systems

Article

On the External Positivity of SISO Linear Dynamic Systems under a Class of Nonzero and Possibly Negative Initial Conditions Eventually Subject to Incommensurate Point Internal and External Delays

Manuel De la Sen

Institute of Research and Development of Processes IIDP, Department of Electricity and Electronics, Faculty of Science and Technology, Campus of Leioa, University of the Basque Country, 48940 Leioa, Spain; manuel.delasen@ehu.eus

Abstract: The property of external positivity of dynamic systems is commonly defined as the non-negativity of the output for all time under zero initial conditions and any given non-negative input for all time. This paper investigates the extension of that property for a structured class of initial conditions of a single-input single-output (SISO) linear dynamic system which can include, in general, certain negative initial conditions. The above class of initial conditions is characterized analytically based on the structure of the transfer function. The basic study is performed in the delay-free case, but extensions are then given for systems subject to a finite number of internal and external, in general incommensurate, point delays and for the closed-loop dynamic systems which incorporate a feedback compensator. The formulation relies on calculating the output based on the impulse responses by considering the relation of the mentioned sets of structured initial conditions with the zero-state response which allows to keep the non-negativity of the zero-input response and that of the total response provided the non-negativity for all time of the zero-state response.

Keywords: internal and external positivity; internal delays; external delays

Citation: De la Sen, M. On the External Positivity of SISO Linear Dynamic Systems under a Class of Nonzero and Possibly Negative Initial Conditions Eventually Subject to Incommensurate Point Internal and External Delays. *Systems* **2022**, *10*, 9. https://doi.org/10.3390/systems10010009

Academic Editor: William T. Scherer

Received: 9 December 2021
Accepted: 21 January 2022
Published: 25 January 2022

1. Introduction

Internally positive, or simply positive, dynamic systems are those whose state and output trajectory solutions have non-negative components for all time under any given non-negative initial conditions and any given input with non-negative components for all time. See, for instance [1–17], and references therein. The concept of externally positive systems refers to the situation when all the components of the output trajectory are non-negative for all time under zero initial conditions and any given input with non-negative components for all time. See, for instance [1,3,4] and some references therein. The above concepts of positivity and external positivity apply in the same way for continuous-time systems, discrete-time systems and hybrid systems. Positive systems appear typically in common problems of the Real World such as, for instance, in biological systems, epidemic systems, medical models and economic systems and also, for instance, in the fields of Telecommunications, Ecology, population dynamics and others. See, for instance [1,2,5]. On the other hand, the properties of observability, controllability and reachability of positive systems have been very widely studied in the background literature. See, for instance [6–10] and some of the references therein. It can be pointed out that the formulations of controllability and reachability problems require the fulfilment of fixing a non-negative targeted state in finite time though either a non-negative control function or a control sequence (in the discrete case) as it is inherent to the own definition of the positivity property. The stability and stabilization properties have been also widely studied in the background literature. See, for instance [10–16], and some references there in, including the problems of finite-time stabilization, stabilization of switched positive systems and stability and stabilization of fractional positive models. However, it can be pointed out that the positivity of the solution is not an intrinsic property to a dynamic system, as it could be for instance, the eigenvalues of a time-invariant linear system, which are not dependent on the particular state space description. In fact, it suffices either a similarity or an equivalence state transformation to loose the positivity property of the initial state space representation. Thus, the positivity property is not invariant, in general, under those kinds of transformations. See, for instance [2]. It is also possible to deal with a kind of the inverse problem to the internal positivity consisting of achieving the internal positivity under certain state transformations perhaps at the expense of destabilizing partially the system. See, for instance [17].

The positivity property of periodic and 2-D systems has been focused on in [1,18,19]. See also some of the references therein while the positivity characterization in the presence of internal (i.e., in the state) and/or external (i.e., in the input and/or output) delays has also received a wide attention in the background literature. See, for instance [20–24]. The reachability and observability properties of a class of fractional positive linear systems which describe certain electrical circuits has been studied in [25] while the positivity of the steady states of sub-homogeneous positive systems has been focused on in [26].

1.1. Specific Contribution

Generally speaking, internal positivity is a property related to the fact that both the state and output of a dynamic system have non-negative state and output components for all time for any given non-negative initial conditions and any given non-negative input for all time. External positivity (in our proposed framework in this paper: "0-external positivity") is the property of all the output components being non-negative for all time under zero initial conditions and any given non-negative input for all time. In particular, a system which is 0-externally positive, but not internally positive, can, eventually, have some negative output component at some time instant even under positive initial conditions. The main objective of this paper is to extend the standard definition of external positivity for a certain class of non-zero initial conditions which can be either positive or eventually negative while belonging to a certain subset of the state space. Under this extension, the output is non-negative for all time for any set of initial conditions in the mentioned set for any non-negative input for all time. The performed basic formal study is performed in Section 2 for the case of delay-free linear time-invariant systems while extensions are given for systems which have, in general, finite sets of incommensurate point internal and/or external delays. Such extensions are given in Section 3. In that case, the initial conditions are given by an absolute continuous function with eventual jumps which is defined on an initialization time interval in the non-positive real axis. The basic mechanism to state and prove the obtained results is the use of the impulse response [27–31], to characterize the zero-state response (that is, the output under zero initial conditions) and then to characterize the zero-input response (that is, the output under zero input) for the defined admissible set of initial conditions which can include some subset outside of the first orthant of the state space, that is, it can include certain negative initial conditions. The extension of the external positivity property for the inclusion of such a set of initial conditions, which extends the standard definition of external positivity, is made through the superposition principle to get the total output from the above two partial responses.

1.2. Notation

$$\mathbf{R}_{+} = \{r \in \mathbf{R} : r > 0\};\ \boldsymbol{R}_{0+} = \boldsymbol{R}_{+} \cup \{0\}$$
$$\boldsymbol{R}_{-} = \{r \in \boldsymbol{R} : r < 0\};\ \boldsymbol{R}_{0-} = \boldsymbol{R}_{-} \cup \{0\}$$
$$\overline{n} = \{1, 2, \cdots, n\};\ \overline{n}_0 = \overline{n} \cup \{0\}$$

L and L^{-1} are the Laplace transform, of argument "s", and the inverse Laplace transform.

The pairs of Laplace transform of a function and such a function as anti-Laplace transform are, respectively, denoted as upper-case and lower-case styles as follows: $M(s) = L(m(t))$ and $m(t) = L^{-1}(M(s))$.

$f(t^+)$ and $f(t^-)$ denote the right and left limits of a discontinuous function $f : \boldsymbol{R}_{0+} \to \boldsymbol{R}$ at $t \in \boldsymbol{R}$.

I_n is the identity matrix of order n.

Consider matrices $X = \left(X_{ij}\right) \in \boldsymbol{R}^{n \times m}$, $Y \in \boldsymbol{R}^{n \times m}$. Then:

$X \succeq 0$, or $X \in \boldsymbol{R}_{0+}^{n \times m}$, denotes a non-negative matrix, that is, $X_{ij} \geq 0$; $\forall i, j \in \overline{n} \times \overline{m}$. $X \succeq Y$ denotes $X - Y \succeq 0$.

$X \succ 0$, or $X \in \boldsymbol{R}_{0+}^{n \times m}$, with $X \neq 0$, denotes a positive matrix, that is, $X_{ij} \geq 0$; $\forall i, j \in \overline{n} \times \overline{m}$ with at least one positive entry. In addition, $X \succ Y$ denotes $X - Y \succ 0$.

$X \succ\succ 0$, or $X \in \boldsymbol{R}_{+}^{n \times m}$, denotes a strictly positive matrix, that is, $X_{ij} > 0$; $\forall i, j \in \overline{n} \times \overline{m}$. In additiion, $X \succ\succ Y$ denotes $X - Y \succ\succ 0$.

Consider column vectors $x = (x_1, x_2, \ldots x_n)^T \in \boldsymbol{R}^n$, $y = (y_1, y_2, \ldots y_n)^T \in \boldsymbol{R}^n$. Then:

$x \succeq 0$, or $x \in \boldsymbol{R}_{0+}^{n}$, denotes a non-negative vector, that is, $x_i \geq 0$; and $x \succeq y$ denotes $x - y \succeq 0$.

$x \succ 0$, or $x(\neq 0) \in \boldsymbol{R}_{0+}^{n}$, denotes a positive vector, that is, $x_i \geq 0$; $\forall i \in \overline{n}$ with at least one positive entry. $x \succ y$ denotes $x - y \succ 0$.

$x \succ\succ 0$, or $x \in \boldsymbol{R}_{+}^{n}$, denotes a strictly positive real vector, that is, $x_i > 0$; $\forall i \in \overline{n}$. $x \succ\succ y$ denotes $x - y \succ\succ 0$.

$x \preceq 0$ (no component of the vector x is positive) denotes $-x \succeq 0$. Similarly, we denote $x \prec 0$ and $x \prec\prec 0$ and close negativity notations are used for matrices X. The negativity concepts or real matrices and vectors can be also denoted by referring them to the sets $\boldsymbol{R}_{-0}$ and $\boldsymbol{R}_{-}$.

The above notation applies also mutatis-mutandis for row vectors $x^T = (x_1, x_2, \ldots x_n)$.

A Metzler matrix A is a real square n-matrix whose off-diagonal entries are non-negative. It is known that if A is a Metzler matrix then the fundamental matrix of the differential system $\dot{x}(t) = Axt$; $\forall t \in \boldsymbol{R}_{0+}$; $x(0) = x_0 \in \boldsymbol{R}^n$, is positive for all time, that is $e^{At} \succ 0$.

2. Results

2.1. External Positivity for a Class of Eventually Non-Zero Initial State Conditions

In the following, we consider state space realizations $R = \left(A, b, c^T, d\right)$ of a transfer function $G(s) = c^T(sI_n - A)^{-1}b + d$, where s is the Laplace transform argument, of a single-input single-output linear time-invariant system, where $A \in \boldsymbol{R}^{n \times n}$ is the matrix of dynamics; b, $c \in \boldsymbol{R}^n$ are the control and output transpose vectors and $d \in \boldsymbol{R}$ is the scalar input-output interconnection gain. The state, input and output of R are $x : \boldsymbol{R}_{0+} \to \boldsymbol{R}^n$, $u : \boldsymbol{R}_{0+} \to \boldsymbol{R}$ and $y : \boldsymbol{R}_{0+} \to \boldsymbol{R}$, respectively, and R describes, in the time domain, a linear time-invariant dynamic system through the equations:

$$\dot{x}(t) = Ax(t) + bu(t);\ y(t) = c^T x(t) + du(t)$$

with initial condition $x_0 = x(0^+) \in \boldsymbol{R}^n$. The rational transfer complex function $G(s)$ is an external description of the same system which is the quotient between the Laplace transforms of the output and input under zero initial conditions.

It is generically assumed that $c^T(sI_n - A)^{-1}b$ is strictly proper so that it has more poles than zeros (then, relative degree of at least one) and, as a result, $c^T(sI_n - A)^{-1}b + d$ is proper (that is, it has no more zeros than poles so that it is realizable in the state space) and also biproper if $d \neq 0$ (that is, it has exactly the same number of zeros and poles so that it is proper and state- space realizable as it is also its inverse). The consideration of a single-input single-output in the study is made just to facilitate the exposition while most of results also apply to multivariable systems under direct trivial generalizations.

The extended external positivity concept dealt with considers a class of, eventually non-zero, initial state conditions which are proportional to the control vector with arbitrary non-negative proportionality slope.

Definition 1. *A state space realization $R = \left(A, b, c^T, 0\right)$ of order n of a single-input single-output transfer function $G(s) = c^T(sI_n - A)^{-1}b$ is externally positive for state initial conditions $x_0 = x(0^+) \in \Xi \subset \boldsymbol{R}^n$ (in brief, Ξ -externally positive) if the output is non-negative for all time for any given non-negative input.* □

By convenience, the following dual definition to Definition 1 is given to be invoked in some results later on:

Definition 2. *A state space realization $R = \left(A, b, c^T, 0\right)$ of order n of a single-input single-output transfer function $G(s) = c^T(sI_n - A)^{-1}b$ is externally negative for state initial conditions $x_0 = x(0^+) \in \Xi \subset \boldsymbol{R}^n$ (in brief, Ξ -externally negative) if the output is non-positive for all time for any given non-negative input.* □

In Definition 1, the impulse response, i.e., the output, under zero initial conditions, for a unity impulse at zero time (which is the Laplace inverse transform of the Laplace transform) is non-negative for all time. In Definition 2, the impulse response is non-positive for all time. Note that if R is externally positive then any realization obtained by changing of sign an odd number of its parameterizing matrices is externally negative and vice-versa.

Kaczorek's standard definition of external positivity corresponds to the above one if $x_0 = 0$, that is to zero-state external positivity [1]. So, it can be said that the standard definition of external positivity corresponds to 0- external positivity according to Definition 1. The reason to the new definition is to extend the external positivity condition from zero initial conditions to some wider sets as it is addressed in the subsequent result.

Note that internal positivity (or simply positivity) is the property of non-negativity of all the components of the state trajectory solution for all time for any everywhere non-negative input for any non-negative components of the initial condition. This property stands if A is a Metzler matrix and $b, c \succ 0$, and it guarantees also the property of 0-external positivity which holds if and only if the impulse response is non-negative for all time.

Note that if a linear system is internally positive then it is also externally positive (or 0-externally positive following Definition 1). In fact, in [1], the definition of internal positivity includes that of the external positivity. However, it has to be pointed out that external positivity does not imply that the output trajectory solution is non-negative for all time and any given everywhere non-negative input, in some case, even the initial conditions are non-negative. This is visualized in the following simple example.

Example 1. *Consider the state space realization $R = \left(A, c, c^T, 0\right)$ of order n , where $A \in \boldsymbol{R}^{n\times n}$ and $c \in \boldsymbol{R}^n$ of the single-input single-output transfer function*

$$G(s) = \left(-c^T\right)(sI_n - A)^{-1}(-c) = c^T(sI_n - A)^{-1}c$$

which is assumed to be 0-externally positive what does not depend of $c \succ 0$ or $c \prec 0$. Now assume that $c \prec\prec 0$ and that, for some $\lambda < 0$, the initial state is $x_0 = -\lambda c \succ\succ 0$. The output of the above realization is:

$$y(t) = -\lambda c^T e^{At} c + c^T \int_0^t e^{A(t-\tau)} u(\tau) d\tau$$

where $u : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is the input. Take a non-negative input $u(t) = \begin{cases} \theta \; for \; t \in [0, T] \\ 0 \; for \; t > T \end{cases}$.

Then,

$$y(t) = -\lambda c^T e^{At} c + c^T \theta\, t \int_0^t e^{A(t-\tau)} c d\tau \; for\; t \in [0\,, T]$$
$$y(t) = -\lambda c^T e^{At} c + c^T \theta\, T \int_0^T e^{A(t-\tau)} c d\tau \; for\; t > T$$

Note that $y(t) < 0$ for $t > T$ if

$$\frac{\lambda}{\theta T} > \frac{c^T \left(\int_0^T e^{A(t-\tau)} d\tau \right) c}{c^T e^{At} c}$$

Take a non-negative input $u(t) = \begin{cases} \theta \; for\; t \in [0, T] \\ 0 \; for\; t > T \end{cases}$ *. Then,*

$$y(t) = \begin{cases} c^T e^{At} \left(\theta t \int_0^t e^{-A\tau} d\tau - \lambda I_n \right) c \; for\; t \in [0,\, T] \\ c^T e^{AT} \left(\theta T \int_0^T e^{-A\tau} d\tau - \lambda I_n \right) c \; for\; t > T \end{cases}$$

Thus, the realization is not $\boldsymbol{R}_{0+}^n$*-external positive.*

Remark 1. *It has been seen that the external positivity for zero initial conditions does not imply the external-positivity for any initial conditions and any given everywhere non-negative input within the first closed orthant of the state space, that is,* $\boldsymbol{R}_{0+}^n$ *-external positivity.* □

Assume that a state space realization $R = \left(A, b, c^T, 0\right)$ of order n of a single-input single-output transfer function $G(s) = c^T (sI_n - A)^{-1} b$ is 0-externally positive. Then, its zero-state response (that is, the output under zero initial conditions) is non-negative for all time for any given non-negative input. However, the zero -input response (that is, the output under identically zero input $y_0(t) = c^T e^{At} x_0$ is not necessarily non-negative since it depends on c and e^{At}. In addition, the superposition of both partial responses could be negative for certain non-negative inputs and non-negative initial conditions without extra conditions on the realization elements. In addition, it is well- known that, if the state space realization fulfils the stronger constraint of being (internally) positive [1,3,4], then the zero-input response, the zero-state response and the total response are non-negative for all time for any non-negative initial conditions and everywhere non-negative input. However, this holds under the extra conditions that the state space realization R satisfies $b \succ 0$, $c \succ 0$ and A being a Metzler matrix.

Note also that the fact that the impulse response is non-negative for all time, for instance, $c^T e^{At} b \geq 0$; $\forall t \in \boldsymbol{R}_{0+}$, in the single-input single-output case, does not imply that the zero-input response $c^T e^{At} x_0 \geq 0$; $\forall t \in \boldsymbol{R}_{0+}$ for any $x_0 \succ 0$. Therefore, 0-external positivity does not imply, in general, neither that the zero-input response is non-negative for all time or that the total response is non-negative for all time. □

The subsequent result states that 0-external positivity implies also external positivity for any initial condition of the form λb for any $\lambda \in \boldsymbol{R}_{0+}$.

Proposition 1. *Assume that a state space realization $R = \left(A, b, c^T, 0\right)$ of order n of a single-input single-output linear time-invariant system of transfer function $G(s)$ is 0 -externally positive. Then, it is Ξ-externally positive where $\Xi = \{x \in \boldsymbol{R}^n : x = \lambda b, \forall \lambda \in \boldsymbol{R}_{0+}\}$. Conversely, if R is Ξ-externally positive then it is 0-externally positive.*

Proof. For given zero initial conditions and any input $u : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$, the output is,

$$y(t) = \int_0^t g(t-\tau) u(\tau) d\tau = \int_0^t g(\tau) u(t-\tau) d\tau \geq 0;\; \forall t \in \boldsymbol{R}_{0+}$$

where the impulse response is $g = L^{-1}(G(s)) : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ since, otherwise if $g(t) < 0$ for some $t \in \boldsymbol{R}_{0+}$, it suffices to take a positive impulsive input $u(0) = \delta(0)$ leading to

$y(t) < 0$ contradicting that zero-state externally positive is zero-sate externally positive. For any initial conditions satisfying $x_0 = \lambda b(\in \Xi)$ with $\lambda \geq 0$, the Laplace transform of the output is:

$$\begin{aligned} Y(s) &= c^T(sI_n - A)^{-1}x_0 + G(s)U(s) \\ &= \lambda c^T(sI_n - A)^{-1}b + G(s)U(s) = \lambda c^T(sI_n - A)^{-1}b + c^T(sI_n - A)^{-1}bU(s) \end{aligned} \tag{1}$$

Then, one gets after taking Laplace inverse transforms in (1) that,

$$y(t) = \lambda L^{-1}\left[c^T(sI_n - A)^{-1}b\right] + L^{-1}\left[c^T(sI_n - A)^{-1}bU(s)\right] = \lambda g(t) + \int_0^t g(\tau)u(t-\tau)d\tau \geq 0;\ \forall t \in \mathbf{R}_{0+} \tag{2}$$

since $g : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$, $\int_0^t g(\tau)u(t-\tau)d\tau \geq 0$ and $\lambda \geq 0$. It has been proved that if R is 0-externally positive then it is Ξ-externally positive. The converse also holds since if R is Ξ-externally positive, since, trivially, $0 \in \Xi$, then R is 0-externally positive. □

Remark 2. *Note that for $\Xi = \{x \in \mathbf{R}^n : x = \lambda b, \forall\lambda \in \mathbf{R}_{0+}\}$, Proposition 1 states that R is Ξ -externally positive if and only if it is 0 -externally positive.* □

Proposition 2. *Assume that the state space realization $R = \left(A, b, c^T, 0\right)$ of order n of $G(s) = c^T(sI_n - A)^{-1}b$ is Ξ-externally positive, where $\Xi = \{x \in \mathbf{R}^n : x = \lambda b, \forall\lambda \in \mathbf{R}_{0+}\}$. Then, the state space realization $R_1 = \left(A, B, c^T, d\right)$ of $G_1(s) = G(s) + d$ is Ξ -externally positive if and only if $d \geq 0$.*

Proof. Now, one has from (2) that,

$$y(t) = \lambda g(t) + \int_0^t g(\tau)u(t-\tau)d\tau + du(t) \geq du(t) \tag{3}$$

Thus, for any given $u : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$, $y(t) \geq 0; \forall t \in \mathbf{R}_{0+}$ since $\lambda \geq 0$ which proves the sufficiency. Now, proceed with contradiction arguments by assuming that $d < 0$. Then, $y(t^+) = \lambda g(t) + \int_0^t g(\tau)u(t-\tau)d\tau - |d|u(t^+) < 0$ for $t \in \mathbf{R}_{0+}$ if $u : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ is such that $u(t^+) > \frac{\lambda g(t) + \int_0^t g(t-\tau)u(\tau)d\tau}{|d|}$; $\forall t \in \mathbf{R}_{0+}$ so that the necessity is proved since the Ξ-external positivity fails since the output is not positive for all time for a particular positive input. □

Remark 3. *Note that the transfer functions $G(s)$ and $G_1(s) = G(s) + d$ associated, respectively, with the realizations $\left(A, b, c^T, 0\right)$ and $\left(A, b, c^T, d\right)$ are strictly proper and non-strictly proper, respectively. Thus, the Ξ -external possibility of a non-strictly proper transfer function is maintained under the incorporation of any direct input-output interconnection gain which converts it into a non-strictly proper transfer function.* □

Note that Proposition 2 states that, if $\Xi = \{x \in \mathbf{R}^n : x = \lambda b, \forall\lambda \in \mathbf{R}_{0+}\}$, then $R = \left(A, b, c^T, 0\right)$ is Ξ-externally positive (equivalently, 0-externally positive) if and only if $R_1 = \left(A, b, c^T, d\right)$ is Ξ-externally positive (equivalently, 0-externally positive) for any $d \geq 0$.

The following result is concerned with the property of 0-external positivity of a closed-loop configuration under the assumption that the feed-forward is Ξ-externally positive. The closed-loop configuration consists of the feed-forward transfer function $G(s)$ and the feedback one $H(s)$ so that the closed-loop error $\varepsilon(t)$ coincides with the input control $u(t)$ to $G(s)$.

Theorem 1. *Assume that the state space realization $R = \left(A, b, c^T, 0\right)$ of order n of the transfer function $G(s) = c^T(sI_n - A)^{-1}b$ is Ξ-externally positive, where $\Xi = \{x \in \mathbf{R}^n : x = \lambda b, \forall\lambda \in \mathbf{R}_{0+}\}$*

and that the state space realization $R_H = \left(A_H, b_H, c_H^T, 0\right)$ *of order* n_H *is of the transfer function* $H(s) = c_H^T\left(sI_{n_H} - A_H\right)^{-1}b_H$ *has initial conditions:*

$$x_H(0) = x_{H0} = \lambda_H b_H \in \hat{\Xi}_H = \{\, x \in \boldsymbol{R}^m \, : x = \lambda_H b_H \, , \forall \lambda_H \in \boldsymbol{R}\}.$$

Assume also that $G(s)$ *and* $H(s)$ *are, respectively, the feed-forward and feedback blocks of a closed-loop transfer function with negative feedback whose reference input is* $r : \boldsymbol{R}_{0+} \to \boldsymbol{R}$*, subject to the constraint* $r(t) \geq \int_0^t g_H(t-\sigma)y(\sigma)d\sigma + \lambda_H g_H(t); \forall t \in \boldsymbol{R}_{0+}$ *where* $g_H : \boldsymbol{R}_{0+} \to \boldsymbol{R}$ *is the impulse response of* $H(s)$*. Then, the following properties hold:*

(i) *The closed-loop state space realization of state* $\left(x^T(t)\, ,\, x_H^T(t)\right)^T$ *has a non-negative output* $y(t) = c^T x(t)$ *for all time so that it is* 0*-externally positive with respect to the closed-loop error* $\varepsilon(t) = u(t) = r(t) - \int_0^t g_H(t-\sigma)y(\sigma)d\sigma - \lambda_H g_H(t)$*. However, it is not* Ξ_{cl}*-externally positive with respect to the closed-loop reference input* $r(t)$*, where:*

$$\Xi_{cl} = \left\{ z = \left(x^T\, ,\, w^T\right)^T \in \boldsymbol{R}^{n+m} \, : x = \lambda b; \; w = \lambda b_H \, , \forall \lambda \in \boldsymbol{R}_{0+} \right\}.$$

(ii) *Assume that* R *is* Ξ*-externally positive,* R_H *is* Ξ*-externally negative and* $r : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$*. Then the closed-loop state space realization of state* $\left(x^T(t)\, ,\, x_H^T(t)\right)^T$ *is* 0*-externally positive and also* Ξ_{cl}*-externally positive.*

(iii) *Assume that* $G(s)$ *and* $H(s)$ *of the Property (i) are replaced, respectively, with* $G_1(s) = G(s) + d$ *and* $H_1(s) = H(s) + d_H$ *with input-output interconnection gains* $d \geq 0$ *and* $d_H \in \boldsymbol{R}$*. Then, Properties (i)–(ii) still hold, for* $d_H \geq 0$ *and* $d_H \leq 0$*, respectively.*

Proof. Since the Lapalce transform of the feedback control is $U(s) = R(s) - H(s)Y(s)$, Equation (1) takes the form:

$$\begin{aligned} & Y(s) = \lambda c^T(sI_n - A)^{-1}b + G(s)U(s) \\ & = \lambda c^T(sI_n - A)^{-1}b + G(s)(R(s) - H(s)Y(s)) \\ & = \lambda c^T(sI_n - A)^{-1}b + c^T(sI_n - A)^{-1}b \\ & \times \left(R(s) - c_H^T(sI_{n_H} - A_H)^{-1}b_H\, Y(s) - c_H^T(sI_{n_H} - A_H)^{-1}\lambda_H b_H\right) \end{aligned} \tag{4}$$

if $x_0 = \lambda b$ and $x_{H0} = x_H(0) = \lambda_H b_H$. From (4), one gets the following identity:

$$\begin{aligned} & P(s) = \left(1 + c^T(sI_n - A)^{-1}bc_H^T(sI_{n_H} - A_H)^{-1}b_H\right)Y(s) \\ & = (1 + G(s)H(s))Y(s) \\ & = Q(s) = c^T(sI_n - A)^{-1}b\left[\lambda - \lambda_H c_H^T(sI_{n_H} - A_H)^{-1}b_H + R(s)\right] \end{aligned} \tag{5}$$

Taking anti-Laplace transforms in the left-hand-side and right-hand-side of the above expression yields:

$$\begin{aligned} & P(t) = L^{-1}\left\{\left(I_n + c^T(sI_n - A)^{-1}bc_H^T(sI_{n_H} - A_H)^{-1}b_H\right)Y(s)\right\} \\ & = y(t) + \int_0^t \int_0^\tau g(t-\tau)g_H(\tau - \sigma)y(\sigma)d\sigma d\tau; \; \forall t \in \boldsymbol{R}_{0+} \end{aligned} \tag{6}$$

and

$$\begin{aligned} & Q(t) = L^{-1}\left\{c^T(sI_n - A)^{-1}b\left[\lambda - \lambda_H c_H^T(sI_{n_H} - A_H)^{-1}b_H + R(s)\right]\right\} \\ & = \lambda g(t) \; - \lambda_H \int_0^t g(\tau)g_H(t-\tau)d\tau + \int_0^t g(t-\tau)r(\tau)d\tau \\ & = \lambda g(t) \; + \int_0^t g(t-\tau)[r(\tau) - \lambda_H g_H(\tau)]d\tau; \; \forall t \in \boldsymbol{R}_{0+} \end{aligned} \tag{7}$$

Since R is Ξ-externally positive, $g : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ and $\lambda g(t) \geq 0$; $\forall t \in \boldsymbol{R}_{0+}$, since $\lambda \geq 0$. By equalizing $P(t) = Q(t)$ in (6)–(7), one gets:

$$y(t) = \lambda g(t) + \int_0^t g(t-\tau)\left[r(\tau) - \int_0^\tau g_H(\tau-\sigma)y(\sigma)d\sigma - \lambda_H g_H(\tau)\right]d\tau;\ \forall t \in \boldsymbol{R}_{0+} \quad (8)$$

With $g : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$. Since $\lambda \geq 0$, $\lambda_H \in \boldsymbol{R}$ and $r(t) \geq \int_0^t g_H(t-\sigma)y(\sigma)d\sigma + \lambda_H g_H(t)$; $\forall t \in \boldsymbol{R}_{0+}$, one concludes from (8) that $y : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ as a result. In addition, it is 0-externally positive related to the closed-loop error (that is, the reference input minus output feedback signal):

$$\varepsilon(t) = u(t) = r(t) - \int_0^t g_H(t-\sigma)y(\sigma)d\sigma - \lambda_H g_H(t)$$

However, it is not Ξ_{cl}-externally positive since the output is not non-negative for any reference input $r : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$. It suffices to take some $r : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ under the constraint $0 \leq r(t) < \lambda g_H(t)$, for some $t \in \boldsymbol{R}_{0+}$, some given $\lambda > 0$ and associated initial state condition of the closed-loop system of the form $\lambda\left(b^T, b_H^T\right)$, so violating the reference constraint of Property (i) to be unable to prove that $y : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$. Property (i) has been proved.

To prove Property (ii), consider again (8) with $g, (-g_H) : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ since R and R_H are, respectively, Ξ-externally positive and Ξ-externally negative. Since $\lambda, \lambda_H \in \boldsymbol{R}_{0+}$, then (8) becomes modified as follows:

$$y(t) = \lambda g(t) + \int_0^t g(t-\tau)\left[r(\tau) + \int_0^\tau |g_H(\tau-\sigma)|y(\sigma)d\sigma + \lambda_H|g_H(\tau)|\right]d\tau;\ \forall t \in \boldsymbol{R}_{0+} \quad (9)$$

so that $y : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ for any reference input $r : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ and Property (ii) is proved.

To prove Property (iii), note that (5) becomes modified as $\hat{P}(s) = Q(s)$ with,

$$\hat{P}(s) = (1 + G_1(s)H_1(s))Y(s) = (1 + (G(s)+d)(H(s)+d_H))Y(s) \quad (10)$$

and the interconnection gains d and d_H can be incorporated to the impulse responses $g_1, g_{H1} : \boldsymbol{R}_{0+} \to \boldsymbol{R}$. The discussion follows as in Property (i) and Property (ii) from which it can be built a similar contradiction to the negativity of the output for some time instant as it was carried out in the proof of Property (i). From there, Property (iii) follows. □

Note that set Ξ_{cl} in Theorem 1 is the subset of $\boldsymbol{R}^{n+m}$ defined by the vectors $z = \lambda\left(b^T, b_H^T\right)^T$, $\lambda \in \boldsymbol{R}_{0+}$.

The following consequence of Theorem 1 is direct.

Corollary 1. *Theorem 1 holds, in particular, if* $G(s) = c^T(sI_n - A)^{-1}b$ *is* 0*-externally positive and the remaining constraints are kept identical.* □

Note that the stipulations of Theorem 1 imply that the closed-loop state space realization is not Ξ_{cl}-externally positive where $\Xi_{cl} = \left\{z = \lambda\left(b^T, b_H^T\right)^T : \lambda \in \boldsymbol{R}_{0+}\right\}$. However, and in view of (7), $Q(t) \geq 0$; $\forall t \in \boldsymbol{R}_{0+}$ if $\lambda_H = 0$, implying zero initial conditions for the feedback block and leading to the reference input condition $r(t) \geq 0$; $\forall t \in \boldsymbol{R}_{0+}$ if the rest of the given stipulations hold. Therefore, Theorem 1 guarantees the Ξ_{cl0}-external positivity of the closed-loop system, where $\Xi_{cl0} = \left\{z = \lambda\left(b^T, 0^T\right)^T : \lambda \in \boldsymbol{R}_{0+}\right\}$, namely, for any initial condition of the feed-forward block being proportional (with non-negative slope) to its corresponding control vector and zero initial condition of the feedback block. Note also that it guarantees, in a similar way, the Ξ_{cl-}-external positivity of the closed-loop system, where $\Xi_{cl-} = \left\{z = \left(\lambda b^T, -\lambda_H b_H^T\right)^T : \lambda \in \boldsymbol{R}_{0+}, \lambda_H \in \boldsymbol{R}_{0-}\right\}$, namely, for any given initial conditions of the feed-forward block proportional (with non-negative slope) to its corresponding control vector and any given non-positive initial conditions of the feedback block. The above considerations are summarized, under the form of corollary to Theorem 1, as follows:

Corollary 2. *Let sets* Ξ_{cl0}, $\Xi_{cl-} \subset \boldsymbol{R}^{n+m}$ *defined by* $\Xi_{cl0} = \left\{ z = \lambda\left(b^T, 0^T\right)^T : \lambda \in \boldsymbol{R}_{0+} \right\}$ *and* $\Xi_{cl-} = \left\{ z = \left(\lambda b^T, -\lambda_H b_H^T\right)^T : \lambda \in \boldsymbol{R}_{0+}, \lambda_H \in \boldsymbol{R}_{0-} \right\}$. *Assume that* $R = \left(A, b, c^T, 0\right)$ *is* Ξ*-externally positive, with* $\Xi = \{x \in \boldsymbol{R}^n : x = \lambda b, \forall \lambda \in \boldsymbol{R}_{0+}\}$ *and that* $R_H = \left(A_H, b_H, c_H^T, 0\right)$ *is* 0 *-externally positive. Then, for any given input-output interconnection gains* $d \geq 0$ *and* $d_H \geq 0$*, the closed-loop systems consisting of any feed-forward and feedback state space realizations* $R_1 = \left(A, b, c^T, d\right)$ *and* $R_H = \left(A_H, b_H, c_H^T, d_H\right)$ *are jointly* Ξ_{cl0}*-externally positive and* Ξ_{cl-} *-externally positive.* □

An important discussion follows in the subsequent remark concerning the irrelevance of the realization being minimal (that is, jointly controllable and observable) or non- minimal if the initial conditions for the extra modes in non-minimal realizations satisfy also the given constraints.

Remark 4. *Note that the orders of the state space realizations are not relevant for the given* Ξ *-external positivity results if the domain* Ξ *of initial conditions is extended in a natural way according to the dimensionality of the non-minimal realization. The key facts are the positivity of the impulse responses and the definition of sets of initial conditions with the indicated proportionality characteristics with respect to the control vectors. This is easily seen by assuming that the transfer function* $G(s) = c^T(sI_n - A)^{-1}b$ *has no zero-pole cancellation, thus, the state space realization* $R = \left(A, b, c^T, 0\right)$ *is controllable and observable, and also a minimal realization of* $G(s)$ *as a result, and the dimension* n *of its state vector* $x(t)$ *of* R *is the degree of the denominator of the transfer function* $G(s)$*. Consider now a transfer function* $\overline{G}(s) = G(s)\frac{p(s)}{p(s)}$*, where* $p(s)$ *is a polynomial of arbitrary degree* $q \geq 1$*. Thus,* $\overline{G}(s)$ *is identical to* $G(s)$ *(after performing the zero-pole cancellations of the extra coincident* q *zeros and poles) but any state space realization* $\overline{R}$ *of* $\overline{G}(s)$ *has* $(n+q)$ *-order (so that its state vector* $\overline{x}(t)$ *is of dimension* $(n+q)$ *and it is either controllable and non-observable, or non-controllable and observable or non -controllable and non-observable, but it cannot be controllable and observable.* □

Now, by inspecting (1), one concludes that,

$$\begin{aligned} G(s) &= \lambda c^T(sI_n - A)^{-1}b + c^T(sI_n - A)^{-1}bU(s) \\ &= \overline{G}(s) = \lambda \overline{c}^T\left(sI_n - \overline{A}\right)^{-1}\overline{b} + \overline{c}^T\left(sI_n - \overline{A}\right)^{-1}\overline{b}U(s) \end{aligned} \tag{11}$$

Thus, the zero-pole cancellations of $\overline{G}(s)$ lead to q extra modes in $\overline{R}$ (implying zero-pole cancellations in the transfer function) related to those of R but this does not affect to any of the given results in Proposition 1 and Proposition 2 if the initial conditions are $x_0 = \lambda b$ and $\overline{x}_0 = \lambda \overline{b}$ for any real number $\lambda \in \boldsymbol{R}_{0+}$. Thus, results for minimal-realizations concerning the given concepts of external positivity are also kept for their corresponding non-minimal realizations of any orders provided the rules on the admissible initial conditions are kept for the extra added modes. Similar conclusions apply to Theorem 1 and its given corollaries for feed-forward/feedback tandems in closed-loop configuration with the appropriate modifications. Thus, if the minimal realization R is Ξ-externally positive, where $\Xi = \{x \in \boldsymbol{R}^n : x = \lambda b, \forall \lambda \in \boldsymbol{R}_{0+}\}$ then the non-minimal one $\overline{R}$ is $\overline{\Xi}$-externally positive, where $\overline{\Xi} = \left\{ \overline{x} \in \boldsymbol{R}^{n+m} : \overline{x} = \lambda \overline{b}, \forall \lambda \in \boldsymbol{R}_{0+} \right\}$. □

2.2. External Positivity for a System Subject to Point Delays for a Class of Eventually Non-Zero Initial State Conditions

It is now considered that the linear time -invariant system is subject to μ and finite sets ν of internal (i.e., in the state) and external (i.e., in the input) point delays h_i and h'_j ordered by $0 = h_0 < h_1 < \ldots < h_\mu = h$ and $0 = h'_0 < h'_1 < \ldots < h'_\nu = h'$ and it is described by the state space realization $R_{\mu\nu} = \left(A_0, A_1, \cdots, A_\mu, b_0, b_1, \cdots, b_\nu, c^T, d\right)$ of associate state and output equations:

$$\dot{x}(t) = \sum_{i=0}^{\mu} A_i x(t-h_i) + \sum_{i=0}^{\nu} b_i u\left(t-h_i'\right) = A_0 x(t) + \left(\sum_{i=1}^{\mu} A_i x(t-h_i) + \sum_{i=0}^{\nu} b_i u\left(t-h_i'\right)\right) \quad (12)$$

$$y(t) = c^T x(t) + du(t) \quad (13)$$

of initial conditions defined by a function $\varphi : [-h, 0] \to \boldsymbol{R}^n$, with $\varphi(0) = x_0$, which consists of an absolutely continuous function plus, eventually, a function of finite jumps on a subset of zero Lebesgue measure of $[-h, 0]$, where $x : [-h, 0] \cup \boldsymbol{R}_{0+} \to \boldsymbol{R}^n$, $u : [-h', 0] \cup \boldsymbol{R}_{0+} \to \boldsymbol{R}$ and $y : \boldsymbol{R}_{0+} \to \boldsymbol{R}$ are the state, input and output, respectively, and $A_i \in \boldsymbol{R}^{n\times n}$; $c,\, b_j \in \boldsymbol{R}^n$; $d \in \boldsymbol{R}$; $i \in \overline{\mu}_0$, $j \in \overline{\nu}_0$ with $x(t) = \varphi(t)$ for $t \in [-h, 0]$ and $u(t) = 0$ for $t \in [-h', 0)$. The transfer function of the above realization is the following one:

$$G_{\mu\nu}(s) = \sum_{i=0}^{\nu} G_i(s) e^{-h_i' s} = c^T \left(sI_n - \sum_{i=0}^{\mu} A_i e^{-h_i s}\right)^{-1} \left(\sum_{i=0}^{\nu} b_i e^{-h_i' s}\right) + d \quad (14)$$

where

$$G_0(s) = c^T \left(sI_n - \sum_{i=0}^{\mu} A_i e^{-h_i s}\right)^{-1} b_0 + d$$
$$G_i(s) = c^T \left(sI_n - \sum_{i=0}^{\mu} A_i e^{-h_i s}\right)^{-1} b_i;\ \forall i \in \overline{\nu} \quad (15)$$

The unique time domain solution of (12) is:

$$x(t) = e^{A_0 t} x_0 + \int_0^t e^{A_0(t-\tau)} \left(\sum_{i=1}^{\mu} A_i x(\tau - h_i) + \sum_{i=0}^{\nu} b_i u\left(\tau - h_i'\right)\right) d\tau \quad (16)$$

$$= \Psi(t, 0) x_0 + \int_{-h}^{0} \Psi(t,\tau)\varphi(\tau) d\tau + \int_0^t \Psi(t,\tau)\left(\sum_{i=0}^{\nu} b_i u\left(\tau - h_i'\right)\right) d\tau; \forall t \in \boldsymbol{R}_{0+} \quad (17)$$

where $\Psi : \boldsymbol{R}_{0+} \times [-h, 0) \cup \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is the unique fundamental matrix of the unforced (12) which satisfies the differential system:

$$\dot{\Psi}(t,\tau) = A_0 \Psi(t,\tau) + \sum_{i=1}^{\mu} A_i \Psi(t-h_i,\, \tau) U(t-h_i-\tau);\ t \ge \tau \ge 0 \quad (18)$$

whose solution is,

$$\Psi(t,\tau) = \sum_{i=0}^{\nu} \Psi_i(t,\tau) = \left[e^{A_0(t-\tau)}\left(I_n + \sum_{i=1}^{\mu} A_i \Psi(t-h_i,\, \tau)\, U(t-h_i-\tau)\right)\right] U(t-\tau);$$
$$\forall (\tau\, (\le t), t) \in ([-h\,, 0) \cup \boldsymbol{R}_{0+}) \times \boldsymbol{R}_{0+} \quad (19)$$

where

$$\Psi_0(t,\tau) = e^{A_0(t-\tau)} U(t-\tau),\ \Psi_i(t,\tau) = e^{A_0(t-\tau)} A_i \Psi(t-h_i,\, \tau)\, U(t-h_i-\tau) U(t,\tau);$$
$$\forall i \in \overline{\mu};\ \forall (\tau\, (\le t), t) \in ([-h\,, 0) \cup \boldsymbol{R}_{0+}) \times \boldsymbol{R}_{0+} \quad (20)$$

where $U(t)$ is the Heaviside function. In particular, $\Psi(t\,,t) = I_n$; $\forall t \in \boldsymbol{R}_{0+}$ and $\Psi(t\,,\tau) = 0$; $\forall \tau(> t)\,,\, t \in \boldsymbol{R}_{0+}$.

The Laplace transform of the output under eventually non-zero initial conditions is:

$$Y(s) = c^T \left(sI_n - \sum_{i=0}^{\mu} A_i e^{-h_i s}\right)^{-1} \varphi(s) + \left[c^T \left(sI_n - \sum_{i=0}^{\mu} A_i e^{-h_i s}\right)^{-1} \left(\sum_{i=0}^{\nu} b_i e^{-h_i' s}\right) + d\right] U(s) \quad (21)$$

Note that,

$$G_{\mu\nu}(s) = L\left(c^T \sum_{i=0}^{\mu} \Psi(t,\, 0) b_i e^{-h_i' s}\right) + d$$

The impulse response is:

$$g_{\mu\nu}(t,\tau) = L^{-1}\left(G_{\mu\nu}(s)\right) = \sum_{i=0}^{\nu} g_i(t,\tau) = \sum_{i=0}^{\nu} L^{-1}(G_i(s)) \forall (\tau\, (\le t), t) \in ([-h\,, 0) \cup \boldsymbol{R}_{0+}) \times \boldsymbol{R}_{0+} \quad (22)$$

and $g_{\mu\nu}(t,\tau) = 0; \forall\tau(> t)$, $t \in \boldsymbol{R}_{0+}$, where $\delta(i,j)$ is the Kronecker delta, that is unity if $i = j$ and zero, otherwise; and $\delta(t-\tau)$ is the Dirac distribution, that is, it tends to infinity for $t = \tau$ and it is zero otherwise. Thus, one gets from (13) and (17),

$$\begin{aligned} y(t) &= c^T\Psi(t,0)x_0 + \sum_{i=1}^{\mu}\int_{-h_i}^{0} c^T\Psi_i(t,\tau)\varphi(\tau)d\tau + \sum_{i=0}^{\nu} c^T\Psi(t,\tau)b_i u(\tau - h'_i)d\tau + du(t) \\ &= c^T\Psi(t,0)x_0 + \int_{-h}^{0} c^T\Psi(t,\tau)\varphi(\tau)d\tau + \sum_{i=0}^{\nu}\int_0^t g_i(t,\tau)u(\tau - h'_i)d\tau \\ &= c^T\Psi(t,0)x_0 + \int_0^{\infty} c^T\Psi(t,\tau - h)\hat{\varphi}(\tau)d\tau + \sum_{i=0}^{\nu}\int_0^t g_i(t-\tau)u(\tau - h'_i)d\tau;\ \forall t \in \boldsymbol{R}_{0+} \end{aligned} \tag{23}$$

where the impulse responses and the extended initial function of initial conditions are:

$$g_0(t,\tau) = g_0(t-\tau) = c^T\Psi(t-\tau)b_0 + \delta(t-\tau)d;\ \forall(\tau\,(\le t), t) \in ([-h,0) \cup \boldsymbol{R}_{0+}) \times \boldsymbol{R}_{0+} \tag{24}$$

$$g_i(t,\tau) = g_i(t-\tau) = c^T\Psi(t\ \tau)b_i;\ \forall(\tau\,(\le t), t) \in ([-h,0) \cup \boldsymbol{R}_{0+}) \times \boldsymbol{R}_{0+};\ \forall i \in \overline{\mu} \tag{25}$$

$$\hat{\varphi}(t) = \begin{cases} \varphi(t-h)\ for\ t \in [0,h] \\ 0\ for\ t > h \end{cases} \tag{26}$$

Note that the function of initial conditions in the second right-hand-side term of (23) plays the role of a forcing function to the solution and that the use of its extended version (26) converts such a contribution as a convolution term similar to the contribution of the inputs. The following external positivity condition for a set of initial conditions extends in a natural way that given previously for the delay-free case.

Definition 3. *A state space realization $R_{\mu\nu} = (A_0, A_1, \cdots, A_\mu, b_0, b_1, \cdots, b_\nu, c^T, d)$ of order n of associate state and output Equations (12)–(13), under initial conditions of initial conditions defined by a function $\varphi : [-h,0] \to \boldsymbol{R}^n$, which consists of an absolutely continuous function plus, eventually, a function of finite jumps on a subset of zero Lebesgue measure of $[-h,0]$ is externally positive for state initial conditions in some set $\{\varphi : [-h,0] \to \Xi_{\mu\nu} \subseteq \boldsymbol{R}^n\}$ (in brief, $\Xi_{\mu\nu}$ -externally positive) if the output is non-negative for all time for any given non-negative input.* □

Note that the transfer function of the system under eventual internal and external delays, referred to in Definition 3, is of the form:

$$G_{\mu\nu}(s) = \sum_{i=0}^{\nu} G_i(s)e^{-h'_i s} = c^T\left(sI_n - \sum_{i=0}^{\mu} A_i e^{-h_i s}\right)^{-1}\left(\sum_{i=0}^{\nu} b_i e^{-h'_i s}\right) + d$$

The following result holds on external positivity for a set of initial conditions:

Theorem 2. *Assume the 0 -external positivity, i.e., the external positivity under zero initial conditions, of the state space realization $R_{\mu\nu} = (A_0, A_1, \cdots, A_\mu, b_0, b_1, \cdots, b_\nu, c^T, d)$ of order n, given by (12)–(13), of a transfer function (14). Then, $R_{\mu\nu}$ is $\Xi_{\mu\nu}$ -externally positive irrespective of the delays sizes, where $\Xi_{\mu\nu} = \{\varphi : [-h,0] \to \boldsymbol{R}^n\ satisfies\ conditions\ IC\}$:*

Conditions IC : $\varphi(t) = \sum_{i=1}^{\nu}\lambda_i(t)b_i;\ \forall t \in [-h,0]$, $x(0^+) = x_0 = \varphi(0^+) = \lambda_0 b_0$, where $\lambda_0 \ge 0$, and $\lambda_i : ([-min(h_i', h), 0) \cup \boldsymbol{R}_{0+}) \to \boldsymbol{R}_{0+};\ \forall i \in \overline{\nu}$ are absolutely continuous functions plus, eventually, functions of finite jumps on subsets of zero Lebesgue measure of their definition domains.

Conversely, if $R_{\mu\nu}$ is $\Xi_{\mu\nu}$-externally positive then it is 0-externally positive.

Proof. The extended initial conditions obtained from *Conditions IC* in the definition of $\Xi_{\mu\nu}$, to write the second right-hand-side of (23) as a convolution term, are as follows:

$$\hat{\varphi}(t) = \sum_{i=1}^{\nu}\hat{\lambda}_i(t)b_i;\ \forall t \in [-h,0) \cup \boldsymbol{R}_{0+} \tag{27}$$

with $\hat{\lambda}_i : \left(\left[-min\left(h_i{}', h\right), 0\right) \cup \boldsymbol{R}_{0+} \right) \to \boldsymbol{R}_{0+}\,; \forall i \in \overline{\nu}$ being defined by,

$$\hat{\lambda}_i(t) = \begin{cases} \lambda_i\left(t - h_i'\right) \ for \ t \in \left[-min\left(h_i{}', h\right), 0\right) \\ 0 \, for \ t \geq 0 \end{cases} ; \forall i \in \overline{\nu} \tag{28}$$

In terms of the impulse responses, (22), subject to (24)–(25), the output (23) is expressed as follows:

$$y(t) = \lambda_0 g_0(t) + \sum_{i=1}^{\nu} \int_0^{\infty} g_i(t - \tau + h)\hat{\lambda}_i(\tau)d\tau + \sum_{i=0}^{\nu} \int_0^{t} g_i(t - \ \tau)u\left(\tau - h'_i\right)d\tau;\ \forall t \in \boldsymbol{R}_{0+} \tag{29}$$

where the total impulse response (22) is,

$$\begin{array}{l} g_{\mu\nu}(t,\tau) = \sum_{j=0}^{\nu} g_i(t - \tau) = \sum_{j=0}^{\nu} c^T \Psi(t,\tau) b_j + \delta(0, j)\delta(t - \tau)d \\ \forall \tau(\leq t) \in [-h, 0] \times \boldsymbol{R}_{0+};\ \forall t \in \boldsymbol{R}_{0+} \end{array} \tag{30}$$

and $g_{\mu\nu}(t,\tau) = 0$ for $\tau > t$. Note that the input-output interconnection gain $d \geq 0$ for the realization to be 0-externally positive. From the 0-external positivity condition, one gets that $g_i : \left([-h, 0] \times \boldsymbol{R}_{0+} \right) \times \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}\,;\ \forall i \in \overline{\nu}_0$. Otherwise, assume that for identically zero function of initial conditions $\varphi : [-h, 0] \to \boldsymbol{R}^n$ and some $i \in \overline{\nu}_0$, there exists some $t \in \boldsymbol{R}_{0+}$ such that $g_i\left(t + h_i'\right) = 0$, then it suffices to take an isolated input impulse of sufficiently large positive amplitude $u(t) = K\delta(0)$ to get a negative $y(t)$. Thus, $g_i : ([-h, 0] \times \boldsymbol{R}_{0+}) \times \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}\,;\ \forall i \in \overline{\nu}_0$. Then, the forced output (or zero-state response) in (29) satisfies:

$$y_f(t) = \sum_{i=0}^{\nu} \int_0^t g_i(t - \ \tau)u\left(\tau - h'_i\right)d\tau \geq 0;\ \forall t \in \boldsymbol{R}_{0+} \tag{31}$$

Note that *Conditions IC* in the definition of $\Xi_{\mu\nu}$ imply that the unforced output (or zero-input response) response in (29) satisfies:

$$y_{uf}(t) = \lambda_0 g_0(t) + \sum_{i=1}^{\nu} \int_0^{\infty} g_i(t - \tau + h)\hat{\lambda}_i(\tau)d\tau \geq 0;\ \forall t \in \boldsymbol{R}_{0+} \tag{32}$$

From (31)–(32), $y(t) \geq 0;\ \forall t \in \boldsymbol{R}_{0+}$. It is direct to prove that, conversely, if $R_{\mu\nu}$ is $\Xi_{\mu\nu}$-externally positive then it is 0-externally positive since $(\varphi \equiv 0) \in \Xi_{\mu\nu}$. □

The subsequent result extends external positivity results Theorem 1, while it is supported by Theorem 2, from the delay-free case to the case of linear feed-forward and feedback linear systems subjects to finite numbers of incommensurate internal and external point delays. The point delays are said to be incommensurate in the subject literature if they are not integer multiples of a basic delay. The commensurate delays are the particular case where the delays are integer multiple of a basic delay. This, the subsequent results for the general case of incommensurate point delays are directly applied to the particular case of commensurate delays.

Theorem 3. *Assume the* 0*-external positivity, irrespective of the delay sizes, of the state space realization* $R_{\mu\nu} = \left(A_0, A_1, \cdots, A_\mu, b_0, b_1, \cdots, b_\nu, c^T, d\right)$ *of order* n*, given by (12)–(13), of a transfer function (14) under the initial conditions:*

IC : $\varphi(t) = \sum_{i=1}^{\nu} \lambda_i(t) b_i;\ \forall t \in [-h, 0]$, $x(0^+) = x_0 = \varphi(0^+) = \lambda_0 b_0$*, where* $\lambda_0 \geq 0$ *and* $\lambda_i : ([-min(h_i?\ h), 0) \cup \boldsymbol{R}_{0+}) \to \boldsymbol{R}_{0+}\,;\ \forall i \in \overline{\nu}$ *are absolutely continuous functions plus, eventually, functions of finite jumps on subsets of zero Lebesgue measure of their definition domains.*

Consider also the state space realization $R_{\mu_H\nu_H} = (A_{0_H}, A_{1_H}, \cdots, A_{\mu_H}, b_{0_H}, b_{1_H}, \cdots,$ $b_{\nu_H}, c_H^T, d_H)$ *of order* n_H *of the transfer function* $H_{\mu_H\nu_H}(s) = c^T \left(sI_{n_H} - \sum_{i=0}^{\mu_H} A_{iH} e^{-h_{iH}s} \right)^{-1}$ $\left(\sum_{i=0}^{\nu_H} b_{iH} e^{-h'_{iH}s} \right) +\ d_H$*, given by similar equations to (12) and (13), with respective* μ_H *and* ν_H

incommensurate nonzero internal and external point delays h_{iH} for $i \in \overline{\mu}_{H0}$ and h'_{jH} for $j \in \overline{\nu}_{H0}$, ordered in size according to their respective subscripts with $h_{0H} = h'_{0H} = 0$ and $h_H = h_{\mu_H H}$, and with initial conditions in the set $\Xi_{\mu_H \nu_H} = \{ \varphi_H : [-h_H, 0] \to \mathbf{R}^{n_H}\ satisfies\ conditions\ IC_H \}$:

Conditions IC_H : $\varphi_H(0) = x_{0H}$ and $\varphi_H(t) = \sum_{i=1}^{\nu_H} \lambda_{iH}(t) b_{iH}$; $\forall t \in [-h_H, 0]$, $x_H(0^+) = x_{0H} = \varphi_H(0^+) = \lambda_{0H} b_{0H}$, where $\lambda_{0H} \in \mathbf{R}$ and $\lambda_{iH} : [-h_H, 0) \to \mathbf{R}$; $\forall i \in \overline{\nu}_H$ are absolutely continuous functions plus, eventually, functions of finite jumps on subsets of zero Lebesgue measure of their definition domains.

Assume also that $G_{\mu\nu}(s)$ and $H_{\mu_H \nu_H}(s)$ are, respectively, the feed-forward and feedback blocks of a closed-loop transfer function with negative feedback whose reference input is $r : \mathbf{R}_{0+} \to \mathbf{R}$, subject to the constraint

$r(t) \geq \sum_{j=0}^{\nu_H} \int_0^t g_{jH}(t-\sigma) y(\sigma)\, d\sigma + \int_{-min(h_H, h_{jH})}^{0} g_{jH}(t-\sigma) \lambda_{jH}(\sigma) d\sigma$; $\forall t \in \mathbf{R}_{0+}$.

Then, the following properties hold:

***(i)** The closed-loop state space realization of state $\left(x^T(t),\, x_H^T(t)\right)^T$ has a non-negative output $y(t) = c^T x(t)$ for all time so that it is 0-externally positive with respect to the closed-loop error $\varepsilon(t) = u(t)$. However, it is not Ξ_{cl}-externally positive with $\Xi_{cl} = \left(\Xi_{\mu\nu},\, \Xi_{\mu_H \nu_H}\right)$ with respect to the closed-loop reference input $r(t)$, where $\Xi_{\mu\nu} = \{ \varphi : [-h, 0] \to \mathbf{R}^n\ satisfies\ conditions\ IC \}$.*

***(ii)** Assume that $R_{\mu\nu}$ is Ξ-externally positive, $R_{\mu_H \nu_H}$ is Ξ-externally negative and $r : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$. Then the closed-loop state space realization of state $\left(x^T(t),\, x_H^T(t)\right)^T$ is 0-externally positive and Ξ_{cl}-externally positive.*

Proof. It is known from Theorem 2 that $R_{\mu\nu}$ *is* $\Xi_{\mu\nu}$-externally positive, where $\Xi_{\mu\nu} = \{ \varphi : [-h, 0] \to \mathbf{R}^n\ satisfies\ conditions\ IC \}$. Now, the extended initial conditions obtained from *Conditions IC* of $\Xi_{\mu\nu}$ for the realization $R_{\mu\nu}$ are defined in (27) and (28) while the extended initial conditions obtained from *Conditions* IC_H of $\Xi_{\mu_H \nu_H}$ for the realization $R_{\mu_H \nu_H}$ are as follows:

$$\hat{\varphi}_H(t) = \sum_{i=1}^{\nu} \hat{\lambda}_{Hi}(t) b_{iH};\ \forall t \in [-h_H, 0) \cup \mathbf{R}_{0+} \tag{33}$$

with $\hat{\lambda}_{Hi} : ([-min(h'_{iH}, h_H), 0) \cup \mathbf{R}_{0+}) \to \mathbf{R}$; $\forall i \in \overline{\nu}$ being defined by

$$\hat{\lambda}_{iH}(t) = \begin{cases} \lambda_{iH}(t - h_H)\ for\ t \in [-min(h'_{iH}, h_H), 0) \\ 0 \quad for\ t \geq 0 \end{cases} \tag{34}$$

□

Proof. Since the Laplace transform of the feedback control is $U(s) = R(s) - H_{\mu_H \nu_H}(s) Y(s)$, Equation (21) becomes:

$$\begin{aligned} Y(s) &= c^T \left(sI_n - \sum_{i=0}^{\mu} A_i e^{-h_i s}\right)^{-1} \varphi(s) + \left[c^T \left(sI_n - \sum_{i=0}^{\mu} A_i e^{-h_i s}\right)^{-1} \left(\sum_{i=0}^{\nu} b_i e^{-h'_i s}\right) + d\right] U(s) \\ &= c^T \left(sI_n - \sum_{i=0}^{\mu} A_i e^{-h_i s}\right)^{-1} \left(\sum_{i=0}^{\nu} b_i e^{-h'_i s} \lambda_i(s)\right) \\ &+ \left[c^T \left(sI_n - \sum_{i=0}^{\mu} A_i e^{-h_i s}\right)^{-1} \left(\sum_{i=0}^{\nu} b_i e^{-h'_i s}\right) + d\right] (R(s) - H_{\mu_H \nu_H}(s) Y(s)) \\ &= c^T \left(sI_n - \sum_{i=0}^{\mu} A_i e^{-h_i s}\right)^{-1} \left(\sum_{i=0}^{\nu} b_i e^{-h'_i s} \lambda_i(s)\right) + \left[c^T \left(sI_n - \sum_{i=0}^{\mu} A_i e^{-h_i s}\right)^{-1} \left(\sum_{i=0}^{\nu} b_i e^{-h'_i s}\right) + d\right] \\ &\times \left(R(s) - \left(c^T \left(sI_{n_H} - \sum_{i=0}^{\mu_H} A_{iH} e^{-h_{iH} s}\right)^{-1} \left(\sum_{i=0}^{\nu_H} b_{iH} e^{-h'_{iH} s}\right) + d_H\right) Y(s) - c_H^T \left(sI_{n_H} - \sum_{i=0}^{\mu_H} A_{iH} e^{-h_{iH}}\right)^{-1} \left(\sum_{i=0}^{\nu_H} b_{iH} e^{-h'_{iH}} \lambda_{iH}(s)\right)\right) \end{aligned} \tag{35}$$

and, by taking Laplace inverse transforms in (35), one gets:

$$
\begin{aligned}
&y(t) = \sum_{i=0}^{\nu} \int_0^t g_i(t-\tau+h)\hat{\lambda}_i(\tau)d\tau \\
&+\sum_{i=0}^{\nu} \int_0^t g_i(t-\tau)\left[r(\tau) - \sum_{j=0}^{\nu_H} \int_0^\tau g_{jH}(\tau-\sigma)y(\sigma)d\sigma - \int_0^\tau g_{jH}(\tau-\sigma+h_H)\lambda_{jH}(\sigma)d\sigma\right]d\tau \\
&= \sum_{i=0}^{\nu} \int_{-min(h,h_i)}^0 g_i(t-\tau)\lambda_i(\tau)d\tau \\
&+\sum_{i=0}^{\nu} \int_0^t g_i(t-\tau)\left[r(\tau) - \sum_{j=0}^{\nu_H} \int_0^\tau g_{jH}(\tau-\sigma)y(\sigma)\,d\sigma - \int_{-min(h_H,h_{jH})}^0 g_{jH}(\tau-\sigma)\lambda_{jH}(\sigma)d\sigma\right]d\tau \\
&\geq \sum_{i=0}^{\nu} \int_0^t g_i(t-\tau)\left[r(\tau) - \sum_{j=0}^{\nu_H} \int_0^\tau g_{jH}(\tau-\sigma)y(\sigma)\,d\sigma - \int_{-min(h_H,h_{jH})}^0 g_{jH}(\tau-\sigma)\lambda_{jH}(\sigma)d\sigma\right]d\tau \geq 0;\ \forall t \in \boldsymbol{R}_{0+}
\end{aligned}
\tag{36}
$$

since $\sum_{i=0}^{\nu} \int_{-\min(h,h_i)}^0 g_i(t-\tau)\lambda_i(\tau)d\tau \geq 0;\ \forall t \in \boldsymbol{R}_{0+}$, since the realization $R_{\mu\nu}$ is $\Xi_{\mu\nu}$-externally positive, and since $r(t) \geq \sum_{j=0}^{\nu_H} \int_0^\tau g_{jH}(\tau-\sigma)y(\sigma)\,d\sigma + \int_{-min(h_H,h_{jH})}^0 g_{jH}(\tau-\sigma)$ $\lambda_{jH}(\sigma)d\sigma;\ \forall t \in \boldsymbol{R}_{0+}$ by hypothesis. Thus, the closed-loop state space realization is 0-externally positive but not Ξ_{cl}-externally positive with respect to the reference signal $r(t)$ since it has to fulfil stronger constraints than its non-negativity. Property (i) has been proved. On the other hand, if, in addition, $R_{\mu_H\nu_H}$ is Ξ-externally negative, then the constraint of Property (i) becomes:

$$
r(t) \geq -\sum_{j=0}^{\nu_H} \int_0^\tau \left|g_{jH}(\tau-\sigma)\right| y(\sigma)\,d\sigma - \int_{-min(h_H,h_{jH})}^0 \left|g_{jH}(\tau-\sigma)\right| \lambda_{jH}(\sigma)d\sigma \tag{37}
$$

is directly fulfilled for any reference signal $r : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ and it also holds, as in Property (i), that $y : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ under the admissibility constraint IC_H for any given function of initial conditions of the feedback transfer function. Property (ii) has been proved. □

Now, some results are got for 0-external positivity of closed-loop configurations involving a feed-forward compensator and a feed-forwards and a feed-back one based of the 0-external positivity of each of the system parts provided they are also strictly stable. It is considered the delay-free case for exposition facility while their extensions to the presence of point delays are direct.

Theorem 4. *Assume a closed-loop system configuration consisting of a of transfer function $G(s)$ with a feed-forward compensator $C(s)$ under unity negative feedback. Assume that both state space realizations of $G(s)$ and $C(s)$ are 0-externally positive and strictly stable. Assume also that the reference signal $r(t)$ is strictly positive and uniformly bounded for all time with $r_0 = \inf_{t\geq 0} r(t)$ and $r_1 = \sup_{t\geq 0} r(t)$ and that the following constraint holds:*

$$
\eta = \frac{k_g k_c}{\rho_c|\rho-\rho_c|}\left(\frac{2}{\rho_c} + \frac{1}{e\rho}\right) \leq \frac{r_0}{r_0+r_1} \tag{38}
$$

where k_g, k_c and $\rho, \rho_c (\neq \rho)$ are positive real constants such that the impulse responses $g(t)$ of $G(s)$ and $g_c(t)$ of $G_c(s)$ are, respectively, upper-bounded by $k_g e^{-\rho t}$ and $k_c e^{-\rho_c t}$; $\forall t \in \boldsymbol{R}_{0+}$.

Then, under zero initial conditions of both space realizations, $y : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ and it is uniformly bounded, and also $(r-y) : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ so that the closed-loop state space realization of transfer function $T(s) = C(s)G(s)/(1+C(s)G(s))$ is 0 -externally positive.

Proof. Under zero initial conditions, the output is given by the response impulses $g : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ of $G(s)$ and $g_c : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ of $C(s)$ as follows:

$$
y(t) = \int_0^t \int_0^\tau g(t-\tau)g_c(\tau-\sigma)\,(r(\sigma)-y(\sigma))d\sigma d\tau;\ \forall t \in \boldsymbol{R}_{0+} \tag{39}
$$

Since $G(s)$ and $G_c(s)$ are stable, that is, they have their poles in $Res < 0$, then there exist positive real constants k_g, k_c and $\rho, \rho_c(\neq \rho)$ such that $g(t) \leq k_g e^{-\rho t}; \forall t \in \mathbf{R}_{0+}$ and $g_c(t) \leq k_c e^{-\rho_c t}; \forall t \in \mathbf{R}_{0+}$. Thus, it follows from (39) that:

$$\begin{aligned}
&|y(t)| \leq \sup_{0\leq\theta\leq t} |y(\theta)| \leq \left(k_g k_c \int_0^t \int_0^\tau e^{-\rho(t-\tau)} e^{-\rho_c(\tau-\sigma)} d\sigma d\tau\right) \sup_{0\leq\tau\leq t} |r(t) - y(t)| \\
&= \left(k_g k_c e^{-\rho t} \int_0^t \int_0^\tau e^{(\rho-\rho_c)\tau} e^{\rho_c\sigma} d\sigma d\tau\right) \sup_{0\leq\tau\leq t} |r(t) - y(t)| \\
&= \left(k_g k_c e^{-\rho t} \int_0^t e^{(\rho-\rho_c)\tau} \left(\int_0^\tau e^{\rho_c\sigma} d\sigma\right) d\tau\right) \sup_{0\leq\tau\leq t} |r(t) - y(t)| \\
&= \left(k_g k_c e^{-\rho t} \frac{e^{(\rho-\rho_c)t}-1}{\rho-\rho_c} \left(\int_0^t \frac{e^{\rho_c\tau}-1}{\rho_c} d\tau\right)\right) \sup_{0\leq\tau\leq t} |r(t) - y(t)| \\
&= \left(k_g k_c \frac{e^{-\rho_c t}-e^{-\rho t}}{(\rho-\rho_c)\rho_c} \left(\frac{e^{\rho_c t}-1}{\rho_c} - t\right)\right) \sup_{0\leq\tau\leq t} |r(t) - y(t)| \\
&= \frac{k_g k_c}{\rho_c|\rho-\rho_c|} \left(\left|\frac{1+e^{-\rho t}-e^{-\rho_c t}-e^{-(\rho-\rho_c)t}}{\rho_c}\right| - t\left(e^{-\rho_c t} - e^{-\rho t}\right)\right) \sup_{0\leq\tau\leq t} |r(t) - y(t)| \\
&\leq \frac{k_g k_c}{\rho_c|\rho-\rho_c|} \left(\frac{2}{\rho_c} + \frac{1}{e\rho}\right) \left(\sup_{0\leq\theta\leq t} r(\theta) + \sup_{0\leq\theta\leq t} |y(\theta)|\right); \ \forall t \in \mathbf{R}_{0+}
\end{aligned} \tag{40}$$

Since $\rho \neq \rho_c$, $r : \mathbf{R}_{0+} \to \mathbf{R}_+$ is bounded, and $\Lambda = \left(\sup_{0\leq\theta\leq+\infty} r(\theta)\right) / \left(\inf_{0\leq\theta\leq+\infty} r(\theta)\right) \geq 1$, if $1 > \eta = \frac{k_g k_c}{\rho_c|\rho-\rho_c|}\left(\frac{2}{\rho_c} + \frac{1}{e\rho}\right)$, one has that

$$|y(t)| \leq \sup_{0\leq\theta\leq t} |y(\theta)| \leq \frac{\eta\,\Lambda}{1-\eta} \inf_{0\leq\theta\leq+\infty} r(\theta) \leq \frac{\eta\,\Lambda}{1-\eta} r(t); \ \forall t \in \mathbf{R}_{0+} \tag{41}$$

Now, if furthermore, $\eta\,\Lambda/(1-\eta) \leq 1$, which is implies the further constraint $\eta \leq 1/(1+\Lambda)$ to the above one $\eta < 1$ used to leads to (39), then $|y(t)| \leq r(t); \forall t \in \mathbf{R}_{0+}$. Since, in addition, $g, g_c : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$, it follows that the output absolute value coincides with $y : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ from $y(r) \leq r(t); \forall t \in \mathbf{R}_{0+}$ and (37). □

Remark 5. *The constraint $\rho \neq \rho_c$ of Theorem 4 is merely instrumental and introduced to facilitate the exposition. If $\rho_c = \rho$ then the third equation of (40) is rearranged as follows before constructing a particular "ad hoc" proof for that particular case:*

$$|y(t)| \leq \sup_{0\leq\theta\leq t} |y(\theta)| \leq \left(k_g k_c e^{-\rho t} \int_0^t \int_0^\tau e^{\rho_c\sigma} d\sigma\right) \sup_{0\leq\tau\leq t} |r(t) - y(t)|; \ \forall t \in \mathbf{R}_{0+} \tag{42}$$

□

Remark 6. *Assume that closed-loop system configuration consists of a of transfer function $G(s)$ with a feed-forward compensator $C(s)$ under a feedback controller of transfer function $H(s)$ and that the state space realizations of the three transfer functions are 0-externally positive and strictly stable. Then, Theorem 4 still holds for the closed-loop transfer function $T_1(s) = C(s)G(s)/(1 + C(s)H(s)G(s))$ by replacing $\eta \to \overline{\eta} = \frac{k_h}{\rho_h}\eta$ in (38), resulting in the modified constraint $\eta \leq \rho_h/\left[(1+\Lambda)\,k_h\right]$, where k_h and ρ_h are positive real constants such that the response impulse of $H(s)$, $g_h : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ is upper-bounded by $k_h e^{-\rho_h t}; \forall t \in \mathbf{R}_{0+}$. The extended proof is direct by re-arranging (40) as:*

$$|y(t)| \leq \frac{k_g k_c}{\rho_c|\rho-\rho_c|} \left(\frac{2}{\rho_c} + \frac{1}{e\rho}\right) \left(\sup_{0\leq\theta\leq t} r(\theta) + \sup_{0\leq\theta\leq t} \left|y_f(\theta)\right|\right); \forall t \in \mathbf{R}_{0+} \tag{43}$$

where $y_f : \boldsymbol{R}_{0+} \to \boldsymbol{R}_0$ *is the output of the feedback block which becomes* $\left|y_f(t)\right| = \int_0^t g_h(t-\tau)y(\tau)$ $d\tau \le \frac{k_h}{\rho_h} \sup_{0\le\theta\le t}\left|y_f(\theta)\right|$; $\forall t \in \boldsymbol{R}_{0+}$ *under zero initial conditions.* □

Example 2. *Consider the transfer function*

$$G(s) = K\frac{a + be^{-\theta s}}{(s+c)(s+d)}$$

Of a linear and time-invariant system of input $u(t)$ *and output* $y(t)$ *under some input delay* $\theta \ge 0$ *with* $c \ne d$*.The associate differential equation is:*

$$\ddot{y}(t) = -(c+d)\dot{y}(t) - cdy(t) + K(au(t) + bu(t-\theta))$$

By decomposing the transfer function with respect to the auxiliary input $v(t) = au(t) + bu(t-\tau)$*in simple fractions* $A/(s+c)$ *and* $B/(s+d)$*, one acquires:*

$$A = K\frac{b-c}{d-c}; B = K\frac{d-b}{d-c}$$

The state space realization obtained for the two above state variables and the output satisfy the subsequent relations for all $t \ge 0$*:*

$$\dot{x}_1(t) = -cx_1(t) + K\tfrac{b-c}{d-c}(au(t) + bu(t-\theta))$$
$$\dot{x}_2(t) = -dx_2(t) + K\tfrac{d-b}{d-c}(au(t) + bu(t-\theta))$$
$$x_1(t) = e^{-ct}x_1(0) + K\textstyle\int_0^t e^{-c(t-\tau)}(au(\tau) + bu(\tau-\theta))d\tau$$
$$x_2(t) = e^{-dt}x_1(0) + K\textstyle\int_0^t e^{-d(t-\tau)}(au(\tau) + bu(\tau-\theta))d\tau$$
$$y(t) = x_1(t) + x_2(t) = e^{-ct}x_1(0) + e^{-dt}x_2(0) + K\textstyle\int_0^t \left(e^{-c(t-\tau)} + e^{-d(t-\tau)}\right)(au(\tau) + bu(\tau-\theta))d\tau$$
$$y(t) = x_1(t) + x_2(t) = e^{-ct}x_1(0) + e^{-dt}x_2(0) + K\textstyle\int_0^t \left(e^{-c(t-\tau)} + e^{-d(t-\tau)}\right)(au(\tau) + bu(\tau-\theta))d\tau$$

Assume that $c \ge 0$, $d < c$, $Kmin(a,b) > 0$, $u(t) = 0$ *for* $-\theta \le t < 0$, $u(t) \ge 0$ *for* $t \ge 0$ *and assume also that* $min(x_1(0), x_2(0)) \ge 0$*. Then,* $x_1(t) \ge 0$, $x_2(t) \ge 0$ *and* $y(t) \ge 0$ *for all* $t \ge 0$*and the state space realization is internally positive and 0-externally positive.*

Now, assume that the initial conditions are changed to $x_1(0) \le 0$, $x_2(0) \ge |x_1(0)|$ *and that* $\sigma_1 = min\left(t \ge 0 : |x_1(0)| \ge K\int_0^t e^{c\tau}(au(\tau) + bu(\tau-\theta))d\tau\right)$ *for some particular everywhere non-negative input. Then,* $x_2(t) \ge 0$ *for all* $t \ge 0$*, and* $x_1(t) = -e^{-ct}(|x_1(0)| - K\int_0^t e^{c\tau}$ $(au(\tau) + bu(\tau-\theta))d\tau) < 0$ *for* $t \in [0, \sigma_1]$ *and*

$$\begin{aligned}
&y(t) = e^{-dt}x_2(0) - e^{-ct}|x_1(0)| + K\textstyle\int_0^t \left(e^{-c(t-\tau)} + e^{-d(t-\tau)}\right)(au(\tau) + bu(\tau-\theta))d\tau \\
&\ge e^{-ct}(x_2(0) - |x_1(0)|) + K\textstyle\int_0^t \left(e^{-c(t-\tau)} + e^{-d(t-\tau)}\right)(au(\tau) + bu(\tau-\theta))d\tau \\
&\ge 0; \forall t \ge 0
\end{aligned}$$

Similarly, one can find in the same way a subset of the non-positive real axis for which $x_2(t)$*is negative on a time interval* $[0, \sigma_2]$ *with,*

$$\sigma_2 = min\left(t \ge 0 : |x_1(0)| \ge K\int_0^t e^{d\tau}(au(\tau) + bu(\tau-\theta))d\tau\right)$$

while the output is non-negative for all time. Thus, the state space realization is externally positive for any initial conditions:

$$x^T(0) \in \Xi = (\boldsymbol{R}_{0+} \times \boldsymbol{R}_{0+}) \cup ((-\Xi_1) \times \Xi_1) \cup (\Xi_2 \times (-\Xi_2)); \forall \Xi_1 \subseteq \boldsymbol{R}_{0+}, \forall \Xi_2 \subseteq \boldsymbol{R}_{0+}$$

irrespectively of the delay size $\theta \geq 0$, *where:*

$$\Xi_1 = \left\{ x \in \boldsymbol{R}_{0+} : x \geq K \int_0^{\sigma_1} e^{c\tau} (au(\tau) + bu(\tau - \theta)) d\tau \right\}$$
$$\Xi_2 = \left\{ x \in \boldsymbol{R}_{0+} : x \geq K \int_0^{\sigma_2} e^{d\tau} (au(\tau) + bu(\tau - \theta)) d\tau \right\}$$

3. Discussion

The external positivity is an important property in dynamic systems which, basically, consists of the output (or measurable solution) being non-negative for all time under zero initial conditions and non-negative control efforts (or input) for all time. It is a weaker property than that of the internal positivity or, simply, positivity under which all the state components and measurable output are non-negative for all time and any non-negative initial conditions and non-negative inputs for all time. The internal positivity typically guarantees also the external positivity and there are relevant properties in many systems and models, such as economic, biological or telecommunications ones.

This paper addresses the extension of the external positivity in the single-input single-output linear time-invariant case to a wider class of initial conditions which can include sets of non-negative initial conditions which are specifically characterized. The discussion is addressed for the case of delay-free systems and also for the eventual presence of a finite number of point internal delays possibly in both, the systems dynamics and the control inputs. The external positivity property is typically related to the non-negativity for all time of the contribution to the response of the forcing terms under zero initial conditions. The basic formal analysis mechanism consists of applying the superposition principle to construct the global output solution, i.e., the response to initial conditions plus the response to the forcing controls or inputs. Based on the primary fact that the standard external positivity property relies on the positivity of the second of the above contribution terms, it is characterized the definitions of the sets of initial conditions which make the first of the above terms (response to initial conditions) to have a similar structure in terms of Laplace transforms, or the equivalent response impulse descriptions, to that of the contribution of the response to the inputs.

Funding: This research was funded by Spanish Government and European Commission, grant number RTI2018-094336-B-I00 (MCIU/AEI/FEDER, UE) and by the Basque Government, grant numberIT1207-19-The APC was funded by Spanish Government and European Commission grant number RTI2018-094336-B-I00 (MCIU/AEI/FEDER, UE).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The author is grateful to the Spanish Government for its support through grant RTI2018-094336-B-I00 (MCIU/AEI/FEDER, UE) and to the Basque Government for its support through Grant IT1207-19. He is also useful to the anonymous Referees by their useful comments and suggestions.

Conflicts of Interest: The author declares no conflict of interests.

References

1. Kaczorek, T. *Positive 1D and 2D Systems*; Springer: Berlin/Heidelberg, Germany, 2002.
2. Bokharaie, V.S. Stability Analysis of Positive Systems with Applications to Epidemiology. Ph.D. Dissertation, Hamilton Institute, National University of Ireland Maynooth, Maynooth, Ireland, January 2012.
3. De la Sen, M. Positivity and stability of a class of time-varying systems. *Nonlinear Anal.-Real World Appl.* **2007**, *8*, 749–768. [CrossRef]
4. De la Sen, M. About the positivity of a class of hybrid dynamic linear systems. *Appl. Math. Comput.* **2007**, *189*, 852–868. [CrossRef]
5. De la Sen, M.; Alonso-Quesada, S. Vaccination strategies based on feedback control techniques for a general SEIR-epidemic model. *Appl. Math. Comput.* **2011**, *218*, 3888–3904. [CrossRef]

6. Bartosiewicz, Z. Positive reachability of discrete-time nonlinear systems. In Proceedings of the IEEE Conference on Control Applications (CCA), Buenos Aires, Argentina, 19–22 September 2016; pp. 1203–1208.
7. Bartosiewicz, Z. Local Positive Reachability of Nonlinear Continuous-Time Systems. *IEEE Trans. Autom. Control* **2015**, *61*, 4217–4221. [CrossRef]
8. Sitli, I.; Naim, M.; Lahmidi, F.; Namir, A. On the controllability and observability of positive nonlinear continuous systems. *Commun. Math. Biol. Neurosci.* **2021**, *2021*, 1–15.
9. Valcher, M.E. Reachability properties of continuous-time positive systems. *IEEE Trans. Autom. Control.* **2009**, *54*, 1586–1590. [CrossRef]
10. Cantó, B.; Coll, C.; Sanchez, E. On stability and reachability of perturbed positive systems. *Adv. Differ. Equ.* **2014**, *2014*, 296. [CrossRef]
11. Yang, L. A new method for judging the stability and boundedness of continuous-time fractional positive systems. In Proceedings of the 2021 4th International Symposium on Big Data and Applied Statistics (ISBDAS 2021), Dali, China, 21–23 May 2021. *J. Phys. Conf. Ser.* **2021**, *1955*, 012058. [CrossRef]
12. Wang, X.; Wen, J. Finite-time Stabilization of Switched Positive Systems with Time-varying Delays and Actuator Saturation. In Proceedings of the 2021 4th International Conference on Advanced Algorithms and Control Engineering (ICAACE 2021), Sanya, China, 29–31 January 2021. *J. Phys. Conf. Ser.* **2021**, *1848*, 012134.
13. Gurvits, L.; Shorten, R.; Mason, O. On the Stability of Switched Positive Linear Systems. *IEEE Trans. Autom. Control* **2007**, *52*, 1099–1103. [CrossRef]
14. Zhao, X.; Zhang, L.; Shi, P.; Liu, M. Stability of switched positive linear systems with average dwell time switching. *Automatica* **2012**, *48*, 1132–1137. [CrossRef]
15. Yu, T.; Zhong, Y.; Chen, T.; Chen, C. Finite-Time Stabilization of Uncertain Switched Positive Linear Systems with Time-Varying Delays. *Discret. Dyn. Nat. Soc.* **2015**, *2015*, 954782. [CrossRef]
16. Cacace, F.; Farina, L.; Germani, A. *Positive Systems: Theory and Applications, Lecture Notes in Control*; Springer: Berlin/Heidelberg, Germany, 2017; Volume 471.
17. Cacace, F.; Germani, A.; Manes, C. Stable Internally Positive Representations of Continuous Time Systems. *IEEE Trans. Autom. Control* **2013**, *59*, 1048–1053. [CrossRef]
18. Zamani, M.; Shafiee, M.; Zamani, I. Stability analysis of singular 2-D positive systems. In Proceedings of the 2021 29th Iranian Conference on Electrical Engineering (ICEE), Tehran, Iran, 18–20 May 2021; pp. 626–631.
19. Zheng, Y.; Wang, Y. Stabilization of positive periodic piecewise linear systems. In Proceedings of the 40th Chinese Control Conference, Shanghai, China, 26–28 July 2021; pp. 932–937.
20. Naim, M. Zero Controllability Criterion for Discrete Positive Systems with Multiple Delays on Both States and Inputs. *Int. J. Math. Math. Sci.* **2021**, *2021*, 1–5. [CrossRef]
21. De la Sen, M. On positivity of singular regular linear time-delay time-invariant systems subject to multiple internal and external incommensurate point delays. *Appl. Math. Comput.* **2007**, *190*, 382–401. [CrossRef]
22. De la Sen, M.; Alonso-Quesada, S.; Ibeas, A. On the stability of an SEIR epidemic model with distributed time-delay and a general class of feedback vaccination rules. *Appl. Math. Comput.* **2015**, *270*, 953–976. [CrossRef]
23. Li, S.; Xiang, Z.R. Stability, l (1)-gain and l (infinity)-gain analysis for discrete-time positive switched singular delayed systems. *Appl. Math. Comput.* **2016**, *275*, 95–106.
24. Phat, V.; Sau, N. On exponential stability of linear singular positive delayed systems. *Appl. Math. Lett.* **2014**, *38*, 67–72. [CrossRef]
25. Yuan, T.; Yang, H. Invariance of reachability and observability for fractional positive linear electrical circuits with delays. *Arch. Electr. Eng.* **2021**, *70*, 513–530.
26. Bokharaie, V.S.; Mason, O.; Wirth, F. Stability and positivity of equilibria for subhomogeneous cooperative systems. *Nonlinear Anal. Theory Methods Appl.* **2011**, *74*, 6416–6426. [CrossRef]
27. Delasen, M. Multirate hybrid adaptive-control. *IEEE Trans. Autom. Control* **2007**, *31*, 210–225.
28. Yamaguchi, I.; Kasai, T.; Igawa, H.; Mitani, S.; Ohtani, T. System Identification Using Impulse Response Including Initial States. *J. Syst. Des. Dyn.* **2009**, *3*, 792–803. [CrossRef]
29. Bruni, C.; Isidori, A.; Ruberti, A. A method of realization based on the moments of the impulse-response matrix. *IEEE Trans. Autom. Control* **1969**, *14*, 203–204. [CrossRef]
30. Lundberg, K.H.; Miller, H.R.; Trumper, D.L. Initial conditions, generalized functions and the Laplace transform. *IEEE Control Syst. Mag.* **2007**, *27*, 22–35.
31. Salamon, D. On controllability and observability of time delay systems. *IEEE Trans. Autom. Control* **1984**, *29*, 432–439. [CrossRef]