



OPEN

Speed-up and slow-down of a quantum particle

X. Gutiérrez de la Cal^{1✉}, M. Pons² & D. Sokolovski^{1,3}

We study non-relativistic propagation of Gaussian wave packets in one-dimensional Eckart potential, a barrier, or a well. In the picture used, the transmitted wave packet results from interference between the copies of the freely propagating state with different spatial shifts (delays), x' , induced by the scattering potential. The Uncertainty Principle precludes relating the particle's final position to the delay experienced in the potential, except in the classical limit. Beyond this limit, even defining an effective range of the delay is shown to be an impracticable task, owing to the oscillatory nature of the corresponding amplitude distribution. Our examples include the classically allowed case, semiclassical tunnelling, delays induced in the presence of a virtual state, and scattering by a low barrier. The properties of the amplitude distribution of the delays, and its pole representation are studied in detail.

A classical particle, crossing in one dimension a short-range potential $V(x)$, goes faster over a well, $V(x) < 0$, and slower over a barrier, $V(x) > 0$. Once it has left the potential, this can be checked either by evaluating the distance x' separating the particle from its freely moving counterpart at a given time t , or by measuring the time interval τ between the two arrivals at a fixed detector. The reason is simple. The particle's velocity inside the potential either exceeds that of the free motion, or is reduced, and $\tau = x'/v_0$, where v_0 is the speed of free motion.

This explanation relies on the existence of a classical trajectory, and can no longer be used in the quantum case. There, a particle described by a wave packet may tunnel across the barrier even if its energy is smaller than the barrier's height. Attempts to ascribe to tunnelling a single duration τ began with McColl's observation that "there is no appreciable delay in transmission of the packet through the barrier"¹, and continue to date, encouraged by the recent progress in atto-second experimental techniques^{2–4}. The discussion has recently become a dispute between those who think that τ should be zero³, and those believing that it should have a non-zero value^{4,5}.

Before estimating the value of a commodity, it may be useful to enquire whether the commodity in question does indeed exist. This is particularly true in the case of tunnelling, where the existence of a well defined "tunnelling time" can be shown to contradict the Uncertainty Principle^{6,7}. Recently, there has been renewed interest in the tunnelling time problem (see^{8–14}). Although tunnelling is, without a doubt, the most interesting example, there are many other situations in which the classical analysis should fail. For example, such would be the case of a slow particle moving across a shallow well, supporting only few bound states, or passing over a low potential barrier. In brief, it would be helpful to have a general approach to quantum scattering, setting clear limits on what can and what cannot be asked of a quantum particle. Such an approach was outlined, and applied to wave packet tunnelling, in¹⁵. In this paper, we discuss a more general application of the method to the transmission across an Eckart potential¹⁶. Unlike the rectangular barriers and wells, often discussed in a similar context, Eckart's potential is amenable to standard semiclassical treatment¹⁶. The corresponding transmission amplitude, $T(p, V)$, is known analytically, as well as the positions and residues of its poles in the complex momentum plane. All this makes this potential an ideal candidate for our demonstration.

The rest of the paper is organised as follows. In "Gaussian wave packets in an Eckart's potential" we briefly discuss Eckart's potential and Gaussian wave packets, used throughout the rest of the paper. Section "The classical limit: spatial advances and delays" discusses the classical limit for passing over an Eckart's well, or barrier. In Section "Apparently "instantaneous" semiclassical tunnelling" we analyse the semiclassical limit of a tunnelling transmission. In Section "Scattering potential as an "interferometer"" a change to the coordinate representation allows us to describe a potential as a kind of an "interferometer" which splits the initial wave packet into the components with different spatial delays, recombined later to produce the transmitted state. In Section "Amplitude distribution of the delays: a pole representation" we introduce a pole representation for the amplitude distribution of the delays. The cases of an Eckart's barrier and an Eckart's well are analysed separately in Sections "An Eckart barrier" and "An Eckart well", respectively. In Section "Is there an effective range of spatial delays?" we ask whether one can define an "effective range" of the delays, and answer the question in the negative. Section

¹Departamento de Química-Física, Universidad del País Vasco, UPV/EHU, Leioa, Spain. ²Departamento de Física Aplicada, Universidad del País Vasco, UPV-EHU, Bilbao, Spain. ³IKERBASQUE, Basque Foundation for Science, 48011 Bilbao, Spain. ✉email: xabig13@gmail.com

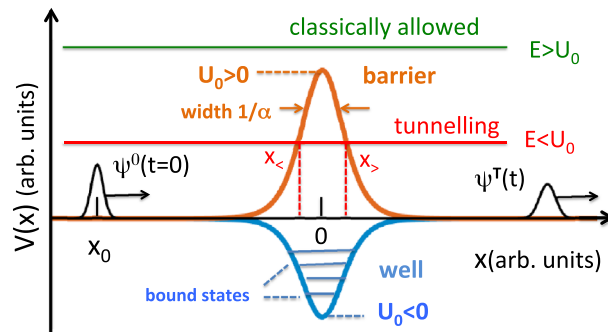


Figure 1. Eckart’s potentials, a barrier and a well. Also shown schematically are the initial, $\psi^0(x, t = 0)$, and the transmitted, $\psi^T(x, t)$, wave packets.

“Spatial delays vs. detection times” discusses the relation between special delays and measurable time intervals. In Section “The “phase time”” we relate the “phase time” to the displacement of the centre of mass of a wave packet, broad in the coordinate space. The delay experienced by a slow particle in a shallow Eckart well is discussed in Section “Scattering by a shallow Eckart well”. The case of a low Eckart barrier is analysed in Section “Scattering by a low Eckart barrier”. Our conclusions are in Section “Conclusions and discussion”.

Gaussian wave packets in an Eckart’s potential

We consider, in one dimension, scattering of a non-relativistic wave packet incident from the left on a smooth potential $V(x)$, $V(x) \rightarrow 0$ for $x \rightarrow \pm\infty$, see Fig. 1.

For the initial wave packet we have $[E(p) = p^2/2\mu]$, where μ is the particle’s mass, and $\hbar = 1$ is used)

$$\psi^0(x, t) = \int A(p - p_0) \exp[ipx - iE(p)t] dp. \tag{1}$$

Its transmitted part is given by

$$\psi^T(x, t) = \int T(p, V) A(p - p_0) \exp[ipx - iE(p)t] dp, \tag{2}$$

where $T(p, V)$ is the transmission amplitude. At a time t , large enough for the scattering to be completed, we will compare the positions of the transmitted wave packet with that of a freely propagating one, in order to determine whether the potential delays the particle or makes it, in some sense, go faster. In particular, we will consider an Eckart potential¹⁶

$$V(x) = \frac{U_0}{\cosh^2(\alpha x)}, \tag{3}$$

a well, or a barrier, depending on the sign of U_0 . For such a potential the transmission amplitude is well known to be¹⁶

$$T(p, V) = \frac{\Gamma(-ip/\alpha - s) \Gamma(-ip/\alpha + s + 1)}{\Gamma(-ip/\alpha) \Gamma(1 - ip/\alpha)}, \tag{4}$$

where $\Gamma(z)$ is the Gamma function¹⁷, and

$$s \equiv 2^{-1} \left[-1 + \sqrt{1 - \frac{8\mu U_0}{\alpha^2}} \right]. \tag{5}$$

The transmission amplitude has the usual property¹⁸ (a star denotes complex conjugation)

$$T(-p^*, V) = T^*(p, V), \tag{6}$$

which can also be obtained directly from (4).

We will be interested in Gaussian wave packets, and choose the momentum distribution in Eq. (1) to be

$$A(p - p_0) = 2^{-1/4} \pi^{-3/4} \Delta p^{-1/2} \times \exp[-(p - p_0)^2 / \Delta p^2 - i(p - p_0)x_0], \tag{7}$$

where $x_0 < 0$. With this, at $t = 0$, the incident wave packet is a Gaussian state with a mean momentum p_0 , of a width $\Delta x = 2/\Delta p$, placed at $t = 0$ a distance $|x_0| \gg 1/\alpha$ to the left of the barrier. In the coordinate representation we, therefore, have

$$\psi^0(x, t) = \exp[ip_0x - iE(p_0)t] G_0(x, t) \tag{8}$$

where the envelope $G_0(x, t)$ is given by Eq. (S2) of the Supplementary Appendix A. It will be convenient to measure the distances in the units of the barrier's width $1/\alpha$, and use dimensionless variables,

$$\begin{aligned} \underline{\mu} &= 1, \quad \underline{\alpha} = 1, \quad \underline{x} = \alpha x, \quad \underline{p} = p/\alpha, \\ \underline{t} &= \alpha^2 t/\mu, \quad \underline{U}_0 = \mu U_0 \alpha^2. \end{aligned} \quad (9)$$

Next we consider the classical limit.

The classical limit: spatial advances and delays

For a classically allowed motion we have $E(p) > V(x)$, and the local momentum, $q(x, p) = \sqrt{p^2 - 2\mu V(x)}$, is a real positive quantity. The classical requirement, that the potential should vary slowly on the scale of a local de Broglie wavelength¹⁶, now translates into a condition $1/\min[q(x, p)] \ll 1/\alpha$. Neglecting the small over-barrier reflection, we write the transmission amplitude (4) as¹⁹

$$T(p, V) \approx \exp \left\{ i \int_{-\infty}^{\infty} [q(x, p) - p] dx \right\} \equiv \exp[i\Phi(p, V)], \quad (10)$$

and expand the phase Φ in a Taylor series around the particle's mean momentum p_0 ,

$$\Phi(p, V) = \Phi(p_0, V) - \int_{-\infty}^{\infty} dx \left[1 - \frac{p_0}{q(x, p_0)} \right] (p - p_0) + \sum_{n=2}^{\infty} \frac{\partial_p^n \Phi(p_0, V)}{n!} (p - p_0)^n / n!. \quad (11)$$

In the classical limit one expects a wave packet of a size $\Delta x \sim 1/\Delta p$, small compared to the size of the potential $\delta x \sim 1/\alpha$, to be transmitted without distortion, and experience a delay or an advancement, depending on whether $V(x)$ is a barrier or a well.

The phase $\Phi(p_0, V)$, and its derivatives in Eq. (11) scale as $1/\alpha$ as the potential becomes broader, $\alpha \rightarrow 0$. The momenta of the initial wave packet (7) lie approximately in the range of $\Delta p \sim 1/\Delta x$ around p_0 , so that we have $(p - p_0)^n \sim 1/\Delta x^n$. Distortion-free transmission will be achieved if the sum in Eq. (11) can be neglected, i.e., for $\sum_{n=2}^{\infty} \frac{\partial_p^n \Phi(p_0, V)}{n!} (p - p_0)^n / n! \ll 1$. This will be the case if we choose

$$\Delta x \sim 1/\alpha^\gamma, \quad 1/2 < \gamma < 1. \quad (12)$$

In Eq. (11), the remaining term in the square brackets is the classical expression for the distance \tilde{x}' separating the particle from its freely propagating counterpart at a given (sufficiently large) time t . Indeed, it can be written as [$v_0 = p_0/\mu$ and $v(x, p_0) = q(x, p_0)/\mu$]

$$\tilde{x}' \equiv v_0 \int_{-\infty}^{\infty} \left[\frac{1}{v_0} - \frac{1}{v(x, p_0)} \right] dx, \quad (13)$$

and Eq. (1) reduces to the desired classical result

$$\psi^T(x, t) = \exp[i\Phi(p_0, V) + ip_0 \tilde{x}'] \psi^0(x - \tilde{x}', t, p_0). \quad (14)$$

In a snapshot taken on a large scale $\sim 1/\alpha$ at a large time t , both the transmitted and the freely propagating probability densities, $|\psi^T(x, t)|^2$ and $|\psi^0(x, t)|^2$, will look like those belonging to point-size particles. As expected, in the case of a well, $V(x) < 0$, $v(x, p) \geq v_0$, and we find the particle lying ahead of its freely propagating counterpart. Similarly, a barrier, $V(x) > 0$, would cause the particle to lag behind the free motion, as is illustrated in Fig. 2a and b. More interesting, however, is the case of transmission not allowed in classical mechanics, which we will discuss next.

Apparently “instantaneous” semiclassical tunnelling

Next we consider the case of a barrier, $V(x) > 0$, and choose all energies, $E(p) < \max[V(x)]$, to lie not too close to the barrier top. Now the local momentum $q(x, p) = \sqrt{p^2 - 2\mu V(x)}$ is imaginary for $x_{<} < x < x_{>}$, where $x_{<(>)}$ are the classical turning points, $q(x_{<(>)}, p) = 0$, and real positive elsewhere. As $\alpha \rightarrow 0$, the semiclassical condition is satisfied everywhere, except in the vicinities of the turning points. Using the standard connection formulae¹⁶ it can be shown that, as before,

$$T(p, V) \approx \exp \left\{ i \int_{-\infty}^{\infty} [q(x, p) - p] dx \right\}, \quad (15)$$

except that now $q(x, p) = i|q(x, p)|$ for $x_{<} \leq x \leq x_{>}$, and $|T(p, V)|^2 \sim \exp \left(-2 \int_{x_{<}}^{x_{>}} |\sqrt{p^2 - 2\mu V(x)}| dx \right) \ll 1$, so that most of the particles are reflected, and only few are transmitted.

Acting as in the previous section, and assuming that (12) holds, we find the transmitted wave packet greatly reduced, narrow compared to the width of the potential, and somewhat distorted. As before, $\psi^T(x, t)$ is given by Eq. (14), but with a complex valued spatial shift, $\tilde{x}' = \tilde{x}'_1 + i\tilde{x}'_2$,

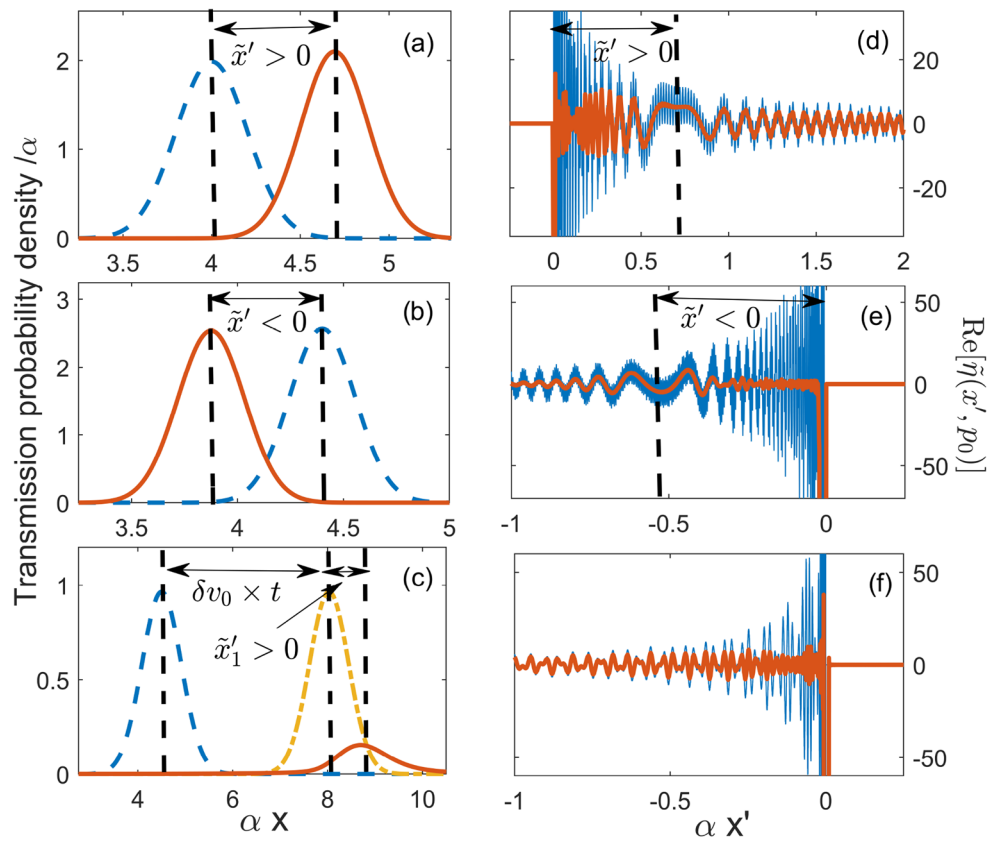


Figure 2. (a) A wave packet is transmitted across an Eckart well, $U_0 = -2 \times 10^4$, $p_0 = 200$, $\Delta p = 6.67$, $x_0 = -4$, and $t = 0.04$. Also shown by the dashed line is its freely propagating counterpart in Eq. (1). The spatial advancement calculated in Eq. (13) is $\alpha \tilde{x}' = 0.6619$ (b) Same as (a) but for a passage above an Eckart barrier, $U_0 = 10^5$, $p_0 = 700$, $\Delta p = 6.67$, $x_0 = -4$, and $t = 0.012$. The spatial delay calculated in Eq. (13) is $\alpha \tilde{x}' = -0.5392$. (c) Wave packet (multiplied by $z = 10^{221}$ for better viewing) tunnels across an Eckart barrier, $U_0 = 10^4$, $p_0 = 50$, $\Delta p = 3.64$, $x_0 = -4$, and $t = 0.17$. Freely propagating wave packets with mean momenta p_0 and $p_0 + \delta p_0$, are shown by dashed and dot-dashed lines, respectively. The spatial advancement in Eq. (16) is $\alpha \tilde{x}' = 0.8515$. (d, e) and (f) show $\text{Re}[\tilde{\eta}(p_0, x')]$ (solid) and its average, $\text{Re}[y^{-1} \int_{x'}^{x+y} \tilde{\eta}(p_0, x'') dx'']$ (thick solid) for the cases in (a, b) and (c), respectively. Where it exists, the classical value \tilde{x}' is marked by a vertical dashed line.

$$\begin{aligned} \tilde{x}'_1 &\equiv v_0 \int_{-\infty}^{\infty} \left\{ \frac{1}{v_0} - \text{Re} \left[\frac{1}{v(x, p_0)} \right] \right\} dx, \\ \tilde{x}'_2 &\equiv -v_0 \int_{x_<}^{x_>} \frac{dx}{|v(x, p_0)|} < 0. \end{aligned} \tag{16}$$

This complex shift has a different effect on the transmitted state. For a Gaussian wave packet (7) evaluation of the integral (2) yields

$$\psi^T(x, t) = T(p_0, V) \exp[\Delta p^2 \tilde{x}'_2 / 4 + ip_0 \tilde{x}'_1] \times \psi^0(x - \tilde{x}'_1, t, p_0 + \delta p_0) \tag{17}$$

where $\psi^0(x - \tilde{x}'_1, t, p_0 + \delta p_0)$ is a free wave packet with a mean momentum $p_0 + \delta p_0$, and $\delta p_0 = \Delta p^2 \tilde{x}'_2 / 2$. Thus, the wave packet is duly delayed by the barrier potential in the classically allowed region. It also appears to cross the classically forbidden region, which does not contribute to \tilde{x}'_1 , “instantaneously”. Furthermore, traversing the region increases the particle’s mean momentum by δp_0 . This is the well known “momentum filtering effect” (see, for example²⁰). Since higher momenta tunnel more easily, the transmitted particle moves faster than the incident one, and the factor multiplying $\psi^0(x - \tilde{x}'_1, t, p_0 + \delta p_0)$ in Eq. (17) is larger than $T(p_0, V)$. In a snapshot taken at a sufficiently large time t the (greatly reduced) tunnelling wave packet would be advanced due to the positive shift induced by the barrier, $\tilde{x}'_1 > 0$, as well as owing to the increase in the particle’s mean velocity (see Fig. 1c).

Scattering potential as an “interferometer”

To get an alternative perspective on the three cases shown in Fig. 2, we change to a different representation by taking a Fourier transform of the transmission amplitude with respect to p , and inserting the result into Eq. (2). This yields an equivalent expression for the transmitted state¹⁵,

$$\psi^T(x, t) = \exp[ip_0x - iE(p_0)t] \times \int_{-\infty}^{\infty} G_0(x - x', t)\eta(p_0, x')dx', \tag{18}$$

where $G_0(x, t)$ is the freely propagating envelope in Eq. (8). The distribution $\eta(p_0, x')$ is a sum of a Dirac delta, and a non-singular smooth function,

$$\eta(p_0, x') = \delta(x') + \tilde{\eta}(p_0, x'), \tag{19}$$

where

$$\tilde{\eta}(p_0, x') \equiv \exp(-ip_0x')\xi(x') = (2\pi)^{-1} \exp(-ip_0x') \int_{-\infty}^{\infty} [T(k, V) - 1] \exp(ikx')dk. \tag{20}$$

With this, the action of a potential $V(x)$ can be understood as follows. There is a continuum of routes, each labelled by the value of x' , via which the particle can reach its final state. Along each route an envelope $G_0(x, t)$ is enhanced, or suppressed, by a factor $|\tilde{\eta}(p_0, x')|$, acquires an additional phase, $\arg[\tilde{\eta}(p_0, x')]$, and is shifted in space by x' . On exit from this “interferometer”, all envelopes are recombined, and added to $G_0(x, t)$ to produce the transmitted wave packet in Eq. (18). In addition, we have

$$T(p_0, V) = \int_{-\infty}^{\infty} \eta(p_0, x')dx', \tag{21}$$

so that $\tilde{\eta}(p_0, x')$ can be understood to be the probability amplitude for a particle with a momentum p_0 to experience a spatial shift x' while crossing a short-range potential $V(x)$ (see Supplementary Appendix B).

To see how a unique shift appears in the classical limit of Section III, we insert the semiclassical approximation (10) into (20), and evaluate the integral over k by the method of steepest descent²¹. This yields

$$\eta(p_0, x') \sim \exp(-ip_0x') \exp[i\Phi(\tilde{k}(x'), V) + ix'\tilde{k}(x')] \tag{22}$$

where $\tilde{k}(x)$ satisfies a condition

$$\partial_k[\Phi(k, V) + kx']|_{k=\tilde{k}(x')} = 0. \tag{23}$$

As a function of x' , $\eta(p_0, x')$ has a critical point at $\tilde{x}'(p_0)$, such that

$$\partial_{x'}[\Phi(\tilde{k}, V) + \tilde{k}x' - p_0x']|_{x'=\tilde{x}'(p_0)} = \tilde{k}(\tilde{x}') - p_0 = 0, \tag{24}$$

or, explicitly,

$$\tilde{x}'(p_0) = -\partial_k\Phi(k, V)|_{k=p_0} = \int_{-\infty}^{\infty} dx \left[1 - \frac{p_0}{q(x, p_0)} \right]. \tag{25}$$

For a particle passing over a well, or above a barrier, $\tilde{x}'(p_0)$ is real, and we recover the result (13), but in a new context. The smooth part of the amplitude distribution, $\tilde{\eta}(p_0, x')$ has two important features. One is a region centred at $\tilde{x}'(p_0)$ where its oscillations are slowed down, the other is a finite narrow dip near $x' = 0$. The purpose of the dip is to cancel the contribution from the δ -function in Eq. (19) (for a similar “Zeno peak” occurring in quantum measurements see²²). The purpose of the stationary region is to replace the now cancelled out $G_0(x, t)$ with $G_0(x - \tilde{x}', t)$. Thus, the classical advancement (or delay), \tilde{x}' , corresponds to a stationary point of the phase of a rapidly oscillating $\eta(p_0, x')$, as shown in Fig. 2d and e. If the incident wave packet is not too narrow [$\Delta x > 1/\sqrt{\partial_p^2\Phi(p_0, V)}$] a single envelope $G_0(x - \tilde{x}', t)$ is selected by the integral (18), and the classical picture is restored.

The case of tunnelling in Section IV is radically different. The position of \tilde{x}' is controlled by the particle’s mean momentum p_0 and, for $E(p_0) < \max[V(x)]$, \tilde{x}' becomes the position of a complex saddle point off the real x' -axis. On the real x' -axis, the amplitude distribution $\eta(p_0, x')$ rapidly oscillates, and it is impossible to choose a single real spatial delay, associated with a classically forbidden tunnelling transition (cf. Fig. 2f). Destructive interference between the delays makes the tunnelling amplitude $T(p_0, V)$ exponentially small, yet $\eta(p_0, x')$ is not itself small, and differs only by a phase $\exp[-i(p_0 - p'_0)x']$ from a distribution for a particle with mean momentum p'_0 passing over the barrier top [cf. Eq. (20)]. This makes semiclassical tunnelling much more delicate, as both the shape and position of the tunnelled wave packet are now determined by the analytic continuation of the envelope $G_0(x, t)$, $G_0(x - \tilde{x}'_1 - i\tilde{x}'_2, t)$, into the complex x' -plane.

In the general case one cannot obtain the transmitted state by a single coordinate shift, be it real or complex valued, and has to sum the contributions of all the delays by evaluating the integral (21). In the next Section we discuss a useful way of doing it.

Amplitude distribution of the delays: a pole representation

One can close the contour of integration in Eq. (20) in the upper and the lower halves of the complex k -plane for $x' > 0$ and $x' < 0$, respectively, and obtain $\eta(p_0, x')$ by summing the contributions of pole singularities of the transmission amplitude. The poles of $T(k, V)$ fall into two categories¹⁸. Those on the positive imaginary semi-axis correspond to the bound (B) states, supported by the potential $V(x)$. The poles in the lower half-plane, located symmetrically on both sides of the negative imaginary semi-axis describe scattering resonances (R). It is sufficient

to know all poles positions, k_n , and the corresponding residues, $\text{Res}(k_n)$, in order to evaluate all the quantities of interest. In particular, we have

$$\eta(p_0, x') = \delta(x') + i \exp(-ip_0x') \times \begin{cases} \sum_{n_B} \text{Res}(k_{n_B}) \exp(ik_{n_B}x'), & x' > 0 \\ -\sum_{n_R} \text{Res}(k_{n_R}) \exp(ik_{n_R}x'), & x' < 0, \end{cases} \tag{26}$$

$$\begin{aligned} \psi^T(x, t) = & \exp[ip_0x - iE(p_0)t] \{ G_0(x, t) + i \sum_{n_B} \text{Res}(k_{n_B}) \int_0^\infty G_0(x - x', t) \exp[i(k_{n_B} - p_0)x'] dx' \\ & - i \sum_{n_R} \text{Res}(k_{n_R}) \int_{-\infty}^0 G_0(x - x', t) \exp[i(k_{n_R} - p_0)x'] dx' \} \end{aligned} \tag{27}$$

and

$$T(p, V) = 1 - \sum_{n_B} \frac{\text{Res}(k_{n_B})}{k_{n_B} - p} - \sum_{n_R} \frac{\text{Res}(k_{n_R})}{k_{n_R} - p}. \tag{28}$$

In the above equations the sums are over the simple poles corresponding to the bound states (n_B), and to the resonances (n_R). The advancement of a classical particle passing over a potential well is already anticipated in Eqs. (26) and (27). Indeed, an envelope in Eq. (18) is advanced relative to free propagation, provided we have $x' > 0$. Such advanced envelopes would be present whenever $V(x)$ is a well, supporting at least one bound state, and would not be there for a barrier where $V(x) > 0$ for all x . Special cases where $T(k, V)$ has higher (second) order poles must be treated differently, as discussed in the Supplementary Appendix C.

[Note that although $\eta(p_0, x')$ is clearly not an analytical function, the asymptotic analysis of the previous Section still applies. In the case of tunnelling, one could construct the asymptote of $\eta(p_0, x')$ e.g., on the negative x' -axis, analytically continue it into the entire x' -plane, and transform the contour of integration $\int_{-\infty}^0 dx \rightarrow \int_\Gamma dx$, making the Γ pass through the complex saddle $\tilde{x}' = \tilde{x}'_1 + i\tilde{x}'_2$]

For an Eckart potential, be it a barrier or a well, the poles' positions and residues are known exactly. They occur when the argument of at least one of the Gamma functions in the numerator of Eq. (4) equals a negative integer, or zero, $-n, n = 0, 1, 2, \dots$. They can, therefore, be divided into two groups,

$$\begin{aligned} k_n^I &= i\alpha(s - n), \quad n = 0, 1, 2, \dots, \\ k_n^{II} &= -i\alpha(n + s + 1), \end{aligned} \tag{29}$$

and the corresponding residues are easily found to be given by

$$\begin{aligned} \text{Res}(k_n^I) &= i \frac{(-1)^n}{n!} \frac{\alpha \Gamma(-ik_n^I/\alpha + s + 1)}{\Gamma(-ik_n^I/\alpha) \Gamma(-ik_n^I/\alpha + 1)}, \\ \text{Res}(k_n^{II}) &= i \frac{(-1)^n}{n!} \frac{\alpha \Gamma(-ik_n^{II}/\alpha - s)}{\Gamma(-ik_n^{II}/\alpha) \Gamma(-ik_n^{II}/\alpha + 1)}. \end{aligned} \tag{30}$$

The cases of a barrier ($U_0 > 0$) and a well ($U_0 < 0$) need to be considered separately, as we will do next.

An Eckart barrier

If $U_0 > \alpha^2/8\mu$, we have $s = [-1 + i\sqrt{8\mu U_0/\alpha^2 - 1}]/2$, and the poles lie on two vertical lines, parallel to the imaginary k -axis in the lower half of the complex k -plane (see Fig. 3a),

$$\begin{aligned} \text{Re}[k_n^{II}] &= 2^{-1}\alpha\sqrt{8\mu U_0/\alpha^2 - 1} = -\text{Re}[k_n^I] \equiv \beta, \\ \text{Im}[k_n^I] &= -\alpha(n + 1/2) = \text{Im}[k_n^{II}], \quad n = 0, 1, 2, \dots \end{aligned} \tag{31}$$

It follows from (6) that $\text{Res}[k_n^I] = -\text{Res}^*[k_n^{II}]$,

and it can be shown (see Supplementary Appendix D) that $\lim_{n \rightarrow \infty} \text{Res}[k_n^I] = -\alpha/2\pi$. Subtracting the limit, and summing the geometric progression we find

$$\tilde{\eta}(p_0, x \leq 0) = \frac{\exp(-ip_0x)}{2\pi} \left[F(x) - \frac{\alpha \sin(\beta x)}{\sinh(\alpha x/2)} \right], \tag{32}$$

where $F(x)$ is given by a convergent series,

$$F(x) \equiv 4\pi \sum_{n=0}^\infty \exp[(n + 1/2)\alpha x] \times \text{Im}\{[\text{Res}(k_n^I) + \alpha/2\pi] \exp(-i\beta x)\}. \tag{33}$$

At $U_0 = \alpha^2/8\mu, s = -1/2$, and the poles coalesce on the negative imaginary semi-axis

$$\begin{aligned} \text{Re}[k_n^I] &= \text{Re}[k_n^{II}] = 0, \\ \text{Im}[k_n^I] &= \text{Im}[k_n^{II}] = -\alpha(n + 1/2). \end{aligned} \tag{34}$$

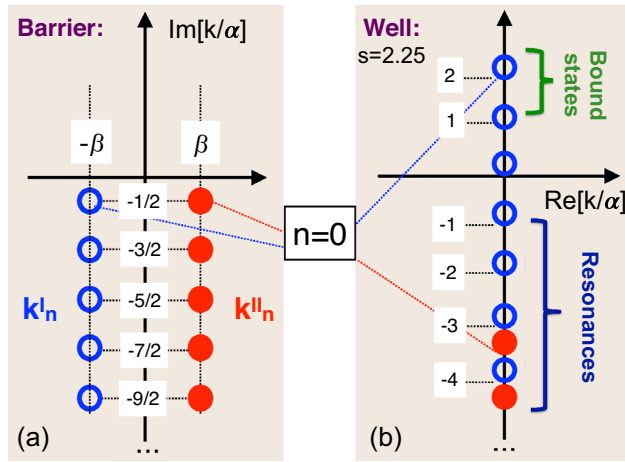


Figure 3. (a) Poles of $T(k, V)$ of the first (open circles) and the second (closed circles) types for an Eckart barrier, $U_0 > 0$. (b) same as (a), but for an Eckart well, $U_0 < 0$, supporting three bound states at $s = 2.25$.

For low barriers, $0 < U_0 < \alpha^2/8\mu$, s remains real and negative. Thus, as U_0 increases, the poles of the first kind, (I), move up the negative imaginary k -axis, while the poles of the second kind, (II), move down the same axis. As $U_0 \rightarrow 0$, we find $k_0^I \rightarrow 0$, and $k_n^{II} \rightarrow -i\alpha n = k_{n+1}^I$. The $k_0^I \rightarrow 0$ pole corresponds to the first virtual state, ready to become the first bound state when the potential becomes a shallow well. Virtual states of this type were described as “long-lived” in¹⁸, and we will return to them in Section “Scattering by a shallow Eckart well”.

An Eckart well

As U_0 decreases further and becomes negative, $V(x)$ forms a one dimensional well, which always supports at least one bound state¹⁶. The poles of the first kind continue their upward motion along the imaginary axis, and for $s > M$, $M = 0, 1, 2, \dots, M + 1$ of them lie on the positive semi-axis, where they correspond to the $M + 1$ bound states, $k_{nB} = k_n^I, n_B = s - n, n = 0, \dots, M$. The poles of the second kind, $k_n^{II} = -i\alpha(n + s + 1)$ move down the imaginary axis (cf. Fig. 3b). We note that some of the poles on the negative semi-axis coalesce whenever s takes an integer, or a semi-integer value.

The analysis is the simplest in the case where s takes an integer value, $s = M$, [$U_0 = -\alpha^2 M(M + 1)/2\mu$], $M = 1, 2, 3, \dots$, and a new bound state is about to enter into the well. The only singularities are the M bound states poles, $k_n^I, n = 0, \dots, M - 1$, and $T(k, V)$ remains finite elsewhere, since the singularities of the numerator in Eq. (4) are cancelled, by the two Gamma functions in the denominator. Thus, the series (26)–(28) become finite sums, and we have

$$\eta(p_0, x') = \delta(x') + i \exp(-ip_0 x') \times \begin{cases} \sum_{n=0}^M \text{Res}(k_n^I) \exp(ik_n^I x'), & x' \geq 0 \\ 0, & x' < 0, \end{cases} \quad (35)$$

where $k_n^I = i\alpha(M - n)$, and

$$\text{Res}(k_n^I) = (-1)^n i\alpha \frac{(2M - n)!}{n!(M - n)!(M - n - 1)!}. \quad (36)$$

Note that at these integer vales of s the well is transparent for all incident momenta, $|T(p, V)| = 1$.

For $s = M + 1/2, M = 0, 1, \dots$ there is a sequence of double poles, whose residues are given in Supplementary Appendix C. Note that the coalescence of the poles does not produce any visible feature in the behaviour of either $\eta(p_0, x')$ or $T(p, V)$.

Is there an effective range of spatial delays?

There is clearly no unique spatial delay describing scattering beyond the classical limit. But is there a characteristic range of delays one can use, e.g., for a qualitative description of quantum tunnelling? Consider first a probability distribution with a well defined range, e.g., $\rho(x) = \exp(-\gamma x)$ for $x \geq 0$, and 0 otherwise. It vanishes for $x \gg 1/\gamma$, and obviously has a “size” $\sim 1/\gamma$. This can be checked by evaluating its moments, $\bar{x}^m = \int x^m \rho(x) dx / \int \rho(x) dx$, $m = 1, 2, \dots$, and noting that $\bar{x} = 1/\gamma$ and $\sigma \equiv \sqrt{x^2 - \bar{x}^2} = \sqrt{2}/\gamma$, yield estimates for the position and the width of the region which contains most of the distribution.

This is no longer true if a distribution is allowed to change sign (for a more detailed discussion see²³). Consider next a different “distribution”,

$$\rho(x) = (1/2 + \epsilon) \exp(-x) - \exp(-2x), \quad x \geq 0, \quad (37)$$

whose size, defined as above, should be of order of 1, since the slowest decaying term in the sum (37) is $\exp(-x)$. An integral $I(z) \equiv \int_0^\infty \exp(-x^2/z^2) \rho(x) dx$, now restricted to an effective range of the order of z , should converge to $I = \int_0^\infty \rho(x) dx = \epsilon$ as $z \rightarrow \infty$. The question is how fast? Expanding $\exp(-x^2/z^2) \approx 1 - x^2/z^2$ and requiring

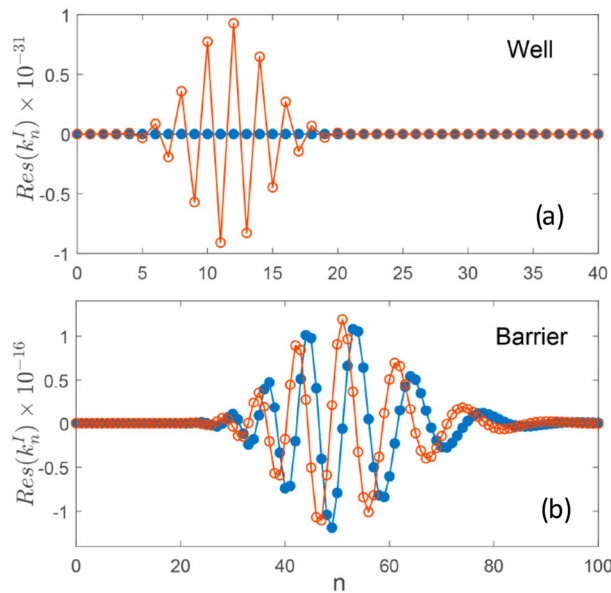


Figure 4. (a) Real (closed circles) and imaginary (open circles) part of the residues for an Eckart well, supporting 42 bound states, $U_0 = -861$, $s = 41$. Note that here are no singularities in the lower half-plane. Note also the scale on the vertical axis. (b) Same as (a) but for a barrier with $U_0 = 2485$, $s \approx -0.5 + 70, 5i$. The poles of $T(k, V)$ are distributed as in Fig. 3a, and there are no singularities in the upper half of the k -plane.

that the contribution of the second term to be negligible, yields an estimate $z^2 \gg |x^2|$. Thus, for $\epsilon \rightarrow 0$ we have $z \gg 1/\sqrt{\epsilon} \rightarrow \infty$, at odds with the expected condition $z \gg 1$. Clearly, a much longer integration range is required for recovering a very small result to a good relative accuracy.

A similar problem would occur in trying to estimate how many delays must be taken into account in order to accurately reproduce the transmission amplitude $T(p, V)$ in the case of tunnelling. Like $\rho(x)$ in Eq. (37), the distribution $\xi(x')$ in Eq. (20) is a sum of exponential terms [cf. Eq. (26)], and for an Eckart barrier with $U_0 > \alpha^2/8\mu$ the $\exp(ik_0^I x')$ and $\exp(ik_0^{II} x')$ have the slowest decay rate of $\text{Im}[k_0^I] = \alpha/2$. Can it then be said that the transmission amplitude is a result of interference between delays in a range $-\alpha \lesssim x' < 0$? The previous example points toward a problem which may arise, especially where a small result is obtained due to cancellations between terms which are not small, as happened in the case of tunnelling (see Fig. 4b). Acting as in the above, we can define complex valued moments of the distribution $\eta(p_0, x')$,

$$\overline{x^m} \equiv \int x^m \eta(p_0, x') dx' / \int \eta(p_0, x') dx' = T(p_0, V)^{-1} (i\partial_p)^m T(p_0, V), \quad m = 1, 2, \dots \quad (38)$$

and obtain a necessary (but not sufficient) condition to have $|T(p_0, V|z) - T(p_0, V)|/|T(p_0, V)| \ll 1$, $T(p_0, V|z) \equiv \int_{-\infty}^0 \exp(-x^2/z^2) \eta(p_0, x') dx'$,

$$z \gg \sqrt{|x^2|} = \sqrt{|\partial_p^2 T(p_0, V)/T(p_0, V)|} \equiv R(p_0, V), \quad (39)$$

For semiclassical tunnelling one finds $z \gg \sqrt{\tilde{x}_1^2 + \tilde{x}_2^2}$, so the range $R(p_0, V)$, defined in this manner, is of the order of the modulus of the complex delay in Eq. (16), $|\tilde{x}|$.

We note that the dependence of $R(p_0, V)$ on p_0 clearly frustrates our attempts to define an effective range of integration in Eq. (21), based on estimating the “size” of $\xi(x')$ in Eq. (20), i.e., by taking into account only the properties of the potential, and ignoring the value of the particle’s momentum. We will return to this subject in Section “Scattering by a shallow Eckart well”.

Spatial delays vs. detection times

One way to quantify the effect produced by a potential on a transmitted particle is to compare, at a given time t , the positions of the centre of mass (COM) of the transmitted density with that of a freely propagating state. For the COM delay we have

$$\delta x_{COM}(t) \equiv x_{COM}^T(t) - x_{COM}^0(t), \quad (40)$$

where

$$x_{COM}^{T,0}(t) = \int x |\psi^{T,0}(x, t)|^2 dx / \int |\psi^{T,0}(x, t)|^2 dx, \tag{41}$$

With the help of Eq. (14) we readily obtain the classical result of Section “The classical limit: spatial advances and delays”, $\delta x_{COM}(t) = \tilde{x}'$. For semiclassical tunnelling of Section “Apparently “instantaneous” semiclassical tunnelling”, from Eq. (17) we find

$$\delta x_{COM}(t) = [\tilde{x}'_1 + (v_0 + \delta v_0)t + x_0] - [v_0 t + x_0] = \tilde{x}'_1 + \delta v_0 t, \tag{42}$$

where $\delta v_0 = \delta p_0 / \mu$ is the increase in the velocity due to the momentum filtering. In the general case the increase in the transmitted particle’s mean velocity can be evaluated as

$$\delta v_0 = \delta p_0 / \mu = \frac{\int (p - p_0) |T(p, V)|^2 |A(p - p_0)|^2 dp}{\mu \int |T(p, V)|^2 |A(p - p_0)|^2 dp}. \tag{43}$$

For a large enough t , the term $\delta v_0 t$ will dominate the r.h.s. of Eq. (42). However, \tilde{x}'_1 can still be determined, by comparing the position of the COM of $\psi^T(x, t)$ with that of a free particle with a higher initial momentum, $\psi^0(x, t, p_0 + \delta p_0)$, and no additional spatial shift. Note that in Eq. (42) one would expect v_0 to be multiplied by the time interval between t and the moment the tunnelling particle enters the classical allowed region $x > x_>$ to the right of the barrier. Yet, t is the *entire* time of motion, as is illustrated in Fig. 2c.

So far we discussed the spatial shifts, since in Eq. (20) we relied on the Fourier transform of the transmission amplitude $T(p, V)$ with respect to the momentum p . Of course, knowing the positions of the COM’s at a time, as well as their velocities, it is easy to compare the times $\tau(p_0, V)$ and $\tau(p_0, V = 0)$ at which the majority of the particles would arrive at a fixed detector with and without the potential in place. In the classically allowed case we have

$$\tau(p_0, V) - \tau(p_0, V = 0) = -\tilde{x}' / v_0. \tag{44}$$

In the case of tunnelling, a comparison with free motion at $p_0 + \delta p_0$, yields

$$\tau(p_0, V) - \tau(p_0 + \delta p_0, V = 0) \equiv -\text{Re}[\tilde{x}' / v_0]. \tag{45}$$

Both Eqs. (44) and (45) refer to time intervals, which can in principle be measured, yet their interpretation is different. The l.h.s. of Eq. (44) can be understood as the “delay experienced by a classical particle in the potential”, since the particle’s position can be determined throughout the transmission sufficiently accurately, and without disturbing the transition. Such an interpretation is not available for Eq. (45), where the transmitted state is seen to be *reshaped* via interference mechanism of the previous Section (for more details see¹⁵). The same is true for any transition resulting from the interference between various delays induced by the potential. If so, the measured time of arrival at a detector cannot reveal the delay induced by the barrier. Its value *must* remain indeterminate⁶ in accordance with the Uncertainty Principle²⁴, just like the slit chosen by a particle in a Young’s double slit experiment.

The “phase time”

Whether surprisingly, or not so surprisingly, it is the \tilde{x}' in Eq. (38) which can be measured in an experiment. By increasing the wave packet’s coordinate width Δx [cf. Eqs. (7) and (S1) in Supplementary Appendix A], one can prepare a wave packet, broad in the coordinate space, and narrow in the momentum representation. Expanding the broad envelope in a Taylor series, $G_0(x - x', t) \approx G_0(x, t) - \partial_x G_0(x, t) x'$ yields¹⁵

$$\delta x_{COM}(t) \approx \text{Re}[\tilde{x}'] + 2\text{Im}[\tilde{x}'] \text{Im} \left[\int x G_0^*(x, t) \partial_x G_0(x, t) dx \right]. \tag{46}$$

The last term in Eq. (46) is recognised as the distance gained due to the momentum filtering, discussed in the previous section, and Eq. (46) can be rewritten

$$\delta x_{COM}(t) \approx v_0 \tau_{phase} + \delta v_0 t, \tag{47}$$

where $\tau_{phase} \equiv v_0^{-1} \partial_p \Phi(p_0, V) [T(p, V) = |T(p, V)| \exp[i\Phi(p, V)]]$ is known as the “phase time” (see, e.g.,²⁵), and δv_0 is obtained by taking the limit $\Delta p \rightarrow 0$ ($\Delta x \rightarrow \infty$) in Eq. (43),

$$\delta v_0 \approx 2 \partial_p \ln |T(p_0, V)| \frac{\int (p - p_0)^2 |A(p - p_0)|^2 dp}{\mu \int |A(p - p_0)|^2 dp} = \text{Im}[\tilde{x}'] \Delta p^2 / 2\mu. \tag{48}$$

Equation (47) is valid for any potential, as long as the incident wave packet is broad enough in the coordinate space. Thus, in the classical limit we recover Eq. (13), and τ_{phase} becomes the excess time, positive or negative, spent by the particle’s trajectory in the potential.

For semiclassical tunnelling $\text{Re}[\tilde{x}'] = \tilde{x}'_1$ is given by the first of Eq. (16), so there is no contribution to τ_{phase} from the classically forbidden region, $x_< < x < x_>$. With the time to cross the forbidden region apparently shorter than the time it takes to cross it at the speed of light, one seems to have a dilemma. Either Einstein’s relativity has no say over what happens in classically forbidden quantum transitions (see, e.g.,²⁶), or there must be a good reason why one should refrain from deducing the duration spent in the barrier from the distance $x_{COM}^T(t) - x_{COM}^0(t)$. Maybe the problem is with the non-relativistic Schrödinger equation used in the calculation? No, the use of the fully relativistic Klein–Gordon^{27,28} and Dirac¹³ showed the same “apparently superluminal”

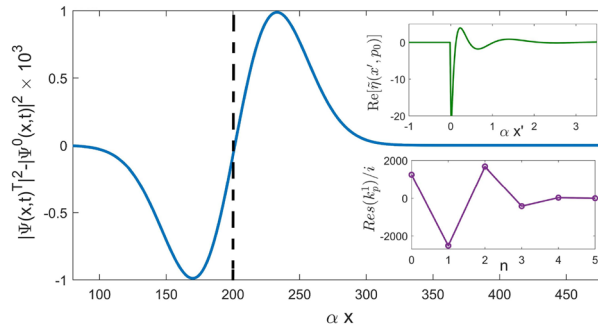


Figure 5. The difference between the probability densities of a free wave packet, $p_0 = 0.5$, $\Delta p = 0.1$, $x_0 = -100$, $t = 650$ and one transmitted across a shallow Eckart well with $s = 5$ ($U_0 = -15$). The COM of the transmitted wave packet is advanced by $\delta x_{COM}(t) = 4.0768$ relative to the COM of a freely propagating one (vertical dashed). Also shown in the insets are the real part of $\bar{\eta}(p_0, x')$ in Eq. (26), and the residues of the five poles, which contribute to Eq. (35).

advancement of the (greatly reduced) transmitted wave packet. There is some consensus that the “superluminal” transmission results from reshaping of the initial wave packet, whereby the transmitted particles come from its front tail^{10,11,27,28,30} and causality is never violated. (The authors of¹³ reject the former suggestion, but agree in that no “superluminal signalling” is possible).

As far as we can see the problem with the “phase time” is as follows. With many envelopes in Eq. (18) interfering destructively, one cannot determine a unique spatial shift induced by the barrier “interferometer”. In all routes across the barrier potential $V(x) > 0$ none of the envelopes $G(x - x', t)$ in equation (18) are advanced even relative to the non-relativistic free motion. The average x' obtained with an alternating distribution in Eq. (38) cannot be used in the same way as the unique classical value of Sect. III. One may recall the double slit conundrum. The interference picture is clearly observable, yet there is no way of telling which of two the holes has been used by the particle (See also Supplementary Appendix E).

Scattering by a shallow Eckart well

The pole representation (26) turns out to be impractical for a deep semiclassical well, supporting many bound states. The magnitudes of the residues in Eq. (30) become prohibitively large (see Fig. 4a), and a large amount of cancellation is required to produce the classical result (14).

The representation is, however, useful in the case of a shallow well, where the semiclassical approximation (14) cannot be applied.

For a Gaussian wave packet (3) the integrals in equation (27) can be expressed in terms of the error function (see Supplementary Appendix A), and for $s = M$ we have

$$\psi^T(x, t) = \exp[ip_0x - iE(p_0)t] \times \{G_0(x, t) + \sum_{n=0}^{M-1} \text{Res}(k_n^I) \mathcal{G}^B(x, k_n^I, p_0)\} \tag{49}$$

where $\mathcal{G}(x, k_n^I, p_0)$ is given by Eq. (S3) in Supplementary Appendix A. We note that equation (49) is exact, and holds for all Gaussian initial states. An example, where a broad Gaussian wave packet ($\Delta x \gg 1/\alpha$) crosses an Eckart well, supporting 5 bound states is shown in Fig. 5.

This analysis can be extended to wells with s close to an integer M , $|s - M| \ll 1$. Below we consider the case $s \sim 1$, where the second bound state is about to enter the well. (The cases $M \sim 2, 3, \dots$ can be analysed in a similar manner, see Supplementary Appendix F.) It is sufficient to include the contributions from the first two poles of the first kind, $k_0^I \approx i\alpha s$ and $k_1^I \approx i\alpha(s - 1)$. Thus, for the delay distribution from Eq. (26) we have

$$\eta(p, x') \approx \delta(x') - 2\alpha \exp[-(\alpha s + ip)x'] \theta(x') + \alpha(s - 1) \exp[-[\alpha(s - 1) + ip]x'] \times [\theta(x')\theta(s - 1) + \theta(-x')\theta(1 - s)], \tag{50}$$

where $\theta(x) = 1$ for $x \geq 0$ and 0 otherwise. Integrating Eq. (50) for $|s - 1|, p/\alpha \ll 1$, yields

$$T(p, V) \approx -\frac{ip/\alpha}{(s - 1) + ip/\alpha} + 2(s + ip/\alpha - 1) \tag{51}$$

The first (Breit–Wigner) term ensures that, for a fixed p , $|T(p, V)|$ is peaked around $s = 1$ with a width $2p/\alpha$. The second term needs to be taken into account when calculating the derivatives with respect to p at $s = 1$. In particular, for the COM delay of a slow broad wave packet from (46) we obtain,

$$\delta x_{COM} - \delta v_0 t \approx \frac{\alpha(s - 1)}{\alpha^2(s - 1)^2 + p_0^2} + \frac{2}{\alpha}, \tag{52}$$

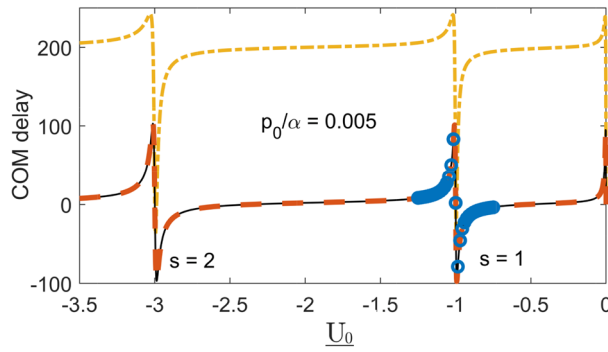


Figure 6. Centre-of-mass delay of a wave packet with $p_0 = 0.005$, $\Delta p = 0.001$, $x_0 = -3 \times 10^3$, at $t = 2 \times 10^6$ vs. the well's depth $U_0 > 0$, calculated using Eqs. (40)–(41) (solid), Eq. (46) (dashed), and Eq. (52) (dot-dashed).

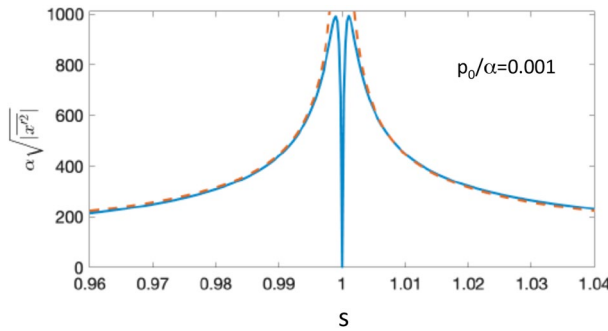


Figure 7. Effective range of integration in equation (21), $R(p_0, V) = \sqrt{|x'^2|} \approx 2/\alpha$ vs. s in equation (2), for a shallow Eckart well, and $p_0/\alpha = 0.001$ (solid). Also shown by the dashed line is an approximation $R(p, V) \approx \sqrt{2/(p_0|s - 1|)}$.

where δv_0 given by Eq. (48) is the increase in the transmitted particle's mean velocity due to momentum filtering, already discussed in Sections “Apparently “instantaneous” semiclassical tunnelling” and “Spatial delays vs. detection times”. Thus, a broad wave packet with $p_0 \ll \alpha$ is advanced relative to free propagation at $p_0 + \delta p_0$ by about the well's width, $\sim 1/\alpha$. However, (see Fig. 6), the largest advancement, $\sim 1/2p_0 \gg 1/\alpha$, is achieved for $s \approx 1 + p_0/\alpha$, where there exists a shallow level with an energy $E \approx -p_0^2/2\mu$, and $\eta(p_0, x')$ has a long tail extending into the $x' > 0$ region. Similarly, the largest delay, $\sim 1/2p_0$ occurs for $s \approx 1 - p_0/\alpha$, where there exists a virtual state¹⁸, and $\eta(p_0, x')$ extends far into the $x' < 0$ region. Finally, for $|s - 1|, p_0/\alpha \ll 1$ and $|s - 1| \gg p_0/\alpha$ we have $\delta x_{COM} - \delta v_0 t \approx 1/\alpha(s - 1)$, in agreement with the slowly decaying exponential term $\exp[-\alpha(s - 1)]$ in Eq. (50).

One can also try to estimate the effective range the delays as proposed in Section IX. With the help of Eqs. (38) and (50) one finds

$$\overline{x'^2} \approx \alpha^{-2} T^{-1}(p_0, V) \left[\frac{-4}{(s + ip_0/\alpha)^3} + \frac{2(s - 1)}{(s - 1 + ip_0/\alpha)^3} \right]. \tag{53}$$

For $s = 1$ and $p_0/\alpha \ll 1$, the range $R(p_0, V)$ in Eq. (39) equals twice the wells width, $R(p_0, V) = \sqrt{|x'^2|} \approx 2/\alpha$. This is to be expected, since there $\eta(p_0, x')$ in Eq. (50) consists of a single exponential with a decay rate α . The range peaks at $|s - 1| = p_0/\alpha$, reaching there the largest value $R(p_0, V) \approx 1/p_0$ as shown in Fig. 7. For $|s - 1| \gg p_0/\alpha$ one may expect the range to be given by the largest decay rate in Eq. (50), i.e., $R(p_0, V) \sim 1/\alpha|s - 1|$. The correct answer is, however, $R(p_0, V) \sim 1/\alpha\sqrt{|s - 1|}$ (see Fig. 7), since the value of $|T(p_0, V)|$, determined by all three terms in Eq. (50), itself falls off as $1/|s - 1|$. This, as we have already seen in Section VII, is a common problem with estimating the effective range of an alternating distribution. An integration range, defined in this manner, depends not only on the apparent “size” of integrand (in this case, $1/|s - 1|$) but also on the value of the integral, which can itself be small due to cancellations.

Scattering by a low Eckart barrier

In the case of a high semiclassical barrier, the pole representation (26) has a similar problem (see Fig. 4b). For a low barrier, however, the pole expansion is more useful, as illustrated in Fig. 8.

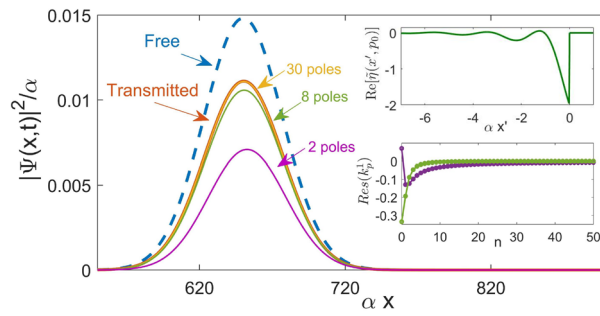


Figure 8. A wave packet with $p_0 = 1.5$, $\Delta p = 0.1$, $x_0 = -100$, at $t = 500$, transmitted across a low barrier, $U_0 = 1$ ($s = -0.5 + 1.323i$). Also shown are the results of calculating $\psi^T(x, t)$ in Eq. (27) using 2, 8, and 30 poles, and the free wave packet (dashed). The insets show the real part of $\bar{\eta}(p_0, x')$ in Eq. (26), and the residues of the first 50 poles.

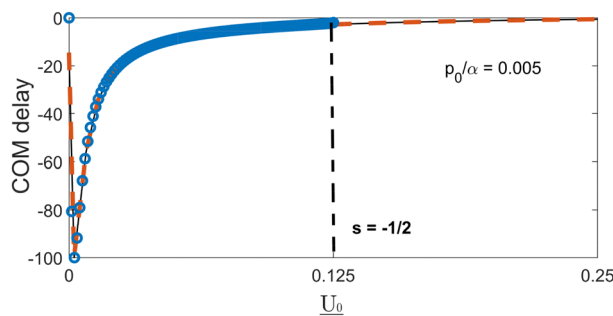


Figure 9. Centre-of-mass delay of a wave packet with $p_0 = 0.005$, $\Delta p = 0.001$, $x_0 = -3 \times 10^3$, at $t = 2 \times 10^6$ vs. the barrier's height $U_0 > 0$, calculated using Eqs. (40)–(41) (solid), Eq. (46) (dashed), and Eq. (56) (dot-dashed).

Low energy scattering by a low barrier can be analysed by the method of the previous section. For $-1/2 < s < 0$, the poles remain on the negative imaginary axis, and for $|s| \ll 1$ the delay is dominated by the presence of the last virtual state, which joins the continuum when the well ceases to exist. Accordingly, we have

$$\eta(p_0, x') \approx \delta(x') - \alpha s \exp[-(\alpha s + ip_0)x']\theta(-x'), \tag{54}$$

$$T(p, V) \approx \frac{ip/\alpha}{s + ip/\alpha}, \tag{55}$$

and from (46)

$$\delta x_{COM} - \delta v_0 t \approx \frac{\alpha s}{\alpha^2 s^2 + p_0^2}. \tag{56}$$

Thus, a slow ($p_0/\alpha \ll 1$) and broad ($\alpha \Delta x \gg 1$) particle experiences the largest delay, $\delta x_{COM} - \delta v_0 t \approx 1/2p_0$, is scattered by a barrier with $s \approx p_0/\alpha$. This simple approximation begins to break down for $s \approx -1/2$ ($U_0 \approx \alpha^2/8\mu$), where the poles of the first and the second kind in Fig. 3 coalesce and, as the barrier increases, begin to move parallel to the real axis. This change in the poles' behaviour has no visible effect on either the transmission amplitude, or the delay in Eq. (42). However, for $U_0 > \alpha^2/8\mu$, a larger number of poles must be taken into account in order to reproduce the transmitted wave packet with sufficient accuracy, as shown in Fig. 9. Note that the role of the poles lying far from the real axis is to cancel the contribution from the δ -term in Eq. (19) and ensure the correct magnitude of the transmitted state (cf. Fig. 8).

Conclusions and discussion

In summary, there are two ways to look at transmission of a quantum particle across a short-range potential, be it a barrier or a well. Firstly, the wave packet can be seen as probing the transmission amplitude, $T(p, V)$, in a range of momenta around its mean momentum, p_0 , determined by the width of $A(p - p_0)$ in Eq. (1). Integration in the momentum space gives the correct answer (2) for the transmitted wave packet, but provides little additional insight.

The second approach helps one to identify the disputed reshaping mechanism (see, e.g.,^{29,31,32}). Equation (18) represents the transmitted state as a sum of freely propagating envelopes, each shifted in space by a distance x' , and weighted by the corresponding probability amplitude $\eta(p_0, x')$. The problem, we argue, is most naturally discussed in terms of the particle's position at a given time (One can, in principle, consider a Fourier transform with respect to the energy E , and represent the transmitted state as a weighted sum of the free envelopes, shifted in time rather than in space. The analysis is the same if the dispersion law is linear, $E = cp$, but becomes more complicated for massive particles, $E = p^2/2\mu$, since $T(E, V)$ is no longer single-valued in the complex energy plane. For this reason we prefer an analysis in terms of spacial delays). One recognises certain features familiar from classical mechanics. For a barrier, all envelopes are delayed relative to free propagation, and it requires a potential well to have some of them advanced. This does not, however, guarantee that the tunnelling particle will be found delayed if compared with a freely propagating one. Since $\eta(p_0, x')$ may change sign, the centre of mass of the transmitted wave packet, constructed from the front tails of the delayed envelopes, may end up advanced, as shown in Fig. 2c.

One of the enduring misconceptions about the subject is the belief that the time at which a transmitted particle arrives at a fixed detector can be related to the duration, spent by the particle in the barrier. As discussed in Section “Spatial delays vs. detection times”, the detection times are easily linked to the transmitted particle's instantaneous position. Yet they provide no further insight into which of the delays in Eq. (19) occurs in the barrier, and for a very fundamental reason. With several spacial delays interfering, the Uncertainty Principle forbids identifying the one occurred in the same sense it leaves indeterminate the slit chosen in a double slit experiment (see Supplementary Appendix E). In general, there is no single spatial delay, associated with quantum transmission. Worse still, since $\eta(p_0, x')$ changes sign, we found no simple way to characterise transmission by an effective range of delays (see Sections “The “phase time”” and “Scattering by a shallow Eckart well”).

One exception is the classical limit. For a classically allowed transmission, the amplitude distribution $\eta(p_0, x')$ selects a single delay \tilde{x}' , and a single envelope $G_0(x - \tilde{x}', t)$, delayed or advanced relative to free propagation, as shown in Fig. 2a and b. This is no longer true for semiclassical tunnelling, where $\eta(p_0, x')$ also has a saddle point, but there the similarity ends. Equation (27) sums the free envelopes along the real x' -axis, but the saddle at $\tilde{x}' = \tilde{x}'_1 - i\tilde{x}'_2$ lies in the complex x' -plane and cannot be seen in Fig. 2. Interference between all real delays produces the resulting envelope, $G_0(x - \tilde{x}'_1 - i\tilde{x}'_2, t)$, which cannot be associated with a single real spatial shift.

Then what is the “phase time”, often associated with the duration of tunnelling³³? Firstly, it is what one obtains by dividing the distance separating the COM of a transmitted wave packet, broad in the coordinate space, from the COM of its free counterpart, by the particle's mean velocity [cf. Eq. (47)]. Secondly, it can be expressed as the real part of the first moment of an alternating “distribution” $\eta(p_0, x')$ [cf. Eq. (38)]. It can, therefore, be measured, but should not be interpreted as the “excess time spent in the barrier” if one wishes to avoid a conflict with special relativity.

Furthermore, the transmitted state $\psi^T(x, t)$ can be written as a discrete sum over the singularities of the transmission amplitude $T(p_0, V)$. This representation is convenient for describing low-energy scattering by shallow wells or low barriers, as discussed in Sections XI and XII. It becomes impractical in the classical or semiclassical limit, where individual terms of the pole sum become very large (see Fig. 4). We note that a similar behaviour occurs in a different model (see Section 6 of¹⁵), using an interference-based reshaping mechanism to advance the transmitted state.

In summary, we provided a detailed analysis of transmission across various Eckart potentials. We conclude that, except in the classical limit, quantum scattering is essentially an interference phenomenon, not amenable to simplistic descriptions in terms of a single delay experienced in the potential, or even of a probability distribution of such delays. This is the fundamental difficulty with the search for a “tunneling time” which began with McColl's paper¹ almost ninety years ago.

Received: 6 November 2021; Accepted: 21 February 2022

Published online: 09 March 2022

References

1. MacColl, L. A. Note on the transmission and reflection of wave packets by potential barriers. *Phys. Rev.* **40**, 621 (1932).
2. Landsman, A. S. & Keller, U. Attosecond science and the tunnelling time problem. *Phys. Rep.* **547**, 1 (2015).
3. Camus, N. *et al.* Experimental evidence for quantum tunneling time. *Phys. Rev. Lett.* **119**, 023201 (2017).
4. Satya Sainadh, U. *et al.* Attosecond angular streaking and tunnelling time in atomic hydrogen. *Nature* **568**, 75–77 (2019).
5. Ramos, R., Spierings, D., Racicot, I. & Steinberg, A. M. Measurement of the time spent by a tunnelling atom within the barrier region. *Nature* **583**, 529–532 (2020).
6. Sokolovski, D. & Akhmatkaya, E. No time at the end of the tunnel. *Commun. Phys.* **1**, 47. <https://doi.org/10.1038/s42005-018-0049-9> (2018).
7. Sokolovski, D. & Akhmatkaya, E. Tunnelling times, Larmor clock, and the elephant in the room. *Sci. Rep.* **11**, 10040. <https://doi.org/10.1038/s41598-021-89247-8> (2021).
8. Pollak, E. Transition path time distribution, tunneling times, friction, and uncertainty. *Phys. Rev. Lett.* **118**, 070401 (2017).
9. Pollak, E. Quantum tunneling: The longer the path, the less time it takes. *J. Phys. Chem. Lett.* **8**(2), 352–356 (2017).
10. Diener, G. Superluminal group velocities and information transfer. *Phys. Lett. A* **223**(5), 327–331 (1996).
11. Petersen, J. & Pollak, E. Tunneling flight time, chemistry, and special relativity. *J. Phys. Chem. Lett.* **8**(17), 4017–4022 (2017).
12. Petersen, J. & Pollak, E. Instantaneous tunneling flight time for wavepacket transmission through asymmetric barriers. *J. Phys. Chem. A* **122**(14), 3563–3571 (2018).
13. Dumont, R. S., Rivlin, T. & Pollak, E. The relativistic tunneling flight time may be superluminal, but it does not imply superluminal signaling. *New J. Phys.* **22**, 093060 (2020).
14. Rivlin, T., Pollak, E. & Dumont, R. S. Determination of the tunneling flight time as the reflected phase time. *Phys. Rev. A* **103**, 012225 (2021).

15. Sokolovski, D. & Akhmatkaya, E. Superluminal paradox in wave packet propagation and its quantum mechanical resolution. *Ann. Phys.* **339**, 307 (2013).
16. Landau, L. D. & Lifshitz, E. M. *Quantum Mechanics* 3rd edn. (Pergamon, 1977).
17. Abramovitz, M. & Stegun, I. *Handbook of Mathematical Functions* (National Bureau of Standards, 1972).
18. Baz, A. I., Perelomov, A. M. & Zeldovich, Y. B. *Scattering, Reactions and Decay in Non-relativistic Quantum Mechanics* (Israel Program for Scientific Translations, 1969).
19. Brink, D. M. *Semiclassical Methods in Nucleus-Nucleus Scattering* (Cambridge University Press, 1985).
20. Sokolovski, D. Causality, apparent superluminality, and reshaping in barrier penetration. *Phys. Rev. A* **081**, 042115 (2010).
21. Fedoryuk, M. V. *Saddle-Point Method* (EMS Press, 2001).
22. Sokolovski, D. Residence time of a two-level system. *Proc. R. Soc. Lond. A* **460**, 1505 (2004).
23. Sokolovski, D. Weak values, negative probability, and the uncertainty principle. *Phys. Rev. A* **76**, 042125 (2007).
24. Feynman, R. P., Leighton, R. & Sands, M. *The Feynman Lectures on Physics III* (Dover Publications Inc, 1989).
25. Hartman, T. E. Tunneling of a wave packet. *J. Appl. Phys.* **33**, 3427 (1962).
26. Nimtz, G. Tunneling confronts special relativity. *Found. Phys.* **41**, 1193 (2011).
27. Deutsch, J. M. & Low, F. E. Barrier penetration and superluminal velocity. *Ann. Phys.* **228**, 184 (1993).
28. Sokolovski, D. Why does relativity allow quantum tunnelling to 'take no time'? *Proc. R. Soc. A* **460**, 499 (2004).
29. Japha, Y. & Kurizki, G. Superluminal delays of coherent pulses in nondissipative media: A universal mechanism. *Phys. Rev. A* **53**, 586 (1996).
30. Sokolovski, D., Msezane, A. Z. & Shaginyan, V. R. Superluminal tunneling as a weak measurement effect. *Phys. Rev. A* **71**, 064103 (2005).
31. Büttiker, M. & Washburn, S. Ado about nothing much. *Nature* **422**, 271 (2003).
32. Winful, H. G. Mechanism for 'superluminal' tunnelling. *Nature* **424**, 638 (2003).
33. Hauge, E. H. & Søveng, J. A. Tunnelling times: A critical review. *Rev. Mod. Phys.* **61**, 917 (1989).

Acknowledgements

Financial support through the grants PGC2018-101355-B-100 funded by MCIN/AEI/ 10.13039/501100011033 and by "ERDF A way of making Europe", PID2019-107609GB-I00 by MCIN, and the Basque Government Grant No IT986-16, is acknowledged by MP and DS.

Author contributions

X.G.d.C, M.P. and D.S. participated equally in the conceptualization, wrote the main manuscript text and prepared the figures. All authors reviewed the manuscript.

Competing interests

The authors declare no competing interests.

Additional information

Supplementary Information The online version contains supplementary material available at <https://doi.org/10.1038/s41598-022-07599-1>.

Correspondence and requests for materials should be addressed to X.G.C.

Reprints and permissions information is available at www.nature.com/reprints.

Publisher's note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

© The Author(s) 2022