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Doubly fractional models for dynamic heteroskedastic cycles

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Abstract

Strong persistence is a common phenomenon that has been documented not only in the levels but also in the volatility of many time series. The class of doubly fractional models is extended to include the possibility of long memory in cyclical (non-zero) frequencies in both the levels and the volatility and a new model, the GARMA-GARMASV (Gegenbauer AutoRegressive Mean Average - Id. Stochastic Volatility) is introduced. A sequential estimation strategy, based on the Whittle approximation to maximum likelihood is proposed and its finite sample performance is evaluated with a Monte Carlo analysis. Finally, a trifactorial in the mean and bifactorial in the volatility version of the model is proved to successfully fit the well-known sunspot index.

Keywords: Stochastic volatility; cycles; long memory; QML estimation; sunspot index.

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1. Introduction

Among all the distinct features of a time series, cycles are one of the most complex and difficult to detect. First of all, in contrast with seasonality, cycles take place at an unknown period that requires to be estimated in order to understand the intrinsic characteristics of the phenomenon under analysis. Moreover the concept of cycle is usually associated with a period longer than that corresponding to a seasonal component, which makes difficult its detection separately from a trend. Secondly, cycles may display all the range of variations as any other feature of the series, for example stationarity versus nonstationarity or deterministic versus stochastic cycles, and in alignment different models have been historically proposed to capture a cyclical evolution.

The original approach to modelling cycles relied on the Fourier or Fixed Frequency Effects regression that can be expressed as

$$X_t = \mu + \alpha\cdot\cos(\omega t) + \beta\cdot\sin(\omega t) + \varepsilon_t$$

where α and β are zero mean uncorrelated random variables with equal variance, ε_t is a stationary sequence of random variables independent of α and β, and ω is the frequency that corresponds to a cyclicity of period $\tau = 2\pi/\omega$. This model can be easily extended to cover a more complicated behaviour by adding more linear components of cosine and sine terms in different frequencies and allows to accommodate cycles that manifest a systematic evolution which strictly repeats every period and which can be more or less apparent depending on the size of the variance of α and β relative to that of ε_t.

Nevertheless, the rigidity of the Fourier regression cannot fit a cyclical behaviour that slightly evolves with time. For that purpose stochastic cyclical models were proposed to allow for some time evolution of the cycle. One of the first proposals was the autoregressive AR(2) process

$$\left(1-\phi_1 L-\phi_2 L^2\right)X_t=\varepsilon_t$$

for $\varepsilon_t \sim iid(0,\sigma^2)$ and L the lag-shift operator ($L^k X_t = X_{t-k}$). Yule (1927) showed that the AR(2) displays a quasi-periodic behaviour when the roots of the polynomial $(1-\phi_1 y-\phi_2 y^2)$ are complex, which implies $\phi_2<-(\phi_1)^2/4$. In contrast with the fixed frequency effects model, the cyclical pattern of this process fades out with time. This difference turns up clearly in the frequency domain. Whereas the fixed frequency effects model possesses a non continuous spectral distribution function with a jump at frequency ω, the autoregressive AR(2) shows a continuous spectral distribution function whose spectral density holds a peak at frequency $\omega=\cos^{-1}[-\phi_1(1-\phi_2)/4\phi_2]$ that can be more or less acute depending on the persistence of the cyclical behaviour. The parameter that governs this persistence is ϕ_2 and the model has a stronger serial dependence as ϕ_2 approaches -1. The stationary complex boundary for AR(2) processes is located at ϕ_2 = -1, where unit root cycles can be generated for any value of ϕ_1 in the range [-2,2]. In unit root cycles, as in any unit root series, the effect of a shock in the innovations is permanent and the variance explodes with time.

Between these two extreme categories of stochastic cycles there is scope for a class of intermediate models whose periodic behaviour is more persistent than that of the AR(2) processes but at the same time their variance does not diverge to infinity and the effect of a stochastic shock finally dies out, although quite slowly. These are the cyclical or Gegenbauer long memory (GLM) models of Andel (1986) or Gray *et al.* (1989), defined as

$$\left(1-2L\cos\omega+L^2\right)^d X_t=\varepsilon_t$$

where ε_t is iid$(0,\sigma^2)$, $\omega \in (0,\pi]$ and d is the memory parameter that can be non-integer and measures the degree of persistence of the cycle at the frequency ω. The GLM models are extended to GARMA(p,d,q) (Gegenbauer AutoRegressive Moving Average) by allowing ε_t to be a stationary and invertible ARMA(p,q). If d>0 the spectral density function diverges at ω indicating a persistent cyclical behaviour. These processes are stationary for d<½ if $|\cos(\omega)|<1$ or d<¼ if $|\cos(\omega)|=1$ and mean reverting for d<1 or d<½ under respectively identical conditions. Its spectral or pseudospectral (in the nonstationary case) density function is defined as

$$f_x(\lambda)=\left[2\left(\cos\omega-\cos\lambda\right)\right]^{-2d}f_\varepsilon(\lambda)$$

which behaves around ω as

$$f_x(\lambda+\omega)\sim C|\lambda|^{-2d}$$

as $\lambda \to 0$, where a ~ b means that a/b→1, and for C a finite positive constant (see Chung, 1996, Arteche and Robinson, 2000, or Arteche, 2002). This behaviour is common in economic series due to a seasonal component and the business cycle, in hydrological series affected by the seasonality and the Joseph effect, similarly in climatology and agronomy and also in astronomy with the periodicity of the sunspots index.

Chung (1996) shows that the autocovariance of a GLM processes with a white noise ε_t is

$$\gamma(k)=\frac{\sigma_\varepsilon^2}{2\sqrt{\pi}}\Gamma(1-2d)\{2\sin\omega\}^{\frac{1}{2}-2d}\left\{P_{k-\frac{1}{2}}^{2d-\frac{1}{2}}(\cos\omega)+(-1)^k P_{k-\frac{1}{2}}^{2d-\frac{1}{2}}(-\cos\omega)\right\} \qquad (1)$$

where $P_a^b(z)$ is the associated Legendre function and k indicates the lag. Figure 1 shows the behaviour of $\gamma(0)$ as a function of d and ω. As we can see, the variance of a GLM increases together with the memory parameter and as ω tends to 0 or π.

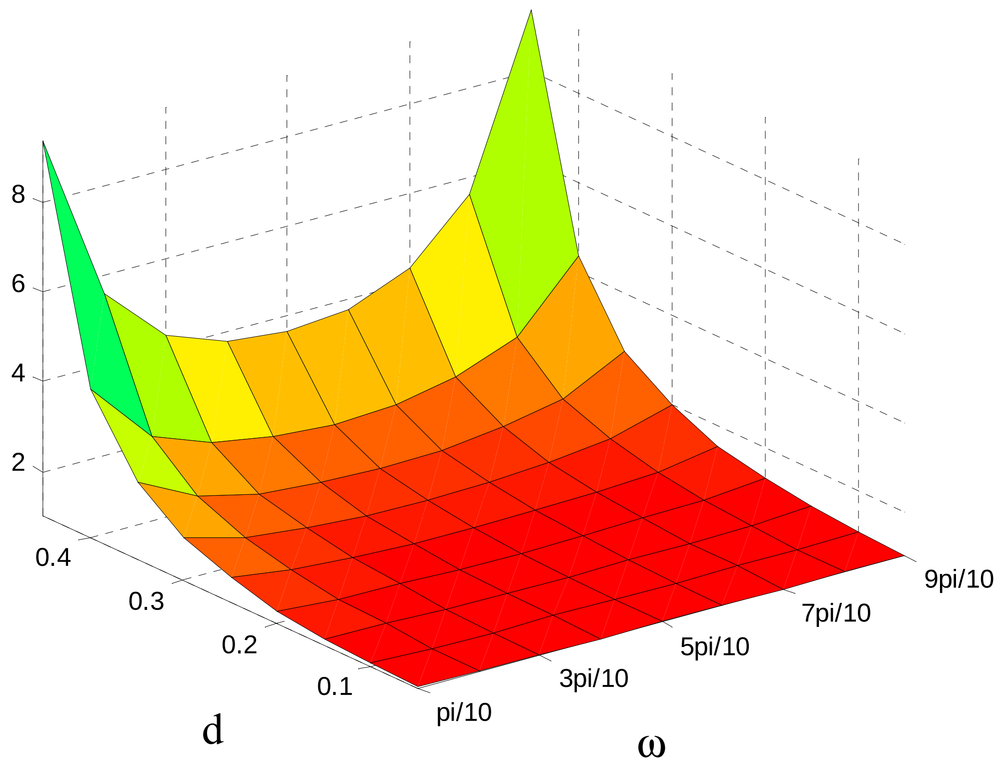

Figure 1. $\gamma(0)$ of a GLM process as a function of d and ω.

Along with the intrinsic features of frequency and persistence, dynamic heteroskedasticity has been a key concept since the seminal paper by Engle (1982), who proposes an autoregressive conditional heteroskedasticity (ARCH) model to capture the time-varying volatility of inflation rates in the United Kingdom. That paper originated a vast number of extensions to conform the volatility characteristics of many economic and financial series. They are characterized by a conditional variance that is fully driven by past observations. The SV models, introduced by Taylor (1986), are a stochastic alternative to the ARCH processes and successive extensions in which the volatility component is driven by different innovations to those affecting the levels.

Recently, particular attention has been paid to the persistence of the volatility, since long memory in second order moments has become one stylized fact in financial time series (Arteche, 2004, 2006, Hurvich *et al.*, 2005, Frederiksen and Nielsen, 2008, Frederiksen et al 2011). Harvey (1998) and Breidt *et al.* (1998) proposed the Long Memory SV model (LMSV) with a long memory structure of the volatility in the frequency $\omega = 0$. However less attention has been paid to the persistence of cyclical or seasonal components of the volatility and traditionally deterministic components have been used if such behaviour is apparent. Recent attempts to allow for a stochastic persistent cyclical component in the volatility are Bordignon et al. (2007) and Arteche (2010). The GARMASV models, introduced in next section, extend this possibility by allowing long memory also at non-zero frequencies. Although dynamic heteroskedasticity has been extensively analyzed in financial and economic time series, such behaviour has also been observed in other areas such as network traffic (Chang and Tsai, 2009) or astronomy (Koenig *et al.*, 1997) where persistence and the existence of cycles are likely to jointly occur not only in the levels but also in higher order moments.

The contribution of this paper is twofold. Section 2 proposes a model that allows for standard and cyclical long memory in levels and volatility, which are characteristics that are likely to jointly occur in some time series (see for example Arteche 2010 for a series of inflation or the application in Section 5), discussing some issues such as stationarity and the existence of moments. The complexity of these models makes difficult an appropriate estimation of the parameters of interest. The second main contribution is a sequential method of estimation of the parameters of the levels and volatility equations. This method is based on the Whittle approximation of maximum likelihood and is described in Section 3. Section 4 shows the finite sample performance of such estimation strategy via Monte Carlo. Section 5 pays particular attention to the persistence and cyclical behaviour of the daily sunspot series, which shows clear evidence of a persistent cycle in both levels and volatility. The former has been analyzed by Gil Alaña (2009) but Section 5 shows that a similar persistent cycle also exists in volatility. Although the sunspot series has been extensively analyzed in a vast number of papers, we are not aware of any literature dealing jointly with cyclical persistence in the levels and volatility, which, as we show below, is a prominent characteristic of the series. Finally Section 6 concludes.

2. GARMA-GARMASV models

The model we propose is based on a GLM for the levels

$$\left(1-2L\cos\omega_{mean}+L^2\right)^{d_{mean}} x_t = y_t ,$$

with y_t a Stochastic Volatility (SV) process defined by

$$y_t = \sigma_* e^{\frac{1}{2}h_t}\varepsilon_t$$

for ε_t an iid(0, σ_ε^2) sequence with σ_* a finite constant and the volatility component h_t being also a GLM model of the form

$$\left(1-2L\cos\omega_{vol}+L^2\right)^{d_{vol}} h_t = \eta_t ,$$

with volatility innovations η_t that are uncorrelated at all leads and lags with ε_t and where the subindices *mean* and *vol* are self explaining. We assume that the means of x_t and h_t are zero. A different from zero mean, either in levels or in volatility, would not affect the results obtained hereafter because the estimation strategy we propose is implicitly mean corrected by ignoring frequency zero. If η_t are iid(0, σ_η^2) we obtain what we call the GLM(d_{mean},ω_{mean})–-GLMSV(d_{vol},ω_{vol}) model where only long memory effects are considered. By contrast, if ε_t

and η_t are allowed to be invertible and stationary ARMA processes we denote such a process as GARMA($p,q,d_{mean},\omega_{mean}$)–GARMASV($s,r,d_{vol},\omega_{vol}$) model, where p,q (s,r) are the orders of the AR and MA polynomials respectively of the mean (volatility) process. Finally, we can extend these models to include the possibility of more than one long memory cyclical frequency in the levels and/or the volatility and thus we get the (k,k*)-factor $GARMA_k(p,q,D_{mean},\Omega_{mean})$–$GARMASV_{k^*}(s,r,D_{vol},\Omega_{vol})$ where $D_{mean}=\{d_{mean(1)}, d_{mean(2)}, \ldots, d_{mean(k)}\}$ $\Omega_{mean}=\{\omega_{mean(1)}, \omega_{mean(2)}, \ldots, \omega_{mean(k)}\}$, $D_{vol}=\{d_{vol(1)}, d_{vol(2)}, \ldots, d_{vol(k^*)}\}$ and $\Omega_{vol}=\{\omega_{vol(1)}, \omega_{vol(2)}, \ldots, \omega_{vol(k^*)}\}$.

In the GARMA-GARMASV model dynamic heteroskedasticity is modelled within a SV framework. Bordignon *et al.* (2007) considered a similar extension for the possibility of a persistent cycle in the volatility but in a GARCH framework. However, as shown by Giraitis *et al.* (2000), stationarity and long memory of the squares are incompatible in ARCH and GARCH models such that a strong persistence in the squared observations necessarily implies an explosive variance. The SV models we propose here allow for a much more flexible specification of persistence in levels and volatility and moments finiteness such that long memory in the levels and volatility is consistent with stationarity. As usual, the existence of more than one innovation complicates the estimation of SV models. To make accessible the estimation of such processes we propose in the next section a simple two step procedure that is easily implementable and has a good behaviour in finite samples.

Under stationarity of the volatility component h_t, which entails $d_{vol}<0.5$ if $|\cos \omega_{vol}| < 1$ or $d_{vol} < 1/4$, if $|\cos \omega_{vol}| = 1$, the existence of the moments of h_t are guaranteed by the finiteness of the moments of η_t and its covariance stationarity. Then, the finiteness of the moments of ε_t and its independence of η_t ensures the finite condition of the moments of powers of y_t and of its absolute value. Consider, for an integer s>0

$$E|y_t|^s = E|\varepsilon_t|^s E e^{0.5sh_t} \pi r^2 .$$

Since

$$e^{0.5sh_t} = e^{0.5}\sum_{j=0}^{\infty}\frac{1}{\sqrt{j!}}H_j(0.5sh_t)$$

for H_j Hermite polynomials, then

$$Ee^{0.5sh_t} = e^{0.5}\sum_{j=0}^{\infty}\frac{1}{\sqrt{j!}}EH_j(0.5sh_t)$$

which is finite under stationarity of h_t with finite moments. Moreover, if ε_t is iid then y_t is a martingale difference sequence. Covariance stationarity of y_t is then guaranteed by the covariance stationarity of h_t and the iid condition (with finite second moments) of ε_t.

The transformation $|y_t|^s$ for some s>0 is commonly used as an approximation to the volatility. The stylized fact of strong persistence in the volatility is then measured by the lag-j correlation of $|y_t|^s$. Robinson (2001) and Surgailis and Viano (2002) show that the lag-j autocovariance of $|y_t|^s$ is proportional to that of the volatility component h_t under some assumptions. Robinson required Gaussianity but Surgailis and Viano relaxed that condition imposing instead stationarity of h_t and finiteness of $E|\varepsilon_t|^u$ and $Ee^{u|\varepsilon t|}$ for all u>0, which in turn holds under Gaussianity of ε_t.

The martingale difference characteristic of y_t ensures the covariance stationarity of x_t in GLM-GLMSV models as long as $d_{mean}<½$ for $|\cos\omega_{mean}|<1$ and $d_{mean}<¼$ for $|\cos\omega_{mean}|=1$, and the usual restrictions for the roots of the AR polynomials.

The GARMASV model can be transformed into a linear model by taking the logarithm of the squares of y_t to obtain

$$\log\left(y_t^2\right) = \mu + h_t + \xi_t$$

where, if ε_t is standard normal, $\xi_t = \log\left(\varepsilon_t^2\right) - E\left[\log\left(\varepsilon_t^2\right)\right]$ is a white noise distributed as the logarithm of χ^2 with one degree of freedom and $\sigma_\xi^2=\pi^2/2$, h_t is as before and μ is a scale parameter. Thus, the logarithm of the squares can be represented as a cyclical long memory process with a non-normal additive perturbation. The effect of the added noise in the estimation of the persistence of the signal has been analyzed in a number of papers, revealing the large bias caused by the noise in the estimators of the memory parameter (Arteche, 2004, Deo and Hurvich, 2001, Haldrup and Nielsen, 2007). Some proposals for bias reduction that account for the added noise have been proposed (Arteche 2006, Hurvich *et al.*, 2005, Frederiksen and Nielsen, 2008, Frederiksen et al 2011).

The GARMA-GARMASV models show a cyclical persistent evolution where the variance of the series also evolves periodically. The most visually identifiable example corresponds to a cyclical series where the variability of the consecutive waves varies in a repetitive manner, which becomes very apparent as it usually affects the amplitude of the cycle (but not necessarily), and at the same time the period remains mostly stable.

3. Estimation of GARMA-GARMASV models

Parametric maximum likelihood estimation of SV models in the time domain is of considerably difficult implementation due to the presence of two different innovations and the nonlinearity of the model, which is aggravated by the complexity of the GARMA-GARMASV models. Substantially simpler implementation is achieved in the frequency domain by the asymptotic approximation to maximum likelihood based on the so called Whittle function as proposed by Breidt *et al.* (1998) and Harvey (1998) for an ARFIMA volatility component with no linear correlation in the levels. We extend here this strategy in two directions. First we consider the case of cyclical long memory, extending the possibility of spectral poles to any frequency in the Nyquist band. Second we allow for a richer structure in the linear dependence of the levels, permitting in particular the possibility of first order cyclical long memory.

Fitting a GARMA-GARMASV model to an observed series can be accomplished by extracting the conditional mean dependence components prior to facing the estimation of the volatility structure. When we move far from economic and financial time series, where the periodicity usually turns up in the form of seasonality, and consequently with known frequencies, the first challenge resides in the identification of the cyclical frequency or frequencies. This can be made by inspection of the periodogram and considering its consecutive maxima as point estimators of potential long memory frequencies, as suggested by Yajima (1996) and Hidalgo and Soulier (2004). Note that consistency of this estimator has only been proved under iid innovations with finite eighth moment ruling out the possibility of a dynamic volatility. Based on the martingale difference characteristic of y_t we believe that the maximizer of the periodogram remains a consistent estimator of the spectral pole in a general GARMA-GARMASV context. Once we have identified the frequency or frequencies with spectral poles we proceed to estimate the GARMA model proposed for the levels by means of the Whittle approach, usually known as Quasi Maximum Likelihood (QML) in the frequency domain. Specifically, in order to estimate the memory parameter in the levels of the series, we will use the discrete version by Graf (1983), simplified for linear models, that can be written as

$$\hat{\Theta}_{QML} = \arg\min_{\Theta} \frac{2\pi}{T} \sum{}' \frac{I_x\left(\lambda_j\right)}{h\left(\lambda_j;\Theta\right)}$$

where Σ' runs for all Fourier frequencies of the form $\lambda_j = 2\pi j/T$, j=1,...,[T/2], (with a finite spectral density function, the omission of the zero frequency in the estimation implies mean correction such that the estimation of an unknown mean is not needed), T is the sample size, $I_x(\lambda_j)$ is the periodogram of x_t, t = 1,...T, defined as

$$I_x\left(\lambda_j\right) = \left| \frac{1}{\sqrt{2\pi T}} \sum_{t=1}^{T} x_t e^{-i\lambda_j t} \right|^2 ,$$

$h(\lambda_j;\Theta)$ is the power transfer function of the process we intend to fit and Θ is the set of parameters to be estimated; in the case of GLM models $\Theta = d_{mean}$ and

$$h\left(\lambda_j;d_{mean}\right) = \left| 4\sin\left(\frac{\lambda_j + \omega_{mean}}{2}\right)\sin\left(\frac{\lambda_j - \omega_{mean}}{2}\right) \right|^{-2d_{mean}} .$$

Such strategy has been proved to be valid by Zaffaroni (2003) for the estimation of the ARMA parameters in the levels when the volatility has the form of a nonlinear moving average, closely related with SV models, that allows for a wide range of forms of strong persistence in the volatility. Shao (2010) extended this result to cover standard long memory in levels but only short memory in volatility. Similarly Diongue and Guegan (2004) show the asymptotic properties of Whittle estimation for k-factor GARMA models in the levels with GARCH innovations that do not allow for long memory in volatility. All of them show consistency and asymptotic normality with a √T rate of convergence. We are not aware of any work dealing with the asymptotic properties of the Whittle estimator when both levels and volatility have long memory. The simulation results in next section shed some light on this issue, showing at least its good finite sample properties although further research in this area seems necessary.

In order to estimate the parameters of the structure of the volatility, we first calculate the residuals of the first order model estimated in the previous step and work on the transformed series $\log\left(\hat{y}_t^2\right)$, which, as we have seen, takes the form of a perturbed long memory model. Therefore, the linear version of the QML estimator cannot be used in this second stage of the estimation process. In this case, we will use the complete discrete QML estimator defined as

$$\hat{\Psi}_{QML} = \arg\min_{\Psi} \frac{2\pi}{T}\left[\sum{}' \frac{I_{\log\left(\hat{y}_t^2\right)}\left(\lambda_j\right)}{f\left(\lambda_j;\Psi\right)} + \sum{}' \log f\left(\lambda_j;\Psi\right)\right]$$

where the sum runs for all Fourier frequencies where h_t (and consequently $\log\left(\hat{y}_t^2\right)$) has a finite spectral density function, $\Psi = \left(\sigma_\eta^2, \sigma_\xi^2, d_{vol}\right)$ and

$$f\left(\lambda_j;\sigma_\eta^2,\sigma_\xi^2,d_{vol}\right) = \frac{\sigma_\eta^2}{2\pi}\left|4\sin\left(\frac{\lambda_j+\omega_{vol}}{2}\right)\sin\left(\frac{\lambda_j-\omega_{vol}}{2}\right)\right|^{-2d_{vol}} + \frac{\sigma_\xi^2}{2\pi}.$$

Again, the need to estimate an unknown mean for the volatility equation is eluded by omitting the zero frequency. If such estimation is required, the sample mean, which remains consistent but looses efficiency with respect to the short memory case, could be used In GLMSV models. As in the analysis of the mean levels, ω_{vol} can be selected by any of the techniques existing in the literature, such as the maximizer of the periodogram of the transformed series. In a sense we follow here the strategy suggested by Breidt *et al.* (1998), Harvey (1998) and Perez and Ruiz (2001) but extended to the cyclical case. The validity of such a procedure has been recently proved by Zaffaroni (2009) for a wide class of stochastic volatility and EGARCH related models which allow for short and long memory, including the cyclical GLMSV model defined for y_t. Under some moment restrictions on $\log \varepsilon_t^2$ and η_t Zaffaroni shows consistency and asymptotic normality of Ψ_{QMLE} with a √T rate of convergence. Note that Gaussianity neither of ε_t nor of η_t are required. We work however with residuals instead of observed series and no asymptotic results are yet available in this case. The √T-consistency of estimators of the parameters in the levels indicates that the consistency of the estimators of the volatility parameters will probably hold, but the asymptotic distribution will surely be affected, at least in the asymptotic covariance matrix, by the use of residuals instead of observables. However, the results of the Monte Carlo in next section suggest that this two step strategy can remain valid in this more complex framework.

Alternatively, semiparametric estimates of the memory parameter can also be obtained by means of a local version of the Whittle function. However, although this estimator remains consistent in a large number of situations, it is subject to a large bias caused by the added noise in the log of squares equation (e.g. Arteche, 2004, 2006; Haldrup and Nielsen, 2007). Moreover its performance relies heavily on a user chosen bandwidth parameter and it is not clear how to optimally select such quantity. With these considerations we focus here on more efficient parametric estimation techniques, acknowledging the risk of inconsistencies caused by misspecification of the models.

4. Finite sample performance

We have explored the following combinations of the four parameters ω_{mean}, ω_{vol}, d_{mean} and d_{vol}. On one hand, the frequencies range from $\pi/10$ to $9\pi/10$, incrementing by $\pi/10$; on the other, the memory parameters range from 0.05 to 0.5, incrementing by 0.05. We have included this last case of non-stationarity as a natural boundary of our parametrical space. Therefore 8100 combinations were tested in two different scenarios of known and unknown frequencies with 500 replications in each, obtaining an overall amount of 8.100.000 estimations of the parameters d_{mean} and d_{vol}. The chosen sample size, T = 3500, is in coherence with the huge availability of astronomical data, that have been registered regularly over four centuries (Ballmoos *et al.*, 2009; Vaquero, 2007) and of financial data where SV models are especially useful. The GLM structures of the samples were generated following the simulating method described by Arteche and Robinson (2000) with η_t and ε_t both standard Gaussian and independent.

In order to assess the performance of the two step QML method proposed in the previous section the following indicators were calculated in the 500 replications of every parametrical combination. We provide below the formulae regarding the estimation of d_{mean}, the corresponding expressions for d_{vol} are similarly defined. The legend is: i = 1 … 9 refer to the different levels of $\omega_{mean} = \pi/10 \ldots 9\pi/10$, j = 1 … 9 to ω_{vol}, k = 1 … 10 to d_{mean} = 0.05 … 0.5 and finally l = 1 … 10 to d_{vol}.

$$\overline{\hat{d}}(i,j,k,l) = \frac{1}{500}\sum_{rep=1}^{500}\hat{d}(i,j,k,l,rep),$$

$$var(i,j,k,l) = \frac{1}{500}\sum_{rep=1}^{500}\left[\hat{d}(i,j,k,l,rep) - \overline{\hat{d}}(i,j,k,l)\right]^2 \quad \text{and}$$

$$bias(i,j,k,l) = \overline{\hat{d}}(i,j,k,l) - d_{mean}(k)_0$$

The estimations were carried out using MATLAB 7.0´s Optimization toolbox, more specifically the minimization rutines use a trust region method. Next tables show a selection of the results. All the remaining results are available upon request.

Tables 1 and 2 display the results of estimation of d_{mean} for binary combinations of parameters and Table 3 the corresponding results for d_{vol}. The figures in the tables are calculated as averages of the values obtained with the parameters not indexed in the tables. For example, in Table 1 the bias is obtained as

$$bias(i,k) = \frac{\sum_j \sum_l bias(i,j,k,l)}{90}.$$

In all these results the frequency of the cycle is assumed to be known. In order to asses if a normal distribution can be a good approximation of the distribution of the QML estimator, we also checked Gaussianity of the distribution of the estimates in every set of 500 replications using a standard Pearson´s χ^2 test at 5% significance level. The proportion of rejections for the combinations of parametrical values in each level of the process is included in Tables 1 and 3.

Some of the conclusions that can be drawn from Table 1 are the following:

- The bias of estimation of d_{mean} seems consistently positive for memories $d_{mean} > 0.40$ and negative below. It is larger in absolute terms for central frequencies and, as can be expected, for less persistent memories.

- The standard error (SE) of estimation shows no dependence on the parametric value of the memory and increases for central frequencies. Central frequencies, therefore, are determinant to worsen the performance of the QML estimator with both higher bias and higher SE.

- Inference based on the Gaussian distribution seems reliable for any parametric combination.

Table 2 shows the results of estimation of d_{mean} for the combinations of values of the volatility parameters, which are neccesary to understand some of the results displayed in Table 3. They can be summarized as follows:

- No combination of the volatility parameters seems to affect the bias of the QML estimator of d_{mean} except for the non-stationary value of $d_{vol} = 0.50$ where a dependence on ω_{vol} also takes place as the bias tends to be smaller for low frequencies and larger in absolute value for high frequencies.

- The standard error of estimation of d_{mean} shows a clear dependence on the variance of the volatility process h_t and so it increases for non central ω_{vol} and the more persistent values of d_{vol}, which correspond to a larger variance of h_t.

d_{mean} \ ω_{mean}	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$	$5\pi/10$	$6\pi/10$	$7\pi/10$	$8\pi/10$	$9\pi/10$
0.05	-0,0013	-0,0016	-0,002	-0,0026	-0,0031	-0,0024	-0,0017	-0,0013	-0,0011
	(0,0116)	(0,0136)	(0,0151)	(0,0169)	(0,0176)	(0,017)	(0,0155)	(0,0133)	(0,0114)
Rejection proportion	0,1222	0,1	0,0444	0,0444	0,1222	0,0778	0,0667	0,0778	0,0667
0.10	-0,0013	-0,0014	-0,0018	-0,0025	-0,0026	-0,0024	-0,0016	-0,0014	-0,001
	(0,0115)	(0,0132)	(0,0152)	(0,0168)	(0,0173)	(0,0166)	(0,0154)	(0,0137)	(0,0114)
Rejection proportion	0,0889	0,1	0,0556	0,1	0,0444	0,0556	0,0778	0,0778	0,1
0.15	-0,0012	-0,0014	-0,0018	-0,0022	-0,0025	-0,0021	-0,0015	-0,0012	-0,001
	(0,0114)	(0,0134)	(0,0149)	(0,0166)	(0,0177)	(0,0168)	(0,0153)	(0,0136)	(0,0114)
Rejection proportion	0,0667	0,0667	0,0889	0,1111	0,0778	0,0778	0,0778	0,1222	0,0778
0.20	-0,0011	-0,0013	-0,0016	-0,0019	-0,0023	-0,0019	-0,0012	-0,0009	-0,0009
	(0,0114)	(0,0129)	(0,0154)	(0,0167)	(0,0175)	(0,0165)	(0,0151)	(0,0134)	(0,0114)
Rejection proportion	0,1111	0,1	0,1	0,0444	0,0667	0,0444	0,1111	0,0889	0,1111
0.25	-0,0008	-0,0010	-0,0014	-0,0017	-0,002	-0,0016	-0,0012	-0,0007	-0,0006
	(0,0116)	(0,0136)	(0,0157)	(0,0169)	(0,0174)	(0,0168)	(0,0155)	(0,0139)	(0,0116)
Rejection proportion	0,0889	0,0778	0,0556	0,0667	0,1	0,1111	0,0444	0,0667	0,0889
0.30	-0,0008	-0,0007	-0,0010	-0,0012	-0,0015	-0,0013	-0,0007	-0,0006	-0,0004
	(0,0116)	(0,0135)	(0,0152)	(0,0168)	(0,0173)	(0,0169)	(0,0156)	(0,0138)	(0,0114)
Rejection proportion	0,0444	0,0667	0,0778	0,0333	0,0667	0,0889	0,0444	0,0778	0,0778
0.35	-0,0005	-0,0004	-0,0006	-0,0007	-0,0008	-0,0007	-0,0003	-0,0002	-0,0001
	(0,0115)	(0,0138)	(0,0151)	(0,0168)	(0,0178)	(0,017)	(0,0151)	(0,0134)	(0,0114)
Rejection proportion	0,0667	0,0556	0,0556	0,0667	0,0667	0,0667	0,0667	0,0444	0,0222
0.40	-0,0002	-0,0002	-0,00003	-0,0002	-0,0002	0,00005	0,0003	0,0003	0,0003
	(0,0119)	(0,0132)	(0,0154)	(0,0172)	(0,018)	(0,0169)	(0,0159)	(0,0136)	(0,0116)
Rejection proportion	0,0889	0,0889	0,1	0,0556	0,0667	0,0889	0,0889	0,0556	0,1222
0.45	0,0002	0,0004	0,0005	0,0008	0,001	0,0012	0,0009	0,0009	0,0007
	(0,0115)	(0,0137)	(0,0152)	(0,0173)	(0,0178)	(0,0172)	(0,0153)	(0,0137)	(0,0119)
Rejection proportion	0,1444	0,0667	0,1	0,1	0,0444	0,0778	0,0889	0,0667	0,1222
0.50	0,0004	0,0009	0,0013	0,0019	0,0022	0,0023	0,0022	0,0017	0,0013
	(0,0117)	(0,0136)	(0,0156)	(0,0174)	(0,0181)	(0,0176)	(0,0157)	(0,014)	(0,0118)
Rejection proportion	0,0667	0,0333	0,0889	0,1	0,0667	0,1111	0,0889	0,0778	0,0444

Table 1. Biases (standard errors) of estimation of d_{mean}. Known frequencies (1).

d_{vol} \ ω_{vol}	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$	$5\pi/10$	$6\pi/10$	$7\pi/10$	$8\pi/10$	$9\pi/10$
0.05	-0.0008	-0.0006	-0.0007	-0.0007	-0.0007	-0.0008	-0.0006	-0.0007	-0.0007
	(0.0113)	(0.011)	(0.0109)	(0.0109)	(0.0109)	(0.0108)	(0.0106)	(0.0108)	(0.0107)
0.10	-0.0007	-0.0007	-0.0008	-0.0007	-0.0007	-0.0007	-0.0007	-0.0007	-0.0008
	(0.0116)	(0.0112)	(0.011)	(0.0109)	(0.0108)	(0.0107)	(0.0106)	(0.0105)	(0.0106)
0.15	-0.0006	-0.0007	-0.0007	-0.0007	-0.0008	-0.0007	-0.0007	-0.0007	-0.0007
	(0.0121)	(0.0115)	(0.0112)	(0.0109)	(0.0108)	(0.0106)	(0.0105)	(0.0105)	(0.0104)
0.20	-0.0007	-0.0007	-0.0006	-0.0007	-0.0006	-0.0007	-0.0007	-0.0007	-0.0007
	(0.0131)	(0.0119)	(0.0113)	(0.0108)	(0.0106)	(0.0103)	(0.0103)	(0.0104)	(0.0105)
0.25	-0.0007	-0.0007	-0.0007	-0.0007	-0.0008	-0.0007	-0.0007	-0. 0008	-0.0007
	(0.0144)	(0.0125)	(0.0114)	(0.0109)	(0.0105)	(0.0103)	(0.0102)	(0.0103)	(0.0105)
0.30	-0. 0007	-0.0006	-0.0006	-0.0007	-0.0006	-0.0008	-0.0007	-0.0008	-0.0007
	(0.0166)	(0.0134)	(0.0117)	(0.011)	(0.0105)	(0.0102)	(0.01)	(0.0102)	(0.0109)
0.35	-0.0006	-0.0006	-0.0007	-0.0007	-0.0008	-0.0007	-0.0008	-0.0008	-0.0007
	(0.0209)	(0.015)	(0.0124)	(0.0111)	(0.0104)	(0.0101)	(0.01)	(0.0104)	(0.0126)
0.40	-0.0008	-0.0006	-0.0007	-0.0007	-0.0007	-0.0008	-0.0007	-0.0008	-0.0008
	(0.029)	(0.0171)	(0.0131)	(0.0114)	(0.0104)	(0.0101)	(0.01)	(0.0108)	(0.0158)
0.45	-0.0003	-0.0005	-0.0006	-0.0008	-0.0008	-0.0008	-0.0008	-0.0009	-0.0009
	(0.041)	(0.0213)	(0.0146)	(0.0119)	(0.0106)	(0.0104)	(0.0106)	(0.0118)	(0.0213)
0.50	0.0001	-0.0004	-0.0007	-0.0008	-0.0009	-0.0009	-0.0009	-0.0010	-0.0009
	(0.0588)	(0.0279)	(0.0171)	(0.0131)	(0.0113)	(0.0113)	(0.0123)	(0.0138)	(0.0295)

Table 2. Biases (standard errors) of estimation of d_{mean}. Known frequencies (2).

d_{vol} \ ω_{vol}	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$	$5\pi/10$	$6\pi/10$	$7\pi/10$	$8\pi/10$	$9\pi/10$
0.05	-0.0023 (0.059)	-0.0037 (0.0782)	-0.0059 (0.0822)	-0.015 (0.1059)	-0.0201 (0.1286)	-0.0161 (0.1103)	-0.0114 (0.0954)	-0.0062 (0.078)	-0.0052 (0.0655)
Rejection proportion	0.4778	0.5667	0.7333	0.9778	0.9889	0.9778	0.8222	0.5889	0.6667
0.10	0.0019 (0.0472)	0.0001 (0.0594)	-0.0038 (0.0696)	-0.0077 (0.0833)	-0.0135 (0.1012)	-0.0108 (0.0949)	-0.0054 (0.0745)	-0.0025 (0.0676)	-0.0005 (0.0549)
Rejection proportion	0.1222	0.1778	0.3556	0.6778	0.9333	0.7556	0.4667	0.3	0.2778
0.15	0.0032 (0.0443)	0.0015 (0.0534)	-0.0008 (0.0606)	-0.004 (0.0677)	-0.0043 (0.0767)	-0.0044 (0.0715)	-0.0023 (0.0679)	0.0002 (0.0617)	0.0016 (0.0469)
Rejection proportion	0.1889	0.1222	0.1333	0.2667	0.4	0.4111	0.1889	0.1667	0.3
0.20	0.0041 (0.0426)	0.0026 (0.0499)	0.0015 (0.0556)	0.0002 (0.0597)	-0.0013 (0.0674)	-0.0007 (0.0613)	0.0008 (0.0591)	0.001 (0.0577)	0.0022 (0.0473)
Rejection proportion	0.3	0.1556	0.0889	0.0778	0.1667	0.1444	0.1556	0.2111	0.2556
0.25	0.0044 (0.0414)	0.0034 (0.0482)	0.0023 (0.0526)	0.0018 (0.056)	0.0023 (0.0576)	0.0023 (0.0555)	0.0021 (0.0566)	0.0028 (0.0514)	0.0028 (0.0475)
Rejection proportion	0.2111	0.0778	0.0444	0.0333	0.0889	0.0778	0.1111	0.1889	0.2778
0.30	0.0047 (0.0409)	0.0037 (0.047)	0.0035 (0.0513)	0.0033 (0.053)	0.0037 (0.0537)	0.0047 (0.0539)	0.0038 (0.051)	0.0041 (0.049)	0.0031 (0.0424)
Rejection proportion	0.1333	0.0444	0.0889	0.0667	0.0889	0.1111	0.0889	0.1111	0.2111
0.35	0.0047 (0.0401)	0.0043 (0.0457)	0.0048 (0.0493)	0.0052 (0.0521)	0.007 (0.0528)	0.0057 (0.0523)	0.0057 (0.053)	0.0053 (0.05)	0.0034 (0.0462)
Rejection proportion	0.1111	0.1	0.0778	0.1111	0.1222	0.0556	0.0889	0.0778	0.1222
0.40	0.0032 (0.0396)	0.0041 (0.0453)	0.0059 (0.0485)	0.0065 (0.0506)	0.0085 (0.0524)	0.0082 (0.0567)	0.0082 (0.0497)	0.0071 (0.049)	0.0004 (0.0459)
Rejection proportion	0.1333	0.0667	0.1	0.0444	0.0556	0.1222	0.0444	0.1111	0.1556
0.45	-0.0055 (0.0383)	0.0005 (0.0449)	0.0056 (0.0488)	0.0087 (0.051)	0.0094 (0.0521)	0.0106 (0.0515)	0.0099 (0.0502)	0.0068 (0.0509)	-0.015 (0.0609)
Rejection proportion	0.0778	0.0111	0.0667	0.1222	0.1333	0.0889	0.0333	0.1556	0.3667
0.50	-0.0328 (0.0376)	-0.0171 (0.0498)	-0.0012 (0.0507)	0.0069 (0.053)	0.0092 (0.0542)	0.011 (0.0544)	0.0087 (0.0519)	0.0013 (0.0594)	-0.0755 (0.1025)
Rejection proportion	0,1111	0,1889	0,1222	0,1	0,1333	0,1111	0,1889	0,3333	0,6444

Table 3. Biases (standard errors) of estimation of d_{vol}. Known frequencies.

In our sequential procedure of estimation, the memory parameter of the volatility is estimated on the log of the squared residuals obtained filtering the original series with the estimated value of d_{mean}. Hence, the properties of the estimation of d_{vol} will be affected not only by the presence of a non normal additive noise term but also by the properties of the previous estimation of d_{mean}. Hereafter we assume for simplicity that the ε_t are standard Gaussian, which is a usual assumption in the SV literature, and therefore the contribution of the perturbation to the spectral density of $\log\left(\hat{y}_t^2\right)$ is expressed as

$$\frac{\sigma_\xi^2}{2\pi} = \frac{\pi}{4}$$

leaving the parameters d_{vol} and σ_η^2 as the arguments of minimization.

In Table 3 we can see the results of the estimation of the memory parameter of the volatility. The main conclusions are:

d_{mean} \ ω_{mean}	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$	$5\pi/10$	$6\pi/10$	$7\pi/10$	$8\pi/10$	$9\pi/10$
0.05	-0,0158 (0,0292)	-0,0216 (0,03)	-0,0256 (0,0291)	-0,0284 (0,027)	-0,0296 (0,0263)	-0,0286 (0,027)	-0,0256 (0,0291)	-0,0211 (0,0298)	-0,0156 (0,0292)
0.10	-0,0053 (0,0159)	-0,0099 (0,0234)	-0,0149 (0,0296)	-0,0194 (0,0335)	-0,0211 (0,0347)	-0,0193 (0,0335)	-0,0148 (0,03)	-0,0096 (0,0235)	-0,005 (0,0163)
0.15	-0,0031 (0,0117)	-0,0042 (0,0153)	-0,006 (0,019)	-0,0079 (0,0218)	-0,009 (0,0234)	-0,0075 (0,0215)	-0,0058 (0,0189)	-0,004 (0,0153)	-0,0028 (0,0118)
0.20	-0,0021 (0,0117)	-0,0025 (0,0135)	-0,0035 (0,0155)	-0,0046 (0,017)	-0,0051 (0,0179)	-0,0044 (0,0172)	-0,0033 (0,0154)	-0,0024 (0,0133)	-0,0019 (0,0116)
0.25	-0,0016 (0,0114)	-0,002 (0,0134)	-0,0025 (0,0149)	-0,0033 (0,0168)	-0,0039 (0,0174)	-0,0034 (0,0168)	-0,0024 (0,0152)	-0,0018 (0,013)	-0,0013 (0,0114)
0.30	-0,0013 (0,0111)	-0,0016 (0,0134)	-0,0019 (0,015)	-0,0026 (0,0166)	-0,003 (0,0174)	-0,0025 (0,0166)	-0,0016 (0,0154)	-0,0012 (0,0131)	-0,0009 (0,0112)
0.35	-0,001 (0,0113)	-0,0011 (0,0133)	-0,0014 (0,0149)	-0,0017 (0,0168)	-0,0019 (0,0175)	-0,0015 (0,0167)	-0,0012 (0,0149)	-0,0007 (0,0134)	-0,0005 (0,0111)
0.40	-0,0006 (0,0111)	-0,0006 (0,0131)	-0,0006 (0,0148)	-0,0009 (0,0168)	-0,0009 (0,0173)	-0,0007 (0,0167)	-0,0003 (0,015)	0 (0,0134)	0 (0,0113)
0.45	-0,0001 (0,0113)	-0,0001 (0,0136)	-0,0001 (0,0151)	0,0003 (0,0168)	0,0003 (0,0178)	0,0006 (0,0169)	0,0007 (0,015)	0,0006 (0,0134)	0,0006 (0,0112)
0.50	0,0002 (0,0114)	0,0005 (0,0131)	0,001 (0,0152)	0,0013 (0,0171)	0,0017 (0,0181)	0,0021 (0,0172)	0,002 (0,0154)	0,0017 (0,0135)	0,0012 (0,0117)

Table 4. Biases (standard errors) of estimation of d_{mean}. Unknown frequencies (1)

- The dependence of the bias of estimation of d_{vol} on the parametrical values of d_{vol} and ω_{vol} resembles what we have already said in the case of the estimation of d_{mean} on d_{mean} and ω_{mean} and so it is larger in absolute terms for central frequencies and increases in real terms with d_{vol}, making it positive for the most persistent memories.

- The large negative biases that appear in the non-stationary value of d_{vol} and extreme values of the frequency ω_{vol} are related to the poor properties of estimation of d_{mean} in these parametrical combinations.

- The size of the bias ranges from twice as big to around 10 times bigger for the most extreme cases than its d_{mean} counterpart as a clear consequence of the presence of the noise and the effects of the estimation of d_{mean}. The standard errors are also larger than what we obtained in the estimation of d_{mean}.

- Again the standard error behaves similarly as in the estimation of d_{mean} (larger SE for central frequencies) but in the case of small values of d_{vol}, the effect of the noise is able to eclipse the long memory process producing strongly antipersistent estimations of the memory parameter that eventually cause a heavily asymmetric distribution and a recurring rejection of Gaussianity.

Tables 4, 5 and 6 show the bias and SE of estimation of d_{mean} and d_{vol} under the empirically more relevant situation of unknown frequencies. In these cases the cyclical frequencies were chosen, following Yajima (1996) and Hidalgo and Soulier (2004), as the values that maximize first the periodogram of x_t for the estimation of d_{mean} and then the periodogram of $\log\left(\hat{y}_t^2\right)$ for the estimation of d_{vol}. In general the two step strategy is quite robust to an unknown frequency, especially with the larger memory parameters. However, the

d_{vol} \ ω_{vol}	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$	$5\pi/10$	$6\pi/10$	$7\pi/10$	$8\pi/10$	$9\pi/10$
0.05	-0,0049 (0,0159)	-0,005 (0,0158)	-0,005 (0,0158)	-0,0048 (0,0155)	-0,005 (0,0155)	-0,0049 (0,0154)	-0,0048 (0,0154)	-0,0049 (0,0156)	-0,0049 (0,0156)
0.10	-0,0047 (0,0158)	-0,0048 (0,0157)	-0,0048 (0,0155)	-0,0049 (0,0156)	-0,0048 (0,0155)	-0,0049 (0,0156)	-0,005 (0,0152)	-0,0049 (0,0154)	-0,0049 (0,0152)
0.15	-0,0048 (0,0163)	-0,0049 (0,0161)	-0,0049 (0,0157)	-0,0048 (0,0155)	-0,005 (0,0156)	-0,005 (0,0154)	-0,0049 (0,0152)	-0,0049 (0,0152)	-0,0049 (0,0156)
0.20	-0,0046 (0,017)	-0,0048 (0,0162)	-0,0049 (0,0158)	-0,005 (0,0157)	-0,0049 (0,0153)	-0,0048 (0,0152)	-0,005 (0,0151)	-0,0051 (0,0156)	-0,005 (0,0151)
0.25	-0,0046 (0,0181)	-0,0048 (0,0167)	-0,0049 (0,0162)	-0,0049 (0,0156)	-0,0049 (0,0151)	-0,0051 (0,0155)	-0,005 (0,0149)	-0,0049 (0,0149)	-0,0048 (0,0153)
0.30	-0,0044 (0,0198)	-0,0048 (0,0173)	-0,0047 (0,0162)	-0,0049 (0,0156)	-0,005 (0,0152)	-0,0051 (0,0151)	-0,0049 (0,015)	-0,005 (0,0151)	-0,0047 (0,0155)
0.35	-0,0039 (0,0236)	-0,0046 (0,0183)	-0,0047 (0,0166)	-0,0049 (0,0155)	-0,0049 (0,0149)	-0,005 (0,0151)	-0,005 (0,0149)	-0,005 (0,0152)	-0,0046 (0,0165)
0.40	-0,0026 (0,0303)	-0,0042 (0,0203)	-0,0044 (0,0169)	-0,0048 (0,0159)	-0,0049 (0,0151)	-0,005 (0,015)	-0,0049 (0,0148)	-0,0048 (0,0154)	-0,0042 (0,019)
0.45	-0,0001 (0,0412)	-0,0037 (0,0237)	-0,0044 (0,0181)	-0,0046 (0,0163)	-0,0047 (0,0152)	-0,0048 (0,0151)	-0,0048 (0,0154)	-0,0045 (0,0156)	-0,0033 (0,0236)
0.50	0,0036 (0,0571)	-0,0026 (0,0292)	-0,0039 (0,02)	-0,0044 (0,0168)	-0,0046 (0,0156)	-0,0047 (0,0157)	-0,0043 (0,0162)	-0,0041 (0,0174)	-0,0017 (0,0304)

Table 5. Biases (standard errors) of estimation of d_{mean}. Unknown frequencies (2).

d_{vol} \ ω_{vol}	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$	$5\pi/10$	$6\pi/10$	$7\pi/10$	$8\pi/10$	$9\pi/10$
0.05	-0,0167 (0,1471)	-0,0164 (0,1279)	-0,0172 (0,1534)	-0,0174 (0,1454)	-0,0165 (0,141)	-0,0166 (0,143)	-0,0161 (0,1484)	-0,0159 (0,1303)	-0,0176 (0,1581)
0.10	-0,0809 (0,2182)	-0,0695 (0,1532)	-0,0622 (0,125)	-0,0583 (0,1141)	-0,0557 (0,1064)	-0,0581 (0,1391)	-0,0623 (0,1306)	-0,0699 (0,1497)	-0,0775 (0,2063)
0.15	-0,1488 (0,3592)	-0,1193 (0,1973)	-0,1039 (0,1325)	-0,094 (0,1088)	-0,0904 (0,0965)	-0,0945 (0,1009)	-0,1031 (0,1312)	-0,1207 (0,2)	-0,1477 (0,3385)
0.20	-0,2004 (0,4925)	-0,1504 (0,2351)	-0,1295 (0,1502)	-0,1181 (0,1092)	-0,1154 (0,099)	-0,119 (0,1094)	-0,1293 (0,1472)	-0,1508 (0,2284)	-0,1958 (0,4737)
0.25	-0,1441 (0,496)	-0,1344 (0,2536)	-0,1233 (0,1633)	-0,1175 (0,1266)	-0,1159 (0,1153)	-0,1169 (0,1262)	-0,1243 (0,1659)	-0,1358 (0,252)	-0,1486 (0,5016)
0.30	-0,0429 (0,287)	-0,0684 (0,2021)	-0,0769 (0,1533)	-0,0801 (0,1307)	-0,0812 (0,1235)	-0,0809 (0,1312)	-0,0765 (0,1514)	-0,0705 (0,2067)	-0,0449 (0,292)
0.35	-0,0067 (0,1025)	-0,0183 (0,103)	-0,0285 (0,1038)	-0,0339 (0,1011)	-0,0349 (0,1012)	-0,0338 (0,1025)	-0,0279 (0,1053)	-0,019 (0,1093)	-0,006 (0,0861)
0.40	-0,0012 (0,0404)	-0,0042 (0,055)	-0,0073 (0,064)	-0,0087 (0,0668)	-0,0091 (0,0689)	-0,0073 (0,067)	-0,0051 (0,063)	-0,0029 (0,0555)	-0,0016 (0,0433)
0.45	-0,0006 (0,0393)	-0,0009 (0,0456)	-0,0003 (0,0497)	0,0004 (0,0529)	0,0012 (0,0544)	0,0021 (0,0534)	0,0023 (0,0504)	0,0023 (0,0472)	-0,0019 (0,0435)
0.50	-0,0081 (0,0378)	-0,003 (0,0447)	0,0022 (0,0487)	0,0039 (0,0515)	0,006 (0,0523)	0,0066 (0,0519)	0,0069 (0,0501)	0,0047 (0,048)	-0,0164 (0,0626)

Table 6. Biases (standard errors) of estimation of d_{vol}. Unknown frequencies.

difficulty to obtain accurate estimates of the frequency of the spectral pole with low memory deteriorates the estimation of the memory parameters, mainly of d_{vol}, because the inaccurate estimation of the frequency adds to the fact that the estimation is based on residuals.

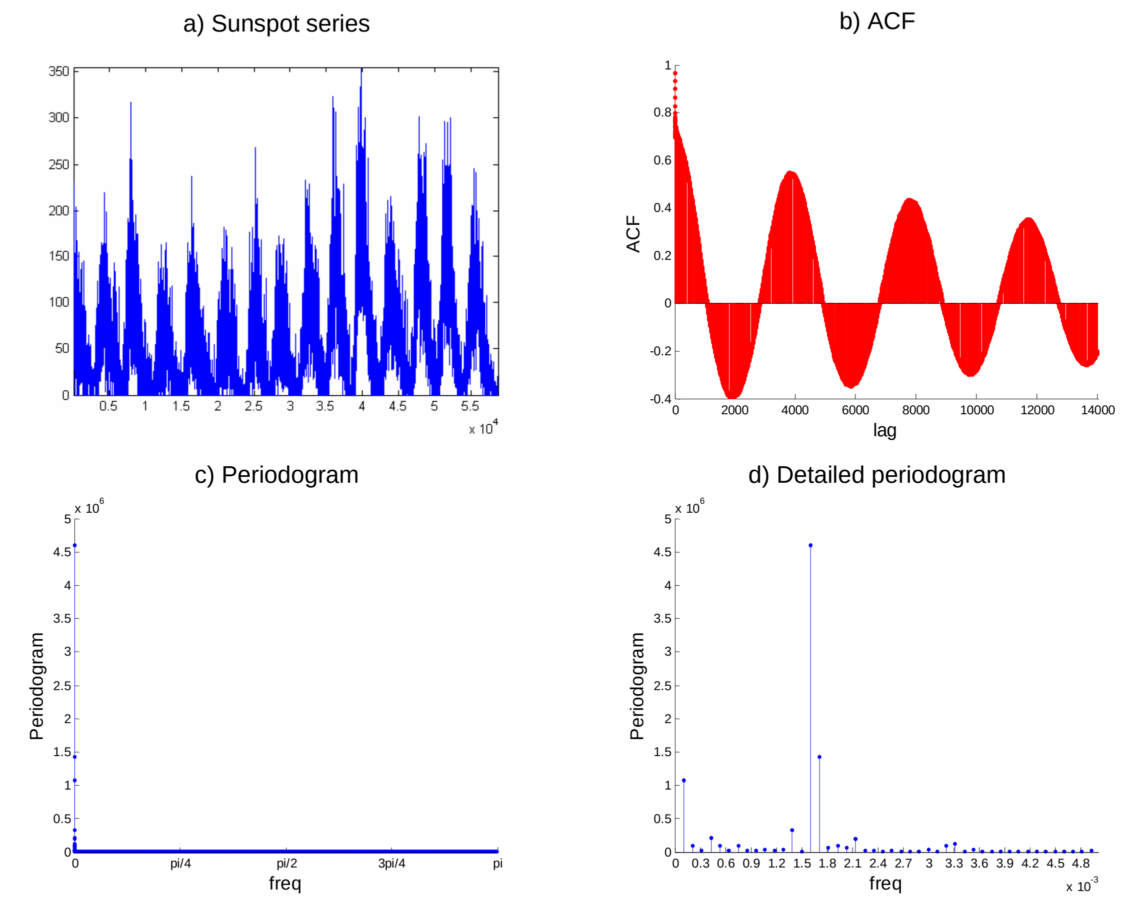

Figure 2. Dynamic features of the Sunspot Index.

5. Empirical application

In this section we will estimate a doubly fractional model for the well-known sunspot index. In particular, we are dealing with a daily version of the index from 1848 December 23th to 2009 September 30th with a sample size T=58721. The data are the sunspot numbers compiled by the US National Oceanic and Atmospheric Administration (NOAA). The series is displayed in Figure 2a and we can clearly observe the evolution of a very stable cycle that completes its period around 15 times.

The periodogram (Figures 2c and 2d) exposes this clear presence of a cycle in the series with a peak at frequency $\hat{\omega}_{mean(1)} = 0.001605$, which corresponds to a period of approximately 10 years and 8.6 months. The ACF (Figure 2b) shows a sinusoidal wave that decays hyperbolically, so we can conclude that there is a persistent stochastic cycle of frequency $\hat{\omega}_{mean(1)}$.

We estimated the GLM for the levels by QML as described in the preceding sections and obtained $\hat{d}_{mean} = 0.4726$, for the memory parameter at $\hat{\omega}_{mean(1)}$, very close to the non-stationary region. The periodogram and ACF of the filtered series $z_t = [1\text{-}L\cos(0.001605)+L^2]^{0.4726}x_t$ are shown in Figure 3 where it turns clearly outstanding that there still remains a cyclical pattern.

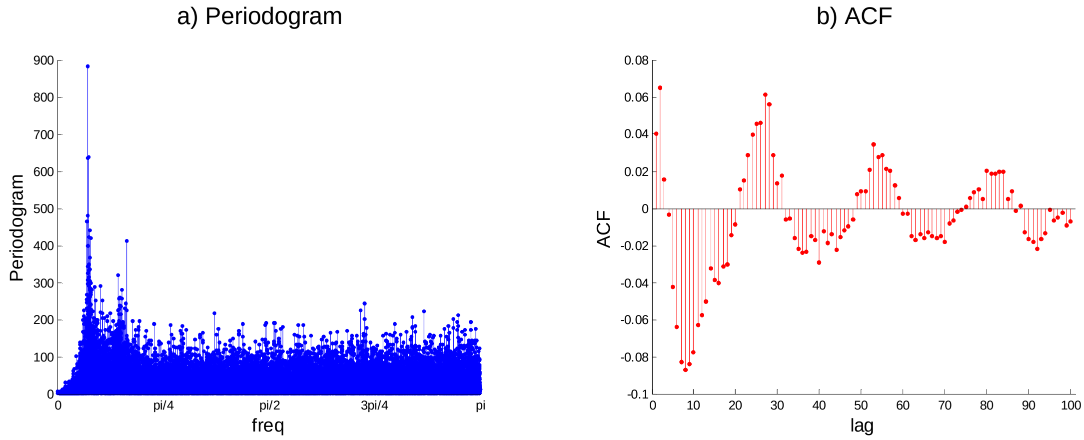

Figure 3. Periodogram and ACF of the residuals of the GLM(d_{mean} = 0.4726, ω_{mean} = 0.001605) process.

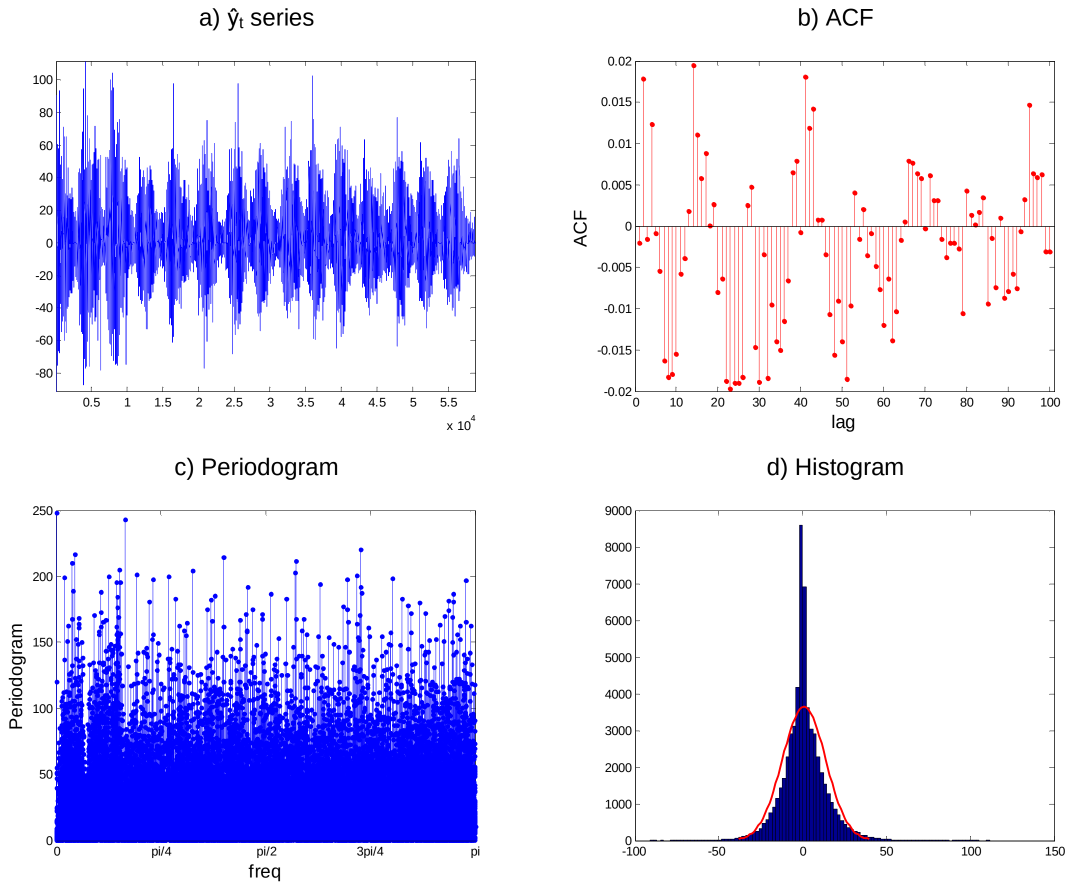

Figure 4. Residuals $\hat{y}_t$ of the trifactorial GARMA$_3$(0,1,D_{mean},Ω_{mean}) process.

Taking into account the ACF and periodogram displayed in Figure 3, we considered a second GLM factor in $\hat{\omega}_{mean(2)}$ = 0.2286 which corresponds to a period of around 27 days that matches the Equator rotation of the sun as seen from the Earth (Beck, 2000). In order to obtain a proper estimation of the memory parameter in $\hat{\omega}_{mean(2)}$, we estimated jointly both memory parameters in the two spectral peaks in the original series. The filtered series of residuals of this bifactorial approach revealed the necessity of a third cyclical long

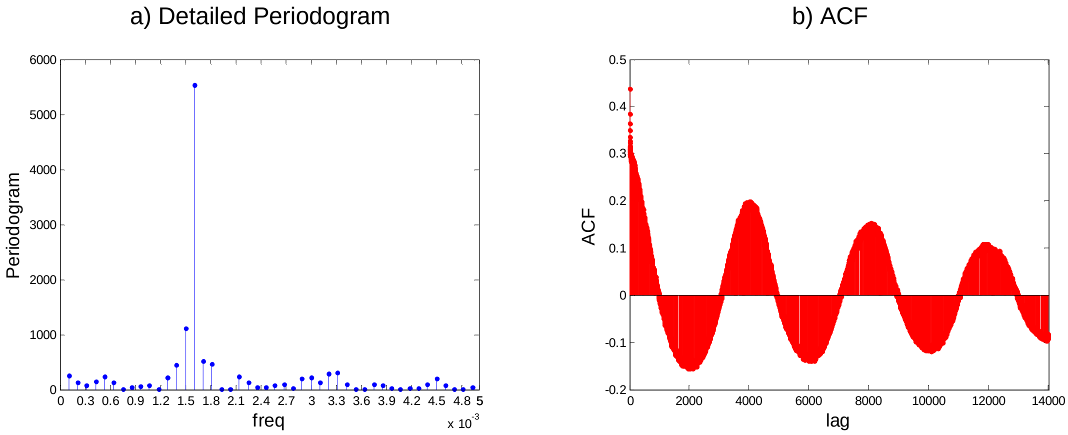

Figure 5. Periodogram and ACF of $\log\left(\hat{y}_t^2\right)$.

memory component at frequency $\hat{\omega}_{mean(3)}$ = 0.5099 ($\hat{\tau}_{mean(3)}$ ≈ 12.3 days) and also a MA(1) process for the remaining weak dependence. The joint estimation of all these parameters were $\hat{d}_{mean(1)}$ = 0.3601, $\hat{d}_{mean(2)}$ =0.4332, $\hat{d}_{mean(3)}$ = 0.1107 and $\hat{\theta}$ = 0.8464. Therefore, our results show that the 27-day cycle has a stronger persistence than the 11-year periodicity, although it accounts for a smaller proportion of the variability of the series due to the dependence of the variance of GLM processes on the frequency ω. Specifically, using Eq. (1), the estimated ratio of variances of the cycle at $\hat{\omega}_{mean(1)}$ over $\hat{\omega}_{mean(2)}$ is 3.8276.

The residuals $\hat{y}_t$ of the whole 3-factor GARMA process are displayed in Figure 4. They are characterized by an evolving dispersion as the alternance of periods of high volatility with periods of low volatility is very recurrent and stable (Figure 4a). Their ACF and periodogram (Figures 4b and 4c) show no more sustainable signs of dependence and their histogram (Figure 4d) shows a large kurtosis (6.7658), which is common to the models of dynamic heteroskedasticity.

The periodogram and ACF of the logarithm of the squares of $\hat{y}_t$ (Figure 5) show again a strong cyclical persistence at frequency $\hat{\omega}_{vol}$ = 0.001605. After a prior estimation of the memory parameter at $\hat{\omega}_{vol}$ a second cyclical long memory factor were found appropriate at frequency $\hat{\omega}_{vol(2)}$ = 0.2347. The estimated memory parameters and variance of innovations for the whole bifactorial model of volatility are $\hat{d}_{vol(1)}$ = 0.2063, $\hat{d}_{vol(2)}$ = 0.1103 and $\hat{\sigma}_{\eta}^2$ = 0.7764.

The estimation of the volatility process h_t can be achieved by means of the smoothing method proposed by Harvey (1998). In this case, the Variance-Covariance Matrix of the process h_t was truncated to a size of 5001×5001. The standardized residuals $\hat{\varepsilon}_t$ of the complete $GARMA_3$-$GLMSV_2$ model can be obtained

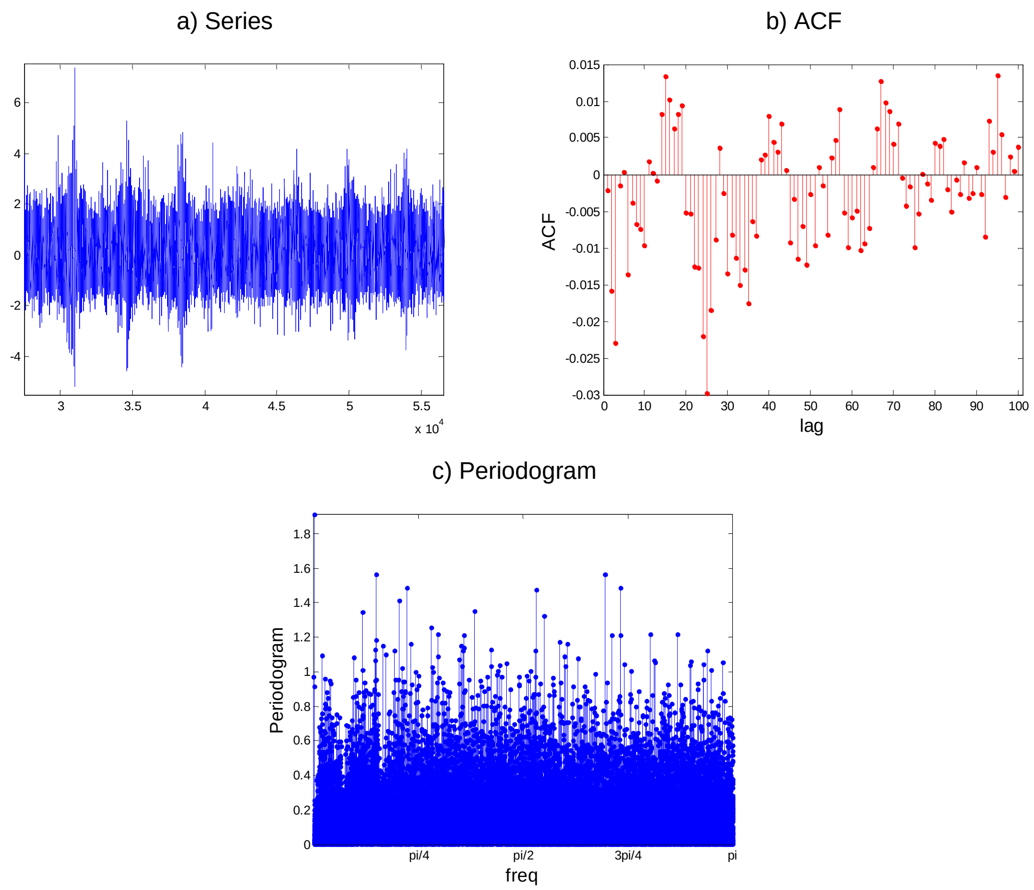

Figure 6. Dynamic features of the residuals $\hat{\varepsilon}_t$ of the GARMA$_3$(0,1,D$_{mean}$,Ω_{mean})-GLMSV$_2$(D$_{vol}$,Ω_{vol}) process.

from this estimated volatility and, assuming $\sigma_\varepsilon^2=1$, the scale parameter is $\hat{\sigma}_* = 2.2595$. Therefore the sunspot index is coherent with the GARMA$_3$(0,1,D$_{mean}$,Ω_{mean})-GLMSV$_2$(D$_{vol}$,Ω_{vol}) process (where D$_{mean}$={0.3601, 0.4332, 0.1107}, Ω_{mean}={0.001605, 0.2286, 0.5099}, D$_{vol}$={0.2063, 0.1103} and Ω_{vol}={0.001605, 0.2347}) expressed by the equations

$$\left(1-2L\cos(0.001605)+L^2\right)^{0.3601}\left(1-2L\cos(0.2286)+L^2\right)^{0.4332}\left(1-2L\cos(0.5099)+L^2\right)^{0.1107}x_t=\hat{\Theta}(L)\hat{y}_t$$

$$\hat{y}_t=2.2595\cdot e^{\left[\frac{1}{2}\left(1-2L\cos(0.001605)+L^2\right)^{-0.2063}\left(1-2L\cos(0.2347)+L^2\right)^{-0.1103}\eta_t\right]}\hat{\varepsilon}_t$$

with $\hat{\Theta}(L)=(1-0.8464L)$.

The residuals $\hat{\varepsilon}_t$, together with their ACF and periodogram displayed in Figure 6, show that our model captures most of the information contained in the original series.

6. Conclusions

In this paper we have introduced a doubly fractional model that presents a cyclical dependence in both first and second order moments that fairly explains the persistent behaviour of the daily sunspot index. Its main innovation consists in allowing a persistent but evolving cyclical behaviour both in levels and volatility, which can become very apparent as it usually affects the amplitude of the consecutive waves.

We have also proposed a sequential strategy of estimation using QML in the frequency domain and checked its finite sample performance in an extensive grid of parameters of the model. Central frequencies have turned out to be determinant to worsen the properties of estimation in both the levels and volatility models and Gaussianity seems a plausible distribution for the QML estimator in any parameter combinations for the levels but is rejected in the volatility model for persistent cycles at low frequencies as the presence of the noise eclipses the long memory process causing recurring antipersistent estimations that can eventually produce a very asymmetric distribution.

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