mu}{}_{\alpha}\underline{P}^{\mu}{}_{\beta}\nabla_{(\mu}\bar{k}_{\nu)} - \frac{1}{2}\underline{P}_{\alpha\beta}\underline{P}^{\mu\nu}\nabla_{\mu}\bar{k}_{\nu} \ . \tag{5.59}$$

This is, of course, the shear associated to $\bar{k}^{\alpha}$. It coincides up to a factor with the shear of m_a,

$$\sqrt{2}\bar{\underline{\sigma}}_{\alpha\beta} \overset{\mathscr{J}}{=} -\underline{P}^{\mu}{}_{\alpha}\underline{P}^{\mu}{}_{\beta}\nabla_{(\mu}m_{\nu)} + \frac{1}{2}\underline{P}_{\alpha\beta}\underline{P}^{\mu\nu}\nabla_{\mu}m_{\nu} \overset{\mathscr{J}}{=} -\underline{\Sigma}_{\alpha\beta}, \tag{5.60}$$

where we have used eqs. (3.31) and (5.1) and $\underline{\Sigma}_{\alpha\beta} := \omega_{\alpha}{}^{a}\omega_{\beta}{}^{b}\underline{\Sigma}_{ab}$. In addition, let us introduce the expansion of $\bar{k}^{\alpha}$,

$$\bar{\theta} := \underline{P}^{\mu\nu}\nabla_{\mu}\bar{k}_{\nu} \ . \tag{5.61}$$

It is possible to formulate an asymptotic 'Goldberg-Sachs'-like theorem:

Lemma 5.4.4. On the neighbourhood of $\mathscr{J}$ where n_{α} is well defined choose an extension of m_{α} such that $n_{\alpha}m^{\alpha} = 0$ and $m_{\alpha}m^{\alpha} = 1$ there. Assume that $d_{\alpha\beta\gamma}{}^{\delta} \overset{\mathscr{J}}{\neq} 0$ and $\overset{+}{k}{}^{\beta}y_{A\beta C} \overset{\mathscr{J}}{=} \bar{k}^{\beta}y_{A\beta C} \overset{\mathscr{J}}{=} 0 \overset{\mathscr{J}}{=} \bar{k}^{\beta}\overset{+}{k}{}^{\gamma}y_{A\beta\gamma}$. Then,

$$\pounds_{\vec{n}}\bar{\underline{D}}_{\alpha\beta} \overset{\mathscr{J}}{=} \bar{\underline{D}}_{\alpha\beta} \overset{\mathscr{J}}{=} 0, \quad \pounds_{\vec{n}}\bar{\underline{D}}_{\alpha} \overset{\mathscr{J}}{=} \bar{\underline{D}}_{\alpha} \overset{\mathscr{J}}{=} 0 \implies \bar{\underline{\sigma}}_{\alpha\beta} \overset{\mathscr{J}}{=} 0.$$

Remark 5.4.8. The condition $\bar{\underline{D}}_{\alpha\beta} \overset{\mathscr{J}}{=} \bar{\underline{D}}_{\alpha} \overset{\mathscr{J}}{=} 0$ is equivalent to $\bar{\underline{\mathcal{Q}}}_{\alpha} = 0$ and, therefore, to saying that m_{α} defines a strong orientation on $\mathscr{J}$ and $\bar{k}^{\alpha}$ is a repeated principal null direction of $d_{\alpha\beta\gamma}{}^{\delta}$.

Remark 5.4.9. The assumption on the components of the Cotton-York tensor, $\overset{+}{k}{}^{\beta}y_{A\beta C} \overset{\mathscr{J}}{=} \bar{k}^{\beta}y_{A\beta C} \overset{\mathscr{J}}{=} 0 \overset{\mathscr{J}}{=} \bar{k}^{\beta}\overset{+}{k}{}^{\gamma}y_{A\beta\gamma}$, is satisfied if the rescaled energy momentum tensor $T_{\alpha\beta}$ fulfils the corresponding equations coming form eq. (3.71). In particular, given eq. (3.72), the assumption is satisfied in vacuum or if the physical energy-momentum tensor $\hat{T}_{\alpha\beta}$ decays towards infinity as $\hat{T}_{\alpha\beta}|_{\mathscr{J}} \sim \mathcal{O}\left(\Omega^{p}\right)$ with $p > 2$.

Proof. We will need the Bianchi identities written in terms of the lightlike components of the rescaled Weyl tensor, which can be found in appendix B –recall that one has to substitute the over ring by an underbar in quantities carrying uppercase Latin indices A, B, C, etc. Under the assumptions above $\bar{\underline{t}}_{ABC} \overset{\mathscr{J}}{=} 0$ and, using eq. (2.44), eq. (B.8) reads

$$
\begin{aligned}
0 \stackrel{\mathscr{J}}{=} & -\underline{E}^{\omega}{}_{A}\underline{E}^{\sigma}{}_{C}\,{}^{-}k^{\mu}\nabla_{\mu}\,{}^{-+}\underline{D}_{\omega\sigma} + \sqrt{2}\,{}^{+}\underline{D}_{C}\underline{E}^{\omega}{}_{A}\,{}^{-}k^{\mu}\nabla_{\mu}\,{}^{-}k_{\omega} \\
& + 2\sqrt{2}\underline{q}_{A[C}\underline{E}^{\sigma}{}_{M]}\,{}^{+}\underline{D}^{M}\,{}^{-}k^{\mu}\nabla_{\mu}\,{}^{-}k_{\sigma} - {}^{-+}\underline{D}_{AC}\underline{P}^{\mu\tau}\nabla_{\mu}\,{}^{-}k_{\tau} \\
& + {}^{-+}\underline{D}_{A\tau}\underline{P}^{\mu\tau}\underline{E}^{\sigma}{}_{C}\nabla_{\mu}\,{}^{-}k_{\sigma} - 2\,{}^{-+}\underline{D}_{[\tau C]}\underline{P}^{\mu\tau}\underline{E}^{\omega}{}_{A}\nabla_{\mu}\,{}^{-}k_{\omega} \\
& - D\left(\underline{E}^{\lambda}{}_{C}\underline{E}^{\mu}{}_{A} - \underline{q}_{AC}\underline{P}^{\lambda\mu}\right)\nabla_{\mu}\,{}^{-}k_{\lambda}\ .
\end{aligned}
\tag{5.62}
$$

Taking the symmetric traceless part of this equation and noting property iii) on page 211, after some manipulation, eq. (5.62) is expressed as

$$
0 \stackrel{\mathscr{J}}{=} -\frac{3}{2}D\,{}^{-}\underline{\sigma}_{AC} + C\frac{3}{2}\underline{E}^{\mu}{}_{D}\underline{\epsilon}_{E(A}\underline{E}^{\sigma}{}_{C)}\underline{q}^{ED}\nabla_{\mu}\,{}^{-}k_{\sigma} - \frac{3}{4}q_{AC}\underline{\epsilon}^{MN}\,{}^{-}\underline{\omega}_{MN}\ , \tag{5.63}
$$

where we have split $\underline{E}^{\mu}{}_{A}\underline{E}^{\nu}{}_{B}\nabla_{\mu}\,{}^{-}k_{\nu}$ into its symmetric and antisymmetric parts, introduced (5.59) and defined ${}^{-}\underline{\omega}_{AB} := \underline{E}^{\mu}{}_{[A}\underline{E}^{\nu}{}_{B]}\nabla_{\mu}\,{}^{-}k_{\nu}$. Note that in two dimensions we have

$$
{}^{-}\underline{\omega}_{AB} = \frac{1}{2}\underline{\epsilon}_{AB}\underline{\epsilon}^{CD}\,{}^{-}\underline{\omega}_{CD}\ , \tag{5.64}
$$

which after substitution into eq. (5.63) leads to

$$
0 \stackrel{\mathscr{J}}{=} -\frac{3}{2}\left(D\,{}^{-}\underline{\sigma}_{AC} + C\,{}^{-}\underline{\sigma}_{E(C}\mathring{\epsilon}_{A)}{}^{E}\right)\ . \tag{5.65}
$$

Equation (5.65) requires either ${}^{-}\underline{\sigma}_{AB} \stackrel{\mathscr{J}}{=} 0$ or

$$
{}^{-}\underline{\sigma}_{AB} \stackrel{\mathscr{J}}{\neq} 0, \quad C \stackrel{\mathscr{J}}{=} D \stackrel{\mathscr{J}}{=} 0. \tag{5.66}
$$

If condition (5.66) holds, then ${}^{-+}\underline{D}_{\alpha\beta} = 0$ by property iii) on page 211, and we have to consider eq. (B.4) –taking into account eq. (5.66) and using eq. (2.44)–,

$$
0 \stackrel{\mathscr{J}}{=} 2\sqrt{2}\,{}^{+}\underline{D}^{M}\underline{E}^{\mu}{}_{(M}\underline{E}^{\omega}{}_{A)}\nabla_{\mu}\,{}^{-}k_{\omega} - \sqrt{2}\,{}^{+}\underline{D}_{A}\underline{P}^{\mu\sigma}\nabla_{\mu}\,{}^{-}k_{\sigma}\ , \tag{5.67}
$$

and in terms of ${}^{-}\underline{\sigma}_{AB}$,

$$
0 \stackrel{\mathscr{J}}{=} {}^{-}\underline{\sigma}_{AM}\,{}^{+}\underline{D}^{M}\ . \tag{5.68}
$$

Because we are working in 2 dimensions and ${}^{-}\underline{\sigma}_{AB}$ is traceless, it cannot have eigenvectors with zero eigenvalue and, thus, eq. (5.68) implies, if ${}^{-}\underline{\sigma}_{AB} \stackrel{\mathscr{J}}{\neq} 0$, that

$$
{}^{-}\underline{\sigma}_{AB} \stackrel{\mathscr{J}}{\neq} 0, \quad {}^{+}\underline{D}_{A} \stackrel{\mathscr{J}}{=} 0\ . \tag{5.69}
$$

If condition (5.69) holds, eq. (B.2) reads $({}^{-}k^{\mu}g^{\alpha\beta}y_{\mu\alpha\beta} \stackrel{\mathscr{J}}{=} -{}^{-}k^{\mu}\,{}^{+}k^{\alpha}\,{}^{-}k^{\beta}y_{\mu\alpha\beta} + {}^{-}k^{\mu}\underline{P}^{\lambda\nu}y_{\mu\lambda\nu} \stackrel{\mathscr{J}}{=}$

$-\bar{k}^\mu {}^+k^\alpha \bar{k}^\beta y_{\mu\alpha\beta} \stackrel{\mathscr{J}}{=} 0)$

$$0 \stackrel{\mathscr{J}}{=} \bar{k}^\mu \nabla_\mu D \;, \tag{5.70}$$

which, recalling eq. (5.66), tells us that $\pounds_{\vec{n}} D \stackrel{\mathscr{J}}{=} 0 \stackrel{\mathscr{J}}{=} {}^+k^\mu \nabla_\mu D$. Considering eqs. (5.66), (5.68) and (5.69), eq. (B.3) gives

$$0 \stackrel{\mathscr{J}}{=} {}^+\underline{D}_{\alpha\tau} \underline{P}^{\tau\mu} \nabla_\mu \bar{k}^\alpha \;, \tag{5.71}$$

or by means of $\bar{\underline{\sigma}}_{AB}$, recalling property vii) on page 211 and ${}^+\underline{D}_{AB} = {}^+\underline{D}_{BA}$,

$$0 \stackrel{\mathscr{J}}{=} {}^+\underline{D}^{CM} \bar{\underline{\sigma}}_{CM} \;. \tag{5.72}$$

Next, taking into account all the quantities that vanish so far, it can be shown that the trace of eq. (B.9) gives eq. (5.72) again, while contracting it with $\underline{\epsilon}^{AC}$ gives (${}^+k^\mu g^{\alpha\beta} y_{\alpha\mu\beta} \stackrel{\mathscr{J}}{=} -{}^+k^\mu \bar{k}^\alpha \bar{k}^\beta y_{\alpha\mu\beta} + {}^+k^\mu \underline{P}^{\lambda\nu} y_{\lambda\mu\nu} \stackrel{\mathscr{J}}{=} -{}^+k^\mu {}^+k^\alpha \bar{k}^\beta y_{\alpha\mu\beta} \stackrel{\mathscr{J}}{=} 0$)

$$0 \stackrel{\mathscr{J}}{=} -{}^+k^\mu \nabla_\mu C + \underline{\epsilon}^{AC}\, {}^+\mathring{D}_A{}^E \left(\bar{\underline{\omega}}_{EC} + \bar{\underline{\sigma}}_{EC} \right), \tag{5.73}$$

where we have taken into account ${}^+\underline{D}_{AB} = {}^+\underline{D}_{BA}$. Equation (5.64) and property vii) on page 211 simplify eq. (5.73) to

$$0 \stackrel{\mathscr{J}}{=} -{}^+k^\mu \nabla_\mu (C) + \underline{\epsilon}^{AC}\, {}^+\mathring{D}_A{}^E \bar{\underline{\sigma}}_{EC} \;. \tag{5.74}$$

Back to eq. (5.62), using eqs. (5.66), (5.69) and (5.70), we arrive at

$$n^\mu \nabla_\mu C \stackrel{\mathscr{J}}{=} 0 \;. \tag{5.75}$$

Then, eq. (5.73) reads simply

$$0 \stackrel{\mathscr{J}}{=} \underline{\epsilon}^{AC}\, {}^+\mathring{D}_A{}^E \bar{\underline{\sigma}}_{EC}. \tag{5.76}$$

It is easily shown, given that $\bar{\underline{\sigma}}_{AB}$ and ${}^+\underline{D}_{AB}$ are both symmetric and traceless, that eqs. (5.72) and (5.76) imply –e.g., by writing these equations in components $A = 2, 3$–

$$\bar{\underline{\sigma}}_{AB} \stackrel{\mathscr{J}}{\neq} 0 \implies {}^+\underline{D}_{AB} \stackrel{\mathscr{J}}{=} 0. \tag{5.77}$$

But ${}^+\underline{D}_{AB} \stackrel{\mathscr{J}}{=} 0$ together with eqs. (5.66) and (5.69) and the assumptions $\bar{\underline{D}}_{AB} \stackrel{\mathscr{J}}{=} 0 \stackrel{\mathscr{J}}{=} \bar{\underline{D}}_A$ leads to $d_{\alpha\beta\gamma}{}^\delta \stackrel{\mathscr{J}}{=} 0$. This follows from the fact that in this case ${}^\mp\mathcal{W} \stackrel{\mathscr{J}}{=} {}^\mp\mathcal{Z} \stackrel{\mathscr{J}}{=} \mathcal{V} \stackrel{\mathscr{J}}{=} 0$ –see eqs. (2.52) to (2.55) and (2.58)– which, by lemma 2.3.1 and eq. (2.19) implies $d_{\alpha\beta\gamma}{}^\delta \stackrel{\mathscr{J}}{=} 0$. Alternatively, use eqs. (2.42) and (2.43) to show that $D_{ab} \stackrel{\mathscr{J}}{=} C_{ab} \stackrel{\mathscr{J}}{=} 0 (\iff d_{\alpha\beta\gamma}{}^\delta \stackrel{\mathscr{J}}{=} 0)$. However, this contradicts one of the assumptions of the theorem. Therefore, the only possibility is

$$\underline{\bar{\sigma}}_{AB} \overset{\mathscr{J}}{=} 0 \; . \tag{5.78}$$

□

Lemma 5.4.4 is in fact a result on m^a as well, noting eq. (5.60):

Corollary 5.4.1. Under the same assumptions of Lemma 5.4.4, its conclusion can be equivalently stated as

$$\underline{\Sigma}_{ab} \overset{\mathscr{J}}{=} 0.$$

5.5 The $\Lambda = 0$ limit

In the preceding sections, we have discussed some differences and analogies of the $\Lambda > 0$ and $\Lambda = 0$ scenarios. Concerning the characterisation of gravitational radiation, one can study the limit to $\Lambda = 0$ of criterion 1. In this subsection we will assume that $\lim_{\Lambda \to 0} g_{\alpha\beta}$ exists and defines a good Lorentzian metric.

The limit of the normal to $\mathscr{J}$, $N_\alpha\Big|_{\Lambda=0}$, coincides with the normal to $\mathscr{J}_0$, the conformal boundary for $\Lambda = 0$. Also, we have already mentioned in section 5.3 that the asymptotic supermomentum p^α (5.34) has a good limit to $\Lambda = 0$,

$$\lim_{\Lambda \to 0} p^\alpha \overset{\mathscr{J}_0}{=} \mathcal{Q}^\alpha \; , \tag{5.79}$$

where $\mathcal{Q}^\alpha$ is the asymptotic *radiant* supermomentum at $\mathscr{J}_0$ –see chapter 4–

$$\mathcal{Q}^\alpha \overset{\mathscr{J}_0}{=} -\left(N^\mu N^\nu N^\rho \mathcal{D}^\alpha{}_{\mu\nu\rho}\right)\Big|_{\Lambda=0} \; . \tag{5.80}$$

Therefore, the absence of gravitational radiation in the $\Lambda > 0$ case according to criterion 1 implies that the asymptotic radiant supermomentum $\mathcal{Q}^\alpha$ vanishes in the $\Lambda = 0$ counterpart and, in consequence, that the news tensor vanishes there so that there is no radiation –see theorem 3. This limit reinforces the validity of criterion 1.

Apart from p^α, it is possible to study the limit of the radiant supermomenta of eqs. (5.48) and (5.49). The first thing to do is to define a couple of lightlike vector fields on $\mathscr{J}$ in a way that their limit to $\Lambda = 0$ is well-behaved. This can be achieved by multiplying the expressions on the right-hand side of eqs. (7.6) and (7.7) by N,

$$\overset{+}{K}{}^\alpha := \frac{1}{\sqrt{2}}\left(N^\alpha + M^\alpha\right) \; , \tag{5.81}$$

$$\overset{-}{K}{}^\alpha := \frac{1}{\sqrt{2}}\left(N^\alpha - M^\alpha\right) \; , \tag{5.82}$$

where $M^\alpha := Nm^\alpha$ with the following normalisations:

$$g_{\mu\nu}\,{}^{-}K^{\mu}\,{}^{+}K^{\nu} = N^2, \quad g_{\mu\nu}M^\mu M^\nu = N^2, \quad g_{\mu\nu}N^\mu M^\nu = 0. \tag{5.83}$$

Vector fields on $\mathscr{J}$ of the kind of M^α obey:

Lemma 5.5.1. Assume that $\lim_{\Lambda\to 0} g_{\alpha\beta}$ exists and let M^α be any vector field on $\mathscr{J}$ whose norm is proportional to a positive power of the cosmological constant Λ. Then,

$$\lim_{\Lambda\to 0} M^\alpha \overset{\mathscr{J}_0}{=} BN^\alpha\Big|_{\Lambda=0}, \tag{5.84}$$

for some function B which may have zeros.

Proof. We know that the limit $N^\alpha\Big|_{\Lambda=0}$ does not vanish and is lightlike at $\mathscr{J}_0$. Then, we have

$$\lim_{\Lambda\to 0}\left(g_{\mu\nu}M^\mu M^\nu\right) = \lim_{\Lambda\to 0} f\Lambda^p = 0\ , \tag{5.85}$$

$$\lim_{\Lambda\to 0}\left(g_{\mu\nu}M^\mu N^\nu\right) = \lim_{\Lambda\to 0} 0 = 0 \overset{\mathscr{J}_0}{=} \lim_{\Lambda\to 0}\left(g_{\mu\nu}M^\mu\right) N^\nu\Big|_{\Lambda=0}\ , \tag{5.86}$$

where f is a function and $p \in \mathbb{R}$, $p > 0$. The first of this formulae implies that the limit of M^α is either lightlike or zero at $\mathscr{J}_0$. Taking this into account, the second formula indicates that, if different from zero, the limit of M^α has to be proportional to N^α –as the scalar product of two non-vanishing lightlike vector fields is zero if and only if they are collinear. □

Then, by lemma 5.5.1, the limit of ${}^{\pm}K^\alpha$ reads

$$\lim_{\Lambda\to 0} {}^{\pm}K^\alpha \overset{\mathscr{J}_0}{=} \frac{1}{\sqrt{2}}\left(1 \pm B\right) N^\alpha\Big|_{\Lambda=0}\ . \tag{5.87}$$

After this, define the radiant supermomenta associated to ${}^{\pm}K^\alpha$,

$$\overset{+}{q}{}^\alpha := -\,{}^{+}K^{\mu}\,{}^{+}K^{\nu}\,{}^{+}K^{\rho}\mathcal{D}^{\alpha}{}_{\mu\nu\rho}\ , \tag{5.88}$$

$$\overset{-}{q}{}^\alpha := -\,{}^{-}K^{\mu}\,{}^{-}K^{\nu}\,{}^{-}K^{\rho}\mathcal{D}^{\alpha}{}_{\mu\nu\rho}\ . \tag{5.89}$$

these are nothing else than the radiant supermomenta given in eqs. (5.48) and (5.49) appropriately rescaled by a factor N^3.

Lemma 5.5.2. Assume that $\lim_{\Lambda\to 0} g_{\alpha\beta}$ exists. The radiant supermomenta (5.88, 5.89) have, respectively, the regular limit

$$\lim_{\Lambda\to 0} \overset{\pm}{q}{}^\alpha \overset{\mathscr{J}_0}{=} \frac{1}{2\sqrt{2}}\left(1 \pm B\right)^3 \mathcal{Q}^\alpha \tag{5.90}$$

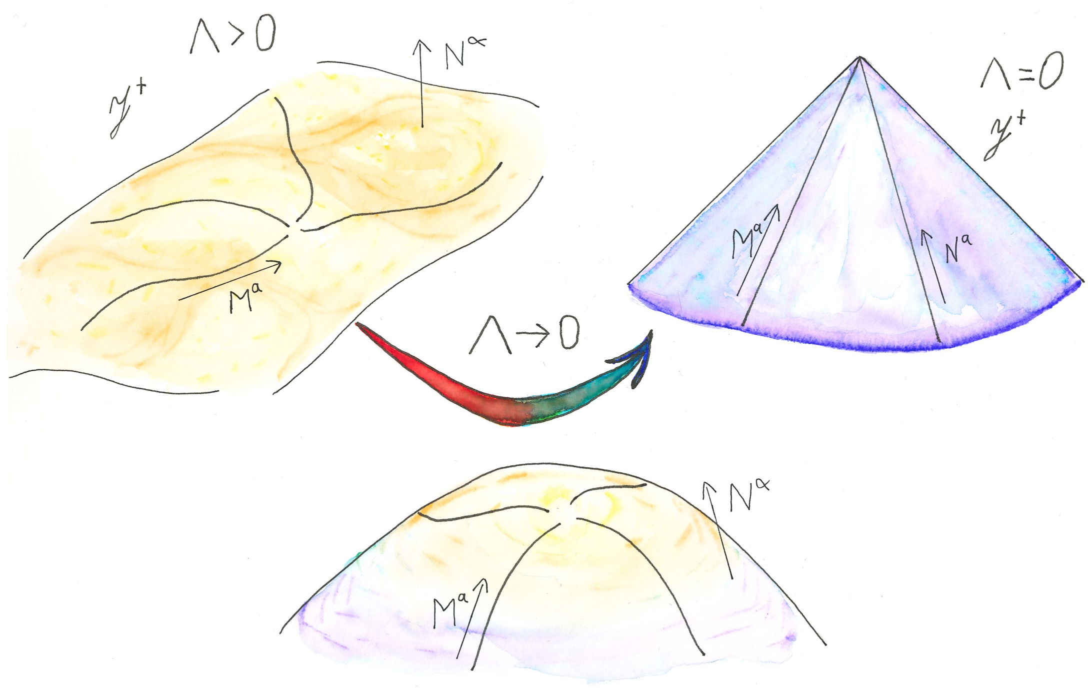

Figure 5.5: In the $\Lambda = 0$ limit vector fields of the class described in lemma 5.5.1 become collinear with $N^\alpha\Big|_{\Lambda=0}$, the vector field tangent to the null generators of $\mathscr{J}_0$.

where $\mathcal{Q}^\alpha$ is the asymptotic radiant supermomentum (5.80) on $\mathscr{J}_0$ for a vanishing cosmological constant and B is a function which may have zeros. Moreover,

$$\lim_{\Lambda\to 0} \overset{+}{q}{}^\alpha = 0 = \lim_{\Lambda\to 0} \overset{-}{q}{}^\alpha \iff \mathcal{Q}^\alpha \overset{\mathscr{J}_0}{=} 0 \iff \text{No gravitational radiation at } \mathscr{J}_0 .$$

Proof. The limit of $\overset{\pm}{q}{}^\alpha$ is computed using eq. (5.87). Then, one notices that for non-vanishing $\overset{\pm}{q}{}^\alpha$, it is not possible that both radiant supermomenta vanish simultaneously in the limit unless $\mathcal{Q}^\alpha \overset{\mathscr{J}_0}{=} 0$. The reason is that

$$B = 1 \implies \lim_{\Lambda\to 0} \overset{-}{q}{}^\alpha = 0, \quad \lim_{\Lambda\to 0} \overset{+}{q}{}^\alpha \overset{\mathscr{J}_0}{=} \frac{1}{2\sqrt{2}} (1+B)^3 \mathcal{Q}^\alpha , \tag{5.91}$$

$$B = -1 \implies \lim_{\Lambda\to 0} \overset{-}{q}{}^\alpha \overset{\mathscr{J}_0}{=} \frac{1}{2\sqrt{2}} (1-B)^3 \mathcal{Q}^\alpha, \quad \lim_{\Lambda\to 0} \overset{+}{q}{}^\alpha = 0 , \tag{5.92}$$

$$B \neq \pm 1 \implies \lim_{\Lambda\to 0} \overset{-}{q}{}^\alpha \overset{\mathscr{J}_0}{=} \frac{1}{2\sqrt{2}} (1-B)^3 \mathcal{Q}^\alpha, \quad \lim_{\Lambda\to 0} \overset{+}{q}{}^\alpha \overset{\mathscr{J}_0}{=} \frac{1}{2\sqrt{2}} (1+B)^3 \mathcal{Q}^\alpha . \tag{5.93}$$

Hence, if we assume $B = 1$, $\lim_{\Lambda\to 0} \overset{+}{q}{}^\alpha = 0 \iff \mathcal{Q}^\alpha \overset{\mathscr{J}_0}{=} 0$. But if $B = -1$, then $\lim_{\Lambda\to 0} \overset{-}{q}{}^\alpha = 0 \iff \mathcal{Q}^\alpha \overset{\mathscr{J}_0}{=} 0$. Finally, if $B \neq \pm 1$ the only possibility is $\lim_{\Lambda\to 0} \overset{-}{q}{}^\alpha = 0 =$

$\lim_{\Lambda\to 0} \overset{+}{q}{}^{\alpha} \Longleftrightarrow \mathcal{Q}^{\alpha} \overset{\mathscr{I}_0}{=} 0.$ □

Corollary 5.5.1. If one (and only one) of the asymptotic radiant supermomenta $\overset{\pm}{q}{}^{\alpha}$ of eqs. (5.88) and (5.89) vanishes, then $B = \mp 1$ and

$$\lim_{\Lambda\to 0} \overset{\pm}{M}{}^{\alpha} \overset{\mathscr{I}_0}{=} \pm N^{\alpha}\Big|_{\Lambda=0} , \tag{5.94}$$

$$\lim_{\Lambda\to 0} \overset{\pm}{K}{}^{\alpha} \overset{\mathscr{I}_0}{=} 0 , \tag{5.95}$$

$$\lim_{\Lambda\to 0} \overset{\mp}{K}{}^{\alpha} \overset{\mathscr{I}_0}{=} \sqrt{2} N^{\alpha}\Big|_{\Lambda=0} . \tag{5.96}$$

Proof. From the proof in lemma 5.5.2, if $\overset{\pm}{q}{}^{\alpha} = 0$, one has $B = \mp 1$. Setting the corresponding value of B in eqs. (5.84) and (5.87) gives eqs. (5.94) to (5.96). □

These results have a particularly interesting interpretation regarding incoming versus outgoing radiation and intrinsic evolution directions that will be presented in section 7.3.

Allí.
En el fondo.
Al filo.
Donde Nietzsche escribía:
"Di tu palabra y rómpete".
Donde nadie te espera.

Benjamín Prado, *Límite*. *Todos nosotros*, 1998.

6 | In the search for news

In sections 5.3 and 5.4, we presented a gravitational radiation condition at infinity and a characterisation of the asymptotic Petrov type of the rescaled Weyl tensor. The latter was related to the directional-dependence problem and with the definition of an intrinsic 'evolution direction' within $\mathscr{J}$. A further step forward in the characterisation of gravitational radiation would be to find a *news tensor*, i.e., an object describing in the *full, covariant* theory the two radiative degrees of freedom of the gravitational field like in in the $\Lambda = 0$ case –see chapter 5. We wonder if a similar tensor may exist in the presence of a positive cosmological constant and, if so, under which conditions.

It is worth recalling that historically the news tensor for $\Lambda = 0$ has been understood from different perspectives: a term in the asymptotic expansion of the metric, the derivative along N^a of the shear tensor of outgoing null geodesics or, the most robust representation, a rank-2 symmetric tensor field *intrinsic* to $\mathscr{J}$, orthogonal to N^a everywhere and depending on the geometry of $(\mathscr{J}, h_{ab})$ only. However, none of these approaches can be completely successful if $\Lambda > 0$, since such a tensor, if it exists, must contain information related to the electric part of the rescaled Weyl tensor; this is studied in sections 6.3 and 6.4 and also in section 7.2.

For the rest of the chapter, we will work with an arbitrary cut, as introduced in section 5.1. We denote by $\{E^\alpha{}_A\}$ any basis of vector fields on $\mathcal{S}$ and by r^a the unit normal to the cut within $\mathscr{J}$. Let us emphasise that r^a is defined at least on $\mathcal{S}$ but not necessarily everywhere on $\mathscr{J}$, neither it is tangent to a congruence of curves on $\mathscr{J}$ in general. The metric q_{AB} is inherited from the ambient metric h_{ab} and we denote the second fundamental form, its trace and the shear by $\mathring{\kappa}_{AB}$, $\mathring{\kappa}$ and $\mathring{\Sigma}_{AB}$, respectively. Also, $\mathring{\epsilon}_{AB}$ is the intrinsic, canonical, volume two-form of the metric q_{AB}. For more details, see appendix A.2. In the same fashion as eq. (5.47), we introduce a pair of vector fields $\overset{\pm}{k}{}^\alpha$, defined at least on $\mathcal{S}$. Notice that $\overset{\pm}{k}{}_\alpha E^\alpha{}_A = 0$. Let us present some useful relations involving the intrinsic Schouten tensor $\overline{S}_{ab}$ and the extrinsic curvature of $\mathcal{S}$. First, define the tangent and orthogonal components to $\mathcal{S}$ –according to the general notation (2.35, 2.38)– as

$$\overline{S}_{ab} \stackrel{\mathcal{S}}{=} \overline{S} r_a r_b + 2\mathring{S}_B r_{(a} W_{b)}{}^B + \mathring{S}_{AB} W_a{}^A W_b{}^B \ . \tag{6.1}$$

After this, project eq. (3.85) thrice with $E^a{}_A$ and once with r^a and use eq. (5.8) to obtain

$$q_{A[C}\mathring{S}_{B]} = \mathcal{D}_{[C}\mathring{\kappa}_{B]A} \ , \tag{6.2}$$

whose trace reads

$$\mathring{S}_B = \mathcal{D}_C \mathring{\kappa}_B{}^C - \mathcal{D}_B \mathring{\kappa} \ . \tag{6.3}$$

Also, by eq. (5.8) and eqs. (A.31) and (A.32), it can be seen that

$$\overline{S} \stackrel{\mathcal{S}}{=} -\frac{1}{2}K + \overline{R}_{ab} r^a r^b \stackrel{\mathcal{S}}{=} \frac{1}{2}K - \mathring{S}^E{}_E \ , \tag{6.4}$$

$$\mathring{S}^E{}_E = K + \frac{1}{2}\mathring{\Sigma}^2 - \frac{1}{4}\mathring{\kappa}^2 \quad . \tag{6.5}$$

Here, K is the Gaussian curvature of the cut, which is related to its scalar curvature as $K = \mathring{R}/2$ —see appendix A.2 for more details.

6.1 General considerations

The news tensor in asymptotically flat space-times vanishes if and only if the radiant asymptotic super-Poynting does so –see chapter 4; indeed, the asymptotic superenergy acts as source for the news tensor. In the presence of a positive cosmological constant, however, the asymptotic supermomentum is not radiant. Thus, a question arises: do we look for a news tensor which can be associated to a radiant supermomentum in a similar fashion as in the $\Lambda = 0$ case or, alternatively, one that vanishes if and only if the asymptotic super-Poynting vanishes? In section 6.3 we will present a general programme valid for both possibilities, while in section 6.4 we will explore thoroughly the first one.

Generically, we expect any news-like object to have some basic properties. First of all, the would-be news tensor must appear at the energy-density level. From this point of view, it is reasonable to think that the gravitational radiative degrees of freedom cannot be extracted by local methods alone — for a discussion in the asymptotically flat case, see [102]. For $\Lambda = 0$, $\mathscr{J}$ is naturally foliated by two dimensional cuts; this is not the case for $\Lambda > 0$ in general, and for that reason we are just considering a single cut $\mathcal{S}$. Another important difference is that in the $\Lambda = 0$ case any cut has a unique, lightlike, orthogonal (outgoing for $\mathscr{J}^+$) direction that escapes from the space-time and is linearly independent of the (incoming for $\mathscr{J}^+$) lightlike direction given by the generators of $\mathscr{J}$.

For $\Lambda > 0$, there are always two independent, future lightlike directions orthogonal to any cut $\mathcal{S}$ on $\mathscr{J}$ pointing out of (or into) the space-time, $\overset{\pm}{k}{}^{\alpha}$. Therefore, a priori there is no reason why there should be only one news-like tensor for each cut instead of two, one for each radiant supermomentum associated to $\overset{\pm}{k}{}^{\alpha}$. Secondly, to describe the radiative sector the would-be news tensor(s) should have two degrees of freedom. The most plausible object is a symmetric, traceless, rank-2 tensor. Thirdly, it has to be gauge invariant to have physical significance. Finally, a key feature that will guide us is that we want it to vanish if and only if some meaningful superenergy quantity vanishes, such as the radiant super-Poynting $\overset{\pm}{\underline{\mathcal{Q}}}{}^{\alpha}$ or the canonical asymptotic super-Poynting $\overline{\mathcal{P}}^{a}$. Thus, according to eqs. (2.15), (2.54) and (2.55), the news tensor has to carry information from *both* the magnetic C_{ab} and the electric D_{ab} parts of the rescaled Weyl tensor.

More concisely, the properties that the would-be news tensor is expected to have are:

i) Rank-2 tensor field on $\mathcal{S}$.

ii) Symmetric.

iii) Traceless.

iv) Gauge invariant.

v) Contain information related to C_{ab} and D_{ab}.

vi) Vanish if and only if some meaningful superenergy quantity does (e.g., $\overset{+}{\mathcal{Z}} = 0$ or/and $\overset{-}{\mathcal{Z}} = 0$, or $\overline{\mathcal{P}}^{a} = 0$).

6.2 A geometric result: the counterpart of Geroch's *tensor* ρ

Here we present an intermediate and crucial step in our search. It begins with the following lemma, where $\mathring{\omega}_A \overset{\mathcal{S}}{:=} \mathcal{D}_A\omega$:

Lemma 6.2.1. Let t_{AB} be any symmetric tensor field on $\mathcal{S}$ whose behaviour under conformal rescalings (C.15) is

$$\tilde{t}_{AB} = t_{AB} - a\frac{1}{\omega}\mathcal{D}_A\mathring{\omega}_B + \frac{2a}{\omega^2}\mathring{\omega}_A\mathring{\omega}_B - \frac{a}{2\omega^2}\mathring{\omega}_C\mathring{\omega}^C q_{AB} \tag{6.6}$$

for some fixed constant $a \in \mathbb{R}$. Then,

$$\tilde{\mathcal{D}}_{[C}\tilde{t}_{A]B} = \mathcal{D}_{[C}t_{A]B} + \frac{1}{\omega}\left(aK - t^{E}{}_{E}\right)\mathring{\omega}_{[C}q_{A]B} \ , \tag{6.7}$$

where K is the Gaussian curvature of $(\mathcal{S}, q_{AB})$. In particular, for any symmetric gauge-invariant tensor field B_{AB} on $\mathcal{S}$,

$$\tilde{\mathcal{D}}_{[C}\tilde{B}_{A]B} = \mathcal{D}_{[C}B_{A]B} - \frac{1}{\omega}B^{E}{}_{E}\mathring{\omega}_{[C}q_{A]B} \tag{6.8}$$

Remark 6.2.1. This results applies locally and it is valid for any Riemannian surface, independently of the topology.

Proof. Using the formulae in appendix C for cuts, a direct calculations yields

$$\tilde{\mathcal{D}}_{[C}\tilde{t}_{A]B} = \mathcal{D}_{[C}t_{A]B} + \frac{1}{\omega}t_{B[C}\mathring{\omega}_{A]} + \frac{1}{\omega}q_{B[C}t_{A]}{}^{D}\mathring{\omega}_{D} + \frac{1}{\omega}aK\mathring{\omega}_{[C}q_{A]B}\ . \tag{6.9}$$

Then, one uses the two-dimensional identity [121]

$$A_{CA}{}^{E} = 2\delta^{E}_{[A}A_{C]D}{}^{D},\ \text{for any tensor such that}\ A_{CA}{}^{E} = -A_{AC}{}^{E} \tag{6.10}$$

in order to write

$$\frac{1}{\omega}t_{B[C}\mathring{\omega}_{A]} + \frac{1}{\omega}q_{B[C}t_{A]}{}^{D}\mathring{\omega}_{D} = -\frac{1}{\omega}t^{E}{}_{E}\mathring{\omega}_{[C}q_{A]B}\ , \tag{6.11}$$

arriving at the final result. For a gauge invariant tensor $a = 0$ in eq. (6.6), therefore one only has to set this value in eq. (6.7) to obtain eq. (6.8). □

Corollary 6.2.1. A symmetric gauge-invariant tensor field m_{AB} on $\mathcal{S}$ satisfies

$$\tilde{\mathcal{D}}_{[C}\tilde{m}_{B]A} = \mathcal{D}_{[C}m_{B]A} \tag{6.12}$$

if and only if $m^{E}{}_{E} = 0$.

Corollary 6.2.2 (The tensor ρ)**.** If $\mathcal{S}$ has $\mathbb{S}^2$-topology, there is a unique symmetric tensor field ρ_{AB} whose behaviour under conformal rescalingseq. (C.15) is as in (6.6) and satisfies the equation

$$\mathcal{D}_{[C}\rho_{A]B} = 0 \tag{6.13}$$

in any conformal frame. This tensor field must have a trace $\rho^{E}{}_{E} = aK$ and obeys

$$\pounds_{\vec{\chi}}\rho_{AB} = -a\mathcal{D}_{A}\mathcal{D}_{B}\phi \tag{6.14}$$

independently of the conformal frame, where χ^{A} is any CKVF of $(\mathcal{S}, q_{AB})$ and $\phi := \mathcal{D}_{M}\chi^{M}/2$. Specifically, it is invariant under transformations generated by KVF (and homothetic Killing vectors) of $(\mathcal{S}, q_{AB})$. Furthermore, it is given for round spheres by $\rho_{AB} = q_{AB}aK/2$.

Proof. Existence is proved by using the (trivial) L^2-orthogonality of the right-hand side of eq. (6.13) with all conformal Killing vectors on $\mathcal{S}$ (see for instance [134], appendix

H) or, more directly, by noticing that $\rho_{AB} = q_{AB} aK/2$ fulfils $\mathcal{D}_A \rho_{BC} = 0$ in the round metric sphere. Concerning uniqueness, notice that lemma 6.2.1 fixes the trace of ρ_{AB} to $\rho^E{}_E = aK$, and recall the assumption that eq. (6.13) holds in any gauge. Then, if two different solutions ${}_1\rho_{AB}$ and ${}_2\rho_{AB}$ exist, $\mathcal{D}_{[C}\left({}_1\rho_{A]B} - {}_2\rho_{A]B}\right) = 0$. However, in that case, the difference ${}_1\rho_{AB} - {}_2\rho_{AB}$ is a traceless, Codazzi tensor on $\mathbb{S}^2$ and, as a consequence of the uniqueness of this kind of tensors [113], ${}_1\rho_{AB} - {}_2\rho_{AB} = 0$. To show eq. (6.14), first define $M_{AB} := \pounds_{\vec{\chi}}\rho_{AB} + a\mathcal{D}_A\mathcal{D}_B\phi$. This tensor field is gauge invariant and using $\mathcal{D}_M\mathcal{D}^M\phi = -\pounds_{\vec{\chi}}K - 2\phi K$ (this formulae can be found in appendix F of [122]) and $\rho^C{}_C = aK$ one derives $M^C{}_C = 0$. Also, write the formula for the commutator $[\pounds_{\vec{\chi}}, \mathcal{D}_A]$ (see e.g. [120]) acting on ρ_{AB}

$$\left(\pounds_{\vec{\chi}}\mathcal{D}_C - \mathcal{D}_C\pounds_{\vec{\chi}}\right)\rho_{AB} = -\rho_{EA}\pounds_{\vec{\chi}}\mathring{\Gamma}^E{}_{CB} - \rho_{EB}\pounds_{\vec{\chi}}\mathring{\Gamma}^E{}_{CA}, \tag{6.15}$$

which, noting that

$$\pounds_{\vec{\chi}}\mathring{\gamma}^E{}_{CB} = \delta^E_C\mathcal{D}_B\phi + \delta^E_B\mathcal{D}_C\phi - q_{BC}q^{FE}\mathcal{D}_E\phi\ , \tag{6.16}$$

can be antisymetrised to get

$$\left(\pounds_{\vec{\chi}}\mathcal{D}_{[C} - \mathcal{D}_{[C}\pounds_{\vec{\chi}}\right)\rho_{A]B} = -\rho_{B[A}\mathcal{D}_{C]}\phi + q_{B[C}\rho_{A]}{}^E\mathcal{D}_E\phi\ . \tag{6.17}$$

Making use of eq. (6.13) one arrives at

$$\mathcal{D}_{[C}M_{A]B} = \rho_{B[A}\mathcal{D}_{C]}\phi - q_{B[C}\rho_{A]}{}^E\mathcal{D}_E\phi + a\frac{1}{2}K\left(\delta^E_{[A}q_{C]B} - q_{B[A}\delta^E_{C]}\right) = 0 \tag{6.18}$$

where the first equality follows by $2\mathcal{D}_{[C}\mathcal{D}_{A]}\mathcal{D}_B\phi = \mathring{R}_{CAB}{}^E\mathcal{D}_E\phi$, and the second using the identity (6.10) together with $\rho^C{}_C = aK$. Because M_{AB} is symmetric, traceless and divergence free (a 'TT-tensor') on the compact two-dimensional $\mathbb{S}^2$ necessarily $M_{AB} = 0$. For (homothetic) KVF, ϕ = constant and eq. (6.14) reads $\pounds_{\vec{\chi}}\rho_{AB} = 0$, i.e., ρ_{AB} is left invariant by (homothetic) KVF. □

Remark 6.2.2. Let $\vec{\chi}$ be a CKVF on $(\mathcal{S}, q_{AB})$,

$$\pounds_{\vec{\chi}}q_{AB} = 2\phi q_{AB}, \tag{6.19}$$

generating a one-parameter group of local conformal transformations $\left\{ \underset{\epsilon}{\Psi} \right\}$ on $\mathcal{S}$ ($\left(\underset{\epsilon}{\Psi}^* q\right)_{AB} = \underset{\epsilon}{\Phi}^2 q_{AB}$) with $\phi := \mathrm{d}\,\underset{\epsilon}{\Phi}/\mathrm{d}\epsilon\Big|_{\epsilon=0}$ and $\underset{\epsilon}{\Phi}\Big|_{\epsilon=0} = 1$. Then, the finite change of ρ_{AB} under these conformal transformations is

$$\tilde{\rho}_{AB} = \rho_{AB} - a\frac{1}{\underset{\epsilon}{\Phi}}\mathcal{D}_A\underset{\epsilon}{\Phi}_B + \frac{2a}{\underset{\epsilon}{\Phi}^2}\underset{\epsilon}{\Phi}_A\underset{\epsilon}{\Phi}_B - \frac{a}{2\,\underset{\epsilon}{\Phi}^2}\underset{\epsilon}{\Phi}_C\underset{\epsilon}{\Phi}^C q_{AB} \tag{6.20}$$

with $\underset{\epsilon}{\Phi}_A := \mathcal{D}_A \underset{\epsilon}{\Phi}$. Expression (6.20) follows from eq. (6.14) and the exponential map from the Lie algebra to the finite group of conformal transformations.

Corollary 6.2.3 (The tensor ρ for non-$\mathbb{S}^2$ manifolds)**.** Let $(\mathcal{S}, q_{AB})$ be a 2-dimensional Riemannian manifold, no necessarily with $\mathbb{S}^2$ topology, and such that there exists a CKVF χ^A with a fixed point. Then, there is a unique symmetric tensor field ρ_{AB} on $\mathcal{S}$ whose behaviour under conformal rescalings (C.15) is as in (6.6) and satisfies the equations

$$\mathcal{D}_{[C}\rho_{A]B} = 0 \ , \tag{6.21}$$

$$\pounds_{\vec{\chi}}\rho_{AB} = -a\mathcal{D}_A\mathcal{D}_B\phi \ , \tag{6.22}$$

in any conformal frame, where $\phi := \mathcal{D}_M\chi^M/2$. Furthermore, this tensor field must have a trace $\rho^E{}_E = aK$, is given for the metric with constant positive Gaussian curvature by $\rho_{AB} = q_{AB}aK/2$, vanishes for the flat Euclidean metric and is invariant under transformations generated by χ^A when this is a KVF (that is, when $\phi = 0$).

Remark 6.2.3. In two dimensions the CKVF χ^A with a fixed point generates an axial symmetry locally around the fixed point (see [135]). The existence of such vector field is ensured for $\mathcal{S} = \mathbb{S}^2$, $\mathcal{S} = \mathbb{S}^2 \setminus \{p_1\} = \mathbb{R}^2$ and $\mathcal{S} = \mathbb{S}^1 \times \mathbb{R}$ –see appendix F in [122].

Remark 6.2.4. The further requirement of eq. (6.22) with respect to corollary 6.2.2 provides the uniqueness of ρ_{AB}. Note that this is a natural condition to be imposed. Actually, the validity of (6.22) for any CKVF would be motivated on physical arguments as well, for it makes the tensor ρ_{AB} respect the symmetries of the cut. This also would fix the behaviour under finite conformal transformations to be of type (6.20).

Proof. Existence is proved by noticing that $\rho_{AB} = q_{AB}aK/2$ fulfils $\mathcal{D}_A\rho_{BC} = 0$ in the metric with constant positive Gaussian curvature, and one can check using the gauge change (6.6) and eq. (6.22) that this gives the vanishing tensor for the flat metric. Concerning uniqueness, the proof follows along the same lines of corollary 6.2.4 and we also arrive at $\mathcal{D}_{[C}\left(\underset{1}{\rho}_{A]B} - \underset{2}{\rho}_{A]B}\right) = 0$, if two different solutions $\underset{1}{\rho}_{AB}$ and $\underset{2}{\rho}_{AB}$ exist. Then, choose the conformal frame such that χ^A becomes a KVF (which necessarily keeps the fixed point). To see that ρ_{AB} is left invariant by χ^A in this conformal frame, one only has to set $\phi = 0$ in eq. (6.22). Now, the difference $\underset{1}{\rho}_{AB} - \underset{2}{\rho}_{AB}$ is trace- and divergence-free, i.e., a TT-tensor which also fulfils the so called KID equations [136] for χ^A because of its invariance by this KVF and we are working in 2 dimensions. Now, a result in [137] states that the only solution to this problem if the KVF has fixed points –as it is the case of χ^A– is the trivial one. Hence, $\underset{1}{\rho}_{AB} - \underset{2}{\rho}_{AB} = 0$. To see that uniqueness holds in any conformal frame, recall that if two solutions exist, they have to coincide in the particular frame(-family) in which χ^A is a KVF. Since the change of any two solutions to that frame is the same (given by eq. (6.6)), the only possibility is $\underset{1}{\rho}_{AB} = \underset{2}{\rho}_{AB}$ in any conformal frame. The proof that ρ_{AB} vanishes for the flat metric will be completed below. □

By these means we can recover, in a direct manner, a non-trivial result on the sphere $\mathbb{S}^2$ (with any metric) —first proven in [138],

Corollary 6.2.4. Let, as before, $(\mathcal{S}, q_{AB})$ be any Riemannian manifold, topologically $\mathbb{S}^2$, with metric q_{AB}. Then, for every conformal Killing vector field χ^A

$$\int_{\mathcal{S}} \pounds_{\vec{\chi}} K \mathring{\epsilon} = 0 \ . \tag{6.23}$$

Proof. Equation (6.13) is equivalent to its trace,

$$\mathcal{D}_C \left(\rho_A{}^C - a\delta^C{}_A K \right) \stackrel{\mathcal{S}}{=} 0, \tag{6.24}$$

which is a gauge invariant equation too. Contracting with χ^A and integrating over $\mathcal{S}$ one obtains the desired result, noting that $2\mathcal{D}_{(A}\chi_{B)} = \mathring{q}_{AB}\mathcal{D}_C\chi^C$. □

From now on, we will use ρ_{AB} to denote this tensor field in the case $a = 1$. We will later need the gauge change of the tensor ρ_{AB} but using the covariant derivative $\tilde{\mathcal{D}}_A$ instead of $\mathcal{D}_A$. To that end, we can use (6.6) with $a = 1$ but applied to the conformal change $q_{AB} = \omega^{-2}\tilde{q}_{AB}$, so that

$$\rho_{AB} = \tilde{\rho}_{AB} - \omega \tilde{\mathcal{D}}_A \tilde{\mathcal{D}}_B \omega^{-1} + 2\omega^2 \tilde{\mathcal{D}}_A \omega^{-1} \tilde{\mathcal{D}}_B \omega^{-1} - \frac{\omega^2}{2} \tilde{q}^{CD} \tilde{\mathcal{D}}_C \omega^{-1} \tilde{\mathcal{D}}_D \omega^{-1} \tilde{q}_{AB} \tag{6.25}$$

and expand the righthand side to get

$$\rho_{AB} = \tilde{\rho}_{AB} + \frac{1}{\omega} \tilde{\mathcal{D}}_A \mathring{\omega}_B - \frac{1}{2\omega^2} \tilde{q}^{CD} \mathring{\omega}_C \mathring{\omega}_D \tilde{q}_{AB} \ . \tag{6.26}$$

Interestingly, ρ_{AB} is closely related to the ρ tensor field defined by Geroch in the asymptotically flat case [17] –see corollary 4.2.2. Indeed, they are the same objects when the latter is restricted to a cut. Its role in the existence of a news tensor will be made clear in section 6.3.

6.2.1 The tensor ρ for axially symmetric 2-dimensional cuts

One can give the explicit form of ρ_{AB} for any 2-dimensional metric with axial symmetry q_{AB}. Lets choose coordinates $x^A = \{p, \varphi\}$ such that

$$q = F(p)\, \mathrm{d}p^2 + G(p)\, \mathrm{d}\varphi^2 \tag{6.27}$$

and ∂_φ is the axial KVF. This metric is locally conformal to the round metric with constant positive Gaussian curvature K

$$\mathring{q} = \frac{1}{K} \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2 \right) \ . \tag{6.28}$$

Assume that the conformal factor ω relating both metrics ($q_{AB} = \omega^2 \mathring{q}_{AB}$) respects the axial symmetry. Then,

$$G\left(p\right) = \frac{\omega^2}{K}\sin^2\theta, \quad F\left(p\right) = \frac{\omega^2}{K}\theta'^2, \quad \theta' = \frac{\mathrm{d}\theta}{\mathrm{d}p}, \tag{6.29}$$

yields

$$\tan\frac{\theta}{2} = Ce^{\epsilon\int\sqrt{F/G}\mathrm{d}p}, \quad \sin\theta = \frac{2Ce^{\epsilon\int\sqrt{F/G}\mathrm{d}p}}{1+C^2e^{2\epsilon\int\sqrt{F/G}\mathrm{d}p}}, \quad \cos\theta = \frac{1-\tan^2\left(\theta/2\right)}{1+\tan^2\left(\theta/2\right)}, \tag{6.30}$$

where C has to be fixed (making the value of p at the fixed point of ∂_ϕ correspond to $\theta = 0$ or $\theta = \pi$) and $\epsilon^2 = 1$. With this, the conformal factor reads

$$\omega = \sqrt{K}\frac{\sqrt{G}}{\sin\theta}. \tag{6.31}$$

Its first and second derivatives (using the connection of q_{AB}) read

$$\frac{1}{\omega}\omega_A = \delta^p_A\psi, \quad \text{with} \quad \psi := \left(\frac{G'}{2G} - \epsilon\sqrt{\frac{F}{G}}\cos\theta(p)\right), \quad G' := \frac{\mathrm{d}G}{\mathrm{d}p}, \tag{6.32}$$

$$\frac{1}{\omega}\mathcal{D}_A\omega_B\mathrm{d}x^A\mathrm{d}x^B = \left(\psi' - \frac{F'}{2F}\psi + \psi^2\right)\mathrm{d}p^2 + \frac{G'}{2F}\psi\mathrm{d}\varphi^2, \quad \text{with} \quad F' := \frac{\mathrm{d}F}{\mathrm{d}p}, \quad \psi' := \frac{\mathrm{d}\psi}{\mathrm{d}p}, \tag{6.33}$$

Setting $\tilde{\rho}_{AB} = \frac{1}{2}K\mathring{q}_{AB}$ and using the inverse conformal behaviour (6.26) one gets

$$\rho_{AB} = \frac{1}{2\omega^2}\left(K + \frac{\omega^2}{F}\psi^2\right)q_{AB} - \frac{1}{\omega}\mathcal{D}_A\omega_B\ . \tag{6.34}$$

Now, inserting eq. (6.31) in this last expression gives the explicit form of ρ_{AB} for any axially symmetric q_{AB} (6.27):

$$\rho_{AB}\mathrm{d}x^A\mathrm{d}x^B = \left(\frac{F}{2G}\sin^2\theta - \psi' + \frac{F'}{2F}\psi - \frac{1}{2}\psi^2\right)\mathrm{d}p^2 + \left(\frac{1}{2}\sin^2\theta + \frac{\psi}{2F}\left(G\psi - G'\right)\right)\mathrm{d}\varphi^2, \tag{6.35}$$

where $\sin^2\theta$ must be understood as the function of p given in eq. (6.30).

We can apply the formula above to compute ρ_{AB} for the flat Euclidean metric written in polar coordinates. This is simply eq. (6.27) with $F(p) = 1$ and $G(p) = p^2$, while it can be checked that the conformal factor is $\omega = 1 + (K/4)p^2$. Application of eq. (6.35), where one must fix $\epsilon = -1$ and $C = 2/\sqrt{K}$, then leads readily to

$$\rho_{AB}|_{flat} = 0.$$

This finishes the proof of corollary 6.2.3

6.3 General approach to gauge-invariant traceless symmetric tensor fields on $\mathscr{J}$

We present a way of constructing equations for tensor fields fulfilling properties i) to iv) on page 103. To that end, we take as starting point eq. (5.14) and contract it with $r^a E^b{}_B$ to obtain

$$N\mathring{C}^A = \mathring{\epsilon}^{CD}\left(\mathcal{D}_{[C}\mathring{S}_{D]}{}^A + \mathring{\kappa}_{[C}{}^A\mathring{S}_{D]}\right) \ . \tag{6.36}$$

By eq. (6.3), the right-hand side of eq. (6.36) can be rearranged as

$$\mathring{\epsilon}^{CD}\left(\mathcal{D}_{[C}\mathring{U}_{D]}{}^A - T_{CD}{}^A\right) \ , \tag{6.37}$$

with

$$U_{AB} := S_{AB} + \frac{1}{2}\mathring{\kappa}\mathring{\Sigma}_{AB} + \left(\frac{1}{8}\mathring{\kappa}^2 - \frac{1}{4}\mathring{\Sigma}^2\right)q_{AB} = U_{(AB)} \ , \tag{6.38}$$

$$T_{AB}{}^C := \frac{1}{2}\left[\delta^C{}_{[A}\mathcal{D}_{B]}\mathring{\Sigma}^2 - \mathcal{D}^C\left(\mathring{\Sigma}^D{}_{[B}\right)\mathring{\Sigma}_{A]D}\right] = T_{[AB]}{}^C \ , \tag{6.39}$$

and we can write eq. (6.36) in the equivalent form

$$\frac{1}{2}N\mathring{\epsilon}_{CA}\mathring{C}_B = \mathcal{D}_{[C}U_{A]B} - T_{CAB} \ . \tag{6.40}$$

Observe that since $T_{AB}{}^C$ is antisymmetric on its two covariant indices, by the identity (6.10) it is completely determined by its trace

$$T_B := T_{CB}{}^C = \frac{1}{4}\left[\mathcal{D}_B\mathring{\Sigma}^2 - \mathring{\Sigma}^{CD}\mathcal{D}_C\left(\mathring{\Sigma}_{DB}\right) + \mathring{\Sigma}_B{}^D\mathcal{D}_C\left(\mathring{\Sigma}_D{}^C\right)\right] \ . \tag{6.41}$$

The point of this decomposition is that T_{ABC} is gauge invariant

$$\tilde{T}_{ABC} = T_{ABC} \tag{6.42}$$

and that U_{AB} transforms as

$$\tilde{U}_{AB} = U_{AB} + \frac{2}{\omega^2}\mathring{\omega}_A\mathring{\omega}_B - \frac{1}{\omega}\mathcal{D}_A\mathring{\omega}_B - \frac{1}{2\omega^2}\mathring{\omega}_P\mathring{\omega}^P q_{AB} \ , \tag{6.43}$$

where we have used the formulae of appendix C. In addition to that, taking eq. (6.5) into account, the trace of U_{AB} reads

$$U^{E}{}_{E} = \frac{\mathring{R}}{2} = K \ . \tag{6.44}$$

Remarkably, these are the gauge behaviour and the trace of the tensor required to prove the following:

Lemma 6.3.1. The tensor field $\mathcal{D}_{[C}U_{D]A}$ is gauge invariant,

$$\tilde{\mathcal{D}}_{[C}\tilde{U}_{D]A} = \mathcal{D}_{[C}U_{D]A} \ . \tag{6.45}$$

Proof. It follows from lemma 6.2.1, noting the gauge behaviour of U_{AB} –eq. (6.43)– and its trace –eq. (6.44). □

Remark 6.3.1. In particular, the combination $\mathcal{D}_{[C}U_{D]A} + T_{DCA}$ is gauge invariant too, as follows from the gauge invariance of T_{ABC}. This can also be proven looking at eq. (6.36) and noting the gauge behaviour of $\mathring{C}^{A}$ (see appendix C).

We write now an important result:

Proposition 6.3.1 (First component of news)**.** Let $(\mathcal{S}, q_{AB})$ be a 2-dimensional Riemannian manifold endowed with the metric q_{AB}. If q_{AB} has a CKVF with a fixed point, the tensor field

$$V_{AB} := U_{AB} - \rho_{AB} \ , \tag{6.46}$$

is symmetric, traceless, gauge invariant and satisfies the gauge-invariant equation

$$\mathcal{D}_{[A}U_{B]C} = \mathcal{D}_{[A}V_{B]C} \ , \tag{6.47}$$

where ρ_{AB} is the tensor field of corollary 6.2.3 (for $a = 1$). Besides, V_{AB} is unique with these properties.

Proof. The tensor field V_{AB} is symmetric, traceless and gauge invariant as a consequence of Equations (6.38), (6.43) and (6.44) and corollary 6.2.3. The uniqueness of V_{AB} follows from corollary 6.2.3 too and Equation (6.47). □

Remark 6.3.2. The existence of a CKVF with a fixed point is warranted for the topologies $\mathcal{S} = \mathbb{S}^2$, $\mathcal{S} = \mathbb{R} \times \mathbb{S}^1$ and $\mathcal{S} = \mathbb{R}^2$ –see appendix F in [122].

In passing, notice the identity that follows taking the trace of eq. (6.40) and applying proposition 6.3.1:

$$\int_{\mathcal{S}} \chi^{B} \left(N \mathring{\epsilon}_{BE} \mathring{C}^{E} - 2T_{B} \right) = 0 \quad \forall \text{ CKVF } \chi^{B} \text{ on a topological-}\mathbb{S}^2 \quad \mathcal{S}. \tag{6.48}$$

We consider V_{AB} an essential component of any news-like tensor as will be justified in section 6.4. In general, one has

Proposition 6.3.2. Let $(\mathcal{S}, q_{AB})$ be a 2-dimensional Riemannian manifold with $\mathbb{S}^2$-topology endowed with the metric q_{AB}. If the equation

$$\mathcal{D}_{[C}Z_{A]B} = Y_{CAB} \tag{6.49}$$

for a given gauge-invariant tensor field $Y_{CAB} = Y_{[CA]B}$ has a solution for $Z_{AB} = Z_{(AB)}$ whose gauge behaviour is given by (6.6) with $a = 1$, then this solution is unique and given by

$$Z_{AB} = U_{AB} + X_{AB} \tag{6.50}$$

where X_{AB} is the unique traceless gauge invariant symmetric tensor field solution of

$$\mathcal{D}_{[C}X_{A]B} = Y_{CAB} - \frac{1}{2}N\mathring{\epsilon}_{CA}\mathring{C}_B - T_{CAB} \ . \tag{6.51}$$

Proof. Lemma 6.3.1 ensures the gauge-invariance of $\mathcal{D}_{[C}Z_{A]B}$, provided X_{AB} is gauge invariant and traceless and applying corollary 6.2.1. For the second part, note that by eq. (6.40)

$$\mathcal{D}_{[C}Z_{A]B} = \mathcal{D}_{[C}X_{A]B} + \frac{1}{2}N\mathring{\epsilon}_{CA}\mathring{C}_B + T_{CAB} \ , \tag{6.52}$$

from where eq. (6.51) follows immediately. If two different solutions ${}_1Z_{AB}$ and ${}_2Z_{AB}$ exist, one has $\mathcal{D}_{[C}\left({}_1Z_{A]B} - {}_2Z_{A]B}\right) = 0$. Then, because their difference is a traceless Codazzi tensor on $\mathbb{S}^2$, the only possibility [113] is ${}_1Z_{AB} - {}_2Z_{AB} = 0$. □

Remark 6.3.3. The $\mathbb{S}^2$ topology can be dropped from the assumptions if $(\mathcal{S}, q_{AB})$ is a 2-dimensional Riemannian manifold such that there exists a CKVF χ^A with a fixed point and Z_{AB} fulfils the KID equations [136]. For proving this, one applies the result of [137] which was used in the proof of corollary 6.2.3 to show that $\mathcal{D}_{[C}\left({}_1Z_{A]B} - {}_2Z_{A]B}\right) = 0$ implies ${}_1Z_{A]B} - {}_2Z_{A]B} = 0$.

Remark 6.3.4. By proposition 6.3.1 (or with different appropriate assumptions, corollary 6.2.3) and eq. (6.50), the general eq. (6.49) is written as

$$\mathcal{D}_{[C}N_{A]B} = Y_{CAB} \ , \tag{6.53}$$

where we have defined the gauge-invariant traceless symmetric tensor field

$$N_{AB} := V_{AB} + X_{AB} \ . \tag{6.54}$$

Equation (6.53) is equivalent to its trace,

$$\mathcal{D}_C N_A{}^C = 2Y_A \ , \tag{6.55}$$

with

$$Y_A := Y_{CA}{}^{A} \ . \tag{6.56}$$

Remark 6.3.5. For $\mathcal{S}$ topologically $\mathbb{S}^2$, solutions to eq. (6.53) exist if and only if

$$\int_{\mathcal{S}} \chi^B Y_B \,\mathring{\epsilon} = 0 \ , \tag{6.57}$$

for any CKVF on $(\mathcal{S}, q_{AB})$ (see [134], appendix H). In general, one can always prescribe Y_{CAB} (equivalently, Y_A) such that solutions exist, one plausible option is

$$Y_B := \Delta y \, \mathcal{D}_B y, \quad \forall y \in C^2(\mathcal{S}) \tag{6.58}$$

which follows from a result proven in [138]: if assumptions in corollary 6.2.4 hold, then

$$\int_{\mathcal{S}} \Delta y \pounds_{\chi} y \mathring{\epsilon} = 0, \quad \forall y \in C^2(\mathcal{S}) \tag{6.59}$$

and this statement is conformally invariant.

N_{AB} as defined in (6.54) is our candidate for the news-like object we are seeking. It has two 'components', one given by V_{AB} which is fully determined on each cut (see next parapgraph), and another component, yet to be uncovered, which depends on the choice of Y_A. Note that N_{AB} fulfils properties i) to iv) on page 103. According to remark 6.3.5, prescriptions of Y_{ABC} are always possible such that these kind of tensor fields exist as solutions to eq. (6.53). Nevertheless, the great difficulty stems in *fixing* Y_{ABC} *such that* N_{AB} *makes a reasonable news tensor* that satisfies all the requirements on page 103, including properties v) and vi) too.

At this stage, there is no reason to ensure that there exists some function y such that the choice (6.58) meets all these points in general. Observe, in this sense, that eq. (6.40) in terms of V_{AB} reads

$$\frac{1}{2} N \mathring{\epsilon}_{CA} \mathring{C}_B = \mathcal{D}_{[C} V_{A]B} - T_{CAB} \tag{6.60}$$

and, therefore, V_{AB} is completely determined by $\mathring{C}_A$ and the intrinsic geometry of $\mathscr{J}$. Hence, in order to achieve a N_{AB} satisfying property v) in addition to the other ones on page 103, the choice of Y_{ABC} has to incorporate the dependence on D_{ab}. Not only that, but it has to vanish in accordance with some meaningful superenergy quantity. As mentioned earlier, there are several options for this quantity, such as the asymptotic super-Poynting, or radiant 1. The problem of in- and out-going radiative sectors seems to make it difficult to find a second component of N_{AB} associated to the former, as it contains information from both sectors. This last difficulty can be connected to the freedom in choosing a radiant 1 at $\mathscr{J}$. Next section deals with these issues by proposing a particular fixing of Y_{ABC}.

6.4 Second component of news

Now we take as a guide property vi) on page 103; throughout this section, in particular we choose to focus on the vanishing of ${}^{\pm}\mathcal{Z}$. In the light of eqs. (2.54) and (2.55), it is clear that it is convenient to work with the quantities ${}^{\pm}\mathring{C}_A$. Observe that eq. (6.60) can be rewritten using properties xxi) and xxii) on page 212 as

$$N\mathring{C}^A = \mathring{\epsilon}^{CD}\left(\mathcal{D}_{[C}V_{D]}{}^{A} + T_{DC}{}^{A}\right) \ , \tag{6.61}$$

$$2N\,{}^{+}\mathring{C}^A + N\mathring{\epsilon}^{AC}\mathring{D}_C = \mathring{\epsilon}^{CD}\left(\mathcal{D}_{[C}V_{D]}{}^{A} + T_{DC}{}^{A}\right) \ , \tag{6.62}$$

$$2N\,{}^{-}\mathring{C}^A - N\mathring{\epsilon}^{AC}\mathring{D}_C = \mathring{\epsilon}^{CD}\left(\mathcal{D}_{[C}V_{D]}{}^{A} + T_{DC}{}^{A}\right) \ , \tag{6.63}$$

or equivalently

$$N\mathring{\epsilon}_{BE}\mathring{C}^E = -\mathcal{D}_E V_B{}^{E} + 2T_B \ , \tag{6.64}$$

$$2N\mathring{\epsilon}_{BE}\,{}^{+}\mathring{C}^E - N\mathring{D}_B = -\mathcal{D}_E V_B{}^{E} + 2T_B \ , \tag{6.65}$$

$$2N\mathring{\epsilon}_{BE}\,{}^{-}\mathring{C}^E + N\mathring{D}_B = -\mathcal{D}_E V_B{}^{E} + 2T_B \ . \tag{6.66}$$

Recall that eqs. (6.62), (6.63), (6.65) and (6.66) are nothing more than eqs. (6.61) and (6.64) expressed in terms of ${}^{\pm}\mathring{C}_A$. It is useful to have them at hand, though.

One approach is to look for the necessary and sufficient conditions such that

$$-2N\mathring{\epsilon}_B{}^{E}\,{}^{+}C_E = \mathcal{D}_C\,{}^{+}n_B{}^{C} \ , \tag{6.67}$$

$$-2N\mathring{\epsilon}_B{}^{E}\,{}^{-}C_E = \mathcal{D}_C\,{}^{-}n_B{}^{C} \ , \tag{6.68}$$

for ${}^{\pm}n_{AB}$ symmetric traceless gauge invariant tensor fields on $\mathcal{S}$. These are the particular versions of the general N_{AB} for the choices (6.67) and (6.68), as we prefer to keep the generic name N_{AB} for the general method. The left-hand side of these equations correspond to two different –compatible– choices of Y_A in eq. (6.55), respectively. Hence, we define

$${}^{+}Y_B := N\mathring{\epsilon}_B{}^{E}\,{}^{+}C_E, \quad {}^{-}Y_B := N\mathring{\epsilon}_B{}^{E}\,{}^{-}C_E. \tag{6.69}$$

Let us emphasise once more that V_{AB} fulfils properties i) to iv) on page 103. According to eq. (6.61), it does not satisfy property v) because it carries no information about D_{ab}. Thus, intuitively one would expect the second component of ${}^{\pm}n_{AB}$, ${}^{\pm}X_{AB}$, to come from an equation for D_{ab}, such that the generic expression (6.54) becomes now

$${}^{+}n_{AB} := V_{AB} + {}^{+}X_{AB} \ , \tag{6.70}$$

$${}^{-}n_{AB} := V_{AB} + {}^{-}X_{AB} \ , \tag{6.71}$$

where ${}^{\pm}X_{AB}$ are unknown symmetric traceless gauge invariant tensor fields on $\mathcal{S}$. Each of them corresponds to a solution X_{AB} in eq. (6.54) of the general approach . It can be checked by direct computation that the necessary and sufficient conditions for eqs. (6.67) and (6.68) to hold are

$$-\frac{1}{2}N\mathring{D}_B = T_B + \frac{1}{2}\mathcal{D}_C\,{}^{+}X_B{}^C \ , \tag{6.72}$$

$$\frac{1}{2}N\mathring{D}_B = T_B + \frac{1}{2}\mathcal{D}_C\,{}^{-}X_B{}^C \ , \tag{6.73}$$

which are satisfied if and only if for any CKVF χ^B on $\mathcal{S}$

$$\int_{\mathcal{S}} \chi^B\,{}^{+}Y_B\,\mathring{\epsilon} = 0 \ , \tag{6.74}$$

$$\int_{\mathcal{S}} \chi^B\,{}^{-}Y_B\,\mathring{\epsilon} = 0 \ , \tag{6.75}$$

which of course fits the general construction –see remark 6.3.5. Summarising, if eqs. (6.72) and (6.73) hold, eqs. (6.67) and (6.68) are satisfied for ${}^{\pm}n_{AB}$ as in eqs. (6.70) and (6.71) and their square produces the following formulae:

$$N^2\,{}^{+}\mathcal{Z} = \mathcal{D}_C\left({}^{+}n_B{}^C\right)\mathcal{D}_D\left({}^{+}n^{BD}\right) \ , \tag{6.76}$$

$$N^2\,{}^{-}\mathcal{Z} = \mathcal{D}_C\left({}^{-}n_B{}^C\right)\mathcal{D}_D\left({}^{-}n^{BD}\right) \ . \tag{6.77}$$

Let us remark that the tensor fields ${}^{\pm}X_{AB}$ satisfying eqs. (6.72) and (6.73) do not necessarily exist in general. They are the two *second components of* ${}^{\pm}n_{AB}$ according to the next result:

Proposition 6.4.1 (Radiant news)**.** If the condition of eq. (6.72) (eq. (6.73)) holds on a cut $\mathcal{S}$ with $\mathbb{S}^2$-topology, then

$${}^{+}n_{AB} = 0 \Longleftrightarrow {}^{+}\mathcal{Z} = 0 \ . \tag{6.78}$$

$$\left({}^{-}n_{AB} = 0 \Longleftrightarrow {}^{-}\mathcal{Z} = 0\right) \ . \tag{6.79}$$

Hence, ${}^{+}n_{AB}$ (${}^{-}n_{AB}$) fulfills properties i) to vi) on page 103. Therefore, ${}^{+}n_{AB}$ (${}^{-}n_{AB}$) can be seen as a news-like tensor for ${}^{+}\underline{\mathcal{Q}}^a$ (${}^{-}\underline{\mathcal{Q}}^k$) and we call them *radiant news.*

Proof. Properties i) to iv) are fulfilled by the definition of ${}^{+}n_{AB}$ (${}^{-}n_{AB}$), see eq. (6.70) (eq. (6.71)) where V_{AB} is the piece of news of proposition 6.3.1. From eqs. (6.62) and (6.72) (eqs. (6.63) and (6.73)), property v) is fulfilled, as well. Since ${}^{+}n_{AB}$ (${}^{-}n_{AB}$) is a symmetric traceless tensor field on the sphere, ${}^{+}n_{AB} = 0 \Longleftrightarrow \mathcal{D}_C\,{}^{+}n_B{}^C = 0$ (${}^{-}n_{AB} = 0 \Longleftrightarrow \mathcal{D}_C\,{}^{-}n_B{}^C = 0$), and that, by eq. (6.67) (eq. (6.68)), this holds if and only if ${}^{+}\mathring{C}_A = 0$ (${}^{-}\mathring{C}_A = 0$) — which by eq. (2.54) (eq. (2.55)) and property iii) on page 15, holds if and only if ${}^{+}\mathcal{Z} = 0$ (${}^{-}\mathcal{Z} = 0$). □

Proposition 6.4.2 (Radiant pseudo-news tensors for non-$\mathbb{S}^2$ cuts)**.** Assume that the condition of eq. (6.72) (eq. (6.73)) holds on a 2-dimensional Riemannian manifold $(\mathcal{S}, q_{AB})$ whose topology is non-necessarily $\mathbb{S}^2$ and that the metric q_{AB} possesses a CKVF χ^A with a fixed point. Then,

$$\overset{+}{n}_{AB} = 0 \implies \overset{+}{\mathcal{Z}} = 0 \; . \tag{6.80}$$

$$\left(\overset{-}{n}_{AB} = 0 \implies \overset{-}{\mathcal{Z}} = 0\right) \; . \tag{6.81}$$

And $\overset{+}{n}_{AB}$ ($\overset{-}{n}_{AB}$) has all the properties i) to v) on page 103 but not property vi). Therefore, we say that $\overset{+}{n}_{AB}$ ($\overset{-}{n}_{AB}$) is a pseudo-news tensor for $\overset{+}{\underline{\mathcal{Q}}}{}^a$ ($\overset{-}{\underline{\mathcal{Q}}}{}^k$).

Proof. The first part of the proof follows the same lines as in proposition 6.4.1, where now the tensor V_{AB} is the one of corollary 6.2.3. Then, by eq. (6.67) (eq. (6.68)), $\overset{+}{\mathring{n}}_{AB} = 0 \implies \overset{+}{\mathring{C}}_A = 0$ ($\overset{-}{n}_{AB} = 0 \implies \overset{-}{\mathring{C}}_A = 0$) —which by eq. (2.54) (eq. (2.55)) and property iii) on page 15, holds if and only if $\overset{+}{\mathcal{Z}} = 0$ ($\overset{-}{\mathcal{Z}} = 0$). The converse is not true in general, as the topology of $\mathcal{S}$ can be other than $\mathbb{S}^2$. □

Remark 6.4.1. Given any cut $\mathcal{S}$ with $\mathbb{S}^2$-topology on $\mathscr{J}$, there always exists a unique (intrinsic) first component of news, V_{AB}, which is determined by the intrinsic geometry of $(\mathscr{J}, h_{ab})$ and the cut. The existence of the entire news-like tensor depends on information extrinsic to $\mathscr{J}$ and, concretely in this section's approach, on $\mathring{D}_A$, from where two different second components, $\overset{\pm}{X}_{AB}$, can emerge. Note that these tensor fields are extrinsic to $\mathscr{J}$, in the sense of not being determined by $(\mathscr{J}, h_{ab})$. Eventually, one ends up with none, one or two different radiant news tensors, $\overset{\pm}{n}_{AB}$, which are each one the sum of the corresponding first and second component — see eqs. (6.70) and (6.71).

There are simple cases in which the conditions for the existence of news, eqs. (6.72) and (6.73) (equivalently, eqs. (6.74) and (6.75)) are fulfilled. The following result shows this:

Lemma 6.4.1. Consider any *umbilical* cut $\mathcal{S}$ with $\mathbb{S}^2$-topology on $\mathscr{J}$ such that r_a defines a strong orientation on $\mathcal{S}$, i. e., $\overset{-}{\mathcal{Z}} \overset{\mathcal{S}}{=} 0$. Then, there always exists the radiant news given by

$$\overset{+}{n}_{AB} = 2V_{AB} \; . \tag{6.82}$$

Proof. On the one hand, the umbilical property ($\mathring{\Sigma}_{AB} = 0$) implies that $T_A = 0$ (see eq. (6.41)). On the other hand, $\overset{-}{\mathcal{Z}} = 0 \iff \mathring{D}_A = 0 \iff \mathring{C}_A = \overset{+}{\mathring{C}}_A$. These two conditions together make eqs. (6.61) and (6.64) read

$$N\,\overset{+}{\mathring{C}}{}^A = \mathring{\epsilon}^{CD}\mathcal{D}_{[C}V_{D]}{}^A \; , \tag{6.83}$$

$$N\mathring{\epsilon}_{BE}\overset{+}{\mathring{C}}{}^E = -\mathcal{D}_E V_B{}^E \; . \tag{6.84}$$

Thus, eq. (6.67) reads

$$2\mathcal{D}_E V_B{}^E = \mathcal{D}_C \overset{+}{n}_B{}^C \ . \tag{6.85}$$

Then, one finds $\mathcal{D}_E\left(\overset{+}{\mathring{n}}_B{}^E - 2V_B{}^E\right) = 0$, which holds if and only if $\overset{+}{n}_{AB} = 2V_{AB}$ —the divergence of a traceless, symmetric, two-dimensional tensor field on the sphere vanishes if and only if the tensor itself vanishes. □

In chapter 7, we will study how to extend these objects to tensor fields on $\mathscr{J}$ by introducing additional structure on $\mathscr{J}$. The relation between the radiant news and the radiation condition of criterion 1 will be analysed in section 7.2.

6.4.1 Possible generalisation

There is a generalisation of the approach we have presented. Since each cut $\mathcal{S}$ is two-dimensional[1], the vanishing of $\overset{\pm}{\mathring{C}}_A$ is trivially equivalent to the vanishing of any linear combination

$$\overset{\pm}{\beta}\,\overset{\pm}{\mathring{C}}_B + \overset{\pm}{\lambda}\mathring{\epsilon}_B{}^E\,\overset{\pm}{\mathring{C}}_E \ , \tag{6.86}$$

where the coefficients $\overset{\pm}{\lambda}$ and $\overset{\pm}{\beta}$ are such that they do not vanish simultaneously, i. e.,

$$\overset{\pm}{\lambda} = 0 \implies \overset{\pm}{\beta} \neq 0 \ , \tag{6.87}$$

$$\overset{\pm}{\beta} = 0 \implies \overset{\pm}{\lambda} \neq 0 \ . \tag{6.88}$$

In other respects, $\overset{\pm}{\lambda}$ and $\overset{\pm}{\beta}$ are arbitrary real functions. One can ask these coefficients to fulfil

$$-2N\left(\overset{+}{\beta}\delta^E{}_B + \overset{+}{\lambda}\mathring{\epsilon}_B{}^E\right)\overset{+}{C}_E = \mathcal{D}_C\overset{+}{n}_B{}^C \ , \tag{6.89}$$

$$-2N\left(\overset{-}{\beta}\delta^E{}_B + \overset{-}{\lambda}\mathring{\epsilon}_B{}^E\right)\overset{-}{C}_E = \frac{1}{2}\mathcal{D}_C\overset{-}{n}_B{}^C \ , \tag{6.90}$$

for $\overset{\pm}{n}_{AB}$ symmetric traceless gauge invariant tensor fields on $\mathcal{S}$. Notice that for $\overset{+}{\lambda} = 1$ and $\overset{+}{\beta} = 0$ we are in the situation described above for eqs. (6.67) and (6.68). Now, one has to define

$$\overset{+}{Y}_B := N\left(\overset{+}{\beta}\delta^E{}_B + \overset{+}{\lambda}\mathring{\epsilon}_B{}^E\right)\overset{+}{C}_E, \quad \overset{-}{Y}_B := N\left(\overset{-}{\beta}\delta^E{}_B + \overset{-}{\lambda}\mathring{\epsilon}_B{}^E\right)\overset{-}{C}_E. \tag{6.91}$$

[1]For higher dimensions the proposed generalisation would not work. Nevertheless, in that other case, one can always define a matrix A_{ab} such that $A_{ap}x^p = 0$ iff $x^p = 0$, where x^p represents a hypothetical vector field playing the role of $\overset{+}{C}_A$ for dimension greater than 2.

Again, one expects the *second components* ${}^{\pm}X_{AB}$ coming from an equation for $\mathring{D}_{ab}$ to be part of ${}^{\pm}n_{AB}$, together with V_{AB} :

$$ {}^{+}n_{AB} := V_{AB} + {}^{+}X_{AB} \ , \tag{6.92} $$
$$ {}^{-}n_{AB} := V_{AB} + {}^{-}X_{AB} \ . \tag{6.93} $$

It can be checked by direct computation that the necessary and sufficient conditions on ${}^{\pm}\lambda$ and ${}^{\pm}\beta$ for eqs. (6.89) and (6.90) to hold are

$$ -\frac{1}{2}N\mathring{D}_B = T_B + \left({}^{+}\lambda - 1\right) N\mathring{\epsilon}_{BC}\,{}^{+}\mathring{C}^C + {}^{+}\beta N\,{}^{+}\mathring{C}_B + \frac{1}{2}\mathcal{D}_C\,{}^{+}X_B{}^C \ , \tag{6.94} $$
$$ \frac{1}{2}N\mathring{D}_B = T_B + \left({}^{-}\lambda - 1\right) N\mathring{\epsilon}_{BC}\,{}^{-}\mathring{C}^C + {}^{-}\beta N\,{}^{-}\mathring{C}_B + \frac{1}{2}\mathcal{D}_C\,{}^{-}X_B{}^C \ , \tag{6.95} $$

which are satisfied if and only if for any CKVF χ^B on $\mathcal{S}$

$$ \int_{\mathcal{S}} \chi^B\,{}^{+}Y_B = 0 \ , \tag{6.96} $$
$$ \int_{\mathcal{S}} \chi^B\,{}^{-}Y_B = 0 \ , \tag{6.97} $$

which again meets the general construction of section 6.3 –in particular, see remark 6.3.5. Notice that equivalent expressions to eqs. (6.94) and (6.95) in terms of $\mathring{C}_A$ are

$$ -N\frac{1}{2}\left({}^{+}\lambda\delta^D{}_B - {}^{+}\beta\mathring{\epsilon}_B{}^D\right)\mathring{D}_D = T_B + N\frac{1}{2}\left[{}^{+}\beta\delta^D{}_B + \left({}^{+}\lambda - 1\right)\mathring{\epsilon}_B{}^D\right]\mathring{C}_D + \frac{1}{2}\mathcal{D}_C\,{}^{+}X_B{}^C \ , \tag{6.98} $$
$$ N\frac{1}{2}\left({}^{-}\lambda\delta^D{}_B - {}^{-}\beta\mathring{\epsilon}_B{}^D\right)\mathring{D}_D = T_B + N\frac{1}{2}\left[{}^{-}\beta\delta^D{}_B + \left({}^{-}\lambda - 1\right)\mathring{\epsilon}_B{}^D\right]\mathring{C}_D + \frac{1}{2}\mathcal{D}_C\,{}^{-}X_B{}^C \ . \tag{6.99} $$

This time, the conclusion is that if eqs. (6.94) and (6.95) hold, eqs. (6.89) and (6.90) are satisfied for the symmetric traceless gauge-invariant tensor fields ${}^{\pm}n_{AB}$ of eqs. (6.92) and (6.93). In that case, one has:

$$ N^2\left({}^{+}\beta^2 + {}^{+}\lambda^2\right){}^{+}\mathcal{Z} = \mathcal{D}_C\left({}^{+}n_B{}^C\right)\mathcal{D}_D\left({}^{+}n^{BD}\right) \ , \tag{6.100} $$
$$ N^2\left({}^{-}\beta^2 + {}^{-}\lambda^2\right){}^{-}\mathcal{Z} = \mathcal{D}_C\left({}^{-}n_B{}^C\right)\mathcal{D}_D\left({}^{-}n^{BD}\right) \ . \tag{6.101} $$

Proposition 6.4.1 can be generalised straightforwardly:

Proposition 6.4.3 (Generalised radiant news)**.** If the condition of eq. (6.94) (eq. (6.95)) holds on a cut $\mathcal{S}$ with $\mathbb{S}^2$-topology, then

$$\overset{+}{n}_{AB} = 0 \Longleftrightarrow \overset{+}{\mathcal{Z}} = 0 \ . \tag{6.102}$$

$$\left(\overset{-}{n}_{AB} = 0 \Longleftrightarrow \overset{-}{\mathcal{Z}} = 0\right) \tag{6.103}$$

and $\overset{+}{n}_{AB}$ ($\overset{-}{n}_{AB}$) fulfils properties i) to vi) on page 103.

Proof. The proof is very much the same as the one of proposition 6.4.1, only that now one uses eqs. (6.89), (6.92) and (6.94) (eqs. (6.90), (6.93) and (6.95)) instead of eqs. (6.67), (6.70) and (6.72) (eqs. (6.68), (6.71) and (6.73)). □

One can generalise lemma 6.4.1 too. In fact, the following result serves to exemplify the role of the coefficients $\overset{\pm}{\lambda}$ and $\overset{\pm}{\beta}$:

Lemma 6.4.2. Consider any *umbilical* cut $\mathcal{S}$ with $\mathbb{S}^2$-topology on $\mathscr{J}$ such that r_a defines a strong orientation on $\mathcal{S}$, i. e., $\overset{-}{\mathcal{Z}} \overset{\mathcal{S}}{=} 0$. Then, there always exists the radiant news $\overset{+}{n}_{AB}$ for $\overset{+}{\lambda} =$ constant and $\overset{+}{\beta} = 0$ given by

$$\overset{+}{n}_{AB} = 2\overset{+}{\lambda} V_{AB} \ . \tag{6.104}$$

Proof. One follows the same steps as in the proof of lemma 6.4.1, this time using eq. (6.89) instead of eq. (6.67), arriving at:

$$-\overset{+}{\lambda}\mathcal{D}_E V_B{}^E + \overset{+}{\beta}\mathring{\epsilon}^{CD}\mathcal{D}_{[C}\left(V_{D]B}\right) = -\frac{1}{2}\mathcal{D}_C\overset{+}{n}_B{}^C \ . \tag{6.105}$$

Setting $\overset{+}{\lambda} =$ constant and $\overset{+}{\beta} = 0$, one finds $\mathcal{D}_E\left(\overset{+}{\mathring{n}}_B{}^E - 2\overset{+}{\lambda} V_B{}^E\right) = 0$, which holds true if and only if $\overset{+}{n}_{AB} = 2\overset{+}{\lambda} V_{AB}$. □

Remark 6.4.2. Given the assumptions of lemma 6.4.2, other solutions exist, for instance $\overset{+}{\beta} =$ constant and $\overset{+}{\lambda} = 0$. The role of $\overset{\pm}{\beta}$ and $\overset{\pm}{\lambda}$ is nothing more but to find solutions, i.e., $\overset{\pm}{X}_{AB}$ tensors, to eqs. (6.94) and (6.95). However, for particular cases –as the one described in lemma 6.4.2– they are pure gauge, in the sense that setting them to one constant value or another provides a combination of the *same* gauge-invariant symmetric traceless tensor field's divergence and its dual, as it is the case of eq. (6.105) where the (two) fundamental degrees of freedom are encoded in V_{AB} . Still in that case, if one considers $\overset{\pm}{\lambda}$ and $\overset{\pm}{\beta}$ as functions it does not affect the fact that the vanishing of $\overset{+}{\mathcal{Z}}$ is equivalent to the vanishing of V_{AB} . This gauge freedom arises as a consequence of asking $\overset{\pm}{\mathcal{Z}}$, a scalar function, to vanish if and only if the divergence of some radiant news tensor does, which allows one to construct combinations of $\overset{\pm}{\mathring{C}}_A$ and its dual, as in eqs. (6.89) and (6.90).

Donde todo
esclarece,
descifra,
echa su red,
dibuja sobre ti su diana.

Benjamín Prado, *Límite. Todos nosotros*, 1998.

7 | Equipped infinity and symmetries

As it has been commented in the introduction to chapter 5, the positivity of Λ spoils any universal structure. For $\Lambda = 0$, such structure consists on the conformal family of degenerate metrics on $\mathscr{J}$ together with a privileged set of curves: the generators. For $\Lambda > 0$, if one wishes to generalise the concept of radiant news tensor presented in chapter 6 for arbitrary cuts of $\mathscr{J}$ to a tensor field on $\mathscr{J}$, study some sort of evolution of the physical fields intrinsically to $\mathscr{J}$ or carry out a close analogy with the $\Lambda = 0$ scenario, one needs to endow $\mathscr{J}$ with further structure.

In particular, a selected family of curves is introduced, trying to keep it as general as possible. In section 7.3 it is seen that such kind of additional structures can be well motivated by physical conditions. Hence, in this section the formalism presented in appendix A.3 is used, where the necessary notation and definitions for the congruences of curves associated to a unit vector field m^a on $\mathscr{J}$ can be found. Also, the same notation as in chapter 6 for objects associated to the decomposition of fields with respect to m^a is followed, only that now underbars will be used instead of over-rings so that they become distinguishable from quantities resulting from the decomposition with respect to r^a. For instance, for the intrinsic Schouten tensor one writes

$$\overline{S}_{ab} \stackrel{\mathcal{S}}{=} \underline{\overline{S}} m_a m_b + 2\underline{S}_B m_{(a} \underline{W}_{b)}{}^B + \underline{S}_{AB} \underline{W}_a{}^A \underline{W}_b{}^B \tag{7.1}$$

and, in general, for any symmetric tensor $B_{\mu\nu}$,

$$B_{\alpha\beta} \stackrel{\mathscr{J}}{=} n^\mu n^\nu B_{\mu\nu} n_\alpha n_\beta + n^\mu \underline{P}^\nu{}_{(\alpha} n_{\beta)} B_{\mu\nu} + 2B_\mu n^\mu m_{(\alpha} n_{\beta)} + m_\alpha m_\beta B + 2\underline{B}_{(\alpha} m_{\beta)} + \underline{B}_{\alpha\beta} \ , \tag{7.2}$$

and

$$\grave{B}_{\alpha\beta} := \underline{B}_{\alpha\beta} - \frac{1}{2} \underline{P}_{\alpha\beta} \underline{P}^{\mu\nu} \underline{B}_{\mu\nu} \ , \tag{7.3}$$

with

$$\underline{B}_{\alpha\beta} := \underline{P}^\mu{}_\alpha \underline{P}^\nu{}_\beta B_{\mu\nu}, \quad B_\alpha := m^\mu B_{\mu\alpha}, \quad \underline{B}_\alpha := \underline{P}^\nu{}_\alpha m^\mu B_{\mu\nu}, \quad B := m^\mu m^\nu B_{\mu\nu}, \tag{7.4}$$

and with capital indices too,

$$\underline{B}_{AB} := \underline{E}^{\alpha}{}_{A}\underline{E}^{\beta}{}_{B}\underline{B}_{\alpha\beta}, \quad \grave{\underline{B}}_{AB} := \underline{B}_{AB} - \frac{1}{2}\underline{q}_{AB}\underline{B}^{C}{}_{C}, \quad \underline{B}_{A} := \underline{E}^{A}{}_{\alpha}m^{\mu}B_{\mu\alpha} \ . \tag{7.5}$$

Also, define a couple of lightlike vector fields on $\mathscr{J}$ as

$$\overset{+}{k}{}^{\alpha} := \frac{1}{\sqrt{2}}\left(n^{\alpha} + m^{\alpha}\right) \ , \tag{7.6}$$

$$\overset{-}{k}{}^{\alpha} := \frac{1}{\sqrt{2}}\left(n^{\alpha} - m^{\alpha}\right) \ . \tag{7.7}$$

This notation applies to all the objects coming from the orthogonal and lightlike decomposition of the rescaled-Weyl tensor (see section 2.2).

Let us start by stating what is meant by additional structure on $\mathscr{J}$.

Definition 7.0.1 (Equipped $\mathscr{J}$)**.** We say that an open, connected, subset $\Delta \subset \mathscr{J}$ with the same topology than $\mathscr{J}$ is *equipped* when it is endowed with a congruence $\mathcal{C}$ of curves characterised by a unit vector field m^{a}. The projected surface $\mathbf{S}_2 := \Delta/\mathcal{C}$ and $\mathcal{C}$ are characterised by the conformal family of pairs

$$\left(\underline{P}_{ab}\,, m_{a}\right) \ , \tag{7.8}$$

where $\underline{P}_{ab}$ is the projector to $\mathbf{S}_2$. Two members belong to the same family if and only if $(\underline{P}'{}_{ab}, m'{}_{a}) = (\Psi^2\underline{P}_{ab}\,, \Psi m_{a})$, where Ψ is a positive function on $\mathscr{J}$.

We will usually assume that Δ is actually one entire connected component of $\mathscr{J}$. The curves of $\mathcal{C}$ are parametrised by any scalar function v such that $\pounds_{\vec{m}}v \neq 0$, and thus it is only defined up to the following changes:

$$v \rightarrow v'\left(v, \zeta^{A}\right), \quad \frac{\partial v'}{\partial v} \neq 0 \ , \tag{7.9}$$

where ζ^{A} label each curve –see appendix A.3 for further details. One can always choose adapted coordinates such that

$$m^{a} = A\delta^{a}_{v} \tag{7.10}$$

for some function A. This form is preserved by (7.9) as long as (A.38) is enforced for the ζ^{A}.

The orthogonal decomposition with respect to m^{a} of $\overline{S}_{ab}$ and C_{ab} gives among other quantities

$$\underset{\sim}{S}_B = \mathcal{D}_C\left(\underset{\sim}{\kappa}_B{}^C + \underset{\sim}{\omega}_B{}^C\right) - \mathcal{D}_B\underset{\sim}{\kappa} - 2\underset{\sim}{a}^C\underset{\sim}{\omega}_{BC} \ , \tag{7.11}$$

$$\underset{\sim}{S}^E{}_E \overset{\mathcal{S}}{=} \underset{\sim}{K} + \frac{1}{2}\underset{\sim}{\Sigma}^2 - \frac{1}{4}\underset{\sim}{\kappa}^2 - \frac{3}{2}\underset{\sim}{\omega}^2 \ , \tag{7.12}$$

$$N\underset{\sim}{C}^A = \underset{\sim}{\epsilon}^{CD}\left(\mathcal{D}_{[C}\underset{\sim}{S}_{D]}{}^A + \underset{\sim}{\kappa}_{[C}{}^A\underset{\sim}{S}_{D]} + \frac{3}{2}\underset{\sim}{S}^A\underset{\sim}{\omega}_{CD}\right) \ . \tag{7.13}$$

Observe that $\mathcal{D}_A$, $\underset{\sim}{q}_{AB}$ and $\underset{\sim}{\epsilon}_{AB}$ represent a one-parameter family of connections, metrics and volume forms on $\mathbf{S}_2$, because they also 'depend on v ' —see appendix A.3. Here, $\underset{\sim}{K}$ is the function appearing in eq. (A.90). Taking these remarks into account, the formulae look aesthetically the same as for any single cut $\mathcal{S}$ if $\mathcal{C}$ is a foliation ($\iff \underset{\sim}{\omega}_{AB} = 0$) in which case we use a different name according to the following definition

Definition 7.0.2 (Strictly equipped $\mathscr{J}$)**.** We say that $\mathscr{J}$ is *strictly equipped* when it is equipped and the unit vector field m^a is surface-orthogonal, providing a foliation by cuts $\mathcal{S}_v$ for $v =$ constant, that is

$$m_a = F\overline{\nabla}_a v$$

for some scalar function F.

Indeed, many of the forthcoming results are considered if this happens, however one has to notice that even when the equations resemble the ones for cuts, they are different. Some insights into the case of general $\mathcal{C}$ will be given in sections 7.2 and 7.4 as well. There is a third level of equipment, the highest one, given by

Definition 7.0.3 (Strongly equipped $\mathscr{J}$)**.** We say that $\mathscr{J}$ is strongly equipped when it is strictly equipped and m^a is shear-free, so that it defines a *foliation* by *umbilical* cuts.

Remark 7.0.1. This definition is the particular case of definition 7.0.1 with $\underset{\sim}{\omega}_{ab} = 0$ (i.e., m^a orthogonal to cuts) and $\underset{\sim}{\Sigma}_{ab} = 0$ (i.e., umbilical cuts).

7.1 Decomposition of the Schouten tensor: kinematic expression

We are going to deduce an expression for $\underset{\sim}{S}_{ab}$ in terms of the kinematic quantities of m^a. To begin with, note that the combination

$$-f_{ab} = \pounds_{\vec{m}}\underset{\sim}{\Sigma}_{ab} - 2\underset{\sim}{\Sigma}_{ad}\underset{\sim}{\Sigma}_b{}^d - \frac{1}{2}\kappa\underset{\sim}{\Sigma}_{ab} \tag{7.14}$$

satisfies properties i) to iv) on page 103. Consider its pullback to $\mathbf{S}_2$ –and use the identity $2\underset{\sim}{\Sigma}_{AD}\underset{\sim}{\Sigma}_B{}^D = \underset{\sim}{q}_{AB}\underset{\sim}{\Sigma}^2$:

$$-f_{AB} \overset{\mathbf{S}_2}{:=} \underset{\sim}{\Sigma}'_{AB} - \underset{\sim}{q}_{AB}\underset{\sim}{\Sigma}^2 - \frac{1}{2}\kappa\underset{\sim}{\Sigma}_{AB} \ , \tag{7.15}$$

with

$$\underline{\Sigma}'_{AB} := \underline{E}^a{}_A \underline{E}^b{}_B \pounds_{\vec{m}} \underline{\Sigma}_{ab} = \pounds_{\vec{m}} \underline{\Sigma}_{AB} \ , \tag{7.16}$$

where the right-hand side follows from eq. (A.56). This term can be replaced using the projection of the Schouten tensor noticing that

$$\begin{aligned}\underline{P}^b{}_f \underline{P}^c{}_d m^e m_d \overline{R}_{ebc}{}^d &= \underline{P}^b{}_f \underline{P}^c{}_d m^e \left(\overline{\nabla}_e \overline{\nabla}_b m_c - \overline{\nabla}_b \overline{\nabla}_e m_c\right)\\ &= \pounds_{\vec{m}} \underline{\Sigma}_{fd} - 2\underline{\Sigma}_{e(d} \overline{\nabla}_{d)} m^e - \underline{\kappa} m_{(f} \underline{a}_{d)} + 2 m_{(d} \underline{\Sigma}_{f)}{}^e \underline{a}_e + \underline{a}_f \underline{a}_d - \underline{P}^b{}_f \underline{P}^c{}_d \overline{\nabla}_b \underline{a}_c\\ &+ \underline{\kappa} m_{(d} \underline{a}_{f)} + \underline{\kappa} \underline{\Sigma}_{df} + \underline{\Sigma}_d{}^e \underline{\Sigma}_{fe} + \left(\frac{1}{4}\underline{\kappa}^2 + \frac{1}{2} m^e \overline{\nabla}_e \underline{\kappa}\right) \underline{q}_{df} \ ,\end{aligned} \tag{7.17}$$

or equivalently

$$\underline{E}^a{}_A \underline{E}^b{}_B m^d m^e \overline{R}_{dabe} = \underline{\Sigma}'_{AB} + \underline{a}_A \underline{a}_B - \underline{\mathcal{D}}_A \underline{a}_B + \underline{q}_{AB} \left(\frac{1}{2} \pounds_{\vec{m}} \underline{\kappa} - \frac{1}{2} \underline{\Sigma}^2 + \frac{1}{4} \underline{\kappa}^2\right) \ . \tag{7.18}$$

Next, use eq. (5.9) to get

$$\underline{P}^b{}_f \underline{P}^c{}_d m^e m_d \overline{R}_{ebc}{}^d = -\underline{P}_{df} \underline{\overline{S}} - \underline{S}_{df} \ . \tag{7.19}$$

Then, take the trace of eq. (7.18) and replace eq. (7.12) in the resulting expression,

$$-\underline{\overline{S}} = \frac{1}{2} \underline{\Sigma}'^C{}_C + \frac{1}{2} \underline{K} - \frac{1}{4} \underline{\Sigma}^2 + \frac{1}{8} \underline{\kappa}^2 + \frac{1}{2} \left(\underline{a}^E \underline{a}_E - \underline{\mathcal{D}}_E \underline{a}^E + \pounds_{\vec{m}} \underline{\kappa}\right) \ . \tag{7.20}$$

After that, use eqs. (7.18) to (7.20) to derive a formula for the projection of the intrinsic Schouten tensor

$$\underline{S}_{AB} = -\underline{\Sigma}'_{AB} + \underline{\mathcal{D}}_A \underline{a}_B - \underline{a}_A \underline{a}_B - \underline{q}_{AB} \left[\frac{1}{2} \left(\underline{\mathcal{D}}_E \underline{a}^E - \underline{a}_E \underline{a}^E\right) - \frac{5}{4} \underline{\Sigma}^2 - \frac{1}{2} \underline{K} + \frac{1}{8} \underline{\kappa}^2\right] \ , \tag{7.21}$$

or more compactly,

$$\underline{S}_{AB} = -\underline{\Sigma}'_{AB} + \underline{\mathcal{D}}_A \underline{a}_B - \underline{a}_A \underline{a}_B - \underline{q}_{AB} \left[\frac{1}{2} \left(\underline{\mathcal{D}}_E \underline{a}^E - \underline{a}_E \underline{a}^E\right) - \frac{1}{2} \underline{\Sigma}'^C{}_C - \frac{1}{2} \underline{S}^C{}_C\right] \ . \tag{7.22}$$

These formulae are interesting on their own and valid for a general foliation on $\mathscr{J}$. It is clear that they have the correct trace and, using the formulae in appendix C, they give the right gauge behaviour —compare with eq. (C.20) in that same appendix. In terms of the gauge invariant quantity f_{AB}, we have

$$\underline{S}_{AB} = f_{AB} + \underline{\mathcal{D}}_A \underline{a}_B - \underline{a}_A \underline{a}_B - \frac{1}{2} \underline{\kappa} \underline{\Sigma}_{AB} - \underline{q}_{AB} \left[\frac{1}{2} \left(\underline{\mathcal{D}}_E \underline{a}^E - \underline{a}_E \underline{a}^E\right) - \frac{1}{4} \underline{\Sigma}^2 - \frac{1}{2} \underline{K} + \frac{1}{8} \underline{\kappa}^2\right] . \tag{7.23}$$

7.2 Radiant news tensor field on equipped $\mathscr{J}$

In chapter 6 we have shown that a gauge invariant traceless symmetric tensor field V_{AB} on any two-dimensional surface (with $\mathbb{S}^2$-topology or, with further assumptions, non-$\mathbb{S}^2$) exists. This applies to cuts with $\mathbb{S}^2$-topology on $\mathscr{J}$ where it represents *a first component of news-like tensors* and actually defines, under suitable conditions –proposition 6.4.1–, radiant news for a radiant supermomentum. However, the question of a news-like tensor field *on* $\mathscr{J}$ is still open. To address it, first we present some geometrical results:

Lemma 7.2.1. Let $(\mathcal{I}, h_{ab})$ be any spacelike hypersurface $\mathcal{I}$ with metric h_{ab} and define $\mathring{P}^c{}_a := \delta^c_a - m^c m_a$ for any unit vector field m^a on $\mathcal{I}$. Then, there are no tensor fields M_{ab} on $\mathcal{I}$ such that $\mathring{M}_{ab} := \mathring{P}^c{}_a \mathring{P}^d{}_b M_{cd}$ is symmetric and traceless ($\mathring{P}^{cd}\mathring{M}_{cd} = 0$) for all possible m^a,

$$\left\{\nexists M_{ab} \quad \Big/ \quad \mathring{M}_{ab} = \mathring{M}_{ba}\,, \quad \mathring{P}^{ab}\mathring{M}_{ab} = 0 \quad \forall m^a\right\}. \tag{7.24}$$

Proof. Given M_{ab}, assume that two different vector fields m^a, m'^a exist such that

$$0 = \mathring{M}'^c{}_c = M^c{}_c - m'^c m'^d M_{cd}\ , \tag{7.25}$$

$$0 = \mathring{M}^c{}_c = M^c{}_c - m^c m^d M_{cd}\ , \tag{7.26}$$

Then, the only possibility for this to happen $\forall m^a$ is

$$m^c m^d M_{cd} = m'^c m'^d M_{cd} \quad \forall m'^a \neq m^a. \tag{7.27}$$

Thus, either $M_{ab} = -M_{ba}$ (which cannot give rise to a symmetric tensor $\mathring{M}_{ab}$) or $M_{ab} = 0$. □

This is in contrast with what happens at the conformal boundary for a vanishing cosmological constant, where any symmetric and traceless tensor field orthogonal to $N^\alpha\Big|_{\Lambda=0}$ on $\mathscr{J}$[1] is a symmetric and traceless tensor field *on any cut*. Precisely, this applies to the news tensor N_{ab} on $\mathscr{J}$ for $\Lambda = 0$; its pullback N_{AB} to *any cut* is symmetric and traceless there –see section 4.2. In any case, in general one has

Lemma 7.2.2. Let $(\mathcal{I}, h_{ab})$ be any spacelike hypersurface $\mathcal{I}$ with metric h_{ab} and M_{ab} and M'_{ab} any couple of symmetric tensor fields on $\mathcal{I}$, each one orthogonal to a unit vector field m^a and m'^a, respectively. Assume that $\mathring{P}^{cd}M_{cd} = 0$ and $\mathring{P}'^{cd}M_{cd} = 0$, where $\mathring{P}^c{}_a$ and $\mathring{P}'^c{}_a$ are the orthogonal projectors associated to m^a and m'^a. Then, the tensor field $B_{ab} := \lambda M_{ab} + \delta M'_{ab}$, for arbitrary coefficients λ and β, is symmetric and traceless, $h^{cd}B_{cd} = 0$.

[1]Recall that $\mathscr{J}$ is lightlike for $\Lambda = 0$, and N^α is the vector field tangent to the generators.

Proof. The tensor field B_{ab} is symmetric and notice that

$$h^{cd}M_{cd} = m^c m^d M_{cd} + \mathring{P}^{cd} M_{cd} = 0 \ , \tag{7.28}$$

$$h^{cd}M'_{cd} = m'^c m'^d M'_{cd} + \mathring{P}'^{cd} M'_{cd} = 0 \ . \tag{7.29}$$

Therefore, $h^{cd}B_{cd} = 0$. □

From lemma 7.2.1 it follows that a unique tensor field on $\mathscr{J}$ cannot generate a would-be news tensor field assigned to every possible cut $\mathcal{S}$ on $\mathscr{J}$. Also, lemma 7.2.2 shows that a linear combination of would-be news tensor fields, associated each one to a different family of cuts, gives rise to a gauge-invariant traceless symmetric tensor field on $\mathscr{J}$. Such a combination will have in general more than two degrees of freedom. All in all, we are led to search for a tensor field on $\mathscr{J}$ associated to the congruence $\mathcal{C}$ of definition 7.0.1.

Having presented the above general results, let us come back to the asymptotic geometry. First, consider the case of a general $\mathcal{C}$ ($\underline{\omega}_{AB} \neq 0$). Equation (7.13) can be written using eq. (7.11) as

$$N\underline{C}^A = \underline{\epsilon}^{CD}\left(\underline{\mathcal{D}}_{[C}\underline{U}_{D]}{}^A + W_{CD}{}^A - S_{CD}{}^A\right) \ , \tag{7.30}$$

where

$$\underline{U}_{AB} := \underline{S}_{AB} + \frac{1}{2}\underline{\kappa}\underline{\Sigma}_{AB} + L_{AB} \ , \tag{7.31}$$

$$L_{AB} := \left(\frac{1}{8}\underline{\kappa}^2 - \frac{1}{4}\underline{\Sigma}^2 + \frac{3}{4}\underline{\omega}^2\right)\underline{q}_{AB} \ , \tag{7.32}$$

$$S_{CAB} := \underline{T}_{CAB} + 3\left[\underline{\mathcal{D}}_D\left(\underline{\omega}_B{}^D\right) - \underline{a}^D\underline{\omega}_{BD}\underline{\omega}_{CA}\right] \ , \tag{7.33}$$

$$\underline{T}_{CAB} := \frac{1}{2}\left[\underline{q}_{B[C}\underline{\mathcal{D}}_{A]}\underline{\Sigma}^2 - \underline{\mathcal{D}}_B\left(\underline{\Sigma}^D{}_{[A}\right)\underline{\Sigma}_{C]D}\right] \ , \tag{7.34}$$

$$W_{CAB} := -\frac{1}{2}\underline{\kappa}^D{}_B\underline{\mathcal{D}}_D\underline{\omega}_{CA} + \underline{a}^D\underline{\kappa}_{DB}\underline{\omega}_{CA} + \frac{3}{2}\underline{\omega}_{CA}\underline{\mathcal{D}}_D\underline{\kappa}_B{}^D - \frac{3}{2}\underline{\omega}_{CA}\underline{\mathcal{D}}_B\underline{\kappa} \ , \tag{7.35}$$

and it will be convenient to define

$$\underline{T}_A := \underline{T}_{CA}{}^C \ . \tag{7.36}$$

The gauge behaviour of these fields follows by direct computation, using the formulae of appendix C,

$$\tilde{\underline{U}}_{AB} = \underline{U}_{AB} - \frac{1}{\omega}\underline{\mathcal{D}}_{(A}\underline{\omega}_{B)} + \frac{2}{\omega^2}\underline{\omega}_A\underline{\omega}_B - \frac{1}{2\omega^2}\underline{\omega}_C\underline{\omega}^C q_{AB} \ , \tag{7.37}$$

$$\tilde{\underline{S}}_{CAB} = \underline{S}_{CAB} \ , \tag{7.38}$$

$$\tilde{\underline{T}}_{CAB} = \underline{T}_{CAB} \ , \tag{7.39}$$

$$\tilde{\underline{W}}_{CAB} = \underline{W}_{CAB} - \frac{1}{2\omega}\pounds_{\vec{m}}\omega\underline{\mathcal{D}}_B\underline{\omega}_{CA} + \frac{1}{\omega}\underline{\omega}_{CA}\left[\underline{\kappa}^D{}_B\underline{\omega}_D - \frac{1}{2}\underline{\mathcal{D}}_B\pounds_{\vec{m}}\omega - \pounds_{\vec{m}}\underline{\omega}_B + \frac{5}{2\omega}\pounds_{\vec{m}}\omega\underline{\omega}_B\right] \ , \tag{7.40}$$

where $\underline{\omega}_A := \underline{\mathcal{D}}_A\omega$. Precisely, the interest of these definitions is that the combination $\underline{\mathcal{D}}_{[C}\underline{U}_{D]A} + \underline{W}_{CDA}$ is gauge invariant,

$$\tilde{\underline{\mathcal{D}}}_{[C}\tilde{\underline{U}}_{D]A} + \tilde{\underline{W}}_{CDA} = \underline{\mathcal{D}}_{[C}\underline{U}_{D]A} + \underline{W}_{CDA} \tag{7.41}$$

and

$$\underline{U}^E{}_E = \underline{K} \ . \tag{7.42}$$

In order to recover a result of the kind of proposition 6.3.1, it is reasonable to consider the splitting

$$\underline{V}_{AB} := \underline{U}_{AB} - \underline{\rho}_{AB} \ , \tag{7.43}$$

for some $\underline{\rho}_{AB}$ fulfilling the gauge-invariant equation

$$\underline{\mathcal{D}}_{[A}\rho_{B]C} + \underline{W}_{ABC} = 0 \ , \tag{7.44}$$

and V_{AB} a two-dimensional gauge-invariant symmetric traceless tensor field on $\mathscr{J}$. This would constitute the *first component of news-like objects* when $\mathscr{J}$ is equipped. However, while existence of general solutions to eq. (7.44) may be provable, uniqueness is in principle a non-trivial task. Indeed, this is an open problem which should be studied carefully.

Now we focus on the case of a strictly equipped $\mathscr{J}$, so that $\mathcal{C}$ defines also a foliation ($\underline{\omega}_{ab} = 0$). In this case, it is always possible to write eq. (A.103), using the freedom (7.9) if needed, as

$$m_a = F\overline{\nabla}_a v \quad \text{with } \frac{1}{F} = \pounds_{\vec{m}} v, \tag{7.45}$$

and each leaf of the foliation $\mathcal{C}$ is a cut $\mathcal{S}_v$, labelled by a constant value of the parameter along the curves, $v = \hat{v} =$ constant, and with basis $\{E^a{}_A\}_v$, $\left\{W_a{}^A\right\}_v$. Therefore, on each leaf we are in the situation described in chapter 6 for any single cut. In other words, proposition 6.3.1 and corollaries 6.2.2 and 6.2.3 apply on each leaf. Hence, one has on each cut

$$N\mathring{C}^A \overset{\mathcal{S}_v}{=} \mathring{\epsilon}^{CD}\left(\mathcal{D}_{[C}V_{D]}{}^A - T_{CD}{}^A\right) \ , \tag{7.46}$$

or, by means of ${}^{\pm}\underline{C}_A$,

$$2N\,{}^{+}\mathring{C}^A + N\mathring{\epsilon}^{AC}\mathring{D}_C \overset{\mathcal{S}_v}{=} \mathring{\epsilon}^{CD}\left(\underline{\mathcal{D}}_{[C}V_{D]}{}^{A} - T_{CD}{}^{A}\right) \,, \tag{7.47}$$

$$2N\,{}^{-}\mathring{C}^A - N\mathring{\epsilon}^{AC}\mathring{D}_C \overset{\mathcal{S}_v}{=} \mathring{\epsilon}^{CD}\left(\underline{\mathcal{D}}_{[C}V_{D]}{}^{A} - T_{CD}{}^{A}\right) \,, \tag{7.48}$$

Moreover, because $\underline{\omega}_{AB} = 0$,

$$S_{ABC} = \underline{T}_{ABC}\,, \quad W_{ABC} = 0\,, \tag{7.49}$$

which makes eq. (7.30) read

$$N\underline{C}^A = \underline{\epsilon}^{CD}\left(\underline{\mathcal{D}}_{[C}\underline{U}_{D]}{}^{A} - \underline{T}_{CD}{}^{A}\right) \,. \tag{7.50}$$

Inserting $\underline{\omega}_{AB} = 0$ in eq. (7.37) too, noting eq. (A.71), it turns out that $\underline{U}_{AB}$ has a recognisable gauge behaviour,

$$\tilde{\underline{U}}_{AB} = \underline{U}_{AB} - a\frac{1}{\omega}\underline{\mathcal{D}}_A\underline{\omega}_B + \frac{2a}{\omega^2}\underline{\omega}_A\underline{\omega}_B - \frac{a}{2\omega^2}\underline{\omega}_C\underline{\omega}^C\underline{q}_{AB}\,. \tag{7.51}$$

Then, one can show some important results for strictly equipped $\mathscr{J}$ (the first two are general i.e., not only for $\mathscr{J}$ but for any Riemannian 3-dimensional hypersurface $\mathcal{I}$ equipped with a foliation),

Lemma 7.2.3. Let $\mathscr{J}$ be strictly equipped and t_{AB} be any symmetric tensor field on $\mathbf{S}_2$ whose behaviour under conformal rescalings (C.23) is

$$\tilde{t}_{AB} = t_{AB} - a\frac{1}{\omega}\underline{\mathcal{D}}_A\underline{\omega}_B + \frac{2a}{\omega^2}\underline{\omega}_A\underline{\omega}_B - \frac{a}{2\omega^2}\underline{\omega}_C\underline{\omega}^C\underline{q}_{AB} \tag{7.52}$$

for some fixed constant $a \in \mathbb{R}$. Then,

$$\tilde{\underline{\mathcal{D}}}_{[C}\tilde{t}_{A]B} = \underline{\mathcal{D}}_{[C}t_{A]B} + \frac{1}{\omega}\left(a\underline{K} - t^E{}_E\right)\underline{\omega}_{[C}\underline{q}_{A]B}\,, \tag{7.53}$$

where now $\underline{K}$ coincides with the Gaussian curvature K of each cut at v =constant. In particular, for any symmetric gauge-invariant tensor field B_{AB} on $\mathcal{S}$,

$$\tilde{\underline{\mathcal{D}}}_{[C}\tilde{B}_{A]B} = \underline{\mathcal{D}}_{[C}B_{A]B} - \frac{1}{\omega}B^E{}_E\underline{\omega}_{[C}\underline{q}_{A]B} \tag{7.54}$$

Proof. One proceeds as in the proof of lemma 6.2.1. □

Corollary 7.2.1. Under the same assumptions of lemma 7.2.3, a symmetric gauge-invariant tensor field B_{AB} on $\mathbf{S}_2$ satisfies

$$\tilde{\underline{\mathcal{D}}}_{[C}\tilde{B}_{B]A} = \underline{\mathcal{D}}_{[C}B_{B]A} \tag{7.55}$$

if and only if $B^E{}_E = 0$.

Now one can prove the following two results:

Corollary 7.2.2 (The tensor field ρ for strictly equipped $\mathscr{J}$ with $\mathbb{S}^2$ leaves)**.** Assume $\mathscr{J}$ is strictly equipped with v the parameter along the curves (7.10) and such that eq. (7.45) holds. If the leaves have $\mathbb{S}^2$-topology, there is a unique tensor field $\underline{\rho}_{ab}$ on $\mathscr{J}$ orthogonal to m^a (equivalently, a one-parameter family of symmetric tensor fields $\underline{\rho}_{AB}(v) := \underline{E}^a{}_A \underline{E}^b{}_B \underline{\rho}_{ab}$ on the projected surface $\mathbf{S}_2$) whose behaviour under conformal rescalings (C.23) is as in eq. (7.52) and satisfies the equation

$$\underline{P}^d{}_a \underline{P}^e{}_b \underline{P}^f{}_c \overline{\nabla}_{[f} \underline{\rho}_{d]e} = 0 \tag{7.56}$$

in any conformal frame. This tensor field must have a trace $\underline{\rho}^e{}_e := \underline{P}^{ae} \underline{\rho}_{ae} = a\underline{K}$ and reduces, at each leaf, to the corresponding tensor of corollary 6.2.2 with all its properties. In particular, it is given for the round-sphere one-parameter family of metrics by $\underline{\rho}_{ab} = \underline{P}_{ab}\, a\underline{K}/2$.

Proof. Let $\mathcal{S}_{\hat{v}}$ represent a leaf of the foliation for constant $v = \hat{v}$. If we evaluate eq. (7.56) at $v = \hat{v}$, i.e., on the leaf $\mathcal{S}_{\hat{v}}$ and contract all the indices with the basis on $\mathcal{S}_{\hat{v}}$, $\{E^a{}_A\}$, we obtain the following equation there

$$\mathcal{D}_{[C}\left(\underline{\rho}_{A]B}\right) \stackrel{\mathcal{S}_{\hat{v}}}{=} 0 \quad , \tag{7.57}$$

where $\mathcal{D}_A$ is the canonical covariant derivative on $\mathcal{S}_{\hat{v}}$. But the solution to this equation exists and is unique

$$\underline{\rho}_{AB} \stackrel{\mathcal{S}_{\hat{v}}}{=} {}_{\hat{v}}\rho_{AB} \quad , \tag{7.58}$$

with ${}_{\hat{v}}\rho_{AB}$ the tensor field of corollary 6.2.2 corresponding to $\mathcal{S}_{\hat{v}}$. Then, one can define $\underline{\rho}_{ab}$ at any leaf simply by

$$\underline{\rho}_{ab} \stackrel{\mathcal{S}_{\hat{v}}}{=} W_a{}^A W_b{}^B\, {}_{\hat{v}}\rho_{AB} \tag{7.59}$$

and this holds on each leaf –i.e., at any value of v. Since $\mathscr{J} = \underset{v}{\cup}\mathcal{S}_v$ and $\mathcal{S}_{\hat{v}_1} \cap \mathcal{S}_{\hat{v}_2} = \emptyset$ for $\hat{v}_1 \neq \hat{v}_2$, at every point on $\mathscr{J}$ eq. (7.56) has a unique solution given by

$$\underline{\rho}_{ab} = \underline{W}_a{}^A \underline{W}_b{}^B\, {}_{v}\rho_{AB}, \quad m^a \underline{\rho}_{ab} = 0 \quad . \tag{7.60}$$

Note that contraction of eq. (7.56) with $\{\underline{E}^a{}_A\}$ gives the equivalent equation on $\mathbf{S}_2$

$$\mathcal{D}_{[C} \underline{\rho}_{A]B} = 0. \tag{7.61}$$

According to lemma 7.2.3, eq. (7.61) is satisfied in any conformal frame if and only if $\underline{\rho}^E{}_E = a\underline{K}$, which using the definition (A.45) of the projector $\underline{P}^a{}_b$ can be recast as $\underline{\rho}^e{}_e := \underline{P}^{ae} \underline{\rho}_{ae} = a\underline{K}$. Finally, notice that by (7.60) and according to corollary 6.2.2, the solution $\underline{\rho}_{AB}$ on each leaf is given by $\underline{\rho}_{AB} \overset{\mathcal{S}_v}{=} q_{AB}aK/2$ for every cut with a round metric and Gaussian constant curvature K. But $\underline{K}$ coincides on each cut with K, and then contraction with $\left\{\underline{W}_a{}^A\right\}$ gives $\underline{\rho}_{ab} = \underline{P}_{ab}a\underline{K}/2$. □

Corollary 7.2.3 (The tensor ρ for strictly equipped $\mathscr{J}$ with non-$\mathbb{S}^2$ leaves)**.** Assume $\mathscr{J}$ is strictly equipped with v the parameter along the curves (7.10) and such that eq. (7.45) holds. Assume that the leaves $(\mathcal{S}_v, q_{AB})$ are non-necessarily topological-$\mathbb{S}^2$ and that there is a vector field χ^a such that $\underline{\chi}^A := \underline{W}_a{}^A\chi^a$ is a CKVF, and has a fixed point, on each leaf. Then, there is a unique tensor field $\underline{\rho}_{ab}$ on $\mathscr{J}$ orthogonal to m^a (equivalently, a one-parameter family of symmetric tensor fields $\underline{\rho}_{AB}(v) := \underline{E}^a{}_A\underline{E}^b{}_B\underline{\rho}_{ab}$ on the projected surface $\mathbf{S}_2$) whose behaviour under conformal rescalings (C.23) is as in eq. (7.52) and satisfies the equations

$$\underline{P}^d{}_a\underline{P}^e{}_b\underline{P}^f{}_c\overline{\nabla}_{[f}\underline{\rho}_{d]e} = 0 \quad , \tag{7.62}$$

$$\pounds_{\vec{\underline{\chi}}}\underline{\rho}_{AB} + a\underline{\mathcal{D}}_A\underline{\mathcal{D}}_B\phi = 0 \quad , \tag{7.63}$$

in any conformal frame, where $\phi := \underline{\mathcal{D}}_C\underline{\chi}^C/2$. Furthermore, this tensor field must have a trace $\underline{\rho}^e{}_e := \underline{P}^{ae}\underline{\rho}_{ae} = a\underline{K}$ and coincides, at each leaf, with the corresponding tensor of corollary 6.2.3 with all its properties.

Remark 7.2.1. An outstanding case for the existence of the vector field χ^a is when this is an *axial* CKVF of h_{ab} orthogonal to m^a, that is, tangent to the leaves, and such that it leaves the equipping congruence conformally invariant ($\pounds_{\vec{\chi}}m_a \propto m_a$). Actually, this could be generalized to symmetries of the type we will introduce later on in definition 7.4.2.

Proof. One follows the same reasoning as in the proof of corollary 7.2.2, this time using corollary 6.2.3 instead of corollary 6.2.2. After the first steps, one finds

$$\mathcal{D}_{[C}\left(\underline{\rho}_{A]B}\right) \overset{\mathcal{S}_{\hat{v}}}{=} 0 \quad , \tag{7.64}$$

on each cut. Taking into account that $\underline{\chi}^A$ has a fixed point on every $\mathcal{S}_v$ and eq. (7.63), the solution to this equation exists and is unique on each cut

$$\underline{\rho}_{AB} \overset{\mathcal{S}_{\hat{v}}}{=} {}_{\hat{v}}\rho_{AB} \quad , \tag{7.65}$$

given by the tensor ρ_{AB} of corollary 6.2.3. The rest of the proof follows the same steps as in corollary 7.2.2. □

Lemma 7.2.4 (No traceless Codazzi tensor fields on $\mathbf{S}_2$ for foliations)**.** Let $\underline{M}_{AB}$ be any one-parameter family of traceless and symmetric tensor fields on $\mathbf{S}_2$ associated to a congruence of curves $\mathcal{C}$ orthogonal to a family of surfaces foliating a 3-dimensional space-like hypersurface $\mathcal{I}$ with topological-$\mathbb{S}^2$ leaves. Then

$$\underline{M}_{AB} = 0 \Longleftrightarrow \underline{\mathcal{D}}_{[C}\underline{M}_{A]B} = 0. \tag{7.66}$$

Proof. Defining on $\mathcal{I}$ $\underline{M}_{ab} := \underline{W}_a{}^A \underline{W}_b{}^B \underline{M}_{AB}$, it vanishes if and only if $\underline{M}_{AB}$ so does, and satisfies $\underline{P}^c{}_d \underline{P}^a{}_e \underline{P}^b{}_f \overline{\nabla}_{[c} \underline{M}_{a]b} = 0$ if and only if $\underline{\mathcal{D}}_{[C} \underline{M}_{A]B} = 0$. Evaluating on each leaf $\mathcal{S}_v$ and contracting the equation $\underline{P}^c{}_d \underline{P}^a{}_e \underline{P}^b{}_f \overline{\nabla}_{[c} \underline{M}_{a]b} = 0$ with $\{E^a{}_A\}$, one finds $\underline{\mathcal{D}}_{[C} \left(\underline{M}_{A]B} \right) \overset{\mathcal{S}_v}{=} 0$ $\forall v$ which using (6.10) is equivalent to its trace and holds if and only if $\underline{M}_{AB} \overset{\mathcal{S}_v}{=} 0$ $\forall v$ because $\underline{M}_{AB}$ is a symmetric and traceless Codazzi tensor on each compact, two-dimensional cut $\mathcal{S}_v$ (see e.g. [113] and references therein). This is equivalent to the vanishing of $\underline{M}_{ab}$ on each $\mathcal{S}_v$ and, since $\mathcal{I} = \underset{v}{\cup} \mathcal{S}_v$ and $\mathcal{S}_{\hat{v}_1} \cap \mathcal{S}_{\hat{v}_2} = \emptyset$ for $\hat{v}_1 \neq \hat{v}_2$, to the vanishing of $\underline{M}_{ab}$ at every point on $\mathcal{I}$ too. □

Let us continue by showing the existence and uniqueness of a *first component of news* on strictly equipped $\mathscr{J}$ with topological $\mathbb{S}^2$ leaves:

Proposition 7.2.1 (The first component of news on strictly equipped $\mathscr{J}$ with $\mathbb{S}^2$ leaves)**.** Assume $\mathscr{J}$ is strictly equipped with v the parameter along the curves (7.10) and such that eq. (7.45) holds. If the leaves have $\mathbb{S}^2$-topology, there is a one-parameter family of symmetric traceless gauge-invariant tensor fields

$$\underline{V}_{AB} := \underline{U}_{AB} - \underline{\rho}_{AB} \quad , \tag{7.67}$$

that satisfies the gauge-invariant equation

$$\underline{\mathcal{D}}_{[A} \underline{U}_{B]C} = \underline{\mathcal{D}}_{[A} \underline{V}_{B]C} \quad , \tag{7.68}$$

where $\underline{\rho}_{AB}$ is the family of tensor fields of corollary 7.2.2 (for $a = 1$). Besides, $\underline{V}_{AB}$ is unique with these properties.

Proof. The one-parameter family of tensor fields $\underline{V}_{AB}$ is symmetric, traceless and gauge invariant as a consequence of eqs. (7.31), (7.37) and (7.42), recalling $\underline{\omega}_{AB} = 0$, and corollary 7.2.2. The uniqueness of $\underline{V}_{AB}$ follows from corollary 7.2.2 too and Equation (7.68). □

Corollary 7.2.4 (The first component of news on strictly equipped $\mathscr{J}$ with non-$\mathbb{S}^2$ leaves)**.** Assume $\mathscr{J}$ is strictly equipped with v the parameter along the curves (7.10) and such that eq. (7.45) holds. Assume that the leaves $(\mathcal{S}_v, q_{AB})$ are non-necessarily topological-$\mathbb{S}^2$ and that there is a vector field χ^a such that $\underline{\chi}^A := \underline{W}_a{}^A \chi^a$ is a CKVF of the metric q_{AB} and has a fixed point on each leaf. Then, there is a one-parameter family of symmetric traceless gauge invariant tensor fields

$$\underline{V}_{AB} := \underline{U}_{AB} - \underline{\rho}_{AB} \quad , \tag{7.69}$$

that satisfies the gauge-invariant equation

$$\underline{\mathcal{D}}_{[A} \underline{U}_{B]C} = \underline{\mathcal{D}}_{[A} \underline{V}_{B]C} \quad , \tag{7.70}$$

where $\underline{\rho}_{AB}$ is the tensor field of corollary 7.2.3 (for $a = 1$). Besides, $\underline{V}_{AB}$ is unique with these properties.

Proof. The proof proceeds as the one of proposition 7.2.1, only that this time one uses corollary 7.2.3 instead of corollary 7.2.2. □

Then, under assumptions of proposition 7.2.1 or corollary 7.2.4, one has on $\mathbf{S}_2$ (equivalently on $\mathscr{J}$ by taking the pullback)

$$N\underline{C}^{A} \overset{\mathbf{S}_2}{=} \underline{\epsilon}^{CD}\left(\underline{\mathcal{D}}_{[C}\underline{V}_{D]}{}^{A} - \underline{T}_{CD}{}^{A}\right) , \tag{7.71}$$

Corollary 7.2.5. The tensor field on $\mathscr{J}$

$$\underline{V}_{ab}(v) := \underline{W}_{a}{}^{A}\underline{W}_{b}{}^{B}\underline{V}_{AB} \tag{7.72}$$

satisfies

$$\underline{V}_{ab} = \underline{W}_{a}{}^{A}\underline{W}_{b}{}^{B}\underset{v}{V}_{AB} , \quad m^{a}\underline{V}_{ab} = 0 , \tag{7.73}$$

where $\underset{v}{V}_{AB}$ is the first component of news associated to each leaf $\mathcal{S}_v$ defined in proposition 6.3.1, respectively.

Proof. One can take the pullback to $\mathscr{J}$ with $\left\{\underline{W}_{a}{}^{A}\right\}$ of eq. (7.69),

$$\underline{V}_{ab} = \underline{U}_{ab} - \underline{\rho}_{ab} , \tag{7.74}$$

and see that

$$\underline{U}_{ab} = \underline{W}_{a}{}^{A}\underline{W}_{b}{}^{B}\underset{v}{U}_{AB} , \quad m^{a}\underline{U}_{ab} = 0 \tag{7.75}$$

where $\underset{v}{U}_{AB}$ is (6.38) for each leaf $\mathcal{S}_v$ –using that m^a and $\underline{P}^{a}{}_{b}$ are the normal and the projector to each cut for constant v, respectively. Now, we have already shown that (see corollary 7.2.2 and eq. (7.60)) $E^{a}{}_{A}E^{b}{}_{B}\underline{\rho}_{ab} \overset{\mathcal{S}_v}{=} \underset{v}{\rho}_{AB}$ where $\underset{v}{\rho}_{AB}$ is the tensor of corollary 6.2.2 or corollary 6.2.3 for each leaf $\mathcal{S}_v$. Hence, one deduces that $E^{a}{}_{A}E^{b}{}_{B}\underline{V}_{ab} \overset{\mathcal{S}_v}{=} \underset{v}{V}_{AB}$ with $\underset{v}{V}_{AB}$ the first component of news of proposition 6.3.1 for each leaf $\mathcal{S}_v$. □

By means of eq. (7.23), a formula for $\underline{V}_{AB}$ in terms of the acceleration $\underline{a}_{A}$, $\underline{K}$, $\underline{\rho}_{AB}$ and the gauge invariant tensor field f_{AB} is obtained for a general foliation:

$$\underline{V}_{AB} = f_{AB} - \underline{\rho}_{AB} + \underline{\mathcal{D}}_{A}\underline{a}_{B} - \underline{a}_{A}\underline{a}_{B} - \frac{1}{2}\underline{q}_{AB}\left(\underline{\mathcal{D}}_{E}\underline{a}^{E} - \underline{a}_{E}\underline{a}^{E} - \underline{K}\right) . \tag{7.76}$$

It is convenient to define the one-parameter family of tensor fields on $\mathbf{S}_2$

$$\tau_{AB} := \underline{\mathcal{D}}_{A}\underline{a}_{B} - \underline{a}_{A}\underline{a}_{B} - \frac{1}{2}\underline{q}_{AB}\left(\underline{\mathcal{D}}_{E}\underline{a}^{E} - \underline{a}_{E}\underline{a}^{E} - \underline{K}\right) . \tag{7.77}$$

Interestingly, its gauge change is the same as the one of $\underline{U}_{AB}$ and $\underline{\rho}_{AB}$ –see eq. (7.52)– and its trace coincides with the trace of $\underline{\rho}_{AB}$, i.e., $\tau^E{}_E = \underline{K}$. Furthermore, taking the covariant derivative of eq. (7.76), antisymmetrising and using eq. (7.56) one has

$$\underline{\mathcal{D}}_{[C}\tau_{A]B} = \underline{\mathcal{D}}_{[C}\left(\underline{V}_{A]B} - f_{A]B}\right) , \tag{7.78}$$

which can be checked to be gauge-invariant, noting that $\underline{V}_{AB} - f_{AB}$ is a symmetric, traceless and gauge-invariant tensor that fulfils corollary 7.2.1, and that $\underline{\tau}_{AB}$ satisfies lemma 7.2.3 for $a = 1$.

Lemma 7.2.5. The vanishing of the *first component of news* $\underline{V}_{AB}$ of proposition 7.2.1 on $\mathscr{J}$ can be written as a relation between the kinematical quantities of m^a (shear, expansion and acceleration) and the curvature $\underline{K}$,

$$\underline{V}_{AB} = 0 \Longleftrightarrow \underline{\mathcal{D}}_{[C}f_{A]B} = \underline{\mathcal{D}}_{[C}\left(\underline{\mathcal{D}}_{A]}\underline{a}_B - \underline{a}_{A]}\underline{a}_B\right) - \frac{1}{2}\underline{q}_{B[A}\underline{\mathcal{D}}_{C]}\left(\underline{\mathcal{D}}_E\underline{a}^E - \underline{a}_E\underline{a}^E - \underline{K}\right) . \tag{7.79}$$

Equivalently,

$$\underline{V}_{AB} = 0 \Longleftrightarrow \underline{\mathcal{D}}_{[C}\left(\tau_{A]B} - f_{A]B}\right) = 0 . \tag{7.80}$$

Corollary 7.2.6. If $\mathscr{J}$ is strongly equipped, ergo the leaves of the foliation are umbilical ($\underline{\Sigma}_{AB}$ vanishes on $\mathscr{J}$) then

$$\underline{V}_{AB} = 0 \Longleftrightarrow \underline{\mathcal{D}}_{[C}\left(\underline{\mathcal{D}}_{A]}\underline{a}_B - \underline{a}_{A]}\underline{a}_B\right) - \frac{1}{2}\underline{q}_{B[A}\underline{\mathcal{D}}_{C]}\left(\underline{\mathcal{D}}_E\underline{a}^E - \underline{a}_E\underline{a}^E - \frac{1}{2}\underline{K}\right) = 0. \tag{7.81}$$

Equivalently,

$$\underline{V}_{AB} = 0 \Longleftrightarrow \underline{\mathcal{D}}_{[C}\tau_{A]B} = 0 . \tag{7.82}$$

Remark 7.2.2. The dependence on $\underline{\kappa}$ and $\underline{\Sigma}_{AB}$ is encoded in f_{AB}, see eq. (7.15).

Proof. Firstly, take the derivative of eq. (7.76) with $\underline{\mathcal{D}}_C$ and then antisymmetrise. Secondly, apply lemma 7.2.4. □

Following the programme developed in chapter 6, we look now for the second components of news and construct a couple of traceless gauge invariant symmetric families of tensor fields $\overset{\pm}{\underline{X}}_{AB}$ on $\mathbf{S}_2$, such that the pullback to $\mathscr{J}$, $\overset{\pm}{\underline{X}}_{ab}(v) := \underline{W}_a{}^A\underline{W}_b{}^B\overset{\pm}{\underline{X}}_{AB}$, satisfies

$$E^a{}_A E^b{}_B \overset{\pm}{\underline{X}}_{ab} \overset{\mathcal{S}_v}{=} \overset{\pm}{\underset{v}{X}}_{AB} , \tag{7.83}$$

where $\overset{\pm}{\underset{v}{X}}_{AB}$ are the (undetermined) second components of news defined in chapter 6 fulfilling eqs. (6.72) and (6.73) on each cut $\mathcal{S}_v$. The pullback of this pair of equations to

$\mathscr{I}$, taken with respect to $\left\{W_b{}^B\right\}_v$, reads

$$-\frac{1}{2}N\underline{P}^e{}_b D_e \overset{\mathcal{S}_v}{=} \underline{T}_b + \frac{1}{2}\underline{P}^d{}_c \overline{\nabla}_d {}^+\underline{X}_b{}^c \ , \tag{7.84}$$

$$\frac{1}{2}N\underline{P}^e{}_b D_e \overset{\mathcal{S}_v}{=} \underline{T}_b + \frac{1}{2}\underline{P}^d{}_c \overline{\nabla}_d {}^-\underline{X}_b{}^c \ . \tag{7.85}$$

Because m^a is orthogonal to each cut, observe that $\underline{P}^a{}_b \overset{\mathcal{S}_v}{=} \mathring{P}^a{}_b$, $\underline{\epsilon}_{ab} = W_a{}^A W_b{}^B \mathring{\epsilon}_{AB}$ and $\underline{T}_b := \underline{W}_b{}^B \underline{T}_B = W_b{}^B T_B$. Thus, remarking that $\mathscr{I} = \bigcup_v \mathcal{S}_v$ and $\mathcal{S}_{\hat{v}_1} \cap \mathcal{S}_{\hat{v}_2} = \emptyset$ for $\hat{v}_1 \neq \hat{v}_2$, eqs. (7.84) and (7.85) hold everywhere on $\mathscr{I}$ and one can take the push-forward to $\mathbf{S}_2$ using $\{\underline{E}^a{}_A\}$,

$$-\frac{1}{2}N\underline{D}_B = \underline{T}_B + \frac{1}{2}\underline{\mathcal{D}}_C {}^+\underline{X}_B{}^C \ , \tag{7.86}$$

$$\frac{1}{2}N\underline{D}_B = \underline{T}_B + \frac{1}{2}\underline{\mathcal{D}}_C {}^-\underline{X}_B{}^C \ , \tag{7.87}$$

Then, one has on $\mathbf{S}_2$ (equivalently, on $\mathscr{I}$ after taking the pull-back)

$$N^2\, {}^+\mathcal{Z} = \underline{\mathcal{D}}_C \left({}^+\underline{n}_B{}^C\right) \underline{\mathcal{D}}_D \left({}^+\underline{n}^{BD}\right) \ , \tag{7.88}$$

$$N^2\, {}^-\mathcal{Z} = \underline{\mathcal{D}}_C \left({}^-\underline{n}_B{}^C\right) \underline{\mathcal{D}}_D \left({}^-\underline{n}^{BD}\right) \ , \tag{7.89}$$

with

$${}^+\underline{n}_{AB} := V_{AB} + {}^+\underline{X}_{AB} \ , \tag{7.90}$$

$${}^-\underline{n}_{AB} := V_{AB} + {}^-\underline{X}_{AB} \ , \tag{7.91}$$

such that ${}^\pm\underline{n}_{ab} := \underline{W}_a{}^A \underline{W}_b{}^B\, {}^\pm\underline{n}_{AB}$ fulfils

$$E^a{}_A E^b{}_B\, {}^\pm\underline{n}_{ab} \overset{\mathcal{S}_v}{=} {}^\pm n_{AB} \quad \forall v \ . \tag{7.92}$$

A generalisation of proposition 6.4.1 can be written for strictly equipped $\mathscr{I}$:

Proposition 7.2.2 (Radiant news on strictly equipped $\mathscr{I}$ with $\mathbb{S}^2$ leaves)**.** Assume that $\mathscr{I}$ is strictly equipped with v the parameter along the curves (7.10) and that the leaves have $\mathbb{S}^2$-topology. Then, if the condition of eq. (7.86) (eq. (7.87)) holds,

$${}^+\underline{n}_{AB} = 0 \Longleftrightarrow {}^+\mathcal{Z} = 0 \quad , \tag{7.93}$$

$$\left({}^-\underline{n}_{AB} = 0 \Longleftrightarrow {}^-\mathcal{Z} = 0\right) \tag{7.94}$$

where ${}^+\underline{n}_{AB}$ (${}^-\underline{n}_{AB}$) is the one-parameter family of tensor fields on $\mathbf{S}_2$ given by eq. (7.90) (eq. (7.91)) that fulfils properties i) to vi) on page 103. Its pullback to $\mathscr{J}$ as

$$\overset{+}{\underline{n}}_{ab} := \underline{W}_a{}^A \underline{W}_b{}^B \overset{+}{\underline{n}}_{AB}, \quad m^a \overset{+}{\underline{n}}_{ab} = 0 \tag{7.95}$$

(analogously for $\overset{-}{\underline{n}}_{ab}$) is a v-dependent tensor field on $\mathscr{J}$ fulfilling eq. (7.92). Hence, we call it *radiant news* on $\mathscr{J}$ for the radiant $\overset{+}{\underline{\mathcal{Q}}}{}^a$ ($\overset{-}{\underline{\mathcal{Q}}}{}^k$).

Proof. By definition eq. (7.90) (eq. (7.91)) the one-parameter family of tensor fields $\overset{+}{\underline{n}}_{AB}$ ($\overset{-}{\underline{n}}_{AB}$) satisfies properties i) to iv) on page 103. Property v) is fulfilled as well, which can be checked by inspection of eqs. (7.71) and (7.86) (eqs. (7.71) and (7.87)). Now, $\overset{+}{\underline{n}}_{AB}$ ($\overset{-}{\underline{n}}_{AB}$) is symmetric and traceless on $\mathbf{S}_2$ and by lemma 7.2.4 $\overset{+}{\underline{n}}_{AB} = 0 \iff \underline{\mathcal{D}}_C \overset{+}{\underline{n}}_B{}^C = 0$ ($\overset{-}{\underline{n}}_{AB} = 0 \iff \underline{\mathcal{D}}_C \overset{-}{\underline{n}}_B{}^C = 0$). But this vanish if and only if $\overset{+}{\mathcal{Z}} = 0$ ($\overset{-}{\mathcal{Z}} = 0$) because of eq. (7.88) (eq. (7.89)). □

Proposition 7.2.3 (Radiant pseudo-news on strictly equipped $\mathscr{J}$ with non-$\mathbb{S}^2$ leaves)**.** Assume $\mathscr{J}$ is strictly equipped with v the parameter along the curves (7.10) and assume the conditions of corollary 7.2.4. Then, if the condition of eq. (7.86) (eq. (7.87)) holds,

$$\overset{+}{\underline{n}}_{AB} = 0 \implies \overset{+}{\mathcal{Z}} = 0 \quad , \tag{7.96}$$

$$(\overset{-}{\underline{n}}_{AB} = 0 \implies \overset{-}{\mathcal{Z}} = 0) \quad , \tag{7.97}$$

where $\overset{+}{\underline{n}}_{AB}$ ($\overset{-}{\underline{n}}_{AB}$) is the one-parameter family of tensor fields on $\mathbf{S}_2$ given by eq. (7.90) (eq. (7.91)) that has the properties properties i) to v) on page 103, but in general it does not fulfils property vi). One defines its pullback to $\mathscr{J}$ as

$$\overset{+}{\underline{n}}_{ab} := \underline{W}_a{}^A \underline{W}_b{}^B \overset{+}{\underline{n}}_{AB}, \quad m^a \overset{+}{\underline{n}}_{ab} = 0 \tag{7.98}$$

and analogously for $\overset{-}{\underline{n}}_{ab}$. The tensor field $\overset{+}{\underline{n}}_{ab}$ ($\overset{-}{\underline{n}}_{ab}$) is a v-dependent tensor field on $\mathscr{J}$ fulfilling eq. (7.92).

Proof. The proof is very much as the one in proposition 7.2.2, except for that now the tensor $\underline{V}_{AB}$ in eq. (7.71) corresponds to the one of corollary 7.2.4. Then, by eq. (7.88) one has $\overset{+}{\mathcal{Z}} = 0 \implies \overset{+}{\underline{n}}_{AB} = 0$. Due to the non-$\mathbb{S}^2$ topology of the cuts, lemma 7.2.4 does not apply and the inverse implications does not follow. □

7.2.1 Relation to the radiation condition

We have shown that under appropriate conditions radiant news $\overset{\pm}{\underline{n}}_{ab}$ for $\overset{\pm}{\underline{\mathcal{Q}}}{}^\alpha$ exist as tensor fields on strictly equipped $\mathscr{J}$. Next task of our programme is to find equations for the derivatives along m^a of these objects. In principle, guiding ourselves by the $\Lambda = 0$ case, the derivative along the 'evolution' direction of a radiant news-like object should be related to $\overset{\pm}{\mathcal{W}}$. The approach that we will follow is similar to the one in section 6.4.

Begin contracting eq. (5.14) with $\underline{E}^a{}_A \underline{E}^b{}_B$ and symmetrising

$$N\underline{\epsilon}_A{}^E \grave{C}_{BE} \stackrel{\mathscr{S}}{=} \underline{S}'_{AB} - \underline{\kappa}_{(A}{}^D \underline{S}_{B)D} - \underline{\mathcal{D}}_{(A}\underline{S}_{B)} - \overline{\underline{S}}\underline{\kappa}_{AB} + 2\underline{S}_{(B}\underline{a}_{A)} \,, \tag{7.99}$$

where we have defined

$$\underline{S}'_{AB} := \underline{E}^a{}_A \underline{E}^b{}_B \pounds_{\vec{m}} \overline{S}_{ab} \,. \tag{7.100}$$

Note that $\underline{\epsilon}_{(A}{}^E C_{B)E} = \underline{\epsilon}_{(A}{}^E \grave{C}_{B)E} = \underline{\epsilon}_A{}^E \grave{C}_{BE}$ and also that $\underline{S}'_{AB} = \pounds_{\vec{m}} \underline{S}_{AB}$ –see eq. (A.56). Equation (7.99) can be expressed in terms of $\overset{\pm}{\underline{C}}_{AB}$ using properties xv) and xvi) on page 212 of appendix D as

$$N\underline{\epsilon}_B{}^E \overset{\pm}{\underline{C}}_{AE} = \underline{S}'_{AB} - \underline{\kappa}_{(A}{}^D \underline{S}_{B)D} - \underline{\mathcal{D}}_{(A}\underline{S}_{B)} - \overline{\underline{S}}\underline{\kappa}_{AB} + 2\underline{S}_{(B}\underline{a}_{A)} \pm N\grave{D}_{AB} \,. \tag{7.101}$$

We can write this equation in terms of $\underline{V}_{AB}$, $\underline{\rho}_{AB}$, L_{AB},

$$\begin{aligned} N\underline{\epsilon}_B{}^E \overset{\pm}{\underline{C}}_{AE} &= V'_{AB} + \underline{\rho}'_{AB} - \underline{L}'_{AB} - \underline{\Sigma}_{(A}{}^D \left(V_{B)D} + \underline{\rho}_{B)D} - \underline{L}_{B)D}\right) \\ &-\frac{1}{2}\underline{\kappa}\left(V_{AB} + \underline{\rho}_{AB} - \underline{L}_{AB}\right) - \underline{\mathcal{D}}_{(A}\underline{S}_{B)} - \overline{\underline{S}}\underline{\kappa}_{AB} + 2\underline{S}_{(B}\underline{a}_{A)} \pm N\grave{D}_{AB} \,, \end{aligned} \tag{7.102}$$

where, in addition, we have expanded $\underline{\kappa}_{AB}$ in terms of $\underline{\Sigma}_{AB}$ and $\underline{\kappa}$ and defined

$$\underline{\rho}'_{AB} := \pounds_{\vec{m}} \underline{\rho}_{AB} \,, \tag{7.103}$$

$$\overset{\pm}{\underline{n}}'_{AB} := \pounds_{\vec{m}} \overset{\pm}{\underline{n}}_{AB} \,. \tag{7.104}$$

A similar expression follows for $\overset{\pm}{\underline{C}}_{AB}$,

$$\begin{aligned} N\overset{\pm}{\underline{C}}_{AB} = \underline{\epsilon}_B{}^E \Big[&\underline{V}'_{AE} + \underline{\rho}'_{AE} - \underline{L}'_{AE} - \underline{\Sigma}_{(A}{}^D \left(\underline{V}_{E)D} + \underline{\rho}_{E)D} - \underline{L}_{E)D}\right) \\ &-\frac{1}{2}\underline{\kappa}\left(\underline{V}_{AE} + \underline{\rho}_{AE} - \underline{L}_{AE}\right) - \underline{\mathcal{D}}_{(A}\underline{S}_{E)} - \overline{\underline{S}}\underline{\kappa}_{AE} + 2\underline{S}_{(E}\underline{a}_{A)} \pm N\grave{D}_{AE} \Big] \,. \end{aligned} \tag{7.105}$$

Now, we propose the following 'transport' equations for $\overset{\pm}{\underline{n}}_{AB}$:

$$N\underline{\epsilon}_B{}^E \overset{+}{C}_{AE} = \overset{+}{n}'_{AB} - \underline{\Sigma}_{(A}{}^C \overset{+}{n}_{B)C} \,, \tag{7.106}$$

$$N\underline{\epsilon}_B{}^E \overset{-}{C}_{AE} = \overset{-}{n}'_{AB} - \underline{\Sigma}_{(A}{}^C \overset{-}{n}_{B)C} \,, \tag{7.107}$$

with $\overset{\pm}{n}_{AB}$ defined as in eqs. (6.70) and (6.71). The square of this expressions reads

$$N^2 \overset{+}{\mathcal{W}} = \left(\overset{+}{n}'_{AB} - \underline{\Sigma}_{(A}{}^C \overset{+}{n}_{B)C}\right)\left(\overset{+}{n}'^{AB} - \underline{\Sigma}^{(A}{}_C \overset{+}{n}^{B)C}\right) \,, \tag{7.108}$$

$$N^2 \overset{-}{\mathcal{W}} = \left(\overset{-}{n}'_{AB} - \underline{\Sigma}_{(A}{}^C \overset{-}{n}_{B)C}\right)\left(\overset{-}{n}'^{AB} - \underline{\Sigma}^{(A}{}_C \overset{-}{n}^{B)C}\right) \,. \tag{7.109}$$

Let us remark that eqs. (7.106) and (7.107) are gauge invariant, which follows from the gauge transformations presented in appendix C, from where the next result is derived as well:

Lemma 7.2.6. Let j_{ab} be any symmetric gauge invariant tensor field on equipped $\mathscr{J}$ orthogonal to m^a, i.e., $m^a j_{ab} = 0$. Then,

$$\tilde{j}'_{ab} = \frac{1}{\omega} j'_{ab} \ , \tag{7.110}$$

where $\underline{j}'_{ab} := \pounds_{\vec{m}} \underline{j}_{ab}$.

The sufficient and necessary conditions for eqs. (7.106) and (7.107) to hold are, respectively:

$$-N\grave{D}_{AB} = -\overset{+}{X}'_{AB} + \overset{+}{X}_{C(B}\underline{\Sigma}_{A)}{}^{C} - \underline{\kappa}_{(A}{}^{D}\left(\underline{\rho}_{B)D} - \underline{L}_{B)D}\right) - \frac{1}{2}\underline{\kappa}\left(V_{AB}\right) - L'_{AB}$$
$$+ \rho'_{AB} - \mathcal{D}_{(A}\underline{S}_{B)} - \overline{S}\underline{\kappa}_{AB} + 2\underline{S}_{(B}\underline{a}_{A)} \ , \tag{7.111}$$

$$N\grave{D}_{AB} = -\overset{-}{X}'_{AB} + \overset{-}{X}_{C(B}\underline{\Sigma}_{A)}{}^{C} - \underline{\kappa}_{(A}{}^{D}\left(\underline{\rho}_{B)D} - \underline{L}_{B)D}\right) - \frac{1}{2}\underline{\kappa}\left(V_{AB}\right) - L'_{AB}$$
$$+ \rho'_{AB} - \mathcal{D}_{(A}\underline{S}_{B)} - \overline{S}\underline{\kappa}_{AB} + 2\underline{S}_{(B}\underline{a}_{A)} \ . \tag{7.112}$$

Therefore, by eqs. (7.108) and (7.109), one has

Lemma 7.2.7. Assume $\mathscr{J}$ is strictly equipped with v the parameter along the curves (7.10) and such that eq. (7.45) holds. Assume that the leaves have $\mathbb{S}^2$-topology and eqs. (7.86) and (7.111) (eqs. (7.87) and (7.112)) hold there. Then,

$$\overset{+}{\underline{n}}_{ab} = 0 \implies \overset{+}{\mathcal{W}} = 0 \ . \tag{7.113}$$

$$\left(\overset{-}{\underline{n}}_{ab} = 0 \implies \overset{-}{\mathcal{W}} = 0\right) \tag{7.114}$$

To see the effects of a vanishing $\overset{+-}{\underline{n}}_{ab}$ on the presence of radiation at $\mathscr{J}$, it is easier if one studies the relation with the radiant supermomenta first

Proposition 7.2.4 (Radiant news and radiant supermomenta)**.** Under the same assumptions of lemma 7.2.7,

$$\overset{+}{\underline{n}}_{ab} = 0 \Longleftrightarrow \overset{+}{\mathcal{Q}}{}^{\alpha} = 0 \ . \tag{7.115}$$

$$\left(\overset{-}{\underline{n}}_{ab} = 0 \Longleftrightarrow \overset{-}{\mathcal{Q}}{}^{\alpha} = 0\right) \ . \tag{7.116}$$

Proof. We give the proof for $\overset{+}{\mathcal{Q}}{}^{\alpha}$. By proposition 7.2.2, one has that $\overset{+}{\mathcal{Z}} = 0 \Longleftrightarrow \overset{+}{\underline{n}}_{AB} = 0$ and, by lemma 7.2.7, that $\overset{+}{\underline{n}}_{AB} = 0 \implies \overset{+}{\mathcal{W}} = 0$, therefore $\overset{+}{\underline{n}}_{AB} = 0 \implies \overset{+}{\mathcal{Q}}{}^{\alpha} = 0$ –see property iii) on page 15. For the converse, $\overset{+}{\mathcal{Q}}{}^{\alpha} = 0 \implies \overset{+}{\mathcal{W}} = 0 = \overset{+}{\mathcal{Z}}$ and, by proposition 7.2.2 again, $\overset{+}{\mathcal{Z}} = 0 \implies \overset{+}{\underline{n}}_{AB} = 0$. □

With this intermediate result, we are able to write a theorem for the asymptotic canonical super-Poynting vector field

Theorem 5 (Asymptotic super-Poynting vector and news)**.** *Assume $\mathscr{J}$ strictly equipped, such that the leaves have $\mathbb{S}^2$-topology and eqs.* (7.86)*,* (7.87)*,* (7.111) *and* (7.112) *hold there. Then,*

$$ {}^{+}\underline{n}_{ab} = 0 = {}^{-}\underline{n}_{ab} \implies \overline{\mathcal{P}}^a = 0 \ . \tag{7.117} $$

Remark 7.2.3. According to criterion 1, the result states that ${}^{+}\underline{n}_{ab} = 0 = {}^{-}\underline{n}_{ab} \implies$ *no radiation at $\mathscr{J}$.*

Proof. The proof follows directly by proposition 7.2.4 and corollary 2.3.1. □

7.2.2 Possible generalisation

We proceed to generalise the above results using the same technique of section 6.4.1. As before, we ask for a couple of families of traceless gauge invariant symmetric tensor fields ${}^{\pm}\underline{X}_{AB}$ on $\mathbf{S}_2$ satisfying eq. (7.83) where this time ${}^{\pm}_{v}X_{AB}$ is the unknown tensor field of chapter 6 appearing in eqs. (6.94) and (6.95) (instead of eqs. (6.72) and (6.73)). The pullback of these equations to $\mathscr{J}$ reads

$$ -\frac{1}{2} N \underline{P}^e{}_b D_e \overset{\mathcal{S}_v}{=} \underline{T}_b + \left({}^{+}\lambda - 1\right) N \underline{\epsilon}_{bc} \underline{P}^c{}_d \, {}^{+}C^d + {}^{+}\beta N \underline{P}^e{}_b \, {}^{+}C_e + \frac{1}{2} \underline{P}^d{}_c \overline{\nabla}_d \, {}^{+}\underline{X}_b{}^c \ , \tag{7.118} $$

$$ \frac{1}{2} N \underline{P}^e{}_b D_e \overset{\mathcal{S}_v}{=} \underline{T}_b + \left({}^{-}\lambda - 1\right) N \underline{\epsilon}_{bc} \underline{P}^c{}_d \, {}^{-}C^d + {}^{-}\beta N \underline{P}^e{}_b \, {}^{-}C_e + \frac{1}{2} \underline{P}^d{}_c \overline{\nabla}_d \, {}^{-}\underline{X}_b{}^c \ . \tag{7.119} $$

On $\mathbf{S}_2$, contracting with $\{\underline{E}^a{}_A\}$,

$$ -\frac{1}{2} N \underline{D}_B = \underline{T}_B + \left({}^{+}\underline{\lambda} - 1\right) N \epsilon_{BC} \, {}^{+}\underline{C}^C + {}^{+}\underline{\beta} N \, {}^{+}\underline{C}_B + \frac{1}{2} \mathcal{D}_C \, {}^{+}\underline{X}_B{}^C \ , \tag{7.120} $$

$$ \frac{1}{2} N \underline{D}_B = \underline{T}_B + \left({}^{-}\underline{\lambda} - 1\right) N \epsilon_{BC} \, {}^{-}\underline{C}^C + {}^{-}\underline{\beta} N \, {}^{-}\underline{C}_B + \frac{1}{2} \mathcal{D}_C \, {}^{-}\underline{X}_B{}^C \ , \tag{7.121} $$

where

$$ \left\{ {}^{\pm}\underline{\lambda} \ \middle/ \ {}^{\pm}\underline{\lambda} \overset{\mathcal{S}_v}{=} {}^{\pm}\lambda \quad \forall v \right\} , \quad \left\{ {}^{\pm}\underline{\beta} \ \middle/ \ {}^{\pm}\underline{\beta} \overset{\mathcal{S}_v}{=} {}^{\pm}\beta \quad \forall v \right\} \tag{7.122} $$

and we assume ${}^{\pm}\underline{\lambda}$ and ${}^{\pm}\underline{\beta}$ differentiable enough. A direct calculation provides on $\mathbf{S}_2$ (equivalently, on $\mathscr{J}$ after taking the pull-back)

$$ N^2 \left({}^{+}\underline{\beta}^2 + {}^{+}\underline{\lambda}^2\right) {}^{+}\mathcal{Z} = \mathcal{D}_C \left({}^{+}\underline{n}_B{}^C\right) \mathcal{D}_D \left({}^{+}\underline{n}^{BD}\right) \ , \tag{7.123} $$

$$ N^2 \left({}^{-}\underline{\beta}^2 + {}^{-}\underline{\lambda}^2\right) {}^{-}\mathcal{Z} = \mathcal{D}_C \left({}^{-}\underline{n}_B{}^C\right) \mathcal{D}_D \left({}^{-}\underline{n}^{BD}\right) \ , \tag{7.124} $$

where the definitions

$$ {}^{+}\underline{n}_{AB} := V_{AB} + {}^{+}\underline{X}_{AB} \ , \tag{7.125} $$

$$ {}^{-}\underline{n}_{AB} := V_{AB} + {}^{-}\underline{X}_{AB} \ , \tag{7.126} $$

were introduced. In a similar fashion, recalling that ${}^{\pm}\mathcal{W}$ vanishes if and only if ${}^{\pm}\underline{C}_{AB}$ does so, we consider combinations of the form

$$
{}^{\pm}\delta\,{}^{\pm}\underline{C}_{AB} + {}^{\pm}\gamma\,\underline{\epsilon}_{B}{}^{C}\,{}^{\pm}\underline{C}_{AC}\ , \tag{7.127}
$$

with δ, γ gauge-invariant, dimensionless, scalar functions obeying

$$
{}^{\pm}\delta = 0 \implies {}^{\pm}\gamma \neq 0\ , \tag{7.128}
$$

$$
{}^{\pm}\gamma = 0 \implies {}^{\pm}\delta \neq 0\ . \tag{7.129}
$$

Now, we propose the following 'transport' equations for ${}^{\pm}\underline{n}_{AB}$:

$$
N\left({}^{+}\delta\,{}^{+}C_{AB} + {}^{+}\gamma\,\underline{\epsilon}_{B}{}^{E}\,{}^{+}C_{AE}\right) = {}^{+}\underline{n}'_{AB} - \underline{\Sigma}_{(A}{}^{C}\,{}^{+}\underline{n}_{B)C}\ , \tag{7.130}
$$

$$
N\left({}^{-}\delta\,{}^{-}C_{AB} + {}^{-}\gamma\,\underline{\epsilon}_{B}{}^{E}\,{}^{-}C_{AE}\right) = {}^{-}\underline{n}'_{AB} - \underline{\Sigma}_{(A}{}^{C}\,{}^{-}\underline{n}_{B)C}\ , \tag{7.131}
$$

The square of this expressions reads

$$
N^{2}\,{}^{+}\mathcal{W}\left({}^{+}\delta^{2} + {}^{+}\gamma^{2}\right) = \left({}^{+}\underline{n}'_{AB} - \underline{\Sigma}_{(A}{}^{C}\,{}^{+}\underline{n}_{B)C}\right)\left({}^{+}\underline{n}'^{AB} - \underline{\Sigma}^{(A}{}_{C}\,{}^{+}\underline{n}^{B)C}\right)\ , \tag{7.132}
$$

$$
N^{2}\,{}^{-}\mathcal{W}\left({}^{-}\delta^{2} + {}^{-}\gamma^{2}\right) = \left({}^{-}\underline{n}'_{AB} - \underline{\Sigma}_{(A}{}^{C}\,{}^{-}\underline{n}_{B)C}\right)\left({}^{-}\underline{n}'^{AB} - \underline{\Sigma}^{(A}{}_{C}\,{}^{-}\underline{n}^{B)C}\right)\ . \tag{7.133}
$$

This time, the sufficient and necessary conditions for eqs. (7.130) and (7.131) to hold are, respectively:

$$
\begin{aligned}
-N\grave{D}_{AB} = &-{}^{+}X'_{AB} + {}^{+}X_{C(B}\underline{\Sigma}_{A)}{}^{C} - \underline{\kappa}_{(A}{}^{D}\left(\underline{\rho}_{B)D} - \underline{L}_{B)D}\right) - \frac{1}{2}\underline{\kappa}\left(V_{AB}\right) - L'_{AB} + \rho'_{AB}\\
&- \underline{\mathcal{D}}_{(A}\underline{S}_{B)} - \overline{\underline{S}}\underline{\kappa}_{AB} + 2\underline{S}_{(B}\underline{a}_{A)} + N\left({}^{+}\gamma - 1\right)\underline{\epsilon}_{B}{}^{E}\,{}^{+}C_{AE} + N\,{}^{+}\delta\,{}^{+}C_{AB}\ ,
\end{aligned} \tag{7.134}
$$

$$
\begin{aligned}
N\grave{D}_{AB} = &-{}^{-}X'_{AB} + {}^{-}X_{C(B}\underline{\Sigma}_{A)}{}^{C} - \underline{\kappa}_{(A}{}^{D}\left(\underline{\rho}_{B)D} - \underline{L}_{B)D}\right) - \frac{1}{2}\underline{\kappa}\left(V_{AB}\right) - L'_{AB} + \rho'_{AB}\\
&- \underline{\mathcal{D}}_{(A}\underline{S}_{B)} - \overline{\underline{S}}\underline{\kappa}_{AB} + 2\underline{S}_{(B}\underline{a}_{A)} + N\left({}^{-}\gamma - 1\right)\underline{\epsilon}_{B}{}^{E}\,{}^{-}C_{AE} + N\,{}^{-}\delta\,{}^{-}C_{AB}\ .
\end{aligned} \tag{7.135}
$$

Finally, one can write a generalised version of lemma 7.2.7, theorem 5, proposition 7.2.2 and proposition 7.2.4 by means of eqs. (7.120), (7.121), (7.134) and (7.135) instead of eqs. (7.86), (7.87), (7.111) and (7.112).

7.3 Incoming radiation

We turn now to investigate possible ways of isolating *outgoing* radiation from *incoming* components. This issue is relevant for characterising isolated sources which on physical grounds one expects to contain no incoming contributions but only to emit gravitational radiation that constitutes the outgoing component. In this section we will consider radiation arriving at the future component of the conformal boundary, $\mathscr{J}^{+}$. The case of

$\mathscr{J}^-$ can be treated similarly. Let us point out that the asymptotically flat scenario automatically has a structure adapted to the outgoing radiation due to the lightlike character of $\mathscr{J}^+$. In simple words, when $\Lambda = 0$ the radiation arriving at infinity and escaping from the space-time necessarily follows lightlike directions transversal to the conformal boundary. Therefore, the generators of $\mathscr{J}^+$ can be considered to point along the direction of propagation of incoming radiation or, from another point of view, incoming radiation never propagates transversally to $\mathscr{J}^+$. In contrast, the $\Lambda > 0$ case presents the following difficulty: *every radiation component, incoming or outgoing, crosses $\mathscr{J}^+$ and escapes from the space-time*. Hence, one is left with the problem of specifying physically reasonable conditions capable of ruling out one of the radiative components –in our setting the incoming one, by definition. This sort of constraints sometimes receives the name of *no incoming radiation conditions*. There is already a proposal [72] in the literature which requires information from the physical space-time. Since according to criterion 1 the presence of radiation at $\mathscr{J}^+$ is determined by the information encoded in $(\mathscr{J}^+, h_{ab}, D_{ab})$ –see property iv) on page 86 and remark 5.3.2–, we believe that absence of incoming radiation should be encoded upon this same data.

Motivated by the $\Lambda = 0$ case (chapter 4), it is reasonable to think that the vanishing of a radiant supermomentum $\overset{\ell}{\mathcal{Q}}{}^\alpha$ is related to the absence of radiation propagating transversally to the null direction ℓ^α. This suggests that in our setup the vanishing of one radiant supermomenta, say $\overset{-}{\mathcal{Q}}{}^\alpha$, could suppress the radiation travelling along transversal directions, in particular along $\overset{+}{k}{}^\alpha$. Looking at the definition in eq. (7.6), this restriction automatically turns m^a into an intrinsic incoming direction field which in particular defines a selected congruence of curves, hence equipping $\mathscr{J}^+$ –or the open portion $\Delta \subset \mathscr{J}^+$ with the same topology where $\overset{-}{\mathcal{Q}}{}^\alpha$ vanishes. In view of these properties, it makes sense to consider m^a as an intrinsic 'evolution direction' on $\mathscr{J}^+$: if we compare with the $\Lambda = 0$ case, the incoming direction given by the generators of $\mathscr{J}^+$ defines the evolution direction; the analogy goes further if we notice that the vector field m^a points towards the region where the worldlines of the isolated sources meet $\mathscr{J}^+$ –see fig. 7.1. As a further positive property, the restriction $\overset{-}{\mathcal{Q}}{}^\alpha = 0$ can be expressed entirely with the information available in $(\mathscr{J}^+, h_{ab}, D_{ab})$:

Lemma 7.3.1. Let $\mathscr{J}$ (or an open portion thereof) be equipped as in definition 7.0.1 and define $\overset{-}{\mathcal{Q}}{}^\alpha$ according to definitions of eqs. (2.49) and (7.7). Then

$$\overset{-}{\mathcal{Q}}{}^\alpha = 0 \iff D_{ab} - \frac{1}{2} D_{ef} m^e m^f \left(3 m_a m_b - h_{ab}\right) = m^d \epsilon_{ed(a} \left(C_{b)}{}^e + m_{b)} m^f C_f{}^e\right) . \quad (7.136)$$

Proof. By eqs. (2.53) and (2.55) and property iii) on page 15, $\overset{-}{\mathcal{Q}}{}^\alpha = 0 \iff \overset{-}{\underline{D}}_{AB} = \overset{-}{\underline{C}}_{AB} = 0 = \overset{-}{\underline{D}}_A = \overset{-}{\underline{C}}_A$. The result is obtained setting these values into eq. (2.42) and using property xxviii) on page 212. □

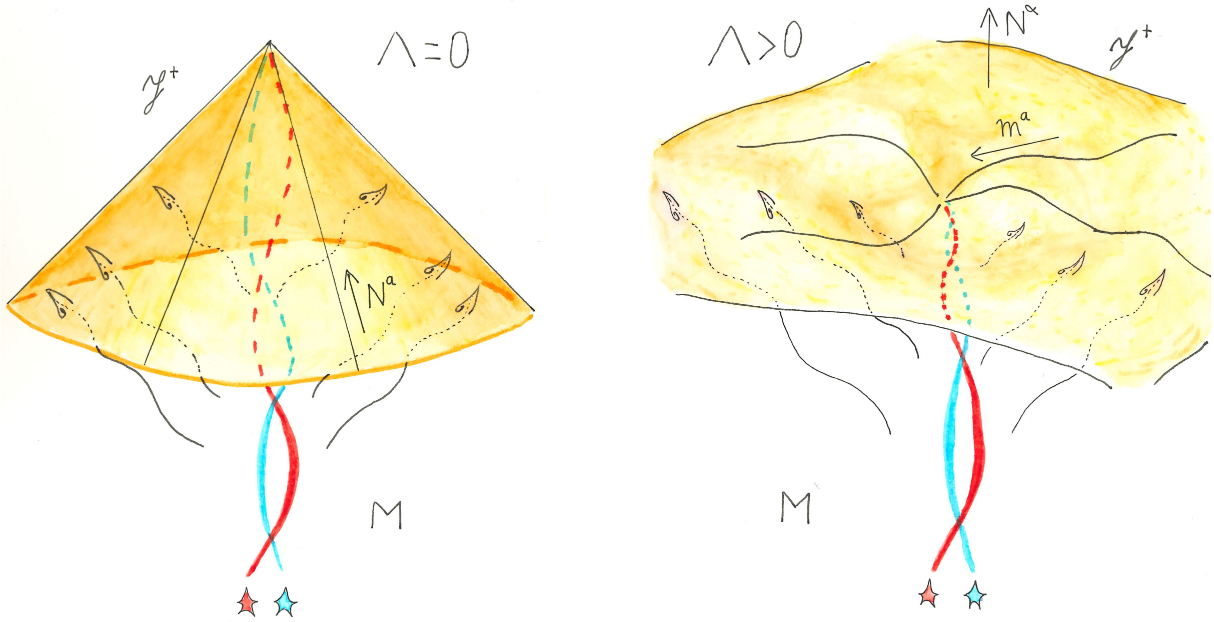

Figure 7.1: On the left: the asymptotically flat case, where the generators of $\mathscr{J}$ rule the natural evolution direction and outgoing radiation crosses $\mathscr{J}^+$ transversally. On the right: the $\Lambda > 0$ scenario, where any direction of propagation of gravitational radiation is transversal to $\mathscr{J}^+$ and criterion 2 selects an intrinsic 'evolution' direction given by m^a, which points towards the region where the source meets $\mathscr{J}^+$.

Then, our proposal to describe absence of incoming radiation reads as follows:

Criterion 2 (No incoming radiation on $\mathscr{J}^+$)**.** We say that there is no incoming radiation at $\mathscr{J}^+$ (or on an open portion $\Delta \subset \mathscr{J}^+$ with the same topology) propagating along a vector field m^a on $\mathscr{J}^+$, if m^a is such that, according to definitions (2.49, 7.7),

$$\bar{\mathcal{Q}}^\alpha = 0 \tag{7.137}$$

or equivalently

$$D_{ab} - \frac{1}{2} D_{ef} m^e m^f \left(3 m_a m_b - h_{ab}\right) = m^d \epsilon_{ed(a} \left(C_{b)}{}^e + m_{b)} m^f C_f{}^e \right) \ . \tag{7.138}$$

Remark 7.3.1. Equivalently, there is no incoming radiation propagating along m^a on $(\Delta \subset)\mathscr{J}^+$ when m^a defines a strong orientation there –see definition 5.4.2.

Remark 7.3.2. If criterion 2 holds, all the components of D_{ab} except $m^a m^b D_{ab}$ are determined by C_{ab}. This is in close analogy to what happens at the conformal boundary for $\Lambda = 0$ where the 'electric' part of the rescaled Weyl tensor defined with respect to the null normal N^α (which algebraically is of the kind (2.25)), is determined by the 'magnetic' part (which is of the sort (2.26)) except for the $N^\alpha N^\beta$ component. In both scenarios, this free component carries the information related to the Coulomb part of the gravitational field (see eq. (D.10)). This evinces that criterion 2 is a constraint that affects the radiative degrees of freedom.

In other words, criterion 2 identifies a class of space-times which can be safely considered to describe situations with only outgoing gravitational radiation arriving at $\mathscr{J}^+$: those where the free data D_{ab} are determined by the intrinsic geometry of $(\mathscr{J}^+, h_{ab})$ according to (7.138) (for unit m^a) except for the one component $m^a m^b D_{ab}$ which remains as the only extra free data independent of $(\mathscr{J}^+, h_{ab})$. It seems interesting to study in deep this class of space-times.

As a consequence of corollary 2.3.1 one has

Corollary 7.3.1. Assume that the no incoming radiation condition of criterion 2 holds. Then,

$$\overset{+}{\mathcal{Q}}{}^{\alpha} = 0 \Longleftrightarrow \overline{\mathcal{P}}{}^{a} = 0. \tag{7.139}$$

Remark 7.3.3. This provides further support to criterion 2 because if condition (7.137) holds, the presence of gravitational waves at $\mathscr{J}^+$ (or on an open portion $\Delta \subset \mathscr{J}^+$) is completely determined, according to our criterion 1, by the *outgoing* components of the radiation —which are associated to $\overset{+}{\mathcal{Q}}{}^{\alpha}$.

Of especial interest is the case of a strictly equipped $\mathscr{J}^+$, so that m^a defines a foliation ($\Longleftrightarrow \underline{\omega}_{ab} = 0$). In particular,

Lemma 7.3.2. Let $\mathscr{J}^+$ by strictly equipped such that m^a satisfies condition (7.137) of criterion 2. Assume that conditions in proposition 7.2.2 and eq. (7.87) hold on $\mathscr{J}^+$. Then,

$$\overset{-}{\underline{n}}{}_{AB} = 0 \tag{7.140}$$

with $\overset{-}{\underline{n}}{}_{AB}$ one of the news tensor fields of proposition 7.2.2.

Proof. On the one hand, condition in criterion 2 is saying that $\overset{-}{\mathcal{Q}}{}^{\alpha} = 0$, which implies $\overset{-}{\mathcal{Z}} = 0$. On the other hand, because of eq. (7.87), proposition 7.2.2 tells us $\overset{-}{\mathcal{Z}} = 0 \Longleftrightarrow \overset{-}{\underline{n}}{}_{AB} = 0$. □

And for the particular case with $\mathscr{J}^+$ strongly equipped,

Lemma 7.3.3. If $\mathscr{J}^+$ is strongly equipped ($\underline{\Sigma}_{ab} = 0$) with $\mathbb{S}^2$ leaves and the condition (7.137) of criterion 2 is satisfied, then there always exists the radiant news $\overset{+}{\underline{n}}{}_{AB}$ of proposition 7.2.2 and is given by

$$\overset{+}{\underline{n}}{}_{AB} = 2\underline{V}_{AB} \quad , \tag{7.141}$$

where $\underline{V}_{AB}$ is the *first component of news* of proposition 7.2.1.

Proof. Note that eq. (7.137) imposes $\overset{+}{\underline{C}}{}^A = \underline{C}^A$ (and $\underline{D}_A\epsilon^{AB} = -\underline{C}^B$, see lemma 2.3.3) and umbilicity implies $\underline{T}_A = 0$ (see (7.36)). It is easy to see that eq. (7.86) is satisfied with $\overset{+}{\underline{X}}_{AB} = \underline{V}_{AB}$. The result follows then by proposition 7.2.2. □

Lemma 7.3.4. Assume $\mathscr{J}^+$ is strongly equipped with leaves that are non-necessarily topological-$\mathbb{S}^2$ and such that condition (7.137) of criterion 2 is satisfied. Assume also that there is a vector field χ^a such that $\underline{\chi}^A := \underline{W}_a{}^A\chi^a$ is a CKVF with a fixed point on each leaf. Then, there always exists the radiant pseudo-news $\overset{+}{\underline{n}}_{AB}$ of proposition 7.2.3 and is given by

$$\overset{+}{\underline{n}}_{AB} = 2\underline{V}_{AB} \quad , \tag{7.142}$$

where $\underline{V}_{AB}$ is the *first component of news* of corollary 7.2.4.

Proof. The proof follows as in lemma 7.3.3, but now one uses proposition 7.2.3 instead of proposition 7.2.2. □

Remark 7.3.4. If instead one uses the generalised approach of section 7.2.2, it is possible to show in a similar fashion that solutions

$$\overset{+}{\underline{n}}_{AB} = 2\overset{+}{\lambda}\underline{V}_{AB} \tag{7.143}$$

always exist, where the values $\overset{+}{\lambda} = \text{constant}$ and $\overset{+}{\beta} = 0$ are fixed. It may be the case that the value of $\overset{+}{\lambda}$ can be fixed by physical arguments.

If we use eqs. (7.86) and (7.111), we end up with a theorem on the presence of radiation,

Theorem 6 (Asymptotic super-Poynting and radiant news under Criterion 2)**.** *Assume that $\mathscr{J}^+$ is strictly equipped with $\mathbb{S}^2$ leaves and that condition (7.137) of criterion 2 holds. Assume also that eqs. (7.86) and (7.111) hold on $\mathscr{J}^+$. Then, radiant news $\overset{+}{\underline{n}}_{AB}$ exists such that*

$$\overset{+}{\underline{n}}_{AB} = 0 \Longleftrightarrow \overline{\mathcal{P}}^a = 0 \Longleftrightarrow \textit{There is no radiation at } \mathscr{J}^+ \quad . \tag{7.144}$$

Proof. Criterion 2 implies that $\overset{-}{\mathcal{Q}}{}^\alpha = 0$, and from corollary 7.3.1 we have $\overset{+}{\mathcal{Q}}{}^\alpha = 0 \Longleftrightarrow \overline{\mathcal{P}}^a = 0$ (which according to criterion 1 occurs if and only if there is no radiation at $\mathscr{J}^+$). Then, proposition 7.2.4 shows that $\overset{+}{\mathcal{Q}}{}^\alpha = 0 \Longleftrightarrow \overset{+}{\underline{n}}_{AB} = 0$ –the existence of $\overset{+}{\underline{n}}_{AB}$ follows from proposition 7.2.2. □

Corollary 7.3.2. Let the assumptions of theorem 6 hold but now with a strongly equipped $\mathscr{J}^+$. Then

$$\underline{V}_{AB} = 0 \Longleftrightarrow \overline{\mathcal{P}}^a = 0 \Longleftrightarrow \text{There is no radiation at } \mathscr{J} \quad . \tag{7.145}$$

Proof. By lemma 7.3.3 $\overset{+}{\underline{n}}_{AB}$ exists such that $\underline{V}_{AB} = 0 \Longleftrightarrow \overset{+}{\underline{n}}_{AB} = 0$. The result follows then by theorem 6. □

7.4 Symmetries

Another field of study is that of symmetries at infinity. Although the asymptotically flat scenario is well understood in this respect, such is not the case for the $\Lambda > 0$ case. For a vanishing cosmological constant, a universal group of symmetries at $\mathscr{J}$ –the so called BMS group– emerges following different approaches. One possibility is to work in the physical space-time and define the symmetries as those transformations preserving some coordinate boundary condition, as in the original work of Bondi, Metzner and Sachs [21, 40, 43]–after whom the symmetry group inherits its name– or by defining 'approximate asymptotic symmetries' [103, 139]. Alternatively one can work in the conformal space-time and define the asymptotic symmetry group as those mappings that leave invariant a particular conformal-gauge fixing, sometimes called 'Bondi systems' [115], or as those transformations which leave invariant certain structure consisting on the degenerate metric and the generators of $\mathscr{J}$ [17, 48]. Moreover, there is an alternative definition of asymptotic symmetries as those which leave unchanged some gauge-invariant tensorial quantity constructed with the elementary objects on $\mathscr{J}$ –an 'asymptotic geometry', to put it in Geroch's words [17]. Indeed, this is the first approach we will consider for $\Lambda > 0$ and, as we will see, it does not lead to the type of enhanced group of symmetries –analogous to the BMS in a broad sense– that one may wish; for this reason we will explore other different paths too, eventually arriving at a proposal providing an infinite-dimensional Lie algebra.

Consider the gauge invariant object

$$\Upsilon_{abcdef} := h_{ab} D_{cd} D_{ef} \tag{7.146}$$

and define the generators of infinitesimal symmetries ξ^a as

$$\pounds_{\vec{\xi}} \Upsilon_{abcde} = 0 \ . \tag{7.147}$$

Expanding this equation we find

$$2\overline{\nabla}_{(a}\xi_{b)} D_{cd} D_{ef} = -h_{ab} \left[D_{cd} \pounds_{\vec{\xi}} D_{ef} + D_{ef} \pounds_{\vec{\xi}} D_{cd} \right] \ , \tag{7.148}$$

from where

$$\pounds_{\vec{\xi}} h_{ab} = 2\overline{\nabla}_{(a}\xi_{b)} = 2\psi h_{ab} \quad \text{with } \psi := \frac{1}{3}\overline{\nabla}_c \xi^c \ . \tag{7.149}$$

Using this back into eq. (7.148) one gets

$$\pounds_{\vec{\xi}} D_{cd} = -\psi D_{cd} \ . \tag{7.150}$$

Equation (7.149) implies that ξ^a is a CKVF of the metric h_{ab}. A result in [136] states that a Killing vector field of a $\Lambda > 0$-vacuum space-time induces a vector field on $\mathscr{J}$ that satisfies precisely eqs. (7.148) and (7.150) and, conversely, a vector field on $\mathscr{J}$ satisfying eqs. (7.148) and (7.150) gives rise, via an initial value problem, to a KVF of the physical space-time. From this point of view, the proposal of preserving (7.146) is fully justified. Importantly, this definition does not require fixing the topology of $\mathscr{J}$ nor requires the metric to be conformally flat –with the high-restrictive aftermath this implies [61]. Also, it includes D_{ab} as a fundamental ingredient, in accordance with our repeated claim that one has to bring D_{ab} into the picture. Nevertheless, it is not completely satisfactory as there may be cases in which no asymptotic symmetries exist. These are the basic asymptotic symmetries

Definition 7.4.1 (Basic infinitesimal asymptotic symmetries)**.** We define the basic infinitesimal asymptotic symmetries as those CKVF ξ^a of $(h_{ab}, \mathscr{J})$ which satisfy

$$\pounds_{\vec{\xi}} D_{cd} = -\frac{1}{3}\overline{\nabla}_m \xi^m D_{cd} \ . \tag{7.151}$$

Nevertheless, definition 7.4.1 is not completely satisfactory as there may be cases in which no such basic asymptotic symmetries exist. Alternatively, we can define other asymptotic symmetries as those which preserve the structure of definition 7.0.1 in the following sense:

Definition 7.4.2 (Equipped $\mathscr{J}$ symmetries.)**.** Consider $\mathscr{J}$ equipped according to definition 7.0.1. The extended asymptotic symmetries are those preserving the conformal class of the one-parameter family of projectors to $\mathbf{S}_2$, and the direction of the congruence $\mathcal{C}$ on $\mathscr{J}$. In other words, these symmetries are the transformations acting on the pairs $(\underline{P}_{ab}, m_a)$ as

$$(\underline{P}_{ab}, m_a) \longrightarrow \left(\Psi^2 \underline{P}_{ab}, \Phi m_a\right) \ .$$

Remark 7.4.1. The infinitesimal version ξ^a of these transformations is

$$\pounds_{\vec{\xi}} \underline{P}_{ab} = 2\psi \underline{P}_{ab} \ , \tag{7.152}$$

$$\pounds_{\vec{\xi}} m_a = \phi m_a \ , \tag{7.153}$$

where ξ^a generates a one-parameter (ϵ) family of transformations of the type definition 7.4.2, with $\phi := \partial_\epsilon \Phi(\epsilon)|_{\epsilon=0}$, $\psi := \partial_\epsilon \Psi(\epsilon)|_{\epsilon=0}$. Note that from these equations it also follows

$$\pounds_{\vec{\xi}} m^a = -\phi m^a \tag{7.154}$$

and

$$\pounds_{\vec{\xi}} h_{ab} = 2\phi m_a m_b + 2\psi \underline{P}_{ab} \ . \tag{7.155}$$

The following gauge-changes follow from eqs. (7.152) and (7.153):

$$\tilde{\psi} = \psi + \pounds_{\vec{\xi}}(\ln\omega) \ , \tag{7.156}$$
$$\tilde{\phi} = \phi + \pounds_{\vec{\xi}}(\ln\omega) \ . \tag{7.157}$$

Observe also that $\tilde{\psi} - \tilde{\phi} = \psi - \phi$ is gauge invariant.

The group of symmetries of definition 7.4.2 will be denoted by B and it constitutes a case of the so called biconformal transformations [140]. Taking into account that the Lie derivative acts linearly and using the property $\pounds_{[\vec{\xi}_1,\vec{\xi}_2]} = \pounds_{\vec{\xi}_1}\pounds_{\vec{\xi}_2} - \pounds_{\vec{\xi}_2}\pounds_{\vec{\xi}_1}$, it can be easily shown that these infinitesimal transformations form a Lie algebra which we denote by $\mathfrak{b}$ and that for $\underset{3}{\xi}{}^a = \left[\underset{1}{\xi}, \underset{2}{\xi}\right]^a$ one has

$$\underset{3}{\psi} = \pounds_{\underset{1}{\vec{\xi}}}\underset{2}{\psi} - \pounds_{\underset{2}{\vec{\xi}}}\underset{1}{\psi} \ , \tag{7.158}$$
$$\underset{3}{\phi} = \pounds_{\underset{1}{\vec{\xi}}}\underset{2}{\phi} - \pounds_{\underset{2}{\vec{\xi}}}\underset{1}{\phi} \ . \tag{7.159}$$

Consider the general decomposition

$$\xi^a = \beta m^a + \chi^a \ , \quad \chi^a m_a = 0. \tag{7.160}$$

We can obtain the necessary and sufficient conditions that β and χ^a have to satisfy so that $\xi^a \in \mathfrak{b}$ by decomposing into tangent and orthogonal parts eqs. (7.152) to (7.154),

$$\begin{cases} \pounds_{\vec{\xi}} \underline{P}_{ab} = 2\psi \underline{P}_{ab} \ , \\ \pounds_{\vec{\xi}} m_a = \phi m_a \ , \end{cases} \Longleftrightarrow \begin{cases} \pounds_{\vec{m}}\beta - \underline{a}^e\chi_e = \phi \ , & (7.161) \\ \beta\underline{a}_b + \underline{D}_b\beta + 2\underline{\omega}_{eb}\chi^e = 0 \ , & (7.162) \\ \pounds_{\vec{m}}\chi^a + \chi^e\underline{a}_e m^a = 0 \ , & (7.163) \\ 2\underline{D}_{(d}\chi_{c)} + 2\beta\underline{\kappa}_{cb} - 2\psi\underline{P}_{cd} = 0 \ . & (7.164) \end{cases}$$

In order to identify some sort of translational subgroup, it seems natural to ask for the existence of a particular class of generators $\tau^a \overset{\mathscr{J}}{:=} \alpha m^a$ completely tangent to $\vec{m}$,

$$\begin{cases} \pounds_{\vec{\tau}} \underline{P}_{ab} = 2\theta \underline{P}_{ab} \ , \\ \pounds_{\vec{\tau}} m_a = \lambda m_a \ , \end{cases} \Longleftrightarrow \begin{cases} \pounds_{\vec{m}}\alpha = \lambda \ , & (7.165) \\ \alpha\underline{a}_b + \underline{D}_b\alpha = 0 \ , & (7.166) \\ \theta\underline{P}_{cd} - \alpha\underline{\kappa}_{cd} = 0 \ . & (7.167) \end{cases}$$

Notice that eq. (7.167) requires m^a to be shear-free. However, this is not an assumption in definition 7.0.1 and in general one has $\underline{\Sigma}_{ab} \neq 0$.

Furthermore, asking for η^a to be a symmetry orthogonal to m_a ($\eta^e m_e = 0$) produces the following set of conditions:

$$\begin{cases} \pounds_{\vec{\eta}} P_{ab} = 2\varphi \underline{P}_{ab} \ , \\ \pounds_{\vec{\eta}} m_a = \mu m_a \ , \end{cases} \iff \begin{cases} \underline{a}^e \eta_e = -\mu \ , & (7.168) \\ \eta^e \underline{\omega}_{eb} = 0 \ , & (7.169) \\ \pounds_{\vec{m}} \eta^a + \eta^e \underline{a}_e m^a = 0 \ , & (7.170) \\ \underline{\mathcal{D}}_{(a}\eta_{b)} - \varphi \underline{P}_{ab} = 0 \ . & (7.171) \end{cases}$$

In this case, eq. (7.169) requires $\underline{\omega}_{ab} = 0$ —to prove this, note that $2\underline{\omega}_{ab} = \underline{\omega}_{cd}\underline{\epsilon}^{cd}\underline{\epsilon}_{ab}$. However, according to definition 7.0.1, the vector field $\vec{m}$ has non-vanishing vorticity, in general. Importantly, to account for the existence of symmetries one has to study integrability conditions too. Then, (multiple) solutions to the above equations may exist or not. The general form of such conditions are out of the scope of this work, but one can study them for each particular metric.

What we have seen is that, in general, definition 7.0.1 is too weak in order to get a notion of translations within $\mathscr{J}$ and along m^a. We will explore the particular case in which this kind of symmetries are present in section 7.4.2. Before that, we present a derivation of the transformations of definition 7.4.2 without further constraints using a different approach which also partly justifies the definition.

7.4.1 Derivation from approximate space-time symmetries

We are about to show that a particular sort of approximate space-time symmetries can lead at infinity to the equipped-$\mathscr{J}$ symmetries of definition 7.0.1. For simplicity, in this subsection we set $T_{\alpha\beta} = 0$ –this does not affect the final result.

Begin by considering a vector field $\hat{\xi}^\alpha$ on the physical space-time $\left(\hat{M}, \hat{g}_{\alpha\beta}\right)$ with a smooth extension to $\mathscr{J}$ on the unphysical space-time $\left(M, g_{\alpha\beta}\right)$, which in this subsection we consider foliated by Ω =constant-hypersurfaces near $\mathscr{J}$. On $\hat{M}$ one has

$$\pounds_{\vec{\xi}} g_{\alpha\beta} = \Omega^2 \pounds_{\vec{\xi}} \hat{g}_{\alpha\beta} + \frac{2}{\Omega} \pounds_{\vec{\xi}} (\Omega)\, \hat{g}_{\alpha\beta} \ . \tag{7.172}$$

We will require that

$$\Omega^2 \pounds_{\vec{\xi}} \hat{g}_{\alpha\beta} = H_{\alpha\beta} \tag{7.173}$$

for some symmetric tensor field $H_{\alpha\beta}$ regular at $\mathscr{J}$. Then, the idea is to ask $H_{\alpha\beta}$ to fulfil certain conditions such that $\hat{\xi}^\alpha$ 'approximates' a symmetry near $\mathscr{J}$. Some obvious examples are:

- If $\hat{\xi}^\alpha$ is a KVF for $\hat{g}_{\alpha\beta}$, then $H_{\alpha\beta} = 0$.

- If $\hat{\xi}^\alpha$ is a CKVF for $\hat{g}_{\alpha\beta}$, then $H_{\alpha\beta} \propto g_{\alpha\beta}$.

Observe that assumption (7.173) and regularity of eq. (7.172) at $\mathscr{J}$ require

$$\pounds_{\vec{\xi}}(\Omega) = \Omega J \tag{7.174}$$

for some scalar function J regular at $\mathscr{J}$. Of course, this implies that

$$\hat{\xi}^\alpha N_\alpha \overset{\mathscr{J}}{=} 0 \ . \tag{7.175}$$

Hence, $\hat{\xi}^\alpha$ has to be tangent to $\mathscr{J}$. For later convenience let us define

$$\xi^\alpha \overset{\mathscr{J}}{:=} \hat{\xi}^\alpha \overset{\mathscr{J}}{=} e^\alpha{}_a \xi^a, \tag{7.176}$$

where $\{e^a{}_\alpha\}$ is a basis on $\mathscr{J}$. Then, eq. (7.172) reads

$$\pounds_{\vec{\xi}} g_{\alpha\beta} = 2J g_{\alpha\beta} + H_{\alpha\beta} \ . \tag{7.177}$$

It is easy to obtain

$$\pounds_{\vec{\xi}} N_\alpha = N_\alpha J + \Omega \nabla_\alpha J, \tag{7.178}$$

from where at $\mathscr{J}$ (recall that $P^\alpha{}_\beta$ is the projector to $\mathscr{J}$ (3.89))

$$\pounds_{\vec{\xi}} P_{\alpha\beta} \overset{\mathscr{J}}{=} 2J P_{\alpha\beta} + H_{\alpha\beta} \ . \tag{7.179}$$

Next, we are going to see whether this equation contains components along N_α or not. Contraction of eq. (7.177) with N^α gives

$$\pounds_{\vec{\xi}} N^\alpha = -J N^\alpha - H^\alpha + \Omega \nabla^\alpha J \tag{7.180}$$

with

$$H^\alpha := N^\mu H_\mu{}^\alpha \ , \tag{7.181}$$

from where,

$$N^\alpha \pounds_{\vec{\xi}} P_{\alpha\beta} = H_\beta + \Omega \left[-\nabla_\beta J + n_\beta n^\alpha \nabla_\alpha J - \frac{2}{N} n_\beta \left(\pounds_{\vec{\xi}} f + f J \right) \right], \tag{7.182}$$

where f is the scalar (3.29). Then, contraction of eq. (7.179) with N^α gives

$$N^\alpha \pounds_{\vec{\xi}} P_{\alpha\beta} \overset{\mathscr{J}}{=} H_\beta \tag{7.183}$$

and contraction of eq. (7.182) with N^β,

$$N^\alpha N^\beta H_{\alpha\beta} = N^\beta H_\beta \overset{\mathscr{J}}{=} 0. \tag{7.184}$$

Hence, $H^\alpha \overset{\mathscr{J}}{=} e^\alpha{}_a H^a$ and by eq. (7.180)

$$H^\alpha \overset{\mathscr{J}}{=} -\left(\pounds_{\vec{\xi}} N^\alpha + J N^\alpha\right) . \tag{7.185}$$

To see if there is any consistency condition for $H_{\alpha\beta}$ compute the following:

$$\pounds_{\vec{\xi}}\pounds_{\vec{N}} g_{\alpha\beta} = 2\pounds_{\vec{\xi}} f g_{\alpha\beta} + 2f\left(2Jg_{\alpha\beta} + H_{\alpha\beta}\right) - \Omega J S_{\alpha\beta} - \Omega\pounds_{\vec{\xi}} S_{\alpha\beta} \ , \tag{7.186}$$

$$\pounds_{\vec{N}}\pounds_{\vec{\xi}} g_{\alpha\beta} = 2\pounds_{\vec{N}} J g_{\alpha\beta} + 2J\left(2f g_{\alpha\beta} - \Omega S_{\alpha\beta}\right) + \pounds_{\vec{N}} H_{\alpha\beta} \ , \tag{7.187}$$

$$\pounds_{\left[\vec{\xi},\vec{N}\right]} g_{\alpha\beta} = \pounds_{\left(-J\vec{N}-\vec{H}-\Omega\vec{\nabla}J\right)} g_{\alpha\beta} = -J\left(2f g_{\alpha\beta} - \Omega S_{\alpha\beta}\right) - \nabla_\alpha H_\beta - \nabla_\beta H_\alpha + 2\Omega\nabla_\alpha\nabla_\beta J \tag{7.188}$$

and then, use the identity $\pounds_{\left[\vec{\xi},\vec{N}\right]} = \pounds_{\vec{\xi}}\pounds_{\vec{N}} - \pounds_{\vec{N}}\pounds_{\vec{\xi}}$ to get

$$0 = 2\left(\pounds_{\vec{\xi}} f - \pounds_{\vec{N}} J + fJ\right) g_{\alpha\beta} + 2fH_{\alpha\beta} - \pounds_{\vec{N}} H_{\alpha\beta} + \nabla_\alpha H_\beta + \nabla_\beta H_\alpha - \Omega\left(2\nabla_\alpha\nabla_\beta J + \pounds_{\vec{\xi}} S_{\alpha\beta}\right) . \tag{7.189}$$

After some computation, it can be checked that the right-hand side of eq. (7.189) does not have components along N^α, therefore this equation contains no information orthogonal to the Ω =constant hypersurfaces. Expanding the Lie derivative of $H_{\alpha\beta}$, eq. (7.189) turns into an expression for the derivative of this tensor along N^α, $\dot{H}_{\alpha\beta} := N^\mu \nabla_\mu H_{\alpha\beta}$,

$$\dot{H}_{\alpha\beta} = \nabla_\alpha H_\beta + \nabla_\beta H_\alpha + \left(\pounds_{\vec{\xi}} f - \pounds_{\vec{N}} J + fJ\right) g_{\alpha\beta} + \Omega\left(H_{\mu(\alpha} S_{\beta)}{}^\mu - 2\nabla_\alpha\nabla_\beta J - \pounds_{\vec{\xi}} S_{\alpha\beta}\right) . \tag{7.190}$$

If one projects this equation to $\mathscr{J}$ with $\{e^\alpha{}_a\}$ and uses eq. (7.184), it reads

$$N\dot{H}_{ab} \overset{\mathscr{J}}{=} 2\overline{\nabla}_{(a} H_{b)} - \pounds_{\vec{N}} \overline{J} h_{ab} \ , \tag{7.191}$$

where $\overline{J} \overset{\mathscr{J}}{=} J$ and we have used $f \overset{\mathscr{J}}{=} 0$ (see eq. (3.30)) and

$$\xi^\alpha \nabla_\alpha f \overset{\mathscr{J}}{=} 0 . \tag{7.192}$$

Next, take the pullback of eq. (7.179) to $\mathscr{J}$,

$$\pounds_{\vec{\xi}} h_{ab} \overset{\mathscr{J}}{=} 2\overline{J} h_{ab} + H_{ab} \ , \tag{7.193}$$

where it is evident that only the tangent part of $H_{\alpha\beta}$ intervenes. Equation (7.193) is important, as the meaning of H_{ab} on $\mathscr{J}$ it is clear here: how we choose H_{ab} defines how we define ξ^a as an asymptotic-symmetry. Our goal is to choose H_{ab} such that one can say that eq. (7.193) comes from an approximate space-time symmetry –as much as possible.

Before entering into this task, let us remark that ξ^a is in one-to-one correspondence with the equivalence class

$$\left\{\grave{\hat{\xi}}^\alpha \in \left[\hat{\xi}^\alpha\right] \quad \Longleftrightarrow \quad \grave{\hat{\xi}}^\alpha - \hat{\xi}^\alpha = \Omega v^\alpha\right\}, \tag{7.194}$$

where v^α is any vector field on M. However, if we want any element of the equivalence class to generate an asymptotic symmetry of the kind (7.193), Ωv^α itself has to satisfy all the equations so far. Calling ${}_0H_{\alpha\beta}$ and ${}_0J$ the $H_{\alpha\beta}$ and J associated to Ωv^α, respectively, from eqs. (7.174) and (7.177) we have

$$\pounds_{\Omega\vec{v}}\Omega = \Omega v^\mu N_\mu = \Omega\, {}_0J \ , \tag{7.195}$$

$$\pounds_{\Omega\vec{v}}g_{\alpha\beta} = 2\Omega\nabla_{(\alpha}v_{\beta)} + 2v_{(\alpha}N_{\beta)} = 2\, {}_0J\, g_{\alpha\beta} + {}_0H_{\alpha\beta} \ . \tag{7.196}$$

Then, putting together these two equations we get a formula for ${}_0H_{\alpha\beta}$:

$${}_0H_{\alpha\beta} = 2\Omega\nabla_{(\alpha}v_{\beta)} + 2v_{(\alpha}N_{\beta)} - 2v^\mu N_\mu g_{\alpha\beta} \ . \tag{7.197}$$

It can also be shown that ${}_0H_\alpha := N^\mu\, {}_0H_{\alpha\mu}$ is tangent at $\mathscr{J}$ and satisfies

$$\begin{aligned} {}_0H_\alpha &\overset{\mathscr{J}}{=} -v^\mu N_\mu - N^2 v_\alpha \overset{\mathscr{J}}{=} -P^\mu{}_\alpha v_\mu \ , \\ &= -\left(\pounds_{\Omega\vec{v}}N_\alpha + {}_0J N_\alpha\right) \end{aligned} \tag{7.198}$$

—compare with eq. (7.185). The combination ${}_0H_{\alpha\beta}$ for arbitrary v^α has no relevance for ξ^a, then, it can be considered as a gauge part in $H_{\alpha\beta}$. Hence, for any $\grave{\hat{\xi}}^\alpha \in \left[\hat{\xi}^\alpha\right]$ one uses

$$\grave{H}_{\alpha\beta} = H_{\alpha\beta} + 2N_{(\alpha}\overline{v}_{\beta)} - 2v^\mu N_\mu P_{\alpha\beta} + 2\Omega\nabla_{(\alpha}v_{\beta)} \ , \tag{7.199}$$

where we have defined $\overline{v}_\alpha := P^\mu{}_\alpha v_\mu$. Note that any term of type $N_\alpha v_\beta + N_\beta v_\alpha$ is pure gauge in $H_{\alpha\beta}$, and the term in $P_{\alpha\beta}$ is the one that makes the definition of ξ^a unambiguous. This is clearly seen projecting to $\mathscr{J}$,

$$\grave{H}_{ab} = H_{ab} - 2\left(v^\mu N_\mu\right)\Big|_{\mathscr{J}} h_{ab} \ . \tag{7.200}$$

Therefore, within $\mathscr{J}$ one gets

$$\pounds_{\grave{\vec{\xi}}}h_{ab} = 2\grave{\overline{J}}h_{ab} + \grave{H}_{ab} = 2\overline{J}h_{ab} + H_{ab} = \pounds_{\vec{\xi}}h_{ab} \ , \tag{7.201}$$

where we have used eqs. (7.193), (7.195) and (7.200) together with $\grave{\overline{J}} = \overline{J} + {}_0\overline{J}$. By typical calculations, it can be proven that the set of such vector fields $\hat{\xi}^\alpha$ on $\left(\hat{M}, \hat{g}_{\alpha\beta}\right)$ form a Lie algebra, as well as their equivalence classes. One should not forget the conformal gauge freedom (3.6). Under such rescalings,

$$\tilde{J} = J + \frac{1}{\omega}\pounds_{\bar{\xi}}\omega \ , \tag{7.202}$$

$$\tilde{H}_{\alpha\beta} = \omega^2 H_{\alpha\beta} \ , \tag{7.203}$$

$$\tilde{H}_{\alpha} = \omega H_{\alpha} + \Omega\omega^{\beta} H_{\alpha\beta} , \tag{7.204}$$

which follow form eqs. (7.174) and (7.177) and the gauge transformations of appendix C.

Now we have to make a choice for $H_{\alpha\beta}$. Should we follow Geroch and Winicour [139], we would have to set $H_{\alpha\beta} \overset{\mathscr{J}}{=} 0$. This fixing makes ξ^a a CKVF of h_{ab}, hence one probably, at the best, recovers the basic symmetries preserving (7.146) of definition 7.4.1. However, as we have argued, these kind of symmetries are not satisfactory. Thus, one is left with the problem of specifying a different kind of $H_{\alpha\beta}$. It makes sense to think that $H_{\alpha\beta}$ should be a rank-1 matrix –at least on $\mathscr{J}$– up to redundancy-correction terms, that is

$$A m_{\alpha} m_{\beta} + \Omega x_{\alpha\beta} \tag{7.205}$$

for some scalar function A and tensor field $x_{\alpha\beta}$. Moreover, the one-form m_{α} has to be tangent to $\mathscr{J}$, i.e. $m_{\alpha} N^{\alpha} \overset{\mathscr{J}}{=} 0$, so that it fulfils eq. (7.184). Still, in order to use (7.205) as $H_{\alpha\beta}$, one has to add the redundancy-correction terms (7.197); the resulting expression reads

$$H_{\alpha\beta} = A m_{\alpha} m_{\beta} + 2N_{(\alpha}\overline{v}_{\beta)} + C P_{\alpha\beta} + \Omega x_{\alpha\beta}, \tag{7.206}$$

where we have set $C := -2v_{\mu}N^{\mu}$. The parameters A and C are general and should not be fixed beforehand, as doing so would restrict the available ξ^a. The pullback to $\mathscr{J}$ is

$$H_{ab} = A m_a m_b + C h_{ab} \ , \tag{7.207}$$

by means of which we can write eq. (7.193) as

$$\pounds_{\vec{\xi}} h_{ab} = \left(2\overline{J} + C\right) h_{ab} + A m_a m_b \ . \tag{7.208}$$

Observe that eqs. (7.204), (C.23) and (C.46) impose

$$\tilde{C} \overset{\mathscr{J}}{=} C, \quad \tilde{A} \overset{\mathscr{J}}{=} A \ . \tag{7.209}$$

Let us define $\underline{P}_{ab} := h_{ab} - m_a m_b$, as in eq. (A.46), to write the last formula as

$$\pounds_{\vec{\xi}} h_{ab} = \left(2\overline{J} + C\right) \underline{P}_{ab} + \left(A + 2\overline{J} + C\right) m_a m_b \ . \tag{7.210}$$

From this expression it becomes manifest that the resulting ξ^a are *biconformal vector fields* *on* $\mathscr{J}$. The Lie-algebra structure of these infinitesimal transformations and eq. (7.210) require

$$\pounds_{\vec{\xi}} \underline{P}_{ab} = 2\psi \underline{P}_{ab} \quad \text{with} \quad \psi := 2\overline{J} + C, \tag{7.211}$$

$$\pounds_{\vec{\xi}} m_a = \phi m_a \quad \text{with} \quad \phi := A + 2\overline{J} + C, \tag{7.212}$$

which are actually eqs. (7.152) and (7.153) for m^a the vector field of definition 7.0.1. Observe that from eqs. (7.202) and (7.209) one can deduce the gauge transformation of ψ and ϕ:

$$\tilde{\psi} = \psi + \frac{2}{\omega} \pounds_{\vec{\xi}} \omega, \quad \tilde{\phi} = \phi + \frac{2}{\omega} \pounds_{\vec{\xi}} \omega \ . \tag{7.213}$$

As a matter of fact, the space-time KVF ($H_{\alpha\beta} = 0$) and CKVF ($H_{\alpha\beta} \propto \hat{g}_{\alpha\beta}$) only generate part of the asymptotic symmetries of definition 7.4.2 –if they also satisfy eq. (7.153).

7.4.2 Strongly equipped $\mathscr{J}$

We consider now the asymptotic symmetries of definition 7.4.2 for strongly equipped $\mathscr{J}$ of definition 7.0.3. Let us keep the notation that was used for denoting general (ξ^a), m^a-orthogonal (η^a) and m^a-tangent (τ^a) symmetries, respectively.

Then, for $\xi^a := \beta m^a + \chi^a$:

$$\begin{cases} \pounds_{\vec{\xi}} \underline{P}_{ab} = 2\psi \underline{P}_{ab} \ , \\ \pounds_{\vec{\xi}} m_a = \phi m_a \ , \end{cases} \iff \begin{cases} \pounds_{\vec{m}} \beta - \underline{a}^e \chi_e = \phi \ , & (7.214) \\ \beta \underline{a}_b + \underline{\mathcal{D}}_b \beta = 0 \ , & (7.215) \\ \pounds_{\vec{m}} \chi^a + \chi^e \underline{a}_e m^a = 0 \ , & (7.216) \\ 2\underline{\mathcal{D}}_{(d} \chi_{c)} + (\underline{\kappa}\beta - 2\psi) \underline{P}_{cd} = 0 \ . & (7.217) \end{cases}$$

For $\tau^a := \alpha m^a$:

$$\begin{cases} \pounds_{\vec{\tau}} \underline{P}_{ab} = 2\theta \underline{P}_{ab} \ , \\ \pounds_{\vec{\tau}} m_a = \lambda m_a \ , \end{cases} \iff \begin{cases} \pounds_{\vec{m}} \alpha = \lambda \ , & (7.218) \\ \alpha \underline{a}_b + \underline{\mathcal{D}}_b \alpha = 0 \ , & (7.219) \\ \theta - \dfrac{1}{2} \alpha \underline{\kappa} = 0 \ . & (7.220) \end{cases}$$

Finally, for η^a ($\eta^e m_e = 0$):

$$\begin{cases} \pounds_{\vec{\eta}} \underline{P}_{ab} = 2\varphi \underline{P}_{ab} \ , \\ \pounds_{\vec{\eta}} m_a = \mu m_a \ , \end{cases} \iff \begin{cases} \underline{a}^e \eta_e = -\mu \ , & (7.221) \\ \pounds_{\vec{m}} \eta^a + \eta^e \underline{a}_e m^a = 0 \ , & (7.222) \\ \underline{\mathcal{D}}_{(a} \eta_{b)} - \varphi \underline{P}_{ab} = 0 \ . & (7.223) \end{cases}$$

It can be shown that all the vector fields τ^a satisfying eqs. (7.218) to (7.220) form a subalgebra $\mathfrak{t}$ which we call 'bitranslations'. Moreover, for any $\xi^a \in \mathfrak{b}$ and any $\tau^a \in \mathfrak{t}$

$$\pounds_{\vec{\tau}}\xi^a = \left(\alpha\phi - \beta\lambda + \alpha\chi^e \underline{a}_e\right) m^a \ . \tag{7.224}$$

Thus, the subalgebra $\mathfrak{t}$ is a Lie ideal of $\mathfrak{b}$. For any two $\vec{\tau}_1, \vec{\tau}_2 \in \mathfrak{t}$,

$$\left[\vec{\tau}_1, \vec{\tau}_2\right]^a = \left(\alpha_1 \pounds_{\vec{m}} \alpha_2 - \alpha_2 \pounds_{\vec{m}} \alpha_1\right) m^a \ , \tag{7.225}$$

therefore, $\mathfrak{t}$ is non-Abelian. Note that $\mathfrak{t}$ has a subalgebra, $\mathfrak{ct}$, of 'conformal translations' defined by those elements of $\mathfrak{t}$ for which $\theta = \lambda$, and that this is Abelian[2]. Furthermore, given one element of $\mathfrak{t}$, multiplying it by a function ν such that $\underline{\mathcal{D}}_a \nu = 0$ produces a new element of $\mathfrak{t}$; the subalgebra $\mathfrak{t}$ is *infinite dimensional* and by eqs. (7.219) and (A.105) the general form of an element $\tau^a \in \mathfrak{t}$ is

$$\tau^a = \nu\left(v\right) F m^a, \quad \text{with} \quad \frac{1}{F} = \pounds_{\vec{m}} v \ , \tag{7.226}$$

where ν is an arbitrary function of v and one has (using obvious notation)

$$\left[\vec{\tau}_1, \vec{\tau}_2\right]^a = \left(\nu_1 \pounds_{\vec{m}} \nu_2 - \nu_2 \pounds_{\vec{m}} \nu_1\right) F m^a \ . \tag{7.227}$$

In the same way, it is easily proven that the vector fields η^a form a subalgebra $\mathfrak{cs}$ and are CKVF of the metric on each cut $\mathcal{S}_v$.

Importantly, wee see that any $\xi^a \in \mathfrak{b}$ is a composition of a $\tau^a \in \mathfrak{t}$, with $\lambda = \phi + \underline{a}^e \xi_e$ and $2\theta = \underline{\mathcal{D}}_d \xi^d + \underline{\kappa}\beta - 2\psi$ (from eqs. (7.214) to (7.216) and (7.218) to (7.220)), and a $\eta^a \in \mathfrak{cs}$, with $\mu = \phi - \pounds_{\vec{m}}\left(\beta\right)$ and $2\varphi = 2\psi - \underline{\kappa}\beta$ (from eqs. (7.214) to (7.216) and (7.221) to (7.223)). Let us denote the groups associated to these algebras by $\mathsf{B}, \mathsf{T}, \mathsf{CS}$, respectively. Then, we have that T is a normal subgroup of B and that it makes sense to define the quotient group B/T whose Lie algebra we denote by $\mathfrak{b}/\mathfrak{t}$. But the elements of $\mathfrak{b}/\mathfrak{t}$ are precisely the elements of $\mathfrak{cs}$: any symmetry η^a modulo a bitranslation is in $\mathfrak{cs}$. Furthermore, since these are the conformal transformations of $\mathcal{S}_v$, if this has $\mathbb{S}^2$-topology, CS is isomorphic to the Lorentz Group $\mathsf{SO}(1,3)$. Another easily verifiable property is that any $\tau^a \in \mathfrak{t}$ commutes with any $\eta^a \in \mathfrak{cs}$,

$$\left[\vec{\tau}, \vec{\eta}\right]^a = 0 \ , \tag{7.228}$$

as one simply has to set $\beta = 0$, $\chi^a = \eta^a$ and $\phi = \mu = -\eta^e a_e$ in eq. (7.224). Note that solutions α to eq. (7.219) always exist because m^a defines a foliation – see eq. (A.105).

[2] And an ideal of the Lie subalgebra of $\mathfrak{b}$ consisting on CKVF ($\phi = \psi$).

Then, given α, one can take eqs. (7.218) and (7.220) as definitions for λ and θ. Also, if we assume $\mathbb{S}^2$-topology for the cuts, there always exist (up to 6) conformal Killing vector fields satisfying eq. (7.223). Equation (7.221) can be taken as the definition for μ and eq. (7.222) is equivalent to

$$m^c \pounds_{\vec{\eta}} \underline{P}_c{}^a = 0, \tag{7.229}$$

which using $\underline{\omega}_{ab} = 0$ can be expressed as

$$\eta^e \underline{\kappa}_e{}^a - \underline{P}^a{}_d m^e \overline{\nabla}_e \eta^d = 0 \tag{7.230}$$

and does not hold in general. Then, given definition 7.0.3, solutions $\xi \in \mathfrak{b}$ to eqs. (7.214) to (7.217) not necessarily exist[3]. In summary:

> *The asymptotic group of symmetries* B *that preserve the strong structure of definition 7.0.3 is the (semidirect) product of the (normal) subgroup of bitranslations* T *and the subgroup of conformal transformations in two dimensions* CS
>
> $$\mathsf{B} = \mathsf{T} \ltimes \mathsf{CS}\ . \tag{7.231}$$
>
> *The subalgebra of bitranslations* $\mathfrak{t}$ *is a non-Abelian Lie ideal and its elements commute with the ones of the algebra* $\mathfrak{cs}$ *of the group of conformal transformations on* $\mathcal{S}_v$*,* CS*.*

Let us conclude this section by briefly commenting on the units of α. If eventually one wished to take the limit of the symmetries to $\Lambda = 0$, assuming the limit exists and $\alpha\big|_{\Lambda=0}$ is regular, one has to rescale any infinitesimal symmetry as –see eq. (5.79)–

$$N\xi^a = \alpha M^a + N\chi^a, \tag{7.232}$$

where $N\xi^a$ should be dimensionless to fit with the asymptotic symmetries of the $\Lambda = 0$ case. Therefore, one has to assign α the dimensions of length, $[\alpha] = L$. Another way of seeing this is that by a conformal rescaling

$$\tilde{m}^a = \frac{1}{\omega} m^a\ , \tag{7.233}$$

and as any infinitesimal symmetry ξ^a must be conformally invariant

$$\tilde{\alpha} = \omega\alpha. \tag{7.234}$$

Because ω is dimensionless and lengths rescale with ω, we arrive at the same conclusion, i.e., $[\alpha] = L$.

[3] One has not only to study the solutions η^a to eq. (7.230) but also the integrability conditions.

7.4.3 Relation between the tensor ρ and asymptotic translations

It is possible to relate the asymptotic symmetries to $\underline{\rho}_{ab}$ and the vanishing of $\underline{V}_{ab}$. For a general foliation, using eq. (7.76), one has that $\underline{V}_{ab} = 0$ if and only if

$$\underline{\rho}_{AB} = f_{AB} + \underline{\mathcal{D}}_A \underline{a}_B - \underline{a}_A \underline{a}_B - \frac{1}{2}\underline{q}_{AB}\left(\underline{\mathcal{D}}_E \underline{a}^E - \underline{a}_E \underline{a}^E - \underline{K}\right) . \tag{7.235}$$

Hence, using the function F of eq. (A.105)

Lemma 7.4.1. Assume $\mathscr{J}$ is strictly equipped. Then

$$\underline{V}_{AB} = 0 \Longleftrightarrow \underline{\rho}_{AB} = f_{AB} - \frac{1}{F}\underline{\mathcal{D}}_A \underline{\mathcal{D}}_B F + \frac{1}{2}\underline{q}_{AB}\left[\frac{1}{F}\underline{\mathcal{D}}_C \underline{\mathcal{D}}^C F + \underline{K}\right] . \tag{7.236}$$

As an immediate consequence,

Corollary 7.4.1. If $\mathscr{J}$ is strongly equipped and $\tau^a = \alpha m^a$ is a bitranslation, then

$$\underline{V}_{AB} = 0 \Longleftrightarrow \underline{\rho}_{AB} = -\frac{1}{\alpha}\underline{\mathcal{D}}_A \underline{\mathcal{D}}_B \alpha + \frac{1}{2}\underline{q}_{AB}\left[\frac{1}{\alpha}\underline{\mathcal{D}}_C \underline{\mathcal{D}}^C \alpha + \underline{K}\right] . \tag{7.237}$$

The last result follows by noting that if $\tau^a \in \mathfrak{t}$ then α satisfies eq. (7.219) and that for umbilical cuts $f_{AB} = 0$ (see eq. (7.15)). Indeed, the equation

$$0 = \frac{1}{\alpha}\underline{\mathcal{D}}_A \underline{\mathcal{D}}_B \alpha + \underline{\rho}_{AB} - \frac{1}{2}\underline{q}_{AB}\left[\frac{1}{\alpha}\underline{\mathcal{D}}_C \underline{\mathcal{D}}^C \alpha + \underline{K}\right] . \tag{7.238}$$

provides us with a neat interpretation for α,

Remark 7.4.2. If the leaves $\mathcal{S}_v$ have topology $\mathbb{S}^2$ then the solutions α correspond to the $l = 0, 1$ *spherical harmonics*; in fact, they are exactly a linear combination of the $l = 0, 1$ *spherical harmonics* in the round gauge with $2\rho_{AB} = Kq_{AB}$, and one obtains for eq. (7.238)

$$\mathcal{D}_A \mathcal{D}_B \mathring{\alpha} - \frac{1}{2} q_{AB} \mathcal{D}_C \mathcal{D}^C \mathring{\alpha} \overset{\mathcal{S}_v}{=} 0 . \tag{7.239}$$

In other words, if $\mathscr{J}$ is strongly equipped and $\tau^a = \alpha m^a$ is a bitranslation, the function F appearing in eq. (7.226) is, on every leaf, a solution of (7.238) if and only if $V_{AB} \overset{\mathcal{S}_v}{=} 0$ there.

In view of this remark, we are induced to distinguish a class of *asymptotic translations*,

Definition 7.4.3 (Asymptotic translations)**.** Let $\mathscr{J}$ be strongly equipped. We say that a bitranslation $\tau^a = \alpha m^a \in \mathfrak{t}$ is an asymptotic translation if and only α satisfies eq. (7.238). In particular, if the leaves have topology $\mathbb{S}^2$, in a round gauge the restriction $\mathring{\alpha}$ of α is a linear combination of the $l = 0, 1$ spherical harmonics.

Observe that this results provide us with a notion of translations intrinsic to $\mathscr{J}$ which, as far as we know, have not been characterised before for $\Lambda > 0$[4]. Although we have required a strongly equipped $\mathscr{J}$ –that is, the existence of a foliation by umbilical cuts– important examples have this structure, as the Kottler, Kerr-de Sitter and Robinson-Trautman metrics or the C-metric. Definition 7.4.3 is supported by the fact that the restriction of the four- dimensional group of translational KVF in de Sitter space-time are asymptotic translations. All the KVF are tangent to $\mathscr{J}$, giving rise to the 10 CKVF of $\mathbb{S}^3$, with a four-dimensional subgroup corresponding to translation in the 4-dimensional cartesian embedding of $\mathbb{S}^3$. The latter are of the form

$$\tau^a = h^{fa}\overline{\nabla}_f F \ , \tag{7.240}$$

with F satisfying

$$\overline{\nabla}_a\overline{\nabla}_b F = -\frac{F}{a^2}h_{ab} \ , \tag{7.241}$$

where a is the constant 'radius' of the round 3-sphere. It is evident that τ_a are surface-orthogonal and that their shear vanishes. Contraction of eq. (7.241) with $\underline{E}^a{}_A m^b$ yields

$$\underline{\mathcal{D}}_A \alpha = 0 \ , \tag{7.242}$$

where $\alpha := \sqrt{\xi^a \xi_a}$ and $m^a := \tau^a/\alpha$. Hence, the restriction $\mathring{\alpha}$ of α to each cut is a constant –α is not a first integral of m^a though. Also, for the cuts associated to each translation, $\underline{V}_{AB} = 0$ –this follows from eq. (7.71) and $C_{ab} = 0$, noting that the cuts are umbilical, thus implying $\Sigma_{AB} = 0$ on each of them.

Proposition 7.4.1. Let $\hat{\xi}^\alpha$ be a CKVF of $\left(M, g_{\alpha\beta}\right)$ with non-vanishing restriction to $\mathscr{J}$ and define

$$\xi^a \overset{\mathscr{J}}{:=} \omega_\alpha{}^a \hat{\xi}^\alpha \tag{7.243}$$

and

$$m^a := \frac{1}{\alpha}\xi^a \quad \text{with} \quad \alpha := \sqrt{\xi_a \xi^a}. \tag{7.244}$$

Assume that ξ^a is orthogonal to cuts with $\mathbb{S}^2$-topology. Then

1. ξ^a is a CKVF of $(\mathscr{J}, h_{ab})$ and a BCKVF of $(m_a, \underline{P}_{ab})$ that belongs to $\mathfrak{t}$.

2. m^a is shear-less ($\underline{\Sigma}_{ab} = 0$).

3. The restriction to the the leaves $\mathring{\alpha}$ of the function α is a solution of (7.238) (and thus proportional to a combination of the first four spherical harmonics in a round gauge) if and only if $\underline{V}_{ab} = 0$ and if and only if $\underline{C}_A := \underline{E}^a{}_A m^b C_{ab} = 0$ (equivalently $\overset{+}{\underline{C}}_A = -\overset{-}{\underline{C}}_A$).

[4]For $\Lambda = 0$, see [17].

Proof. Point 1 is trivial. Point 2 follows by noting that bitranslations satisfy eq. (7.167). Hence, $\underline{\Sigma}_{ab} = 0$, which together with the assumption that m_a is surface-orthogonal make $\mathscr{J}$ strongly equipped. Then, by corollary 7.4.1 and its remark it follows that the restriction $\mathring{\alpha}$ of α to the leaves is a solution of (7.238) if and only if $V_{ab} = 0$. Now, the fact that $T_{AB}{}^{C} = 0$ (which follows from $\underline{\Sigma}_{ab} = 0$) eqs. (7.47) and (7.48), together with the $\mathbb{S}^2$-topology of the cuts gives

$$\underline{V}_{ab} = 0 \Longleftrightarrow \underline{C}_{A} = 0. \tag{7.245}$$

□

7.5 Conserved charges and balance laws

We treat two type of charges and conserved currents associated with symmetries. The first class is defined using symmetric tensor fields and symmetries intrinsic to $\mathscr{J}$; the second, employs the rescaled Bel-Robinson tensor $\mathcal{D}_{\alpha\beta\gamma}{}^{\delta}$ together with conformal symmetries of $\left(M, g_{\alpha\beta}\right)$ and/or asymptotic symmetries. We comment on why the first or second class currents presented below cannot give the right answer for a gravitational energy on $\mathscr{J}$. The use of this charges can be fruitful in other investigations though.

7.5.1 First class charges

Let t_{ab} be any rank-two, symmetric tensor field on $\mathscr{J}$ and η^a a CKVF of $(\mathscr{J}, h_{ab})$. Define the current

$$j^a := t^{ab}\eta_b. \tag{7.246}$$

The divergence of this current reads

$$\overline{\nabla}_a j^a = \eta_b \overline{\nabla}_a t^{ab} + \lambda h_{ab} t^{ab} \ , \tag{7.247}$$

where $3\lambda := \overline{\nabla}_a \eta^a$. If instead one uses a biconformal infinitesimal symmetry of definition 7.4.2 and defines the current

$$y^a := t^{ab}\xi_b \ , \tag{7.248}$$

its divergence gives

$$\overline{\nabla}_a y^a = \xi_b \overline{\nabla}_a t^{ab} + \psi \underline{P}_{ab} t^{ab} + \phi m_a m_b t^{ab}. \tag{7.249}$$

Particular cases of conserved j^a-currents include those constructed with any TT-tensor (symmetric traceless divergence-free tensor), and for them the charge

$$\mathcal{J} := \int_{\mathcal{S}} j^a r_a \mathring{\epsilon} \tag{7.250}$$

is conserved, where $\mathcal{S}$ is *any* cut with normal r_a and volume form ϵ. This follows from Stokes theorem, assuming a region Δ bounded by two such cuts . An example of conserved

y^a-currents is obtained using TT-tensors satisfying $m_a m_b t^{ab} = 0$. For any such current,

$$\mathcal{Y} := \int_{\mathcal{S}} y^a r_a \mathring{\epsilon} \tag{7.251}$$

is a conserved charge. Observe that in this particular case $\mathcal{Y}$ is *trivial* for cuts orthogonal to m_a -if they exist-, that is, when $m_a = r_a$ because of the requirement $y_d r^d = 0$. Another case, is a y^a-current constructed with a TT-tensor and a BCKVF with $\phi = 0$. Despite being obvious, it is necessary to remark that charges defined with y^a or j^a may be conserved even when the current itself is not divergence-free – it is enough that the integral over the region Δ of the divergence of the current vanishes. In that sense, one may also obtain conserved charges only for a particular family of regions Δ, as it is the trivial case of Δ bounded with cuts such that the normal r^a is orthogonal to the current.

It is tempting to define charges using D_{ab} or C_{ab} –or a linear combination thereof, or ${}^{\pm}D_{ab}$, etc– for the tensor t_{ab}, as it has been already proposed in the literature for D_{ab} [61]. The balance law associated to these charges that results from the application of Stokes theorem is not affected by the presence of gravitational radiation, and to illustrate this with our formalism consider the specific case of a strongly equipped $\mathscr{J}$ (definition 7.0.3). Let ξ^a be a member of the algebra of biconformal transformations $\mathfrak{b}$. We know that, in general, it will be composed by a member τ^a of the bitranslations $\mathfrak{t}$ and an element χ^a of the CKVF $\mathfrak{cs}$ of the projector $\underline{P}_{ab}$ –see the end part of section 7.4. For $t_{ab} = D_{ab}$ in eq. (7.248),

$$\mathcal{Y} = \int_{\mathcal{S}} y^a m_a \underline{\epsilon} = \int_{\mathcal{S}} \left\{ \alpha \underline{D} + \chi_a \underline{D}^a \right\} \underline{\epsilon} \; . \tag{7.252}$$

Let Δ be a region bounded by two cuts $\mathcal{S}_{1,2}$ of the foliation given by m^a, then

$$\mathcal{Y}\Big|_{\mathcal{S}_2} - \mathcal{Y}\Big|_{\mathcal{S}_1} = \int_{\Delta} \overline{\nabla}_a y^a \epsilon = \int_{\Delta} \left[\left(\xi_a \overline{\nabla}_d D^{da} \right) + (\phi - \psi) \, \underline{D} \right] \epsilon \; , \tag{7.253}$$

The divergence $\overline{\nabla}_d D^{da}$ is sourced by the matter fields (see eq. (3.119)), whereas the second term only contains Coulomb contributions and vanishes identically for conformal symmetries of $(\mathscr{J}, h_{ab})$, in particular for the asymptotic basic symmetries of definition 7.4.1. Unfortunately, there is no contribution by gravitational radiation even when gravitational waves *can be arriving at* $\mathscr{J}$ according to criterion 1. The same formula holds interchanging D_{ab} by C_{ab}, only that now the first term in the integrand vanishes identically due to eq. (3.120). And similar results can be found for linear combinations of D_{ab} and C_{ab}, and for ${}^{+}D_{ab}$. This is surprising, as the charge (7.252), or the analogous ones using linear combinations of D_{ab} and C_{ab} or ${}^{\pm}D_{ab}$ etcetera, include terms of type $\chi_a \underline{D}^a$ which are associated to the radiative sector of the gravitational field. This opens the door for modifications of these currents associated to D_{ab} and C_{ab} by adding extra terms that may lead to a more satisfactory balance law. This is work in progress. Next, assume that criterion 2 holds, and thus $-m^a$ points in the spatial projection of the propagation direction of radiation, as discussed in sections 5.4 and 7.3. Then, charges defined on the 'natural' cuts orthogonal to m^a might be sensible to radiative contributions. But now $\underline{D}^a$ is the divergence of the symmetric traceless tensor field $\underline{V}_{ab}$, the first component of news,

$$N\underline{D}_b = -\underline{\mathcal{D}}_c \underline{V}_b{}^c \ , \tag{7.254}$$

which on any cut of the foliation is written by means of the intrinsic connection as

$$N\underline{D}_B = -\mathcal{D}_C \underline{V}_B{}^C \ . \tag{7.255}$$

Then, for topological spheres, and in general for compact cuts, the term $\chi_a \underline{D}^a$ integrates out on using (7.217) and the charge reads

$$\mathcal{Y} = \int_{\mathcal{S}} y^a m_a \underline{\epsilon} = \int_{\mathcal{S}} \alpha \underline{D} \underline{\epsilon} \ , \tag{7.256}$$

which only contains the Coulomb contribution $\underline{D}$. A very similar cancellation occurs if one uses a CKVF of $\mathscr{J}$ because the tangent part to an umbilical cut of the conformal symmetry is a CKVF of the metric on that cut too. Hence, neither these charges nor their difference, given by the general eq. (7.253), contain explicit radiative terms. Of course, the discussion of sections 5.4 and 7.3 on the interpretation of the Coulomb and radiative terms as such depends on the choice of m_a. Still, the fact that a general firs-class current y^a is identically conserved in the absence of matter fields and for any conformal transformation shows that the associated charges $\mathcal{Y}$ for any choice of cut are insensible to gravitational radiation. Indeed, for $\mathscr{J} = \mathbb{S}^3$ or $\mathscr{J} = \mathbb{R}^3$ the radius of the topological 2-spheres can be shrunk to 0, hence making these charges to vanish identically. This is not the case for $\mathbb{R} \times \mathbb{S}^2$ and thus one could consider the vanishing of these charges as a topological feature.

Of course, the interest of having conserved charges is not only related to the existence of gravitational radiation, and in that sense the above charges may be very useful in different contexts.

7.5.2 Second class charges

When dealing with fields other than gravity, the standard approach is to consider charges associated to the energy-momentum tensor of the field theory. As it is already well-known, there is not such thing in General Relativity. Now, we define a second class of charges that result from using the rescaled Bel-Robinson tensor. One has to be aware of the dimensionality of such charges and currents, since they are of tidal nature and do not carry, in general, units of energy-momentum.

Consider first a triplet of CKVF $\left(\underset{(i)}{\hat{\xi}}{}^{\alpha},\ \underset{(j)}{\hat{\xi}}{}^{\alpha},\ \underset{(k)}{\hat{\xi}}{}^{\alpha}\right)$ of the space-time $\left(M, g_{\alpha\beta}\right)$, which can contain repeated elements. Assume that in a neighbourhood of $\mathscr{J}$ $T_{\alpha\beta}=0$ –note that this is a more restrictive condition that the one taken in the rest of the work, see property iv) on page 22. Then, in that neighbourhood of $\mathscr{J}$

$$\nabla_{\mu}\mathcal{D}^{\mu}{}_{\alpha\beta\gamma}=0\ . \tag{7.257}$$

It is easy to check that the current [85, 87]

$$\mathcal{B}^{\alpha}:=\underset{(i)}{\hat{\xi}}{}^{\mu}\ \underset{(j)}{\hat{\xi}}{}^{\nu}\ \underset{(k)}{\hat{\xi}}{}^{\rho}\mathcal{D}^{\alpha}{}_{\mu\nu\rho} \tag{7.258}$$

is divergence-free in that region of the space-time (including $\mathscr{J}$)

$$\nabla_{\mu}\mathcal{B}^{\mu}=0\ . \tag{7.259}$$

Then, the quantity defined on any spacelike hypersurface Σ orthogonal to a timelike t_{α}

$$\mathcal{B}_{\Sigma}:=\int_{\Sigma}t_{\mu}\mathcal{B}^{\mu}\epsilon \tag{7.260}$$

is conserved in a space-time region Δ_{M} bounded by any two Σ_1 and Σ_2 orthogonal to any two future-pointing timelike $\underset{1,2}{t}{}_{\alpha}$ and with Σ_2 to the future of Σ_1,

$$0=\int_{\Delta_{M}}\nabla_{\mu}\mathcal{B}^{\mu}\eta=\mathcal{B}_{\Sigma_2}-\mathcal{B}_{\Sigma_1}\ . \tag{7.261}$$

In particular, Σ can be chosen to be $\mathscr{J}$.

Suppose first that $\underset{(i)}{\hat{\xi}}{}^{\alpha}$ are completely tangent to $\mathscr{J}$. Then

$$\mathcal{B}_{\mathscr{J}}=\int_{\mathscr{J}}Q_{abc}\ \underset{(i)}{\hat{\xi}}{}^{a}\ \underset{(j)}{\hat{\xi}}{}^{b}\ \underset{(k)}{\hat{\xi}}{}^{c}\epsilon\ , \tag{7.262}$$

where Q_{abc} is defined for the rescaled Bel-Robinson tensor on $\mathscr{J}$ as in eq. (2.17), and one can write

$$\mathcal{B}_{\mathscr{J}}=\int_{\mathscr{J}}\left(\underset{(i)}{\xi}{}^{m}\ \underset{(j)}{\xi}{}_{m}\overline{\mathcal{P}}_{d}\ \underset{(k)}{\xi}{}^{d}-4\ \underset{(i)}{\xi}{}^{e}C_{ec}\ \underset{(j)}{\xi}{}^{f}D_{fd}\epsilon^{pcd}\ \underset{(k)}{\xi}{}_{p}\right)\epsilon\ . \tag{7.263}$$

If the condition in criterion 2 holds on $\mathscr{J}$ and there is no radiation $\overline{\mathcal{P}}^{a}=0$, $\mathcal{B}_{\mathscr{J}}=0$.

Suppose now that $\underset{(i)}{\hat{\xi}}{}^{\alpha}\overset{\mathscr{J}}{=}\underset{(i)}{\beta}n^{\alpha}$ for some non-vanishing functions $\underset{(i)}{\beta}$,

$$\mathcal{B}_{\mathscr{J}}=\int_{\mathscr{J}}\underset{(i)}{\beta}\ \underset{(j)}{\beta}\ \underset{(k)}{\beta}\mathcal{W}:=\mathcal{C}_{\Sigma}\ , \tag{7.264}$$

where $\mathcal{W}$ is the asymptotic canonical 1 density –see eq. (5.36). Observe that $\mathcal{C}_\Sigma$ vanishes if and only if $\mathcal{W} = 0$ at $\mathscr{J}$, i.e., $d_{\alpha\beta\gamma\delta} = 0$ there. Because the charge is conserved in Δ_M, $\mathcal{C}_\Sigma = 0$ for all Σ in Δ_M. In particular, de Sitter space-time has $\mathcal{C}_\Sigma = 0$ everywhere, including at $\mathscr{J}$, and any other space-time having $\mathcal{B}_{\mathscr{J}} = 0$ is de Sitter space-time in the domain of dependence of $\mathscr{J}$.

Now, let us focus on a strongly equipped $\mathscr{J}$ and consider a bitranslation $\tau^a = \alpha m^a$. Let us define the current

$$\mathcal{R}^a \overset{\mathscr{J}}{:=} -\mathcal{D}^{\alpha}{}_{\mu\nu\rho}\omega_\alpha{}^a e^\mu{}_b e^\nu{}_c e^\rho{}_d \tau^b\tau^c\tau^d. \tag{7.265}$$

Notice that according to the discussion at the end of section 7.4.2, the dimensions of this quantity are.

$$[\mathcal{R}^a] = L^{-1} , \tag{7.266}$$

despite of being constructed with a superenergy tensor –indeed, $\mathcal{R}^a$ has physical units of MT^{-2}. Its integral over any cut gives a charge with units of energy. If m^a is orthogonal to cuts $\mathcal{S}_i$, the divergence of the current $\mathcal{R}^a$ integrated over the compact region Δ bounded by $\mathcal{S}_{1,2}$ gives a balance law

$$\int_\Delta \overline{\nabla}_a \mathcal{R}^a \epsilon = \mathcal{R}_{\mathcal{S}_2} - \mathcal{R}_{\mathcal{S}_1} , \tag{7.267}$$

where

$$\mathcal{R}_{\mathcal{S}_i} := \int_{\mathcal{S}_i} \mathcal{R}^a m_a \mathring{\epsilon} = \int_{\mathcal{S}_i} \alpha^3 \left[\frac{1}{4}\left(\overset{+}{\mathcal{W}} + \overset{-}{\mathcal{W}}\right) - \left(\overset{+}{\mathcal{Z}} + \overset{-}{\mathcal{Z}}\right) + \frac{3}{2}\mathcal{V}\right]\mathring{\epsilon} . \tag{7.268}$$

The left-hand side follows by decomposing $\sqrt{2}m_\alpha = \overset{+}{k}_\alpha - \overset{-}{k}_\alpha$ and introducing the definitions of eqs. (2.52) to (2.55) and (2.58). This charge contains both radiative and Coulomb contributions. In order to compute the intrinsic divergence of $\mathcal{R}^a$, one has to know the Lie derivative of $\mathcal{D}_{\alpha\beta\gamma\delta}$ along n^α at $\mathscr{J}$.

7.5.3 Balance law from the divergence property of the asymptotic supermomentum

The divergence of the asymptotic supermomentum given by eq. (5.43) can be integrated over a compact region Δ bounded by $\mathcal{S}_{1,2}$ to give

$$\int_\Delta \left(\pounds_{\vec{n}}\mathcal{W} - N\varkappa \underset{1}{T}_{ab} D^{ab}\right)\epsilon = \underset{\Lambda}{\Phi}[\mathcal{S}_2] - \underset{\Lambda}{\Phi}[\mathcal{S}_1] , \tag{7.269}$$

where $\underset{\Lambda}{\Phi}[\mathcal{S}]$ is the *asymptotic superenergy density flux on* $\mathcal{S}$, defined as

$$\underset{\Lambda}{\Phi}[\mathcal{S}] := -\int_{\mathcal{S}} m_a \overline{\mathcal{P}}^a \mathring{\epsilon} \tag{7.270}$$

where m_a is the normal to $\mathcal{S}$. Remarkably, *if* m_a *defines a strong orientation*, then

$$\underset{\Lambda}{\Phi}[\mathcal{S}] \geq 0 \tag{7.271}$$

and

$$\underset{\Lambda}{\Phi}[\mathcal{S}] = 0 \Longleftrightarrow \overline{\mathcal{P}}^a \stackrel{\mathcal{S}}{=} 0 \tag{7.272}$$

which follows from lemma 5.4.3. Compare eq. (7.269) with eq. (4.233).

8 | Examples

This last chapter[1] collects some examples of application of the main results presented in this thesis. Important ideas are put to test, such as the determination of gravitational radiation at $\mathscr{J}$ or the existence of the first component of news V_{ab} when the conditions are met –see proposition 7.2.1. Thus, the asymptotic super-Poynting vector field $\overline{\mathcal{P}}^a$ and V_{ab} are computed, and also the additional symmetries of definition 7.4.2 associated to the curves selected by strong orientation –see definition 5.4.2. The easy and topology-independent calculation of $\overline{\mathcal{P}}^a$ allows to determine if a given metric contains gravitational radiation at $\mathscr{J}$ in a very straightforward manner. The outcome of this calculation for the metrics considered here agrees with what one would expect in each case.

8.1 The Kerr-de Sitter and Kottler metrics

Let us start with the conformal Kerr-de Sitter metric

$$\begin{aligned}\mathrm{d}s^2 = \frac{1}{r^2}\Bigg\{ &\left(-\frac{\Delta_r}{\rho^2} + \frac{\Delta_\theta}{\rho^2}a^2\sin^2\theta\right)\mathrm{d}t^2 + \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 + \frac{1}{\Xi^2}\left[-\frac{\Delta_r}{\rho^2}a^2\sin^4\theta + \frac{\Delta_\theta}{\rho^2}(r^2+a^2)^2\sin^2\theta\right]\mathrm{d}\phi^2 + \\ &+\frac{1}{\Xi}\left[\frac{\Delta_r}{\rho^2}a\sin^2\theta - \frac{\Delta_\theta}{\rho^2}a\sin^2\theta(r^2+a^2)\right](\mathrm{d}\phi\mathrm{d}t + \mathrm{d}t\mathrm{d}\phi) + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 \Bigg\}.\end{aligned} \tag{8.1}$$

These are Boyer-Lindquist-type coordinates, with

$$t \in \mathbb{R}, \quad r \in \mathbb{R}, \quad \theta \in [0,\pi), \quad \phi \in [0, 2\pi] \tag{8.2}$$

and

$$\Lambda > 0, \quad a \in \mathbb{R}, \quad m \in \mathbb{R}\setminus 0 \quad . \tag{8.3}$$

[1]Some of the calculations presented in this chapter were performed using the computer algebra system Maxima –distributed under GNU GPL license.

The metric functions are defined as

$$\rho^2 := r^2 + a^2\cos^2\theta \quad , \tag{8.4}$$
$$\Delta_r := (a^2 + r^2)\left(1 - \frac{\Lambda}{3}r^2\right) - 2mr \quad , \tag{8.5}$$
$$\Delta_\theta := 1 + \frac{\Lambda}{3}a^2\cos^2\theta \quad , \tag{8.6}$$
$$\Xi := 1 + \frac{\Lambda}{3}a^2 \quad . \tag{8.7}$$

The particular case with $a = 0$ gives the Kottler (sometimes called Schwarzschil-de Sitter) spherically symmetric conformal metric.

Infinity is located at $r \to \infty$, and we have chosen

$$\Omega := \frac{A}{r} \quad , \tag{8.8}$$

with A =constant with dimensions $[A] = L$, so that $[\Omega] = 1$. From now on we set $A = 1$. This choice of Ω indeed belongs to the divergence-free family of conformal gauges (3.92). Hence, the normal to $\mathscr{J}$ is

$$N_\alpha = -\frac{1}{r^2}\nabla_\alpha r \quad . \tag{8.9}$$

Notice that $\mathrm{d}r^2 = r^4\mathrm{d}\Omega^2$ and that

$$N^2 \overset{\mathscr{J}}{=} \frac{\Lambda}{3} \quad , \tag{8.10}$$
$$\frac{\rho^2}{\Delta_r}\Omega^2 r^4 \overset{\mathscr{J}}{=} -\frac{1}{N^2} \quad , \tag{8.11}$$
$$\frac{\rho^2}{\Delta_\theta}\Omega^2 \overset{\mathscr{J}}{=} \frac{1}{1 + N^2a^2\cos^2\theta} \quad , \tag{8.12}$$
$$\frac{\Delta_r}{\rho^2}\Omega^2 \overset{\mathscr{J}}{=} -N^2 \quad , \tag{8.13}$$
$$\frac{\Delta_\theta}{\rho^2}\Omega^2 \overset{\mathscr{J}}{=} 0 \quad , \tag{8.14}$$
$$\frac{\Delta_\theta}{\rho^2}\left(r^2 + a^2\right)^2\Omega^2 \overset{\mathscr{J}}{=} 1 + N^2a^2\cos^2\theta \quad . \tag{8.15}$$

Using these formulae, one can write the metric of $\mathscr{J}$ as

$$h = N^2\mathrm{d}t^2 + \frac{1}{\Xi^2}\left(1 + N^2a^2\right)\sin^2\theta\mathrm{d}\phi^2 - \frac{1}{\Xi}N^2a\sin^2\theta\left(\mathrm{d}\phi\mathrm{d}t + \mathrm{d}t\mathrm{d}\phi\right) + \left(1 + N^2a^2\cos^2\theta\right)^{-1}\mathrm{d}\theta^2 \quad . \tag{8.16}$$

The electric and magnetic parts of the rescaled Weyl tensor at $\mathscr{J}$ read respectively

$$C_{ab} = 0 \quad , \tag{8.17}$$
$$\begin{aligned} D_{ab} = &-\frac{2}{3}\Lambda m \overline{\nabla}_a t \overline{\nabla}_b t + \frac{2}{\Xi} N^2 a m \sin^2\theta \left(\overline{\nabla}_a \phi \overline{\nabla}_b t + \overline{\nabla}_a t \overline{\nabla}_b \phi\right) + \\ &+ m\left(1 + a^2 N^2 \cos^2\theta\right)^{-1} \overline{\nabla}_a \theta \overline{\nabla}_b \theta + \frac{1}{\Xi^2} m \sin^2\theta \left(1 + a^2 \Lambda \cos^2\theta - \frac{2}{3} a^2 \Lambda\right) \overline{\nabla}_a \phi \overline{\nabla}_b \phi \quad . \end{aligned} \tag{8.18}$$

The intrinsic Ricci tensor, scalar curvature and Schouten tensor have the following expressions:

$$\begin{aligned} \overline{R}_{ab} = &\, 2a^2 N^4 \cos^2\theta \overline{\nabla}_a t \overline{\nabla}_b t - \frac{2}{\Xi} a N^2 \sin^2\theta \left(1 + 3a^2 N^2 \cos^2\theta\right) \overline{\nabla}_{(a} t \overline{\nabla}_{b)} \phi \\ &+ \frac{1}{\Xi^2}\left[1 - \left(1 - 3a^2 N^2\right)\cos^2\theta - 3a^2 N^2 \cos^4\theta\right]\left(1 + a^2 N^2\right) \overline{\nabla}_a \phi \overline{\nabla}_b \phi \\ &+ \frac{1 + 4a^2 N^2 \cos^2\theta - a^2 N^2}{1 + a^2 N^2 \cos^2\theta} \overline{\nabla}_a \theta \overline{\nabla}_b \theta \quad , \end{aligned} \tag{8.19}$$
$$\overline{R} = 2 - 2a^2 N^2 + 10 N^2 a^2 \cos^2\theta \quad , \tag{8.20}$$
$$\begin{aligned} \overline{S}_{ab} = &\, \frac{1}{2} N^2 \left(a^2 N^2 - a^2 N^2 \cos^2\theta - 1\right) \overline{\nabla}_a t \overline{\nabla}_b t - \frac{1}{\Xi} a N^2 \sin^2\theta \left(1 + a^2 N^2 + a^2 N^2 \cos^2\theta\right) \overline{\nabla}_{(a} t \overline{\nabla}_{b)} \phi \\ &+ \frac{1}{2\Xi^2}\left(1 + a^2 N^2\right)\sin^2\theta\left(1 + a^2 N^2 + a^2 N^2 \cos^2\theta\right) \overline{\nabla}_a \phi \overline{\nabla}_b \phi \\ &+ \frac{1 - a^2 N^2 + 3a^2 N^2 \cos^2\theta}{2\left(1 + a^2 N^2 \cos^2\theta\right)} \overline{\nabla}_a \theta \overline{\nabla}_b \theta \quad . \end{aligned} \tag{8.21}$$

There are two repeated PND $\underset{1}{\ell}_\alpha$ and $\underset{2}{\ell}_\alpha$ which read at $\mathscr{J}$

$$\underset{1}{\ell}_\alpha \overset{\mathscr{J}}{=} \frac{1}{\sqrt{2}}\left(-\frac{1}{N r^2}\nabla_\alpha r - N \nabla_\alpha t + \frac{1}{\Xi} a N \sin^2\theta \nabla_\alpha \phi\right) \quad , \tag{8.22}$$
$$\underset{2}{\ell}_\alpha \overset{\mathscr{J}}{=} \frac{1}{\sqrt{2}}\left(-\frac{1}{N r^2}\nabla_\alpha r + N \nabla_\alpha t - \frac{1}{\Xi} a N \sin^2\theta \nabla_\alpha \phi\right) \quad . \tag{8.23}$$

Accordingly, there are two different strong orientations (see definition 5.4.2 and remark 5.4.4). We choose one of them by defining

$$\overset{-}{k}_\alpha \overset{\mathscr{J}}{:=} \underset{1}{\ell}_\alpha \quad , \tag{8.24}$$
$$m_\alpha \overset{\mathscr{J}}{:=} n_\alpha - \sqrt{2}\,\overset{-}{k}_\alpha = N\left(\nabla_\alpha t - \frac{1}{\Xi} a \sin^2\theta \nabla_\alpha \phi\right) \quad , \tag{8.25}$$
$$\overset{+}{k}_\alpha \overset{\mathscr{J}}{:=} \frac{1}{\sqrt{2}}\left(n_\alpha + m_\alpha\right) = \frac{1}{\sqrt{2}}\left(-\frac{1}{N r^2}\nabla_\alpha r + N \nabla_\alpha t - \frac{1}{\Xi} a N \sin^2\theta \nabla_\alpha \phi\right) = \underset{2}{\ell}_\alpha \quad , \tag{8.26}$$

where $N n_\alpha \overset{\mathscr{J}}{:=} N_\alpha$, such that $\overset{-}{k}{}^\alpha \overset{+}{k}_\alpha \overset{\mathscr{J}}{=} -1$, $m^\alpha \overset{-}{k}_\alpha \overset{\mathscr{J}}{=} -1/\sqrt{2}$ and $m^\alpha \overset{+}{k}_\alpha \overset{\mathscr{J}}{=} 1/\sqrt{2}$. Notice that both repeated PND are coplanar with the normal N_α which makes the two strong orientations equivalent from the viewpoint of $\mathscr{J}$, in the sense that they define, *up to sign*, the same vector field m^a there. The pullback to $\mathscr{J}$ of m_α is

$$m_a \overset{\mathscr{J}}{=} N\left(\overline{\nabla}_a t - \frac{1}{\Xi} a \sin^2\theta \overline{\nabla}_a \phi\right), \tag{8.27}$$

$$m^a \overset{\mathscr{J}}{=} \frac{1}{N}\delta^a_t \quad , \tag{8.28}$$

∂_t being a KVF of $(\mathscr{J}, h_{ab})$. The non-vanishing intrinsic connection coefficients are

$$\overline{\Gamma}^\theta{}_{t\phi} = \frac{1}{\Xi} a N^2 \cos\theta \sin\theta \left(1 + a^2 N^2 \cos^2\theta\right) \quad , \tag{8.29}$$

$$\overline{\Gamma}^\theta{}_{\phi\phi} = -\frac{1}{\Xi^2}\left(1 + a^2 N^2\right)\cos\theta\sin\theta\left(1 + a^2N^2\cos^2\theta\right), \tag{8.30}$$

$$\overline{\Gamma}^\theta{}_{\theta\theta} = \frac{a^2 N^2 \cos\theta \sin\theta}{1 + a^2 N^2 \cos^2\theta} \quad , \tag{8.31}$$

$$\overline{\Gamma}^\phi{}_{\theta t} = -\frac{\Xi a N^2 \cos\theta}{\sin\theta\left(1 + a^2 N^2 \cos^2\theta\right)} \quad , \tag{8.32}$$

$$\overline{\Gamma}^\phi{}_{\phi\theta} = \frac{\cos\theta}{\sin\theta} \quad , \tag{8.33}$$

$$\overline{\Gamma}^t{}_{\theta t} = -\frac{a^2 N^2 \cos\theta \sin\theta}{1 + a^2 N^2 \cos^2\theta} \quad . \tag{8.34}$$

One does not need them to compute the kinematics of m_a (see definitions in appendix A.3) though; noting the fact that m^a is a KVF, $\underline{\kappa}_{ab}$ vanishes[2], whereas $\underline{a}_b$ vanishes by symmetrising in eq. (A.51) and contracting once with m^a, and $\underline{\omega}_{ab}$ does not involve the connection:

$$\underline{a}_b = 0 \quad , \tag{8.35}$$

$$\underline{\kappa}_{ab} = 0 \quad , \tag{8.36}$$

$$\underline{\omega}_{ab} = \frac{2}{\Xi} a \sin\theta\cos\theta \overline{\nabla}_{[a}\phi \overline{\nabla}_{b]}\theta \quad . \tag{8.37}$$

Equation (8.37) implies that m_a is not surface-orthogonal, that is, it does not give a foliation. The projector to $\mathbf{S}_2$ (see appendix A.3) reads

$$\underline{P}_{ab} = \frac{1}{1 + a^2N^2\cos^2\theta}\overline{\nabla}_a\theta\overline{\nabla}_b\theta + \frac{1}{\Xi^2}\left(1 + a^2N^2\cos^2\theta\right)\sin^2\theta\overline{\nabla}_a\phi\overline{\nabla}_b\phi \quad . \tag{8.38}$$

The pair $(\underline{P}_{ab}, m_a)$ characterises the congruence of curves given by m^a and the projected surface $\mathbf{S}_2$; we say, according to definition 7.0.1, that $\mathscr{J}$ is equipped.

[2] One can check that this is the case by doing the explicit calculation using the non-vanishing components of $\overline{\Gamma}^a{}_{cb}$.

All the quantities corresponding to the decomposition of D_{ab} and C_{ab} (see section 2.2 and eqs. (7.4) and (7.5)) vanish except for

$$D = -\underline{D}^M{}_M = -2m \tag{8.39}$$

and thus we have

$$D_{ab} = -m\left(3m_a m_b - h_{ab}\right) \ . \tag{8.40}$$

8.1.1 Asymptotic symmetries

It is known that due to the $\mathbb{R} \times \mathbb{S}^2$ topology of $\mathscr{J}$ [109] the group of CKVF of $(\mathscr{J}, h_{ab})$ is 4-dimensional [61]. This, however, misses the TT-tensor D_{ab}, which must be taken into account as essential part of the asymptotic structure. Taking the Lie derivative of (8.40) one easily finds that the definition 7.4.1 requires the solutions to be actually KVF of $(\mathscr{J}, h_{ab})$. In this sense, the generators of the basic symmetries are given by ∂_t and ∂_ϕ, which indeed are KVF of $(\mathscr{J}, h_{ab})$. Hence, this group is just 2-dimensional —unless in the Kottler metric case, $a = 0$, which is 4-dimensional. In addition, we can study the asymptotic symmetries of (definition 7.4.2). The algebra of biconformal transformations $\mathfrak{b}$ is consituted by elements of the form

$$\xi^a = \beta m^a + \chi^a \quad , \tag{8.41}$$

where β and χ^a satisfy eqs. (7.161) to (7.164), that is,

$$\pounds_{\vec{m}}\chi^a = 0, \quad \underline{\mathcal{D}}_b\beta = -2\underline{\omega}_{eb}\chi^e, \quad 2\underline{\mathcal{D}}_{(a}\chi_{b)} = 2\psi \underline{P}_{ab} \tag{8.42}$$

and one defines $\phi := m^e\overline{\nabla}_e\beta$. On the one hand, from eqs. (7.165) to (7.167), it follows that the elements of the subalgebra of bitranslations $\mathfrak{t}$ have the form

$$\tau^a = \alpha m^a \quad \text{with} \quad \mathcal{D}_a\alpha = 0 \quad , \tag{8.43}$$

and $\lambda := m^e\overline{\nabla}_e\alpha$. However, according to eq. (A.71), $\underline{\mathcal{D}}_{[a}\underline{\mathcal{D}}_{b]}\alpha = -\underline{\omega}_{ab}m^e\overline{\nabla}_e\alpha$. Thus the only possibility is α =constant. In other words, there is just one element of $\mathfrak{t}$ and this is the KVF ∂_t. On the other hand, it is easily seen that the non-vanishing $\underline{\omega}_{ab}$ spoils the existence of a subalgebra of conformal transformations of the projector $\mathfrak{cs}$ –see comments below eq. (7.171). Finally, a more detailed calculation shows that the remaining general biconformal symmetries $\xi^a \in \mathfrak{b}$ associated to the orientation given by eq. (8.27) are of the form:

$$\xi^a = \alpha\delta^a_t + b\delta^a_\phi \quad , \tag{8.44}$$

where b is a constant. Therefore, the basic infinitesimal symmetries are precisely biconformal infinitesimal symmetries of the pairs $(m_a, \underline{P}_{ab})$ that define strong orientation, unless in the particular Kottler metric with $a = 0$, where the equipped symmetries constitute an infinite-dimensional algebra. This algebra will be given as a particular case of the different equipments that we are going to consider next.

8.1.2 Strong equipment

There are other equipments on $\mathscr{J}$, that is, other choices for m_a. Specifically, the vector field

$$m_a := \frac{N}{\Xi}\left(1 + N^2 a^2 \cos^2\theta\right)^{\frac{1}{2}} \overline{\nabla}_a t \quad , \tag{8.45}$$

$$m^a = \Xi \frac{1}{N\left(1 + N^2 a^2 \cos^2\theta\right)^{\frac{1}{2}}}\left(\delta^a_t + aN^2\delta^a_\phi\right) \tag{8.46}$$

is worth attention. All its kinematic quantities of the vector field m^a vanish except the acceleration,

$$\underline{a}_b = a^2 N^2 \frac{\cos\theta\sin\theta}{1 + a^2N^2\cos^2\theta}\underline{\mathcal{D}}_b\theta \quad . \tag{8.47}$$

Hence, it is orthogonal to a foliation of umbilical cuts with metric

$$\underline{q}_{AB} = \frac{\sin^2\theta}{\Xi}\underline{\mathcal{D}}_A\phi\underline{\mathcal{D}}_B\phi + \frac{1}{1 + a^2N^2\cos^2\theta}\underline{\mathcal{D}}_A\theta\underline{\mathcal{D}}_B\theta \tag{8.48}$$

and Gaussian curvature

$$\underline{K} = 1 + 2a^2N^2\cos^2\theta \quad . \tag{8.49}$$

The projector to these cuts is written as

$$\underline{P}_{ab} = \frac{\sin^2\theta}{\Xi}\left(aN^2\overline{\nabla}_a t - \overline{\nabla}_a\phi\right)\left(aN^2\overline{\nabla}_b t - \overline{\nabla}_b\phi\right) + \frac{1}{1 + a^2N^2\cos^2\theta}\overline{\nabla}_a\theta\overline{\nabla}_b\theta \quad . \tag{8.50}$$

Observe that this provides a *strongly* equipped $\mathscr{J}$ –definition 7.0.3.

The tensor $\underline{\rho}_{ab}$ can be computed using the general expressions for axially-symmetric metrics of section 6.2.1. These yield

$$\begin{aligned}\underline{\rho}_{AB} = {} & \frac{\sin^2\theta}{2\Xi}\left(1 + a^2N^2 + a^2N^2\cos^2\theta\right)\underline{\mathcal{D}}_A\phi\underline{\mathcal{D}}_B\phi \\ & + \frac{1}{2\left(1 + a^2N^2\cos^2\theta\right)}\left[1 + 3a^2N^2\cos^2\theta - a^2N^2\right]\underline{\mathcal{D}}_A\theta\underline{\mathcal{D}}_B\theta.\end{aligned} \tag{8.51}$$

Next, noting that $\underline{T}_{AB}{}^C = 0$ and that this implies $\underline{U}_{AB} = \underline{S}_{AB}$, one can compute $\underline{V}_{AB}$ using eq. (8.21):

$$\underline{V}_{AB} = 0 \quad . \tag{8.52}$$

This is the expected result taking into account $C_{ab} = 0$ and eq. (7.71). There is still more to say on this. The biconformal symmetries acting on the new pairs $(m_a, \underline{P}_{ab})$ have the form

$$\xi^a = \alpha m^a + \eta^a \quad , \tag{8.53}$$

where the restriction to the cuts $\underline{\eta}^A$ of η^a are CKVF of $\left(\underline{q}_{AB}, \mathcal{S}\right)$, m^a is given in eq. (8.46) and

$$\alpha = \nu(v)\left(1 + N^2 a^2 \cos^2\theta\right)^{\frac{1}{2}} . \tag{8.54}$$

Here v is the parameter of the foliation as in eq. (A.103), in this case given by $v = t$, and $\nu(v)$ an arbitrary function depending on v only which makes the dimension of the subalgebra $\mathfrak{t}$ infinite. Indeed, this is the canonical form of a bitranslation, see eq. (7.226), with $F = (N/\Xi)\left(1 + N^2 a^2 \cos^2\theta\right)^{1/2}$. Now, since $\mathcal{S}$ is topologically $\mathbb{S}^2$, $\underline{\eta}^A$ are the infinitesimal symmetries of the Lorentz Group $\mathsf{SO}(1,3)$. This agrees with the general results of section 7.4.2. As a further remark, for going from the round metric to the current one one has to rescale the metric on the cuts by $\omega = K\Xi\left(1 + N^2 a^2 \cos^2\theta\right)$ with K =constant, which shows that the restriction $\mathring{\alpha}$ to the cuts of α is constant in the round gauge (that is, a constant times the $l = 0$ spherical harmonic). This agrees with corollary 7.4.1, since as we will see later on $\underline{V}_{ab}$ vanishes for these cuts, hence the subgroup of bitranslations given by $\tau^a = \alpha m^a$ acting on the strong equipment given by these cuts correspond to infinitesimal asymptotic translations of definition 7.4.3. This structure is also the general solution for the Kottler metric with $a = 0$, for which both equipments are actually the same.

8.1.3 Asymptotic supermomentum

We compute the asymptotic canonical super-Poynting vector field $\overline{P}^a$ and canonical superenergy density $\mathcal{W}$ with the following outcome:

$$\overline{\mathcal{P}}^a \stackrel{\mathscr{J}}{=} 0 \quad , \tag{8.55}$$

$$\mathcal{W} \stackrel{\mathscr{J}}{=} 6m^2 \quad . \tag{8.56}$$

The vanishing of $\overline{\mathcal{P}}^a$ indicates that the space-time contains no gravitational radiation at infinity. This agrees with the fact that the two repeated PND $\underset{1}{\ell}^\alpha$ and $\underset{2}{\ell}^\alpha$ are coplanar with N_α –see remarks 5.3.1 and 5.3.4– and that h_{ab} is conformally flat.

There is an interesting feature of the canonical superenergy $\mathcal{W}$: it does not depend on a at $\mathscr{J}$, so that it has the same constant value as the one for $a = 0$.[3] We can compute

[3]That is, the Kottler metric.

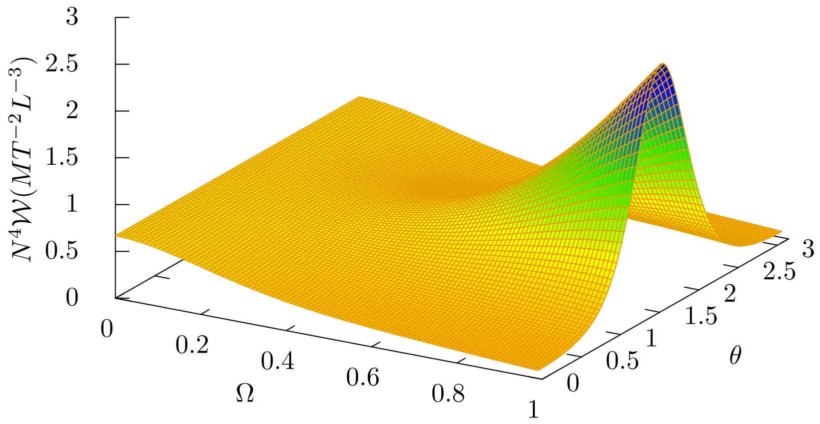

Figure 8.1: Asymptotic superenergy for the Kerr-dS metric around $\mathscr{J}$ for $a = m = \Lambda = 1$. The dependence on the angular parameter a fades away as approaching $\mathscr{J}$. The peak in the superenergy, then, occurs at the equator ($\theta = \pi/2$).

the superenergy density associated to N^α outside $\mathscr{J}$, its expression is

$$N^4\mathcal{W} = \frac{6\Delta_r^2 m^2 \Omega^4}{\rho^4\left(1 + a^2\Omega^2\cos^2\theta\right)^3} \quad , \tag{8.57}$$

and its Taylor expansion around $\Omega = 0$ yields

$$N^4\mathcal{W} \overset{\mathscr{J}}{=} \frac{2}{3}\Lambda^2 m^2 - \frac{2}{3}\Lambda m^2\left[6 + 3a^2\Lambda - 5a^2\Lambda\sin^2\theta\right]\Omega^2 + \ldots \quad . \tag{8.58}$$

Therefore, we see that a enters only at second leading order. This effect can be appreciated in fig. 8.1.

8.2 The C-metric

The existence of exact solutions of Einstein's Field Equations containing gravitational radiation at infinity when $\Lambda = 0$ was demonstrated in [106] by showing that the so called C-metric has a non-vanishing news tensor at $\mathscr{J}$. For $\Lambda > 0$, the first proof of an exact solution having gravitational waves at infinity according to criterion 1 was presented in [75] using precisely the C-metric but now with $\Lambda > 0$. In the present work, we expand that analysis in several directions and in particular we suggest that two news tensors on $\mathscr{J}$ exist using the results of chapter 6.

The C-metric with $\Lambda > 0$ describes two accelerating black holes in a de Sitter background [141]. As such, one expects the presence of gravitational radiation at $\mathscr{J}$. We will consider this metric in the form of a particular sub-case of the accelerating, charged, rotating Plebański-Demiański solution [142] –see also [143]. The conformal metric, selecting a gauge according to eq. (3.22), reads

$$\mathrm{d}s^2 = \frac{(\eta f(\eta))^2}{S}\left(-T\mathrm{d}\tau^2 + \frac{1}{T}\mathrm{d}q^2 + \frac{1}{S}\mathrm{d}p^2 + S\mathrm{d}\sigma^2\right) \quad , \tag{8.59}$$

where

$$T(q) := (q^2 - a^2)(1 + 2mq) - \Lambda/3 \quad , \tag{8.60}$$
$$S(p) := (1 - p^2)(1 - 2amp) \quad , \tag{8.61}$$

η is a conformal-gauge function and $f(\eta)$ an arbitrary function regular and different from zero at $\eta = 0$, both to be specified. The conformal boundary $\mathscr{I}$ is defined by $q = -ap$ the conformal factor being

$$\Omega^2 := \frac{(\eta f(\eta))^2}{S}(q + ap)^2 \tag{8.62}$$

and the normal to $\mathscr{I}$

$$N_\alpha \overset{\mathscr{I}}{=} -\frac{\eta f(\eta)}{\sqrt{S}}\left(\nabla_\alpha q + a\nabla_\alpha p\right). \tag{8.63}$$

The gauge function η is a first integral of N^α and we choose it to be

$$\eta := e^{a(1-2am)F(q)}\frac{(1-p)^{\frac{1}{2}}}{(1-2amp)^{2am/(1+2am)}(1+p)^{(1-2am)/2(1+2am)}} \quad , \tag{8.64}$$

with

$$F(q) = -\int \frac{1}{T(q)}\mathrm{d}q \quad . \tag{8.65}$$

It is possible to set

$$\frac{(\eta f(\eta))^2}{S} \overset{\mathscr{I}}{=} 1, \tag{8.66}$$

and for that we take

$$f(\eta) := e^{-a(1-2am)F\left(-aP^{-1}(\eta^2)\right)}\left(1 + P^{-1}(\eta^2)\right)^{\frac{1}{(1+2am)}}\left(1 - 2amP^{-1}(\eta^2)\right)^{\frac{(1+6am)}{2(1+2am)}} \tag{8.67}$$

where $P^{-1}(\eta^2)$ is the inverse function of $P(p)$ such that $P(p) \overset{\mathscr{I}}{=} \eta^2$.

There are four constant parameters, namely the acceleration a, the mass m , Λ and C. The metric may present two conical singularities at $p = 1$ and/or at $p = -1$. One can fix C to cure one of these singularities but never both of them at the same time. Additionally, this fixing defines the range of the coordinate $\sigma \in [0, 2\pi C)$ [143]. Since both singularities are not curable at the same time, one has to restrict the range of the coordinate p in order to exclude the persisting one. We fix C to

$$C = \frac{1}{(1-2am)} \tag{8.68}$$

so that $p = 1$ defines a regular axis for the KVF ∂_σ and restricts the range $p \in (-1, 1]$ in order to avoid the singular point at $p = -1$. This, together with the further condition

$$2am < 1, \tag{8.69}$$

makes $S \geq 0$, only vanishing at $p = 1$ –thus preserving the signature of the metric.

There are two KVF of $g_{\alpha\beta}$: ∂_τ and ∂_σ. The former has $\mathbb{R}$-orbits whereas the latter has cyclic orbits. Observe that $T(q) < 0$ at $\mathscr{J}$ and g_{qq} becomes negative there, hence the space-time is non-stationary around the conformal boundary, as one expects. Because we are interested in studying $\mathscr{J}$, we further restrict ourselves to $q \in (-a, a)$ –which keeps $T(q)$ between two roots and negative. One more feature is that the roots of $T(q)$ represent horizons which, by our previous remarks, do not meet $\mathscr{J}$. The Weyl tensor has two repeated principal null directions (which also become repeated PND of $d_{\alpha\beta\gamma}{}^{\delta}$ at $\mathscr{J}$) given by:

$$\underset{1}{\ell}_\alpha = \frac{1}{\sqrt{2}}\frac{N}{T}\left(-T\nabla_\alpha\tau + \nabla_\alpha q\right) \quad , \tag{8.70}$$

$$\underset{2}{\ell}_\alpha = \frac{1}{\sqrt{2}}\frac{N}{T}\left(T\nabla_\alpha\tau + \nabla_\alpha q\right) \quad . \tag{8.71}$$

Notice that we have chosen them such that $\underset{1}{\ell}^\alpha \underset{2}{\ell}_\alpha \overset{\mathscr{J}}{=} -1$, but this does not hold outside $\mathscr{J}$.

From now on we focus on $\mathscr{J}$. Note that $\overline{T} := -a^2 S - N^2 \overset{\mathscr{J}}{=} T$. The metric there reads

$$h = (a^2 S + N^2)\mathrm{d}\tau^2 + \frac{N^2}{S(a^2 S + N^2)}\mathrm{d}p^2 + S\mathrm{d}\sigma^2 \quad . \tag{8.72}$$

This is positive definite and has a regular limit when $\Lambda \to 0$ leading to a degenerate metric $\overline{g}_{ab}$. The intrinsic connection on $\mathscr{J}$ is

$$\overline{\Gamma}^\tau{}_{ab} \overset{\mathscr{J}}{=} -\frac{\alpha^2}{\bar{T}}\partial_p S \overline{\nabla}_{(a}\tau\overline{\nabla}_{b)}p \quad , \tag{8.73}$$

$$\overline{\Gamma}^p{}_{ab} \overset{\mathscr{J}}{=} \frac{3}{2\Lambda}S\bar{T}\alpha^2\partial_p S\overline{\nabla}_a\tau\overline{\nabla}_b\tau + \frac{6S\alpha^2 + \Lambda}{6S\bar{T}}\partial_p S\overline{\nabla}_a p\overline{\nabla}_b p + \frac{3}{2\Lambda}\bar{T}S\partial_p S\overline{\nabla}_a\sigma\overline{\nabla}_b\sigma \quad . \tag{8.74}$$

and the intrinsic Ricci tensor of $\mathscr{J}$ reads

$$\overline{R}_{ab} = \frac{a^2}{2N^2}\overline{T}\left[S\partial_p^2 S + (\partial_p S)^2\right]\overline{\nabla}_a\tau\overline{\nabla}_b\tau + \frac{1}{S\overline{T}}\left[a^2 S\partial_p^2 S + \frac{1}{2}a^2\left(\partial_p S\right)^2 + \frac{1}{2}N^2\partial_p^2 S\right]\overline{\nabla}_a p\overline{\nabla}_b p$$
$$- \frac{S}{2N^2}\left[a^2 S\partial_p^2 S + a^2\left(\partial_p S\right)^2 + N^2\partial_p^2 S\right]\overline{\nabla}_a\sigma\overline{\nabla}_b\sigma \quad , \tag{8.75}$$
$$\overline{R} = -\frac{1}{2N^2}\left[4Sa^2\partial_p^2 S + 3a^2\left(\partial_p S\right)^2 + 2N^2\partial_p^2 S\right] \quad , \tag{8.76}$$

thus the intrinsic Schouten tensor follows

$$\overline{S}_{ab} \overset{\mathscr{J}}{=} \frac{3\overline{T}}{8N^2}\left[\left(\partial_p S\right)^2 a^2 - 2N^2\partial_p^2 S\right]\overline{\nabla}_a\tau\overline{\nabla}_b\tau + \frac{1}{8S\overline{T}}\left[4a^2 S\partial_p^2 S + a^2\left(\partial_p S\right)^2 + 2N^2\partial_p^2 S\right]\overline{\nabla}_a p\overline{\nabla}_b p$$
$$- \frac{S}{8N^2}\left[\left(\partial_p S\right)^2 a^2 + 2N^2\partial_p^2 S\right]\overline{\nabla}_a\sigma\overline{\nabla}_b\sigma \quad . \tag{8.77}$$

One can also obtain the electric and magnetic parts of the rescaled Weyl tensor on $\mathscr{J}$ whose non-vanishing components are

$$C_{ab} = \frac{6}{\Lambda}amS(3Sa^2 + \Lambda)\overline{\nabla}_{(a}\tau\overline{\nabla}_{b)}\sigma \quad , \tag{8.78}$$
$$D_{ab} = -\frac{m}{\Lambda}\left(9S^2a^4 + 5\Lambda Sa^2 + \frac{2}{3}\Lambda^2\right)\overline{\nabla}_a\tau\overline{\nabla}_b\tau$$
$$+ \frac{m\Lambda}{S\left(\Lambda + 3Sa^2\right)}\overline{\nabla}_a p\overline{\nabla}_b p + \frac{m}{\Lambda}S\left(\Lambda + 9Sa^2\right)\overline{\nabla}_a\sigma\overline{\nabla}_b\sigma \quad . \tag{8.79}$$

Now we make a choice of *strong orientation* –see definition 5.4.2. For that, define

$$\overset{-}{k}_\alpha \overset{\mathscr{J}}{:=} {}_1\ell_\alpha \quad , \tag{8.80}$$
$$m_\alpha \overset{\mathscr{J}}{:=} n_\alpha - \sqrt{2}\,\overset{-}{k}_\alpha = \left[N\nabla_\alpha\tau - \left(\frac{N}{T} + \frac{1}{N}\right)\nabla_\alpha q - \frac{a}{N}\nabla_\alpha p\right] \quad , \tag{8.81}$$
$$\overset{+}{k}_\alpha \overset{\mathscr{J}}{:=} \frac{1}{\sqrt{2}}\left(n_\alpha + m_\alpha\right) = \frac{1}{\sqrt{2}}\left[N\nabla_\alpha\tau - \left(\frac{N}{T} + \frac{2}{N}\right)\nabla_\alpha q - \frac{2a}{N}\nabla_\alpha p\right] \quad , \tag{8.82}$$

where, as usual, n_α is the unit version of the normal to $\mathscr{J}$, in this case given by (8.63). Notice that $\overset{-}{k}^\alpha\overset{+}{k}_\alpha = -1$, $m^\alpha\overset{-}{k}_\alpha = -1/\sqrt{2}$, $m^\alpha\overset{+}{k}_\alpha = 1/\sqrt{2}$. This choice of null directions constitutes an example of the lightlike set up of section 5.4 with m_α defining a strong orientation. The pullback of m_α to $\mathscr{J}$ is

$$m_a = N\left(\overline{\nabla}_a\tau + \frac{a}{\overline{T}}\overline{\nabla}_a p\right) \quad , \tag{8.83}$$
$$m^a = \left(-\frac{N}{\overline{T}}\delta^a_\tau - \frac{aS}{N}\delta^a_p\right) \quad . \tag{8.84}$$

The projector to the $\mathbf{S}_2$ and its 'metric' read, respectively,

$$\underline{P}_{ab} = Sa^2\overline{\nabla}_a\tau\overline{\nabla}_b\tau + \frac{N^4}{\mathring{T}^2 S}\overline{\nabla}_a p\overline{\nabla}_b p - \frac{N^2 a}{\mathring{T}}\left(\overline{\nabla}_a\tau\overline{\nabla}_b p + \overline{\nabla}_a p\overline{\nabla}_b\tau\right) + S\overline{\nabla}_a\sigma\overline{\nabla}_b\sigma \quad , \tag{8.85}$$

$$\underline{q} = \frac{1}{S}\mathrm{d}p^2 + S\mathrm{d}\sigma^2 \quad . \tag{8.86}$$

Recall that eqs. (8.83) and (8.85) characterise the projected surface $\mathbf{S}_2$ –see definition 7.0.3. After this, we can study the kinematics of m_a namely the acceleration, vorticity and expansion tensor (see eqs. (A.52) to (A.54)):

$$\underline{a}_b = 0 \quad , \tag{8.87}$$

$$\underline{\kappa}_{ab} = -\partial_p S\frac{a}{2N}\underline{q}_{ab} \quad , \tag{8.88}$$

$$\underline{\omega}_{ab} = 0 \quad . \tag{8.89}$$

Equation (8.89) tells us that m_a is surface-orthogonal, and thus defines a foliation by cuts; eq. (8.88) indicates that $\underline{\Sigma}_{ab} = 0$, therefore the cuts are umbilical; eq. (8.87) is consistent with m_a defining a foliation and in addition shows that m^a is geodesic. From eq. (A.105) we deduce that the function $1/F = m^a\overline{\nabla}_a v$ is constant on the cuts, $\underline{\mathcal{D}}_A F = 0$, where v is the parameter selecting the leaves (A.103). From eq. (8.83) one deduces

$$v = \tau + a\int\frac{1}{\underline{\underline{T}}}\mathrm{d}p \quad , \tag{8.90}$$

with F set to $F = N$. Therefore, with this choice of m_a the C-metric possesses a *strongly equipped* $\mathscr{J}$ –see definition 7.0.3. On each cut one has the following non-vanishing connection symbols

$$\mathring{\Gamma}^p{}_{pp} \overset{\mathcal{S}_v}{=} -\frac{1}{2S}\partial_p S, \quad \mathring{\Gamma}^\sigma{}_{p\sigma} \overset{\mathcal{S}_v}{=} \frac{1}{2S}\partial_p S, \quad \mathring{\Gamma}^p{}_{\sigma\sigma} \overset{\mathcal{S}_v}{=} -\frac{1}{2}S\partial_p S. \tag{8.91}$$

and the Gaussian curvature

$$\underline{K} = -\frac{1}{2}\partial_p^2 S = 1 - 6amp \quad , \tag{8.92}$$

The projections of eq. (8.77) to any cut give

$$\underline{S}_{AB} \overset{\mathcal{S}_v}{=} -\frac{1}{8N^2 S}\left[(\partial_p S)^2\alpha^2 + 2N^2\partial_p^2 S\right]\underline{\mathcal{D}}_A p\underline{\mathcal{D}}_B p - \frac{S}{8N^2}\left[a^2(\partial_p S)^2 + 2N^2\partial_p^2 S\right]\underline{\mathcal{D}}_A\sigma\underline{\mathcal{D}}_B\sigma. \tag{8.93}$$

Note that all these quantities defined on each cut of the foliation hold on $\mathscr{J}$, thus one can underline them as they belong to $\mathbf{S}_2$ (see appendix A.3). Another important consequence of having $\underaccent{\tilde}{\sigma}_{AB} = 0 = \underaccent{\tilde}{\omega}_{AB}$ is that from eqs. (7.31) to (7.35)

$$\underaccent{\tilde}{T}_{ABC} = \underaccent{\tilde}{S}_{ABC} = \underaccent{\tilde}{W}_{ABC} = 0 \quad , \tag{8.94}$$

$$\underaccent{\tilde}{L}_{AB} = \frac{1}{8}\underaccent{\tilde}{\kappa}^2 \underaccent{\tilde}{q}_{AB} \quad , \tag{8.95}$$

$$\underaccent{\tilde}{U}_{AB} = \underaccent{\tilde}{S}_{AB} + \underaccent{\tilde}{L}_{AB} = -\frac{1}{4}\partial_p^2 S \left(\frac{1}{S}\underaccent{\tilde}{\mathcal{D}}_A p \underaccent{\tilde}{\mathcal{D}}_B p + S \underaccent{\tilde}{\mathcal{D}}_A \sigma \underaccent{\tilde}{\mathcal{D}}_B \sigma \right) \quad . \tag{8.96}$$

A quick check shows that $\underaccent{\tilde}{U}^C{}_C = -\partial_p^2 S/2 = \underaccent{\tilde}{K}$. We are interested in the lightlike projections of the rescaled Weyl tensor defined in section 2.2 because they are extensively used in the search of news and very useful for computing the asymptotic radiant superenergy. The non-vanishing ones, written in our notation for congruences (7.4),(7.5), are:

$$D = -2m \quad , \tag{8.97}$$

$$\underaccent{\tilde}{C}_A = {}^{+}\underaccent{\tilde}{C}_A \overset{\mathcal{S}}{=} \frac{3}{N} a m S \underaccent{\tilde}{\mathcal{D}}_A \sigma \quad , \tag{8.98}$$

$$\underaccent{\tilde}{D}_A = {}^{+}\underaccent{\tilde}{D}_A \overset{\mathcal{S}}{=} -\frac{3}{N} a m \underaccent{\tilde}{\mathcal{D}}_A p \quad , \tag{8.99}$$

$$\grave{\underaccent{\tilde}{C}}_{AB} = \frac{1}{2}{}^{+}\underaccent{\tilde}{C}_{AB} = -\frac{9}{N^2} S a^2 m \underaccent{\tilde}{\mathcal{D}}_{(A} p \underaccent{\tilde}{\mathcal{D}}_{B)} \sigma \quad , \tag{8.100}$$

$$\grave{\underaccent{\tilde}{D}}_{AB} = \frac{1}{2}{}^{+}\underaccent{\tilde}{D}_{AB} = -\frac{3}{N^2} m a^2 \underaccent{\tilde}{\mathcal{D}}_A p \underaccent{\tilde}{\mathcal{D}}_B p + \frac{3}{N^2} S^2 a^2 m \underaccent{\tilde}{\mathcal{D}}_A \sigma \underaccent{\tilde}{\mathcal{D}}_B \sigma \quad . \tag{8.101}$$

8.2.1 Asymptotic symmetries

The metric h_{ab} at $\mathscr{J}$ inherits as KVF ∂_τ and ∂_σ which, in addition, leave invariant m_a and thus belong to the algebra of biconformal symmetries $\mathfrak{b}$ –see section 7.4. Apart from those, we can study more general asymptotic symmetries of definition 7.4.2. Because we are in the case in which $\mathscr{J}$ is strongly equipped (definition 7.0.3), we can use eq. (7.226) and $1/F = \sqrt{S}$ to write the general form of the elements of the subalgebra of bitranslations $\mathfrak{t}$:

$$\tau^a = \frac{\nu}{\sqrt{S}} m^a \quad , \tag{8.102}$$

with $\nu(v)$ an arbitrary function depending on v (8.90). As we have shown in section 7.4, the general elements of the biconformal symmetries $\mathfrak{b}$ preserving the strongly equipped $\mathscr{J}$ are the sum of an element of $\mathfrak{t}$ and an element η^a of the conformal transformations $\mathfrak{cs}$ of the projector, which in this case is given by (8.85). On each cut $\mathcal{S}_v$ we can project eq. (7.223) to give

$$\mathcal{D}_{(A}\eta_{B)} \overset{\mathcal{S}_v}{=} 2\varphi q_{AB} \quad , \tag{8.103}$$

where $\eta_B \overset{\mathcal{S}_v}{=} E^b{}_B \eta_b$ on each cut. Thus the restriction to each cut of η_b gives locally the CKVF of the flat metric q_{AB}; globally the topology rules out a subset of these vector fields. If one makes the change of variable $p = 1 - y^2$, the 2-dimensional metric has a regular KVF with a fixed point at $y = 0$. Notice that the topology of the cuts is $\mathbb{R}^2$ —since we have to remove the point at $p = -1$. By following the discussion in appendix F in [122], the topology and the periodicity of σ force the CKVFs that survive to be constructed with any periodic function $f(z)$ in the complex plane. Thus, there is still an infinite number of CKVF on each cut.

8.2.2 Asymptotic supermomentum

The asymptotic canonical super-Poynting vector and superenergy are represented in fig. 8.2 and have the following expressions:

$$\overline{\mathcal{P}}^a \stackrel{\mathscr{J}}{=} \sqrt{\frac{3}{\Lambda}} 18am^2 S \left(1 + \frac{6}{\Lambda} S a^2\right) \delta_p^a \quad , \tag{8.104}$$

$$\mathcal{W} \stackrel{\mathscr{J}}{=} 6m^2 \left(1 + \frac{54}{\Lambda^2} S^2 a^4 + \frac{18}{\Lambda} S a^2\right) \quad . \tag{8.105}$$

Observe that the super-Poynting vector field does not vanish anywhere on $\mathscr{J}$. This fact, according to criterion 1, indicates that *there is gravitational radiation at* $\mathscr{J}$. This is the expected result. Note that the canonical asymptotic super-Poynting (8.104) vanishes if and only if the acceleration parameter a is zero (which implies the absence of radiation in that case). However, the canonical superenergy density eq. (8.105) is different from zero even for $a = 0$. Another feature characterising strong orientation is eq. (5.56), which can be easily verified for the present example contracting eq. (8.104) with m_a given by eq. (8.83)

$$m_a \overline{\mathcal{P}}^a \stackrel{\mathscr{J}}{=} -\frac{54a^2 m^2 S}{\Lambda \left(a^2 S + \Lambda/3\right)} \left(1 + \frac{6}{\Lambda} S a^2\right) \leq 0 \quad . \tag{8.106}$$

Now we can take the limit to $\Lambda = 0$ –see section 5.5. For that one has to use the asymptotic supermomentum (5.34)

$$p^\alpha \stackrel{\mathscr{J}}{=} 2m^2 \left[\left(a^2 S + \frac{\Lambda}{3}\right)\left(\Lambda + 9Sa^2\right)\delta_q^\alpha + aS\left(2\Lambda + 9Sa^2\right)\delta_p^\alpha\right] \quad . \tag{8.107}$$

Then, we set $\Lambda = 0$ in eq. (8.107) which by eq. (5.79) gives the asymptotic radiant supermomentum,

$$\mathcal{Q}^\alpha \stackrel{\mathscr{J}_0}{=} 18m^2 S^2 a^3 \left(a\delta_q^\alpha + \delta_p^\alpha\right) \quad . \tag{8.108}$$

The manifestly non-vanishing asymptotic radiant supermomentum for $\Lambda = 0$ implies the presence of a non-vanishing news tensor [76] and, in consequence, that gravitational waves arrive at infinity.

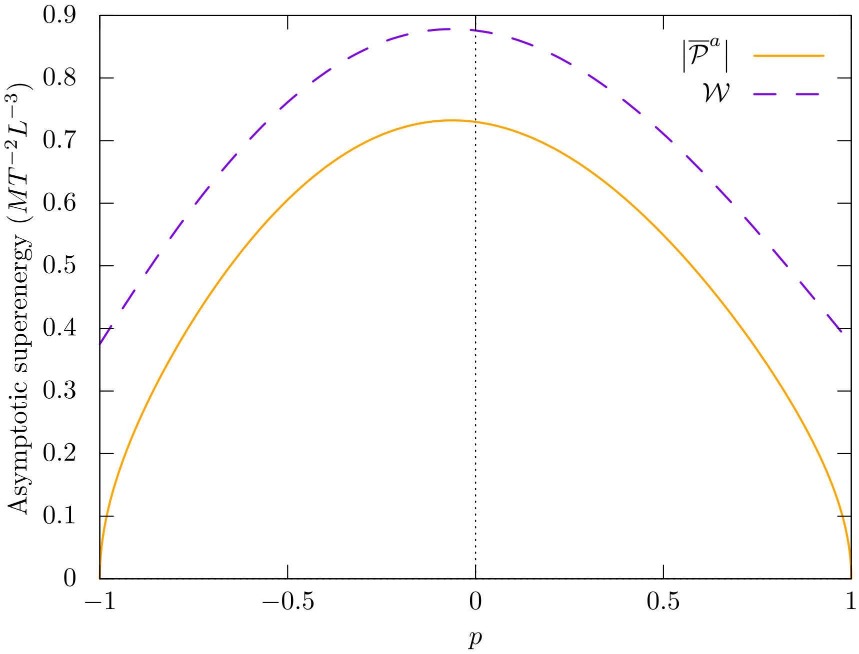

Figure 8.2: Canonical asymptotic superenergy $\mathcal{W}$ and super-Poynting vector $\overline{\mathcal{P}}^a$ for the C-metric with $\Lambda > 0$. The constant parameters have been set to $\Lambda = 1$, $a = 1/4$, $m = 1/4$.

8.2.3 Radiant quantities

We turn now to the study of the radiant asymptotic superenergy. Following sections 2.2 and 2.3, we compute the quantities associated to the $\overset{\pm}{k}{}^{\alpha}$ of eqs. (8.80) and (8.82). The procedure is straight-forward using eqs. (8.98), (8.99) and (8.101) and recalling eqs. (2.52) to (2.55) and (2.58). The non-vanishing quantities are:

$$\overset{+}{\mathcal{W}} = \frac{1296}{\Lambda}a^4m^2S^2 \quad , \tag{8.109}$$

$$\overset{+}{\mathcal{Z}} = \frac{108}{\Lambda}Sa^2m^2 \quad , \tag{8.110}$$

$$\begin{aligned}\overset{+}{\mathcal{Q}}{}^{\alpha} = &-\sqrt{\frac{3}{\Lambda}}\frac{36}{\sqrt{2}}a^2m^2\frac{S}{\overline{\overline{T}}}\delta^{\alpha}_{\tau} + 2\sqrt{2}\sqrt{\frac{3}{\Lambda}}\frac{a^2m^2}{\Lambda}\left(-54S\overline{T} - 9\Lambda S\right)\delta^{\alpha}_{q} \\ &+ \sqrt{2}108\sqrt{\frac{3}{\Lambda}}\frac{a^3m^2}{\Lambda}S^2\delta^{\alpha}_{p} \quad ,\end{aligned} \tag{8.111}$$

$$\mathcal{V} = 4m^2 \quad , \tag{8.112}$$

while $\overset{-}{\mathcal{Q}}{}^{\alpha} = 0$ –hence $\overset{-}{\mathcal{W}} = \overset{-}{\mathcal{Z}} = 0$. Another useful check is to note that

$$4\mathcal{W} - \overset{+}{\mathcal{W}} - 4\overset{+}{\mathcal{Z}} - 6\mathcal{V} \overset{\mathscr{J}}{=} 0 \quad , \tag{8.113}$$

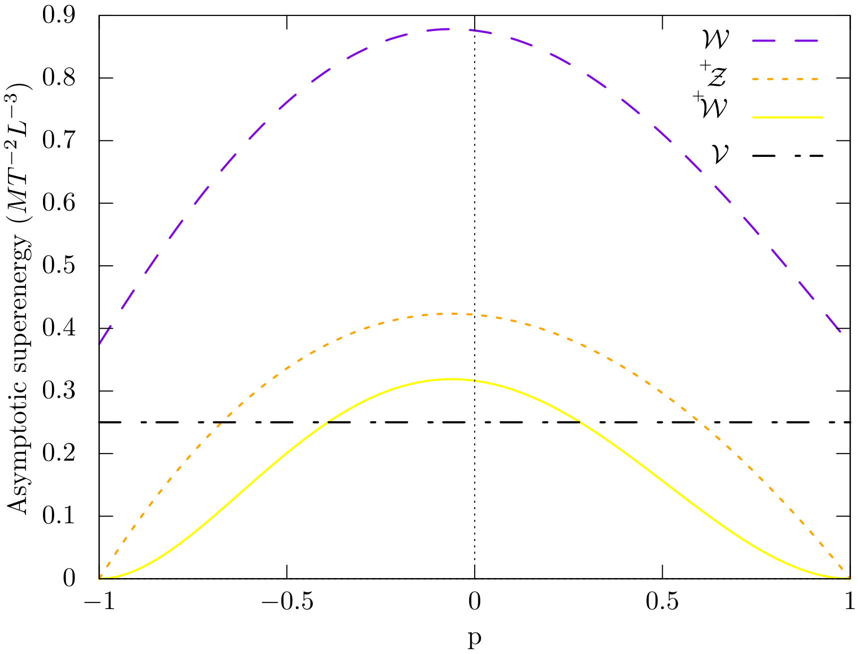

Figure 8.3: Radiant and Coulomb components of the asymptotic superenergy on $\mathscr{J}$ together with the canonical supernergy density $\mathcal{W}$ for the C-metric with $\Lambda > 0$. The constant parameters have been set to $\Lambda = 1$, $a = 1/4$, $m = 1/4$.

which shows that eq. (2.60) is satisfied. Note that criterion 2 is fulfilled too, i.e., there is no incoming radiation along m^a. Then, lemma 7.3.4 tells us that the first component of news tensor exists.

8.2.4 Radiant news

If we want to find a news tensor as proposed in section 7.2, the first thing to notice is that due to eqs. (8.94), (8.96) and (8.99) one has on each cut

$$N\mathring{\epsilon}_{BE}\overset{+}{C}{}^{E} \overset{\mathcal{S}_v}{=} N\mathring{F}_B \overset{\mathcal{S}_v}{=} -\mathcal{D}_E V_B{}^{E} \quad . \tag{8.114}$$

We know that the solution V_{AB} to this equation gives the first component of news, see proposition 6.3.1. To compute it, write eq. (8.114) explicitly in terms of the right-hand side of eq. (8.99),

$$-3am\mathcal{D}_A p \overset{\mathcal{S}}{=} -S\partial_p V_{Ap} - \frac{1}{S}\partial_\sigma V_{A\sigma} + S\mathring{\Gamma}^{C}{}_{pA}V_{Cp} + S\mathring{\Gamma}^{C}{}_{pp}V_{CA} + \frac{1}{S}\mathring{\Gamma}^{C}{}_{\sigma\sigma}V_{CA} + \frac{1}{S}\mathring{\Gamma}^{C}{}_{A\sigma}V_{C\sigma} \,. \tag{8.115}$$

We have to set V_{AB} traceless ($V_{\sigma\sigma} + S^2 V_{pp} \overset{\mathcal{S}}{=} 0$) and symmetric ($V_{p\sigma} \overset{\mathcal{S}}{=} V_{\sigma p}$) and we further assume that V_{AB} is left invariant by the axial KVF ∂_σ, that is

$$\partial_\sigma V_{AB} \overset{\mathcal{S}}{=} 0. \tag{8.116}$$

The solution reads

$$V_{p\sigma} \overset{\mathcal{S}}{=} \frac{c_1}{S} \quad \text{with} \quad c_1 = \text{constant} \quad , \tag{8.117}$$

$$V_{pp} \overset{\mathcal{S}}{=} \frac{H}{S^2}, \tag{8.118}$$

where

$$H :\overset{\mathcal{S}}{=} \int 3amSdp \overset{\mathcal{S}}{=} 3am\left(\frac{1}{2}amp^4 - \frac{1}{3}p^3 - amp^2 + p\right) + c_2 \quad \text{with} \quad c_2 = \text{constant}. \tag{8.119}$$

This function H must be positive where $S > 0$ and because we assume $a > 0$, $m > 0$. Regularity at $p = \mu$ with $\mu = \pm 1$ requires

$$c_1 \overset{\mathcal{S}}{=} 0, \quad c_2 \overset{\mathcal{S}}{=} \frac{3}{2}m^2a^2 - \mu 2am \quad , \tag{8.120}$$

and cannot be achieved on both poles, $p = -1, 1$, simultaneously. Because with our gauge fixing $p \in (-1, 1]$, we have to choose $\mu = 1$. Then,

$$\underline{V}_{AB} = \frac{H}{S^2}\underline{\mathcal{D}}_A p \underline{\mathcal{D}}_B p - H\underline{\mathcal{D}}_A \sigma \underline{\mathcal{D}}_B \sigma \quad , \tag{8.121}$$

$$\underline{V}_{AB}\bigg|_{p=1} = 0. \tag{8.122}$$

It is possible now to deduce what $\underline{\rho}_{AB}$ is:

$$\begin{aligned}\underline{\rho}_{AB} &= U_{AB} - V_{AB} \\ &= -\frac{3a^2m^2p^4 - 2amp^3 - 6a^2m^2p^2 + 18amp + 3a^2m^2 - 4am - 2}{4(p-1)(p+1)(2amp-1)}\underline{\mathcal{D}}_A p \underline{\mathcal{D}}_B p \\ &+ \frac{1}{4}(p-1)(p+1)(2amp-1)(3a^2m^2p^4 - 2amp^3 - 6a^2m^2p^2 - 6amp \\ &+ 3a^2m^2 - 4am + 2)\underline{\mathcal{D}}_A \sigma \underline{\mathcal{D}}_B \sigma.\end{aligned} \tag{8.123}$$

The radiant news tensor on each leaf of this strong equipment is simply given by $\overset{+}{\underline{n}}_{AB} = 2\underline{V}_{AB}$. Observe that

$$\overset{+}{\underline{n}}_{AB} \overset{\mathcal{S}_v}{=} 0 \quad \forall v \Longleftrightarrow \overline{\mathcal{P}}^a \overset{\mathscr{I}}{=} 0 \Longleftrightarrow {}^{+}\mathcal{Q}^\alpha = 0 \quad . \tag{8.124}$$

8.2.5 The other strong orientation

If we choose $\bar{k}^{\alpha}$ aligned with the other repeated PND, that is

$$\bar{k}_{\alpha} \stackrel{\mathscr{J}}{:=} {}_{2}\ell_{\alpha} \quad , \tag{8.125}$$

$$m_{\alpha} \stackrel{\mathscr{J}}{:=} n_{\alpha} - \sqrt{2}\bar{k}_{\alpha} = \left[-N\nabla_{\alpha}\tau - \left(\frac{N}{\overline{\overline{T}}} + \frac{1}{N}\right)\nabla_{\alpha}q - \frac{a}{N}\nabla_{\alpha}p\right] \quad , \tag{8.126}$$

$$\overset{+}{k}_{\alpha} \stackrel{\mathscr{J}}{:=} \frac{1}{\sqrt{2}}\left(n_{\alpha} + m_{\alpha}\right) = \frac{1}{\sqrt{2}}\left[-N\nabla_{\alpha}\tau - \left(\frac{N}{\overline{\overline{T}}} + \frac{2}{N}\right)\nabla_{\alpha}q - \frac{2a}{N}\nabla_{\alpha}p\right] \quad , \tag{8.127}$$

neither the asymptotic super-Poynting nor the asymptotic superenergy change, as they do not depend on this choice. The radiant superquantities $\overset{+}{\mathcal{Z}}$ and $\overset{+}{\mathcal{W}}$ in general would be different, nevertheless for the new $\overset{+}{k}^{\alpha}$ they have the same value as for the old $\overset{+}{k}^{\alpha}$; one also finds $\bar{\mathcal{Q}}^{\alpha} = 0$. There is a change in the direction of the radiant supermomentum $\overset{+}{\mathcal{Q}}^{\alpha}$ though –compare with eq. (8.111) –

$$\begin{aligned}\overset{+}{\mathcal{Q}}^{\alpha} \stackrel{\mathscr{J}}{=} & \sqrt{\frac{3}{\Lambda}}\frac{36}{\sqrt{2}}a^{2}m^{2}\frac{S}{\overline{\overline{T}}}\delta^{\alpha}_{\tau} + 2\sqrt{2}\sqrt{\frac{3}{\Lambda}}\frac{a^{2}m^{2}}{\Lambda}\left(-54S\overline{T} - 9\Lambda S\right)\delta^{\alpha}_{q} + \\ & + \sqrt{2}108\sqrt{\frac{3}{\Lambda}}\frac{a^{3}m^{2}}{\Lambda}S^{2}\delta^{\alpha}_{p} \quad .\end{aligned} \tag{8.128}$$

An intuitive interpretation of this difference is that on the first case, with $\bar{k}_{\alpha} = {}_{1}\ell_{\alpha}$, $-m_{a}$ points along the spatial propagation direction of the gravitational radiation coming from one of the two black holes, while with $\bar{k}_{\alpha} = {}_{2}\ell_{\alpha}$, $-m_{a}$ gives the propagation direction of the radiation coming from the other one. Notice that in each case the no incoming radiation condition holds, a fact that is compatible with the existence of two different propagation directions: with $\bar{k}_{\alpha} = {}_{1}\ell_{\alpha}$, criterion 2 tells that there is no radiation travelling along the spatial direction m_{α} of eq. (8.83); with $\bar{k}_{\alpha} = {}_{2}\ell_{\alpha}$, criterion 2 tells that there is no radiation travelling along the spatial direction m_{α} of eq. (8.126).

8.3 The Robinson-Trautman type N metric

We explore now the Robinson-Trautman family of solutions to vacuum EFE with a positive cosmological constant and admiting $\mathscr{J}$ –for details, see [143, 144] and also [145]. We write the conformal metric as

$$\mathrm{d}s^{2} = P^{2}\left(\mathrm{d}u\mathrm{d}\ell + \mathrm{d}\ell\mathrm{d}u - \left(2\ell^{2}H\right)\mathrm{d}u^{2} + \frac{1}{P^{2}}\left(\mathrm{d}\zeta\mathrm{d}\bar{\zeta} + \mathrm{d}\bar{\zeta}\mathrm{d}\zeta\right)\right) \quad , \tag{8.129}$$

where u is a retarded time coordinate, ℓ an inverse radius and $\zeta, \bar{\zeta}$ a couple of complex stereographic coordinates. The gauge has been chosen such that

$$\nabla_\mu N^\mu \overset{\mathscr{I}}{=} 0. \tag{8.130}$$

The metric functions are defined as

$$-2\ell^2 H := \frac{\Lambda}{3} + 2\ell\partial_u \ln P - \ell^2 K + 2m\ell^3 \quad , \tag{8.131}$$

$$K := 2P^2\partial_\zeta\partial_{\bar{\zeta}} \ln P \quad . \tag{8.132}$$

The function $P = P(u, \zeta, \bar{\zeta})$ and the function $m(u)$ satisfy the so called Robinson-Trautman equation, which is a fourth-order differential equation. Infinity is located at $\Omega := \ell = 0$, therefore the normal to $\mathscr{I}$ is

$$N_\alpha \overset{\mathscr{I}}{:=} \nabla_\alpha \ell \tag{8.133}$$

and the metric at $\mathscr{I}$ reads

$$h = N^2 P^2 \mathrm{d}u^2 + \mathrm{d}\zeta\mathrm{d}\bar{\zeta} + \mathrm{d}\bar{\zeta}\mathrm{d}\zeta \quad , \tag{8.134}$$

where for simplicity we use the same letter u to denote the restriction to $\mathscr{I}$ of the retarded time. So far, this applies to the general Robinson-Trautman Λ-vacuum solution. From now on, we concentrate on the Petrov type-N case and set the conditions that particularise the metric to that subfamily of space-times:

$$m = 0 \quad , \quad K = K(u) \quad . \tag{8.135}$$

For type N, the general solution for P is (see [145])

$$P = \frac{1}{\sqrt{\partial_{\bar{\zeta}}\bar{F}\partial_\zeta F}}\left(1 + \epsilon F\bar{F}\right), \tag{8.136}$$

with $\epsilon = -1, 0, 1$ and $F(u, \zeta)$ any function analytic on ζ. The non-vanishing components of the intrinsic connection in these coordinates are

$$\overline{\Gamma}^u{}_{au} = \partial_a \ln P \quad , \tag{8.137}$$

$$\overline{\Gamma}^\zeta{}_{uu} = -N^2 P\partial_{\bar{\zeta}} P \quad , \tag{8.138}$$

$$\overline{\Gamma}^{\bar{\zeta}}{}_{uu} = -N^2 P\partial_\zeta P \quad , \tag{8.139}$$

and the curvature and Schouten tensor are given by

$$\overline{R}_{ab} = -2N^2 P\partial_\zeta\partial_{\bar{\zeta}}P\overline{\nabla}_a u\overline{\nabla}_b u - \frac{2}{P}\partial_\zeta\partial_{\bar{\zeta}}P\overline{\nabla}_{(a}\zeta\overline{\nabla}_{b)}\bar{\zeta}$$
$$- \frac{1}{P}\partial_\zeta\partial_\zeta P\overline{\nabla}_a\zeta\overline{\nabla}_b\zeta - \frac{1}{P}\partial_{\bar{\zeta}}\partial_{\bar{\zeta}}P\overline{\nabla}_a\bar{\zeta}\overline{\nabla}_b\bar{\zeta} \quad , \tag{8.140}$$

$$\overline{R} = -\frac{4}{P}\partial_\zeta\partial_{\bar{\zeta}}P \quad , \tag{8.141}$$

$$\overline{S}_{ab} = -N^2 P\partial_\zeta\partial_{\bar{\zeta}}P\overline{\nabla}_a u\overline{\nabla}_b u - \frac{1}{P}\partial_\zeta\partial_\zeta P\overline{\nabla}_a\zeta\overline{\nabla}_b\zeta - \frac{1}{P}\partial_{\bar{\zeta}}\partial_{\bar{\zeta}}P\overline{\nabla}_a\bar{\zeta}\overline{\nabla}_b\bar{\zeta}. \tag{8.142}$$

and the electric and magnetic parts of the rescaled Weyl tensor at $\mathscr{J}$ read

$$D_{ab} = \frac{1}{N^2P^3}\partial_\zeta\left(P^2\partial_u\partial_\zeta \ln P\right)\overline{\nabla}_a\zeta\overline{\nabla}_b\zeta + \frac{1}{N^2P^3}\partial_{\bar{\zeta}}\left(P^2\partial_u\partial_{\bar{\zeta}}\ln P\right)\overline{\nabla}_a\bar{\zeta}\overline{\nabla}_b\bar{\zeta} \quad , \tag{8.143}$$

$$C_{ab} = i\frac{1}{N^2P^3}\partial_\zeta\left(P^2\partial_u\partial_\zeta \ln P\right)\overline{\nabla}_a\zeta\overline{\nabla}_b\zeta - i\frac{1}{N^2P^3}\partial_{\bar{\zeta}}\left(P^2\partial_u\partial_{\bar{\zeta}}\ln P\right)\overline{\nabla}_a\bar{\zeta}\overline{\nabla}_b\bar{\zeta} \quad . \tag{8.144}$$

There is one quadruple PND of the Weyl tensor (and hence of $d_{\alpha\beta\gamma}{}^{\delta}$), as it is of Petrov type-N, which at $\mathscr{J}$ reads

$$\underset{1}{\ell}^\alpha = -\frac{N}{P\sqrt{2}}\delta^\alpha_\ell \quad . \tag{8.145}$$

We choose strong orientation (definition 5.4.2) by setting

$$\bar{k}^\alpha \overset{\mathscr{J}}{:=} \underset{1}{\ell}^\alpha \quad , \tag{8.146}$$

$$m^\alpha \overset{\mathscr{J}}{:=} n^\alpha - \sqrt{2}\,\bar{k}^\alpha = \frac{1}{NP}\delta^\alpha_u \quad , \tag{8.147}$$

$$\overset{+}{k}{}^\alpha \overset{\mathscr{J}}{:=} \frac{1}{\sqrt{2}}\left(n^\alpha + m^\alpha\right) = -\frac{N}{\sqrt{2}P}\delta^\alpha_\ell + \frac{\sqrt{2}}{NP}\delta^\alpha_u \quad , \tag{8.148}$$

where as usual $Nn^\alpha := N^\alpha$. The pullback of m_α to $\mathscr{J}$ is

$$m^a = \frac{1}{NP}\delta^a_u \quad , m_a = NP\overline{\nabla}_a u \quad . \tag{8.149}$$

Using the connection coefficients one can compute the kinematic quantities of m_a; they read

$$\underline{\omega}_{ab} = 0, \quad \underline{\kappa}_{ab} = 0, \quad \underline{a}_b = -\underline{D}_b \ln P \quad . \tag{8.150}$$

Therefore, m_a is orthogonal to a foliation of umbilical cuts –see eqs. (A.52) to (A.54). This was expected, as the conditions in corollary 5.4.1 are met –see remarks 5.4.8 and 5.4.9. The cuts, in general, contain singularities, this depends on the choice of the function $F(u,\zeta)$ in eq. (8.136). The projector to the cuts is

$$\underline{P}_{ab} = \overline{\nabla}_a\zeta\overline{\nabla}_b\bar{\zeta} + \overline{\nabla}_a\bar{\zeta}\overline{\nabla}_b\zeta \quad . \tag{8.151}$$

With this choice of m_a, $\mathscr{J}$ is strongly equipped –see definition 7.0.3–, characterised by $(m_a, \underline{P}_{ab})$.

8.3.1 Asymptotic symmetries

The infinitesimal biconformal symmetries acting on the pair $(m_a, \underline{P}_{ab})$ are of the form

$$\xi^a = \alpha m^a + \chi^a \quad , \tag{8.152}$$

with

$$\alpha = \nu(u) P \quad , \tag{8.153}$$

$\nu(u)$ an arbitrary function of the coordinate u and the restriction χ^A of χ^a to each cut a CKVF of the flat metric. This time, the dimension of the algebra of biconformal infinitesimal symmetries can be 'doubly infinite', depending on the topology of the cuts. For instance, this is the case for $\mathbb{R}^2$ and $\mathbb{R} \times \mathbb{S}^1$ topologies, which also warrants the existence of CKVFs with fixed points –see remark 6.3.2. In general, there are no CKVF that are infinitesimal basic symmetries, what is to be expected as there are no KVFs for Robinson-Trautman type N in the generic case.

8.3.2 Asymptotic supermomentum

Gravitational waves are expected at infinity. Indeed, this is the case according to criterion 1 because the asymptotic canonical super-Poynting vector field and superenergy read

$$\overline{\mathcal{P}}^a \overset{\mathscr{J}}{=} -\frac{4}{N^4 P^7} \partial_\zeta \left(P^2 \partial_u \partial_\zeta \ln P \right) \partial_{\bar{\zeta}} \left(P^2 \partial_u \partial_{\bar{\zeta}} \ln P \right) m^a \quad , \tag{8.154}$$

$$\mathcal{W} \overset{\mathscr{J}}{=} \frac{4}{N^4 P^6} \partial_\zeta \left(P^2 \partial_u \partial_\zeta \ln P \right) \partial_{\bar{\zeta}} \left(P^2 \partial_u \partial_{\bar{\zeta}} \ln P \right) \quad , \tag{8.155}$$

thus $\overline{\mathcal{P}}^a$ is non-vanishing everywhere on $\mathscr{J}$, pointing along $-m^a$.

8.3.3 Radiant quantities

There is just one radiant quantity different from 0, as corresponds to a $d_{\alpha\beta\gamma}{}^{\delta}$ of Petrov type-N when strong orientation is chosen –see fig. 5.4:

$$\overset{+}{\mathcal{W}} \overset{\mathscr{J}}{=} \frac{16}{N^4 P^6} \partial_\zeta \left(P^2 \partial_u \partial_\zeta \ln P \right) \partial_{\bar{\zeta}} \left(P^2 \partial_u \partial_{\bar{\zeta}} \ln P \right) \quad . \tag{8.156}$$

From this expression and eq. (8.155), clearly

$$4\mathcal{W} = \overset{+}{\mathcal{W}} \quad , \tag{8.157}$$

fulfilling eq. (2.60), and by eq. (8.154)

$$\overline{P}^{a} \overset{\mathscr{J}}{=} -\mathcal{W}m^{a}, \tag{8.158}$$

which is fine with the general expression (5.55) of $\overline{\mathcal{P}}^{a}$ for algebraically special $d_{\alpha\beta\gamma}{}^{\delta}$.

8.3.4 News tensor

It is easy to see what the radiant news tensor is in this case. Since the cuts are umbilical, the tensor f_{AB} of eq. (7.15) vanishes. The same argument applies to $\underline{T}_{ABC}$ of eq. (7.34). Furthermore, because $\underline{\kappa} = 0$, from eq. (7.31) one has

$$\underline{U}_{AB} = \underline{S}_{AB}, \tag{8.159}$$

or, by means of the decomposition of corollary 7.2.3,

$$\underline{V}_{AB} + \underline{\rho}_{AB} = \underline{S}_{AB}. \tag{8.160}$$

As it is pointed out in section 8.3.1, if we assume $\mathbb{R}^2$ or $\mathbb{R}\times\mathbb{S}^1$ topology for the cuts, the existence a CKVF with a fixed point on each cut is ensured. In that case, by corollary 7.2.3, we now that a flat metric on the cuts, as it is the case, implies $\underline{\rho}_{AB} = 0$. Hence, the radiant news tensor of lemma 7.3.4, given by

$$\overset{+}{\underline{n}}_{AB} = 2\underline{V}_{AB} \tag{8.161}$$

is simply the tangent part of $\overline{S}_{ab}$ eq. (8.142) to the cuts, that is

$$\overset{+}{\underline{n}}_{AB} = -\frac{2}{P}\partial_{\zeta}\partial_{\zeta}P\underline{\mathcal{D}}_{A}\zeta\underline{\mathcal{D}}_{B}\zeta - \frac{2}{P}\partial_{\bar{\zeta}}\partial_{\bar{\zeta}}P\underline{\mathcal{D}}_{A}\bar{\zeta}\underline{\mathcal{D}}_{B}\bar{\zeta} \quad . \tag{8.162}$$

It is possible indeed to write the asymptotic canonical 1 and super-Poynting in terms of $\overset{+}{\underline{n}}_{AB}$,

$$\overline{\mathcal{P}}^{a} = -\frac{1}{N^4P^6}\left[-\partial_u P + P\partial_u\right]\left(\overset{+}{\underline{n}}_{\zeta\zeta}\right)\left[-\partial_u P + P\partial_u\right]\left(\overset{+}{\underline{n}}_{\bar{\zeta}\bar{\zeta}}\right)m^{a}. \tag{8.163}$$

$$\mathcal{W} = \frac{1}{N^4P^6}\left[-\partial_u P + P\partial_u\right]\left(\overset{+}{\underline{n}}_{\zeta\zeta}\right)\left[-\partial_u P + P\partial_u\right]\left(\overset{+}{\underline{n}}_{\bar{\zeta}\bar{\zeta}}\right). \tag{8.164}$$

From these expressions we find

$$\overset{+}{\underline{n}}_{AB} = 0 \implies \overline{\mathcal{P}}^{a} = 0, \quad \mathcal{W} = 0. \tag{8.165}$$

Camino hacia nosotros dos,
regreso
donde todo comienza.
Y tú dices:
–Volver es una forma de llegar al final.
Volver es una forma de que nada termine.

Benjamín Prado, *Límite. Todos nosotros*, 1998.

9 | Conclusions

The end of this dissertation consists of a concise set of conclusions and open questions. It is not intended to be a list of the results put forward in the memoir; a table of contents with theorems is placed at the beginning to that end. Rather, this last chapter aims at a comprehensive overview of achievements, a common canvas of ideas to display their mutual features.

Tidal characterisation of gravitational radiation at infinity

- The novel characterisation of the asymptotic structure with a non-negative cosmological constant relies on the application of tidal methods. This is a new perspective and technique in the study of asymptotics, different from traditional methods employed so far. It naturally suits the tidal nature of gravitational-wave measurements.

- Based on the rescaled Bel-Robinson tensor $\mathcal{D}_{\alpha\beta\gamma\delta}$ at infinity, the asymptotic super-momentum (which is radiant for $\Lambda = 0$) determines the presence of gravitational radiation escaping from –or entering into– the space-time. At the same time, it provides a direct connection between the existence of gravitational radiation and the algebraic classification of the rescaled Weyl tensor ${d_{\alpha\beta\gamma}}^{\delta}$ at $\mathscr{J}$.

- The radiation criteria thus defined have a neat correspondence in the two considered scenarios, $\Lambda > 0$ and $\Lambda = 0$, and share the same geometric and algebraic meaning: there is no gravitational radiation on $\mathscr{J}$ if and only if $N^{\alpha}|_{\mathscr{J}}$ is a principal vector of ${d_{\alpha\beta\gamma}}^{\delta}$ in the sense of Pirani. In fact, it is possible to take the limit from $\Lambda > 0$ to $\Lambda = 0$ explicitly. This feature exceeds the capability of traditional methods, which either do not have a direct correspondence in $\Lambda > 0$ (e.g. the news tensor for $\Lambda = 0$) or when they do, they do not tell the presence of radiation (e.g. asymptotic shear).

- The presence of gravitational radiation on the conformal boundary according to the criteria based on tidal energies does not depend on the choice of gauge nor on the observer –which is fixed by the geometry– and does not need of a choice of foliation.

- The computational effort in obtaining the asymptotic supermomentum is very low, in comparison to other characterisations which need of a suitable choice of conformal frame –e.g., in the $\Lambda = 0$, the computation of the news tensor N_{ab} as the time derivative of the asymptotic shear in a Bondi gauge, or the determination of ρ_{ab}.

The scenario with vanishing cosmological constant

- The classical criterion by means of the news tensor field N_{ab} has been shown to give the same answer as the tidal-based criterion. This served as a test of the tidal techniques.

- Indeed, the news tensor is sourced by the asymptotic radiant superenergy quantities. This fact motivated the search of the radiant news in the $\Lambda > 0$ scenario.

- The peeling behaviour has been derived from a robust geometric construction. The result is an endomorphism $L_{\alpha}{}^{\beta}$ –actually, a family of automorphisms– at the tangent space of any point in $\mathscr{J}^+$. It gives the asymptotic behaviour of physical fields approaching $\mathscr{J}^+$ along null geodesics. The endomorphism depends on the selected curve and is defined at its endpoint at $\mathscr{J}$. In particular, a nice feature emerges: the alignment of physical supermomenta in the direction of the asymptotic radiant supermomenta at leading order in their expansion along null geodesics.

- The asymptotic group of symmetries BMS emerges from the universal structure on $\mathscr{J}$, consisting of the conformal class of pairs $(\overline{g}_{ab}, N^a)$.

- The determination of the two degrees of freedom of the gravitational field does not follows alone from $(\overline{g}_{ab}, N^a)$. Another ingredient is needed, ${}^{N}\!D^{ab}\Big|_{\mathscr{J}}$.

- The news tensor N_{ab} and ρ_{ab} and the asymptotic group of symmetries hold an interplay: ρ_{ab} selects a subgroup of translations, which in combination with N_{ab} are the building blocks of the energy-momentum of the gravitational field at $\mathscr{J}$.

The scenario with positive cosmological constant

- One of the main ideas emphasised in the thesis is that any dynamics of the gravitational field at $\mathscr{J}$ must be encoded in the triplet $(\mathscr{J}, h_{ab}, D_{ab})$, in consonance with the fundamental results by Friedrich [112, 132]. The asymptotic supermomentum depends on these three elements, and the asymptotic radiation condition shows that the gravitational radiation is an interplay of the three of them.

- A general method has been presented for computing news-like tensors at $\mathscr{J}$. In particular, necessary conditions for the existence of a class of such tensors –the radiant news– has been found. The radiant news contain a first component V_{AB}

which has a very similar origin as that of the news tensor field in $\Lambda = 0$. More concretely, its existence relies on a tensor field ρ_{AB} that has been found for general Riemannian two-dimensional manifolds. The tensor V_{AB} is determined by $(\mathscr{J}, h_{ab})$ and a cut $(\mathcal{S}, q_{AB})$. There is a pair of second components, ${}^{\pm}X_{AB}$. Although having the algebraic properties of V_{AB}, their origin is different. When they exist, they are determined by D_{ab} and the extrinsic curvature of the cuts where they are defined.

- The radiant news tensors ${}^{\pm}n_{AB}$ associated to a cut, if they exist, are sourced by the asymptotic radiant superenergy quantities associated to that cut.

- The introduction of a congruence of curves on $\mathscr{J}$ serves to define a structure consisting of the conformal class of pairs $(\underline{P}_{ab}, m^a)$, where m^a is the unit vector field tangent to the curves. The structure has three degrees of specialisation –equipped, strictly equipped, and strongly equipped $\mathscr{J}$, respectively– and allows for promoting the radiant news and their components to tensor fields on $\mathscr{J}$.

- Novel symmetries are introduced as those transformations preserving a given equipment of $\mathscr{J}$. There is an interplay between those symmetries, the first component of news V_{ab} and ρ_{ab} associated to that equipment. Remarkably, ρ_{ab} serves to define a set of 'translations' on $\mathscr{J}$.

- The radiant news are determined by (h_{ab}, m^a) (first component) together with D_{ab} (second component).

- Conserved quantities can be defined using the equipments of $\mathscr{J}$ and the basic and new symmetries.

Further research and open questions

The work presented in this thesis opens the window to further research. Some of these matters and open problems are:

1. A more general class of news-like tensor in space-times with $\Lambda > 0$ can be sought by means of the general method here presented. Also, a refined study of the radiant news tensor and their connection with the radiation condition is possible. Particularly, a transport equation for ρ_{ab} along m^a would shed light on these two issues.

2. The definition of an energy-momentum at $\mathscr{J}$ with $\Lambda > 0$ is still an open problem. The exploration of conserved charges and the study of the $\Lambda = 0$ scenario suggest that a definition of momentum associated to the symmetries of an equipped $\mathscr{J}$ is plausible.

3. The exploration of symplectic methods is to be done. In particular, their application to an equipped $\mathscr{J}$ may help in the search of an energy-momentum.

4. Given an equipped $\mathscr{J}$, classes of equivalence of connections on $\mathscr{J}$ emerge. Their characterisation should be put in connection with the first component of news and with the associated symmmetries.

5. It is natural to think on applying the geometrical approach to the peeling behaviour and, in general, to the asymptotic propagation of fields in the $\Lambda > 0$ scenario. Also, it would be interesting to see if the endomorphism ${L_\alpha}^\beta$ can be derived for general curves other than null geodesics.

6. The application of the tidal approach to the $\Lambda < 0$ scenario has not been considered in this thesis and should be addressed. If the outcome is successful, one could then talk of a *universal* radiation condition at infinity.

The hope is that this work contributes to the understanding of infinity and to a deeper comprehension of gravitational radiation.

A | Geometry of spatial hypersurfaces, cuts and congruences

We introduce some geometric tools for a general 3-dimensional, spacelike hypersurface $\mathcal{I}$ embedded in a 4-dimensional space-time $\left(M, g_{\alpha\beta}\right)$. We will also consider the geometrical objects associated to a single cut $\mathcal{S}$ on $\mathcal{I}$ and to a general congruence $\mathcal{C}$ given by a vector field r^a on $\mathcal{I}$.

A.1 Induced connection

Consider a general spacelike hypersurface $\mathcal{I}$ embedded in a 4-dimensional space-time $\left(M, g_{\alpha\beta}\right)$. Let n_α be the timelike normal one-form at each point of $\mathcal{I}$ normalised to $n_\mu n_\nu g^{\mu\nu} = -1$. Also, at each point, consider a set of linearly independent tangent vector fields $\{\vec{e}_a\}$, $a = 1, 2, 3$. By definition, $n_\mu e^\mu{}_a = 0$ and $\{\vec{e}_a\}$ constitutes a basis for $\mathfrak{X}_{\mathcal{I}}$, the set of vector fields of $\mathcal{I}$. Use the inverse space-time metric to define the normal vector $n^\alpha := g^{\alpha\mu} n_\mu$. This field completes a basis, $\{\vec{n}, \vec{e}_a\}$, for the set of vector fields of M, $\mathfrak{X}_M$, at $\mathcal{I}$. Analogously, consider a set of linearly independent one-forms orthogonal to $\vec{n}$, $\{\boldsymbol{\overline{\omega}}^a\}$. They constitute a basis for the set of one-forms of $\mathcal{I}$, $\Lambda_{\mathcal{I}}$, and $\{-\boldsymbol{n}, \boldsymbol{\overline{\omega}}^a\}$, for the set of one-forms of M, Λ_M, at $\mathcal{I}$.

The hypersurface $\mathcal{I}$ is endowed with an intrinsic Riemannian metric h_{ab}, given by the pullback of the space-time metric to $\mathcal{I}$ — the first fundamental form of $\mathcal{I}$:

$$h_{ab} \stackrel{\mathcal{I}}{=} e^\mu{}_a e^\nu{}_b g_{\mu\nu} \ . \tag{A.1}$$

It is non-degenerate and its inverse is uniquely defined by

$$h^{ab} = \omega_\mu{}^a \omega_\nu{}^b g^{\mu\nu} \ . \tag{A.2}$$

The second fundamental form of $\mathcal{I}$ is defined by

$$\kappa_{ab} = e^{\mu}{}_{a} e^{\nu}{}_{b} \nabla_{\mu} n_{\nu} \ . \tag{A.3}$$

Any space-time vector field v^{α} can be decomposed into parts tangent and normal to $\mathcal{I}$,

$$v^{\alpha} \overset{\mathcal{I}}{=} -n^{\alpha} n_{\mu} v^{\mu} + \overline{v}^{\alpha}, \quad \overline{v}^{\mu} n_{\mu} = 0, \quad \overline{v}^{\alpha} = e^{\alpha}{}_{a} \overline{v}^{a} \ , \tag{A.4}$$

with $\vec{v} \in \mathfrak{X}_{\mathcal{I}}$. This decomposition and notation can be generalised to any tensor (field). The tangent part $\overline{v}^{\alpha}$ can be obtained by the action of the projector

$$P^{\alpha}{}_{\beta} := e^{\alpha}{}_{p} \omega_{\beta}{}^{p}, \quad P^{\alpha}{}_{\beta} n_{\alpha} = 0, \quad \overline{v}^{\alpha} = P^{\alpha}{}_{\mu} v^{\mu} \ . \tag{A.5}$$

Its covariant version reads

$$P_{\alpha\beta} \overset{\mathcal{I}}{=} P^{\mu}{}_{\beta} P^{\nu}{}_{\beta} g_{\mu\nu} \overset{\mathcal{I}}{=} g_{\alpha\beta} + n_{\alpha} n_{\beta} \overset{\mathcal{I}}{=} P_{\beta\alpha} \ . \tag{A.6}$$

The intrinsic volume form of $(\mathcal{I}, h_{ab})$ is determined by

$$-n_{\alpha} \epsilon_{abc} = \eta_{\alpha\mu\nu\rho} e^{\mu}{}_{a} e^{\nu}{}_{b} e^{\rho}{}_{c} \ , \tag{A.7}$$
$$-n^{\alpha} \epsilon^{abc} = \eta^{\alpha\mu\nu\rho} \omega_{\mu}{}^{a} \omega_{\nu}{}^{b} \omega_{\rho}{}^{c} \ , \tag{A.8}$$

such that $\epsilon^{abc}\epsilon_{abc} = 6$. This also fixes the orientation[1] to $\overline{\epsilon}_{123} = 1$, and $\overline{\epsilon}_{abc}$ is the canonical volume element defined by h_{ab}.

Given the space-time connection, one can define an intrinsic covariant derivative on $\mathcal{I}$ as

$$v^{m} \overline{\nabla}_{m} u^{a} \overset{\mathcal{I}}{:=} \omega_{\mu}{}^{a} v^{\nu} \nabla_{\nu} u^{\mu}, \text{ for } u^{\alpha} n_{\alpha} = v^{\alpha} n_{\alpha} = 0, \quad u^{\alpha} = e^{\alpha}{}_{a} u^{a}, \quad v^{\alpha} = e^{\alpha}{}_{a} v^{a} \ , \tag{A.9}$$

and extend this operator to act on any field on $\mathcal{I}$. For any tensor field $T^{\alpha_1 \ldots \alpha_r}{}_{\beta_1 \ldots \beta_q}$ defined *at least* on $\mathcal{I}$, one has

$$\begin{aligned} &\omega_{\mu_1}{}^{a_1} \ldots \omega_{\mu_r}{}^{a_r} e^{\nu_1}{}_{b_q} \ldots e^{\nu_q}{}_{b_q} e^{\rho}{}_{c} \nabla_{\rho} T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_q} = \overline{\nabla}_{c} \overline{T}^{a_1 \ldots a_r}{}_{b_1 \ldots b_q} \\ &- \sum_{i=1}^{r} T^{a_1 \ldots a_{i-1} \sigma a_{i+1} \ldots a_r}{}_{b_1 \ldots b_q} n_{\sigma} \kappa^{a_i}{}_{c} - \sum_{i=1}^{q} T^{a_1 \ldots a_r}{}_{b_1 \ldots b_{i-1} \sigma b_{i+1} \ldots b_q} \kappa_{c b_i} n^{\sigma} \end{aligned}$$

where $\overline{T}^{a_1 \ldots a_r}{}_{b_1 \ldots b_q} \overset{\mathcal{I}}{:=} \omega_{\mu_1}{}^{a_1} \ldots \omega_{\mu_r}{}^{a_r} e^{\nu_1}{}_{b_q} \ldots e^{\nu_q}{}_{b_q} T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_q}$. The new derivative operator is torsion-less, metric and volume preserving —the underlying connection is the Levi-Civita

[1] According to the orientation for the unphysical space-time, $\eta_{0123} = 1$. This coincides with the one we chose for the physical space-time —see the conventions at the end of chapter 1.

connection associated to h_{ab},

$$\overline{\nabla}_a h_{bc} = 0, \tag{A.10}$$

$$\overline{\nabla}_a \epsilon_{bcd} = 0 \ . \tag{A.11}$$

The intrinsic curvature is defined by means of $\overline{\nabla}_a$ as

$$\left(\overline{\nabla}_a\overline{\nabla}_b - \overline{\nabla}_b\overline{\nabla}_a\right) v_c = \overline{R}_{abc}{}^{m} v_m, \quad \boldsymbol{v} \in \Lambda_{\mathcal{I}} \tag{A.12}$$

and the intrinsic Ricci tensor and scalar curvature by $\overline{R}_{ab} := \overline{R}_{amb}{}^{m}$, $\overline{R} := \overline{R}^{m}{}_{m}$. The relation with the space-time curvature is given by the Gauss equation and its traces:

$$\overline{R}_{abc}{}^{d} = e^{\alpha}{}_{a} e^{\beta}{}_{b} e^{\gamma}{}_{c} R_{\alpha\beta\gamma}{}^{\delta} \omega_{\delta}{}^{d} + \kappa_{bc}\kappa_{a}{}^{d} - \kappa_{ac}\kappa_{b}{}^{d} \ , \tag{A.13}$$

$$\overline{R}_{ac} = e^{\alpha}{}_{a} e^{\gamma}{}_{c} R_{\alpha\gamma} + n^{\beta} n_{\delta} e^{\alpha}{}_{a} e^{\gamma}{}_{c} R_{\alpha\beta\gamma}{}^{\delta} + \kappa_{cd}\kappa_{a}{}^{d} - \kappa_{ac}\kappa \ , \tag{A.14}$$

$$\overline{R} = R + 2n^{\alpha} n^{\gamma} R_{\alpha\gamma} + \kappa_{cd}\kappa^{cd} - \kappa^2 \ , \tag{A.15}$$

with $\kappa := \kappa^{c}{}_{c}$, and the space-time curvature and the second fundamental form are related by the Codazzi equation:

$$e^{\alpha}{}_{a} e^{\beta}{}_{b} e^{\gamma}{}_{c} R_{\alpha\beta\gamma}{}^{\delta} n_{\delta} = 2\overline{\nabla}_{[a}\kappa_{b]c} \ . \tag{A.16}$$

A.2 Cuts

Let $\mathcal{S}$ be any two-dimensional submanifold embedded in $\mathcal{I}$ and assume that it has $\mathbb{S}^2$-topology. Generically, we will refer to these kind of surfaces as 'cuts'. Let r_a be the (spacelike) normal one-form to the cut within $\mathcal{I}$ —n_α is orthogonal to the cut too, of course. In a similar fashion as we have done above, we introduce a couple of linearly independent vector fields $\{E^{a}{}_{A}\}$, $A = 2, 3$, orthogonal to r_a and tangent to $\mathcal{I}$, such that they constitute a basis for the set $\mathfrak{X}_{\mathcal{S}}$ of vector fields of $\mathcal{S}$. Also, rise an index to the normal one-form using h^{ab} and define a dual basis $\{W_{a}{}^{A}\}$ orthogonal to r^a. These sets of vector fields, being completely tangent to $\mathcal{I}$, can be written as space-time fields: $r^{\alpha} \overset{\mathcal{I}}{:=} e^{\alpha}{}_{a} r^{a}$, $E^{\alpha}{}_{A} \overset{\mathcal{S}}{:=} E^{a}{}_{A} e^{\alpha}{}_{a}$ and $W_{\alpha}{}^{A} \overset{\mathcal{S}}{:=} W_{a}{}^{A} \omega_{\alpha}{}^{a}$. The triads $\{\vec{r}, \vec{E_A}\}$ and $\{\boldsymbol{r}, \boldsymbol{W}^{A}\}$ constitute a basis for $\mathfrak{X}_{\mathcal{I}}$ and $\Lambda_{\mathcal{I}}$ at $\mathcal{S}$, respectively. Pushforwards/pullbacks of intrinsic objects to $\mathcal{S}$ can be written in terms of $W_{\alpha}{}^{A}$ and $E^{\alpha}{}_{A}$.

The intrinsic metric of $\mathcal{S}$ is given by the pullback of the metric of $\mathcal{I}$ —the first fundamental form of $\mathcal{S}$,

$$q_{AB} \overset{\mathcal{S}}{:=} E^{a}{}_{A} E^{b}{}_{B} h_{ab} \ , \tag{A.17}$$

which concides with the pullback of the metric of M with $E^{\alpha}{}_{A}$,

$$q_{AB} \overset{\mathcal{S}}{=} E^{\alpha}{}_{A} E^{\beta}{}_{B} g_{\alpha\beta} \ . \tag{A.18}$$

The second fundamental form of $\mathcal{S}$ in $\mathcal{I}$ is defined as

$$\mathring{\kappa}_{AB} \overset{\mathcal{S}}{:=} E^{a}{}_{A} E^{b}{}_{B} \overline{\nabla}_{a} r_{b} \ , \tag{A.19}$$

and the projector to the cut as

$$\mathring{P}^{a}{}_{b} \overset{\mathcal{S}}{:=} E^{a}{}_{A} W_{b}{}^{A} \overset{\mathcal{S}}{=} \delta^{a}{}_{b} - r^{a} r_{b} \ . \tag{A.20}$$

Its covariant version is symmetric

$$\mathring{P}_{ab} \overset{\mathcal{S}}{=} h_{ab} - r_{a} r_{b} \tag{A.21}$$

and

$$\mathring{P}_{\alpha\beta} \overset{\mathcal{S}}{:=} \omega_{\alpha}{}^{a} \omega_{\beta}{}^{b} \mathring{P}_{ab} \overset{\mathcal{S}}{=} g_{\alpha\beta} + n_{\alpha} n_{\beta} - r_{\alpha} r_{\beta} \ . \tag{A.22}$$

Any $\vec{v} \in \mathfrak{X}_{\mathcal{I}}$ can be split into a normal and tangent part to $\mathcal{I}$ as before (see eq. (A.4)). Now, in addition to that, the tangent part to $\mathcal{I}$ is decomposed into its tangent and normal parts to $\mathcal{S}$:

$$v^{\alpha} = -n_{\mu} v^{\mu} n^{\alpha} + \overline{v}^{\alpha} = -n_{\mu} v^{\mu} n^{\alpha} + r_{\mu} v^{\mu} r^{\alpha} + \mathring{v}^{\alpha}, \quad \text{with} \quad r_{\mu} \mathring{v}^{\mu} = 0 = n_{\mu} \mathring{v}^{\mu} \ , \tag{A.23}$$

where, $\mathring{P}^{\alpha}{}_{\mu} v^{\mu} \overset{\mathcal{S}}{=} \mathring{v}^{\alpha} = \mathring{v}^{A} E^{\alpha}{}_{A}$, with $\vec{\mathring{v}} \in \mathfrak{X}_{\mathcal{S}}$.

Also, the intrinsic volume two-form of $(\mathcal{S}, q_{AB})$ is determined by

$$r_{a} \mathring{\epsilon}_{AB} \overset{\mathcal{S}}{=} \epsilon_{amn} E^{m}{}_{A} E^{n}{}_{B} \ , \tag{A.24}$$

$$r^{a} \mathring{\epsilon}^{AB} \overset{\mathcal{S}}{=} \epsilon^{amn} W_{m}{}^{A} W_{n}{}^{B} \ , \tag{A.25}$$

such that $\mathring{\epsilon}^{AB} \mathring{\epsilon}_{AB} = 2$ and fixing the orientation to $\mathring{\epsilon}_{23} = 1$. Notice that using eq. (A.7) one can write the space-time version of this two-form as $\mathring{\epsilon}_{\alpha\beta} \overset{\mathcal{S}}{=} \mathring{P}^{\sigma}{}_{\alpha} \mathring{P}^{\rho}{}_{\beta} \eta_{\mu\nu\sigma\rho} n^{\mu} r^{\nu}$, $\mathring{\epsilon}^{\alpha\beta} \overset{\mathcal{S}}{=} \mathring{P}^{\alpha}{}_{\sigma} \mathring{P}^{\beta}{}_{\rho} \eta^{\mu\nu\sigma\rho} n_{\mu} r_{\nu}$.

An intrinsic connection on the cut can be defined as

$$V^{M} \mathcal{D}_{M} U^{A} \overset{\mathcal{S}}{:=} W_{m}{}^{A} V^{n} \overline{\nabla}_{n} U^{m}, \quad \text{where} \quad U^{a} \overset{\mathcal{S}}{=} E^{a}{}_{A} U^{A}, \quad V^{a} \overset{\mathcal{S}}{=} W^{a}{}_{A} V^{A} \ . \tag{A.26}$$

Or, equivalently, by

$$V^M \mathcal{D}_M U^A \overset{\mathcal{S}}{:=} W_\mu{}^A V^\nu \nabla_\nu U^\mu, \quad \text{where} \quad U^\alpha = E^\alpha{}_A U^A, \quad V^\alpha = W^\alpha{}_A V^A \ . \tag{A.27}$$

The intrinsic covariant derivative of a tensor field $T^{a_1\ldots a_r}{}_{b_1\ldots b_q}$ defined *at least* on $\mathcal{S}$ is written as

$$\begin{aligned} &W_{m_1}{}^{A_1}\ldots W_{m_r}{}^{A_r} E^{n_1}{}_{B_q}\ldots E^{n_q}{}_{B_q} E^r{}_C \overline{\nabla}_r T^{m_1\ldots m_r}{}_{n_1\ldots n_q} = \mathcal{D}_C \mathring{T}^{A_1\ldots A_r}{}_{B_1\ldots B_q} \\ &+ \sum_{i=1}^{r} T^{A_1\ldots A_{i-1} s A_{i+1}\ldots A_r}{}_{B_1\ldots B_q} r_s \mathring{k}^{A_i}{}_C + \sum_{i=1}^{q} T^{A_1\ldots A_r}{}_{B_1\ldots B_{i-1} s B_{i+1}\ldots B_q} \mathring{k}_{CB_i} r^s \ , \end{aligned}$$

where $\mathring{T}^{A_1\ldots A_r}{}_{B_1\ldots B_q} \overset{\mathcal{S}}{:=} W_{m_1}{}^{A_1}\ldots W_{m_r}{}^{A_r} E^{n_1}{}_{B_q}\ldots E^{n_q}{}_{B_q} T^{m_1\ldots m_r}{}_{n_1\ldots n_q}$. Again, the underlying connection is the Levi-Civita connection associated to q_{AB}:

$$\mathcal{D}_A q_{BC} = 0 \ , \tag{A.28}$$

$$\mathcal{D}_A \mathring{\epsilon}_{BC} = 0 \ . \tag{A.29}$$

The Gauss equation and its traces read

$$\mathring{R}_{ABC}{}^D = E^a{}_A E^b{}_B E^c{}_C \overline{R}_{abc}{}^d W_d{}^D - \mathring{k}_{BC}\mathring{k}_A{}^D + \mathring{k}_{AC}\mathring{k}_B{}^D \ , \tag{A.30}$$

$$\mathring{R}_{AC} = E^a{}_A E^c{}_C \overline{R}_{ac} + r^b r_d E^a{}_A E^c{}_C \overline{R}_{abc}{}^d - \mathring{k}_{CD}\mathring{k}_A{}^D + \mathring{k}_{AC}\mathring{k} \quad , \tag{A.31}$$

$$\mathring{R} = \overline{R} + 2 r^a r^c \overline{R}_{ac} - \mathring{k}_{CD}\mathring{k}^{CD} + \mathring{k}^2 \quad , \tag{A.32}$$

and the Codazzi equation,

$$E^a{}_A E^b{}_B E^c{}_C \overline{R}_{abc}{}^d r_d = 2\mathcal{D}_{[A}\mathring{k}_{B]C} \ . \tag{A.33}$$

A.3 Congruences

Assume $\mathcal{I}$, or at least an open connected portion[2] $\Delta \subset \mathcal{I}$ with the same topology as $\mathcal{I}$, and let $\mathcal{C}$ be a congruence of curves there locally defined by

$$x^a = X^a\left(v, \zeta^A\right), \tag{A.34}$$

where X^a are invertible functions such that

$$v = V(x^a), \quad \zeta^A = Z^A(x^a) \ . \tag{A.35}$$

Each curve of $\mathcal{C}$ is marked by constant values of ζ^A and parametrised by v. The unit

[2]In which case, the results below apply only to that region.

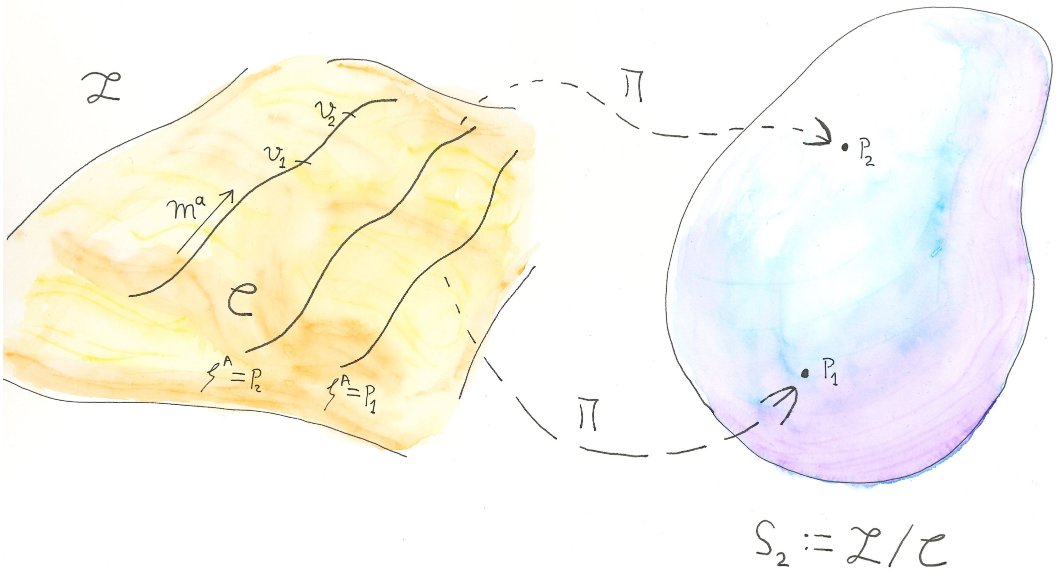

Figure A.1: The space-like hypersurface $\mathcal{I}$ equipped with a congruence $\mathcal{C}$ of curves. The canonical projection Π maps each curve to a point on the projected 'surface' $\mathbf{S}_2$.

vector field m^a tangent to the curves can be written in the local basis $(\partial/\partial x^a)$,

$$m^a = \left(h_{cd}\frac{\partial X^c}{\partial v}\frac{\partial X^d}{\partial v}\right)^{-\frac{1}{2}}\frac{\partial X^a}{\partial v}, \quad m^a m_a = 1 \; . \tag{A.36}$$

It is easily checked that $m^a\overline{\nabla}_a V \neq 0$ and $m^a\overline{\nabla}_a Z^A = 0$. Notice that there is the following freedom in reparametrising and changing the markers of the curves:

$$v \to v'\left(v, \zeta^A\right), \quad \frac{\partial v'}{\partial v} \neq 0 \; , \tag{A.37}$$

$$\zeta^A \to \zeta'^A\left(\zeta^A\right), \quad \left|\frac{\partial \zeta^A}{\partial \zeta^B}\right| \neq 0 \; . \tag{A.38}$$

The quotient $\mathbf{S}_2 := \mathcal{I}/\mathcal{C}$ is called the projected 'surface'. It is a two-dimensional differential manifold although, in general, it is not Riemannian because it is not endowed with a natural metric as it will become clear later on. One can define a canonical projection Π that maps all points on a curve of $\mathcal{C}$ to the same point on $\mathbf{S}_2$. In this sense, each point on $\mathbf{S}_2$ represents a curve of $\mathcal{C}$ and ζ^A are local coordinates on $\mathbf{S}_2$ –indeed eq. (A.38) can be regarded as a local change of coordinates on $\mathbf{S}_2$. The one-forms

$$\underline{W}_a{}^A(x) := \left(\Pi^*\left(\mathrm{d}\zeta^A\right)\right)_a = \frac{\partial Z^A(x)}{\partial x^a}, \quad m^a \underline{W}_a{}^A = 0, \quad \pounds_{\vec{m}}\underline{\boldsymbol{W}}^A = 0 \; , \tag{A.39}$$

allow us to write the pullback Π^* to $\mathcal{I}$ of any covariant tensor field $T_{A_1 \dots A_p}$ on $\mathbf{S}_2$ as

$$\underline{T}_{a_1 \dots a_p}(x) := \left[\Pi^* T(\zeta)\right]_{a_1 \dots a_p} = T_{A_1 \dots A_p}(Z(x))\, \underline{W}_{a_1}{}^{A_1}(x) \dots \underline{W}_{a_p}{}^{A_p}(x) \ . \tag{A.40}$$

The objects $\underline{T}_{a_1 \dots a_p}$ are covariant tensor fields on $\mathcal{I}$ with no dependence on v and fully orthogonal to m^a. Thus there exists an isomorphism between covariant tensor fields on $\mathcal{S}$ and covariant tensor fields on $\mathcal{I}$ that have vanishing Lie derivative along m^a and are orthogonal to m^a.

Also, one can take the push-forward Π' of any contravariant tensor field $T^{a_1 \cdots a_p}$ at a point $q \in \mathcal{I}$ to a point $\Pi(q)$ on $\mathbf{S}_2$,

$$\underline{T}^{A_1 \dots A_p}(\zeta)\bigg|_{\Pi(q)} := \left[\Pi' T(x)\right]^{A_1 \dots A_p} = \left[T^{a_1 \dots a_p}(x)\, \underline{W}_{a_1}{}^{A_1}(x) \dots \underline{W}_{a_p}{}^{A_p}(x)\right]\bigg|_q \ . \tag{A.41}$$

Because $T^{a_1 \cdots a_p}$ is defined everywhere on $\mathcal{I}$ and Π' acts pointwise, the quantities $\underline{T}^{A_1 \dots A_p}$ are well defined at each point on $\mathcal{I}$ and, thus, they can be considered as a set of *scalar fields* on $\mathcal{I}$. However, even though they change tensorially under the transformations (A.38), they do not constitute tensor fields on $\mathcal{S}_2$, in the sense that $T^{a_1 \cdots a_p}$ can give rise to different tensor fields on $\mathbf{S}_2$ due to the dependence of $\underline{T}^{A_1 \dots A_p}$ on v. Furthermore, as $T^{a_1 \cdots a_p}$ may contain transversal components along m^a, multiple tensor fields on $\mathcal{I}$ can project to the same family of scalars $\underline{T}^{A_1 \dots A_p}$. In any case, there exists an isomorphism between contravariant tensor fields on $\mathcal{I}$ completely orthogonal to m^a and with vanishing Lie derivative along m^a and contravariant tensor fields on $\mathbf{S}_2$.

We can define a couple of linearly independent vector fields on $\mathcal{I}$, $(\underline{E}^a{}_A)$, satisfying

$$m_a \underline{E}^a{}_A = 0, \quad \underline{E}^a{}_A \underline{W}_a{}^B = \delta^B_A \ . \tag{A.42}$$

Then, $(m^a, \underline{E}^a{}_A)$, $\left(m_a, \underline{W}_a{}^A\right)$ constitute a pair of dual bases. On the one hand, it is possible to lift contravariant tensor fields on $\mathbf{S}_2$ to contravariant tensor fields on $\mathscr{J}$ by

$$\underline{T}^{a_1 \dots a_p}(x) := T^{A_1 \dots A_p}(Z(x))\, \underline{E}^{a_1}{}_{A_1}(x) \dots \underline{E}^{a_p}{}_{A_p}(x) \tag{A.43}$$

which are orthogonal to m_a and have, in general, non-vanishing Lie derivative along m^a. On the other hand, given a covariant tensor field on $\mathscr{J}$, one can construct pointwise a set of scalar fields on $\mathscr{J}$ as

$$\underline{T}_{A_1 \dots A_p}(x)\bigg|_{\Pi(q)} := \left[T_{a_1 \dots a_p}(x)\, \underline{E}^{a_1}{}_{A_1}(x) \dots \underline{E}^{a_p}{}_{A_p}(x)\right]\bigg|_q \ . \tag{A.44}$$

The projector orthogonal to m^a is defined as

$$\underline{P}^a{}_b := \underline{E}^a{}_C \underline{W}_b{}^C, \quad \underline{P}^c{}_b m_c = 0 = \underline{P}^a{}_c m^c, \quad \underline{P}^c{}_c = 2 \ , \tag{A.45}$$

and in terms of m_a its covariant version reads

$$\underline{P}_{ab} = h_{ab} - m_a m_b \ . \tag{A.46}$$

This object gives a scalar product on $\mathcal{I}$ of vectors orthogonal to m^a. It is possible to introduce a *family* of inverse metric tensor fields on $\mathbf{S}_2$ as

$$\underline{q}^{AB} := \underline{W}_a{}^A \underline{W}_b{}^B h^{ab} \ , \tag{A.47}$$

while the covariant version is given by the condition $\underline{q}_{AC}\underline{q}^{BC} = \delta^B_A$. Alternatively, using $\underline{E}^a{}_A$

$$\underline{q}_{AB} = \underline{E}^a{}_A \underline{E}^b{}_B h_{ab} \ . \tag{A.48}$$

Although one can use $\underline{q}^{AB}$ and $\underline{q}_{AB}$ to rise and lower indices on $\mathbf{S}_2$, it *is not a metric tensor*, since it depends on v. More appropriately, it represents a one-parameter family of metric tensors. One can induce a 'volume two-form' in a simple way,

$$m_a \underline{\epsilon}_{AB} \stackrel{\mathcal{S}}{=} \epsilon_{amn} \underline{E}^m{}_A \underline{E}^n{}_B \ , \tag{A.49}$$

$$m^a \underline{\epsilon}^{AB} \stackrel{\mathcal{S}}{=} \epsilon^{amn} \underline{W}_m{}^A \underline{W}_n{}^B \ , \tag{A.50}$$

which is completely antisymmetric and satisfies $\underline{\epsilon}^{AB}\underline{\epsilon}_{AB} = 2$. However, this object depends on v too in general and, therefore, it constitutes a one-parameter family of volume forms. Observe that we have fixed the orientation to $\mathring{\epsilon}_{23} = 1$.

The covariant derivative on $\mathcal{I}$ of m_a is decomposed as

$$\overline{\nabla}_a m_b = m_a \underline{a}_b + \underline{\kappa}_{ab} + \underline{\omega}_{ab} \ , \tag{A.51}$$

where

$$\underline{a}_b := m^c \overline{\nabla}_c m_b \quad \text{is the acceleration,} \tag{A.52}$$

$$\underline{\kappa}_{ab} := \underline{P}^c{}_a \underline{P}^d{}_b \overline{\nabla}_{(c} m_{d)} \quad \text{is the expansion tensor,} \tag{A.53}$$

$$\underline{\omega}_{ab} := \underline{P}^c{}_a \underline{P}^d{}_b \overline{\nabla}_{[c} m_{d]} \quad \text{is the vorticity,} \tag{A.54}$$

and the shear of m_a is defined as the traceless part of $\underline{\kappa}_{ab}$,

$$\underline{\Sigma}_{ab} := \underline{\kappa}_{ab} - \frac{1}{2}\underline{P}_{ab}\underline{\kappa}, \quad \underline{\kappa} := \underline{P}^{cd}\underline{\kappa}_{cd} \ . \tag{A.55}$$

It is easy to show that

$$\pounds_{\vec{m}} \underline{E}^a{}_A = -m^a \underline{E}^c{}_A \underline{a}_c \ , \tag{A.56}$$
$$\pounds_{\vec{m}} \underline{P}^a_{\ b} = -\underline{a}_b m^a \ , \tag{A.57}$$
$$\pounds_{\vec{m}} \underline{P}_{ab} = 2\underline{\kappa}_{ab} \ . \tag{A.58}$$

Also, defining

$$\underline{\epsilon}_{ab} := m^e \underline{P}^c_{\ a} \underline{P}^d_{\ b} \epsilon_{ecd} = \underline{W}_a{}^A \underline{W}_b{}^B \underline{\epsilon}_{AB} \ , \tag{A.59}$$

and using eq. (A.57) one derives

$$\pounds_{\vec{m}} \underline{\epsilon}_{ab} = \underline{\kappa} \underline{\epsilon}_{ab} \ . \tag{A.60}$$

Incidentally,

$$\pounds_{\vec{m}} \epsilon_{abc} = \underline{\kappa} \epsilon_{abc} \ . \tag{A.61}$$

As all the kinematic tensors are orthogonal to m^a, they are univocally determined by the one-parameter family of scalar fields on $\mathcal{I}$ –which can be seen as objects on $\mathbf{S}_2$–

$$\underline{a}_A := \underline{E}^a{}_A \underline{a}_a \ , \tag{A.62}$$
$$\underline{\kappa}_{AB} := \underline{E}^a{}_A \underline{E}^b{}_B \underline{\kappa}_{ab} \ , \tag{A.63}$$
$$\underline{\Sigma}_{AB} := \underline{E}^a{}_A \underline{E}^b{}_B \underline{\Sigma}_{ab} \ , \tag{A.64}$$
$$\underline{\omega}_{AB} := \underline{E}^a{}_A \underline{E}^b{}_B \underline{\omega}_{ab} \ . \tag{A.65}$$

The scalar fields $T^{A_1 \dots A_q}{}_{B_1 \dots B_p}$ associated to an arbitrary tensor field $T^{a_1 \dots a_q}{}_{b_1 \dots b_p}$ on $\mathcal{I}$ can be differentiated along m^a:

$$\begin{aligned}
\pounds_{\vec{m}} T^{A_1 \dots A_q}{}_{B_1 \dots B_p} &= m^j \overline{\nabla}_j T^{A_1 \dots A_q}{}_{B_1 \dots B_p} \\
&= \underline{E}^{b_1}{}_{B_1} \dots \underline{E}^{b_p}{}_{B_p} \underline{W}_{a_1}{}^{A_1} \dots \underline{W}_{a_q}{}^{A_q} \pounds_{\vec{m}} \left(T^{a_1 \dots a_q}{}_{b_1 \dots b_p} \right) \\
&- \sum_{i=1}^{p} T^{A_1 \dots A_q}{}_{B_1 \dots B_{i-1} \sigma B_{i+1} \dots B_p} m^\sigma \underline{a}_{B_i} \ ,
\end{aligned} \tag{A.66}$$

where we have used the Leibniz property of the Lie derivative together with eqs. (A.39) and (A.56). Then, the Lie derivative along m^a of the one-parameter family of 'metrics' $\underline{q}_{AB}$ for fixed A, B can be computed to give

$$\pounds_{\vec{m}} \underline{q}_{AB} = \underline{\kappa}_{AB}. \tag{A.67}$$

Now, let U be a function such that

$$m^c s_c = 1, \quad s_a := (\mathrm{d}U)_a \tag{A.68}$$

and expand s_a in the $\left(m_a, \underline{W}_a{}^A\right)$ basis,

$$s_a = m_a + M_A \underline{W}_a{}^A \ . \tag{A.69}$$

Taking the Lie derivative along m^a of the functions $\underline{M}_A$ for fixed A one finds

$$\underline{a}_A = -\pounds_{\vec{m}} M_A . \tag{A.70}$$

Instead, if one takes de exterior derivative of s_a –which vanishes by definition– one gets the relation

$$\left[\vec{\underline{E}}_A, \vec{\underline{E}}_B\right]^a = -2\underline{\omega}_{AB} m^a, \tag{A.71}$$

which allows us to derive

$$\pounds_{\vec{\underline{E}}_A} \underline{\boldsymbol{W}}^B = 0. \tag{A.72}$$

Also,

$$\pounds_{\vec{\underline{E}}_A} m_a = -\underline{a}_A m_a + 2\underline{\omega}_{AC} \underline{W}_a{}^C. \tag{A.73}$$

So far we have not introduced a connection on $\mathbf{S}_2$, nor a covariant derivative. Note that in the basis $(m^a, \underline{E}^a{}_A)$ one has

$$\underline{E}^c{}_A \overline{\nabla}_c \underline{E}^a{}_B = -\left(\underline{\kappa}_{AB} + \underline{\omega}_{AB}\right) m^a + \underline{\gamma}^C{}_{AB} \underline{E}^a{}_C \ , \tag{A.74}$$

where $\underline{\gamma}^C{}_{AB}$ are functions such that $\underline{\gamma}^C{}_{AB} = \underline{\gamma}^C{}_{BA}$, as one can check computing the commutator and using eq. (A.71). Taking this into account it follows that

$$m^c \overline{\nabla}_c \underline{E}^a{}_A = -\underline{a}_A m^a + \left(\underline{\kappa}_A{}^C + \underline{\omega}_A{}^C\right) \underline{E}^a{}_C \ . \tag{A.75}$$

In addition, it can be shown that

$$\underline{E}^c{}_A \overline{\nabla}_c \underline{W}_a{}^B = -\left(\underline{\kappa}_A{}^B + \underline{\omega}_A{}^B\right) m_a - \underline{\gamma}^B{}_{AC} \underline{W}_a{}^C \ . \tag{A.76}$$

Contracting eq. (A.60) with $\underline{E}^a{}_A \underline{E}^b{}_B$ and using eq. (A.66) one derives

$$\pounds_{\vec{m}} \underline{\epsilon}_{AB} = \underline{\kappa} \underline{\epsilon}_{AB}. \tag{A.77}$$

Under the change in eq. (A.38) , $\underline{\gamma}^C{}_{AB}$ behaves like a connection. However, due to the dependence on v, it *is not a connection*, but a one-parameter family of such objects. Nevertheless, we can define a 'covariant derivative' operator by

$$\underline{\mathcal{D}}_A v^B := \underline{E}^a{}_A \partial_a v^B + \underline{\gamma}^B{}_{AC} v^C, \text{ with } v^A = v^a \underline{W}_a{}^A, \quad v^a m_a = 0. \tag{A.78}$$

For the same reasons stated above, this is not a tensor field on $\mathbf{S}_2$. The definition can be extended to arbitrary-rank contravariant and covariant tensor objects. Its relation to the covariant derivative on $\mathcal{I}$ acting on a tensor field $T^{a_1 \dots a_r}{}_{b_1 \dots b_q}$ is written as

$$
\begin{aligned}
&\underline{W}_{m_1}{}^{A_1} \dots \underline{W}_{m_r}{}^{A_r} \underline{E}^{n_1}{}_{B_q} \dots \underline{E}^{n_q}{}_{B_q} \underline{E}^{r}{}_{C} \overline{\nabla}_r T^{m_1 \dots m_r}{}_{n_1 \dots n_q} = \underline{\mathcal{D}}_C \underline{T}^{A_1 \dots A_r}{}_{B_1 \dots B_q} \\
&+ \sum_{i=1}^{r} T^{A_1 \dots A_{i-1} s A_{i+1} \dots A_r}{}_{B_1 \dots B_q} m_s \left(\underline{\kappa}_C{}^{A_i} + \underline{\omega}_C{}^{A_i} \right) + \sum_{i=1}^{q} T^{A_1 \dots A_r}{}_{B_1 \dots B_{i-1} s B_{i+1} \dots B_q} \left(\underline{\kappa}_{C B_i} + \underline{\omega}_{C B_i} \right) m^s .
\end{aligned}
\tag{A.79}
$$

Then, for $T^{m_1 \dots m_r}{}_{n_1 \dots n_q}$ completely orthogonal to m^a and m_a one has

$$
\underline{W}_{m_1}{}^{A_1} \dots \underline{W}_{m_r}{}^{A_r} \underline{E}^{n_1}{}_{B_q} \dots \underline{E}^{n_q}{}_{B_q} \underline{E}^{r}{}_{C} \overline{\nabla}_r T^{m_1 \dots m_r}{}_{n_1 \dots n_q} = \underline{\mathcal{D}}_C \underline{T}^{A_1 \dots A_r}{}_{B_1 \dots B_q} \ . \tag{A.80}
$$

This 'covariant derivative' is 'metric' and 'volume-preserving' in the sense that

$$
\underline{\mathcal{D}}_A \underline{\epsilon}_{BC} = 0 \ , \tag{A.81}
$$
$$
\underline{\mathcal{D}}_A \underline{q}_{BC} = 0 \ , \tag{A.82}
$$

and a typical calculation leads to an expression in terms of $\underline{q}_{AB}$

$$
\underline{\gamma}^C{}_{AB} = \frac{1}{2} \underline{q}^{CD} \left(\underline{E}^a{}_A \partial_a \underline{q}_{BD} + \underline{E}^b{}_B \partial_b \underline{q}_{AD} - \underline{E}^d{}_D \partial_d \underline{q}_{AB} \right) \ . \tag{A.83}
$$

Define a one-parameter family of tensor fields on $\mathbf{S}_2$ by

$$
\underline{R}_{BAC}{}^D := \underline{E}^a{}_A \partial_a \underline{\gamma}^D{}_{BC} - \underline{E}^b{}_B \partial_b \underline{\gamma}^D{}_{AC} + \underline{\gamma}^D{}_{AE} \underline{\gamma}^E{}_{BC} - \underline{\gamma}^D{}_{BE} \underline{\gamma}^E{}_{AC}, \tag{A.84}
$$

which by construction has the symmetries

$$
\underline{R}_{ABC}{}^D = -\underline{R}_{BAC}{}^D, \quad \underline{R}_{ABC}{}^D + \underline{R}_{BCA}{}^D + \underline{R}_{CAB}{}^D = 0 \ . \tag{A.85}
$$

A direct calculation gives

$$
\left(\underline{\mathcal{D}}_A \underline{\mathcal{D}}_B - \underline{\mathcal{D}}_B \underline{\mathcal{D}}_A \right) V^D = -\underline{R}_{ABC}{}^D V^C - 2 \underline{\omega}_{AB} \pounds_{\vec{m}} V^D \ , \tag{A.86}
$$

where $\pounds_{\vec{m}} V^C$ is computed according to eq. (A.66). One can define a covariant version,

$$
\underline{R}_{ABCD} := \underline{q}_{ED} \underline{R}_{ABC}{}^E . \tag{A.87}
$$

Note that this object does not have the antisymmetry property in the second pair of indices:

$$
\underline{R}_{AB(CD)} = 2 \underline{\omega}_{AB} \underline{\kappa}_{CD} \ , \tag{A.88}
$$

where we have used eqs. (A.81) and (A.86). Hence

$$
\underline{R}_{ABCD} = 2 \underline{\omega}_{AB} \underline{\kappa}_{CD} + \underline{R}_{AB[CD]} \ . \tag{A.89}
$$

The second term, since we are in 2 dimensions, can be identically written as

$$\underline{R}_{AB[CD]} = \underline{K}\left(\underline{q}_{AC}\underline{q}_{BD} - \underline{q}_{AD}\underline{q}_{BC}\right) , \tag{A.90}$$

for some scalar function $\underline{K}$. The relation between the curvature tensor on $\mathcal{I}$ and this one-parameter family of 'curvature tensors' on $\mathbf{S}_2$ can be determined by typical calculations. The result is a Gauss-like relation,

$$\begin{aligned}\underline{R}_{ABC}{}^{D} =& E^{a}{}_{A}E^{b}{}_{B}E^{c}{}_{C}\overline{R}_{abc}{}^{d}W_{d}{}^{D} + 2\underline{\omega}_{AB}\left(\underline{\kappa}_{C}{}^{D} + \underline{\omega}_{C}{}^{D}\right) - \left(\underline{\kappa}_{BC} + \underline{\omega}_{BC}\right)\left(\underline{\kappa}_{A}{}^{D} + \underline{\omega}_{A}{}^{D}\right)\\ &+ \left(\underline{\kappa}_{AC} + \underline{\omega}_{AC}\right)\left(\underline{\kappa}_{B}{}^{D} + \underline{\omega}_{B}{}^{D}\right) ,\end{aligned} \tag{A.91}$$

which lowering an index can be written also as

$$\begin{aligned}\underline{R}_{AB[CD]} =& E^{a}{}_{A}E^{b}{}_{B}E^{c}{}_{C}\overline{R}_{abcd}E^{d}{}_{D} + 2\underline{\omega}_{AB}\underline{\omega}_{CD} - \left(\underline{\kappa}_{BC} + \underline{\omega}_{BC}\right)\left(\underline{\kappa}_{AD} + \underline{\omega}_{AD}\right)\\ &+ \left(\underline{\kappa}_{AC} + \underline{\omega}_{AC}\right)\left(\underline{\kappa}_{BD} + \underline{\omega}_{BD}\right) ,\end{aligned} \tag{A.92}$$

and a Codazzi-like equation

$$\underline{E}^{a}{}_{A}\underline{E}^{b}{}_{B}\underline{E}^{c}{}_{C}\overline{R}_{abc}{}^{d}m_{d} = 2\underline{\mathcal{D}}_{[A}\left(\underline{\kappa}_{B]C} + \underline{\omega}_{B]C}\right) + 2\underline{\omega}_{AB}\underline{a}_{C} . \tag{A.93}$$

Now we are going to give an expression for the intrinsic Schouten tensor on $\mathcal{I}$. Equation (5.9) is valid in general for dimension 3, i.e., valid for $\mathcal{I}$,

$$\overline{R}_{abcd} = 2h_{a[c}\overline{S}_{d]b} - 2h_{b[c}\overline{S}_{d]a} . \tag{A.94}$$

Using this expression in eq. (A.93) one arrives at

$$2\underline{q}_{C[A}\underline{S}_{B]} = 2\underline{\mathcal{D}}_{[A}\left(\underline{\kappa}_{B]C} + \underline{\omega}_{B]C}\right) + 2\underline{\omega}_{AB}\underline{a}_{C} , \tag{A.95}$$

which is equivalent to its trace,

$$\underline{S}_{B} = \mathcal{D}_{C}\left(\underline{\kappa}_{B}{}^{C} + \underline{\omega}_{B}{}^{C}\right) - \underline{\mathcal{D}}_{B}\underline{\kappa} + 2\underline{\omega}_{CB}\underline{a}^{C} . \tag{A.96}$$

Notice also that using the same relations and contracting with $\underline{q}^{AC}\underline{q}^{BD}$ in eq. (A.90) one gets

$$\underline{S}^{E}{}_{E} = \underline{K} + \frac{1}{2}\underline{\Sigma}^{2} - \frac{1}{4}\underline{\kappa}^{2} - \frac{3}{2}\underline{\omega}^{2} . \tag{A.97}$$

A direct calculation together with eqs. (A.51) and (A.93) leads to

$$\pounds_{\vec{m}}\underline{\gamma}^{C}{}_{AB} = 2\underline{\mathcal{D}}_{(A}\underline{\kappa}_{B)}{}^{C} - \underline{q}^{EC}\underline{\mathcal{D}}_{E}\underline{\kappa}_{AB} + \underline{a}^{C}\underline{\kappa}_{AB} - 2\underline{a}_{(A}\underline{\kappa}_{B)}{}^{C} . \tag{A.98}$$

This last equation provides a condition for the vanishing of $m^{e}\partial_{e}\underline{\gamma}^{C}{}_{AB}$ which appears below in eq. (A.101). In general $\mathbf{S}_2$ is endowed with a one-parameter family of geometrical objects; only in the cases in which these quantities have vanishing derivative along m^a –i.e., when they do not depend on v– they are a true metric, connection or volume form, respectively. Summarising, from eqs. (A.67), (A.77) and (A.98),

$$\pounds_{\vec{m}} \underline{q}_{AB} = 0 \Longleftrightarrow \underline{\kappa}_{AB} = 0 \ , \tag{A.99}$$

$$\pounds_{\vec{m}} \underline{\epsilon}_{AB} = 0 \Longleftrightarrow \underline{\kappa} = 0 \ , \tag{A.100}$$

$$\pounds_{\vec{m}} \underline{\gamma}^{C}{}_{AB} = 0 \Longleftrightarrow \underline{\mathcal{D}}_C \underline{\kappa}_{AB} = \underline{a}_C \underline{\kappa}_{AB} \ . \tag{A.101}$$

Notice that $\underline{\kappa} = 0$ is not a conformally-invariant equation; one can always achieve this condition by a conformal transformation of h_{ab}[3]. Observe that, for $\underline{\kappa} = 0$, $\underline{\kappa}_{AB} = 0$ if and only if m^a is *shear-free*[4], i.e., $\underline{\Sigma}_{AB} = 0$ –see eq. (A.55). Additionally, in that case, the condition on the right-hand side of eq. (A.101) is trivially satisfied. Hence, for umbilical m^a there is a conformal class of metrics $\{h_{ab}\}$ for which $\pounds_{\vec{m}} \underline{q}_{AB} = \pounds_{\vec{m}} \underline{\epsilon}_{AB} = \pounds_{\vec{m}} \underline{\gamma}^{C}{}_{AB} = 0$.

Finally, there is a particular case of interest:

$$\underline{\omega}_{AB} = 0 \Longleftrightarrow \underline{m}^a \text{ orthogonal to cuts.} \tag{A.102}$$

This is the case of a foliation, in which each leaf is a surface (a cut). Under this condition, the normal form can always be written as

$$m_a = F \overline{\nabla}_a v \tag{A.103}$$

for some scalar function F such that

$$\frac{1}{F} = \pounds_{\vec{m}} v. \tag{A.104}$$

The calculation of the acceleration produces

$$\underline{a}_b = -\underline{P}^c{}_b \overline{\nabla}_c \ln F \ . \tag{A.105}$$

Let us point out that the geometrical objects induced by m^a still depend on v and coincide on each leaf (v =constant) with the intrinsic geometric quantities of the cuts, but only there. In general they are fields on $\mathcal{I}$ associated to the particular family of curves.

To end up with this appendix, let us mention that a very similar construction for congruences as the one above can be developed using the so called Cattaneo operator in

[3] A conformal transformation $h_{ab} \to \Psi^2 h_{ab}$ implies $m_a \to \Psi m_a$ according to eq. (A.36), as well as $\underline{q}_{AB} \to \underline{\Psi}^2 \underline{q}_{AB}$, with $\underline{\Psi} := \Pi^*(\Psi)$.

[4] Note that the shear-free property is a conformally-invariant property.

substitution of the derivative operator $\underline{\mathcal{D}}_A$,

$$\underline{\mathcal{D}}_c \underline{T}^{a_1 \ldots a_r}{}_{b_1 \ldots b_q} := \underline{P}_{m_1}{}^{a_1} \ldots \underline{P}_{m_r}{}^{a_r} \underline{P}^{n_1}{}_{b_q} \ldots \underline{P}^{n_q}{}_{b_q} \underline{P}^{r}{}_{c} \overline{\nabla}_r T^{m_1 \ldots m_r}{}_{n_1 \ldots n_q} \ , \tag{A.106}$$

which is defined for arbitrary tensor fields $T^{a_1 \ldots a_r}{}_{b_1 \ldots b_q}$ on $\mathcal{I}$.

B | Bianchi identities

Assume that a general congruence of curves with tangent vector field m^a exists, and define $m^\alpha := e^\alpha{}_a m^a$ at $\mathscr{J}$ and on a neighbourhood —this allows us to take its derivative along n^α, though no particular extension of m^α is required. The plan is to write de components of eq. (3.84) in terms of the lightlike projections of the rescaled Weyl tensor, i.e., the quantities appearing in section 2.2. Objects that carry an over-ring will be substituted by objects carrying an underbar, for the same reason explained in section 5.4. Also, quantities originally defined with indices A, B, C, etc will be written fully/partially with space-time indices α, β, γ, etc indicating that they have been contracted with $W_\alpha{}^A$ and $E^\alpha{}_A$ conveniently. The same mixed notation can appear in space-time tensors that have been contracted in some of their indices. As a couple of examples, one may find $\underline{C}_A$ in substitution of $\mathring{C}_A$ and $\underline{C}_\alpha$ in correspondence to $\underline{C}_A W_\alpha{}^A$, or $y_{\alpha AB}$ corresponding to $\underline{W}^\mu{}_A \underline{W}^\nu{}_B y_{\alpha\mu\nu}$.

Then, recast eq. (3.84) as

$$\begin{aligned}
-y_{\alpha\beta\gamma} &= g^{\mu\tau}\nabla_\mu d_{\alpha\beta\gamma\tau} = \left(-{}^{+}k^\mu\,{}^{-}k^\tau - {}^{-}k^\mu\,{}^{+}k^\tau + \underline{P}^{\mu\tau}\right)\nabla_\mu d_{\alpha\beta\gamma\tau} \\
&= -{}^{+}k^\mu\nabla_\mu\left(d_{\alpha\beta\gamma\tau}\,{}^{-}k^\tau\right) - d_{\alpha\beta\gamma\tau}\,{}^{-}k^\tau\,{}^{+}k_\lambda\,{}^{+}k^\mu\nabla_\mu\,{}^{-}k^\lambda + d_{\alpha\beta\gamma\tau}\underline{P}^\tau{}_\lambda\,{}^{+}k^\mu\nabla_\mu\,{}^{-}k^\lambda \\
&\quad - {}^{-}k^\mu\nabla_\mu\left(d_{\alpha\beta\gamma\tau}\,{}^{+}k^\tau\right) - d_{\alpha\beta\gamma\tau}\,{}^{+}k^\tau\,{}^{-}k_\lambda\,{}^{-}k^\mu\nabla_\mu\,{}^{+}k^\lambda + d_{\alpha\beta\gamma\tau}\underline{P}^\tau{}_\lambda\,{}^{-}k^\mu\nabla_\mu\,{}^{+}k^\lambda \\
&\quad + \underline{P}^{\mu\tau}\nabla_\mu\left(d_{\alpha\beta\gamma\tau}\right) \qquad \text{(B.1)}
\end{aligned}$$

here ${}^{\pm}k^\alpha$ are defined as in eq. (5.47), noting that this time they are extended outside $\mathscr{J}$. Next step is to contract this equation with ${}^{\pm}k^\alpha$ and the basis spanning the space of vectors orthogonal to m_a, $\{\underline{E}^\alpha{}_A\}$ —uppercase, Latin indices denote projections with this basis. This process is a straight-forward calculation. It is very long, though, and we just write down here the final outcome evaluated at $\mathscr{J}$:

$$
\begin{aligned}
-{}^{-}k^{\alpha}\,{}^{+}k^{\beta}\,{}^{-}k^{\gamma}y_{\alpha\beta\gamma} \overset{\mathscr{J}}{=}\; & -{}^{-}k^{\mu}\nabla_{\mu}D-2\sqrt{2}\,{}^{+}\underline{D}_{\gamma}\,{}^{-}k^{\mu}\nabla_{\mu}{}^{-}k^{\gamma} \\
& +2\sqrt{2}\,{}^{-}\underline{D}_{\beta}\,{}^{-}k^{\mu}\nabla_{\mu}{}^{+}k^{\beta}+\sqrt{2}\underline{P}^{\mu\tau}\nabla_{\mu}{}^{-}\underline{D}_{\tau}-\underline{P}^{\mu\tau}D\nabla_{\mu}{}^{-}k_{\tau} \\
& -{}^{-}\underline{D}_{\beta\tau}\underline{P}^{\mu\tau}\nabla_{\mu}{}^{+}k^{\beta}+\sqrt{2}\,{}^{-}\underline{D}^{\mu}\,{}^{-}k^{\beta}\nabla_{\mu}{}^{+}k_{\beta} \\
& +{}^{-+}\underline{D}_{\tau\alpha}\underline{P}^{\tau\mu}\nabla_{\mu}{}^{-}k^{\alpha}-2\,{}^{-+}\underline{D}_{[\gamma\tau]}\underline{P}^{\mu\tau}\nabla_{\mu}{}^{-}k^{\gamma}\quad,
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
-{}^{-}k^{\alpha}\,{}^{+}k^{\beta}\,{}^{+}k^{\gamma}y_{\alpha\beta\gamma} \overset{\mathscr{J}}{=}\; & -2\sqrt{2}\,{}^{-}\underline{D}_{\gamma}\,{}^{+}k^{\mu}\nabla_{\mu}{}^{+}k^{\gamma}+2\sqrt{2}\,{}^{+}\underline{D}_{\alpha}\,{}^{+}k^{\mu}\nabla_{\mu}{}^{-}k^{\alpha}+\sqrt{2}\underline{P}^{\mu\tau}\nabla_{\mu}{}^{+}\underline{D}_{\tau} \\
& D\underline{P}^{\mu\tau}\nabla_{\mu}{}^{+}k_{\tau}+\sqrt{2}\,{}^{+}\underline{D}^{\mu}\,{}^{+}k^{\alpha}\nabla_{\mu}{}^{-}k_{\alpha}+{}^{+}\underline{D}_{\alpha\tau}\underline{P}^{\mu\tau}\nabla_{\mu}{}^{-}k^{\alpha} \\
& -{}^{-+}\underline{D}_{\beta\tau}\underline{P}^{\mu\tau}\nabla_{\mu}{}^{+}k^{\beta}-{}^{-+}\underline{D}_{[\gamma\tau]}\underline{P}^{\mu\tau}\nabla_{\mu}{}^{+}k^{\gamma}+{}^{+}k^{\mu}\nabla_{\mu}D\quad,
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
-{}^{-}k^{\beta}\,{}^{+}k^{\gamma}\underline{y}_{A\beta\gamma} \overset{\mathscr{J}}{=}\; & -\sqrt{2}\underline{E}^{\omega}{}_{A}\,{}^{+}k^{\mu}\nabla_{\mu}{}^{-}\underline{D}_{\omega}+D\,{}^{+}k^{\mu}\underline{E}^{\omega}{}_{A}\nabla_{\mu}{}^{-}k_{\omega}+{}^{-}\underline{D}_{A\gamma}\,{}^{+}k^{\mu}\nabla_{\mu}{}^{+}k^{\gamma} \\
& -\sqrt{2}\,{}^{-}\underline{D}_{A}\,{}^{+}k^{\lambda}\,{}^{+}k^{\mu}\nabla_{\mu}{}^{-}k_{\lambda}+2\,{}^{-+}\underline{D}_{[\beta A]}\,{}^{+}k^{\mu}\nabla_{\mu}{}^{-}k^{\beta}-{}^{-+}\underline{D}_{A\lambda}\,{}^{+}k^{\mu}\nabla_{\mu}{}^{-}k^{\lambda} \\
& -\underline{P}^{\mu\tau}\underline{E}^{\omega}{}_{A}\nabla_{\mu}{}^{-+}\underline{D}_{\omega\tau}-\sqrt{2}\,{}^{-}\underline{D}_{A}\underline{P}^{\mu\tau}\nabla_{\mu}{}^{+}k_{\tau}+\sqrt{2}\,{}^{+}\underline{D}_{\tau}\underline{P}^{\mu\tau}\underline{E}^{\omega}{}_{A}\nabla_{\mu}{}^{-}k_{\omega} \\
& -{}^{-}\underline{t}_{\gamma\tau A}\underline{P}^{\mu\tau}\nabla_{\mu}{}^{+}k^{\gamma}+{}^{+}\underline{t}_{A\beta\tau}\underline{P}^{\mu\tau}\nabla_{\mu}{}^{-}k^{\beta}\quad,
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
-{}^{+}k^{\beta}\,{}^{-}k^{\gamma}\underline{y}_{A\beta\gamma} \overset{\mathscr{J}}{=}\; & \sqrt{2}\underline{E}^{\omega}{}_{A}\,{}^{-}k^{\mu}\nabla_{\mu}{}^{+}\underline{D}_{\omega}+D\,{}^{-}k^{\mu}\underline{E}^{\omega}{}_{A}\nabla_{\mu}{}^{+}k_{\omega}+{}^{+}\underline{D}_{A\gamma}\,{}^{-}k^{\mu}\nabla_{\mu}{}^{-}k^{\gamma} \\
& +\sqrt{2}\,{}^{+}\underline{D}_{A}\,{}^{-}k^{\lambda}\,{}^{-}k^{\mu}\nabla_{\mu}{}^{+}k_{\lambda}+2\,{}^{-+}\underline{D}_{[A\beta]}\,{}^{-}k^{\mu}\nabla_{\mu}{}^{+}k^{\beta}-{}^{-+}\underline{D}_{\lambda A}\,{}^{-}k^{\mu}\nabla_{\mu}{}^{+}k^{\lambda} \\
& -\underline{P}^{\mu\tau}\underline{E}^{\omega}{}_{A}\nabla_{\mu}{}^{-+}\underline{D}_{\tau\omega}+\sqrt{2}\,{}^{+}\underline{D}_{A}\underline{P}^{\mu\tau}\nabla_{\mu}{}^{-}k_{\tau}-\sqrt{2}\,{}^{-}\underline{D}_{\tau}\underline{P}^{\mu\tau}\underline{E}^{\omega}{}_{A}\nabla_{\mu}{}^{+}k_{\omega} \\
& -{}^{+}\underline{t}_{\gamma\tau A}\underline{P}^{\mu\tau}\nabla_{\mu}{}^{-}k^{\gamma}+{}^{-}\underline{t}_{A\beta\tau}\underline{P}^{\mu\tau}\nabla_{\mu}{}^{+}k^{\beta}\quad,
\end{aligned}
\tag{B.5}
$$

$$
\begin{aligned}
-{}^{+}k^{\beta}\,{}^{+}k^{\gamma}\underline{y}_{A\beta\gamma} \overset{\mathscr{J}}{=}\; & -\sqrt{2}\,{}^{+}k^{\mu}\underline{E}^{\omega}{}_{A}\nabla_{\mu}{}^{+}\underline{D}_{\omega}+\sqrt{2}\,{}^{+}D_{A}\,{}^{+}k^{\mu}\,{}^{+}k^{\lambda}\nabla_{\mu}{}^{-}k_{\lambda}+{}^{-+}\underline{D}_{\gamma A}\,{}^{+}k^{\mu}\nabla_{\mu}{}^{+}k^{\gamma} \\
& +2\,{}^{-+}\underline{D}_{[\beta A]}\,{}^{+}k^{\mu}\nabla_{\mu}{}^{+}k^{\beta}-D\,{}^{+}k^{\mu}\underline{E}^{\omega}{}_{A}\nabla_{\mu}{}^{+}k_{\omega}-{}^{+}\underline{D}_{A\lambda}\,{}^{+}k^{\mu}\nabla_{\mu}{}^{-}k^{\lambda} \\
& -\underline{P}^{\mu\tau}\underline{E}^{\omega}{}_{A}\nabla_{\mu}{}^{+}\underline{D}_{\omega\tau}-\sqrt{2}\,{}^{+}\underline{D}_{A}\underline{P}^{\mu\tau}\nabla_{\mu}{}^{+}k_{\tau}-\sqrt{2}\,{}^{+}\underline{D}_{\tau}\underline{P}^{\tau\mu}\underline{E}^{\omega}{}_{A}\nabla_{\mu}{}^{+}k_{\omega} \\
& -{}^{+}\underline{t}_{\lambda\tau A}\underline{P}^{\mu\tau}\nabla_{\mu}{}^{+}k^{\lambda}-{}^{+}\underline{t}_{\lambda A\tau}\underline{P}^{\tau\mu}\nabla_{\mu}{}^{+}k^{\lambda}\quad,
\end{aligned}
\tag{B.6}
$$

$$
\begin{aligned}
-{}^{-}k^{\beta}\,{}^{-}k^{\gamma}\underline{y}_{A\beta\gamma} \overset{\mathscr{J}}{=}\; & \sqrt{2}\,{}^{-}k^{\mu}\underline{E}^{\omega}{}_{A}\nabla_{\mu}{}^{-}\underline{D}_{\omega}-\sqrt{2}\,{}^{-}D_{A}\,{}^{-}k^{\mu}\,{}^{-}k^{\lambda}\nabla_{\mu}{}^{+}k_{\lambda}+{}^{-+}\underline{D}_{A\gamma}\,{}^{-}k^{\mu}\nabla_{\mu}{}^{-}k^{\gamma} \\
& +2\,{}^{-+}\underline{D}_{[A\beta]}\,{}^{-}k^{\mu}\nabla_{\mu}{}^{-}k^{\beta}-D\,{}^{-}k^{\mu}\underline{E}^{\omega}{}_{A}\nabla_{\mu}{}^{-}k_{\omega}-{}^{-}\underline{D}_{A\lambda}\,{}^{-}k^{\mu}\nabla_{\mu}{}^{+}k^{\lambda} \\
& -\underline{P}^{\mu\tau}\underline{E}^{\omega}{}_{A}\nabla_{\mu}{}^{-}\underline{D}_{\omega\tau}+\sqrt{2}\,{}^{-}\underline{D}_{A}\underline{P}^{\mu\tau}\nabla_{\mu}{}^{-}k_{\tau}+\sqrt{2}\,{}^{-}\underline{D}_{\tau}\underline{P}^{\tau\mu}\underline{E}^{\omega}{}_{A}\nabla_{\mu}{}^{-}k_{\omega} \\
& -{}^{-}\underline{t}_{\lambda\tau A}\underline{P}^{\mu\tau}\nabla_{\mu}{}^{-}k^{\lambda}-{}^{-}\underline{t}_{\lambda A\tau}\underline{P}^{\tau\mu}\nabla_{\mu}{}^{-}k^{\lambda}\quad,
\end{aligned}
\tag{B.7}
$$

$$
\begin{aligned}
-{}^{-}k^{\beta}\underline{y}_{A\beta C} \overset{\mathscr{I}}{=} & -\underline{E}^{\omega}{}_{A}\underline{E}^{\sigma}{}_{C}\,{}^{+}k^{\mu}\nabla_{\mu}\,{}^{-}D_{\omega\sigma} + \sqrt{2}\,{}^{-}\underline{D}_{A}\underline{E}^{\sigma}{}_{C}\,{}^{+}k^{\mu}\nabla_{\mu}\,{}^{-}k_{\sigma} + \sqrt{2}\,{}^{-}\underline{D}_{C}\underline{E}^{\omega}{}_{A}\,{}^{+}k^{\mu}\nabla_{\mu}\,{}^{-}k_{\omega} \\
& - 2\,{}^{-}\underline{D}_{AC}\,{}^{+}k^{\lambda}\,{}^{+}k^{\mu}\nabla_{\mu}\,{}^{-}k_{\lambda} + {}^{-}\underline{t}_{A\lambda C}\,{}^{+}k^{\mu}\nabla_{\mu}\,{}^{-}k^{\lambda} + {}^{-}\underline{t}_{C\lambda A}\,{}^{+}k^{\mu}\nabla_{\mu}\,{}^{-}k^{\lambda} \\
& - \underline{E}^{\omega}{}_{A}\underline{E}^{\sigma}{}_{C}\,{}^{-}k^{\mu}\nabla_{\mu}\,{}^{-+}\underline{D}_{\omega\sigma} - \sqrt{2}\,{}^{-}\underline{D}_{A}\underline{E}^{\sigma}{}_{C}\,{}^{-}k^{\mu}\nabla_{\mu}\,{}^{+}k_{\sigma} + \sqrt{2}\,{}^{+}\underline{D}_{C}\underline{E}^{\omega}{}_{A}\,{}^{-}k^{\mu}\nabla_{\mu}\,{}^{-}k_{\omega} \\
& + {}^{+}\underline{t}_{A\lambda C}\,{}^{-}k^{\mu}\nabla_{\mu}\,{}^{-}k^{\lambda} + {}^{-}\underline{t}_{C\lambda A}\,{}^{-}k^{\mu}\nabla_{\mu}\,{}^{+}k^{\lambda} + \underline{E}^{\omega}{}_{A}\underline{E}^{\sigma}{}_{C}\underline{P}^{\mu\tau}\nabla_{\mu}\,{}^{-}\underline{t}_{\sigma\tau\omega} \\
& - {}^{-}\underline{D}_{AC}\underline{P}^{\tau\mu}\nabla_{\mu}\,{}^{+}k_{\tau} - {}^{-+}\underline{D}_{AC}\underline{P}^{\mu\tau}\nabla_{\mu}\,{}^{-}k_{\tau} + {}^{-}\underline{D}_{A\tau}\underline{P}^{\tau\mu}\underline{E}^{\sigma}{}_{C}\nabla_{\mu}\,{}^{+}k_{\sigma} \\
& + {}^{-+}\underline{D}_{A\tau}\underline{P}^{\mu\tau}\underline{E}^{\sigma}{}_{C}\nabla_{\mu}\,{}^{-}k_{\sigma} - 2\,{}^{-+}\underline{D}_{[\tau C]}\underline{P}^{\mu\tau}\underline{E}^{\omega}{}_{A}\nabla_{\mu}\,{}^{-}k_{\omega} \\
& + {}^{-}\underline{t}_{C\tau A}\,{}^{+}k^{\lambda}\underline{P}^{\mu\tau}\nabla_{\mu}\,{}^{-}k_{\lambda} - D\left(\underline{E}^{\lambda}{}_{C}\underline{E}^{\mu}{}_{A} - \underline{q}_{AC}\underline{P}^{\lambda\mu}\right)\nabla_{\mu}\,{}^{-}k_{\lambda} \quad ,
\end{aligned}
\tag{B.8}
$$

$$
\begin{aligned}
-{}^{+}k^{\beta}\underline{y}_{A\beta C} \overset{\mathscr{I}}{=} & -\underline{E}^{\omega}{}_{A}\underline{E}^{\sigma}{}_{C}\,{}^{-}k^{\mu}\nabla_{\mu}\,{}^{+}\underline{D}_{\omega\sigma} - \sqrt{2}\,{}^{+}\underline{D}_{A}\underline{E}^{\sigma}{}_{C}\,{}^{-}k^{\mu}\nabla_{\mu}\,{}^{+}k_{\sigma} - \sqrt{2}\,{}^{+}\underline{D}_{C}\underline{E}^{\omega}{}_{A}\,{}^{-}k^{\mu}\nabla_{\mu}\,{}^{+}k_{\omega} \\
& - 2\,{}^{+}\underline{D}_{AC}\,{}^{-}k^{\lambda}\,{}^{-}k^{\mu}\nabla_{\mu}\,{}^{+}k_{\lambda} + {}^{+}\underline{t}_{A\lambda C}\,{}^{-}k^{\mu}\nabla_{\mu}\,{}^{+}k^{\lambda} + {}^{+}\underline{t}_{C\lambda A}\,{}^{-}k^{\mu}\nabla_{\mu}\,{}^{+}k^{\lambda} \\
& - \underline{E}^{\omega}{}_{A}\underline{E}^{\sigma}{}_{C}\,{}^{+}k^{\mu}\nabla_{\mu}\,{}^{-+}\underline{D}_{\sigma\omega} + \sqrt{2}\,{}^{+}\underline{D}_{A}\underline{E}^{\sigma}{}_{C}\,{}^{+}k^{\mu}\nabla_{\mu}\,{}^{-}k_{\sigma} - \sqrt{2}\,{}^{-}\underline{D}_{C}\underline{E}^{\omega}{}_{A}\,{}^{+}k^{\mu}\nabla_{\mu}\,{}^{+}k_{\omega} \\
& + {}^{-}\underline{t}_{A\lambda C}\,{}^{+}k^{\mu}\nabla_{\mu}\,{}^{+}k^{\lambda} + {}^{+}\underline{t}_{C\lambda A}\,{}^{+}k^{\mu}\nabla_{\mu}\,{}^{-}k^{\lambda} + \underline{E}^{\omega}{}_{A}\underline{E}^{\sigma}{}_{C}\underline{P}^{\mu\tau}\nabla_{\mu}\,{}^{+}\underline{t}_{\sigma\tau\omega} \\
& - {}^{+}\underline{D}_{AC}\underline{P}^{\tau\mu}\nabla_{\mu}\,{}^{-}k_{\tau} - {}^{-+}\underline{D}_{CA}\underline{P}^{\mu\tau}\nabla_{\mu}\,{}^{+}k_{\tau} + {}^{+}\underline{D}_{A\tau}\underline{P}^{\tau\mu}\underline{E}^{\sigma}{}_{C}\nabla_{\mu}\,{}^{-}k_{\sigma} \\
& + {}^{-+}\underline{D}_{\tau A}\underline{P}^{\mu\tau}\underline{E}^{\sigma}{}_{C}\nabla_{\mu}\,{}^{+}k_{\sigma} - 2\,{}^{-+}\underline{D}_{[C\tau]}\underline{P}^{\mu\tau}\underline{E}^{\omega}{}_{A}\nabla_{\mu}\,{}^{+}k_{\omega} \\
& + {}^{+}\underline{t}_{C\tau A}\,{}^{-}k^{\lambda}\underline{P}^{\mu\tau}\nabla_{\mu}\,{}^{+}k_{\lambda} - D\left(\underline{E}^{\lambda}{}_{C}\underline{E}^{\mu}{}_{A} - \underline{q}_{AC}\underline{P}^{\lambda\mu}\right)\nabla_{\mu}\,{}^{+}k_{\lambda} \quad .
\end{aligned}
\tag{B.9}
$$

The number of independent components of the (rescaled) Cotton tensor $y_{\alpha\beta\gamma}$ in four dimensions is 16 (see [146]). Here we have written a total of 18 of which two can be expanded in terms of other ones: it is possible to write the $(2,-,2)$ component of $y_{\alpha\beta\gamma}$ in terms of the $(3,-,3)$ and $(+,-,-)$, using $y^{\mu}{}_{\alpha\mu}=0$ and $y_{\alpha\beta\gamma}=-y_{\beta\alpha\gamma}$; the same for $(2,+,2)$ in terms of $(3,+,3)$ and $(-,+,+)$.

C | Conformal-gauge transformations

We present a collection of formulae giving the gauge behaviour of fields on $\mathscr{J}$ ($\Lambda \geq 0$) and also of those associated with single cuts $\mathcal{S}$ and with the projected space $\mathbf{S}_2$ ($\Lambda > 0$) associated to a general congruence. Recall that the gauge changes are residual transformations of the conformal factor,

$$\Omega \to \tilde{\Omega} = \omega \Omega \ , \tag{C.1}$$

with ω a positive definite function such that $N^\mu \nabla_\mu \omega \overset{\mathscr{J}}{=} 0$ —according to our partial gauge fixing.

C.1 Metric, connection, volume form and curvature

Quantities of $\left(M, g_{\alpha\beta}\right)$:

$$\tilde{g}_{\alpha\beta} = \omega^2 g_{\alpha\beta} \ , \tag{C.2}$$

$$\tilde{\eta}_{\alpha\beta\gamma\delta} = \omega^4 \eta_{\alpha\beta\gamma\delta} \ , \tag{C.3}$$

$$\tilde{\Gamma}^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} + C^\alpha{}_{\beta\gamma} \ , C^\alpha{}_{\beta\gamma} = \frac{1}{\omega} g^{\gamma\tau}\left(2 g_{\tau(\beta}\omega_{\alpha)} - g_{\gamma\beta}\omega_\tau\right) \tag{C.4}$$

$$\tilde{R}_{\alpha\beta} = R_{\alpha\beta} - 2\frac{1}{\omega}\nabla_\alpha \omega_\beta - \frac{1}{\omega^2} g_{\alpha\beta}\omega_\mu\omega^\mu - \frac{1}{\omega} g_{\alpha\beta}\nabla_\mu\omega^\mu + 4\frac{1}{\omega^2}\omega_\alpha\omega_\beta \ , \tag{C.5}$$

$$\tilde{R} = \frac{1}{\omega^2} R - 6\frac{1}{\omega^3}\nabla_\mu\omega^\mu \ , \tag{C.6}$$

$$\tilde{N}_\alpha = \omega N_\alpha + \Omega\omega_\alpha \ , \tag{C.7}$$

$$\tilde{N} = \frac{1}{\omega}\left(\omega^2 N^2 - \Omega^2\omega_\mu\omega^\mu - 2\Omega N_\mu\omega^\mu\right)^{1/2} \ . \tag{C.8}$$

$$\Lambda > 0$$

Quantities of $(\mathscr{J}, h_{ab})$:

$$\tilde{h}_{ab} \stackrel{\mathscr{J}}{=} \omega^2 h_{ab} \ , \tag{C.9}$$

$$\tilde{\epsilon}_{abc} \stackrel{\mathscr{J}}{=} \omega^3 \epsilon_{abc} \ , \tag{C.10}$$

$$\tilde{\overline{\Gamma}}^a{}_{bc} \stackrel{\mathscr{J}}{=} \overline{\Gamma}^a{}_{bc} + \overline{C}^a{}_{bc} \ , \overline{C}^a{}_{bc} = \frac{1}{\omega} h^{at} \left(2 h_{t(b} \overline{\varpi}_{c)} - h_{cb} \overline{\varpi}_t \right) \tag{C.11}$$

$$\tilde{\overline{R}}_{ab} \stackrel{\mathscr{J}}{=} \overline{R}_{ab} - \frac{1}{\omega} \overline{\nabla}_a \overline{\varpi}_b - \frac{1}{\omega^2} h_{ab} \overline{\varpi}_m \overline{\varpi}^m - \frac{1}{\omega} h_{ab} \overline{\nabla}_m \overline{\varpi}^m + 2 \frac{1}{\omega^2} \overline{\varpi}_a \overline{\varpi}_b \ , \tag{C.12}$$

$$\tilde{\overline{R}} \stackrel{\mathscr{J}}{=} \frac{1}{\omega^2} \overline{R} - 4 \frac{1}{\omega^3} \overline{\nabla}_m \overline{\varpi}^m - 2 \frac{1}{\omega^4} \overline{\varpi}_m \overline{\varpi}^m \ , \tag{C.13}$$

$$\tilde{\overline{S}}_{ab} \stackrel{\mathscr{J}}{=} \overline{S}_{ab} + 2 \frac{1}{\omega^2} \overline{\varpi}_a \overline{\varpi}_b - \frac{1}{\omega} \overline{\nabla}_a \overline{\varpi}_b - \frac{1}{2\omega^2} \overline{\varpi}_s \overline{\varpi}^s h_{ab} \ . \tag{C.14}$$

Quantities associated to a cut $(\mathcal{S}, q_{AB})$:

$$\tilde{q}_{AB} \stackrel{\mathcal{S}}{=} \omega^2 q_{AB} \ , \tag{C.15}$$

$$\tilde{\mathring{\epsilon}}_{AB} \stackrel{\mathcal{S}}{=} \omega^2 \mathring{\epsilon}_{AB} \ , \tag{C.16}$$

$$\tilde{\mathring{\Gamma}}^A{}_{BC} \stackrel{\mathcal{S}}{=} \mathring{\Gamma}^A{}_{BC} + \mathring{C}^A{}_{BC} \ , \mathring{C}^A{}_{BC} \stackrel{\mathcal{S}}{=} \frac{1}{\omega} q^{AT} \left(2 q_{T(B} \mathring{\omega}_{A)} - q_{AB} \mathring{\omega}_T \right) \tag{C.17}$$

$$\tilde{\mathring{R}}_{AB} \stackrel{\mathcal{S}}{=} \mathring{R}_{AB} + \frac{1}{\omega^2} q_{AB} \mathring{\omega}_M \mathring{\omega}^M - \frac{1}{\omega} q_{AB} \mathcal{D}_M \mathring{\omega}^M \ , \tag{C.18}$$

$$\tilde{\mathring{R}} \stackrel{\mathcal{S}}{=} \frac{1}{\omega^2} \mathring{R} + 2 \frac{1}{\omega^4} \mathring{\omega}_M \mathring{\omega}^M - 2 \frac{1}{\omega^3} \mathcal{D}_M \mathring{\omega}^M \ , \tag{C.19}$$

$$\tilde{\mathring{S}}_{AB} \stackrel{\mathcal{S}}{=} \mathring{S}_{AB} + 2 \frac{1}{\omega^2} \mathring{\omega}_A \mathring{\omega}_B - \frac{1}{\omega} \mathcal{D}_A \mathring{\omega}_B - \frac{1}{2\omega^2} \mathring{\omega}_P \mathring{\omega}^P q_{AB} - r^e \overline{\varpi}_e \left(\frac{1}{\omega} \mathring{\kappa}_{AB} + \frac{1}{2\omega^2} r^d \overline{\varpi}_d q_{AB} \right) \ , \tag{C.20}$$

$$\tilde{\mathring{S}}_A \stackrel{\mathcal{S}}{=} \frac{1}{\omega} \left[\mathring{S}_A - \frac{1}{\omega} \mathring{D}_A \left(r^e \overline{\varpi}_e \right) + 2 \frac{1}{\omega^2} \mathring{\omega}_A r^e \overline{\varpi}_e + \frac{1}{\omega} \mathring{\omega}_E \mathring{\kappa}_A{}^E \right] \ , \tag{C.21}$$

$$\tilde{\overline{S}} \stackrel{\mathcal{S}}{=} \frac{1}{\omega^2} \left[\overline{S} - \frac{1}{\omega} r^a r^b \overline{\nabla}_a \overline{\varpi}_b + 2 \frac{1}{\omega^2} \left(r^e \overline{\varpi}_e \right)^2 - \frac{1}{2\omega^2} \overline{\varpi}_e \overline{\varpi}^e \right] \ . \tag{C.22}$$

Quantities associated to $\mathbf{S}_2$:

$$\tilde{\underline{q}}_{AB} \stackrel{\mathbf{S}_2}{=} \omega^2 \underline{q}_{AB} \quad , \tag{C.23}$$

$$\tilde{\underline{\epsilon}}_{AB} \stackrel{\mathbf{S}_2}{=} \omega^2 \underline{\epsilon}_{AB} \ , \tag{C.24}$$

$$\tilde{\underline{\Gamma}}^A{}_{BC} \stackrel{\mathbf{S}_2}{=} \underline{\Gamma}^A{}_{BC} + \underline{C}^A{}_{BC} \ , \underline{C}^A{}_{BC} \stackrel{\mathbf{S}_2}{=} \frac{1}{\omega} \underline{q}^{AT} \left(2 \underline{q}_{T(B} \underline{\omega}_{A)} - \underline{q}_{AB} \underline{\omega}_T \right) \tag{C.25}$$

(C.26)

$$\tilde{\underline{S}}_{AB} \overset{\mathbf{S_2}}{=} \underline{S}_{AB} + 2\frac{1}{\omega^2}\underline{\omega}_A\underline{\omega}_B - \frac{1}{\omega}\underline{\mathcal{D}}_{(A}\underline{\omega}_{B)} - \frac{1}{2\omega^2}\underline{\omega}_P\underline{\omega}^P\underline{q}_{AB} - r^e\overline{\omega}_e\left(\frac{1}{\omega}\underline{\kappa}_{AB} + \frac{1}{2\omega^2}r^d\overline{\omega}_d\underline{q}_{AB}\right) \ , \tag{C.27}$$

$$\tilde{\underline{S}}_A \overset{\mathbf{S_2}}{=} \frac{1}{\omega}\left[\underline{S}_A - \frac{1}{\omega}\underline{D}_A\left(r^e\overline{\omega}_e\right) + \frac{2}{\omega^2}\underline{\omega}_A r^e\overline{\omega}_e + \frac{1}{\omega}\underline{\omega}_E\left(\underline{\kappa}_A{}^E + \underline{\omega}_A{}^E\right)\right] \ , \tag{C.28}$$

$$\tilde{\overline{S}} \overset{\mathscr{J}}{=} \frac{1}{\omega^2}\left[\overline{S} - \frac{1}{\omega}r^a r^b\overline{\nabla}_a\overline{\omega}_b + \frac{2}{\omega^2}\left(r^e\overline{\omega}_e\right)^2 - \frac{1}{2\omega^2}\overline{\omega}_e\overline{\omega}^e\right] \ . \tag{C.29}$$

$\Lambda = 0$

Quantities of $(\mathscr{J}, \overline{g}_{ab})$:

$$\tilde{\overline{g}}_{ab} \overset{\mathscr{J}}{=} \omega^2\overline{g}_{ab} \ , \tag{C.30}$$

$$\tilde{\epsilon}_{abc} \overset{\mathscr{J}}{=} \omega^3\epsilon_{abc} \ , \tag{C.31}$$

$$\tilde{\overline{\Gamma}}^a{}_{bc} \overset{\mathscr{J}}{=} \overline{\Gamma}^a{}_{bc} + \overline{C}^a{}_{bc} \ , \overline{C}^a{}_{bc} = \frac{1}{\omega}\left(2\delta^a_{(b}\overline{\omega}_{c)} - \overline{g}_{bc}\overline{\omega}^a\right) \tag{C.32}$$

$$\tilde{\overline{R}}_{ab} \overset{\mathscr{J}}{=} \overline{R}_{ab} - \frac{1}{\omega}\overline{\nabla}_a\overline{\omega}_b - \frac{1}{\omega}\overline{g}_{ab}\overline{\nabla}_m\overline{\omega}^m + 2\frac{1}{\omega^2}\overline{\omega}_a\overline{\omega}_b \ , \tag{C.33}$$

$$\tilde{\overline{R}} \overset{\mathscr{J}}{=} \frac{1}{\omega^2}\overline{R} - 2\frac{1}{\omega^3}\overline{g}^{mp}\overline{\nabla}_m\overline{\omega}_p + \frac{2}{\omega^4}\overline{\omega}_m\overline{\omega}^m \ , \tag{C.34}$$

$$\tilde{\overline{S}}_{ab} \overset{\mathscr{J}}{=} \overline{S}_{ab} + 2\frac{1}{\omega^2}\overline{\omega}_a\overline{\omega}_b - \frac{1}{\omega}\overline{\nabla}_a\overline{\omega}_b - \frac{1}{2\omega^2}\overline{\omega}_s\overline{\omega}^s\overline{g}_{ab} \ , \tag{C.35}$$

where $g^{\alpha\mu}\omega_\mu = e^\alpha{}_a\overline{\omega}^a$.
Quantities associated to a cut $(\mathcal{S}, q_{AB})$:

$$\tilde{q}_{AB} \overset{\mathcal{S}}{=} \omega^2 q_{AB} \ , \tag{C.36}$$

$$\tilde{\mathring{\epsilon}}_{AB} \overset{\mathcal{S}}{=} \omega^2\mathring{\epsilon}_{AB} \ , \tag{C.37}$$

$$\tilde{\mathring{\Gamma}}^A{}_{BC} \overset{\mathcal{S}}{=} \mathring{\Gamma}^A{}_{BC} + \mathring{C}^A{}_{BC} \ , \mathring{C}^A{}_{BC} \overset{\mathcal{S}}{=} \frac{1}{\omega}q^{AT}\left(2q_{T(B}\mathring{\omega}_{A)} - q_{AB}\mathring{\omega}_T\right) \tag{C.38}$$

$$\tilde{\mathring{R}}_{AB} \overset{\mathcal{S}}{=} \mathring{R}_{AB} + \frac{1}{\omega^2}q_{AB}\mathring{\omega}_M\mathring{\omega}^M - \frac{1}{\omega}q_{AB}\mathcal{D}_M\mathring{\omega}^M \ , \tag{C.39}$$

$$\tilde{\mathring{R}} \overset{\mathcal{S}}{=} \frac{1}{\omega^2}\mathring{R} + 2\frac{1}{\omega^4}\mathring{\omega}_M\mathring{\omega}^M - 2\frac{1}{\omega^3}\mathcal{D}_M\mathring{\omega}^M \ , \tag{C.40}$$

$$\tilde{\mathring{S}}_{AB} \overset{\mathcal{S}}{=} \mathring{S}_{AB} + 2\frac{1}{\omega^2}\mathring{\omega}_A\mathring{\omega}_B - \frac{1}{\omega}\mathcal{D}_A\mathring{\omega}_B - \frac{1}{2\omega^2}\mathring{\omega}_P\mathring{\omega}^P q_{AB} \ , \tag{C.41}$$

C.2 Extrinsic geometry and kinematic quantities of cuts for $\Lambda > 0$

For a cut $\mathcal{S}$:

$$\tilde{r}^a \overset{\mathcal{S}}{=} \frac{1}{\omega} r^a \ , \tag{C.42}$$

$$\tilde{\mathring{\kappa}}_{AB} \overset{\mathcal{S}}{=} \omega \mathring{\kappa}_{AB} + q_{AB} r^e \overline{\omega}_e \ , \tag{C.43}$$

$$\tilde{\mathring{\Sigma}}_{AB} \overset{\mathcal{S}}{=} \omega \mathring{\Sigma}_{AB} \ , \tag{C.44}$$

$$\tilde{\mathring{\kappa}} \overset{\mathcal{S}}{=} \frac{1}{\omega} \mathring{\kappa} + 2 \frac{1}{\omega^2} r^e \overline{\omega}_e \ . \tag{C.45}$$

For $\mathbf{S}_2$:

$$\tilde{m}^a \overset{\mathscr{J}}{=} \frac{1}{\omega} m^a \quad , \tag{C.46}$$

$$\tilde{\underline{\kappa}}_{AB} \overset{\mathbf{S}_2}{=} \omega \underline{\kappa}_{AB} + q_{AB} m^e \overline{\omega}_e \ , \tag{C.47}$$

$$\tilde{\underline{\Sigma}}_{AB} \overset{\mathbf{S}_2}{=} \omega \underline{\Sigma}_{AB} \ , \tag{C.48}$$

$$\tilde{\mathring{\kappa}} \overset{\mathbf{S}_2}{=} \frac{1}{\omega} \mathring{\kappa} + 2 \frac{1}{\omega^2} m^e \overline{\omega}_e \ , \tag{C.49}$$

$$\tilde{\underline{a}}_A \overset{\mathbf{S}_2}{=} \underline{a}_A - \frac{1}{\omega} \underline{\mathcal{D}}_A \omega \ , \tag{C.50}$$

$$\tilde{\underline{\omega}}_{AB} \overset{\mathbf{S}_2}{=} \omega \underline{\omega}_{AB} \ . \tag{C.51}$$

C.3 (rescaled) Weyl decomposition

$$\Lambda > 0$$

Let r^a be the vector field giving a congruence on $\mathscr{J}$, changing as

$$\tilde{m}^a \overset{\mathscr{J}}{=} \frac{1}{\omega} r^a . \tag{C.52}$$

The parts of the rescaled Weyl tensor in the decomposition with respect to this vector field on $\mathscr{J}$ transform as:

$$\tilde{D}_{ab} \overset{\mathscr{J}}{=} \frac{1}{\omega} D_{ab} \;, \qquad \tilde{C}_{ab} \overset{\mathscr{J}}{=} \frac{1}{\omega} C_{ab} \;, \tag{C.53}$$

$$\tilde{\underline{D}}_{AB} \overset{\mathscr{J}}{=} \frac{1}{\omega} \underline{D}_{AB} \;, \qquad \tilde{\underline{C}}_{AB} \overset{\mathscr{J}}{=} \frac{1}{\omega} \underline{C}_{AB} \;, \tag{C.54}$$

$$\tilde{\underline{D}}_{A} \overset{\mathscr{J}}{=} \frac{1}{\omega^2} \underline{D}_{A} \;, \qquad \tilde{\underline{C}}_{A} \overset{\mathscr{J}}{=} \frac{1}{\omega^2} \underline{C}_{A} \;, \tag{C.55}$$

$$\tilde{D} \overset{\mathscr{J}}{=} \frac{1}{\omega^3} D \;, \qquad \tilde{C} \overset{\mathscr{J}}{=} \frac{1}{\omega^3} C \;, \tag{C.56}$$

$${}^{\pm}\tilde{D}_{\alpha\beta} \overset{\mathscr{J}}{=} \frac{1}{\omega} {}^{\pm}D_{\alpha\beta} \;, \qquad {}^{\pm}\tilde{C}_{\alpha\beta} \overset{\mathscr{J}}{=} \frac{1}{\omega} {}^{\pm}C_{\alpha\beta} \;, \tag{C.57}$$

$${}^{\pm}\tilde{\underline{D}}_{AB} \overset{\mathscr{J}}{=} \frac{1}{\omega} {}^{\pm}\underline{D}_{AB} \;, \qquad {}^{\pm}\tilde{\underline{C}}_{AB} \overset{\mathscr{J}}{=} \frac{1}{\omega} {}^{\pm}\underline{C}_{AB} \;, \tag{C.58}$$

$${}^{\pm}\tilde{\underline{D}}_{A} \overset{\mathscr{J}}{=} \frac{1}{\omega^2} {}^{\pm}\underline{D}_{A} \;, \qquad {}^{\pm}\tilde{\underline{C}}_{A} \overset{\mathscr{J}}{=} \frac{1}{\omega^2} {}^{\pm}\underline{C}_{A} \;, \tag{C.59}$$

$${}^{\pm}\tilde{D} \overset{\mathscr{J}}{=} \frac{1}{\omega^3} {}^{\pm}D \;, \qquad {}^{\pm}\tilde{C} \overset{\mathscr{J}}{=} \frac{1}{\omega^3} {}^{\pm}C \;. \tag{C.60}$$

$\Lambda = 0$

The lightlike projections of the rescaled Weyl tensor on $\mathscr{J}$, calculated with respect to N^a, have the following gauge transformations:

$${}^{N}\tilde{D}^{ab} \overset{\mathscr{J}}{=} \frac{1}{\omega^5} {}^{N}D^{ab} \;, \qquad {}^{N}\tilde{C}^{ab} \overset{\mathscr{J}}{=} \frac{1}{\omega^5} {}^{N}C^{ab} \;, \tag{C.61}$$

$${}^{N}\tilde{D}_{ab} \overset{\mathscr{J}}{=} \frac{1}{\omega} {}^{N}D_{ab} \;, \qquad {}^{N}\tilde{C}_{ab} \overset{\mathscr{J}}{=} \frac{1}{\omega} {}^{N}C_{ab} \;, \tag{C.62}$$

$${}^{N}\tilde{\underline{D}}_{A} \overset{\mathscr{J}}{=} \frac{1}{\omega^2} {}^{N}\underline{D}_{A} \;, \qquad {}^{N}\tilde{\underline{C}}_{A} \overset{\mathscr{J}}{=} \frac{1}{\omega^2} {}^{N}\underline{C}_{A} \;. \tag{C.63}$$

D | Lightlike projections of a Weyl-tensor candidate

D.1 Properties of the lightlike projections of a Weyl-tensor candidate

The following is a list of properties of the quantities defined in section 2.2 :

i) ${}^{+}\!k^{\mu}\,{}^{-+}\!C_{\mu\nu} = -\,{}^{-}\!k^{\mu}\,{}^{+}\!C_{\mu\nu}$, ${}^{+}\!k^{\mu}\,{}^{-+}\!D_{\mu\nu} = -\,{}^{-}\!k^{\mu}\,{}^{+}\!D_{\mu\nu}$.

ii) $C = -\mathring{C}^{E}{}_{E} = {}^{*}\!C^{EF}{}_{EF}$, $D = -\mathring{D}^{E}{}_{E} = -C^{EF}{}_{EF}$.

iii) ${}^{-+}\!\mathring{D}_{AB} = -\frac{1}{2}Dq_{AB} - \frac{1}{2}C\mathring{\epsilon}_{AB}$.

iv) ${}^{-+}\!\mathring{C}_{AB} = \frac{1}{2}Cq_{AB} - \frac{1}{2}D\mathring{\epsilon}_{AB}$.

v) ${}^{+-}\!\mathring{D}_{AB} = -\frac{1}{2}Dq_{AB} + \frac{1}{2}C\mathring{\epsilon}_{AB}$.

vi) ${}^{+-}\!\mathring{C}_{AB} = \frac{1}{2}Cq_{AB} + \frac{1}{2}D\mathring{\epsilon}_{AB}$.

vii) ${}^{+}\!\mathring{C}^{B}{}_{B} = 0$, ${}^{+}\!\mathring{D}^{B}{}_{B} = 0$.

viii) ${}^{+}\!\mathring{C}^{AB}\,{}^{-+}\!\mathring{C}_{AB} = 0 = {}^{+}\!\mathring{C}^{AB}\,{}^{-+}\!\mathring{D}_{AB}$, ${}^{+}\!\mathring{D}^{AB}\,{}^{-+}\!\mathring{D}_{AB} = 0 = {}^{+}\!\mathring{D}^{AB}\,{}^{-+}\!\mathring{C}_{AB}$.

ix) ${}^{-}\!k^{\mu}\,{}^{+}\!C_{A\mu} = -\sqrt{2}\,{}^{+}\!\mathring{C}_{A}$, ${}^{+}\!k^{\mu}\,{}^{-}\!C_{A\mu} = \sqrt{2}\,{}^{-}\!\mathring{C}_{A}$.

x) ${}^{-}\!k^{\mu}\,{}^{+}\!D_{A\mu} = -\sqrt{2}\,{}^{+}\!\mathring{D}_{A}$, ${}^{+}\!k^{\mu}\,{}^{-}\!D_{A\mu} = \sqrt{2}\,{}^{-}\!\mathring{D}_{A}$.

xi) $\mathring{C}_{A} = {}^{+}\!\mathring{C}_{A} + {}^{-}\!\mathring{C}_{A}$, $\mathring{D}_{A} = {}^{+}\!\mathring{D}_{A} + {}^{-}\!\mathring{D}_{A}$.

xii) ${}^{+}\!\mathring{D}_{A}\mathring{\epsilon}^{AB}\,{}^{-}\!\mathring{D}_{B} = \frac{1}{2}\mathring{D}_{A}\mathring{C}^{A}$.

xiii) $4\,{}^{+}\!\mathring{D}_{A}\,{}^{-}\!\mathring{D}^{A} = \mathring{D}_{A}\mathring{D}^{A} - \mathring{C}_{A}\mathring{C}^{A}$.

xiv) $\mathring{D}_A \mathring{\epsilon}^{AB} \mathring{C}_B = {}^{+}D_A {}^{+}D^A - {}^{-}D_A {}^{-}D^A$.

xv) ${}^{+}\mathring{C}_{AB} = \grave{C}_{AB} - \mathring{D}^T{}_{(B} \mathring{\epsilon}_{A)T}$.

xvi) ${}^{+}\mathring{D}_{AB} = \grave{D}_{AB} + \mathring{C}^T{}_{(B} \mathring{\epsilon}_{A)T}$.

xvii) ${}^{-}\mathring{C}_{AB} = \grave{C}_{AB} + \grave{D}^T{}_{(B} \mathring{\epsilon}_{A)T}$.

xviii) ${}^{-}\mathring{D}_{AB} = \grave{D}_{AB} - \grave{C}^T{}_{(B} \mathring{\epsilon}_{A)T}$.

xix) $\grave{C}_{AB} = \frac{1}{2} \left({}^{+}\mathring{C}_{AB} + {}^{-}\mathring{C}_{AB} \right)$.

xx) $\grave{D}_{AB} = \frac{1}{2} \left({}^{+}\mathring{D}_{AB} + {}^{-}\mathring{D}_{AB} \right)$.

xxi) ${}^{+}\mathring{C}_A = \frac{1}{2} \left(\mathring{C}_A - \mathring{\epsilon}_A{}^E \mathring{D}_E \right)$.

xxii) ${}^{-}\mathring{C}_A = \frac{1}{2} \left(\mathring{C}_A + \mathring{\epsilon}_A{}^E \mathring{D}_E \right)$.

xxiii) ${}^{+}\mathring{D}_A = \frac{1}{2} \left(\mathring{D}_A + \mathring{\epsilon}_A{}^E \mathring{C}_E \right)$.

xxiv) ${}^{-}\mathring{D}_A = \frac{1}{2} \left(\mathring{D}_A - \mathring{\epsilon}_A{}^E \mathring{C}_E \right)$.

xxv) ${}^{+}\mathring{D}_A + {}^{-}\mathring{D}_A = \mathring{\epsilon}_A{}^B \left({}^{+}\mathring{C}_B - {}^{-}\mathring{C}_B \right)$.

xxvi) $r^a \, {}^{-+}D_{aA} = {}^{+}\mathring{D}_A$.

xxvii) $r^a \, {}^{+-}D_{aA} = {}^{-}\mathring{D}_A$.

xxviii) $2 r^a r^b \, {}^{+-}D_{ab} = 2 r^a r^b \, {}^{-+}D_{ab} = {}^{\pm}k^\mu \, {}^{\pm}k^\nu \, {}^{\mp}D_{\mu\nu} = r^a r^b D_{ab}$.

xxix) $\mp \, {}^{\mp}k^\mu E^\alpha{}_A \, {}^{\pm}D_{\mu\alpha} = \sqrt{2} \, {}^{\pm}\mathring{D}_A$.

xxx) $\mp \, {}^{\mp}k^\mu E^\alpha{}_A \, {}^{\pm}C_{\mu\alpha} = \sqrt{2} \, {}^{\pm}\mathring{C}_A$.

xxxi) $\mathring{\epsilon}^{AB} \mathring{D}_B = r_\mu r_\rho u_\sigma W_\gamma{}^A \, {}^{*}C^{\sigma\rho\gamma\mu}$.

xxxii) $\mathring{\epsilon}^{AB} \mathring{C}_B = -r_\mu r_\rho u_\sigma W_\gamma{}^A C^{\sigma\rho\gamma\mu}$.

xxxiii) $\mathring{\epsilon}^A{}_B \, {}^{+}\mathring{C}_A = - {}^{+}\mathring{D}_B$.

xxxiv) $\mathring{\epsilon}^A{}_B \, {}^{-}\mathring{C}_A = {}^{-}\mathring{D}_B$.

xxxv) ${}^{+}\mathring{D}^A \, {}^{+}\mathring{C}_A = 0 = {}^{-}\mathring{D}^A \, {}^{-}\mathring{C}_A$.

xxxvi) ${}^{+}\mathring{C}^A \, {}^{-}\mathring{D}_A = {}^{-}\mathring{C}^A \, {}^{+}\mathring{D}_A$.

D.2 NP formulation

Writing the lightlike projections of a Weyl candidate tensor (section 2.2) and the superenergy quantities (sections 2.1 and 2.3) in terms of the Weyl candidate scalars allows to have a first-glance interpretation of their significance. Not only that but having their components at hand can be helpful to check calculations. Consider a lightlike vierbein $\left({}^{-}\!k^{\alpha}, {}^{+}\!k^{\alpha}, m^{\alpha}, \bar{m}^{\alpha}\right)$ such that ${}^{+}\!k^{\alpha}\,{}^{-}\!k_{\alpha} = -1$, $m^{\alpha}\bar{m}_{\alpha} = 1$, ${}^{\pm}\!k^{\alpha}m_{\alpha} = 0 = m^{\alpha}m_{\alpha}$, with orientation fixed to $\eta_{\hat{0}\hat{1}\hat{2}\hat{3}} = i$. We use the following definitions for the Weyl candidate scalars:

$$\begin{aligned} &\phi_0 := C_{\hat{0}\hat{2}\hat{0}\hat{2}} && \phi_1 := C_{\hat{0}\hat{1}\hat{0}\hat{2}} && \phi_2 = \frac{1}{2}\left(C_{\hat{0}\hat{1}\hat{0}\hat{1}} - C_{\hat{0}\hat{1}\hat{2}\hat{3}}\right) \\ &\phi_3 := -C_{\hat{0}\hat{1}\hat{1}\hat{3}} && \phi_4 := C_{\hat{1}\hat{3}\hat{1}\hat{3}} \end{aligned} \tag{D.1}$$

Be aware that all the formulae below hold only with these definitions and choice of orientation. In this subsection, hatted Greek and Latin characters $\hat{\alpha}$, $\hat{A}$ represent basis indices.

The lightlike 'magnetic' and 'electric' parts associated to ${}^{+}\!k^{\alpha}$ and ${}^{-}\!k^{\alpha}$ respectively.

$${}^{+}C^{\hat{\alpha}\hat{\beta}} = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -i2\Im(\phi_2) & -\phi_3 & \bar{\phi}_3 \\ 0 & -\phi_3 & \phi_4 & 0 \\ 0 & \bar{\phi}_3 & 0 & -\bar{\phi}_4 \end{pmatrix}, \quad {}^{+}D^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\Re(\phi_2) & -\phi_3 & -\bar{\phi}_3 \\ 0 & -\phi_3 & \phi_4 & 0 \\ 0 & -\bar{\phi}_3 & 0 & \bar{\phi}_4 \end{pmatrix}. \tag{D.2}$$

$${}^{-}C^{\hat{\alpha}\hat{\beta}} = i\begin{pmatrix} -i2\Im(\phi_2) & 0 & \bar{\phi}_1 & -\phi_1 \\ 0 & 0 & 0 & 0 \\ \bar{\phi}_1 & 0 & -\bar{\phi}_0 & 0 \\ -\phi_1 & 0 & 0 & \phi_0 \end{pmatrix}, \quad {}^{-}D^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} 2\Re(\phi_2) & 0 & -\bar{\phi}_1 & -\phi_1 \\ 0 & 0 & 0 & 0 \\ -\bar{\phi}_1 & 0 & \bar{\phi}_0 & 0 \\ -\phi_1 & 0 & 0 & \phi_0 \end{pmatrix}. \tag{D.3}$$

The two dimensional components,

$${}^{+}\mathring{C}^{\hat{A}\hat{B}} = i\begin{pmatrix} s\phi_4 & 0 \\ 0 & -\bar{\phi}_4 \end{pmatrix}, \quad {}^{+}\mathring{D}^{\hat{A}\hat{B}} = \begin{pmatrix} \phi_4 & 0 \\ 0 & \bar{\phi}_4 \end{pmatrix}. \tag{D.4}$$

$${}^{-}\mathring{C}^{\hat{A}\hat{B}} = i\begin{pmatrix} -\bar{\phi}_0 & 0 \\ 0 & \phi_0 \end{pmatrix}, \quad {}^{-}\mathring{D}^{\hat{A}\hat{B}} = \begin{pmatrix} \bar{\phi}_0 & 0 \\ 0 & \phi_0 \end{pmatrix}. \tag{D.5}$$

The traceless, two dimensional magnetic and electric parts,

$$\grave{C}^{\hat{A}\hat{B}} = \frac{i}{2}\begin{pmatrix} \phi_4 - \bar{\phi}_0 & 0 \\ 0 & -(\bar{\phi}_4 - \phi_0) \end{pmatrix}, \quad \grave{D}^{\hat{A}\hat{B}} = \frac{1}{2}\begin{pmatrix} \phi_4 + \bar{\phi}_0 & 0 \\ 0 & \bar{\phi}_4 + \phi_0 \end{pmatrix}. \tag{D.6}$$

The two dimensional vectors,

$$
{}^{+}\mathring{C}^{\hat{A}} = \frac{i}{\sqrt{2}} \begin{pmatrix} -\phi_3 \\ \bar{\phi}_3 \end{pmatrix} \ , \quad {}^{+}\mathring{D}^{\hat{A}} = -\frac{1}{\sqrt{2}} \begin{pmatrix} \phi_3 \\ \bar{\phi}_3 \end{pmatrix} \ . \tag{D.7}
$$

$$
{}^{-}\mathring{C}^{\hat{A}} = \frac{i}{\sqrt{2}} \begin{pmatrix} -\bar{\phi}_1 \\ \phi_1 \end{pmatrix} \ , \quad {}^{-}\mathring{D}^{\hat{A}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{\phi}_1 \\ \phi_1 \end{pmatrix} \ . \tag{D.8}
$$

$$
\mathring{C}^{\hat{A}} = \frac{i}{\sqrt{2}} \begin{pmatrix} -(\phi_3 + \bar{\phi}_1) \\ \bar{\phi}_3 + \phi_1 \end{pmatrix} \ , \quad \mathring{D}^{\hat{A}} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\phi_3 + \bar{\phi}_1 \\ -\bar{\phi}_3 + \phi_1 \end{pmatrix} \ . \tag{D.9}
$$

And the traces,

$$
C = -\mathring{C}^{\hat{E}}{}_{\hat{E}} = 2\Im(\phi_2) \ , \quad D = -\mathring{D}^{\hat{E}}{}_{\hat{E}} = 2\Re(\phi_2) \ . \tag{D.10}
$$

For any lightlike tetrad $\left({}^{+}\!k^{\alpha}, {}^{-}\!k^{\alpha}, m^{\alpha}, \bar{m}^{\alpha}\right)$ and a general unit timelike vector field v^{α},

$$
v^{\alpha} := \left(a\,{}^{-}\!k^{\alpha} + b\,{}^{+}\!k^{\alpha} + c m^{\alpha} + \overline{c m}^{\alpha}\right), \quad v^{\alpha} v_{\alpha} = -1, \quad \forall a, b, c, \bar{c} \quad / \quad ab - c\bar{c} = \frac{1}{2} \ , \tag{D.11}
$$

it is a matter of direct calculation to get the expression of a basic superenergy tensor

$$
\begin{aligned}
\mathcal{T}^{\alpha\beta\gamma\delta} = 4\Big\{ & \phi_0\overline{\phi}_0\Big[{}^{+}\!k^{\alpha}\,{}^{+}\!k^{\beta}\,{}^{+}\!k^{\gamma}\,{}^{+}\!k^{\delta}\Big] + \phi_0\overline{\phi}_1\Big[-2\,{}^{+}\!k^{\alpha}\,{}^{+}\!k^{\beta}\overline{m}^{(\gamma}\,{}^{+}\!k^{\delta)} - 2\,{}^{+}\!k^{\delta}\,{}^{+}\!k^{\gamma}\overline{m}^{(\alpha}\,{}^{+}\!k^{\beta)}\Big] + \text{c.c.} \\
& + \phi_0\overline{\phi}_2\Big[\overline{m}^{\alpha}\overline{m}^{\beta}\,{}^{+}\!k^{\gamma}\,{}^{+}\!k^{\delta} + {}^{+}\!k^{\alpha}\,{}^{+}\!k^{\beta}\overline{m}^{\gamma}\overline{m}^{\delta} + 4\overline{m}^{(\alpha}\,{}^{+}\!k^{\beta)}\overline{m}^{(\gamma}\,{}^{+}\!k^{\delta)}\Big] + \text{c.c.} \\
& + \phi_0\overline{\phi}_3\Big[-2\overline{m}^{\alpha}\overline{m}^{\beta}\overline{m}^{(\gamma}\,{}^{+}\!k^{\delta)} - 2\overline{m}^{\delta}\overline{m}^{\gamma}\overline{m}^{(\alpha}\,{}^{+}\!k^{\beta)}\Big] + \text{c.c.} + \phi_0\overline{\phi}_4\Big[\overline{m}^{\alpha}\overline{m}^{\beta}\overline{m}^{\gamma}\overline{m}^{\delta}\Big] + \text{c.c.} \\
& + \phi_1\overline{\phi}_1\Big[{}^{+}\!k^{\alpha}\,{}^{+}\!k^{\beta}\left(2\,{}^{+}\!k^{(\gamma}\,{}^{-}\!k^{\delta)} + 2m^{(\gamma}\overline{m}^{\delta)}\right) + 4\overline{m}^{(\alpha}\,{}^{+}\!k^{\beta)}m^{(\gamma}\,{}^{+}\!k^{\delta)} + {}^{+}\!k^{\gamma}\,{}^{+}\!k^{\delta}\left(2\,{}^{+}\!k^{(\alpha}\,{}^{-}\!k^{\beta)} + 2m^{(\alpha}\overline{m}^{\beta)}\right) \\
& + 4\overline{m}^{(\gamma}\,{}^{+}\!k^{\delta)}m^{(\alpha}\,{}^{+}\!k^{\beta)}\Big] + \phi_1\overline{\phi}_2\Big[-2\overline{m}^{\alpha}\overline{m}^{\beta}m^{(\gamma}\,{}^{+}\!k^{\delta)} - 2\,{}^{+}\!k^{\alpha}\,{}^{+}\!k^{\beta}\overline{m}^{(\gamma}\,{}^{-}\!k^{\delta)} - 2\overline{m}^{(\alpha}\,{}^{+}\!k^{\beta)}\left(2\,{}^{+}\!k^{(\gamma}\,{}^{-}\!k^{\delta)} + 2m^{(\gamma}\overline{m}^{\delta)}\right) \\
& - 2\overline{m}^{\gamma}\overline{m}^{\delta}m^{(\alpha}\,{}^{+}\!k^{\beta)} - 2\,{}^{+}\!k^{\gamma}\,{}^{+}\!k^{\delta}\overline{m}^{(\alpha}\,{}^{-}\!k^{\beta)} - 2\overline{m}^{(\gamma}\,{}^{+}\!k^{\delta)}\left(2\,{}^{+}\!k^{(\alpha}\,{}^{-}\!k^{\beta)} + 2m^{(\alpha}\overline{m}^{\beta)}\right)\Big] + \text{c.c.} \\
& + \phi_1\overline{\phi}_3\Big[\overline{m}^{\alpha}\overline{m}^{\beta}\left(2\,{}^{+}\!k^{(\gamma}\,{}^{-}\!k^{\delta)} + 2m^{(\gamma}\overline{m}^{\delta)}\right) + \overline{m}^{\gamma}\overline{m}^{\delta}\left(2\,{}^{+}\!k^{(\alpha}\,{}^{-}\!k^{\beta)} + 2m^{(\alpha}\overline{m}^{\beta)}\right) + 4\overline{m}^{(\alpha}\,{}^{+}\!k^{\beta)}\overline{m}^{(\gamma}\,{}^{-}\!k^{\delta)} \\
& + 4\overline{m}^{(\gamma}\,{}^{+}\!k^{\delta)}\overline{m}^{(\alpha}\,{}^{-}\!k^{\beta)}\Big] + \text{c.c.} + \phi_1\overline{\phi}_4\Big[-2\overline{m}^{\alpha}\overline{m}^{\beta}\overline{m}^{(\gamma}\,{}^{-}\!k^{\delta)} - 2\overline{m}^{\gamma}\overline{m}^{\delta}\overline{m}^{(\alpha}\,{}^{-}\!k^{\beta)}\Big] + \text{c.c.} \\
& + \phi_2\overline{\phi}_2\Big[{}^{-}\!k^{\alpha}\,{}^{-}\!k^{\beta}\,{}^{+}\!k^{\gamma}\,{}^{+}\!k^{\delta} + m^{\alpha}m^{\beta}\overline{m}^{\gamma}\overline{m}^{\delta} + 4m^{(\alpha}\,{}^{-}\!k^{\beta)}\overline{m}^{(\gamma}\,{}^{+}\!k^{\delta)} + 4\overline{m}^{(\alpha}\,{}^{-}\!k^{\beta)}m^{(\gamma}\,{}^{+}\!k^{\delta)} + {}^{-}\!k^{\gamma}\,{}^{-}\!k^{\delta}\,{}^{+}\!k^{\alpha}\,{}^{+}\!k^{\beta} \\
& + m^{\gamma}m^{\delta}\overline{m}^{\alpha}\overline{m}^{\beta} + 4m^{(\gamma}\,{}^{-}\!k^{\delta)}\overline{m}^{(\alpha}\,{}^{+}\!k^{\beta)} + 4\overline{m}^{(\gamma}\,{}^{-}\!k^{\delta)}m^{(\alpha}\,{}^{+}\!k^{\beta)} + \left(2\,{}^{+}\!k^{(\alpha}\,{}^{-}\!k^{\beta)} + 2m^{(\alpha}\overline{m}^{\beta)}\right)\Big(2\,{}^{+}\!k^{(\gamma}\,{}^{-}\!k^{\delta)} \\
& + 2m^{(\gamma}\overline{m}^{\delta)}\Big)\Big] + \phi_2\overline{\phi}_3\Big[-2\,{}^{-}\!k^{\alpha}\,{}^{-}\!k^{\beta}\overline{m}^{(\gamma}\,{}^{+}\!k^{\delta)} - 2\overline{m}^{\gamma}\overline{m}^{\delta}m^{(\alpha}\,{}^{-}\!k^{\beta)} - 2\overline{m}^{(\alpha}\,{}^{-}\!k^{\beta)}\left(2\,{}^{+}\!k^{(\gamma}\,{}^{-}\!k^{\delta)} + 2m^{(\gamma}\overline{m}^{\delta)}\right) \\
& - 2\,{}^{-}\!k^{\gamma}\,{}^{-}\!k^{\delta}\overline{m}^{(\alpha}\,{}^{+}\!k^{\beta)} - 2\overline{m}^{\alpha}\overline{m}^{\beta}m^{(\gamma}\,{}^{-}\!k^{\delta)} - 2\overline{m}^{(\gamma}\,{}^{-}\!k^{\delta)}\left(2\,{}^{+}\!k^{(\alpha}\,{}^{-}\!k^{\beta)} + 2m^{(\alpha}\overline{m}^{\beta)}\right)\Big] + \text{c.c.} + \phi_2\overline{\phi}_4\Big[{}^{-}\!k^{\alpha}\,{}^{-}\!k^{\beta}\overline{m}^{\gamma}\overline{m}^{\delta} \\
& + {}^{-}\!k^{\gamma}\,{}^{-}\!k^{\delta}\overline{m}^{\alpha}\overline{m}^{\beta} + 4\overline{m}^{(\alpha}\,{}^{-}\!k^{\beta)}\overline{m}^{(\gamma}\,{}^{-}\!k^{\delta)}\Big] + \text{c.c.} + \phi_3\overline{\phi}_3\Big[{}^{-}\!k^{\alpha}\,{}^{-}\!k^{\beta}\left(2\,{}^{+}\!k^{(\gamma}\,{}^{-}\!k^{\delta)} + 2m^{(\gamma}\overline{m}^{\delta)}\right) + 4m^{(\alpha}\,{}^{-}\!k^{\beta)}\overline{m}^{(\gamma}\,{}^{-}\!k^{\delta)} \\
& + {}^{-}\!k^{\gamma}\,{}^{-}\!k^{\delta}\left(2\,{}^{+}\!k^{(\alpha}\,{}^{-}\!k^{\beta)} + 2m^{(\alpha}\overline{m}^{\beta)}\right) + 4m^{(\gamma}\,{}^{-}\!k^{\delta)}\overline{m}^{(\alpha}\,{}^{-}\!k^{\beta)}\Big] + \phi_3\overline{\phi}_4\Big[-2\,{}^{-}\!k^{\alpha}\,{}^{-}\!k^{\beta}\overline{m}^{(\gamma}\,{}^{-}\!k^{\delta)} - 2\,{}^{-}\!k^{\delta}\,{}^{-}\!k^{\gamma}\overline{m}^{(\alpha}\,{}^{-}\!k^{\beta)}\Big] \\
& + \text{c.c.} + \phi_4\overline{\phi}_4\Big[{}^{-}\!k^{\alpha}\,{}^{-}\!k^{\beta}\,{}^{-}\!k^{\gamma}\,{}^{-}\!k^{\delta}\Big]\Big\}\,,
\end{aligned}
\tag{D.12}
$$

the super-Poynting vector field

$$
\begin{aligned}
\overline{\mathcal{P}}^{a} = -4\omega_{\alpha}{}^{a}\Bigg\{ & \phi_0\overline{\phi}_0\Big[-a^3\,{}^{+}\!k^{\alpha}\Big] + \phi_0\overline{\phi}_1\Big[-3a^2c\,{}^{+}\!k^{\alpha} + a^3\overline{m}^{\alpha}\Big] + \text{c.c.} + \phi_0\overline{\phi}_2\Big[-3ac^2\,{}^{+}\!k^{\alpha} + 3a^2c\overline{m}^{\alpha}\Big] \\
& + \text{c.c.} + \phi_0\overline{\phi}_3\Big[-c^3\,{}^{+}\!k^{\alpha} + 3ac^2\overline{m}^{\alpha}\Big] + \text{c.c.} + \phi_0\overline{\phi}_4\Big[c^3\overline{m}^{\alpha}\Big] + \text{c.c.} + \phi_1\overline{\phi}_1\Big[-a^3\,{}^{-}\!k^{\alpha} \\
& + (-3a^2b - 6ac\overline{c})\,{}^{+}\!k^{\alpha} + 3a^2cm^{\alpha} + 3a^2\overline{c}\overline{m}^{\alpha}\Big] + \phi_1\overline{\phi}_2\Big[-3a^2c\,{}^{-}\!k^{\alpha} + (-6abc - 3c^2\overline{c})\,{}^{+}\!k^{\alpha} \\
& + 3ac^2m^{\alpha} + (6ac\overline{c} + 3a^2b)\overline{m}^{\alpha}\Big] + \text{c.c.} + \phi_1\overline{\phi}_3\Big[-3ac^2\,{}^{-}\!k^{\alpha} - 3bc^2\,{}^{+}\!k^{\alpha} + c^3m^{(\alpha} + (6abc \\
& + 3c^2\overline{c})\overline{m}^{\alpha}\Big] + \text{c.c.} + \phi_1\overline{\phi}_4\Big[-c^3\,{}^{-}\!k^{\alpha} + 3bc^2\overline{m}^{\alpha}\Big] + \text{c.c.} + \phi_2\overline{\phi}_2\Big[(-3a^2b - 6ac\overline{c})\,{}^{-}\!k^{\alpha} \\
& + (-3ab^2 - 6bc\overline{c})\,{}^{+}\!k^{\alpha} + (6abc + 3c^2\overline{c})m^{\alpha} + (6ab\overline{c} + 3c\overline{c}^2)\overline{m}^{\alpha}\Big] + \phi_2\overline{\phi}_3\Big[(-6abc - 3c^2\overline{c})\,{}^{-}\!k^{\alpha} \\
& - 3b^2c\,{}^{+}\!k^{\alpha} + 3bc^2m^{\alpha} + (3ab^2 + 6bc\overline{c})\overline{m}^{\alpha}\Big] + \text{c.c.} + \phi_2\overline{\phi}_4\Big[-3bc^2\,{}^{-}\!k^{\alpha} + 3b^2c\overline{m}^{\alpha}\Big] + \text{c.c.} \\
& + \phi_3\overline{\phi}_3\Big[(-3ab^2 - 6bc\overline{c})\,{}^{-}\!k^{\alpha} - b^3\,{}^{+}\!k^{\alpha} + 3b^2cm^{\alpha} + 3b^2\overline{c}\overline{m}^{\alpha}\Big] + \phi_3\overline{\phi}_4\Big[-3b^2c\,{}^{-}\!k^{\alpha} + b^3\overline{m}^{\alpha}\Big] + \text{c.c.} \\
& + \phi_4\overline{\phi}_4\Big[-b^3\,{}^{-}\!k^{\alpha}\Big]\Bigg\}\,,
\end{aligned}
\tag{D.13}
$$

and the superenergy density

$$
\begin{aligned}
\mathcal{W} = 4\Bigg\{ & \phi_0\overline{\phi}_0\Big[a^4\Big] + \phi_0\overline{\phi}_1\Big[4a^3c\Big] + \text{c.c.} + \phi_0\overline{\phi}_2\Big[6a^2c^2\Big] + \text{c.c.} + \phi_0\overline{\phi}_3\Big[4ac^3\Big] + \text{c.c.} \\
& + \phi_0\overline{\phi}_4\Big[c^4\Big] + \text{c.c.} + \phi_1\overline{\phi}_1\Big[4a^3b + 12a^2c\overline{c}\Big] + \phi_1\overline{\phi}_2\Big[12a^2bc + 12ac^2\overline{c}\Big] + \text{c.c.} \\
& + \phi_1\overline{\phi}_3\Big[12abc^2 + 4c^3\overline{c}\Big] + \text{c.c.} + \phi_1\overline{\phi}_4\Big[4bc^3\Big] + \text{c.c.} + \phi_2\overline{\phi}_2\Big[6a^2b^2 + 24abc\overline{c} + 6c^2\overline{c}^2\Big] \\
& + \phi_2\overline{\phi}_3\Big[12ab^2c + 12bc^2\overline{c}\Big] + \text{c.c.} + \phi_2\overline{\phi}_4\Big[6b^2c^2\Big] + \text{c.c.} + \phi_3\overline{\phi}_3\Big[4ab^3 + 12b^2c\overline{c}\Big] \\
& + \phi_3\overline{\phi}_4\Big[4b^3c\Big] + \text{c.c.} + \phi_4\overline{\phi}_4\Big[b^4\Big]\Bigg\}.
\end{aligned}
\tag{D.14}
$$

The radiant and Coulomb superenergy densities read

$$ {}^{+}\mathcal{Z} = 4\phi_3\bar{\phi}_3 \quad , \tag{D.15} $$

$$ {}^{-}\mathcal{Z} = 4\phi_1\bar{\phi}_1 \quad , \tag{D.16} $$

$$ {}^{+}\mathcal{W} = 4\phi_4\bar{\phi}_4 \quad , \tag{D.17} $$

$$ {}^{-}\mathcal{W} = 4\phi_0\bar{\phi}_0 \quad , \tag{D.18} $$

$$ {}^{+}\mathcal{Q}^A = -4\left(\phi_4\bar{\phi}_3 m^A + \phi_3\bar{\phi}_4\bar{m}^A\right) \quad , \tag{D.19} $$

$$ {}^{-}\mathcal{Q}^A = -4\left(\phi_1\bar{\phi}_0 m^A + \phi_0\bar{\phi}_1\bar{m}^A\right) \quad , \tag{D.20} $$

$$ \mathcal{V} = 4\phi_2\bar{\phi}_2 \quad . \tag{D.21} $$

From here and eqs. (2.48) and (2.49) it is easy to write the radiant supermomenta,

$$ {}^{+}\mathcal{Q}^\alpha = 4\left(\phi_4\bar{\phi}_4\,{}^{-}k^\alpha + \phi_3\bar{\phi}_3\,{}^{+}k^\alpha - \phi_4\bar{\phi}_3 m^\alpha - \bar{\phi}_4\phi_3\bar{m}^\alpha\right) \ , \tag{D.22} $$

$$ {}^{-}\mathcal{Q}^\alpha = 4\left(\phi_0\bar{\phi}_0\,{}^{+}k^\alpha + \phi_1\bar{\phi}_1\,{}^{-}k^\alpha - \bar{\phi}_0\phi_1 m^\alpha - \phi_0\bar{\phi}_1\bar{m}^\alpha\right) \ . \tag{D.23} $$

Finally, the vector defined in eq. (2.87) has the expression

$$ d^A = \sqrt{2}\left(\phi_1\bar{\phi}_2 + \bar{\phi}_3\phi_2\right)\bar{m}^A + \sqrt{2}\left(\bar{\phi}_1\phi_2 + \phi_3\bar{\phi}_2\right)m^A \ . \tag{D.24} $$

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