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On Nieh-Yan transport

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Abstract: We study nondissipative transport induced by the Nieh-Yan anomaly. After computing the torsional terms in the equilibrium partition function using transgression, we find the constitutive relations for the covariant axial-vector, heat, stress, and spin currents. A number of new transport effects are found, driven by background torsion and the spin chemical potential. Torsional constitutive relations in two-dimensional systems are also analyzed.

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1 Introduction

Geometric torsion, a common ingredient of speculative scenarios in high energy physics [1, 2], has important physical applications in the effective description of lattice dislocations in solid state physics [3] (see [4, 5] for some early proposals, and [6] for a review). The basic underlying idea is to regard the lattice as a discretization of continuous space, such that at long distances the lattice vectors at each point span a continuous vielbein e^a. Lattice dislocations cause charge carriers circulated around then to undergo spatial translations. This phenomenon is modeled in the continuous theory by introducing a background geometric torsion, which causes a "parallelogram" defined by two nonparallel vectors not to close. In differential geometry, torsion is quantified by the two-form[1]

$$T^a = de^a + \omega^a{}_b e^b, \tag{1.1}$$

where $\omega^a{}_b$ is the spin connection. Curvature is in turn encoded by

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \omega^c{}_b, \tag{1.2}$$

and effectively describes disclinations in the underlying lattice. An important technical point is that, in the presence of nonvanishing torsion, the vielbein e^a and the spin connection $\omega^a{}_b$ have to be regarded as independent fields.

For many purposes, e^a can be considered an Abelian gauge field, with the torsion two-form its corresponding (Lorentz covariant) field strength. An important difference, however, is that unlike a standard *bona fide* gauge field the vielbein is dimensionless. At the level of the action, torsion minimally couples to fundamental fermions via the axial-vector

[1]In what follows, we denote Lorentz indices by $a, b, c, \ldots$ Space-time indices are indicated by Greek fonts, whereas $i, j, k, \ldots$ are reserved for spatial indices. In addition, to unclutter expressions, we drop the wedge product symbol $\wedge$ throughout the paper.

current [1, 2, 7, 8] and contributes to the axial anomaly through the correlation function of two axial-vector currents at one loop which, being quadratically divergent, keeps memory of the relevant UV energy scale Λ of theory. Using functional methods, it was shown in [9] that the axial anomaly receives a torsion-dependent contribution proportional to the so-called Nieh-Yan topological invariant [10, 11]

$$d\langle\star J_5\rangle = a_F\mathcal{F}^2 + a_R\mathrm{tr}\,R^2 + a_{\mathrm{NY}}\Lambda^2\eta_{ab}\Big(T^aT^b - e^aR^b{}_ce^c\Big), \tag{1.3}$$

where $\mathcal{F} = d\mathcal{A}$ is the electromagnetic field strength, "tr" indicates the trace over Lorentz indices, and the three coefficients a_F, a_R, and a_{NY} are dimensionless quantities. In fact, using the two Cartan structure equations (1.1) and (1.2), the Nieh-Yan term in (1.3) can be written as an exact four-form

$$\eta_{ab}\Big(T^aT^b - e^aR^b{}_ce^c\Big) = d(e_aT^a). \tag{1.4}$$

Despite its multiple confirmations in the literature [12–14] and more formal analyses [15–17], the torsional contribution in (1.3) remains somewhat puzzling (see, for example, [18]). The presence of the energy scale Λ, that is traced back to the peculiar dimensions of the vielbein as a putative gauge field, seems to conflict with the topological and consequently infrared origin of anomalies. Moreover, while in condensed matter scenarios there is a natural built-in cutoff, it is not clear what the appropriate scale might be in the case of fundamental torsion.

Torsional anomalies have been widely studied in the context of solid state and fluid physics [19–35]. In this paper we want to further this program by analyzing the consequences for transport phenomena of the Nieh-Yan contribution to the 't Hooft anomaly of the axial-vector current in four dimensions. Our strategy is to construct the equilibrium partition function from the torsional anomaly polynomial using transgression [36–38]. From it, we compute the different constitutive relations for axial-vector, stress, heat, and spin covariant current. In the case of the axial-vector current, besides a vortical term, we find chiral separation effects sourced by the magnetic component of the background torsion and the spin chemical potential. As for the constitutive relations for the heat and stress covariant currents, we find that they differ from each other only by a term proportional to the curl of the axial-vector external gauge field, and are therefore equal in the limit in which the spatial components of the external axial-vector gauge field vanish.

We also analyze the torsional contributions to the constitutive relations of the various covariant currents in a $(1+1)$-dimensional fluid in thermal equilibrium. In this case, the anomaly polynomial does not couple to the external gauge fields, so it is zero for theories with the same number of right- and left-handed fermion species. For this reason, we analyze the case of a fluid of right-handed fermions coupled to an external gauge field. In particular, we find that the heat and stress currents are proportional to each other.

The remaining of the paper is organized as follows. In section 2 we discuss the descent formalism for the covariant Nieh-Yan anomaly of the axial-vector current. Section 3 is devoted to the computation of the equilibrium partition function of a four-dimensional fluid coupled to an external axial-vector gauge field, and in the presence of nonvanishing torsion.

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The torsional contributions to the constitutive relations for this theory are analyzed in section 4, while in section 5 the two-dimensional case is studied. Finally, we summarize and discuss our results in section 6.

2 Descent formalism for the Nieh-Yan anomaly

We begin by presenting a general study the Nieh-Yan anomaly using the differential geometry methods employed in the standard analysis of quantum field theory anomalies [39]. The starting point is the anomaly polynomial in $D+2=6$ dimensions, obtained by adding the contributions of right- and left-handed fields with a relative minus sign

$$\begin{aligned}\mathcal{P}_6(\mathcal{F}_{R,L},d\mathcal{H}) &\equiv \left(-\frac{i}{24\pi^2}\mathcal{F}_R^3+\frac{c_H}{2}\mathcal{F}_R d\mathcal{H}\right)-\left(-\frac{i}{24\pi^2}\mathcal{F}_L^3+\frac{c_H}{2}\mathcal{F}_L d\mathcal{H}\right) \\ &= -\frac{i}{24\pi^2}\Big(\mathcal{F}_R^3-\mathcal{F}_L^3\Big)+\frac{c_H}{2}\Big(\mathcal{F}_R-\mathcal{F}_L\Big)d\mathcal{H}. \end{aligned} \tag{2.1}$$

Here $\mathcal{F}_{R,L}\equiv d\mathcal{A}_{R,L}$, with $\mathcal{A}_{R,L}$ the external gauge fields coupling to microscopic right- and left-handed chiral fermions and c_H is a nonuniversal constant with dimensions of (energy)^2. The torsion-dependent part of the anomaly polynomial, on the other hand, is written in terms of the torsional Chern-Simons three form [17]

$$\mathcal{H}\equiv e_a T^a. \tag{2.2}$$

The torsional term in (2.1) is in fact the only closed six-form that can be constructed from $\mathcal{H}$ and $\mathcal{F}_{R,L}$. A similar term was considered also in refs. [20, 21] coupled to the vector field strength.

The right and left gauge fields $\mathcal{A}_{R,L}$ can be written as the following combinations of the vector and axial-vector gauge fields $\mathcal{V}$ and $\mathcal{A}$

$$\begin{aligned}\mathcal{A}_R &= \mathcal{V}+\mathcal{A}, \\ \mathcal{A}_L &= \mathcal{V}-\mathcal{A},\end{aligned} \tag{2.3}$$

in terms of which the anomaly polynomial reads

$$\mathcal{P}_6(\mathcal{F}_{V,A},\mathcal{H}) = -\frac{i}{4\pi^2}\left(\mathcal{F}_A\mathcal{F}_V^2+\frac{1}{3}\mathcal{F}_A^3\right)+c_H\mathcal{F}_A d\mathcal{H}, \tag{2.4}$$

with $\mathcal{F}_V = d\mathcal{V}$ and $\mathcal{F}_A = d\mathcal{A}$. This expression shows that torsion only couples to the axial-vector gauge field. Another important feature of this anomaly polynomial is that, besides its gauge invariance under vector and axial-vector gauge transformations

$$\begin{aligned}\mathcal{V} &\longrightarrow \mathcal{V}+d\alpha, \\ \mathcal{A} &\longrightarrow \mathcal{A}+d\beta,\end{aligned} \tag{2.5}$$

it also remains invariant under shifts of the torsional Chern-Simons form by an arbitrary exact three-form

$$\mathcal{H}\longrightarrow\mathcal{H}+d\gamma. \tag{2.6}$$

In the hydrodynamical context to be explored in the following sections, the field $\mathcal{H}$ can be regarded as an external source coupling to a higher-order current form $\mathcal{J}_H$ [40].

When constructing the Chern-Simons five-form $\omega_5^0(\mathcal{V},\mathcal{A},\mathcal{H})$ using the methods presented in [37], we have the freedom of adding local counterterms. This we use to secure the invariance under vector gauge transformations, since the field $\mathcal{V}$ will be eventually identified with the physical electromagnetic potential. In our case, given that the torsional term only depends on $\mathcal{A}$ and is automatically invariant under vector gauge transformations, we only need to add the standard Abelian Bardeen counterterm, whose explicit expression can be obtained from appendix B of ref. [37].

This settled, we have the further choice of whether to preserve the invariance under torsional gauge transformations (2.6). In fact, it is possible to construct a family of local counterterms that shifts the invariance of the torsional piece of the Chern-Simons form from axial-vector to torsional gauge invariance. Taking into account what was said in the previous paragraph, we write the following Chern-Simons five-form

$$\omega_5^0(\mathcal{V},\mathcal{A},\mathcal{H}) = -\frac{i}{4\pi^2}\mathcal{A}\left(\mathcal{F}_V^2+\frac{1}{3}\mathcal{F}_A^2\right)+(1-a)\frac{c_H}{2}\mathcal{F}_A\mathcal{H}+(1+a)\frac{c_H}{2}\mathcal{A}d\mathcal{H}, \tag{2.7}$$

where $-1\leq a\leq 1$. This expression is invariant under vector gauge transformations, while breaking axial-vector gauge invariance. The torsion-dependent part, on the other hand, exhibits the tension between axial-vector and torsional gauge transformations: by tuning the a parameter, we can shift its invariance from the first class of transformations ($a=-1$) to the second ($a=1$). Notice that there is no value of a for which the torsional part of the Chern-Simons five-form remains invariant under both kinds of transformations. Incidentally, the pure-gauge part of the Chern-Simons form (2.7) leads to the Abelian version of the Bardeen anomaly [41] (see also section 2.2 of ref. [37] for the relevant explicit expressions).

The nonlocal effective action is obtained then by integrating the Chern-Simons form given in (2.7) on a five-dimensional manifold $\mathcal{M}_5$, whose boundary is identified with the physical spacetime

$$\Gamma[\mathcal{V},\mathcal{A},\mathcal{H}]_{\rm CS} = \int\limits_{\mathcal{M}_5}\left[-\frac{i}{4\pi^2}\mathcal{A}\left(\mathcal{F}_V^2+\frac{1}{3}\mathcal{F}_A^2\right)+(1-a)\frac{c_H}{2}\mathcal{F}_A\mathcal{H}+(1+a)\frac{c_H}{2}\mathcal{A}d\mathcal{H}\right]. \tag{2.8}$$

To compute the consistent axial anomaly, we evaluate the variation of the action under axial-vector gauge transformations, $\delta_\beta\mathcal{A}=d\beta$

$$\delta_\beta\Gamma[\mathcal{V},\mathcal{A},\mathcal{H}]_{\rm CS} = -\int\limits_{\partial\mathcal{M}_5}\beta d\langle\star\mathcal{J}_5\rangle_{\rm cons}, \tag{2.9}$$

and get the result

$$d\langle\star\mathcal{J}_5\rangle_{\rm cons} = \frac{i}{4\pi^2}\left(\mathcal{F}_V^2+\frac{1}{3}\mathcal{F}_A^2\right)-(1+a)\frac{c_H}{2}d\mathcal{H}. \tag{2.10}$$

As expected, the consistent anomaly becomes torsion-independent for $a=-1$.

To find the covariant anomaly, we evaluate first the Bardeen-Zumino (BZ) term for the axial-vector current

$$\begin{aligned}\langle\star\mathcal{J}_5\rangle_{\rm BZ} &\equiv -\frac{\delta\Gamma_{\rm CS}}{\delta\mathcal{F}_A} \\ &= \frac{i}{6\pi^2}\mathcal{A}\mathcal{F}_A - (1-a)\frac{c_H}{2}\mathcal{H},\end{aligned} \tag{2.11}$$

and apply the relation between the consistent and covariant axial-vector currents

$$\langle\star\mathcal{J}_5\rangle_{\rm cov} = \langle\star\mathcal{J}_5\rangle_{\rm cons} + \langle\star\mathcal{J}_5\rangle_{\rm BZ}. \tag{2.12}$$

Taking the exterior differential, we arrive at the covariant form of the axial anomaly

$$d\langle\star\mathcal{J}_5\rangle_{\rm cov} = \frac{i}{4\pi^2}\left(\mathcal{F}_V^2 + \mathcal{F}_A^2\right) - c_H d\mathcal{H}. \tag{2.13}$$

As we see, the Nieh-Yan contribution to the axial anomaly [cf. (1.4)] has a coefficient which is independent of the parameter a. This serves as a good check of our calculation since, unlike its consistent counterpart, the covariant anomaly is insensitive to any local counterterms in the nonlocal effective action.[2]

A comparison of the results shown in eqs. (2.10) and (2.13) also highlights an intriguing feature of the Nieh-Yan anomaly. Although the torsional contribution to the consistent axial anomaly can be cancelled by a local counterterm without affecting the conservation of the vector current (taking $a=-1$), the covariant anomaly retains the Nieh-Yan term independently of the value of the parameter a. Since the anomaly inflow argument [36] implies that physical transport of charge is governed by the covariant current [38], torsional contributions to the constitutive relations of physical currents are independent of any local counterterms added to the effective action.

This brings about another important issue concerning the overall normalization c_H of the torsional part of the anomaly polynomial (2.4). In the general theory of quantum field theory anomalies, it is known that the descent method does not fix the overall normalization of the anomaly polynomial. This overall factor has to be determined by a diagrammatic or functional calculation of the anomaly, or by applying the Atiyah-Singer index theorem. In the case of the axial anomaly, this coefficient is universal, a fact that is linked to its infrared origin. Since it is determined by the residue of zero momentum pole in the expectation value of the axial-vector current, it is therefore independent of the UV structure of the theory or model.[3]

[2] An alternative way of finding the BZ current and the covariant anomaly is by calculating the variation of the Chern-Simons effective action under $\delta_B\mathcal{A} = B$, using (see [37])

$$\delta_B\Gamma[\mathcal{V},\mathcal{A},\mathcal{H}]_{\rm CS} = \int_{\mathcal{M}_5} B\langle\star\mathcal{J}_5\rangle_{\rm bulk} - \int_{\partial\mathcal{M}_5} B\langle\star\mathcal{J}_5\rangle_{\rm BZ}.$$

The covariant anomaly is then given by the value of $-\langle\star\mathcal{J}_5\rangle_{\rm bulk}$ at the boundary $\partial\mathcal{M}_5$. This calculation highlights the fact that boundary terms in the effective action do contribute to the Bardeen-Zumino current but not to the covariant anomaly.

[3] The nonvanishing residue of this IR pole is precisely what explains the electromagnetic decay of the neutral pion, despite the suppression implied by the Sutherland-Veltman theorem in the context of PCAC.

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This is indeed not the case of the torsional contribution to the axial anomaly (2.13). The microscopic calculations of the Nieh-Yan anomaly [9] shows a strong sensitivity to the UV structure of the theory, which leads to the fact that its overall normalization is not universal but rather depends quadratically on the relevant UV scale of the theory. An immediate consequence is that the dimensionfull global factor of the torsional part of the anomaly polynomial depends on the microscopic details of the corresponding model. This makes this quantity nonuniversal in the sense of being model-dependent. In fact, it is set by the energy scale of the physical effects associated to the background geometric torsion (see also [20, 21] for a related discussion in the context of a calculation of the effects of the Nieh-Yan anomaly in condensed matter systems using a Pauli-Villars regularization method).

This fact is actually connected with a basic mathematical issue that has been analyzed in detail in [21]. Whereas both contributions on the right-hand side of the anomaly equation (2.13) are exact

$$d\langle\star\mathcal{J}_5\rangle_{\text{cov}} = \frac{i}{4\pi^2}d\Big(\mathcal{V}\mathcal{F}_V + \mathcal{A}\mathcal{F}_A\Big) - c_H d\mathcal{H}, \tag{2.14}$$

there is a glaring difference between the two terms. The gauge part is not the differential of a globally well-defined form, so upon integrating it over the compact manifold $\partial\mathcal{M}_5$ it is quantized in terms of the winding number of the corresponding gauge fields. In the case of the torsional contribution, on the other hand, the three-form $\mathcal{H}$ is globally well-defined and therefore $d\mathcal{H}$ gives zero when integrated over $\partial\mathcal{M}_5$. This crucial fact preserves the topological character of the integrated axial anomaly, despite the presence of a dimensionfull nonuniversal quantity in the expression of the anomaly density.

After this long but necessary digression, we close our discussion in this section with the computation of the anomaly associated with the torsional gauge transformations (2.6). The Ward identity for the consistent current is obtained from the variation of the Chern-Simons action under $\delta_\gamma\mathcal{H} = d\gamma$, with the result

$$d\langle\star\mathcal{J}_H\rangle_{\text{cons}} = -(1-a)\frac{c_H}{2}\mathcal{F}_A. \tag{2.15}$$

Thus, the current $\mathcal{J}_H$ remains anomalous whenever $a \neq 1$. The corresponding Bardeen-Zumino term is in turn given by (minus) the functional derivative of the action with respect to $d\mathcal{H}$, leading to

$$\langle\star\mathcal{J}_H\rangle_{\text{BZ}} = -(1+a)\frac{c_H}{2}\mathcal{A}. \tag{2.16}$$

This gives the following expression for the covariant anomaly

$$d\langle\star\mathcal{J}_H\rangle_{\text{cov}} = -c_H\mathcal{F}_A, \tag{2.17}$$

which, as expected, is also independent of the parameter a. At this point it should be stressed however that, although intriguing, the invariance under (2.6) seems to be in our context a purely accidental symmetry.

3 The equilibrium partition function

The Chern-Simons form (2.7) is the basic ingredient in the construction of the equilibrium partition function of a four-dimensional electron fluid coupled to vector and axial-vector gauge sources and in the presence of torsion. Since the effects of the gauge part have been widely studied in the literature, here we will focus our attention entirely on the torsional contributions. Furthermore, since physical charge transport is described by the covariant currents [37], which are determined by the a-independent bulk part of the effective action, we will set $a = -1$ from now on in order to simplify expressions. This being so, our starting point is the torsional Chern-Simons form

$$\omega_5^0(\mathcal{A}, \mathcal{H})_H = c_H \mathcal{F}_A \mathcal{H}. \tag{3.1}$$

To compute the partition function, we need to evaluate the transgression form [36–38]

$$\mathcal{T}_5(\mathcal{A}, \mathcal{H}; \widehat{\mathcal{A}}, \widehat{\mathcal{H}}) = \omega_5^0(\mathcal{A}, \mathcal{H})_H - \omega_5^0(\widehat{\mathcal{A}}, \widehat{\mathcal{H}})_H, \tag{3.2}$$

where the two background configurations $\{\mathcal{A}, e^a, \omega^a{}_b\}$ and $\{\widehat{\mathcal{A}}, \widehat{e}^a, \widehat{\omega}^a{}_b\}$ are related by

$$\begin{aligned} \mathcal{A} &= \widehat{\mathcal{A}} - \mu_5 u, \\ e^a &= \widehat{e}^a - \chi^a u, \\ \omega^a{}_b &= \widehat{\omega}^a{}_b - \mu^a{}_b u. \end{aligned} \tag{3.3}$$

Here $u = u_\mu dx^\mu$ is the fluid velocity one-form and μ_5 is the chiral chemical potential, while χ^a and $\mu^a{}_b$ are respectively interpreted as the stress and spin chemical potentials.[4] Similar decompositions can be written for the gauge field strength and the torsion as

$$\begin{aligned} \mathcal{F}_A &= \widehat{\mathcal{F}}_A - 2\mu_5 \omega + u(d + \mathfrak{a})\mu_5, \\ T^a &= \widehat{T}^a - 2\omega\chi^a + u\Big[(\widehat{D} + \mathfrak{a})\chi^a - \mu^a{}_b \widehat{e}^b\Big], \end{aligned} \tag{3.4}$$

with $\mathfrak{a} = \imath_u u$ the fluid acceleration and ω the vorticity two-form, which is implicitly defined by the identity

$$du = 2\omega - u\mathfrak{a}. \tag{3.5}$$

In addition to this, we have introduced the covariant derivative of the stress chemical potential with respect to the hatted connection, $\widehat{D}\chi^a = d\chi^a + \widehat{\omega}^a{}_b \chi^b$. Corresponding expressions can be derived also for the curvature two-form

$$R^a{}_b = \widehat{R}^a{}_b - 2\mu^a{}_b \omega + u(\widehat{D} + \mathfrak{a})\mu^a{}_b, \tag{3.6}$$

with $\widehat{D}\mu^a{}_b = d\mu^a{}_b + \widehat{\omega}^a{}_c \mu^c{}_b - \mu^a{}_c \widehat{\omega}^c{}_b$, as well as for the torsional three-form $\mathcal{H}$

$$\mathcal{H} = \widehat{e}_a\Big(\widehat{T}^a - 2\omega\chi^a\Big) - u\Big[\chi_a \widehat{T}^a + \widehat{e}_a(\widehat{D} + \mathfrak{a})\chi^a - 2\omega\chi_a\chi^a - \mu^a{}_b \widehat{e}_a \widehat{e}^b\Big]. \tag{3.7}$$

[4] This follows from the fact that e^a and $\omega^a{}_b$ respectively couple to the stress and spin currents.

To find an explicit expression for the transgression five-form (3.2), we consider the following single-parameter family of tetrads and connections interpolating between the field configuration $\{\mathcal{A}, e^a, \omega^a{}_b\}$ and its hatted counterpart [cf. (3.3)]

$$\begin{aligned}\mathcal{A}_t &= \widehat{\mathcal{A}} - t\mu_5 u,\\ e^a_t &= \widehat{e}^a - t\chi^a u,\\ (\omega_t)^a{}_b &= \widehat{\omega}^a{}_b - t\mu^a{}_b u,\end{aligned}\tag{3.8}$$

where $0 \leq t \leq 1$. Then, we apply the Mañes-Stora-Zumino generalized transgression formula [42]

$$\int_{\partial T} \frac{\ell_t^p}{p!}\mathscr{Q} = \int_T \frac{\ell_t^{p+1}}{(p+1)!}d\mathscr{Q} + (-1)^{p+q} d\int_T \frac{\ell_t^{p+1}}{(p+1)!}\mathscr{Q},\tag{3.9}$$

where the even operator ℓ_t acts by replacing exterior differential d by

$$d_t = dt\frac{d}{dt},\tag{3.10}$$

and in our case the integration domain is $T = [0,1]$. Taking $\mathscr{Q} = \omega_5^0(\mathcal{A}_t, \mathcal{H}_t)_H$ with $p = q = 0$, we arrive at the following expression for the transgression five-form (3.2)

$$\mathcal{T}_5(\mathcal{A}, \mathcal{H}; \widehat{\mathcal{A}}, \widehat{\mathcal{H}}) = \int_0^1 \ell_t \mathcal{P}_6(\mathcal{F}_{A,t}, \mathcal{H}_t)_H + d\int_0^1 \ell_t \omega_5^0(\mathcal{A}_t, \mathcal{H}_t)_H,\tag{3.11}$$

where we have used that $\mathcal{P}_6(\mathcal{F}_A, \mathcal{H})_H \equiv c_H \mathcal{F}_A d\mathcal{H} = d\omega_5^0(\mathcal{A}, \mathcal{H})_H$.

At equilibrium, all hatted quantities are transverse to the fluid four-velocity u [36], so the relations (3.3) provide the electric-magnetic decompositions of the two connections and the vielbein with respect to this one-form. To make the notation more transparent, from now on all magnetic components will be denoted by boldface fonts. In particular, we set

$$\widehat{\mathcal{A}} \equiv \boldsymbol{A}, \qquad \widehat{e}^a \equiv \boldsymbol{e}^a, \qquad \widehat{\omega}^a{}_b \equiv \boldsymbol{\omega}^a{}_b.\tag{3.12}$$

This equilibrium condition leads to a number of identities for the magnetic parts of the various quantities, which can be changed by tuning the external sources, while the electric parts are fixed by the different chemical potentials and their gradients

$$\begin{aligned}\mathcal{F}_A &= d\boldsymbol{A} - 2\mu_5\omega + u(d + \mathfrak{a})\mu_5\\ &\equiv \boldsymbol{B}_A + uE_A,\\ T^a &= d\boldsymbol{e}^a + \boldsymbol{\omega}^a{}_b\boldsymbol{e}^b - 2\omega\chi^a + u\Big[(\boldsymbol{D} + \mathfrak{a})\chi^a - \mu^a{}_b\boldsymbol{e}^b\Big].\\ &\equiv \boldsymbol{B}^a + uE^a.\end{aligned}\tag{3.13}$$

Here, we denoted by $\boldsymbol{D}$ the covariant derivative with respect to the magnetic part of the spin connection $\boldsymbol{\omega}^a{}_b$. For the curvature, we obtain the corresponding electric-magnetic decomposition

$$\begin{aligned}R^a{}_b &= d\boldsymbol{\omega}^a{}_b + \boldsymbol{\omega}^a{}_c\boldsymbol{\omega}^c{}_b - 2\mu^a{}_b\omega + u(\boldsymbol{D} + \mathfrak{a})\mu^a{}_b.\\ &\equiv \boldsymbol{B}^a{}_b + uE^a{}_b,\end{aligned}\tag{3.14}$$

as well as for the three-form $\mathcal{H}$

$$\begin{aligned}\mathcal{H} &= \boldsymbol{e}_a \boldsymbol{B}^a - u\Big(\boldsymbol{e}_a E^a + \chi_a \boldsymbol{B}^a\Big) \\ &\equiv \boldsymbol{\mathcal{H}} + uE_H. \end{aligned} \tag{3.15}$$

Incidentally, the Bianchi identity for the torsion $dT^a + \omega^a{}_b T^b = R^a{}_b e^b$ can be recast as

$$\begin{aligned}\boldsymbol{D}\boldsymbol{B}^a &= \boldsymbol{B}^a{}_b \boldsymbol{e}^b - 2\omega E^a, \\ (\boldsymbol{D}+\mathfrak{a})E^a &= \boldsymbol{B}^a{}_b \chi^b - E^a{}_b \boldsymbol{e}^b - \mu^a{}_b \boldsymbol{B}^b, \end{aligned} \tag{3.16}$$

where all electric parts are given by their values at equilibrium, shown in eqs. (3.13) and (3.14).

A further consequence of setting the electric parts of the hatted quantities to zero is that the second term on the right-hand side of eq. (3.2) vanishes

$$\omega_5^0(\boldsymbol{A}, \boldsymbol{\mathcal{H}})_H = 0, \tag{3.17}$$

since it is a purely magnetic differential form of maximal rank. Plugging this into eq. (3.11), we get the following expression for the equilibrium partition function

$$\begin{aligned} W_{\text{eq}} &\equiv \int\limits_{\mathcal{M}_5} \omega_5^0(\boldsymbol{A} - \mu_5 u, \boldsymbol{\mathcal{H}} + uE_H)_H \\ &= \int\limits_{\mathcal{M}_5} \int_0^1 \ell_t \mathcal{P}_6(\mathcal{A}_t, \mathcal{H}_t)_H + \int\limits_{\partial\mathcal{M}_5} \int_0^1 \ell_t \omega_5^0(\mathcal{A}_t, \mathcal{H}_t)_H, \end{aligned} \tag{3.18}$$

where in the last term we have used the Stokes theorem to write a boundary integral. This identity provides the standard decomposition of the equilibrium partition function into a gauge invariant bulk piece and an anomalous boundary part, $W_{\text{eq}} = W_{\text{bulk}} + W_{\text{bdy}}$ [36–38]. To find explicit expressions for both terms, we need to evaluate the action of ℓ_t on the anomaly polynomial and the Chern-Simons form. After a bit of algebra, we get

$$\begin{aligned} W_{\text{bulk}} = c_H \int\limits_{\mathcal{M}_5} u \Big[&\mu_5\Big(\boldsymbol{e}_a \boldsymbol{B}^a{}_c \boldsymbol{e}^c - \boldsymbol{B}_a \boldsymbol{B}^a\Big) - \chi_a \boldsymbol{B}^a \Big(\boldsymbol{B}_A + 3\mu_5\omega\Big) + \mu_5 \mu^a{}_b \boldsymbol{e}_a \boldsymbol{e}^b \omega \\ &- 2\mu_5 \chi_a \chi^a \omega^2 - \boldsymbol{e}_a E^a \Big(\boldsymbol{B}_A + \mu_5 \omega\Big) + \boldsymbol{e}_a \chi^a \omega E \Big], \\ W_{\text{bdy}} = -c_H \int\limits_{\partial\mathcal{M}_5} &u\mu_5 \boldsymbol{e}_a \Big(\boldsymbol{B}^a + \chi^a \omega\Big). \end{aligned} \tag{3.19}$$

In writing the first identity, we have used that

$$-2\int\limits_{\mathcal{M}_5} \boldsymbol{e}_a \chi^a \Big(\boldsymbol{B}_A + \mu_5 \omega\Big)\omega = 0, \tag{3.20}$$

since the integrand has no u-component. In addition, we also applied the Stokes theorem to shift a total derivative term in W_{bulk} into an integral over $\partial\mathcal{M}_5$, which cancels an analogous term in the boundary piece W_{bdy}.

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Constraints from equilibrium. In equilibrium, our system can be regarded as defined on a generic background static metric of the form [43]

$$ds^2 = -u \otimes u + g_{ij} dx^i \otimes dx^j, \tag{3.21}$$

where u again is the fluid four-velocity one-form and all metric functions are independent of the time coordinate x^0. Imposing that $ds^2 = e_a \otimes e^a$ with $e^a = \boldsymbol{e}^a - u\chi^a$, we derive the conditions

$$\begin{aligned} \chi_a \chi^a &= -1, \\ \chi_a \boldsymbol{e}^a &= 0, \\ \boldsymbol{e}_a \otimes \boldsymbol{e}^a &= g_{ij} dx^i \otimes dx^j. \end{aligned} \tag{3.22}$$

In fact, the first two identities fix the stress chemical potential in terms of the four-velocity

$$\chi_a e^a = u, \tag{3.23}$$

from where we find $\chi_a = u_a = (-1, 0, 0, 0)$ and $\chi^a = u^a = (1, 0, 0, 0)$.

The condition (3.23) can be recast in a more useful form. Plugging it on the left-hand side of eq. (3.5), and separating the magnetic and electric components of the resulting equation, we find

$$\begin{aligned} 2\omega &= (\boldsymbol{D}\chi_a)\boldsymbol{e}^a + \chi_a \boldsymbol{B}^a, \\ \mathfrak{a} &= -\chi_a E^a - \mu^a{}_b \chi^b \boldsymbol{e}_a. \end{aligned} \tag{3.24}$$

In order to arrive at the second identity we have applied metric compatibility, $\omega_{(ab)} = 0$, as well as the relation $\chi_a \chi^a = -1$. Although these constraints lead to some simplifications in the bulk and boundary partition functions (3.19), we should not be too hasty in implementing them. This should be done only after taking the appropriate variations to compute the different currents.

Finally, the thermal partition function is computed by taking the five-dimensional manifold to have topology $\mathcal{M}_5 = \mathcal{M}_4 \times S^1_\beta$, with β the circle's length, and carring out the dimensional reduction onto the thermal cycle through the substitution (cf. [37, 38])

$$\int_{\mathcal{M}_4 \times S^1_\beta} u[\ldots] \longrightarrow \int_{\mathcal{M}_4} \frac{1}{T}[\ldots], \tag{3.25}$$

where T is the local temperature. A similar replacement has to be carried out on the boundary integral onto $\partial\mathcal{M}_4 \times S^1_\beta$. At the level of the currents, on the other hand, the prescription works by replacing u with $1/T$ and the four-dimensional Hodge dual on $\partial\mathcal{M}_4 \times S^1_\beta$ with its three-dimensional counterparts on $\partial\mathcal{M}_4$.

4 Torsional constitutive relations

The constitutive relations for the various covariant currents are obtained by taking variations of the bulk partition function (3.19) with respect to the appropriate sources, keeping only the boundary contributions [36–38]. Let us begin with the axial-vector current. Varying W_{bulk} with respecto to the axial-vector gauge field $\boldsymbol{A}$, we find

$$\delta W_{\text{bulk}} = -c_H \int\limits_{\partial\mathcal{M}_5} \delta\boldsymbol{A} u\Big(\chi_a \boldsymbol{B}^a + \boldsymbol{e}_a E^a\Big) + \text{bulk terms}, \tag{4.1}$$

from where we read the expression of the axial-vector covariant current

$$\langle \star \boldsymbol{J}_5 \rangle_{\text{cov}} = -c_H u\Big(\chi_a \boldsymbol{B}^a + \boldsymbol{e}_a E^a\Big). \tag{4.2}$$

At this point we should recall that in equlibrium the electric parts of the axial-vector field strength, the torsion, and the curvature are fixed in terms of the chemical potentials and their gradients by the expressions given in eqs. (3.13) and (3.14). The magnetic parts, on the other hand, are arbitrary in the sense that they are determined by the corresponding external sources. In fact, since we are computing torsional corrections in thermodynamical equilibrium, all transport coefficients in our expressions are nondissipative.[5] This in particular means that it is necessary to impose the first equilibrium condition in eq. (3.24) to express the electric component of the torsion in terms of its magnetic part, the vorticity, and the spin chemical potential. Indeed, combining the identity mentioned with the expression of E^a given in (3.13), we find the relation

$$-u\boldsymbol{e}_a E^a = u\Big(2\omega + \mu^a{}_b \boldsymbol{e}_a \boldsymbol{e}^b - \chi_a \boldsymbol{B}^a\Big), \tag{4.3}$$

where we have implemented the constraint $\chi_a \boldsymbol{e}^a = 0$.

Substituting (4.3) into eq. (4.2), we arrive at the following form of the covariant axial-vector current

$$\langle \star \boldsymbol{J}_5 \rangle_{\text{cov}} = c_H u\Big(2\omega + \mu^a{}_b \boldsymbol{e}_a \boldsymbol{e}^b - 2\chi_a \boldsymbol{B}^a\Big). \tag{4.4}$$

This result shows the existence of a vortical separation effect, together with transport of chiral charge mediated by the spin chemical potential and the magnetic part of the torsion. As a nontrivial check of our result for the axial-vector covariant current, we evaluate the corresponding anomalous Ward identity. Implementing the Bianchi identities (3.16) and after some algebra, we retrieve the Nieh-Yan term

$$\begin{aligned} d\langle \star \boldsymbol{J}_5 \rangle_{\text{cov}} &= c_H u\Big\{2\boldsymbol{B}_a\Big[(\boldsymbol{D}+\mathfrak{a})\chi^a - \mu^a{}_b \boldsymbol{e}^b\Big] + 2\chi_a \boldsymbol{B}^a{}_b \boldsymbol{e}^b - \big[(\boldsymbol{D}+\mathfrak{a})\mu^a{}_b\big]\boldsymbol{e}_a \boldsymbol{e}^b\Big\} \\ &= c_H\Big(T_a T^a - e_a R^a{}_b e^b\Big). \end{aligned} \tag{4.5}$$

To write these expressions we have used that the electric parts of both the torsion and the curvature take their equilibrium values given in eqs. (3.13) and (3.14), as well as implemented the equilibrium constraint (3.23).

[5] This suggests that the dissipative constitutive relations should include terms proportional to the combinations $E_A - \big(d + \boldsymbol{a}\big)\mu_5$ and $E^a - \big(\boldsymbol{D} + \boldsymbol{a}\big)\chi^a + \mu^a{}_b \boldsymbol{e}^b$, which vanish in equilibrium.

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Our next task is the calculation of the heat and stress currents. These are obtained by computing the boundary variation of the bulk partition function induced by δu, $\delta\chi^a$, and δe^a

$$\begin{aligned}\delta W_{\rm bulk} &= \int\limits_{\partial\mathcal{M}_5}\left[\delta u\frac{\delta W_{\rm bulk}}{\delta(2\omega)}+\delta\chi^a\frac{\delta W_{\rm bulk}}{\delta E^a}+\delta e^a\frac{\delta W_{\rm bulk}}{\delta\boldsymbol{B}^a}\right]+\text{bulk terms}\\ &= c_H\int\limits_{\partial\mathcal{M}_5}\bigg\{\delta u\, u\left(-\frac{3}{2}\mu_5\chi_a\boldsymbol{B}^a+\frac{1}{2}\mu_5\mu^a{}_b\boldsymbol{e}_a\boldsymbol{e}^b-2\mu_5\chi_a\chi^a\omega-\frac{1}{2}\mu_5\boldsymbol{e}_aE^a+\frac{1}{2}\boldsymbol{e}_a\chi^aE\right)\\ &\quad+\delta\chi^a\boldsymbol{e}_a u\Big(\boldsymbol{B}_A+\mu_5\omega\Big)-\delta e^a u\Big[2\mu_5\boldsymbol{B}_a+\chi_a\Big(\boldsymbol{B}_A+3\mu_5\omega\Big)\Big]\bigg\}+\text{bulk terms.}\end{aligned} \tag{4.6}$$

At this point, however, we need to realize that δu, $\delta\chi^a$, and δe^a are not independent. This is a consequence of the equilibrium constrain (3.23), which implies $u\chi_a e^a=0$ and gives the following relation between the three variations

$$\delta u u-\delta\chi^a e_a u-\delta e^a\chi_a u=0. \tag{4.7}$$

Eliminating $\delta\chi^a\boldsymbol{e}_a u=\delta\chi^a e_a u$ in favor of δu and δe^a, we rewrite eq. (4.6) as

$$\begin{aligned}\delta W_{\rm bulk} = c_H\int\limits_{\partial\mathcal{M}_5}&\bigg\{\delta u u\left(-\frac{3}{2}\mu_5\chi_a\boldsymbol{B}^a+\frac{1}{2}\mu_5\mu^a{}_b\boldsymbol{e}_a\boldsymbol{e}^b-\frac{1}{2}\mu_5\boldsymbol{e}_aE^a+\boldsymbol{B}_A+3\mu_5\omega\right)\\ &-\delta e^a u\Big[2\mu_5\boldsymbol{B}_a+2\chi_a\Big(\boldsymbol{B}_A+2\mu_5\omega\Big)\Big]\bigg\}+\text{bulk terms.}\end{aligned} \tag{4.8}$$

The covariant heat current is given by the coefficient of the variation δu

$$\langle\star\boldsymbol{q}\rangle_{\rm cov}=c_H u\left(-\frac{3}{2}\mu_5\chi_a\boldsymbol{B}^a+\frac{1}{2}\mu_5\mu^a{}_b\boldsymbol{e}_a\boldsymbol{e}^b-\frac{1}{2}\mu_5\boldsymbol{e}_aE^a+\boldsymbol{B}_A+3\mu_5\omega\right). \tag{4.9}$$

Here again we eliminate the electric component of the torsion using the equilibrium condition given in eq. (4.3), to arrive at the more compact result

$$\langle\star\boldsymbol{q}\rangle_{\rm cov}=c_H u\Big(\boldsymbol{B}_A+4\mu_5\omega+\mu_5\mu^a{}_b\boldsymbol{e}_a\boldsymbol{e}^b-2\mu_5\chi_a\boldsymbol{B}^a\Big). \tag{4.10}$$

Interestingly, using the expression of the axial-vector current found in eq. (4.4), we can recast the covariant heat current in terms of the axial-vector current as (cf. [44])

$$\langle\star\boldsymbol{q}\rangle_{\rm cov}=\mu_5\langle\star\boldsymbol{J}_5\rangle_{\rm cov}+c_H u\Big(\boldsymbol{B}_A+2\mu_5\omega\Big). \tag{4.11}$$

In particular, taking $\boldsymbol{A}=0$ we have $\boldsymbol{B}_A=-2\mu_5\omega$ and the heat current becomes proportional to the axial-vector current

$$\langle\boldsymbol{q}\rangle_{\rm cov}=\mu_5\langle\boldsymbol{J}_5\rangle_{\rm cov}\qquad\qquad(\boldsymbol{A}=0). \tag{4.12}$$

The coefficient of δe^a in (4.8), on the other hand, renders the value of the stress current

$$\langle\star\boldsymbol{\mathfrak{F}}_a\rangle_{\rm cov}=-2c_H u\Big[\mu_5\boldsymbol{B}_a+\chi_a\Big(\boldsymbol{B}_A+2\mu_5\omega\Big)\Big]. \tag{4.13}$$

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Unlike previous expressions, this is already written in terms of the chemical potentials and the magnetic parts of the axial-vector field and the torsion, so no equilibrium constraint needs to be applied. Taking again $\boldsymbol{A} = 0$, the stress current becomes

$$\langle \star \boldsymbol{\mathfrak{F}}_a \rangle_{\text{cov}} = -2c_H \mu_5 u \boldsymbol{B}_a \qquad (\boldsymbol{A} = 0), \tag{4.14}$$

which vanishes in the limit of zero magnetic torsion.

To conclude the analysis in this section, we take variations in the bulk partition function with respect to the spin connection

$$\delta W_{\text{bulk}} = c_H \int_{\partial \mathcal{M}_5} \delta \boldsymbol{\omega}^a{}_b u \mu_5 \boldsymbol{e}_a \boldsymbol{e}^b + \text{bulk terms}, \tag{4.15}$$

which gives the components of the covariant spin current

$$\langle \star \boldsymbol{\mathfrak{S}}_a{}^b \rangle_{\text{cov}} = c_H \mu_5 u \boldsymbol{e}_a \boldsymbol{e}^b. \tag{4.16}$$

A first thing to notice in this result is that it is completely independent of the value of torsion, being fully generated by chiral imbalance. The corresponding anomalous Ward identity is written in terms of the energy-momentum current (4.13) as

$$D \langle \star \boldsymbol{\mathfrak{S}}_a{}^b \rangle_{\text{cov}} + e^{[a} \langle \star \boldsymbol{\mathfrak{F}}^{b]} \rangle_{\text{cov}} = -c_H e^a e^b \mathcal{F}_A, \tag{4.17}$$

with D the covariant derivative associated with the full connection $\omega^a{}_b$.

Although all the previous currents have been computed from the boundary variation of the bulk effective action, they can be alternatively obtained by adding to the appropriate variations of the boundary partition function in (3.19) the corresponding BZ terms. For the axial-vector and the stress currents, these are given by

$$\begin{aligned} \langle \star \boldsymbol{J}_5 \rangle_{\text{BZ}} &= -c_H T^a e_a, \\ \langle \star \boldsymbol{\mathfrak{F}}_a \rangle_{\text{BZ}} &= -c_H \mathcal{F}_A e_a. \end{aligned} \tag{4.18}$$

5 The two-dimensional case

Torsional constitutive relations in two dimensions can be computed along similar lines. An important difference, however, is that unlike in the previous case now the torsional four-form anomaly polynomial is proportional to $d\mathcal{H}$ and does not couple at all to the gauge fields. As a consequence of this, the contributions from right- and left-handed fermions cancel each other. This is the reason why here we will consider a single right-handed fermion, so the anomaly polynomial takes the form

$$\mathcal{P}(\mathcal{F}_R, \mathcal{H}) = -\frac{1}{4\pi} \mathcal{F}_R^2 + \frac{c_H}{2} d\mathcal{H}, \tag{5.1}$$

where c_H has again dimensions of $(\text{energy})^2$. The effective action constructed by integrating the corresponding Chern-Simons form $\omega_3^0(\mathcal{A}_R, \mathcal{H})$ over a three-dimensional manifold with boundary reads

$$\Gamma[\mathcal{A}_R, \mathcal{H}]_{\text{CS}} = \int_{\mathcal{M}_3} \left(-\frac{1}{4\pi} \mathcal{A}_R \mathcal{F}_R + \frac{c_H}{2} \mathcal{H} \right). \tag{5.2}$$

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Due to the peculiar form of the torsional contribution, the two-dimensional consistent anomaly does not include torsion-dependent terms

$$d\langle\star\mathcal{J}_R\rangle_{\text{cons}} = \frac{1}{4\pi}\mathcal{F}_R, \tag{5.3}$$

and neither does the right-handed BZ current

$$\langle\star\mathcal{J}_R\rangle_{\text{BZ}} = \frac{1}{4\pi}\mathcal{A}_R. \tag{5.4}$$

Thus, unlike in the four-dimensional case, the covariant anomaly does not pick up any torsional contributions either

$$d\langle\star\mathcal{J}_R\rangle_{\text{cov}} = \frac{1}{2\pi}\mathcal{F}_R, \tag{5.5}$$

a fact that can understood on general grounds by noticing the absence of a two-form counterpart of the Nieh-Yan term.

All this notwithstanding, the torsional term in the Chern-Simons effective action (5.2) does have effects on transport, as it will be seen once we compute the equilibrium partition function and derive the corresponding covariant currents. We repeat the analysis presented in section 3, taking into account that on $\mathcal{M}_3$ the magnetic component of the three-form $\mathcal{H}$ in eq. (3.15) is identically zero, $\boldsymbol{\mathcal{H}} = 0$, since maximal rank forms are purely electrical. The generic static two-dimensional metric, on the other hand, has the form

$$ds^2 = -u\otimes u + \boldsymbol{v}\otimes\boldsymbol{v}, \tag{5.6}$$

where $\boldsymbol{v}$ is a spatial one-form. Consistency with the vielbein e^a results in the constraints

$$\begin{aligned} \eta_{ab}\chi^a\chi^b &= -1, \\ \eta_{ab}\chi^a\boldsymbol{e}^b &= 0, \\ \eta_{ab}\boldsymbol{e}^a\otimes\boldsymbol{e}^b &= \boldsymbol{v}\otimes\boldsymbol{v}. \end{aligned} \tag{5.7}$$

The first two identities lead again to the equilibrium constraint (3.23). Notice that in this case the constraints (5.7) are solved by $\chi^a = (\pm 1, 0)$ and $\boldsymbol{e}^a = (0, \boldsymbol{v})$.

Focusing on the torsional part of the Chern-Simons form

$$\omega_3^0(\mathcal{H})_H = \frac{c_H}{2}\mathcal{H}, \tag{5.8}$$

and keeping in mind that in three dimensions $\mathcal{H} = -u(\boldsymbol{e}_a E^a + \chi_a\boldsymbol{B}^a)$ [cf. (3.15)], we find that the torsional part of the equilibrium partition function has no boundary piece

$$W_{\text{eq}} = W_{\text{bulk}} = -\frac{c_H}{2}\int\limits_{\mathcal{M}_3} u\Big(\boldsymbol{e}_a E^a + \chi_a\boldsymbol{B}^a\Big). \tag{5.9}$$

The form of the bulk partition function shows that there are no torsional contributions either to $\langle\star\boldsymbol{J}_R\rangle_{\text{cov}}$ or the covariant spin current $\langle\star\boldsymbol{\mathfrak{S}}_a{}^b\rangle_{\text{cov}}$. As for the heat and stress

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currents, we take again variations with respect to χ^a, e^a, and u, picking up the resulting boundary terms. Since there are no explicit terms depending on du, we find

$$\delta W_{\text{bulk}} = -\frac{c_H}{2} \int\limits_{\partial\mathcal{M}_3} \Big(\delta e^a \chi_a u - \delta\chi^a \boldsymbol{e}_a u\Big). \tag{5.10}$$

As in four dimensions, we have to take into account the constraint (4.7). Eliminating $\delta\chi^a \boldsymbol{e}_a u$, we get

$$\delta W_{\text{bulk}} = \frac{c_H}{2} \int\limits_{\mathcal{M}_3} \Big(\delta u u - 2\delta e^a \chi_a u\Big). \tag{5.11}$$

From here we arrive at very simple expressions for both the covariant heat and stress currents

$$\begin{aligned} \langle \star \boldsymbol{q} \rangle_{\text{cov}} &= \frac{c_H}{2} u, \\ \langle \star \boldsymbol{\mathfrak{F}}_a \rangle_{\text{cov}} &= -c_H u \chi_a. \end{aligned} \tag{5.12}$$

Moreover, using the first constraint in (5.7), we derive a suggestive relation between the torsional contributions to the stress and heat currents

$$\langle \boldsymbol{\mathfrak{F}}^a \rangle_{\text{cov}} = -2\chi^a \langle \boldsymbol{q} \rangle_{\text{cov}}, \tag{5.13}$$

where Hodge duals have been dropped on both sides. All these torsional terms in the constitutive relations of the various currents should be added to any other torsion-independent contributions, such as the ones studied in [45, 46].

6 Closing remarks

In this paper we have studied the effects of the Nieh-Yan anomaly on the constitutive relations of an electron fluid axially coupled to an external gauge field and in the presence of torsion. Beginning with the six-dimensional anomaly polynomial, we carried out the descent analysis to write the boundary (local) and bulk (nonlocal) contributions to the equilibrium partition function. The covariant currents were then computed by varying the latter with respect to the appropriate external sources.

Our results show the existence of Nieh-Yan-induced terms in the various covariant currents. To begin with, we pointed out the existence of chiral separation effect driven by the magnetic part of the torsion, the vorticity, and the spin chemical potential. In the presence of torsion, the two latter terms are independent, but cancel each other for torsionless backgrounds. None of the terms in the torsional constitutive relations for the axial-vector current actually depends on chiral imbalance.

As for the heat and stress currents, their expressions are particularly simple in the limit in which the magnetic part of the axial-vector gauge field vanishes: they are proportional to the axial-vector and the magnetic torsion respectively. The spin current, on the other hand, is independent of torsion and proportional to the chemical potential governing chiral

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imbalance, as well as to the Nieh-Yan energy scale c_H. We have also analyzed the two-dimensional case and found that the stress current is proportional to the heat current.

Our result for the covariant axial-vector current of the $(3+1)$-dimensional theory in (4.2) has some bearings on the results of ref. [27]. It seems in principle incompatible with the ansatz for the current used in this reference, namely

$$\begin{aligned}\langle \star \boldsymbol{J}_5\rangle_{\text{cov}} &= c_V u du + c_T^{\parallel} u_a u_b T^a e^b + c_T^{\perp} P_{ab} T^a e^b \\ &= c_T^{\perp} \boldsymbol{B}_a \boldsymbol{e}^a + u\Big(c_V du + c_T^{\parallel} \chi_a \boldsymbol{B}^a - c_T^{\perp} \boldsymbol{e}_a E^a\Big),\end{aligned} \tag{6.1}$$

where $P_{ab} = \eta_{ab} + u_a u_b$ is the projector onto the hypersurfaces orthogonal to the four-velocity. In addition, in writing the second line we have used the constraint (3.23) in the form $u_a = \chi_a$, which makes some terms vanish after implementing the first two identities in (3.22). It is clear that there are no values of the coefficients $c_T^{\parallel}$ and $c_T^{\perp}$ that can reproduce the expression found in eq. (4.2).

This apparent problem is clarified once we remember that the presence of the four-velocity u breaks the invariance of our static background spacetime (3.21) down to the subgroup preserving this vector. This means that in writing the most general form of the current, tensor structures built from the electric and magnetic components of both the vierbein and the torsion have to be regarded as *independent*. To be more specific, we have to take a general linear combination of the longitudinal and transverse projections of all three-form terms that are linear in the torsion and have the correct T-parity:[6] $\boldsymbol{e}^a\boldsymbol{B}^b$, $u\chi^a\boldsymbol{B}^b$, and $u\boldsymbol{e}^a E^b$. In addition, we implement the equilibrium constraint $u_a = \chi_a$, which implies $u_a\boldsymbol{e}^a = 0$ and $P_{ab}\boldsymbol{e}^b = \boldsymbol{e}_a$, as well as $u_a\chi^a = -1$ and $P_{ab}\chi^b = 0$. The consequence is that the structures $\boldsymbol{B}^a\boldsymbol{e}^b$ and $u\boldsymbol{e}^a E^b$ are transverse, whereas $u\chi^a\boldsymbol{B}^b$ is longitudinal. Thus, the most general structure of the axial-vector current has the form

$$\langle \star \boldsymbol{J}_5\rangle_{\text{cov}} = b_{TB}^{\perp} \boldsymbol{B}_a \boldsymbol{e}^a + u\Big(c_V du + c_{TB}^{\parallel} \chi_a \boldsymbol{B}^a + c_{TE}^{\perp} \boldsymbol{e}_a E^a\Big). \tag{6.2}$$

Alternatively, we can reach the same result for the structure of the covariant axial-vector current by taking into account that, being the spin chemical potential of the same order as the torsion, the most general expression of the current at equilibrium is a linear combination of the four structures: $\boldsymbol{B}^a\boldsymbol{e}_a$, udu, $u\chi^a\boldsymbol{B}_a$, and $u\mu^a{}_b\boldsymbol{e}_a\boldsymbol{e}^b$. We have again a total of four independent coefficients, which are linear combinations of the ones in eq. (6.2).

Comparing the ansatz (6.2) with the one in (6.1), we find that the reduced symmetry does not force the identification of the coefficients $b_{TB}^{\perp}$ and $-c_{TE}^{\perp}$, as it is was the case there. This additional freedom is however crucial, since our explicit result for the axial-vector current (4.2) exhibits precisely the structure shown in (6.2), with the following values of the coefficients

$$\begin{aligned}b_{TB}^{\perp} &= 0, \\ c_{TB}^{\parallel} &= c_{TE}^{\perp} = -c_H.\end{aligned} \tag{6.3}$$

[6] We recall that $\boldsymbol{e}^a$ and $\boldsymbol{B}^a$ are T-even, whereas u, χ^a and E^a are T-odd. In addition, the current $\boldsymbol{J}_5$ is T-odd, which implies that its Hodge dual $\star\boldsymbol{J}_5$ is T-even.

Notice that in addition to the explicit term proportional to the vorticity in (6.2), there are similar contributions coming from $u\chi^a \boldsymbol{B}_a$ and $u\mu^a{}_b \boldsymbol{e}_a \boldsymbol{e}^b$, after applying the equilibrium condition (4.3). In fact, the vortical term in (4.4) entirely comes from implementing the thermal equilibrium constraint (4.3) and it is therefore induced by the very presence of torsion (or, in other words, our results show that $c_V = 0$).

The analysis presented here has provided us with a list of terms induced by the Nieh-Yan anomaly in the constitutive relations for the different currents, and therefore with a series of potentially new transport effects associated with an effective background torsion. Notice, however, that the values of all the corresponding transport coefficients are proportional to the global normalization of the torsional part of the anomaly polynomial c_H whose value, as we discussed above, depends on the UV details of the concrete models. Given the relevance of torsion for the effective description of condensed matter systems, it would be interesting to go beyond a general analysis and study these effects in specific models where a quantitative estimation of c_H is possible. This would allow to make precise predictions as to the possible experimental signatures to be expected in realistic materials. These and other issues will be addressed elsewhere.

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