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Aingeru Fernández-Bertolin, Karlheinz Gröchenig, Philippe Jaming

FROM HEISENBERG UNIQUENESS PAIRS TO PROPERTIES OF THE HELMHOLTZ AND LAPLACE EQUATIONS

AINGERU FERNÁNDEZ-BERTOLIN, KARLHEINZ GRÖCHENIG, AND PHILIPPE JAMING

ABSTRACT. The aim of this paper is to establish uniqueness properties of solutions of the Helmholtz and Laplace equations. In particular, we show that if two solutions of such equations on a domain of $\mathbb{R}^d$ agree on two intersecting $d - 1$-dimensional submanifolds in generic position, then they agree everywhere.

1. INTRODUCTION

The aim of this paper is to bridge two topics: unique determination of measures from restrictions of their Fourier transform and uniqueness properties of solutions of Helmholtz and Laplace equations.

Our starting point is the recent notion of Heisenberg uniqueness pairs introduced by H. Hedenmalm and A. Montes-Rodríguez [HMR] that we slightly extended in [GrJ]:

Definition 1.1. Let $\mathcal{M} \subset \mathbb{R}^d$ be a manifold and $\Sigma \subset \mathbb{R}^d$ be a set. We say that $(\mathcal{M}, \Sigma)$ is a *Heisenberg uniqueness pair* if the only finite measure μ supported on $\mathcal{M}$ with Fourier transform vanishing on Σ is $\mu = 0$.

The main focus of [HMR] was 2-dimensional with $\mathcal{M}$ being a hyperbola and Σ a discrete set. The setting was slightly more restrictive as the measure was supposed to be absolutely continuous with respect to arc length. The property of being a Heisenberg uniqueness pair has then been established for various curves: P. Sjölin [Sj1, Sj2] considered the parabola and the circle, N. Lev [Le] the circle. Ph. Jaming and K. Kellay [JK] introduced new geometric methods that allowed to treat many curves when Σ consists of 2 intersecting lines. Those methods were used in [GS] to provide more examples. For results in higher dimensions we refer to our previous work [GrJ]. For links with the problem of determining point distributions in $\mathbb{R}^d$ (or more generally the determination of finite measures on $\mathbb{R}^d$) from their projections onto lower dimensional spaces, we refer to [FBGJ].

So far all results on the subject treated Heisenberg uniqueness pairs as a topic in Fourier analysis related to the uncertainty principle and with methodological input from dynamical systems. In this work we change our point of view and study Heisenberg uniqueness pairs from the perspective of partial differential equations. To explain the connection to PDEs, we recall the results of N. Lev and P. Sjölin [Le, Sj1] (*see* also [JK]). To $f \in L^1(\mathbb{S}^1)$ (where $\mathbb{S}^{d-1}$ is the unit sphere of $\mathbb{R}^d$) we associate the measure

on $\mathbb{R}^2$ given by

$$\int \varphi \, \mathrm{d}\mu = \int_0^{2\pi} \varphi(\cos s, \sin s) f(\cos s, \sin s) \, \mathrm{d}s,$$

$\varphi \in \mathcal{C}(\mathbb{R}^2)$, i.e., μ is a measure supported on $\mathbb{S}^1$ and is absolutely continuous with respect to arc length on $\mathbb{S}^1$. The Fourier transform of μ is then defined as

$$\widehat{\mu}(\xi, \eta) = \int \exp\bigl(-i(x\eta + y\xi)\bigr) \, \mathrm{d}\mu(x, y).$$

Let $\theta_1, \theta_2 \in \mathbb{R}^2$ be two unit vectors and define $\theta = \arccos \langle \theta_1, \theta_2 \rangle$. Assume that $\widehat{\mu} = 0$ on the lines $\theta_1^\perp$, $\theta_2^\perp$. The main result of [Le, Sj1] asserts that $\mu = 0$ if and only if $\theta \notin \pi\mathbb{Q}$. The connection to partial differential equations is established by the following observation. If $\operatorname{supp} \mu \subseteq \mathbb{S}^{d-1}$, then $u = \widehat{\mu}$ is a solution of the Helmholtz equation $\Delta u + u = 0$. The result of Lev and Sjölin can be recast as follows: If a solution of the Helmholtz equation is the Fourier transform of a measure and vanishes on two lines whose angle of intersection is an irrational multiple of π, then the solution is identically 0. It turns out that this statement is a special case of an older result about nodal sets by S.Y. Cheng [Ch, Thm. 2.5].

Theorem. (Cheng) *Let h be a $\mathcal{C}^\infty$ function on a domain $\Omega \subset \mathbb{R}^2$. Let u be a solution of $(\Delta + h)u = 0$ and assume that two nodal lines of u intersect at a point $x_0 \in \Omega$, i.e., there exists two smooth curves Γ_1, Γ_2 such that $u = 0$ on $\Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap \Gamma_2 = \{x_0\}$. Then the set of nodal lines through x_0 forms an equiangular system. In particular, the angle between the tangents of Γ_1, Γ_2 at x_0 is a rational multiple of π.*

Cheng's proof is rather involved and is based on a subtle local representation formula of solutions of elliptic equations due to L. Bers [Be]. One of our goals is to provide two much simpler proofs of Cheng's result in the particular case of the Helmholtz ($k > 0$) and the Laplace ($k = 0$) equation $\Delta u + k^2 u = 0$. These proofs are only marginally more involved than the ones in [Le] and [Sj1] and work under much less restrictive assumptions.

The first proof is simple and based on the Schwarz reflection principle for harmonic and analytic functions and yields the following extension of the result of Lev and Sjölin.

Theorem A. *Let $d \geq 2$ and Ω be a domain in $\mathbb{R}^d$ with $0 \in \Omega$. Let $\theta_1, \theta_2 \in \mathbb{S}^{d-1}$ be such that $\arccos \langle \theta_1, \theta_2 \rangle \notin \pi\mathbb{Q}$. Let $k \in \mathbb{R}$ and let u be a solution of the Laplace-Helmholtz equation on Ω*

$$\Delta u + k^2 u = 0.$$

If u satisfies one of the following boundary conditions

$$\begin{cases} u = 0 & on \; \theta_1^\perp \cap \Omega \\ u = 0 & on \; \theta_2^\perp \cap \Omega \end{cases}, \quad or \quad \begin{cases} u = 0 & on \; \theta_1^\perp \cap \Omega \\ \partial_n u = 0 & on \; \theta_2^\perp \cap \Omega \end{cases},$$

then $u = 0$.

We remark that Theorem A covers general solutions of the Helmholtz equation, not only solutions that are Fourier transforms of a finite measure. Furthermore, Theorem A allows for a mixture of Dirichlet and Neumann conditions on the given hyperplanes. Finally, the result is valid for higher dimensions for which part of Cheng's argument is not valid.

It is natural to replace hyperplanes by more general manifolds. However, in that case, the reflection principle becomes substantially more involved, as it requires not only a point-to-point reflection, but also an additional integral operator (see, e.g., [EK, Sa] and references therein). This is a major limitation to carry such a method to a more general setting. To overcome this limitation, we pursue a second idea of proof based on the fact that a solution of the Helmholtz equation $\Delta u + k^2 u = 0$ ($k \neq 0$) in a neighborhood of 0 has an expansion in spherical harmonics of the form

$$(1.1) \qquad u(r\theta) \sim (2\pi)^{1/2}(kr)^{-(d-2)/2} \sum_{m=0}^{\infty} \sum_{j=1}^{N(m)} a_{m,j} J_{\nu(m)}(kr) Y_m^j(\theta)$$

for $r \geq 0$ and $\theta \in S^{d-1}$. Here J_ν is the Bessel function, $\nu(m) = m + (d-2)/2$ and $\{Y_m^j\}_{j=1,\ldots,N(m)}$ is a basis for the spherical harmonics of degree m in $\mathbb{R}^d$. The solution of the Laplace equation $\Delta u = 0$ ($k = 0$) possesses a similar local expansion with r^m in place of the Bessel function $(kr)^{-(d-2)/2} J_{\nu(m)}(kr)$.

In dimension 2, we use this formula to show that two Robin conditions on curves whose angle of intersection is an irrational multiple of π are essentially incompatible for Laplace-Helmholtz equations (see Theorem 3.1 for the precise statement). In this context, we prove a version of Cheng's theorem for functions that have a Fourier expansion in polar coordinates of the form

$$u(r\theta) \sim \sum_{m=-\infty}^{\infty} c_m \varphi_{|m|}(r) e^{im\theta},$$

where the φ_k's are a family of functions such that $\varphi_k(r) \sim r^{\alpha_k}$ for some (strictly) increasing sequence of non-negative reals. We call this result the *non-crossing lemma* as it states that nodal lines of such functions cannot cross at an arbitrary angle. As a corollary, we obtain a simpler proof of Cheng's result for solutions of the equation $\Delta u + h(x)u = 0$ for radial h (in particular for constant h, we obtain the Helmholtz-Laplace equation). Exploiting explicit formulas for bases of spherical harmonics, this proof extends to arbitrary dimension and yields the following statement.

Theorem B. *Let $d \geq 3$ and let Ω be a domain in $\mathbb{R}^d$. Let $\mathcal{M}_1, \mathcal{M}_2$ be two $d-1$ dimensional submanifolds of $\mathbb{R}^d$ that intersect at 0 and let θ_1, θ_2 be the normal vectors to $\mathcal{M}_1, \mathcal{M}_2$ at 0. Assume that $\arccos \langle \theta_1, \theta_2 \rangle \notin \pi\mathbb{Q}$.*

Let $k \in \mathbb{R}$ and let u be a solution of $\Delta u + k^2 u = 0$ on Ω such that $u = 0$ on $\mathcal{M}_1 \cup \mathcal{M}_2$. Then $u = 0$ on Ω.

In other words, under the stated conditions on $\mathcal{M}_1, \mathcal{M}_2$ as above, $(\mathbb{S}^{d-1}, \mathcal{M}_1 \cup \mathcal{M}_2)$ is a Heisenberg uniqueness pair.

Theorem B is a variation on the unique continuation of solutions of the Helmholtz equation. It states that a solution of that equation is uniquely determined by its restriction to two generic $(d-1)$-dimensional intersecting submanifolds and can therefore be continued to the full domain.

From the point of view of PDEs, it is perhaps more natural to say that a solution of the Helmholtz equation can be uniquely continued from one side of the manifold $\mathcal{M}_1$ to the other (by some sort of reflection) and likewise for $\mathcal{M}_2$. Theorem B says that, in

general, these continuations are not compatible and thus a solution vanishing on $\mathcal{M}_1$ and $\mathcal{M}_2$ vanishes everywhere.

This picture makes it plausible that the extension/continuation of a solution from its restrictions to lower dimensional manifolds, though unique, is inherently unstable. Indeed, we will prove several statements in this regard (see, e.g., Propositions 3.7 and 4.2).

The paper is organized as follows. In Section 2, we give two simple proofs of Theorem A and prove the non-crossing Lemma and its corollaries. Section 3 treats the more technical extension of the uniqueness properties under Robin-type conditions in dimension 2. Section 4 is devoted to some results in higher dimensions.

2. Two simple proofs

2.1. **Proofs using reflection principles.** In this section we give a simple proof of Theorem A based on the reflection principle for harmonic functions and its extension to the Helmholtz equation. At the same time we remove the artificial condition in [Le, Sj1, JK, GrJ] where the solution of the Helmholtz equation is assumed to be the Fourier transform of a measure.

Theorem 2.1. *Let $d \geq 2$, $k \in \mathbb{R}$, and Ω be a domain (an open, connected set) in $\mathbb{R}^d$ with $0 \in \Omega$. Let $\theta_1, \theta_2 \in \mathbb{S}^{d-1}$ be such that $\arccos \langle \theta_1, \theta_2 \rangle \notin \pi\mathbb{Q}$. Let u be a solution of the Laplace-Helmholtz equation on Ω*

$$\Delta u + k^2 u = 0$$

satisfying one of the following boundary conditions

$$\begin{cases} u = 0 & on \ \theta_1^\perp \cap \Omega \\ u = 0 & on \ \theta_2^\perp \cap \Omega \end{cases} \quad or \quad \begin{cases} u = 0 & on \ \theta_1^\perp \cap \Omega \\ \partial_n u = 0 & on \ \theta_2^\perp \cap \Omega \end{cases}.$$

Then $u = 0$.

Proof. Let r be such that the ball $\overline{B(0,r)} \subset \Omega$. As u is real analytic, it is enough to show that $u = 0$ on $B(0,r)$.

For $j = 1, 2$, let R_j be the reflection with respect to the hyperplane $\theta_j^\perp$. The Schwarz reflection principle (see, e.g., [DL]) implies that $u(R_j x) = -u(x)$ in the case of the Dirichlet condition $u\big|_{H_j} = 0$ and $u(R_j x) = u(x)$ in the case of the Neumann condition $\partial_n u\big|_{H_j} = 0$. It follows that $u((R_1 R_2)^{2n} x) = u(x)$.

In particular, if $u = 0$ on $\theta_1^\perp \cap B(0,r)$, then $u\big((R_1 R_2)^{2n} x\big) = 0$ for every $x \in \theta_1^\perp \cap B(0,r)$. But $R_1 R_2$ is a rotation by the angle $2 \arccos \langle \theta_1, \theta_2 \rangle$ in the affine plane $\mathrm{span}(\theta_1, \theta_2)$. As the angle is an irrational multiple of π, the orbit of $\theta_1^\perp \cap B(0,r)$ under $(R_1 R_2)^2$ is dense in $B(0,r)$, thus $u = 0$ on a dense set and, as u is continuous, $u = 0$ everywhere. $\qquad\square$

Theorem 2.1 assumes either two Dirichlet conditions or a mixture of Dirichlet and Neumann conditions. For the case of two Neumann conditions it is not difficult to see

that, if u is radial, then it satisfies two Neumann conditions, $\begin{cases} \partial_n u = 0 & \text{on } \theta_1^\perp \\ \partial_n u = 0 & \text{on } \theta_2^\perp \end{cases}$ on two arbitrary hyperplanes $\theta_1^\perp$ and $\theta_2^\perp$.

Moreover, in dimension 2, the above proof shows that only radial functions can occur. It is then not difficult to see that u is constant if it satisfies the Laplace equation $\Delta u = 0$ and that u is a constant multiple of $J_0(k|x|)$ if u is a solution of the Helmholtz equation $\Delta u + k^2 u = 0$.

2.2. **The non-crossing lemma and applications.** In the previous section we used reflection principles to show that a solution of the Laplace and Helmholtz equation can not vanish on two lines that intersect with irrational angle. When replacing lines by more general curves, reflection principles become substantially more involved [EK, Sa]. In order to tackle that case, we develop an alternative proof that is based on Fourier expansions in the angular variable in polar coordinates. The results will follow from the following lemma.

Lemma 2.2 (Non-crossing lemma). *Let Γ_1, Γ_2 be two $\mathcal{C}^1$-curves in the plane intersecting at 0. Let γ_1' (resp. γ_2') be the vector tangent to Γ_1 (resp. Γ_2) at 0 and assume that the angle between γ_1' and γ_2' is not a rational multiple of π, $\arccos \langle \gamma_1', \gamma_2' \rangle \notin \pi\mathbb{Q}$.*

Let k_m be a strictly increasing sequence of non-negative numbers and $(\varphi_m)_{m \geq 0}$ be a sequence of functions such that $\varphi_m(r) = r^{k_m}\big(1 + o(1)\big)$ uniformly in a fixed neighborhood of 0. Let u be a function on $\mathbb{R}^2$ given by an expansion in polar coordinates of the form

$$u(r, \theta) = \sum_{m \in \mathbb{Z}} c_m \varphi_{|m|}(r) e^{im\theta},$$

with $\sum_{m \in \mathbb{Z}} |c_m| r_0^{k_{|m|}} < +\infty$ for some $r_0 > 0$.
If u vanishes on Γ_1 and on Γ_2, then $u \equiv 0$ in $\{z \in \mathbb{C} : |z| < r_0\}$.

Remark 2.3. The picture below illustrates the conditions in the lemma. Also, the curves do not need to intersect, it is enough that they are rays emanating from the origin. The lemma is typically applied to a function u that is real analytic in a domain Ω and that has a representation of the above form in the neighborhood of 0. In that case we can even conclude that u vanishes in all of Ω.
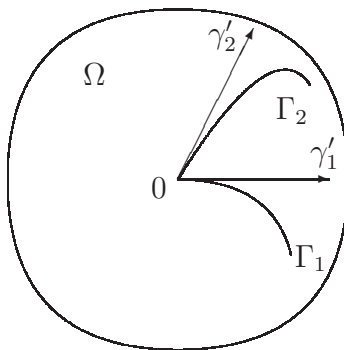

FIGURE 1. In this lemma $\gamma_1' = (1, 0)$ and $\gamma_2' = (1, 2)$.

Proof. Without loss of generality, we may parametrize the curves in polar coordinates by the radius r so that $\gamma_1(r) = (r, \theta_1(r))$ and $\gamma_2(r) = (r, \theta_2(r))$ with $\theta_1(0) = 0$, $\theta_2(0) = \eta$ and $\eta = \arccos \langle \gamma_1', \gamma_2' \rangle$. (If necessary, we shrink the given neighborhood of 0 so that $\theta_j(t)$ is well-defined.) Since the curves intersect at the origin, we must have $u(0,0) = c_0 = 0$.

Now assume that $c_m = 0$ for $m = 0, \pm 1, \ldots, \pm(n-1)$, then by induction we have

$$
\begin{aligned}
u(r, \theta) &= \sum_{|m| \geq n} c_m \varphi_{|m|}(r) e^{im\theta} = \left(c_n e^{in\theta} + c_{-n} e^{-in\theta} \right) r^{k_n} \left(1 + o(1) \right) + \sum_{|m| > n} c_m \varphi_m(r) e^{im\theta} \\
&= \left(c_n e^{in\theta} + c_{-n} e^{-in\theta} \right) r^{k_n} + o(r^{k_n})
\end{aligned}
$$

In other words, we have

$$
\lim_{r \to 0} \frac{u\left(r, \theta_j(r) \right)}{r^{k_n}} = 0
$$

for $j = 1, 2$, and the hypothesis implies that

$$
(2.2) \qquad \begin{cases} c_n & + & c_{-n} & = & 0, \\ c_n e^{in\eta} & + & c_{-n} e^{-in\eta} & = & 0. \end{cases}
$$

This is a system of two equations with two unknowns, and the determinant of its coefficient matrix is $-2i \sin(n\eta) \neq 0$ since $\eta \notin \pi\mathbb{Q}$. Hence, $c_{\pm n} = 0$ and by induction we conclude that $c_n = 0$ for all $n \in \mathbb{Z}$, thus $u = 0$. $\qquad \square$

Remark 2.4. The condition $u = 0$ on Γ_2 may be replaced by $\partial_\theta u = 0$ on Γ_2. Then $\partial_\theta u(r, \theta) = \sum_{m \in \mathbb{Z}} im c_m \varphi_{|m|}(r) e^{im\theta}$, and the system (2.2) is replaced by

$$
\begin{cases} c_n & + & c_{-n} & = & 0 \\ nc_n e^{in\eta} & - & nc_{-n} e^{-in\eta} & = & 0. \end{cases}
$$

Since the determinant is $-2n \cos n\eta \neq 0$ for $n \neq 0$, the induction procedure is the same.

Corollary 2.5. *Let Ω be a domain of $\mathbb{R}^2$. Let Γ_1, Γ_2 be two $\mathcal{C}^1$-curves in the plane intersecting at some $\omega \in \Omega$. Let γ_1' (resp. γ_2') be the unit vector tangent to Γ_1 (resp. Γ_2) at ω and assume that the angle between γ_1' and γ_2' is not a rational multiple of π, $\arccos \langle \gamma_1', \gamma_2' \rangle \notin \pi\mathbb{Q}$.*

Let g be a continuous radial function and define h by $h(x) = g(x - \omega)$. Then the only solution u of $\Delta u + h\, u = 0$ in Ω such that $u = 0$ on $\Gamma_1 \cup \Gamma_2$ is $u = 0$ in Ω.

Proof. Without loss of generality, we may assume that $\omega = 0$. As u is real-analytic, it is enough to prove that u is 0 in a neighborhood of 0 in Ω. In such a neighborhood, u has a Fourier expansion of the form

$$
u(r \cos \theta, r \sin \theta) = \sum_{m \in \mathbb{Z}} c_m e^{im\theta} \varphi_{|m|}(r),
$$

where φ_m is the unique normalized solution of

$$
\varphi_m''(r) + \frac{1}{r} \varphi_m'(r) + \left(h(r) - \frac{m^2}{r^2} \right) \varphi_m(r) = 0
$$

that is smooth at 0. It is not hard to see that $\varphi_m(r) \simeq r^m$ when $r \to 0$. Thus the non-crossing lemma applies and yields the stated conclusion. $\qquad \square$

3. Robin type conditions

We next study the Helmholtz equation with Robin-type conditions on two intersecting curves Γ_1 and Γ_2 in $\mathbb{R}^2$.

3.1. **The main result.** We write the solution of the Helmholtz-Laplace equation in polar coordinates

$$u_p(r, \theta) = u(r\cos\theta, r\sin\theta).$$

Recall that the gradient in polar coordinates is given by $\nabla F = (\partial_r F, r^{-1}\partial_\theta F)$.

We parametrize the curves in polar coordinates as $\gamma(r) = (r\cos\theta(r), r\sin\theta(r))$ with the radius r as the variable. By $\partial_t u_p$ and $\partial_n u_p$ we denote the tangential and normal derivatives of u_p along a curve. A simple computation shows that

$$(3.3) \qquad \partial_t u_p\big(r, \theta(r)\big) = \partial_r u_p\big(r, \theta(r)\big) + \theta'(r)\partial_\theta u_p\big(r, \theta(r)\big)$$

and

$$(3.4) \qquad \partial_n u_p\big(r, \theta(r)\big) = -r\theta'(r)\partial_r u_p\big(r, \theta(r)\big) + \frac{1}{r}\partial_\theta u_p\big(r, \theta(r)\big).$$

We will work with Robin type conditions of the form $\alpha u_p + \beta\partial_t u_p + \tilde{\beta}\partial_n u_p = 0$ on two curves Γ_1 and Γ_2 and allow the coefficients $\alpha, \beta, \tilde{\beta}$ to be functions of r.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a domain with $B(0, r_0) \subset \Omega$ for some $r_0 > 0$. Let $k \in \mathbb{R}$ and for $j = 1, 2$ let $\alpha_j, \beta_j, \tilde{\beta}_j$ be $\mathcal{C}^1$-functions $(-r_0, r_0) \to \mathbb{R}$.*

When $\beta_j(0) = \tilde{\beta}_j(0) = 0$, we impose the following non-degeneracy: $\alpha_j(0) \neq 0$ and $\beta_j(r) = \tilde{\beta}_j(r) = o(r)$ when $r \to 0$.

Next define $\varphi_j = 0$ if $\beta_j(0) = \tilde{\beta}_j(0) = 0$ and $\varphi_j = \arg\big(\beta_j(0) + i\tilde{\beta}_j(0)\big)$ otherwise.

Let θ_1, θ_2 be two $\mathcal{C}^1$-functions $(-r_0, r_0) \to \mathbb{R}$ and assume that $\varphi_1 - \varphi_2 \notin \big(\theta_1(0) - \theta_2(0)\big)\mathbb{Z} + \pi\mathbb{Z}$. Let $\Gamma_j = \{(r\cos\theta_j(r), r\sin\theta_j(r)) : r \in (-r_0, r_0)\}$ be the corresponding curves in polar coordinates in Ω.

Let u be a solution of the Helmholtz-Laplace equation

$$(3.5) \qquad \Delta u(x, y) + k^2 u(x, y) = 0, \qquad (x, y) \in \Omega,$$

with Robin-type conditions

$$(3.6) \qquad \alpha_j(r)u\big(r, \theta_j(r)\big) + \beta_j(r)\partial_t u\big(r, \theta_j(r)\big) + \tilde{\beta}_j(r)\partial_n u\big(r, \theta_j(r)\big) = 0$$

on Γ_j for $j = 1, 2$. If $u(0, 0) = 0$ then $u = 0$.

Remark 3.2. The condition $\varphi_1 - \varphi_2 \notin \big(\theta_1(0) - \theta_2(0)\big)\mathbb{Z} + \pi\mathbb{Z}$ can be seen as either excluding a countable set of initial conditions on $\varphi_1 - \varphi_2$ or a countable set of directions $\theta_1(0) - \theta_2(0)$ at the origin. For instance, if $\beta_1(0) = \beta_2(0) = 0$ or $\tilde{\beta}_1(0) = \tilde{\beta}_2(0) = 0$ we obtain the condition $\theta_1(0) - \theta_2(0) \notin \mathbb{Q}\pi$.

The statement would be more natural with $\alpha_j, \beta_j, \tilde{\beta}_j$ constants. However, in order to apply Theorem 3.1 to the Helmholtz equation on manifolds, we need this more general condition. Also, the non-degeneracy is then needed, since any solution of course satisfies a Robin-type condition of the type (3.6).

Proof. First note that, as u is real-analytic, it is enough to show that $u = 0$ in $B(0, r_0)$.

Case 1: $k = 0$. We first consider the case $k = 0$ corresponding to the Laplace equation. Locally in a neighborhood of 0 and in polar coordinates, its solution has the following series expansion [MF] (with $u_p(r, \theta) = u(r\cos\theta, r\sin\theta)$):

$$(3.7) \qquad u_p(r, \theta) = \sum_{m \in \mathbb{Z}} c_m r^{|m|} e^{im\theta}.$$

Let us note right away that $c_0 = u(0, 0) = 0$.

It follows that

$$\partial_r u_p(r, \theta) = \sum_{m=1}^{\infty} m\big(c_m e^{im\theta} + c_{-m} e^{-im\theta}\big) r^{m-1},$$

$$\frac{1}{r}\partial_\theta u_p(r, \theta) = \sum_{m=1}^{\infty} im\big(c_m e^{im\theta} - c_{-m} e^{-im\theta}\big) r^{m-1}.$$

Now suppose that $c_{-m_0} = \cdots = c_{m_0} = 0$ for some $m_0 \geq 0$. Then by looking at the lowest order terms in r we see from (3.3) and (3.4) that

$$(3.8) \quad u_p\big(r, \theta_j(r)\big) = \big(c_{m_0+1} e^{i(m_0+1)\theta_j} + c_{-(m_0+1)} e^{-i(m_0+1)\theta_j}\big) r^{m_0+1} + o(r^{m_0+1})$$

$$\partial_t u_p\big(r, \theta_j(r)\big) = (m_0 + 1)\big(c_{m_0+1} e^{i(m_0+1)\theta_j} + c_{-(m_0+1)} e^{-i(m_0+1)\theta_j}\big) r^{m_0} + o(r^{m_0})$$

$$\partial_n u_p\big(r, \theta_j(r)\big) = i(m_0 + 1)\big(c_{m_0+1} e^{i(m_0+1)\theta_j} - c_{-(m_0+1)} e^{-i(m_0+1)\theta_j}\big) r^{m_0} + o(r^{m_0}).$$

It follows that

$$\alpha_j(r) u_p\big(r, \theta_j(r)\big) + \beta_j(r)\partial_t u_p\big(r, \theta_j(r)\big) + \tilde\beta_j(r)\partial_n u_p\big(r, \theta_j(r)\big)$$

$$= \alpha_j(r)\big(c_{m_0+1} e^{i(m_0+1)\theta_j} + c_{-(m_0+1)} e^{-i(m_0+1)\theta_j}\big) r^{m_0+1}$$

$$+ (m_0 + 1)\big(\beta_j(r) + i\tilde\beta_j(r)\big) c_{m_0+1} e^{i(m_0+1)\theta_j} r^{m_0}$$

$$+ (m_0 + 1)\big(\beta_j(r) - i\tilde\beta_j(r)\big) c_{-(m_0+1)} e^{-i(m_0+1)\theta_j} r^{m_0}$$

$$+ o(r^{m_0+1}) + o\Big(\big(\beta_j(r) + i\tilde\beta_j(r)\big) r^{m_0}\Big).$$

We now distinguish two cases:

– either $\big(\beta_j(0), \tilde\beta_j(0)\big) \neq (0, 0)$ and then $\kappa_j := \beta_j(0) + i\tilde\beta_j(0) \neq 0$ so that

$$\alpha_j(r) u_p\big(r, \theta_j(r)\big) + \beta_j(r)\partial_t u_p\big(r, \theta_j(r)\big) + \tilde\beta_j(r)\partial_n u_p\big(r, \theta_j(r)\big)$$

$$= (m_0 + 1)\big(c_{m_0+1}\kappa_j e^{i(m_0+1)\theta_j} + c_{-(m_0+1)}\overline{\kappa_j} e^{-i(m_0+1)\theta_j}\big) r^{m_0} + o(r^{m_0}),$$

– or $\big(\beta_j(0), \tilde\beta_j(0)\big) = (0, 0)$, and then we assumed that $\beta_j(r) \pm i\tilde\beta_j(r) = o(r)$, thus

$$\alpha_j(r) u_p\big(r, \theta_j(r)\big) + \beta_j(r)\partial_t u_p\big(r, \theta_j(r)\big) + \tilde\beta_j(r)\partial_n u_p\big(r, \theta_j(r)\big)$$

$$= \alpha_j(0)\big(c_{m_0+1} e^{i(m_0+1)\theta_j} + c_{-(m_0+1)} e^{-i(m_0+1)\theta_j}\big) r^{m_0+1} + o(r^{m_0+1}).$$

It follows that each Robin condition

$$\alpha_j(r) u_p\big(r, \theta_j(r)\big) + \beta_j(r)\partial_t u_p\big(r, \theta_j(r)\big) + \tilde\beta_j(r)\partial_n u_p\big(r, \theta_j(r)\big) = 0$$

implies that

$$\begin{cases} c_{m_0+1}\kappa_j e^{i(m_0+1)\theta_j(0)} + c_{-(m_0+1)}\overline{\kappa_j}e^{-i(m_0+1)\theta_j(0)} = 0 & \text{if } \kappa_j \neq 0, \\ c_{m_0+1}e^{i(m_0+1)\theta_j(0)} + c_{-(m_0+1)}e^{-i(m_0+1)\theta_j(0)} = 0 & \text{if } \kappa_j = 0. \end{cases}$$

Now, if we impose two Robin conditions, then every pair $c_{-(m_0+1)}, c_{m_0+1}$ is the solution of one of the following linear systems.

(i) If $\big(\beta_j(0), \tilde{\beta}_j(0)\big) \neq (0,0)$ for $j = 1$ and $j = 2$, then

$$\begin{cases} c_{m_0+1}\kappa_1 e^{i(m_0+1)\theta_1(0)} + c_{-(m_0+1)}\overline{\kappa_1}e^{-i(m_0+1)\theta_1(0)} = 0, \\ c_{m_0+1}\kappa_2 e^{i(m_0+1)\theta_2(0)} + c_{-(m_0+1)}\overline{\kappa_2}e^{-i(m_0+1)\theta_2(0)} = 0, \end{cases}$$

which has determinant $2i\text{Im}\,\kappa_1\overline{\kappa_2}e^{i(m_0+1)(\theta_1-\theta_2)} = 2i|\kappa_1\kappa_2|\sin\big(\varphi_1-\varphi_2+(m_0+1)\big(\theta_1(0)-\theta_2(0)\big)$. This determinant is never 0 by the assumption on $\theta_1(0) - \theta_2(0)$. Therefore $c_{m_0+1} = c_{-(m_0+1)} = 0$.

(ii) If $\big(\beta_j(0), \tilde{\beta}_j(0)\big) = (0,0)$ for $j = 1, 2$, then

$$\begin{cases} c_{m_0+1}e^{i(m_0+1)\theta_1(0)} + c_{-(m_0+1)}e^{-i(m_0+1)\theta_1(0)} = 0, \\ c_{m_0+1}e^{i(m_0+1)\theta_2(0)} + c_{-(m_0+1)}e^{-i(m_0+1)\theta_2(0)} = 0. \end{cases}$$

Again the determinant of this system is $2i\sin(m_0 + 1)\big(\theta_1(0) - \theta_2(0)\big) \neq 0$ by our assumption on $\theta_1(0) - \theta_2(0)$. Therefore $c_{m_0+1} = c_{-(m_0+1)} = 0$.

(iii) If $\big(\beta_1(0), \tilde{\beta}_1(0)\big) = (0,0)$ and $\big(\beta_2(0), \tilde{\beta}_2(0)\big) \neq (0,0)$ (without loss of generality), then

$$\begin{cases} c_{m_0+1}e^{i(m_0+1)\theta_1(0)} + c_{-(m_0+1)}e^{-i(m_0+1)\theta_1(0)} = 0, \\ c_{m_0+1}\kappa_2 e^{i(m_0+1)\theta_2(0)} + c_{-(m_0+1)}\overline{\kappa_2}e^{-i(m_0+1)\theta_2(0)} = 0, \end{cases}$$

which has determinant $2i\text{Im}\,\overline{\kappa_2}e^{i(m_0+1)(\theta_1(0)-\theta_2(0))} = 2i|\kappa_2|\sin\big(-\phi_2 + (m_0 + 1)\big(\theta_1(0) - \theta_2(0)\big) \neq 0$ by assumption on $\theta_1(0) - \theta_2(0)$. Therefore $c_{m_0+1} = c_{-(m_0+1)} = 0$.

In all three cases, we obtain that $c_{m_0+1} = c_{-(m_0+1)} = 0$, and by induction, it follows that $c_m = 0$ for every m, thus $u = 0$.

Case 2: $k \neq 0$. We next treat the Helmholtz equation with Robin-type conditions on two curves. After a change of variables, we may assume without loss of generality that $k = 1$. Then in a neighborhood of 0, the solution written in polar coordinates possesses a Bessel expansion of the form

$$(3.9) \qquad u_p(r, \theta) = \sum_{m \in \mathbb{Z}} c_m J_{|m|}(r)e^{im\theta},$$

where J_m is the Bessel function of the first kind and order m.

Since $J_m(0) = 0$ for $m \neq 0$, we obtain that $c_0 = u(0,0) = 0$.

Using the relation for the Bessel functions

$$\frac{m}{r}J_m(r) = \frac{1}{2}(J_{m+1}(r) + J_{m-1}(r)),$$

we obtain

$$
\begin{aligned}
\frac{1}{r}\partial_\theta u_p(r,\theta) &= \sum_{m\geq 1}(c_m e^{im\theta} - c_{-m}e^{-im\theta})\frac{im}{r}J_m(r) \\
&= \sum_{m\geq 1}(c_m e^{im\theta} - c_{-m}e^{-im\theta})\frac{i}{2}(J_{m+1}(r) + J_{m-1}(r)) \\
&= \frac{i}{2}\sum_{m\geq 0}(c_{m+1}e^{i(m+1)\theta} - c_{-m-1}e^{-i(m+1)\theta})J_m(r) \\
&\quad + \frac{i}{2}\sum_{m\geq 2}(c_{m-1}e^{i(m-1)\theta} - c_{-m+1}e^{-i(m-1)\theta})J_m(r) \\
&= \frac{i}{2}\left(c_1 e^{i\theta} - c_{-1}e^{-i\theta}\right)J_0(r) + \frac{i}{2}\left(c_2 e^{2i\theta} - c_{-2}e^{-2i\theta}\right)J_1(r) \\
&\quad + \frac{i}{2}\sum_{k\geq 2}\Bigl(c_{k+1}e^{i(k+1)\theta} + c_{k-1}e^{i(k-1)\theta} \\
&\qquad\qquad -c_{-(k-1)}e^{-i(k-1)\theta} - c_{-(k+1)}e^{-i(k+1)\theta}\Bigr)J_k(r).
\end{aligned}
$$

In the same way, using the relation for the Bessel functions

$$
\frac{\mathrm{d}}{\mathrm{d}r}J_m(r) = \frac{1}{2}(J_{m+1}(r) - J_{m-1}(r))
$$

for $m \neq 0$, we obtain

$$
\begin{aligned}
\partial_r u_p(r,\theta) &= \frac{1}{2}\sum_{m\geq 1}(c_m e^{im\theta} + c_{-m}e^{-im\theta})\left(J_{m+1}(r) - J_{m-1}(r)\right) \\
&= -\frac{1}{2}(c_1 e^{i\theta} + c_{-1}e^{-i\theta})J_0(r) - \frac{1}{2}(c_2 e^{2i\theta} + c_{-2}e^{-2i\theta})J_1(r) \\
&\quad + \frac{1}{2}\sum_{k\geq 2}(-c_{k+1}e^{i(k+1)\theta} + c_{k-1}e^{i(k-1)\theta} \\
&\qquad\qquad + c_{-(k-1)}e^{-i(k-1)\theta} - c_{-(k+1)}e^{-i(k+1)\theta})J_k(r).
\end{aligned}
$$

Recall that $J_m(r) = \dfrac{r^m}{2^m m!} + o(r^m)$ when $r \to 0$. Suppose now that $c_{-m_0} = \cdots = c_{m_0} = 0$ for some $m_0 \geq 0$, then the terms of lowest order in r in (3.3) and (3.4) yield

$$
u_p\bigl(r,\theta_j(r)\bigr) = \frac{1}{2^{m_0+1}(m_0+1)!}\left(c_{m_0+1}e^{i(m_0+1)\theta_j} + c_{-(m_0+1)}e^{-i(m_0+1)\theta_j}\right)r^{m_0+1} + o(r^{m_0+1})
$$

$$
\partial_t u_p\bigl(r,\theta_j(r)\bigr) = \frac{-1}{2^{m_0+1}m_0!}\left(c_{m_0+1}e^{i(m_0+1)\theta_j} + c_{-(m_0+1)}e^{-i(m_0+1)\theta_j}\right)r^{m_0} + o(r^{m_0})
$$

$$
\partial_n u_p\bigl(r,\theta_j(r)\bigr) = \frac{i}{2^{m_0+1}m_0!}\left(c_{m_0+1}e^{i(m_0+1)\theta_j} - c_{-(m_0+1)}e^{-i(m_0+1)\theta_j}\right)r^{m_0} + o(r^{m_0}).
$$

Thus the terms of lowest order in r are exactly a fixed multiple of the terms in (3.8) for the Laplace equation.

Consequently the Robin condition (3.6) implies the same systems of equations as in the previous case. Thus the remainder of the proof is exactly the same and we conclude that $u = 0$. $\qquad\square$

Remark 3.3. If we replace the exact condition

$$\alpha_j(r)u\big(r, \theta_j(r)\big) + \beta_j(r)\partial_t u\big(r, \theta_j(r)\big) + \tilde{\beta}_j(r)\partial_n u\big(r, \theta_j(r)\big) = 0,$$

by an approximate one

$$\alpha_j(r)u\big(r, \theta_j(r)\big) + \beta_j(r)\partial_t u\big(r, \theta_j(r)\big) + \tilde{\beta}_j(r)\partial_n u\big(r, \theta_j(r)\big) = o(r^N),$$

and assume that $(\beta_j(0), \tilde{\beta}_j(0)) \neq (0,0)$, then we may still conclude that $c_{-N-1} = \cdots = c_{N+1} = 0$ in the decompositions (3.7)-(3.9). As a consequence, $u(x) = o(|x|^{N+1})$.

The case of the Helmholtz equation on a manifold with Robin conditions can be deduced by following Cheng's argument.

Corollary 3.4. *Let $\mathcal{M}$ be a two-dimensional $\mathcal{C}^\infty$-Riemann manifold without boundary and $h \in \mathcal{C}^\infty(M)$.*

Let $\omega \in \mathcal{M}$ and γ_1, γ_2 be two geodesics on $\mathcal{M}$ that intersect at ω such that the angle of their tangents at ω is an irrational multiple of π. Let $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that $(a_j, b_j) \neq (0,0)$.

If u is a solution of the Helmholtz equation $\Delta u + hu = 0$ with $u(\omega) = 0$ and satisfies the two Robin conditions

$$(3.10) \qquad\qquad a_j u + b_j \partial_n u = 0 \quad \text{on } \gamma_j, \quad j = 1, 2,$$

then $u = 0$.

Proof. The proof follows the path of Cheng's result on nodal domains [Ch, Thm. 2.2]. First, we use normal coordinates on $\mathcal{M}$ which allows us to restrict u to a small neighborhood of 0, $u(0) = 0$, such that γ_1, γ_2 are two straight lines intersecting at 0 with angle that is an irrational multiple of π.

Further, Bers' result [Be] implies that we can write

$$u(x,y) = p(x,y) + (x^2 + y^2)^{m/2+\epsilon}q(x,y)$$

where p is an harmonic polynomial of degree m and q is smooth. Further, u cannot vanish to infinite order at 0 unless $u = 0$ uniformly. Thus we may assume that $p \neq 0$ if $u \neq 0$.

One then easily sees that the Robin condition (3.10) implies a Robin condition of the type given by (3.6) with $\alpha_j(0) = a_j$, $\beta_j(0) = 0$ and $\tilde{\beta}_j(0) = b_j$. Moreover, if $b_j = 0$ then $\beta_j(r) = \tilde{\beta}_j(r) = 0$ so that the non-degeneracy condition is satisfied.

Now, according to Remark 3.3, this Robin condition implies that $u(x,y) = (x^2 + y^2)^{m/2+\epsilon}q(x,y)$ and $p = 0$. We thus conclude that $u = 0$. $\qquad\square$

3.2. The condition $u(0,0) = 0$.
One may ask whether the condition $u(0,0) = 0$ is really needed in Theorem 3.1. This hypothesis was the basis for the induction and therefore crucial to obtain $u = 0$ in the previous proof.

We now show that this condition may not always be removed, not even in the simple case of constant coefficients in the Robin condition.

Proposition 3.5. *Let $\Omega \subset \mathbb{R}^2$ be a domain with $0 \in \Omega$. Let $k \in \mathbb{R}$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ with $\beta_1, \beta_2 \neq 0$. Let $\theta_1, \theta_2 \in \mathbb{R}$ be such that $\theta_1 - \theta_2 \notin \pi\mathbb{Q}$. Let $l_{\theta_i} = \mathbb{R}(\cos\theta_j, \sin\theta_j)$ be lines passing through 0. Then the space of solutions of the Helmholtz equation with Robin boundary conditions:*

$$
(3.11) \qquad \begin{cases}
\Delta u(x,y) + k^2 u(x,y) = 0, & (x,y) \in \Omega, \\
\alpha_1 u(x,y) + \beta_1 \partial_n u(x,y) = 0, & (x,y) \in \ell_{\theta_1} \\
\alpha_2 u(x,y) + \beta_2 \partial_n u(x,y) = 0, & (x,y) \in \ell_{\theta_2},
\end{cases}
$$

has dimension at most 1.

The solution space has dimension exactly one, if $\dfrac{\theta_1 - \theta_2}{\pi}$ is badly approximable by rationals in the sense that there is a $c > \sqrt{5}$ such that for all but finitely many integers k, ℓ,

$$
(3.12) \qquad |k(\theta_1 - \theta_2) - \ell\pi| \geq \frac{1}{ck}.
$$

Remark 3.6. The condition $c > \sqrt{5}$ comes from Hurwitz's theorem. If $\theta_1 - \theta_2$ is an algebraic number of degree 2, then it is badly approximable by rationals.

As already noticed, if $\Delta u + k^2 u = 0$ and $\partial_n u = 0$ on ℓ_{θ_1} and on ℓ_{θ_2}, then u is radial. It is well known that the only radial solutions of $\Delta u + k^2 u = 0$ are constant when $k = 0$ and constant multiples of $J_0(k\sqrt{x^2 + y^2})$ when $k \neq 0$. The above result extends this to more general Robin conditions.

Proof. We only treat the case $k = 1$ and leave the scaling to obtain $k \neq 0$ and the easier case $k = 0$ to the reader. Again, writing u in polar coordinates as $u_p(r, \theta) = u(r\cos\theta, r\sin\theta)$, the solution to the Helmholtz equation is of the form

$$
u_p(r, \theta) = \sum_{m \in \mathbb{Z}} c_m J_{|m|}(r) e^{im\theta}.
$$

The previous proof shows that the coefficients of J_0 and J_1 in the Robin boundary condition for $j = 1, 2$ are given by

$$
(3.13) \qquad \alpha_j c_0 + \frac{i\beta_j}{2}(c_1 e^{i\theta_j} - c_{-1} e^{-i\theta_j}) = 0,
$$

$$
(3.14) \qquad \alpha_j(c_1 e^{i\theta_j} + c_{-1} e^{-i\theta_j}) + \frac{i\beta_j}{2}(c_2 e^{2i\theta_j} - c_{-2} e^{-2i\theta_j}) = 0,
$$

and, for $m \geq 2$,

$$
\alpha_j(c_m e^{im\theta_j} + c_{-m} e^{-im\theta_j})
$$
$$
+ \frac{i\beta_j}{2}\Big(c_{m+1} e^{i(m+1)\theta_j} - c_{-m-1} e^{-i(m+1)\theta_j}
$$
$$
(3.15) \qquad + c_{m-1} e^{i(m-1)\theta_j} - c_{-m+1} e^{-i(m-1)\theta_j} \Big) = 0.
$$

Writing $\gamma_j = 2i\dfrac{\alpha_j}{\beta_j}$, (3.13) implies that

$$
\begin{cases}
c_1 e^{i\theta_1} - c_{-1} e^{-i\theta_1} = \gamma_1 c_0 \\
c_1 e^{i\theta_2} - c_{-1} e^{-i\theta_2} = \gamma_2 c_0,
\end{cases}
$$

from which follows

$$c_1 = -\frac{\gamma_2 e^{-i\theta_1} - \gamma_1 e^{-i\theta_2}}{2i\sin(\theta_1 - \theta_2)}c_0 \quad , \quad c_{-1} = -\frac{\gamma_2 e^{i\theta_1} - \gamma_1 e^{i\theta_2}}{2i\sin(\theta_1 - \theta_2)}c_0.$$

Next, (3.14) implies

$$\begin{cases} c_2 e^{2i\theta_1} - c_{-2}e^{-2i\theta_1} = \gamma_1(c_1 e^{i\theta_1} + c_{-1}e^{-i\theta_1}) \\ c_2 e^{2i\theta_2} - c_{-2}e^{-2i\theta_2} = \gamma_2(c_1 e^{i\theta_2} + c_{-1}e^{-i\theta_2}) \end{cases},$$

that is

$$\begin{cases} c_2 e^{2i\theta_1} - c_{-2}e^{-2i\theta_1} = \frac{\gamma_1}{i\sin(\theta_1-\theta_2)}(\gamma_1\cos(\theta_1-\theta_2) - \gamma_2)c_0, \\ c_2 e^{2i\theta_2} - c_{-2}e^{-2i\theta_2} = \frac{\gamma_2}{i\sin(\theta_1-\theta_2)}(\gamma_1 - \gamma_2\cos(\theta_1-\theta_2))c_0. \end{cases}$$

This may be written in the form

$$\begin{cases} c_2 e^{2i\theta_1} - c_{-2}e^{-2i\theta_1} = \frac{\mu_2}{\sin(\theta_1-\theta_2)}c_0 \\ c_2 e^{2i\theta_2} - c_{-2}e^{-2i\theta_2} = \frac{\nu_2}{\sin(\theta_1-\theta_2)}c_0 \end{cases},$$

with $|\mu_2|, |\nu_2| \le (|\gamma_1| + |\gamma_2|)^2$. It follows that

$$c_2 = \frac{\mu_2 e^{-2i\theta_2} - \nu_2 e^{-2i\theta_1}}{2i\sin(\theta_1 - \theta_2)\sin 2(\theta_1 - \theta_2)}c_0 \quad , \quad c_{-2} = -\frac{\nu_2 e^{2i\theta_1} - \mu_2 e^{2i\theta_2}}{2i\sin(\theta_1 - \theta_2)\sin 2(\theta_1 - \theta_2)}c_0.$$

One can then write (3.15) in the form

$$c_{m+1}e^{i(m+1)\theta_j} - c_{-m-1}e^{-i(m+1)\theta_j} = \gamma_j(c_m e^{im\theta_j} + c_{-m}e^{-im\theta_j})$$
$$-c_{m-1}e^{i(m-1)\theta_j} + c_{-m+1}e^{-i(m-1)\theta_j},$$

from which we get that $c_{\pm(m+1)}\sin(m+1)(\theta_1 - \theta_2)$ is a linear combination of $c_{\pm m}$ and $c_{\pm(m-1)}$ with coefficients bounded by $\max(1, \gamma_1, \gamma_2)$ so that

$$c_{\pm(m+1)} = \frac{\kappa_{\pm(m+1)}}{\prod_{k=1}^{m+1}\sin k(\theta_1 - \theta_2)}c_0,$$

with $|\kappa_{\pm(m+1)}| \le (2 + |\gamma_1| + |\gamma_2|)^{m+1}$.

Further, the Bessel's function satisfies the bound

(3.16) $$|J_m(r)| \le \frac{r^m}{2^m m!}.$$

Thus, if $\frac{\theta_1 - \theta_2}{\pi}$ is badly approximable by rationals,

$$\left(\prod_{k=1}^{m+1}\sin k(\theta_1 - \theta_2)\right)^{-1} = O\big((m+1)!\big)$$

so that

(3.17) $$u(r, \theta) = \sum_m c_m J_{|m|}(r)e^{im\theta}$$

converges to a solution u of the Helmholtz equation.

$\square$

We suspect that in some cases the series (3.17) diverges so that the solution space is still of dimension 0. For instance, if $\alpha_1 = 0$, $\beta_1 = 1$, $\theta_1 = 0$, that is if u satisfies a Neumann equation on the x-axis, then the Schwarz reflection principle shows that $c_{-m} = c_m$. Then, the Robin condition on ℓ_{θ_2} implies

$$\gamma_2 c_m \cos m\theta_2 = c_{m-1} \sin(m-1)\theta_2 + c_{m+1} \sin(m+1)\theta_2.$$

We think that for some numbers θ_2 (in particular when θ_2/π is a Liouville number), this would lead to a series (3.17) that is divergent.

3.3. **Stability.** A natural extension of Remark 3.3 would be to replace the Robin condition by an approximate Robin condition and ask whether the corresponding solution is small. In general this is not true. We illustrate the instability in the simplest case, namely for the Laplace equation with Dirichlet conditions on lines.

Proposition 3.7. *Let $\theta_1, \theta_2 \in \mathbb{R}$, be such that $\theta_1 - \theta_2 \notin \pi\mathbb{Q}$, and let $\varepsilon > 0$.*
There exist infinitely many $n \in \mathbb{N}$ such that the solution of the Laplace equation

$$\begin{cases} \Delta u(x,y) = 0, & (x,y) \in D(0,1), \\ u(r,\theta_1) = \varepsilon|r|^n, & |r| < 1, \\ u(r,\theta_2) = 2\varepsilon|r|^n, & |r| < 1, \end{cases}$$

satisfies $\|u\|_{L^\infty(D(0,1))} \geq \frac{n\varepsilon}{8}$.

Proof. Write u in polar coordinates as

$$u(r,\theta) = \sum_{m\in\mathbb{Z}} c_m r^{|m|} e^{im\theta} = c_0 + \sum_{m=1}^{\infty} (c_m e^{im\theta} + c_{-m} e^{-im\theta}) r^m.$$

Then each Dirichlet condition is equivalent to the set of equations

$$c_m e^{im\theta_j} + c_{-m} e^{-im\theta_j} = u_{m,j},$$

where $u_{n,1} = \varepsilon$, $u_{n,2} = 2\varepsilon$ and $u_{m,j} = 0$ if $m \neq n$.

For any m the resulting system of equations has determinant $2i \sin m(\theta_1 - \theta_2) \neq 0$. If $m \neq n$, then $c_m = c_{-m} = 0$. For $m = n$ we obtain

$$c_n = \frac{1 - 2e^{-in(\theta_1-\theta_2)}}{2i \sin n(\theta_1 - \theta_2)} e^{-in\theta_2} \varepsilon \quad , \quad c_{-n} = -\frac{1 - 2e^{in(\theta_1-\theta_2)}}{2i \sin n(\theta_1 - \theta_2)} e^{in\theta_2} \varepsilon.$$

It follows that

$$u(r,\theta) = \frac{\varepsilon}{\sin n(\theta_1 - \theta_2)} \text{Im}\left((1 - 2e^{-in(\theta_1-\theta_2)}) e^{in(\theta-\theta_2)}\right) r^n.$$

But now, according to Dirichlet's theorem, there exist infinitely many $n, p \in \mathbb{Z} \setminus \{0\}$ such that

$$\left|\frac{\theta_1 - \theta_2}{\pi} n - p\right| \leq \frac{1}{n},$$

and thus $|\cos n(\theta_1 - \theta_2) - \cos p\pi| \leq \pi/n$ and $|\sin n(\theta_1 - \theta_2) - \sin p\pi| \leq \pi/n$.

Now choose θ such that $n(\theta - \theta_2) = \pi/2$ and $e^{in(\theta-\theta_2)} = i$. Then for $n > 6$

$$|\text{Im}\left((1 - 2e^{-in(\theta_1-\theta_2)}) e^{in(\theta-\theta_2)}\right) = |\text{Re}\,(1 - 2e^{-in(\theta_1-\theta_2)})|$$

$$= |1 - 2\cos n(\theta_1 - \theta_2)| \geq |1 - 2(-1)^p| - \frac{\pi}{n} \geq \frac{1}{2}.$$

It follows that $|u(r, \theta)| \geq \frac{n}{2\pi} \varepsilon r^n$, from which the result follows immediately. $\qquad\square$

Proposition 3.7 implies that the reconstruction of a harmonic function from its restriction to two lines is always unstable. In particular, if u_1 and u_2 are harmonic functions such that $\|(u_1 - u_2)|_{\Gamma_j}\|_\infty < \epsilon$, we cannot make any assertion on the error $\|u_1 - u_2\|_{L^\infty(D(0,1))}$.

Note that Dirichlet's theorem allows to simultaneously approximate arbitrarily many irrational numbers. Consider the Laplace equation

$$\begin{cases} \Delta u(x, y) = 0, & (x, y) \in D(0, 1), \\ u(r, \theta_j) = \alpha_j \varepsilon |r|^n, & r \in \mathbb{R}, j = 1, \ldots, N. \end{cases}$$

For this system to have a solution, the α_j's must satisfy some compatibility condition. Consequently, this system either has no solution, or, if the α_j's are compatible, the norm of the solution can be arbitrarily large.

4. RESULTS IN HIGHER DIMENSION

In this section we study possible extensions of the uniqueness property to higher dimensions. The general problem is the following: For $1 \leq m < d$, let $\Gamma_1, \ldots, \Gamma_N$ be m-dimensional analytic submanifolds (or just linear subspaces) in $\mathbb{R}^d$ intersecting at 0. Under which conditions is $\Gamma_1 \cup \cdots \cup \Gamma_N$ a set of uniqueness for solutions of $\Delta u + k^2 u = 0$? In other words, when is it true that $u|_{\Gamma_1 \cup \cdots \cup \Gamma_N} = 0$ implies that $u = 0$.

We will treat the case of two hypersurfaces ($m = d - 1$) and give a counter-example for an arbitrary finite collection of lines.

4.1. **An extension of the results of Section 2.2.** We will now extend some of the results of Section 3 to higher dimensions.

Theorem 4.1. *Let $d \geq 3$ and let Ω be a domain in $\mathbb{R}^d$. Let $\mathcal{M}_1, \mathcal{M}_2$ be two $d - 1$ dimensional submanifolds of $\mathbb{R}^d$ that intersect at 0 and let θ_1, θ_2 be the unit normal vectors to $\mathcal{M}_1$ and $\mathcal{M}_2$ at 0. Assume that $\arccos \langle \theta_1, \theta_2 \rangle \notin \pi\mathbb{Q}$.*

If u is a solution of $\Delta u + k^2 u = 0$ on Ω such that $u = 0$ on $\mathcal{M}_1 \cup \mathcal{M}_2$, then $u = 0$ on Ω.

Proof. We first describe a particular basis of spherical harmonics and fix some notation taken from *e.g.* [DX].

First, we introduce the Gegenbauer polynomials

$$C_n^\lambda(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{\Gamma(n - k + \lambda)}{\Gamma(\lambda) k! (n - 2k)!} (2x)^{n-2k}.$$

They satisfy the orthogonality relation $\int_0^\pi C_m^\lambda(\cos \theta) C_n^\lambda(\cos \theta) \sin(\theta)^{2\lambda} \, \mathrm{d}\theta = 0$ if $m \neq n$.

We use spherical coordinates in $\mathbb{R}^d$,

$$
\begin{cases}
x_1 &= r\sin\theta_{d-2}\cdots\sin\theta_1\cos\varphi \\
x_2 &= r\sin\theta_{d-2}\cdots\sin\theta_1\sin\varphi \\
x_3 &= r\sin\theta_{d-2}\cdots\sin\theta_2\cos\theta_1 \\
\vdots & \vdots \\
x_{d-1} &= r\sin\theta_{d-2}\cos\theta_{d-3} \\
x_d &= r\cos\theta_{d-2}
\end{cases}
$$

with $r \geq 0$, $\theta := (\theta_1, \ldots, \theta_{d-2}) \in [0,\pi)^{d-2}$, $\varphi \in [0, 2\pi)$.

After a suitable translation and rotation, we may assume that, in a neighborhood V of 0, both $\mathcal{M}_1 \cap V$ and $\mathcal{M}_2 \cap V$ are parametrized as

$$
\mathcal{M}_j \cap V = \left\{ \left(r, \theta, \psi_j(r,\theta)\right) : 0 < r < \varepsilon, \theta \in [0,\pi)^{d-2} \right\}
$$

where the ψ_j's are smooth functions. Moreover, if we define

$$
\varphi_j = \psi_j(0,\theta) = \lim_{r\to 0}\psi_j(r,\theta)
$$

(which does not depend on r), then our hypothesis on $\mathcal{M}_1, \mathcal{M}_2$ implies that $\varphi_1 - \varphi_2 \notin \pi\mathbb{Q}$.

Next, we define a basis of spherical harmonics. For $\alpha \in \mathcal{N} := \mathbb{N}_0^{d-2} \times \mathbb{Z}$ we set

$$
\begin{aligned}
Y_\alpha(r,\theta_1,\ldots,\theta_{d-2},\varphi) &= r^{|\alpha|}e^{i\alpha_{d-1}\varphi}\prod_{j=1}^{d-2}(\sin\theta_{d-j})^{|\alpha|^{j+1}}C_{\alpha_j}^{\lambda_j}(\cos\theta_j) \\
&:= r^{|\alpha|}e^{i\alpha_{d-1}\varphi}\tilde{Y}_\alpha(\theta_1,\ldots,\theta_{d-2})
\end{aligned}
$$

where $|\alpha|^j = \alpha_j + \cdots + \alpha_{d-1}$ and $\lambda_j = |\alpha|^{j+1} + (d-j-1)/2$. Then $\{Y_\alpha : \alpha \in \mathcal{N}\}$ forms a basis of spherical harmonics on $\mathbb{R}^d$. Further, for each $n \geq 1$, the set

$$
\{\tilde{Y}_{\beta,m} : (\beta,m) \in \mathbb{N}_0^{d-1}, |\beta| + m = n\}
$$

is linearly independent.

In a neighborhood of 0, an arbitrary solution u of the Laplace equation $\Delta u = 0$ can then be written in spherical coordinates in the form

$$
(4.18)\quad u(r,\theta_1,\ldots,\theta_{d-2},\varphi) = \sum_{\alpha\in\mathcal{N}}c_\alpha Y_\alpha(r,\theta_1,\ldots,\theta_{d-2},\varphi)
$$

$$
= \sum_{n=0}^\infty r^n \sum_{m=-n}^n \left(\sum_{\substack{\beta\in\mathbb{N}_0^{d-2} \\ |\beta|+|m|=n}} c_{\beta,m}\tilde{Y}_{\beta,m}(\theta_1,\ldots,\theta_{d-2}) \right) e^{im\varphi}
$$

The local solution of the Helmholtz equation $\Delta u + u = 0$ can be written in the form [MF]

$$
(4.19)\quad u(r,\theta_1,\ldots,\theta_{d-2},\varphi)
$$

$$
= r^{-\frac{d-2}{2}}\sum_{n=0}^\infty J_{n+\frac{d-2}{2}}(r) \sum_{m=-n}^n \left(\sum_{\substack{\beta\in\mathbb{N}_0^{d-2} \\ |\beta|+|m|=n}} c_{\beta,m}\tilde{Y}_{\beta,m}(\theta_1,\ldots,\theta_{d-2}) \right) e^{im\varphi}.
$$

Now let u be a solution of the Laplace equation, thus given by (4.18).

First $c_0 = u(0, \theta, \varphi_j) = 0$. But then

$$u(r, \theta, \psi_j(r, \theta)) = r \sum_{m=-1}^{1} \left(\sum_{\substack{\beta \in \mathbb{N}_0^{d-2} \\ |\beta| + |m| = 1}} c_{\beta,m} \tilde{Y}_{\beta,m}(\theta) \right) e^{im\psi_j(r,\theta)} + o(r).$$

As $e^{im\psi_j(r,\theta)} = e^{im\varphi_j} + o(1)$, we get

$$\sum_{m=-1}^{1} \left(\sum_{\substack{\beta \in \mathbb{N}_0^{d-2} \\ |\beta| + |m| = 1}} c_{\beta,m} \tilde{Y}_{\beta,m}(\theta) \right) e^{im\varphi_j} = 0.$$

Thus for all $\theta \in [0, \pi)^{d-2}$

$$(c_{0,1} e^{im\varphi_j} + c_{0,-1} e^{im\varphi_j}) \tilde{Y}_{0,1}(\theta) + \sum_{\substack{\beta \in \mathbb{N}_0^{d-2} \\ |\beta| = 1}} c_{\beta,0} \tilde{Y}_{\beta,0}(\theta) = 0.$$

The linear independence of the $\tilde{Y}_{\beta,m}$'s implies that $c_{\beta,0} = 0$ when $|\beta| = 1$ and

$$\begin{cases} c_{0,-1} e^{-i\varphi_1} + c_{0,1} e^{i\varphi_1} = 0, \\ c_{0,-1} e^{-i\varphi_2} + c_{0,1} e^{i\varphi_2} = 0. \end{cases}$$

As the determinant of this system is $2i \sin(\varphi_2 - \varphi_1) \neq 0$ we get $c_{0,-1} = c_{0,1} = 0$.

Assume now that $c_{\beta,m} = 0$ for every β, m with $|\beta| + |m| \leq n_0 - 1$ then, as previously

$$u(r, \theta, \psi_j(r, \theta)) = r^{n_0} \sum_{m=-n_0}^{n_0} \left(\sum_{\substack{\beta \in \mathbb{N}_0^{d-2} \\ |\beta| + |m| = n_0}} c_{\beta,m} \tilde{Y}_{\beta,m}(\theta) \right) e^{im\varphi_j} + o(r^{n_0}).$$

It follows again that

$$(4.20) \quad \left(\sum_{\substack{\beta \in \mathbb{N}_0^{d-2} \\ |\beta| = n_0}} c_{\beta,0} \tilde{Y}_{\beta,0}(\theta) \right)$$

$$+ \sum_{m=1}^{n_0} \left(\sum_{\substack{\beta \in \mathbb{N}_0^{d-2} \\ |\beta| + m = n_0}} (c_{\beta,m} e^{im\varphi_j} + c_{\beta,-m} e^{-im\varphi_j}) \tilde{Y}_{\beta,m}(\theta) \right) = 0.$$

This implies that $c_{\beta,0} = 0$ if $|\beta| = n_0$ and that for each $m = 1, \ldots, n_0$ and each β with $|\beta| = n_0 - m$,

$$\begin{cases} c_{\beta,-m} e^{-im\varphi_1} + c_{\beta,m} e^{im\varphi_1} = 0, \\ c_{\beta,-m} e^{-im\varphi_2} + c_{\beta,m} e^{im\varphi_2} = 0. \end{cases}$$

The determinant of this system is $2i \sin m(\varphi_2 - \varphi_1) \neq 0$, thus $c_{\beta,-m} = c_{\beta,m} = 0$. We have thus proved the result.

The case of u of the form (4.19) is similar.

$\square$

4.2. **Negative results.** Finally we show that there is no unique continuation of the Helmholtz equation for a finite set of intersecting lines.

Proposition 4.2. *Let $d \geq 3$ and $k \in \mathbb{R}$. For every $\theta_1, \ldots, \theta_N \in \mathbb{S}^{d-1}$ there exists a non-zero solution of $\Delta u + k^2 u = 0$ such that $u = 0$ on $\mathbb{R}\theta_1 \cup \cdots \cup \mathbb{R}\theta_N$.*

Proof. It suffices to find a non-zero spherical harmonic Y of suitable degree m such that $Y(\pm\theta_j) = 0$, $j = 1, \ldots, N$. Then the functions $u(r\theta) = r^m Y(\theta)$ (for $k = 0$) and $u(r\theta) = r^{-(d-2)/2} J_{m+(d-2)/2}(kr)Y(\theta)$ (for $k \neq 0$) vanish on the lines $\mathbb{R}\theta_j, j = 1, \ldots, N$, and thus the solution of the Helmholtz-Laplace equation is not uniquely determined by its restriction to a finite number of lines.

Let $\mathcal{H}_m^d$ denote the subspace of spherical harmonics of degree m in $\mathbb{R}^d$. Its dimension is $\binom{m+d-1}{m} \geq 2m + 1$ for $d \geq 3$.

Now consider the linear forms $L_j : \mathcal{H}_m^d \to \mathbb{C}$ given by $L_j(Y) = Y(\theta_j)$ for $j = 1, \ldots, N$ and $L_j(Y) = Y(-\theta_j)$ for $j = N+1, \ldots, 2N$. A dimension count yields $\bigcap_j \ker L_j \geq \dim \mathcal{H}_m^d - 2N \geq 1$ for m large enough. So $\bigcap_j \ker L_j \neq \{0\}$ and there exists a $Y \in \mathcal{H}_m^d$ satisfying $Y(\theta_j) = 0$ for $j = 1, \ldots, N$.

Thus the solution of the Helmholtz equation is not uniquely determined by its restriction to any finite set of intersecting lines. $\square$

We do not know what happens for restrictions of the Helmholtz equation to k-dimensional subspaces when $2 \leq k \leq d - 2$ and $d \geq 4$.

Proposition 4.2 can be turned into a statement about Heisenberg uniqueness pairs.

Corollary 4.3. *The sphere $\mathbb{S}^{d-1}$ for $d \geq 3$ and an arbitrary finite set $\mathbb{R}\theta_j, j = 1, \ldots, N$ of lines through the origin cannot be a Heisenberg uniqueness pair. There always exist two distinct finite positive measures such that μ_+, μ_- are supported on $\mathbb{S}^{d-1}$ and $\widehat{\mu_+}|_{\mathbb{R}\theta_j} = \widehat{\mu_-}|_{\mathbb{R}\theta_j}$ for $j = 1, \ldots, N$.*

Proof. Let $Y \in \mathcal{H}_m^d$ be a non-zero spherical harmonic such that $Y(\pm\theta_j) = 0$ for $j = 1, \ldots, N$ and define two measures $\mu_\pm$ by

$$\mu_\pm = \left(1 \pm \frac{Y(\theta)}{\|Y(\theta)\|_{L^\infty(\mathbb{S}^{d-1})}}\right) \, \mathrm{d}\sigma(\theta).$$

Clearly $\mu_\pm$ are positive and absolutely continuous with respect to the surface measure on $\mathbb{S}^{d-1}$. Furthermore, the Hecke-Funck Formula [SW] shows that $\widehat{\mu_+ - \mu_-}(r\theta) = cr^{-(d-2)/2} J_{m+(d-2)/2}(r)Y(\theta)$ where c is a non-zero constant. Thus $\widehat{\mu_+}|_{\mathbb{R}\theta_j} = \widehat{\mu_-}|_{\mathbb{R}\theta_j}$. $\square$

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Univ. Bordeaux, IMB, UMR 5251, F-33400 Talence, France. CNRS, IMB, UMR 5251, F-33400 Talence, France.
E-mail address: aingeru.fernandez-bertolin@u-bordeaux.fr

Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria
E-mail address: karlheinz.groechenig@univie.ac.at

Univ. Bordeaux, IMB, UMR 5251, F-33400 Talence, France. CNRS, IMB, UMR 5251, F-33400 Talence, France.
E-mail address: Philippe.Jaming@math.u-bordeaux.fr